GRAVITATIONAL LENSING

18 - AXIALLY SYMMETRIC LENSES

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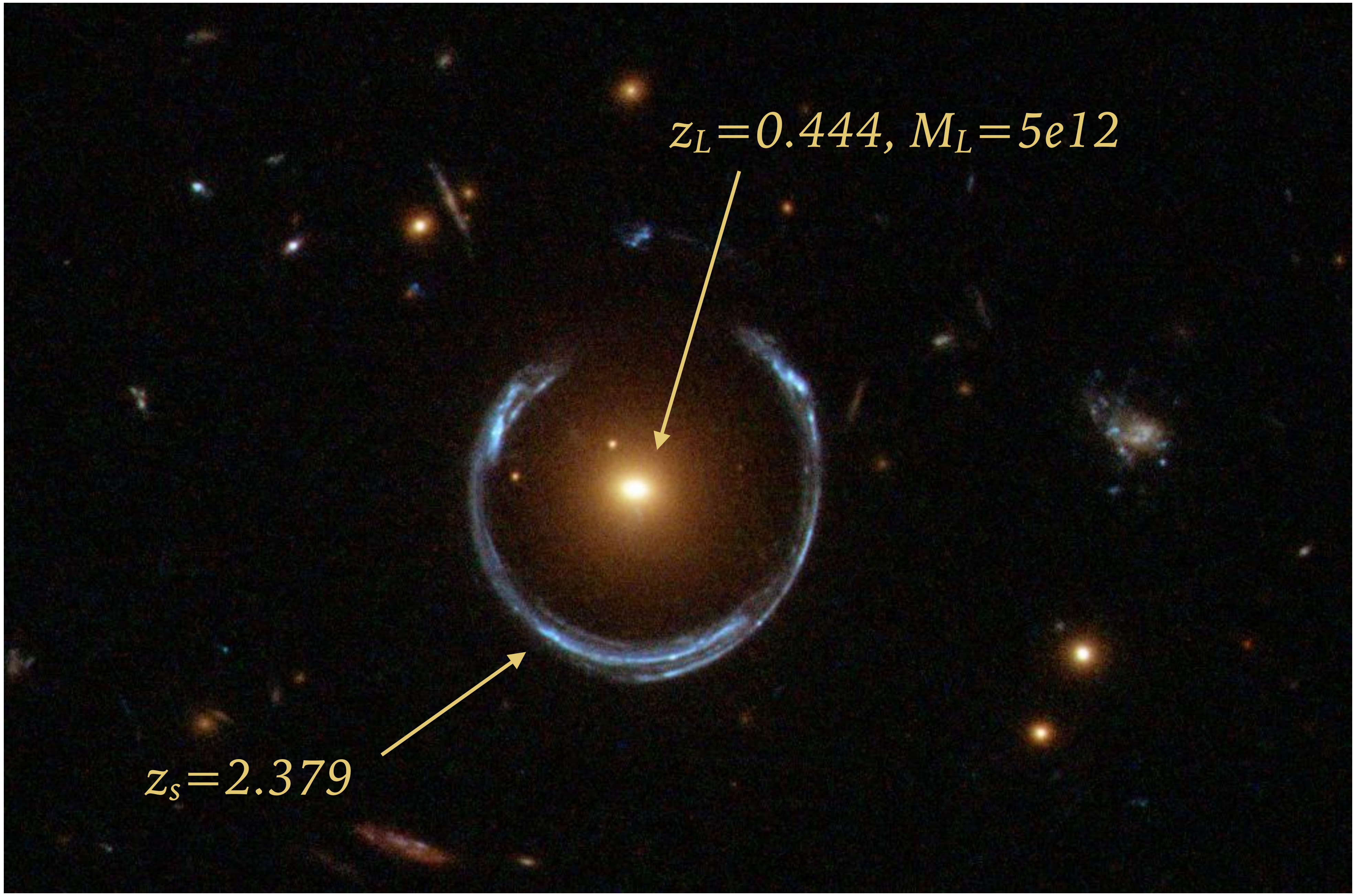
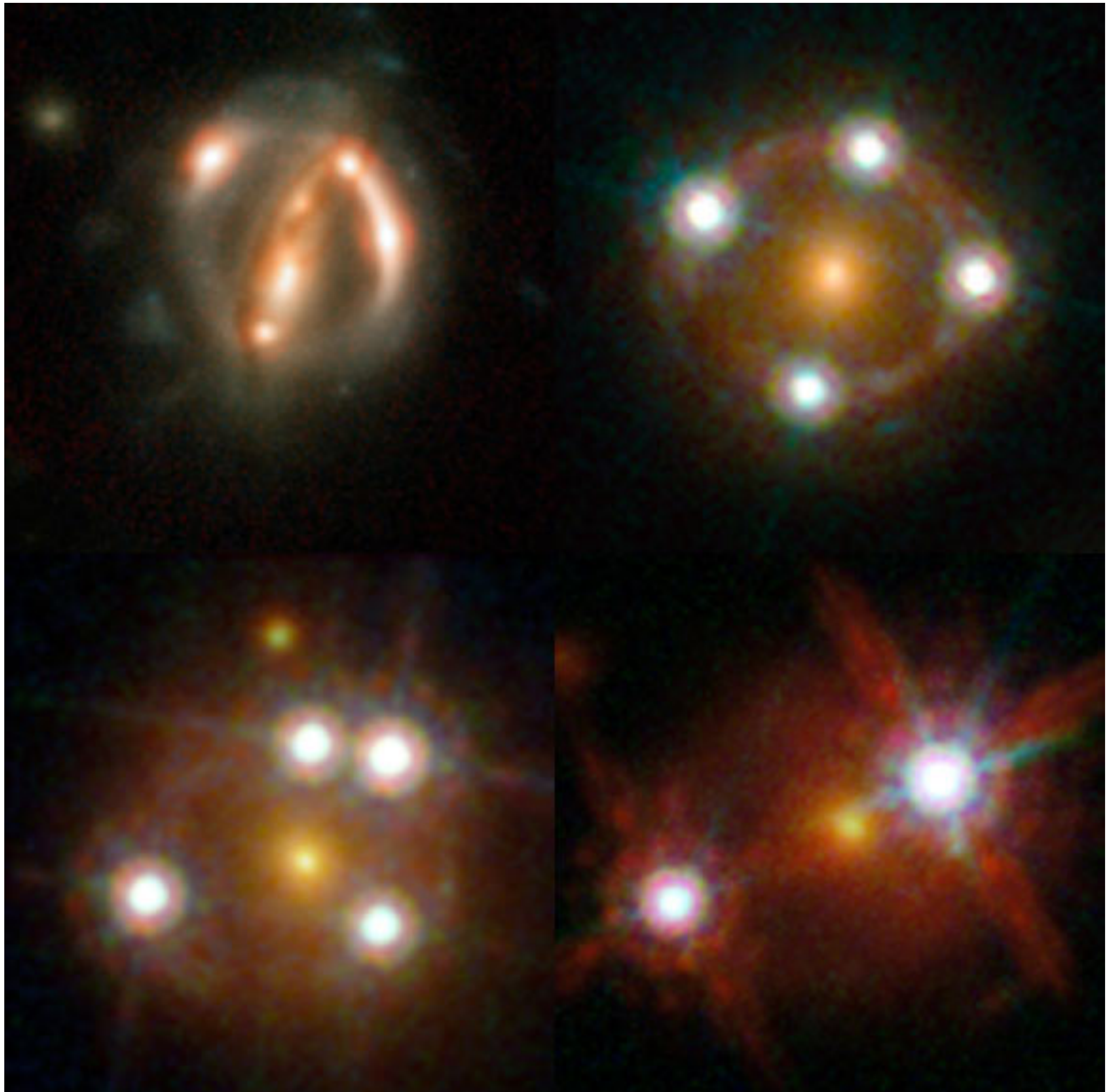
EXTENDED LENSES

