# GRAVITATIONAL LENSING

## 3 - DEFLECTION OF LIGHT AND LENS EQUATION

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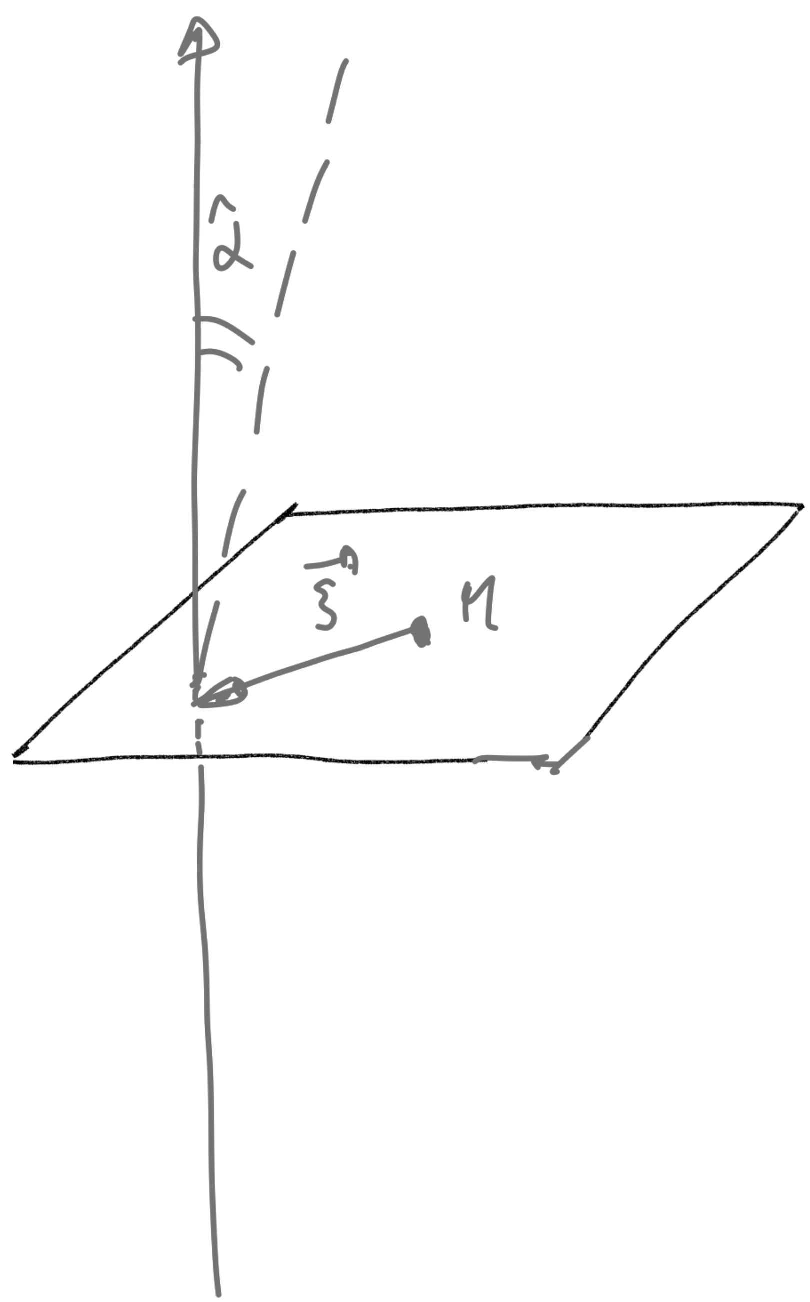
*AA 2018-2019*

