GRAVITATIONAL LENSING

17 - POWER LAW LENSES

Massimo Meneghetti

AA 2017-2018

SUMMARY FROM THE LAST LESSON

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\alpha(x) &= \frac{D_\mathrm{L} D_\mathrm{LS}}{\xi_0 D_\mathrm{S}} \hat{\alpha}(\xi_0 x) \\
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&= \frac{M(\xi_0 x)}{\pi \xi_0^2 \Sigma_\mathrm{cr}} \frac{1}{x} \equiv \frac{m(x)}{x} ,
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$\lambda_t(x)$ $\lambda_r(x)$

POWER-LAW MASS PROFILE

$$\boxed{m(x) = x^{3-n}}$$

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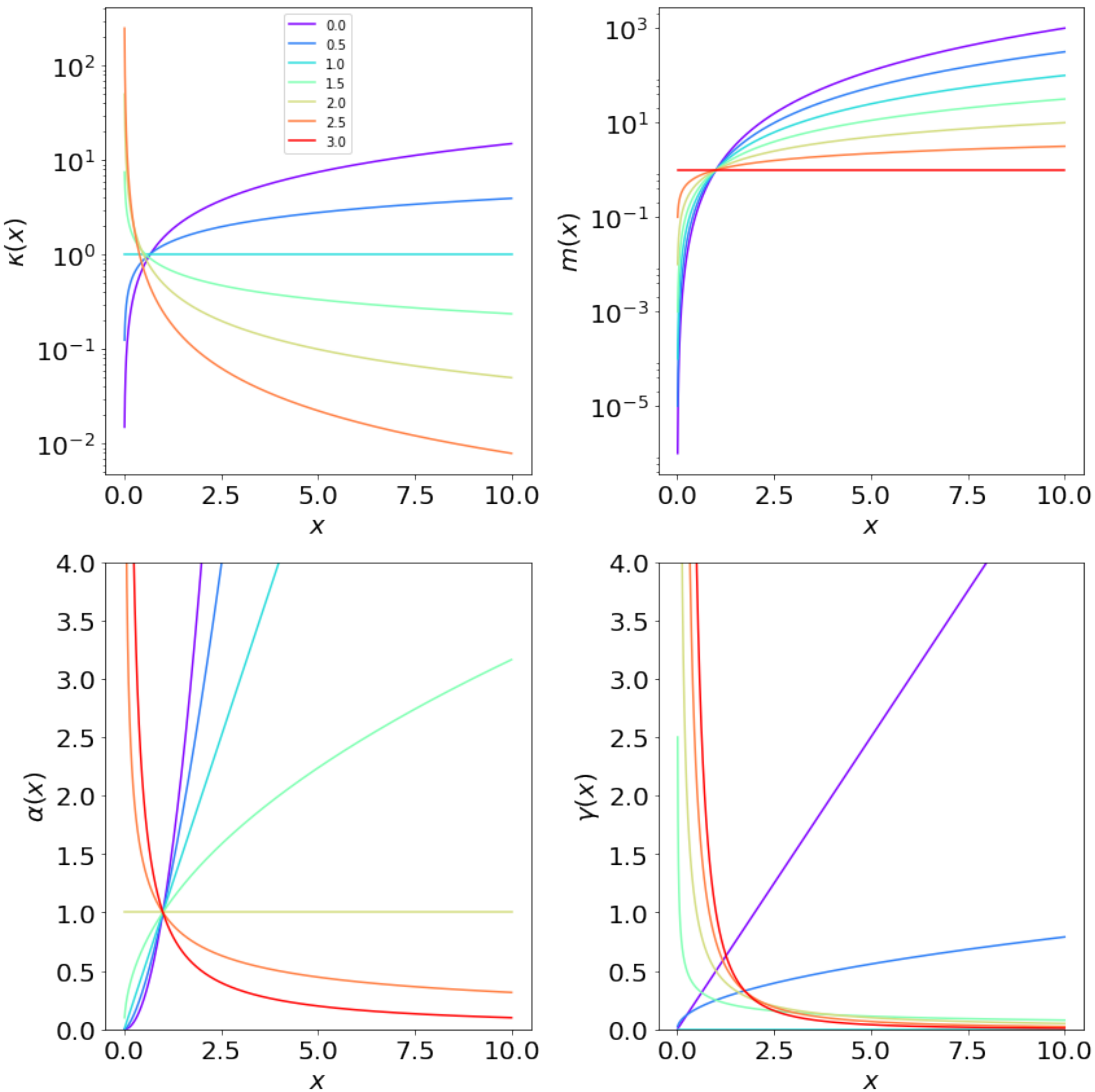
$$\gamma(x) = \frac{m(x)}{x^2} - \kappa(x) = \frac{n-1}{2}x^{1-n}$$

POWER-LAW LENS



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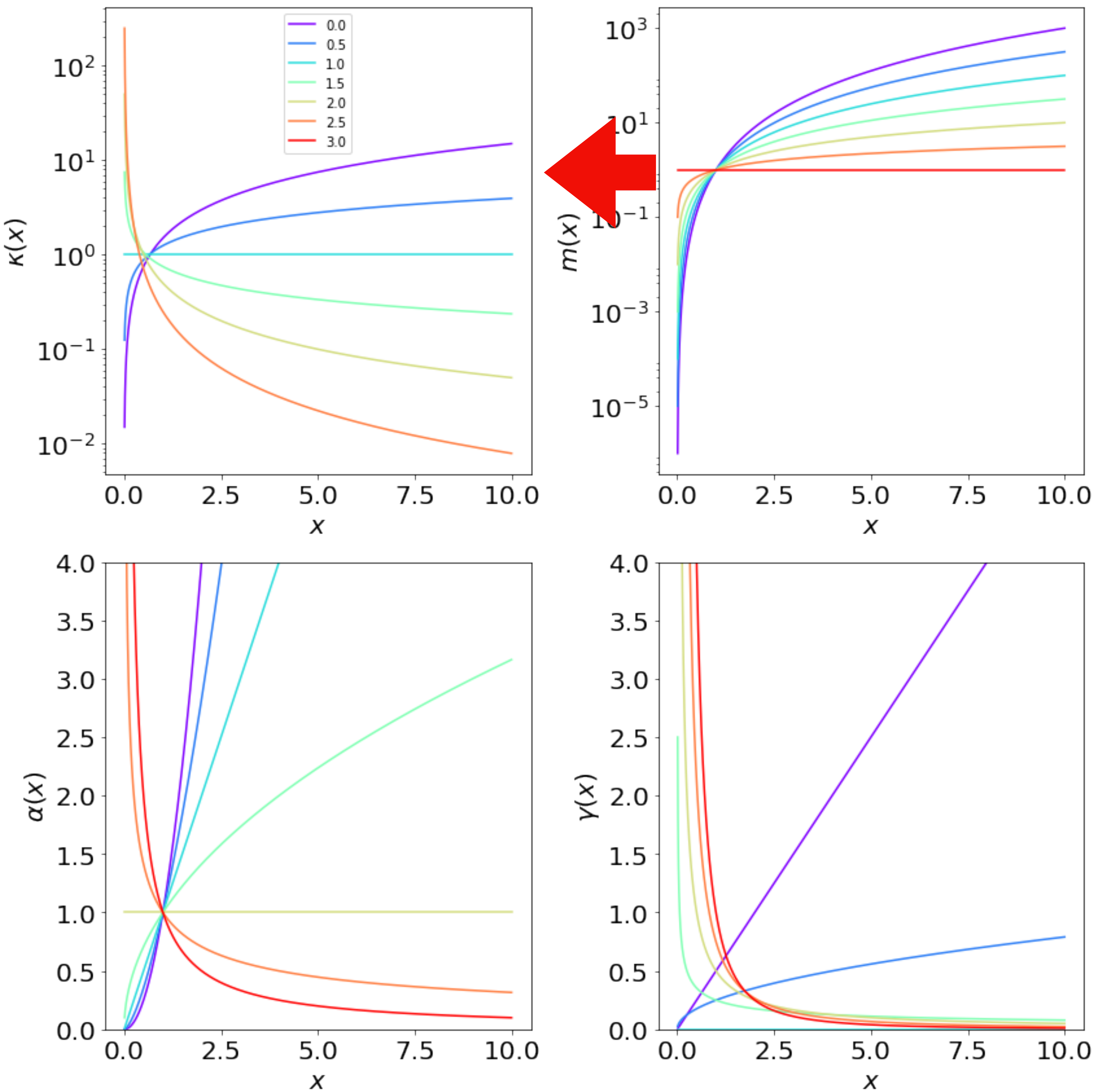

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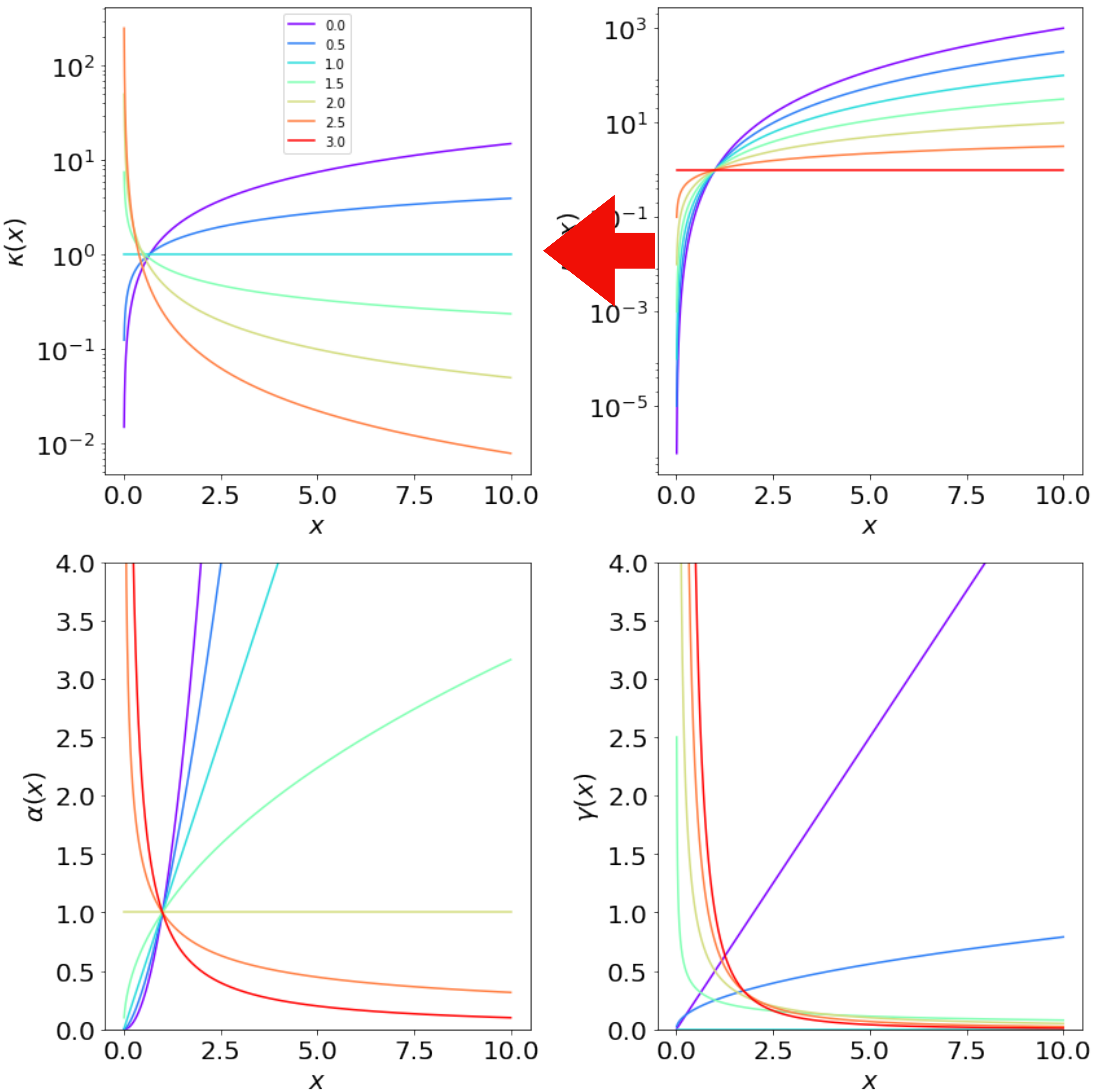

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- $n=1$: the convergence profile is flat;