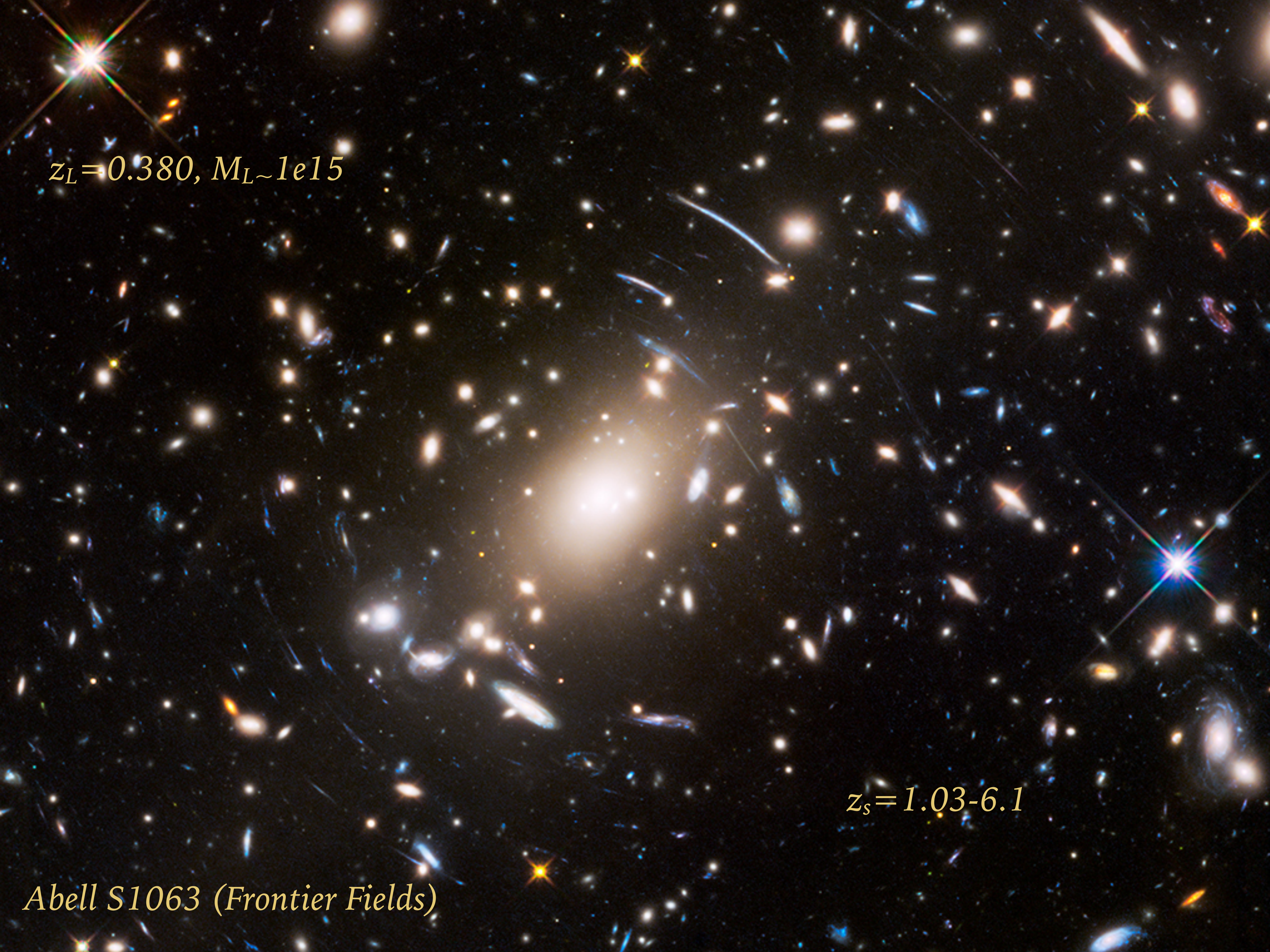


Cosmic horseshoe (Belokurov et al. 2007)

EXTENDED LENSES



Suyu et al. (H0LiCOW team)


$z_L = 0.380$, $M_L \sim 1e15$
$z_s = 1.03\text{-}6.1$
Abell S1063 (Frontier Fields)

EXTENDED LENSES

- Cosmic structures like galaxies and galaxy clusters are characterized by bound mass distributions, which cannot be approximated by point lenses
- Indeed these are *extended lenses*, and their lensing properties are determined by e.g. their surface mass density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2}\ \mathrm{d}^2\xi'$$

EXTENDED LENSES

- Recall that the surface density is related to the lensing potential by

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

WHAT ARE THE RELEVANT PROPERTIES OF THE LENSES?

- The surface density distribution of a lens (and its potential) can be characterized by means of
  - the profile
  - the shape of the iso-density (iso-potential) contours
  - the smoothness
  - the environment where the lens resides
- In this and in the following lessons, we will study how these features determine the ability of a mass distribution to produce lensing effects.
- We will do that by building analytical models with increasing level of complexity.