# DEFLECTION BY A POINT MASS

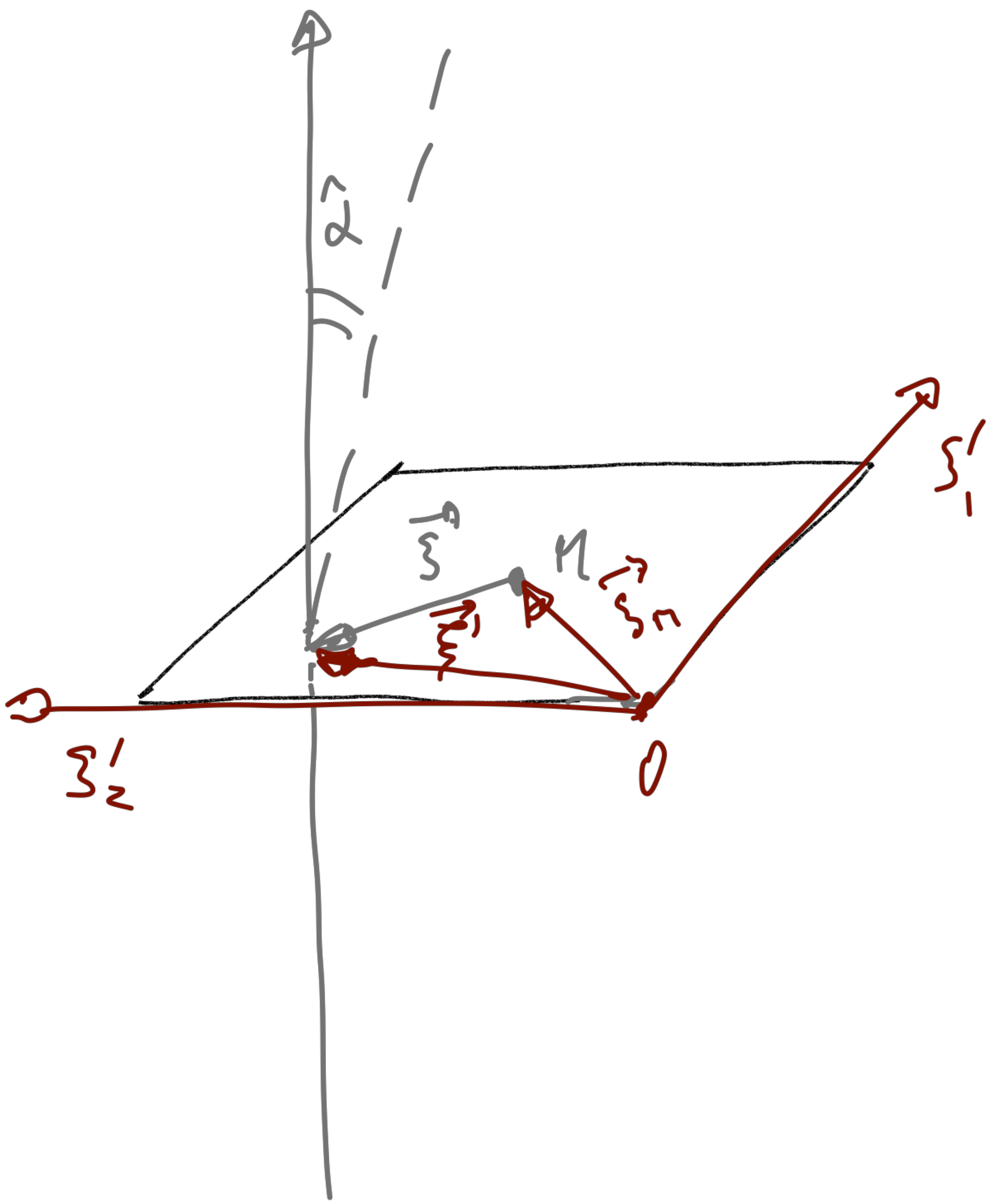



$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{2}{c^2}\int_{-\infty}^{\infty} \vec{\nabla}_{\perp}\Phi(\vec{\xi}, z)dz$$