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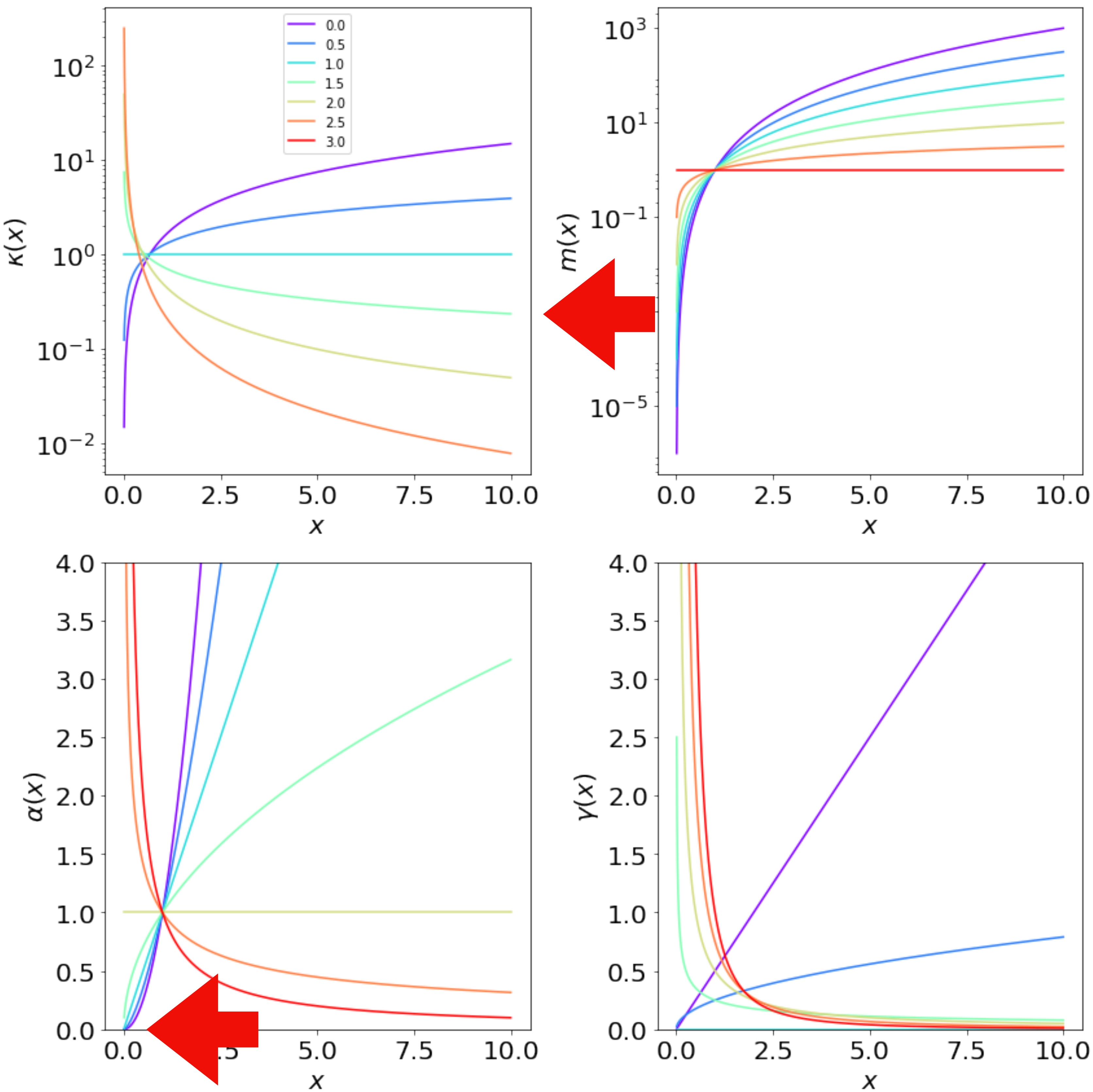

- $n<1$: the power-law lens has a monotonically increasing convergence profile;
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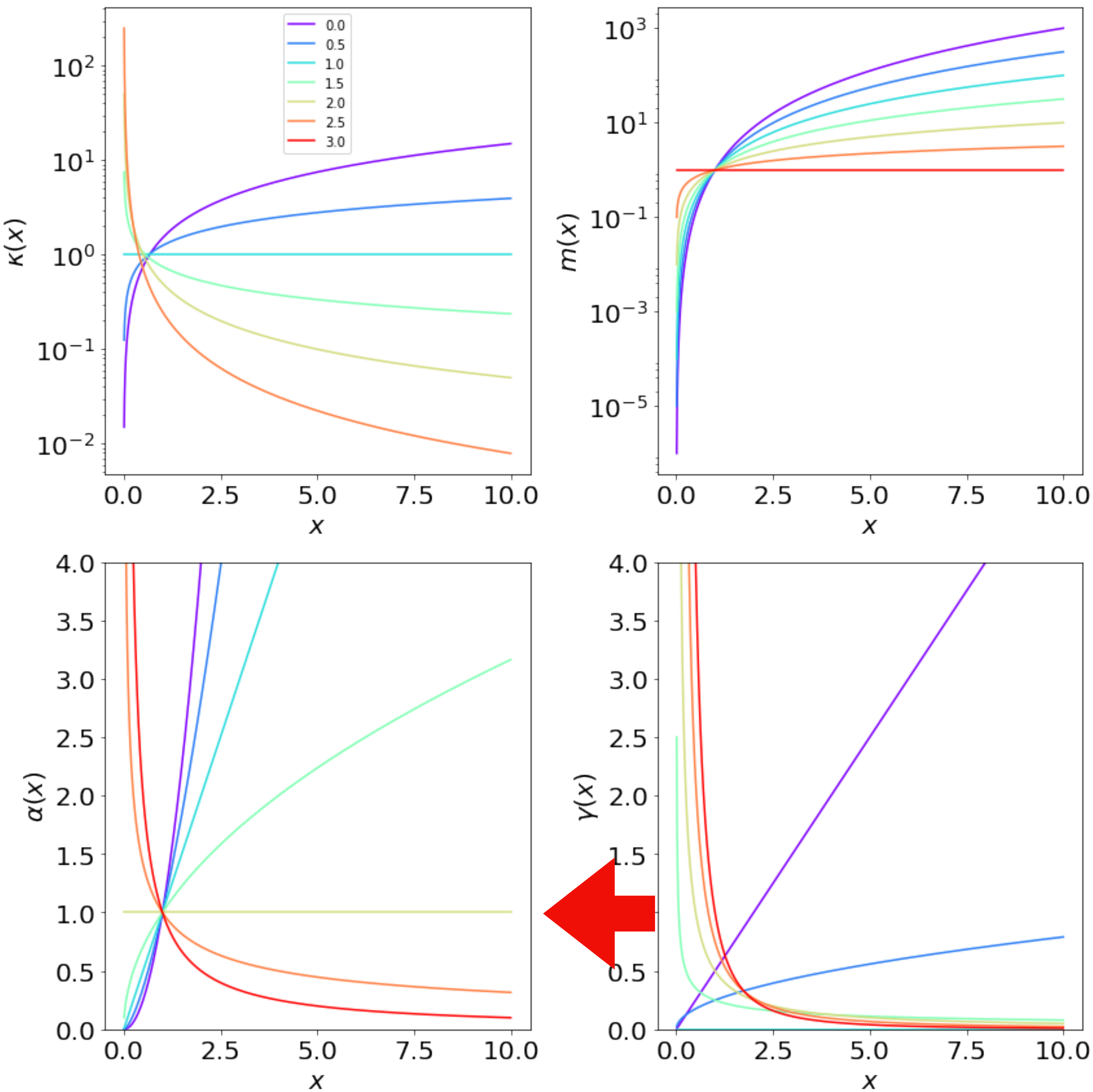

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- $n=2$: $\alpha(x)=const$;

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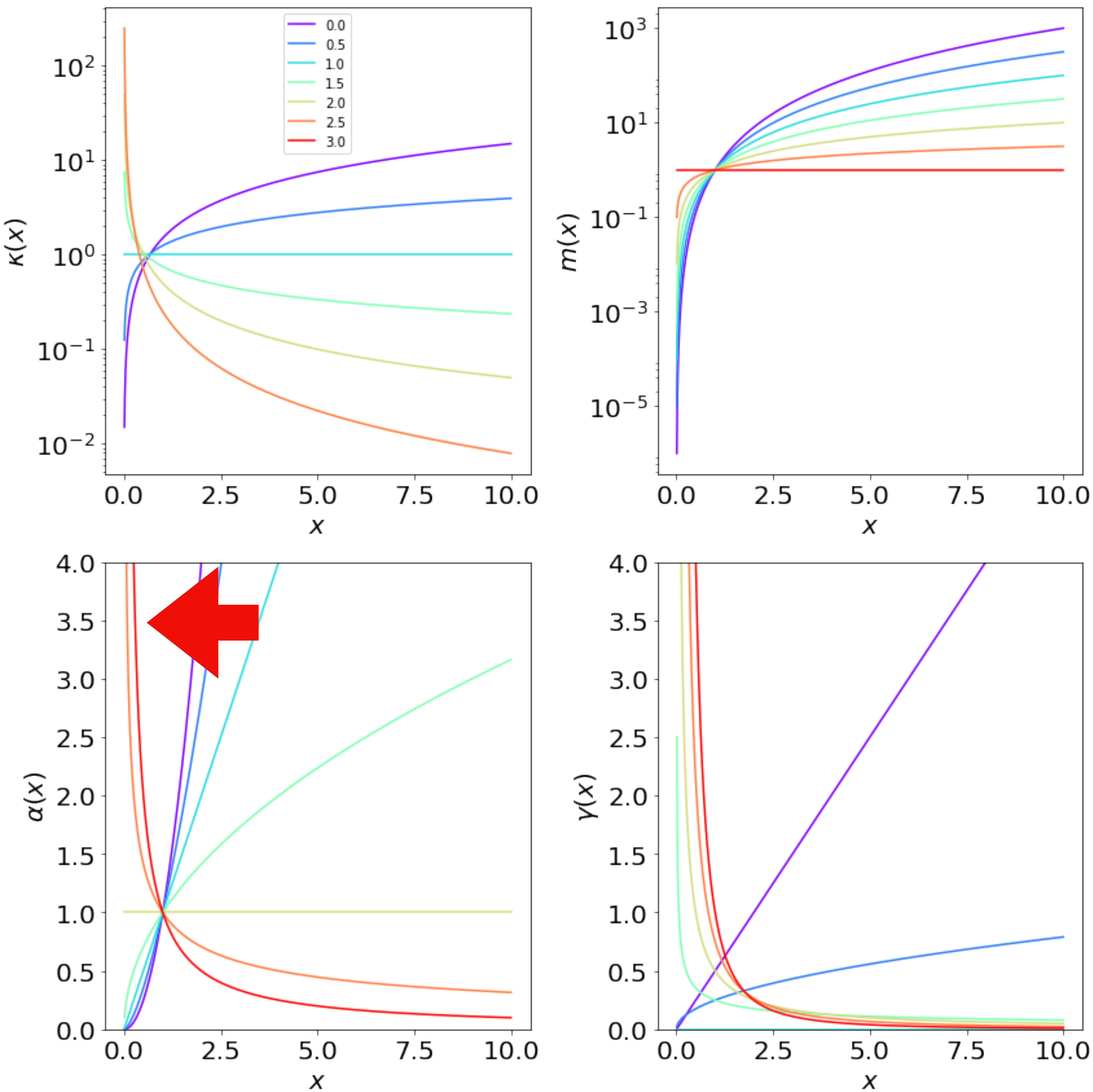

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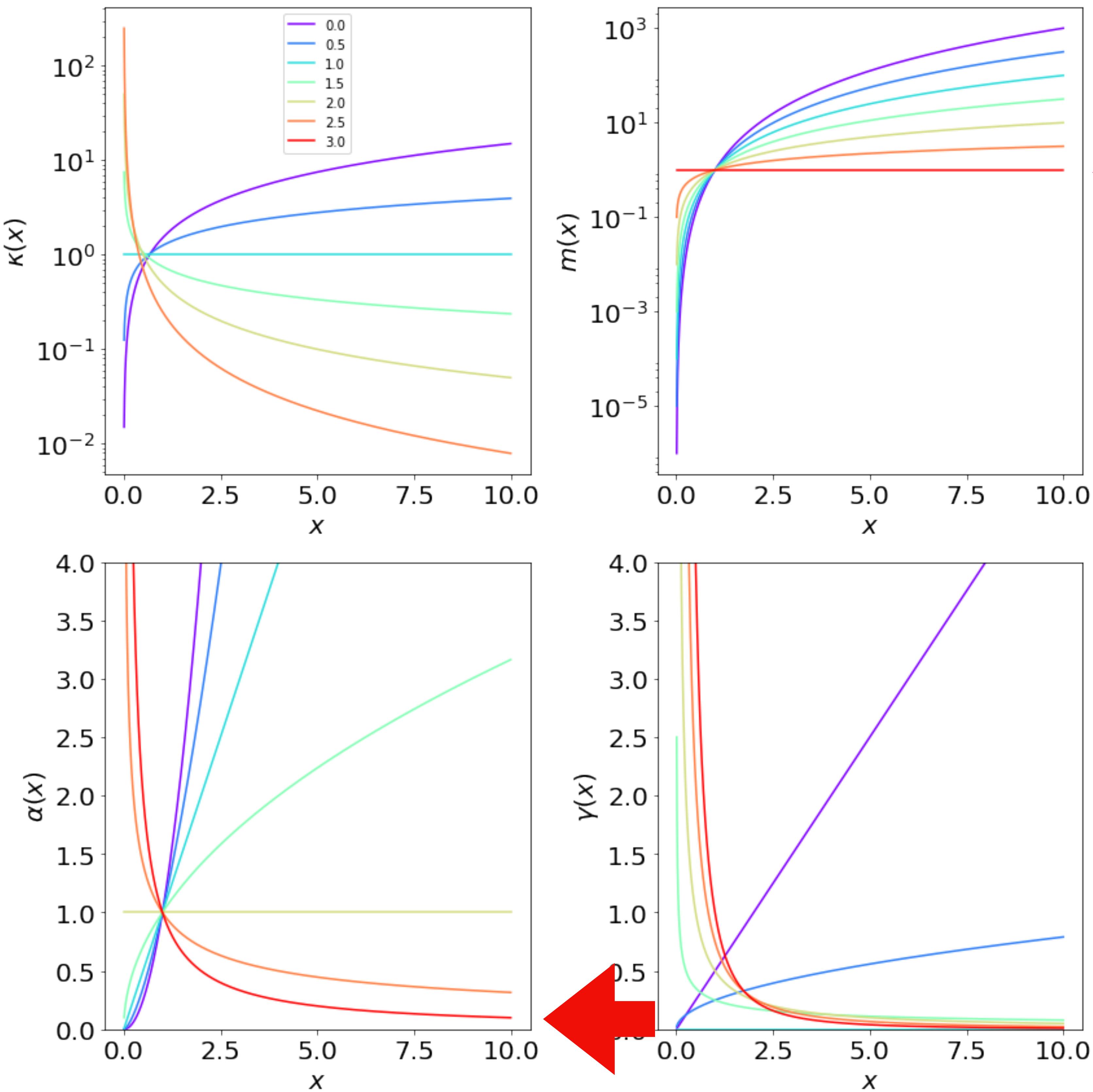