AXIALLY SYMMETRIC, CIRCULAR LENSES

- Axially symmetric, circular models are the simplest lens models for describing extended mass distributions
- For these lenses $\hat{\Psi}(\vec{\theta}) = \hat{\Psi}(\theta)$
- Several quantities relevant for lensing can be derived in a simple manner by using the symmetry properties of the lens.
- One example is the deflection angle…

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\nabla}_\theta \equiv D_\mathrm{L}\left(\frac{\partial}{\partial\xi}\vec{e}_\xi + \frac{1}{\xi}\frac{\partial}{\partial\phi}\vec{e}_\phi\right) = \left(\frac{\partial}{\partial\theta}\vec{e}_\theta + \frac{1}{\theta}\frac{\partial}{\partial\phi}\vec{e}_\phi\right)$$

$$\nabla_\theta\hat{\Psi}(\vec{\theta}) = \hat{\Psi}'(\theta)\vec{e}_\theta = \vec{\alpha}(\vec{\theta}) = \alpha(\theta)\vec{e}_\theta$$

For an axially symmetric lens, the deflection is "radial": it depends only on the distance from the lens center.

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\nabla_\theta^2 = \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

From this equation, we obtain

$$\begin{aligned}
\alpha(\theta) &= \frac{2\int_0^\theta \kappa(\theta')\theta' d\theta'}{\theta} \\
&= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\mathrm{cr}}} \\
&= \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\frac{4GM(\theta)}{c^2 D_{\mathrm{L}}\theta} \\
&= \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\alpha}(\theta)\,.
\end{aligned}$$

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\nabla_\theta^2 = \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\theta^2}\frac{\partial^2}{\partial\phi^2}$$

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\alpha(\theta) &= \frac{2\int_0^\theta \kappa(\theta')\theta' d\theta'}{\theta} \\
&= \frac{2\int_0^\theta \Sigma(\theta')\theta' d\theta'}{\theta\Sigma_{\rm cr}} \\
&= \frac{D_{\rm LS}}{D_{\rm S}}\frac{4GM(\theta)}{c^2 D_{\rm L}\theta} \\
&= \frac{D_{\rm LS}}{D_{\rm S}}\hat{\alpha}(\theta)\,.
\end{aligned}$$

Identical to point-mass lens!

DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

Dimensionless form:

$$
\begin{aligned}
\alpha(x) &= \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}} \hat{\alpha}(\xi_0 x) \\
&= \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}} \frac{4GM(\xi_0 x)}{c^2 \xi} \frac{\pi \xi_0}{\pi \xi_0} \\
&= \frac{M(\xi_0 x)}{\pi \xi_0^2 \Sigma_{\mathrm{cr}}} \frac{1}{x} \equiv \frac{m(x)}{x} \,,
\end{aligned}
$$

Dimensionless mass

$$
\alpha(x) = \frac{2}{x} \int_0^x x' \kappa(x') \mathrm{d}x' \Rightarrow m(x) = 2 \int_0^x x' \kappa(x') \mathrm{d}x'
$$

LENS EQUATION

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \qquad\qquad \vec{\alpha}(\vec{x}) = \frac{m(\vec{x})}{x^2}\vec{x}$$

Given that the deflection angle and x are parallel, so will be y!

$$y = x - \frac{m(x)}{x}$$

CONVERGENCE

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right)\hat{\Psi}(\theta) = 2\kappa(\theta)$$

$$\kappa(\theta) = \frac{1}{2}\left(\hat{\Psi}''(\theta) + \frac{\hat{\Psi}'(\theta)}{\theta}\right)$$

$$\hat{\Psi}'(\theta) = \alpha(\theta)$$

$$\kappa(\theta) = \frac{1}{2}\left(\alpha'(\theta) + \frac{\alpha(\theta)}{\theta}\right)$$

$$\alpha'(x) = \frac{m'(x)}{x} - \frac{m(x)}{x^2}$$

$$\kappa(x) = \frac{1}{2}\frac{m'(x)}{x}$$

SHEAR

The shear components are derived from the second derivatives of the potential or from the first derivatives of the deflection angle components:

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta_2} = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_1 = \alpha\cos\phi$$

$$\alpha_2 = \alpha\sin\phi$$

SHEAR

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

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$$\alpha_1 = \alpha\cos\phi$$

$$\alpha_2 = \alpha\sin\phi$$

$$\begin{aligned}
\gamma_1(\theta) &= \frac{1}{2}\left[\frac{\partial}{\partial\theta_1}\alpha_1(\theta) - \frac{\partial}{\partial\theta_2}\alpha_2(\theta)\right] \\
&= \frac{1}{2}\left[(\cos^2\phi - \sin^2\phi)\alpha'(\theta) - (\cos^2\phi - \sin^2\phi)\frac{\alpha(\theta)}{\theta}\right] \\
&= \frac{\cos 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] ,
\end{aligned}$$