$$= \frac{4GM}{c^2\xi^2}\vec{\xi}$$

$$\Phi(\xi, z) = -\frac{GM}{\sqrt{\xi^2 + z^2}}$$

# DEFLECTION BY A POINT MASS



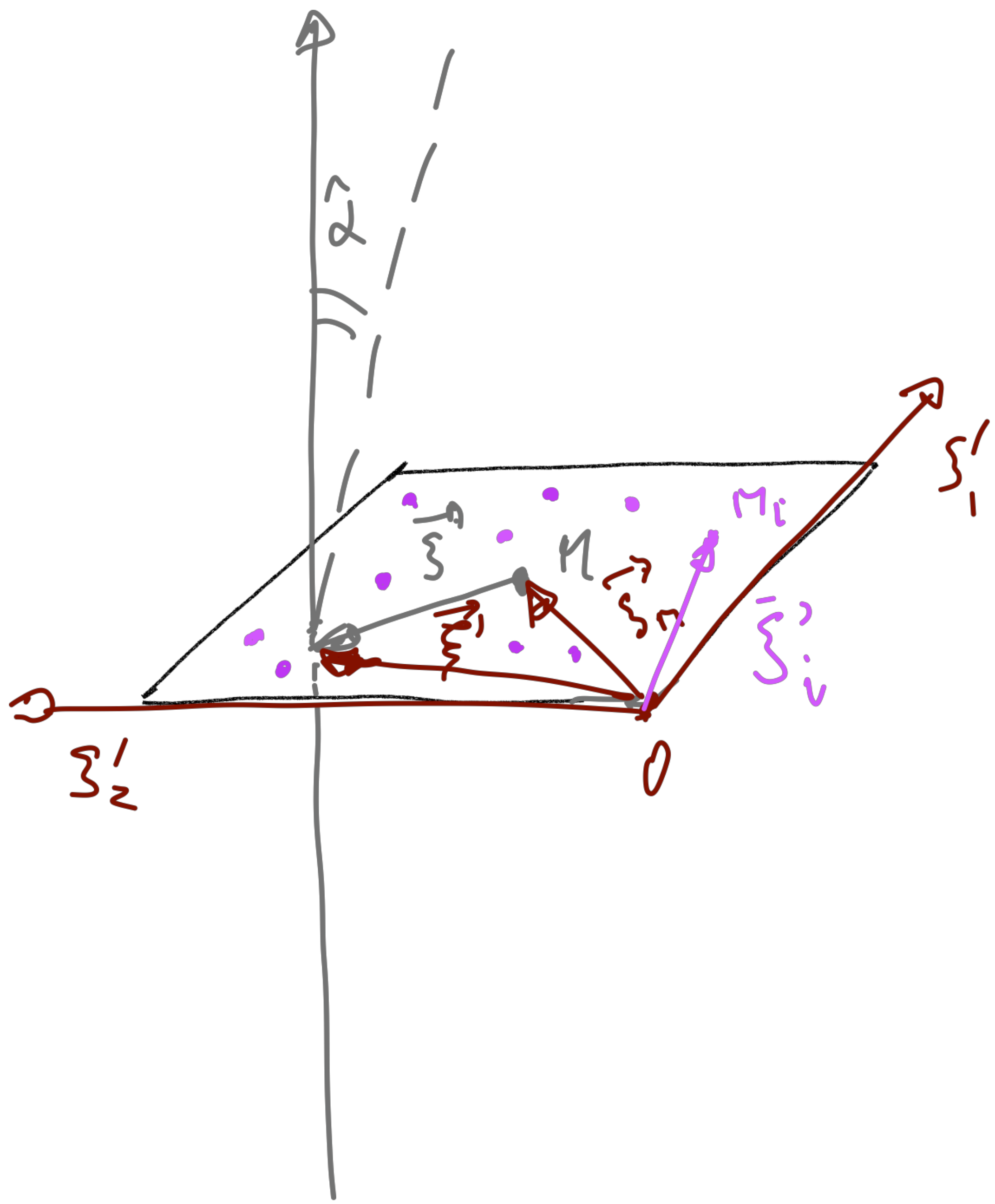
$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{2}{c^2}\int_{-\infty}^{\infty} \vec{\nabla}_\perp \Phi(\vec{\xi}, z)\,dz$$

$$\hat{\alpha}(\bar{\xi}') = \frac{4GM}{c^2\xi^2}\vec{\xi} = \frac{4GM}{c^2}\frac{\bar{\xi}' - \bar{\xi}'_M}{|\bar{\xi}' - \bar{\xi}'_M|^2}$$

$$\Phi(\xi, z) = -\frac{GM}{\sqrt{\xi^2 + z^2}}$$

$$\bar{\xi} = \bar{\xi}' - \bar{\xi}'_M$$

# DEFLECTION BY AN ENSEMBLE OF POINT MASSES




$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{2}{c^2}\int_{-\infty}^{\infty}\vec{\nabla}_{\perp}\Phi(\vec{\xi}, z)dz$$

$$\hat{\alpha}(\bar{\xi}') = \frac{4GM}{c^2\xi^2}\vec{\xi} = \frac{4GM}{c^2}\frac{\bar{\xi}' - \bar{\xi}'_m}{|\bar{\xi}' - \bar{\xi}'_m|^2}$$