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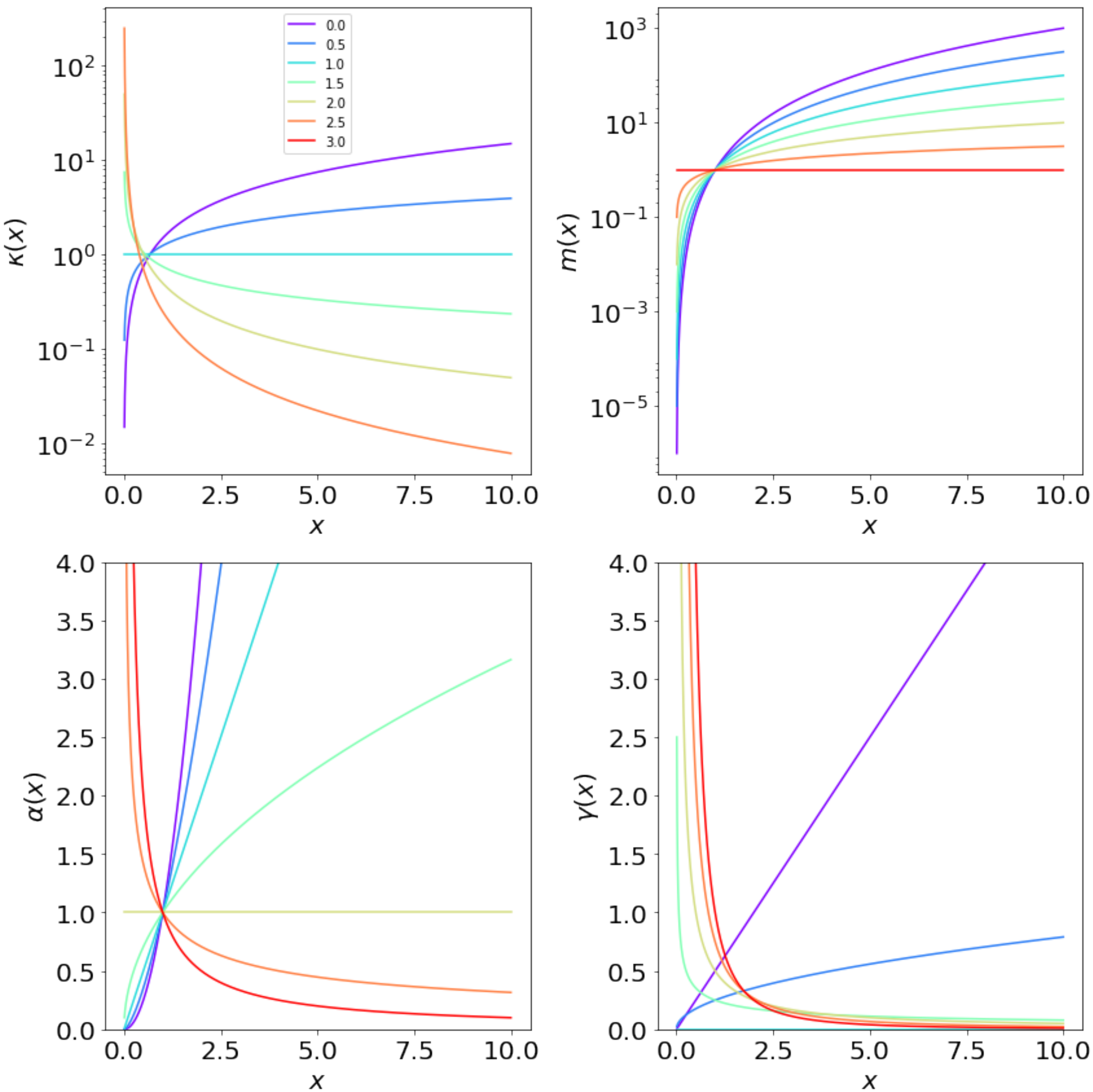

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- $n=3$: $m(x)=1$; $\alpha(x)=1/x$;
- $n>3$: $m(x)$ decreasing with x..

POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

The tangential critical line has equation x=1 for any value of the slope parameter n.

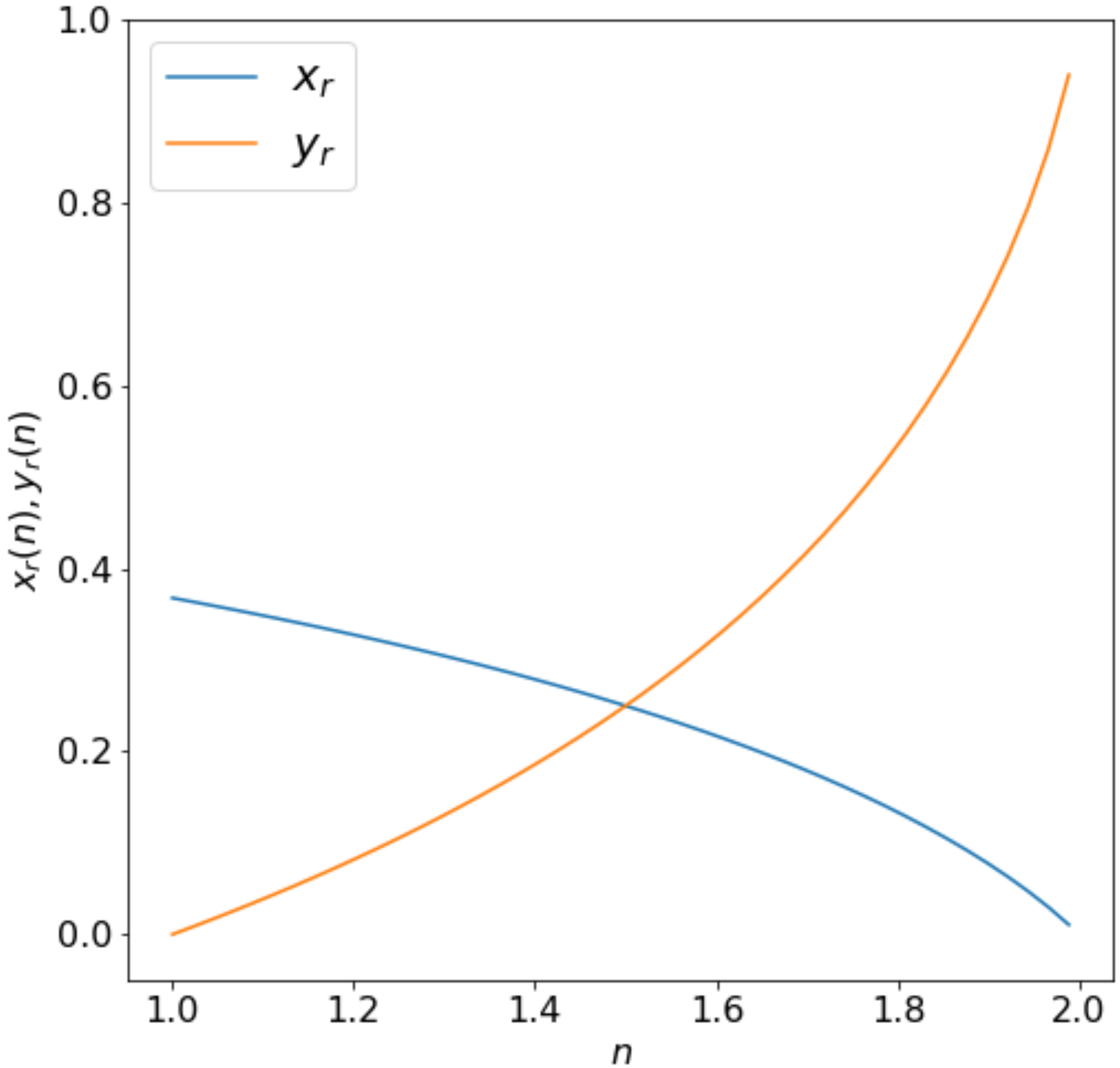
The caustic is the point y=0

Instead, the size of the radial critical line depends on n:

$$(2-n)x_r^{1-n} = 1$$

$$x_r = (2-n)^{1/(n-1)}$$

RADIAL CRITICAL LINE



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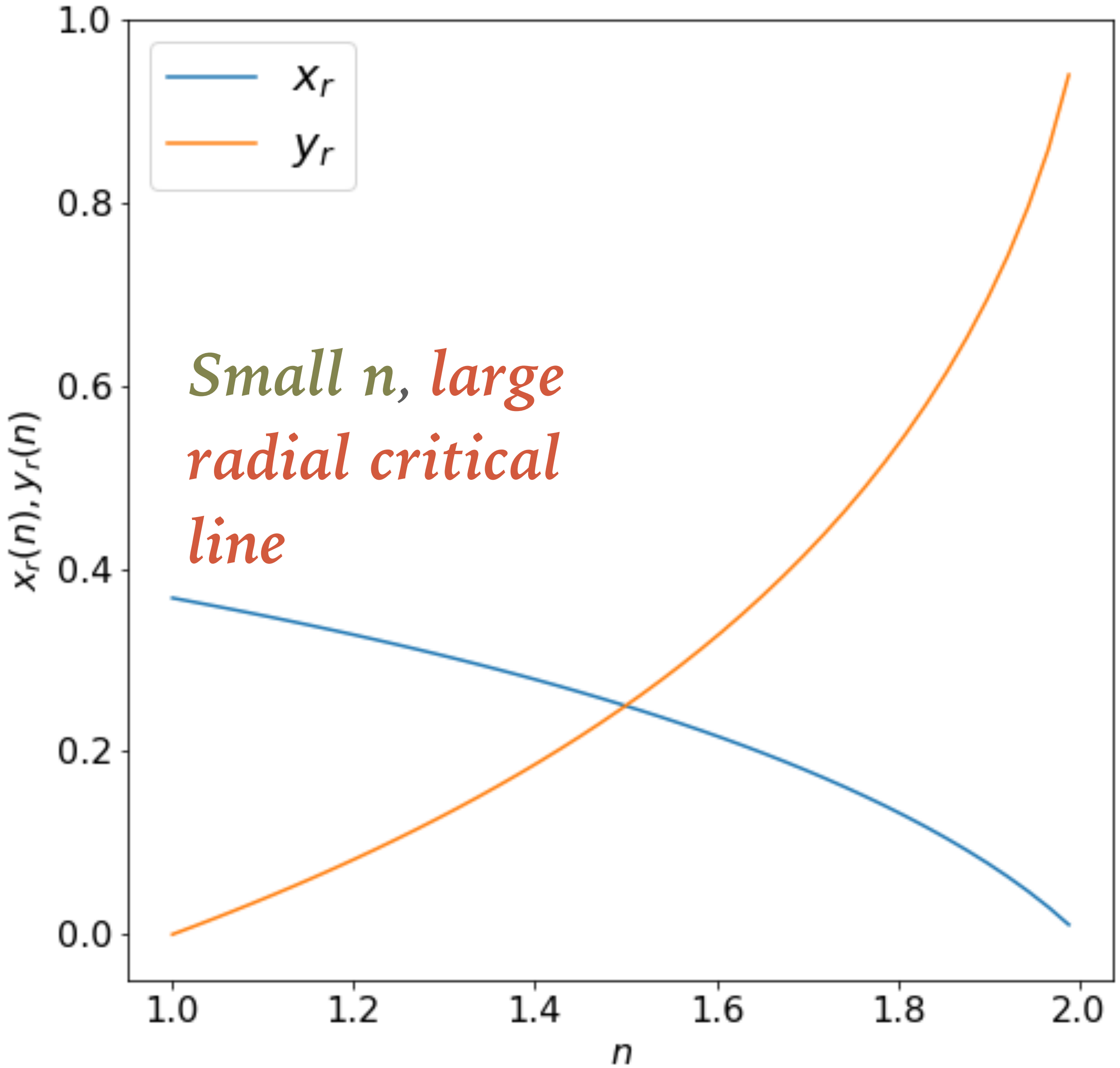