SHEAR

$$\frac{\partial}{\partial\theta_1} = \cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_1 = \alpha\cos\phi$$

$$\frac{\partial}{\partial\theta_2} = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi}$$

$$\alpha_2 = \alpha\sin\phi$$

$$\begin{aligned}
\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta) \\
&= \left[\sin\phi\cos\phi\,\alpha'(\theta) - \sin\phi\cos\phi\,\frac{\alpha(\theta)}{\theta}\right] \\
&= \frac{\sin 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] .
\end{aligned}$$

SHEAR

$$\begin{aligned}\gamma_1(\theta) &= \frac{1}{2}\left[\frac{\partial}{\partial\theta_1}\alpha_1(\theta) - \frac{\partial}{\partial\theta_2}\alpha_2(\theta)\right] \\ &= \frac{1}{2}\left[(\cos^2\phi - \sin^2\phi)\alpha'(\theta) - (\cos^2\phi - \sin^2\phi)\frac{\alpha(\theta)}{\theta}\right] \\ &= \frac{\cos 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] ,\end{aligned}$$

$$\begin{aligned}\gamma_2(\theta) &= \frac{\partial}{\partial\theta_2}\alpha_1(\theta) \\ &= \left[\sin\phi\cos\phi\,\alpha'(\theta) - \sin\phi\cos\phi\,\frac{\alpha(\theta)}{\theta}\right] \\ &= \frac{\sin 2\phi}{2}\left[\alpha'(\theta) - \frac{\alpha(\theta)}{\theta}\right] .\end{aligned}$$

$$\begin{aligned}\alpha(x) &= \frac{D_\mathrm{L} D_\mathrm{LS}}{\xi_0 D_\mathrm{S}}\hat{\alpha}(\xi_0 x) \\ &= \frac{D_\mathrm{L} D_\mathrm{LS}}{\xi_0 D_\mathrm{S}}\frac{4GM(\xi_0 x)}{c^2\xi}\frac{\pi\xi_0}{\pi\xi_0} \\ &= \frac{M(\xi_0 x)}{\pi\xi_0^2\Sigma_\mathrm{cr}}\frac{1}{x} \equiv \frac{m(x)}{x} ,\end{aligned}$$

$$\alpha'(x) = \frac{m'(x)}{x} - \frac{m(x)}{x^2}$$

$$\begin{aligned}\gamma(x) &= \frac{1}{2}\left|\frac{m'(x)}{x} - \frac{2m(x)}{x^2}\right| \\ &= \left|\kappa(x) - \overline{\kappa}(x)\right| ,\end{aligned}$$

$$\overline{\kappa}(x) = \frac{m(x)}{x^2} = 2\pi\frac{\int_0^x x'\kappa(x')\mathrm{d}x'}{\pi x^2}$$

LENSING JACOBIAN

$$A = \left[1 - \frac{m'(x)}{2x}\right] I - \frac{1}{2}\left[\frac{m'(x)}{x} - \frac{2m(x)}{x^2}\right] \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$$

$$A = I + \frac{m}{x^2} C(\phi) - \frac{m'(x)}{2x} [I + C(\phi)]$$

$$\begin{aligned} C(\phi) &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} \end{aligned}$$

$$\begin{aligned} I + C(\phi) &= \begin{pmatrix} 1 + \cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1 - \cos 2\phi \end{pmatrix} \\ &= 2\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \end{aligned}$$

LENSING JACOBIAN

$$A = I + \frac{m}{x^2}C(\phi) - \frac{m'(x)}{2x}[I + C(\phi)]$$

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$$\begin{aligned} I + C(\phi) &= \begin{pmatrix} 1+\cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1-\cos 2\phi \end{pmatrix} \\ &= 2\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} A &= I + \frac{m(x)}{x^2}\begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} - \\ &\quad \frac{m'(x)}{x}\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}. \end{aligned}$$

LENSING JACOBIAN (CARTESIAN COORDINATES)

$$A = I + \frac{m(x)}{x^2}\begin{pmatrix} \cos^2\phi - \sin^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \sin^2\phi - \cos^2\phi \end{pmatrix} - \frac{m'(x)}{x}\begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}.$$

$$(x_1, x_2) = (x\cos\phi, x\sin\phi),$$

$$A = I + \frac{m(x)}{x^4}\begin{pmatrix} x_1^2 - x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_2^2 - x_1^2 \end{pmatrix} - \frac{m'(x)}{x^3}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}.$$