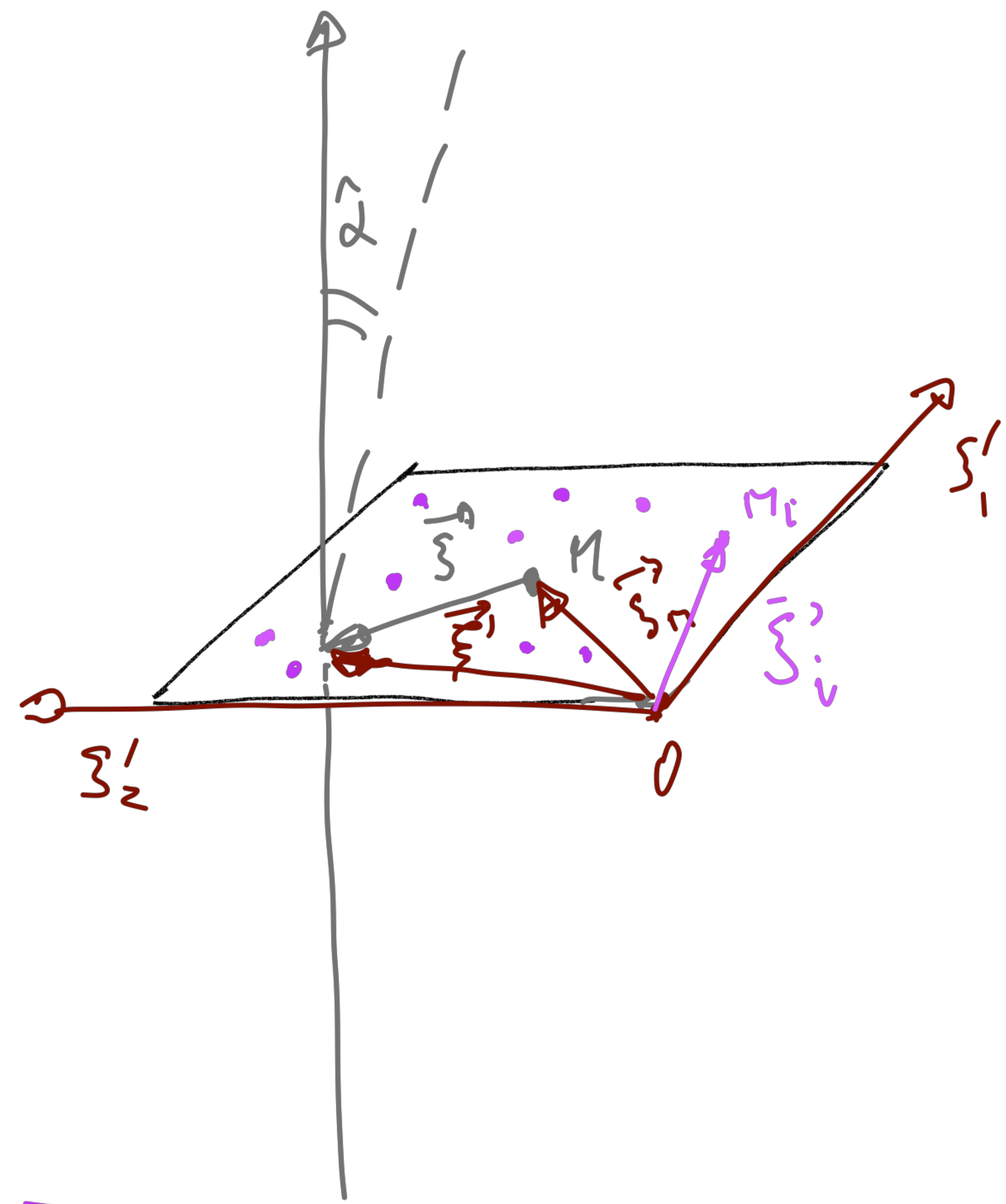
$$\Phi(\xi, z) = -\frac{GM}{\sqrt{\xi^2 + z^2}}$$