Large n, small radial critical line

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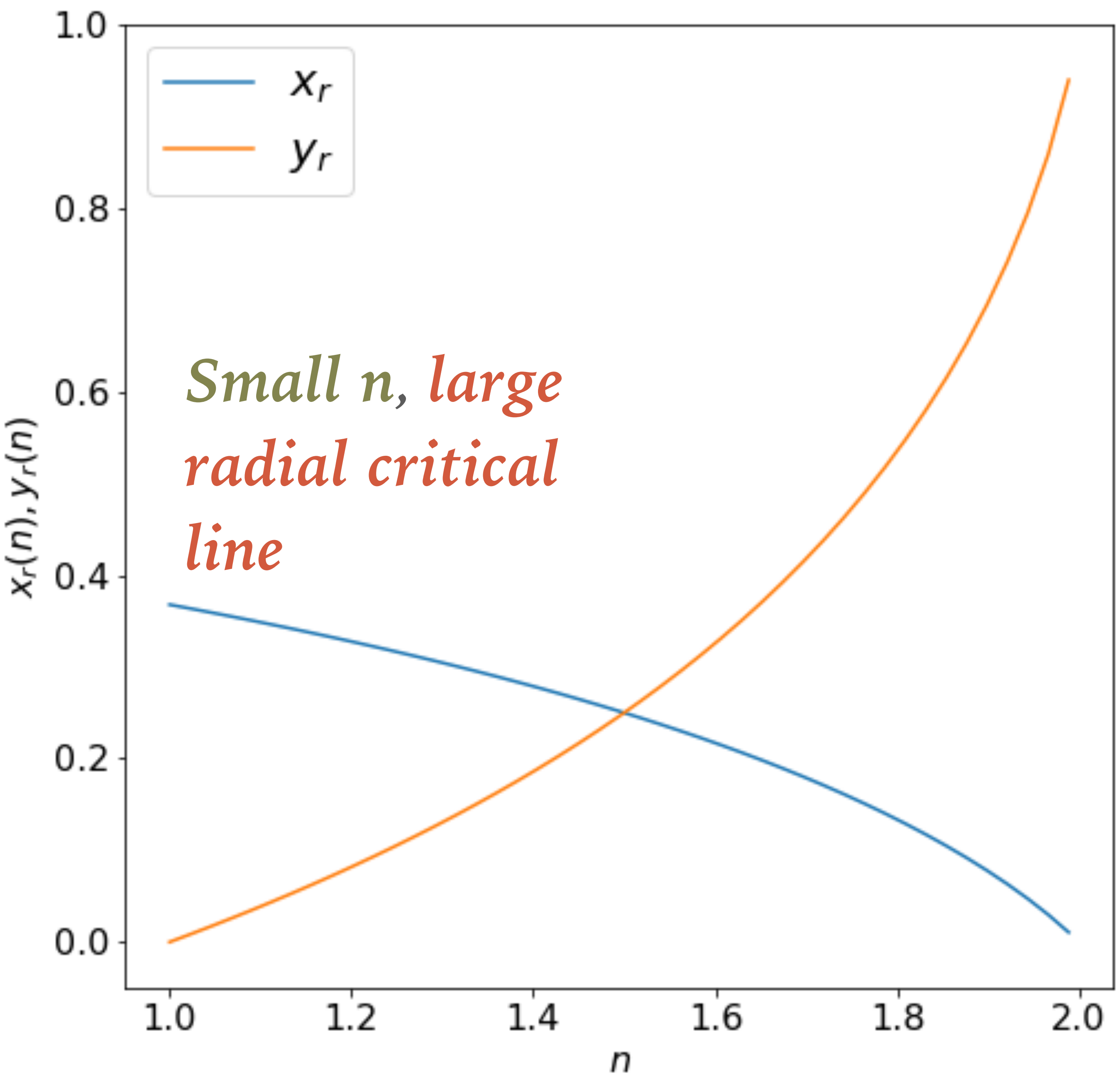
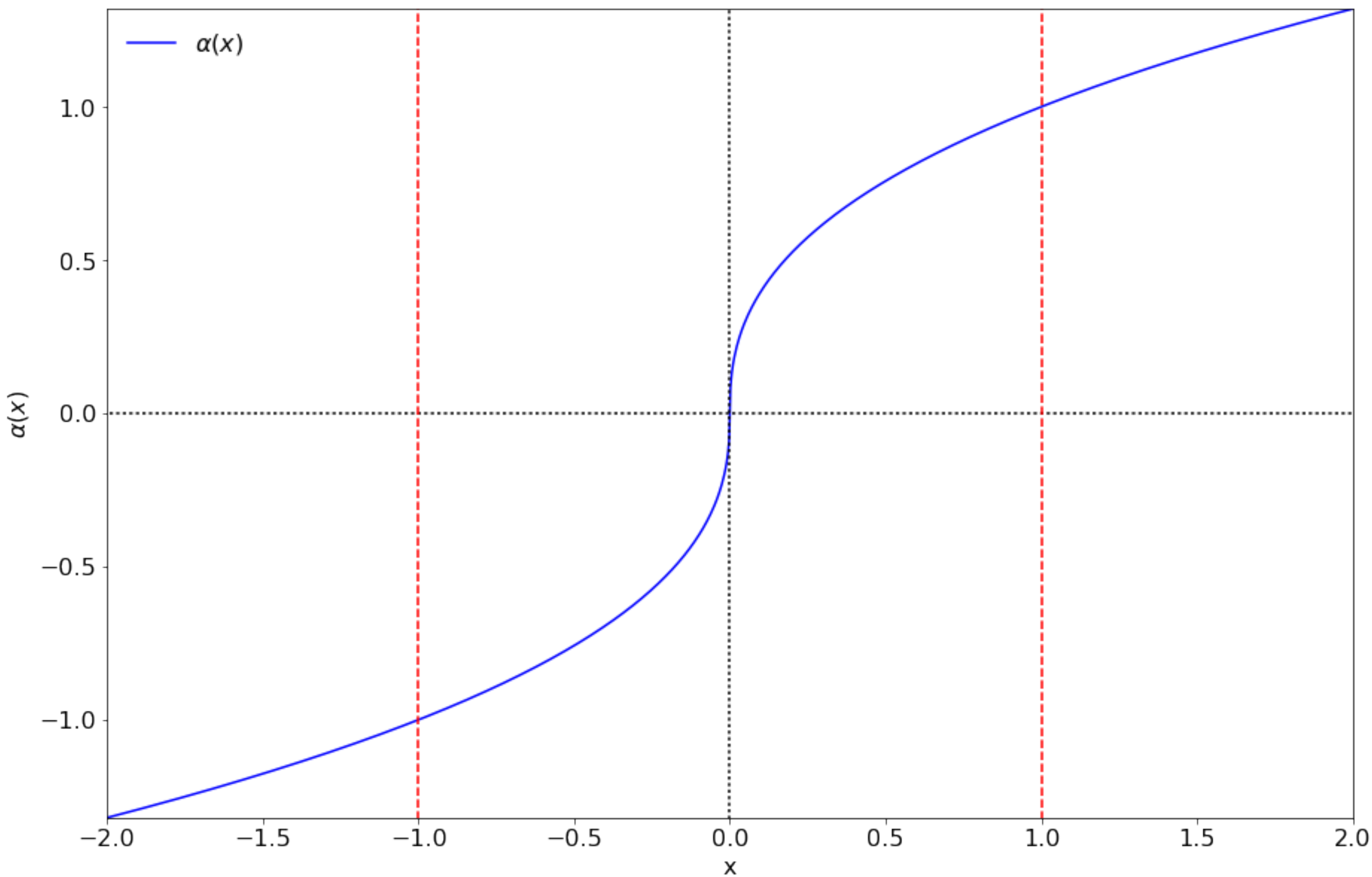


Small n, large radial critical line

n>=2, no radial critical line

Large n, small radial critical line

IMAGE DIAGRAM (EXAMPLE: N=1.7)



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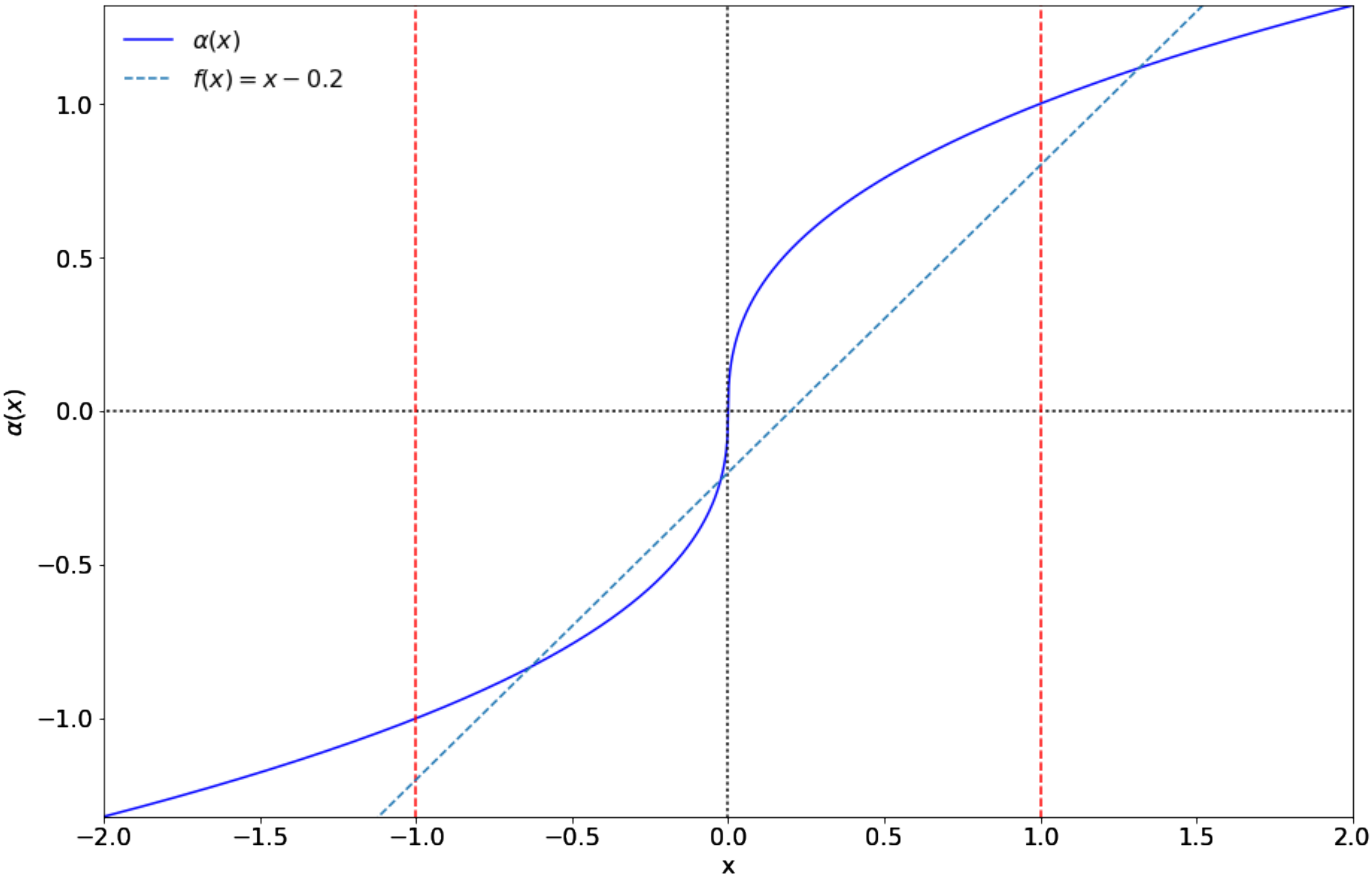

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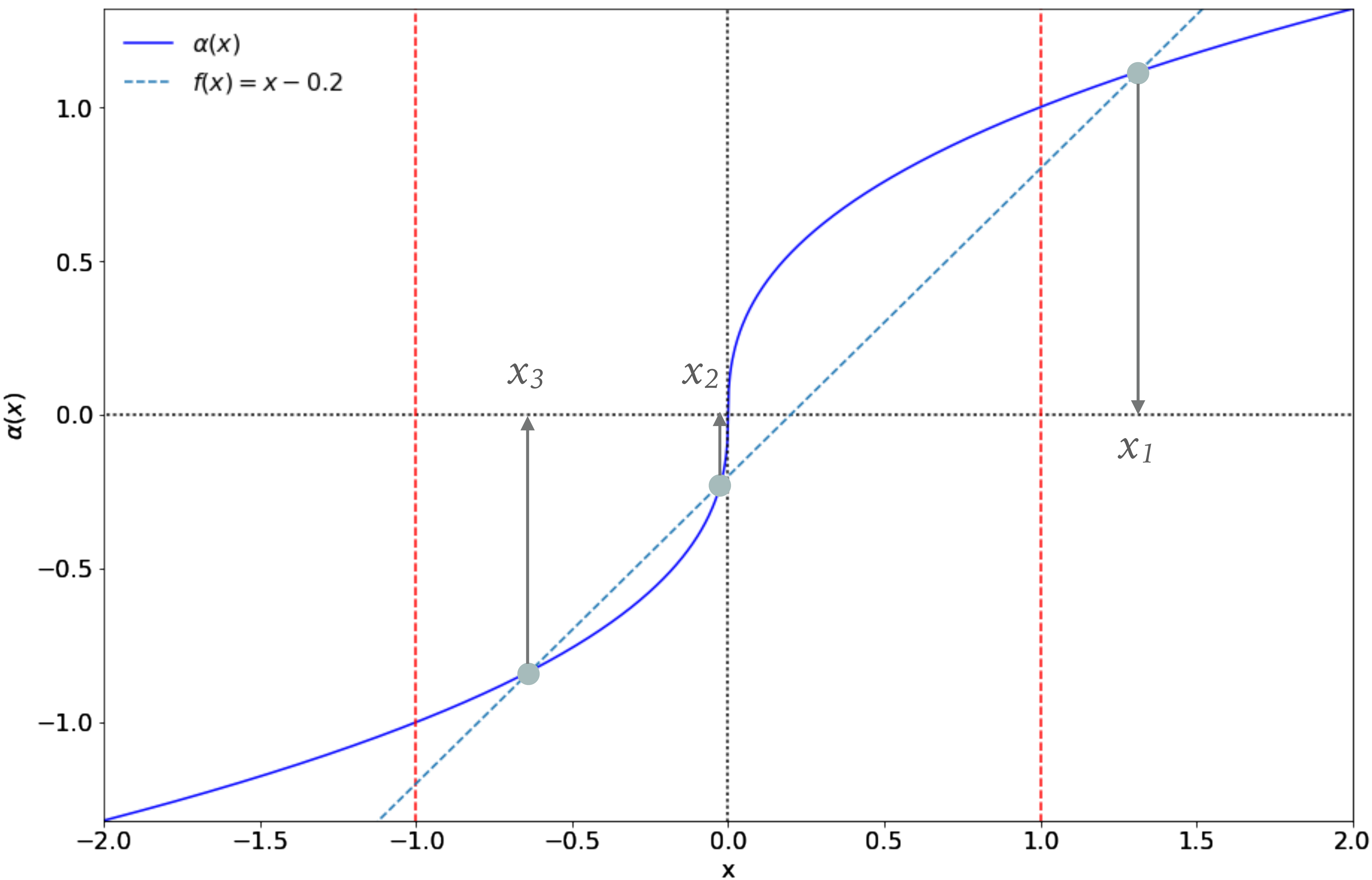