DETERMINANT OF THE LENSING JACOBIN

$$y = x - \frac{m(x)}{x}$$

$$\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] \ .
\end{aligned}$$

CRITICAL LINES

$$\begin{aligned}\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\ &= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\ &= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] \, .\end{aligned}$$

First critical line: $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

Second critical line: $\alpha'(x) = m'(x)/x - m/x^2 = 2\kappa(x) - \overline{\kappa}(x) = 1$

CRITICAL LINES

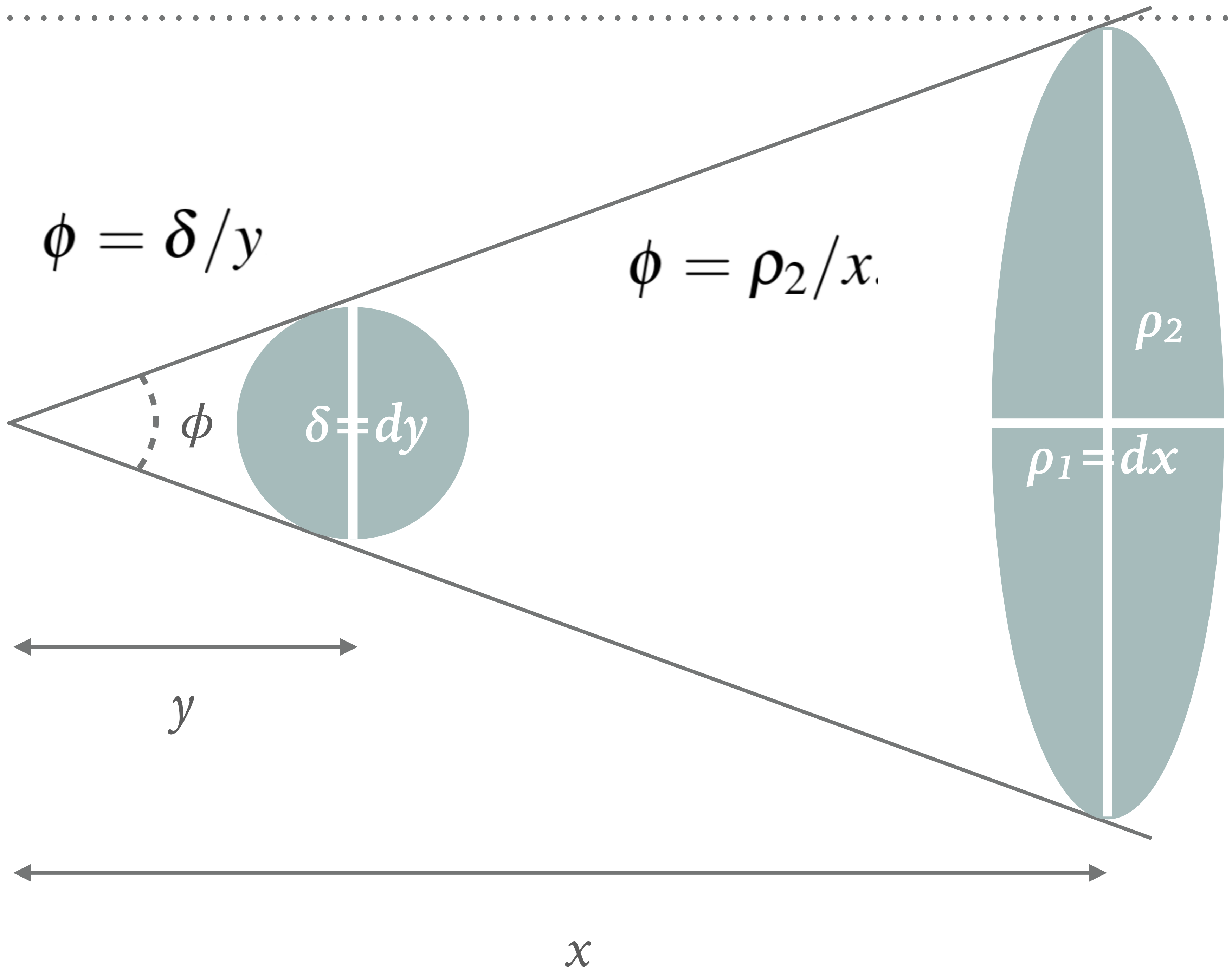
The tangential critical line occurs where $\alpha(x)/x = m(x)/x^2 = \overline{\kappa}(x) = 1$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L}D_{\rm LS}}$$

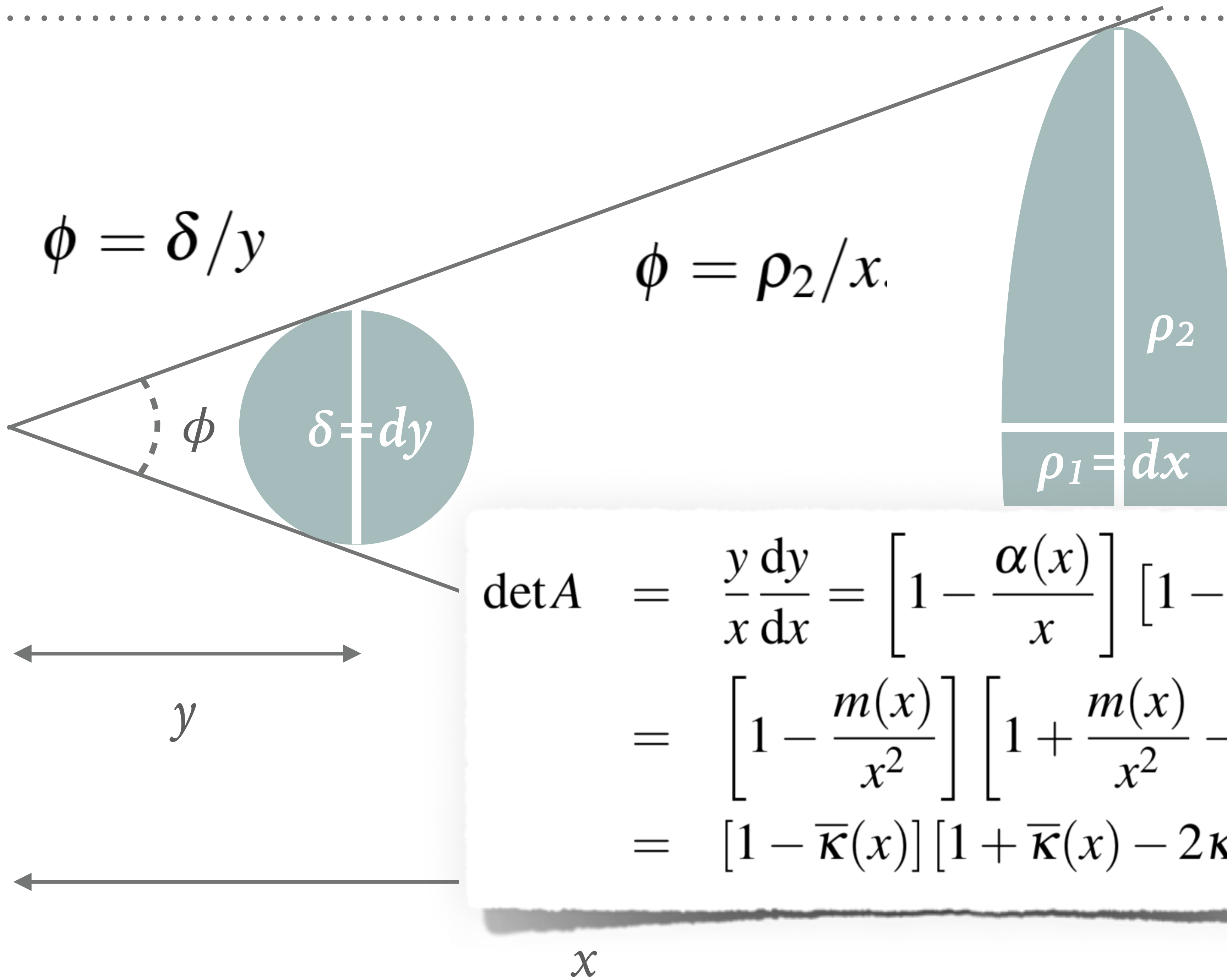
$$M(\theta_E) = \pi\Sigma_{\rm cr}\theta_E^2 D_{\rm L}^2$$

$$\theta_E = \sqrt{\frac{4GM(\theta_E)}{c^2}\frac{D_{\rm LS}}{D_{\rm L}D_{\rm S}}}$$