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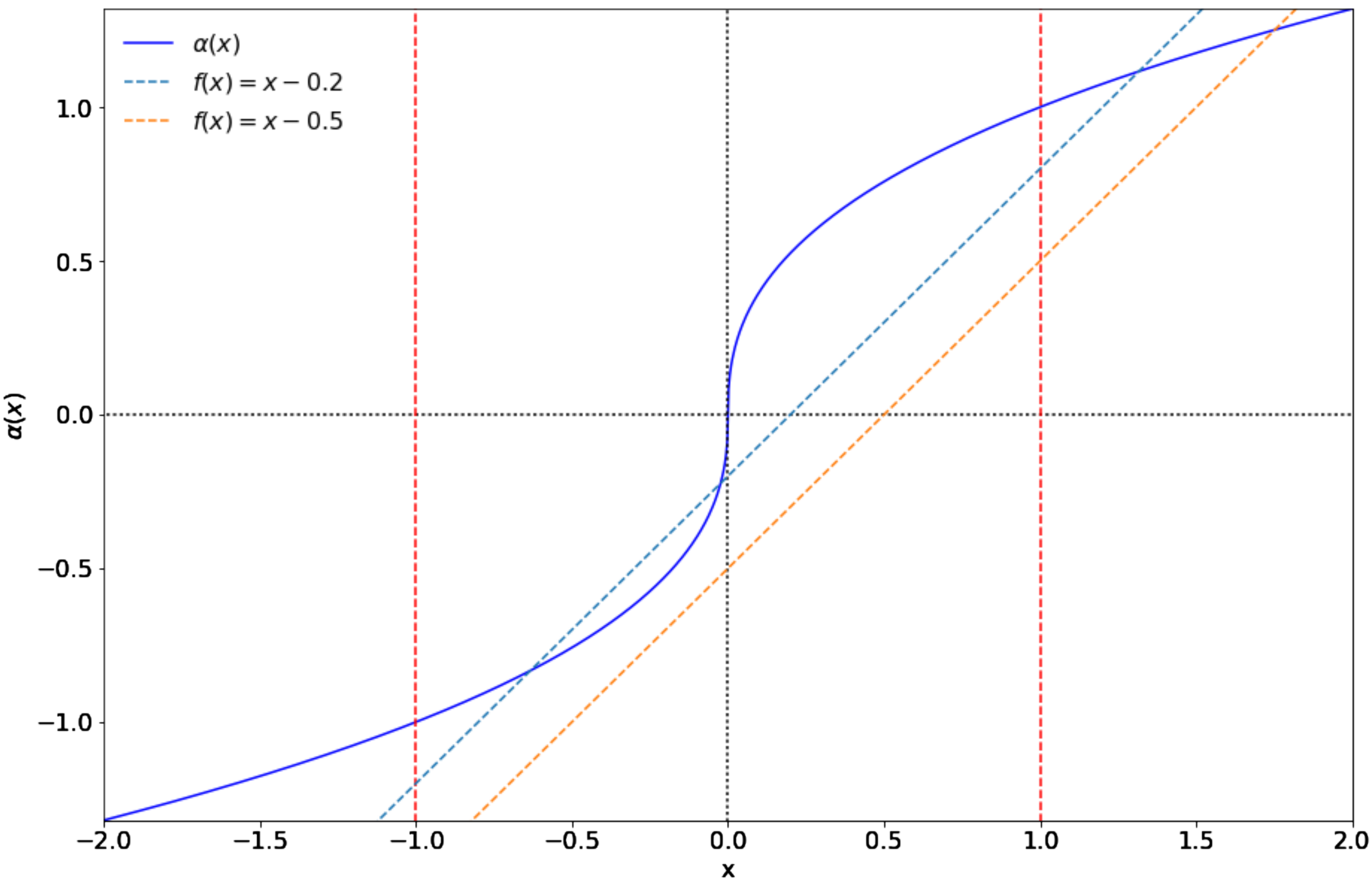

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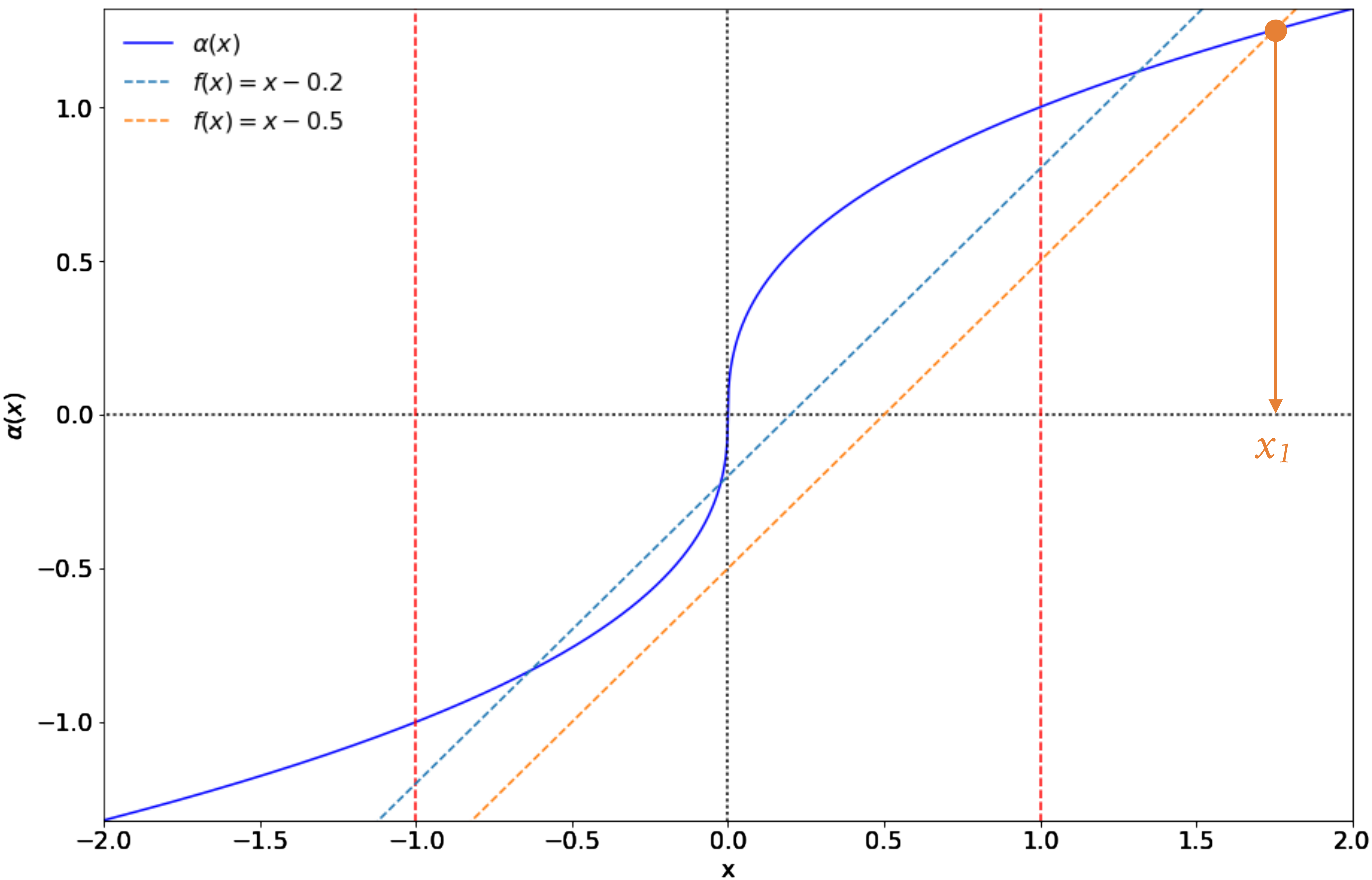

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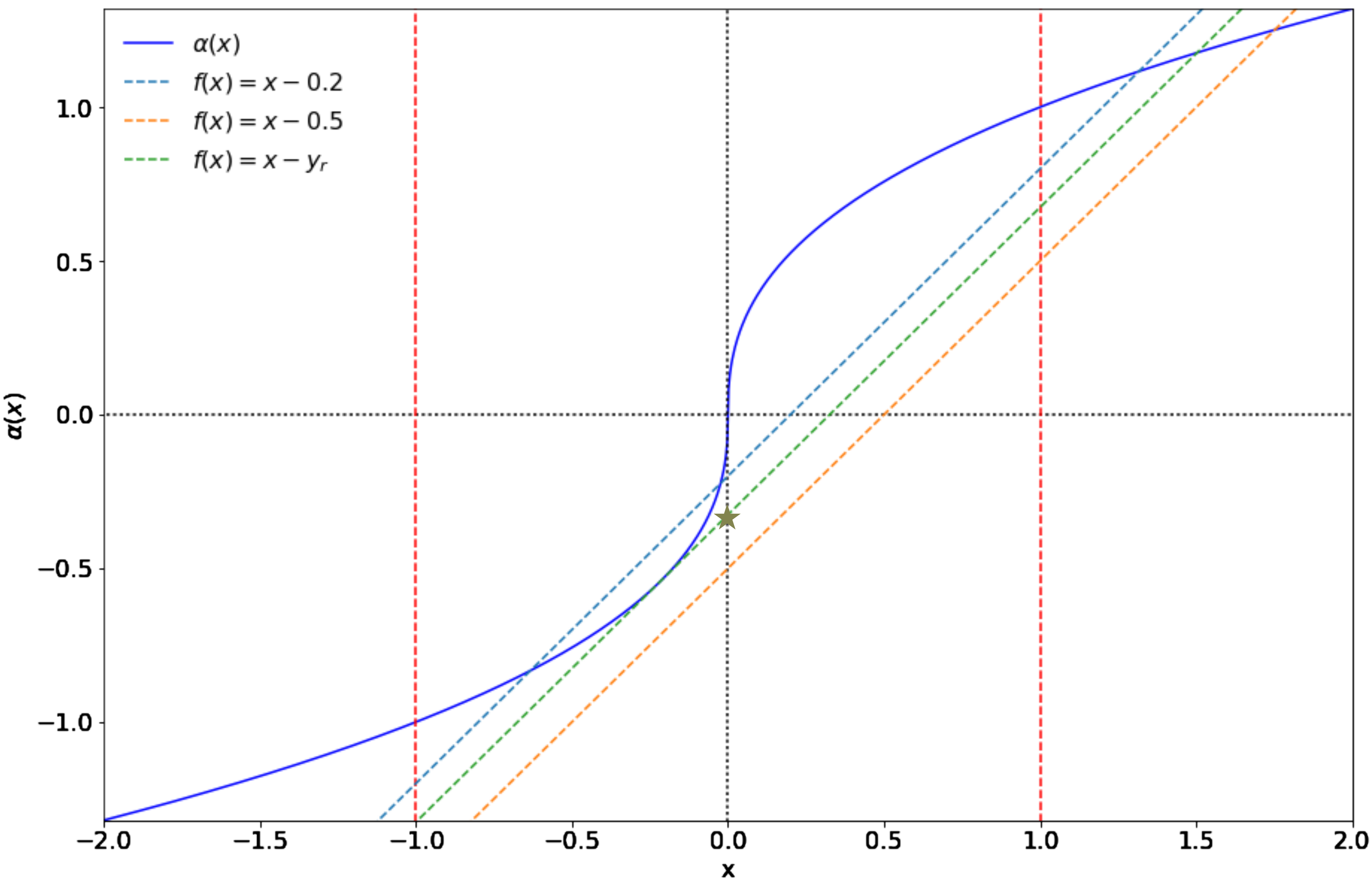

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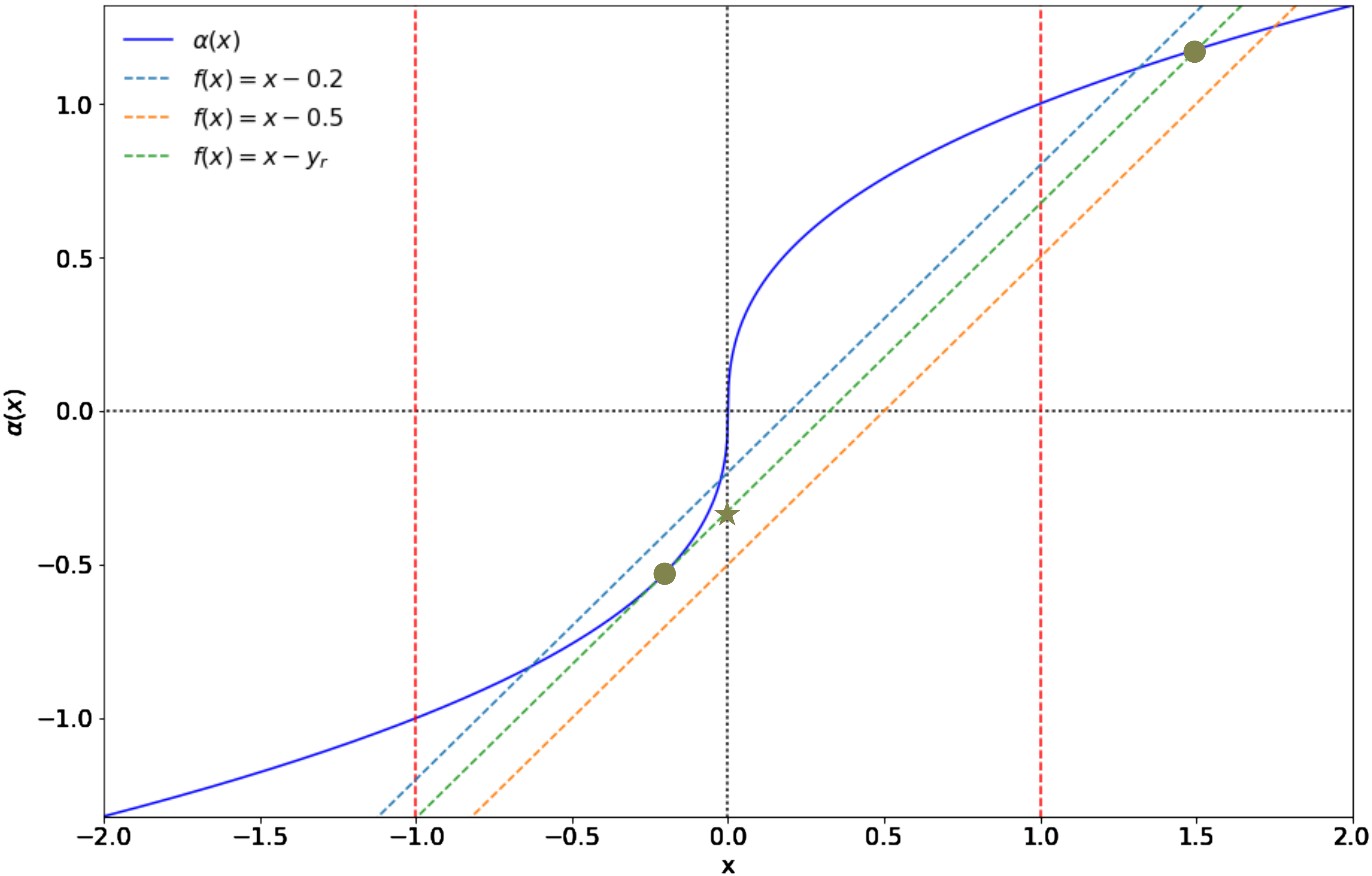

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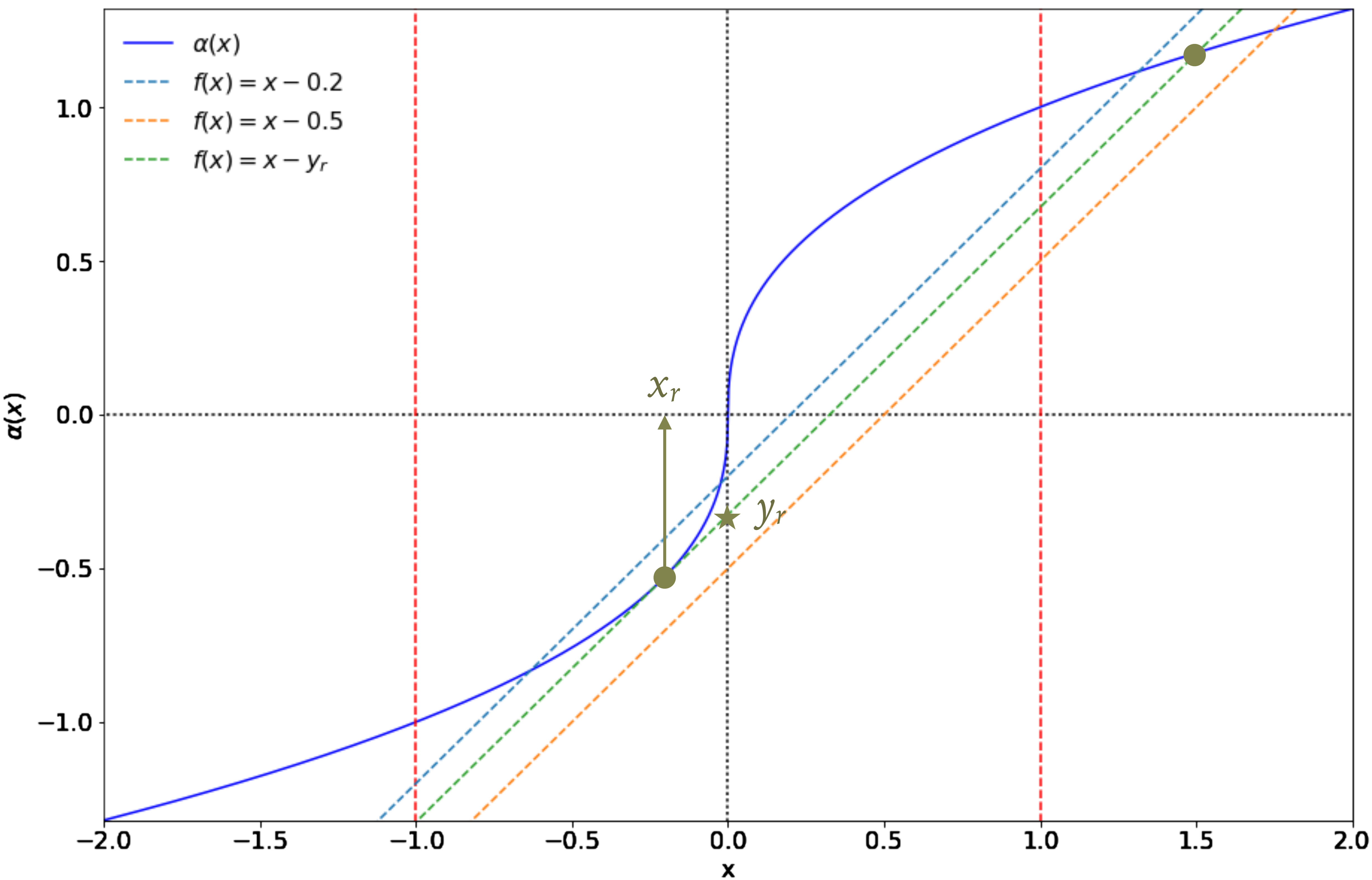

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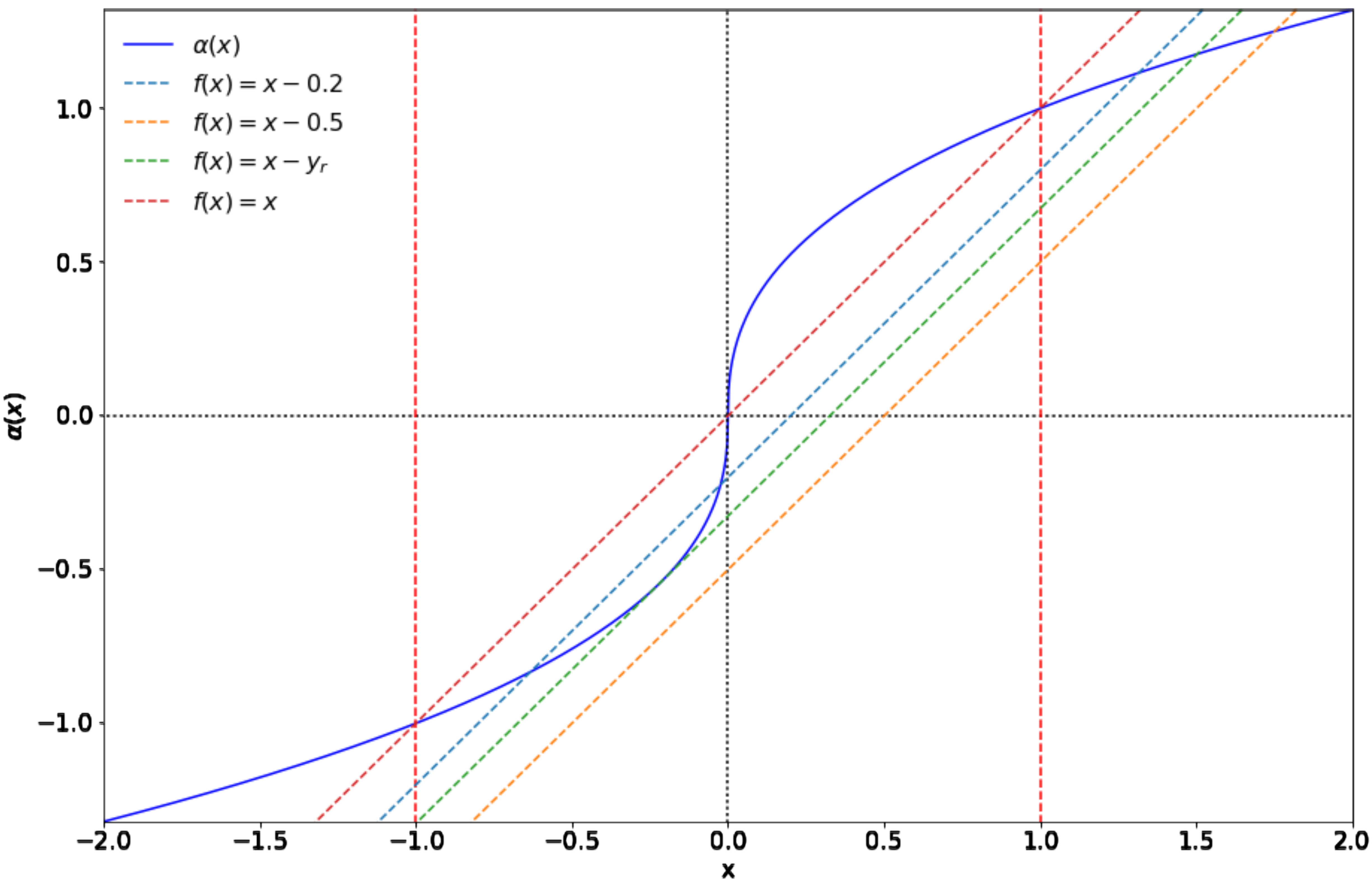
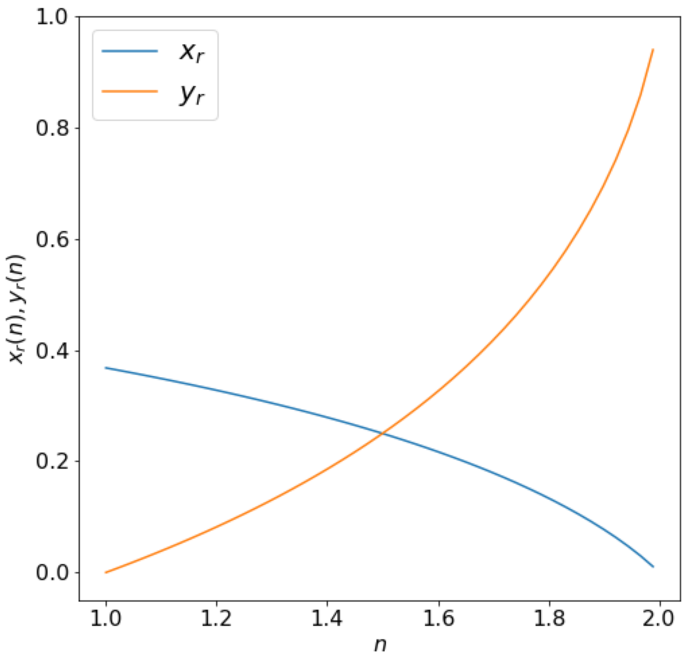
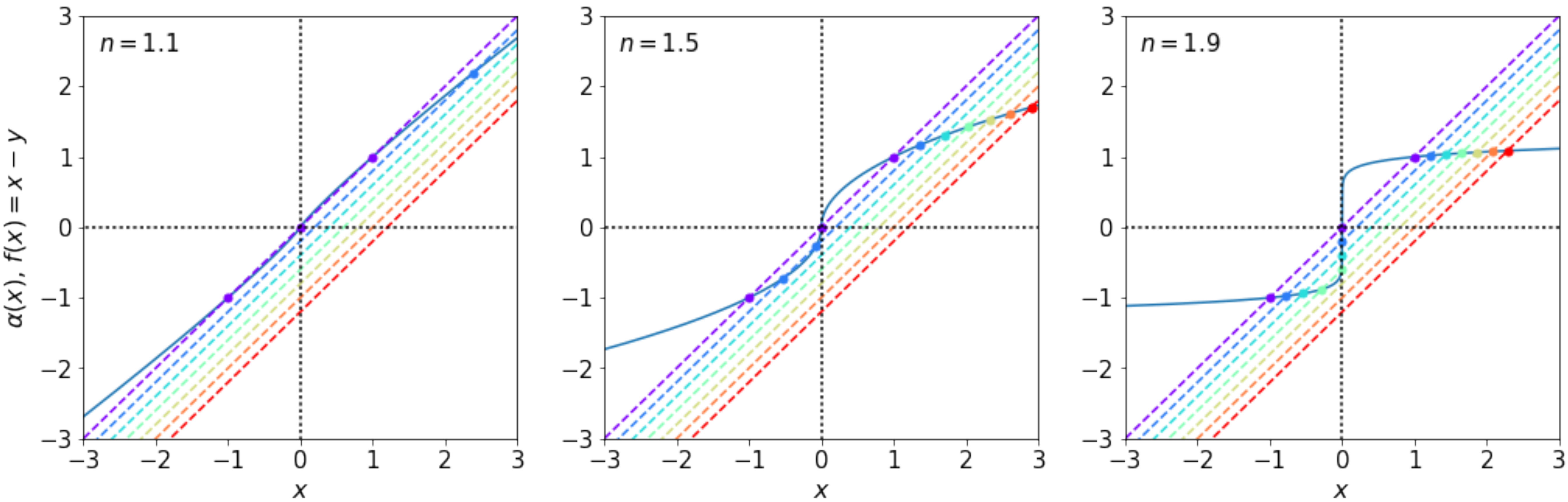