RADIAL AND TANGENTIAL MAGNIFICATION



RADIAL AND TANGENTIAL MAGNIFICATION

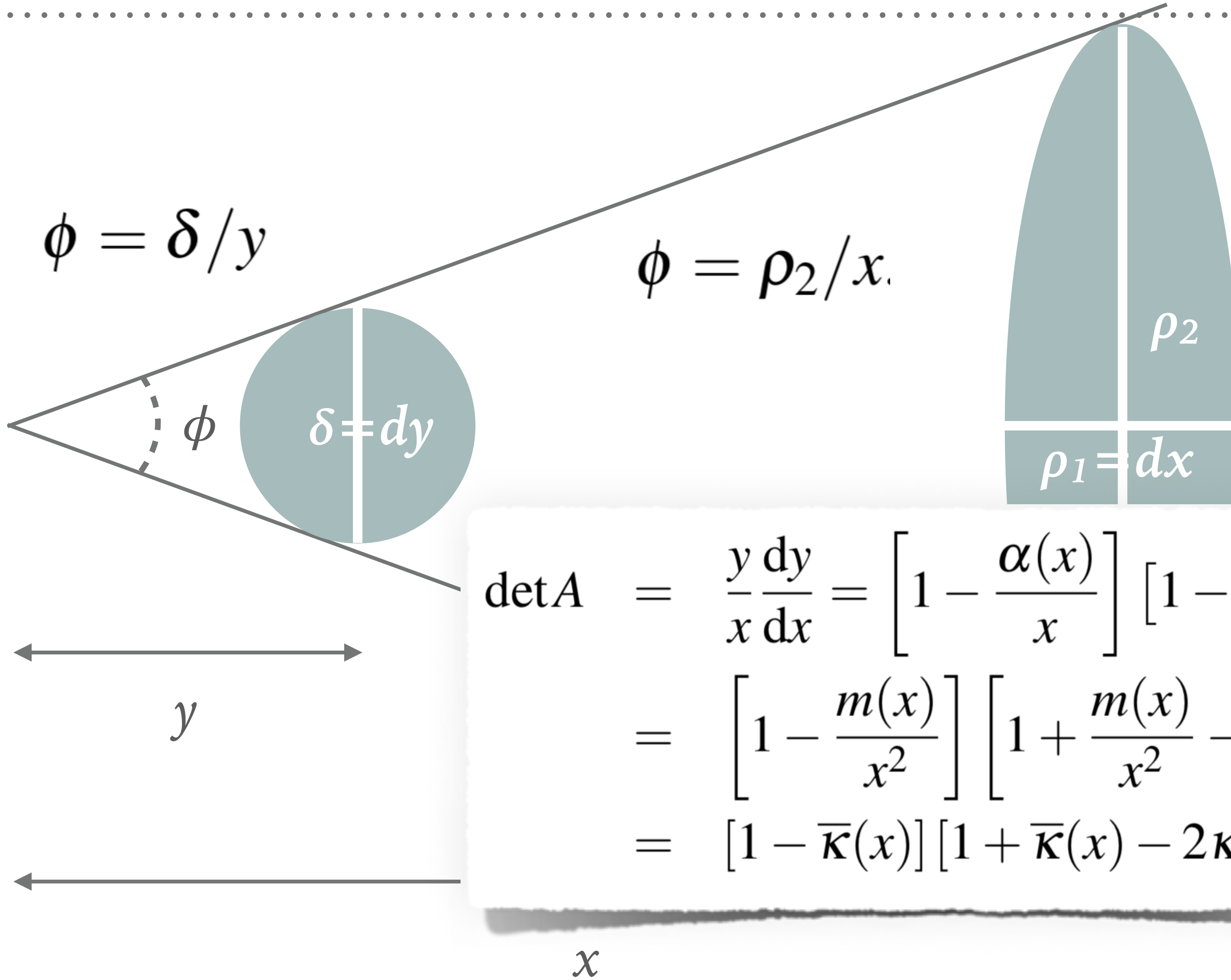


$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}
$$

RADIAL AND TANGENTIAL MAGNIFICATION

$$\phi = \delta / y$$

$$\phi = \rho_2 / x.$$



$$\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left[1 - \frac{\alpha(x)}{x}\right]\left[1 - \alpha'(x)\right] \\
&= \left[1 - \frac{m(x)}{x^2}\right]\left[1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x}\right] \\
&= \left[1 - \overline{\kappa}(x)\right]\left[1 + \overline{\kappa}(x) - 2\kappa(x)\right] .
\end{aligned}$$

$$\frac{\delta}{\rho_2} = 1 - \frac{m(x)}{x^2} \qquad \frac{\delta}{\rho_1} = 1 + \frac{m(x)}{x^2} - 2\kappa(x)$$