SOLUTIONS OF THE LENS EQUATION: IMAGE DIAGRAM



$1 < n < 2$



Small cross-section for multiple images

Large cross section for multiple images

IMAGE MAGNIFICATION

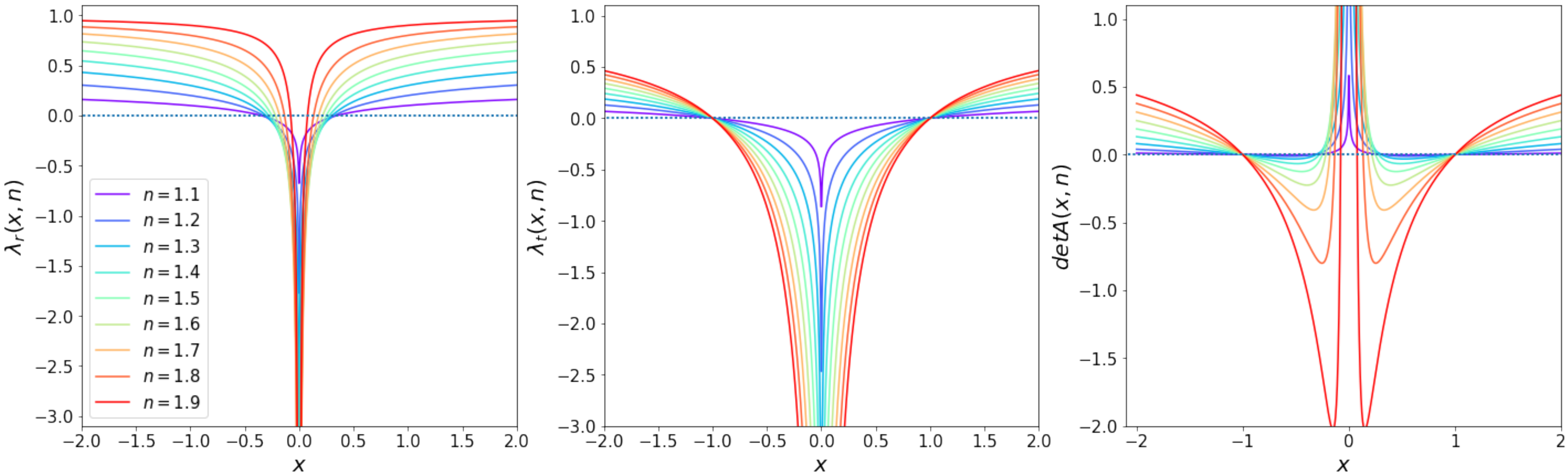
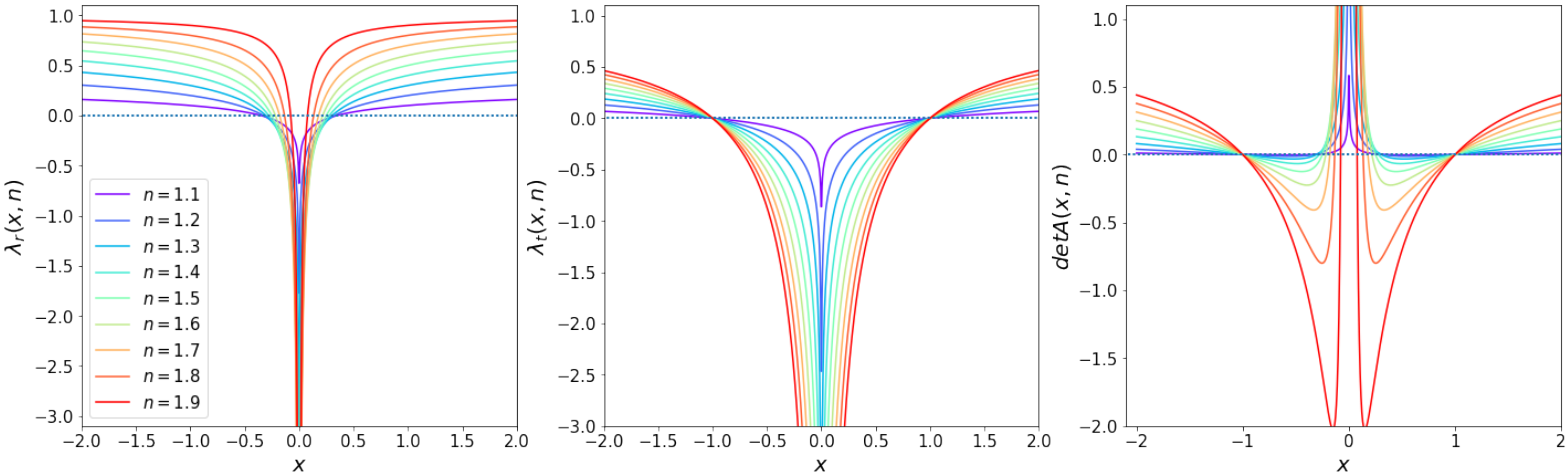


IMAGE MAGNIFICATION

higher radial magnification when n is small!



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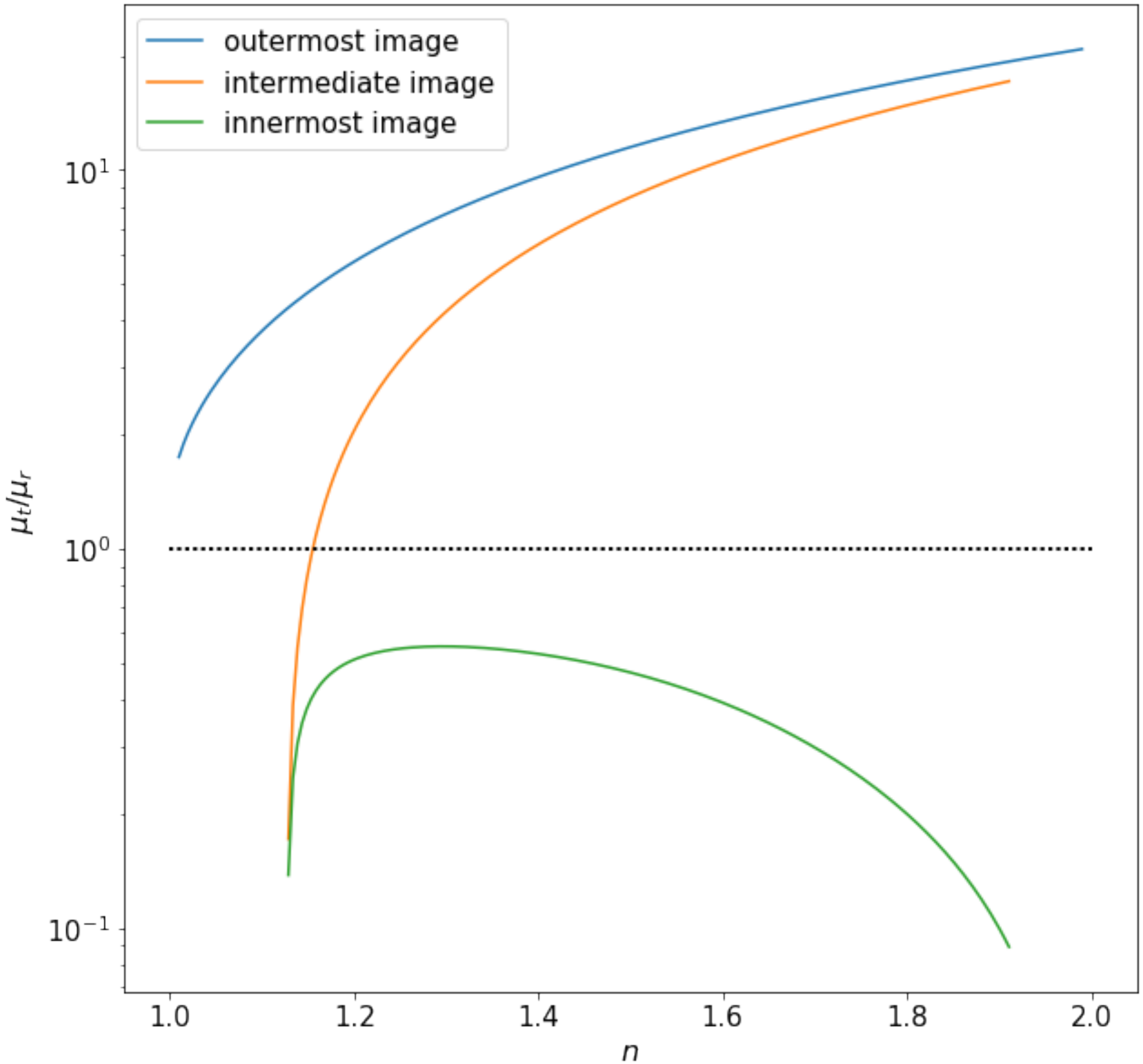
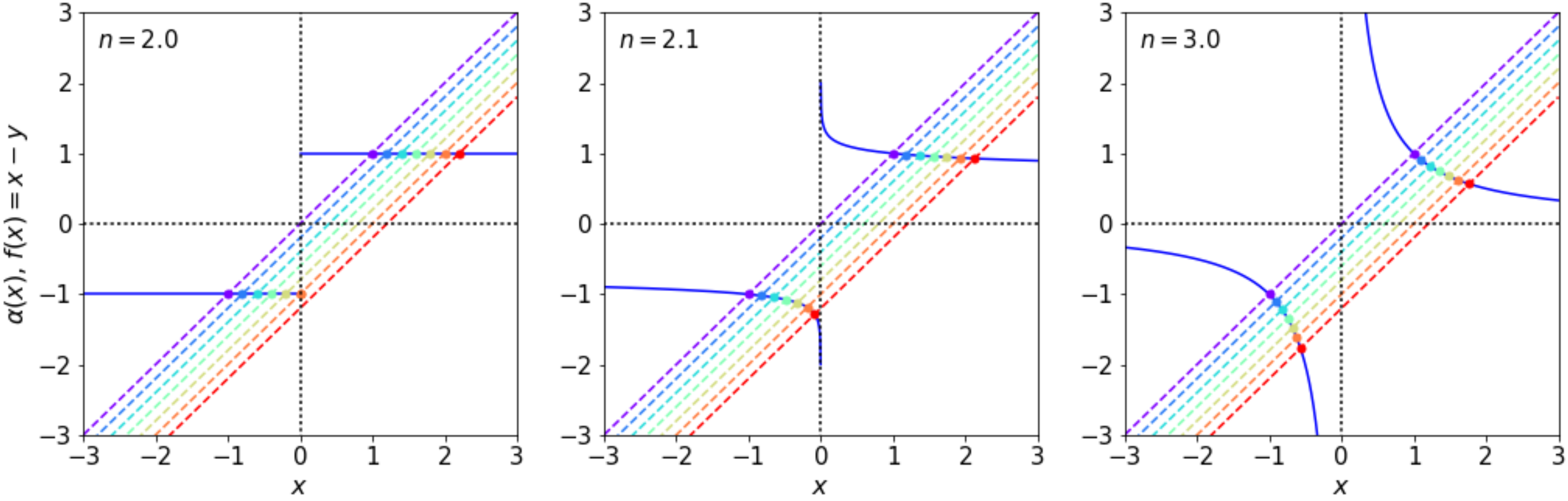


$y_s=0.05$

$$\begin{aligned}\lambda_t(x) &= 1-x^{1-n}\\ \lambda_r(x) &= 1-(2-n)x^{1-n}\end{aligned}$$

IMAGE DIAGRAM (N>2)



$$\lambda_t(x) = 1 - x^{1-n}$$
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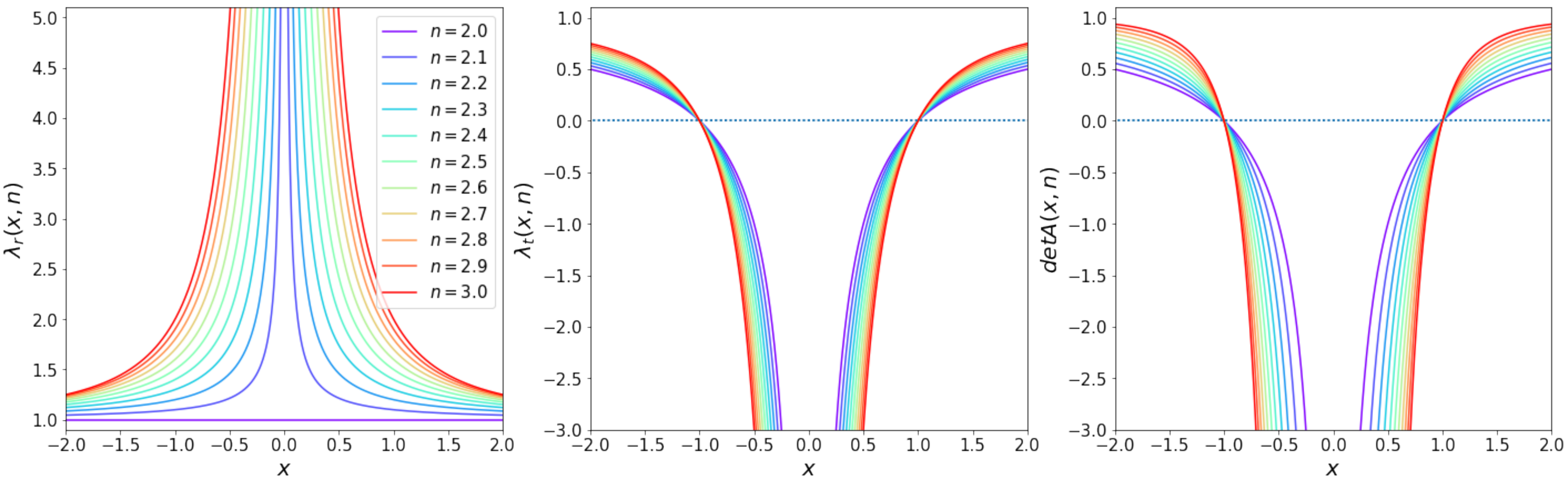

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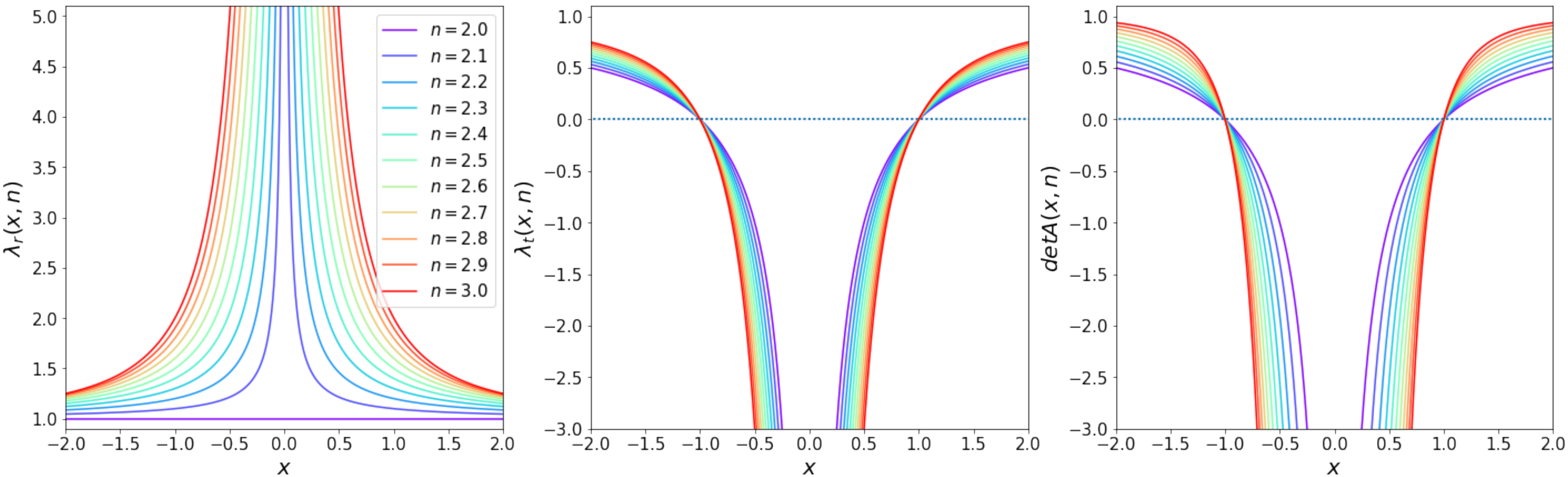
no radial critical line if n>=2!



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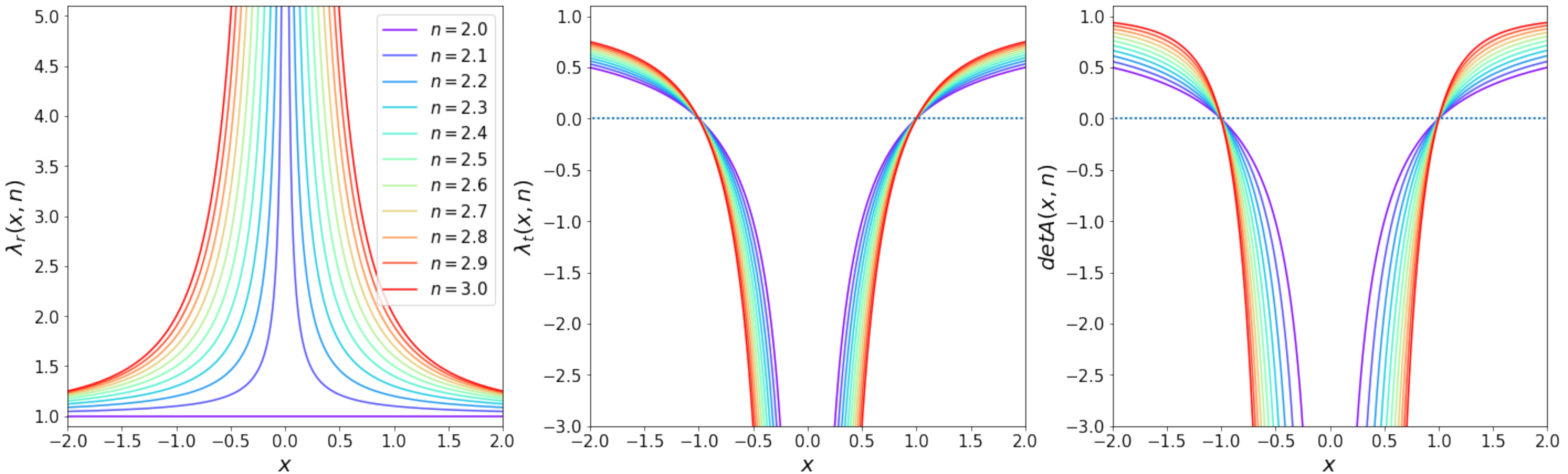

radial de-magnification!

SOME OBSERVATIONAL CONSEQUENCES

- $1<n<2$:
  - radial critical line exists
  - becomes smaller as n increases
  - images can be radially magnified
  - 1-3 images
  - cross section increases with n
- $n>2$:
  - no radial critical line
  - images are radially de-magnified
  - 2 images
- Therefore:
  - we see a radial arc? => $n<2$
  - do we observe the central image? => $n<2$
  - we see radially de-magnified images? => $n>2$

THE SINGULAR ISOTHERMAL SPHERE

The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is in isothermal and hydrostatic equilibrium, its density profile is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

velocity dispersion of the gas particles

The profile is "unphysical"

- *singularity near the center*
- *mass is infinite*

THE SINGULAR ISOTHERMAL SPHERE

For lensing purposes, we are interested in the projection of this profile:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

$$\begin{aligned}
\Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\
&= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\
&= \frac{\sigma_v^2}{2G\xi}\,.
\end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

As usual, we can switch to dimensionless units.

Let's take $$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

Then: $$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}} = \frac{1}{2x}\Sigma_{\rm cr} .$$

$$\kappa(x) = \frac{1}{2x}$$

Thus, the SIS lens is a power-law lens with n=2!

THE SINGULAR ISOTHERMAL SPHERE

The mass profile is readily computed:

$$m(x) = |x|$$

as well as the deflection angle:

$$\alpha(x) = \frac{x}{|x|}$$

The lens equation reads

$$y = x - \frac{x}{|x|}$$

How many solutions does this equation have?

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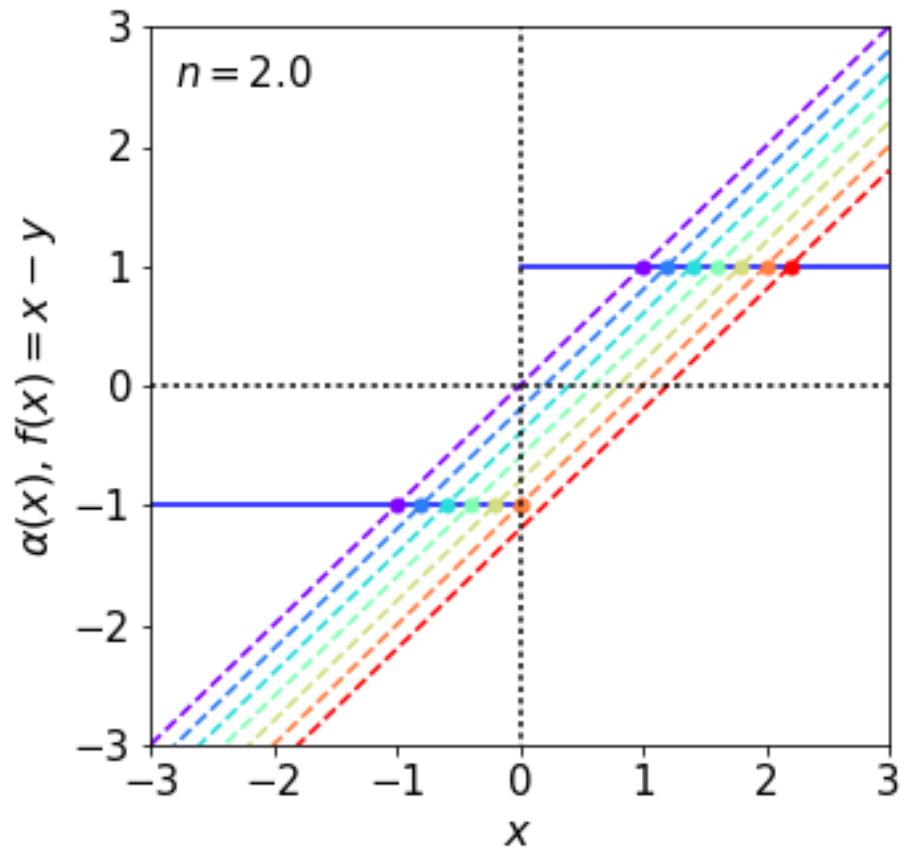

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THE SINGULAR ISOTHERMAL SPHERE

If $0<y<1$, the solution are two:

$$x_- = y - 1$$

$$x_+ = y + 1$$

$$\theta_\pm = \beta \pm \theta_E$$

Otherwise, there is only one solution at

$$x_+ = y + 1$$

Thus, the circle of radius $y=1$ is called "cut" and plays the same role of the radial caustic for the power-law lens with $n<2$, separating the source plane into regions with different image multiplicity.

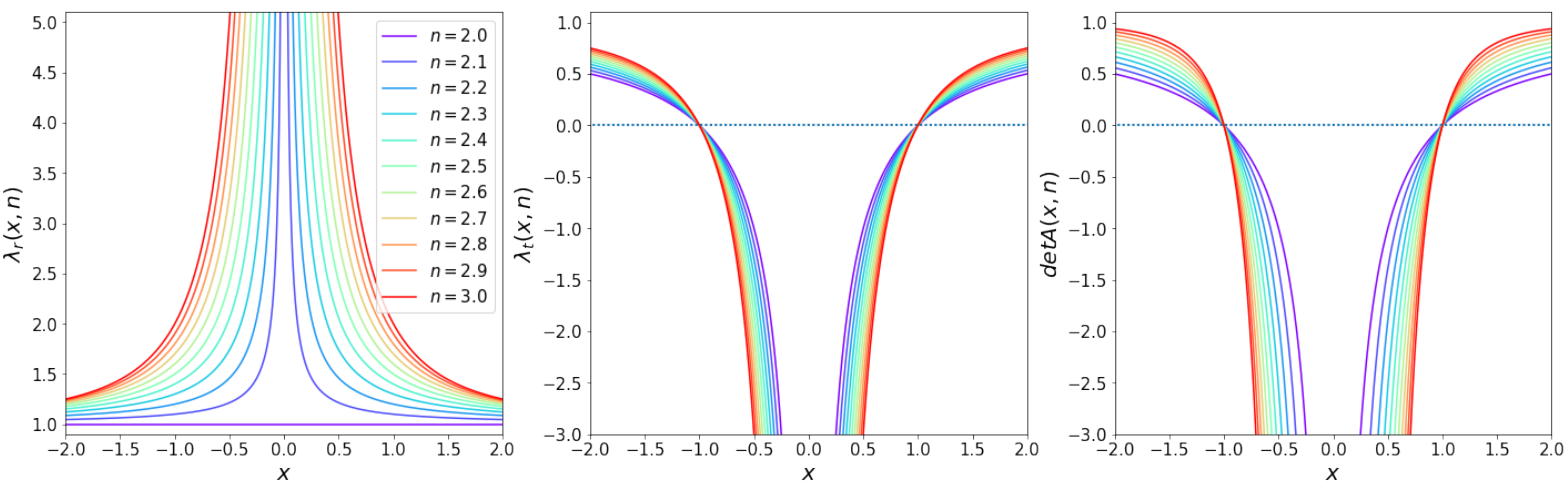
$$y_c = \lim_{x \to 0} y(x) = -\alpha(x)$$

THE SINGULAR ISOTHERMAL SPHERE

On the other hand, for the SIS: $\alpha'(x) = 0$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1$

Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.



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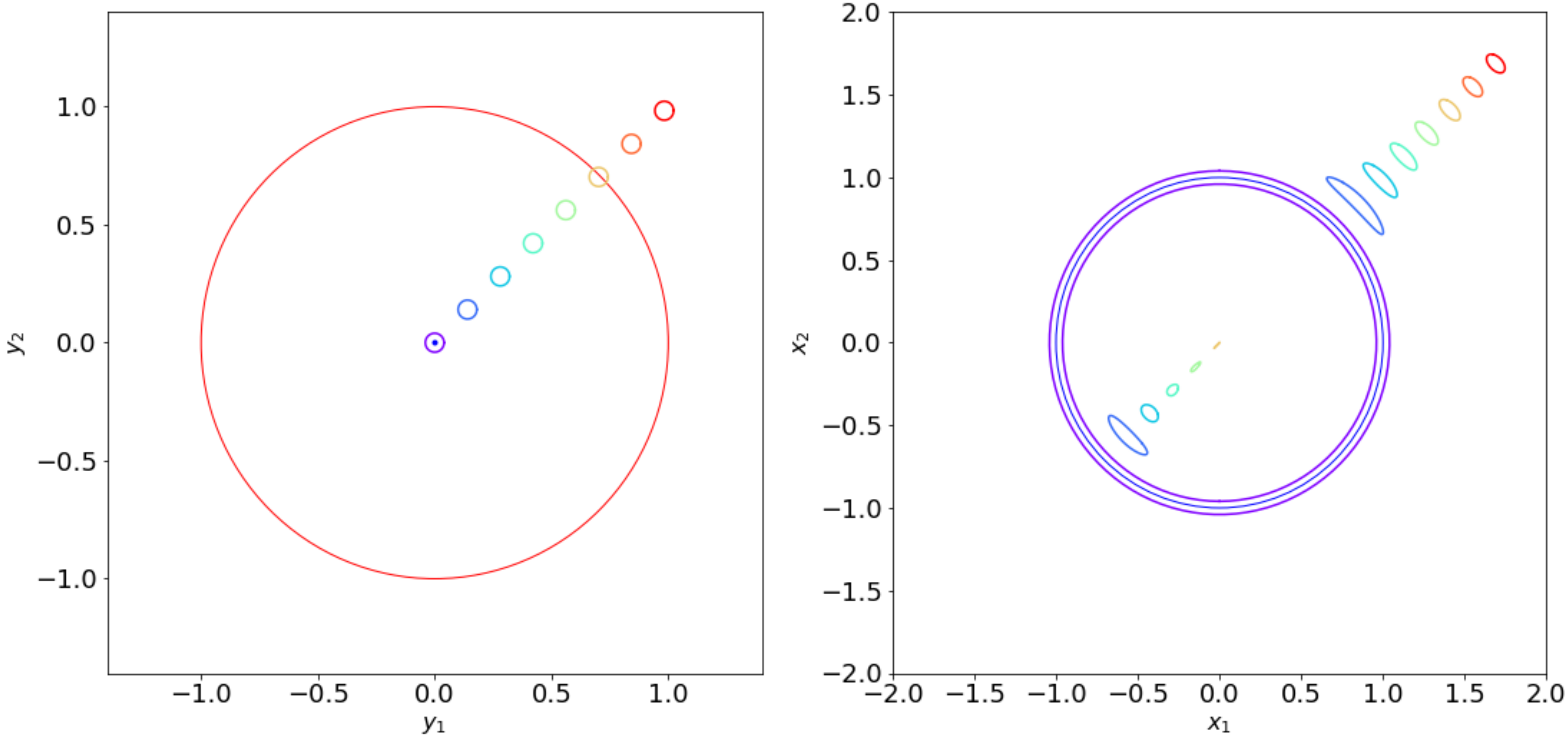

THE SINGULAR ISOTHERMAL SPHERE

The shear can be computed easily:

$$\gamma(x) = \frac{m(x)}{x} - \kappa(x) = \frac{1}{2x}$$

$$\begin{aligned} \gamma_1 &= \frac{1}{2}\frac{\cos 2\phi}{x}, \\ \gamma_2 &= \frac{1}{2}\frac{\sin 2\phi}{x}. \end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

as well as the magnification

$$\mu(x) = \frac{|x|}{|x|-1}$$

$$\mu_+(y) = \frac{y+1}{y} = 1 + \frac{1}{y} \quad ; \quad \mu_-(y) = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

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tends to unity for large y!

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zero for y=1!