$$\bar{\xi} = \bar{\xi}' - \bar{\xi}'_m$$

*What is the total deflection angle?*

# DEFLECTION BY AN ENSEMBLE OF POINT MASSES

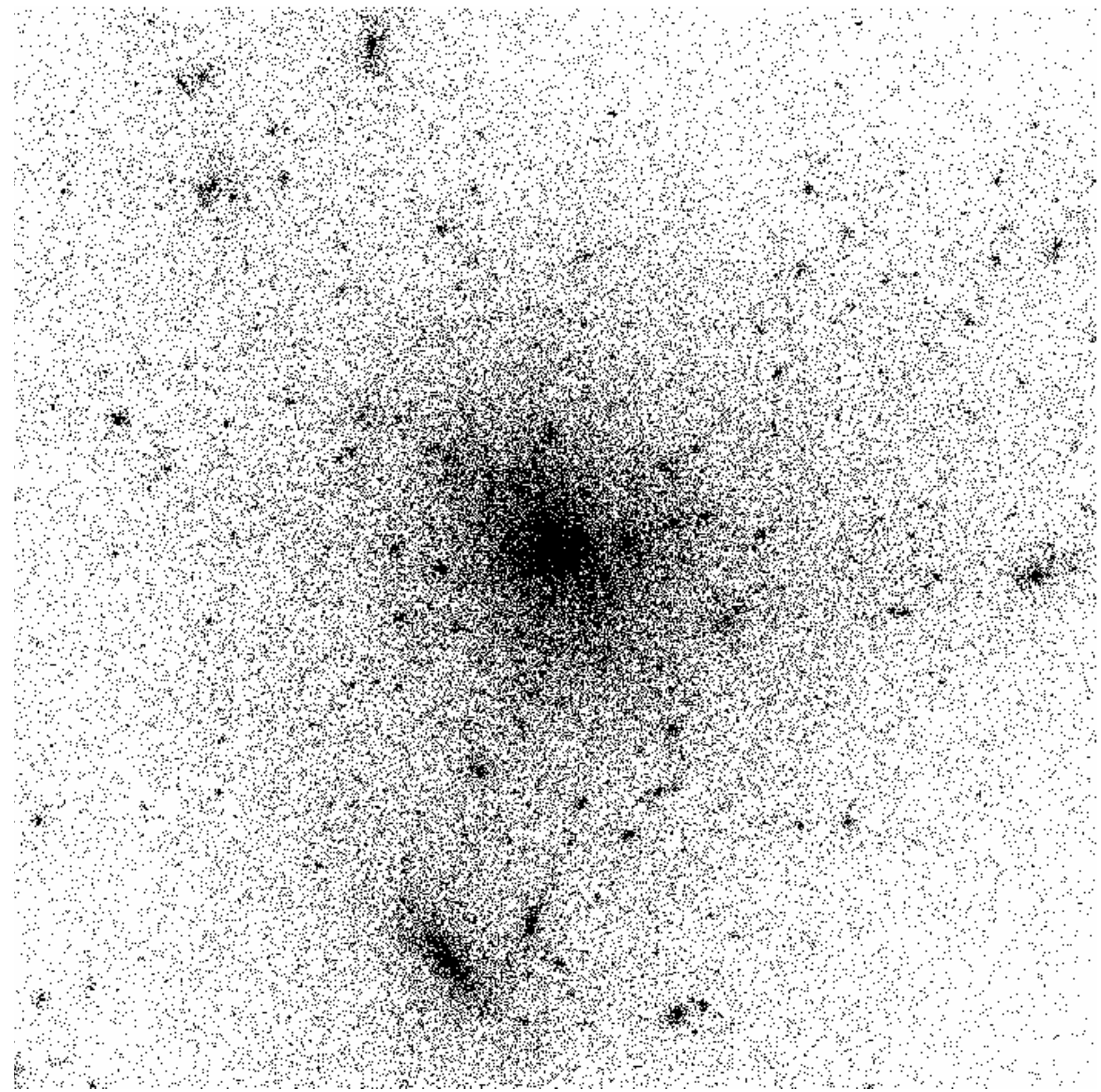
- Thin screen approximation
- Remaining in the weak field limit, one can use the superposition principle
- The deflection angle by a system of point masses is the vectorial sum of the deflection angles of the single lenses
- Applications: studying the deflection by mass distributions obtained from N-body/hydrodynamical simulations; lensing by an ensemble of stars




$$\hat{\vec{\alpha}}(\vec{\xi}') = \frac{4G}{c^2}\sum_{i=1}^{N} M_i \frac{\vec{\xi}' - \vec{\xi}'_i}{|\vec{\xi}' - \vec{\xi}'_i|^2}$$

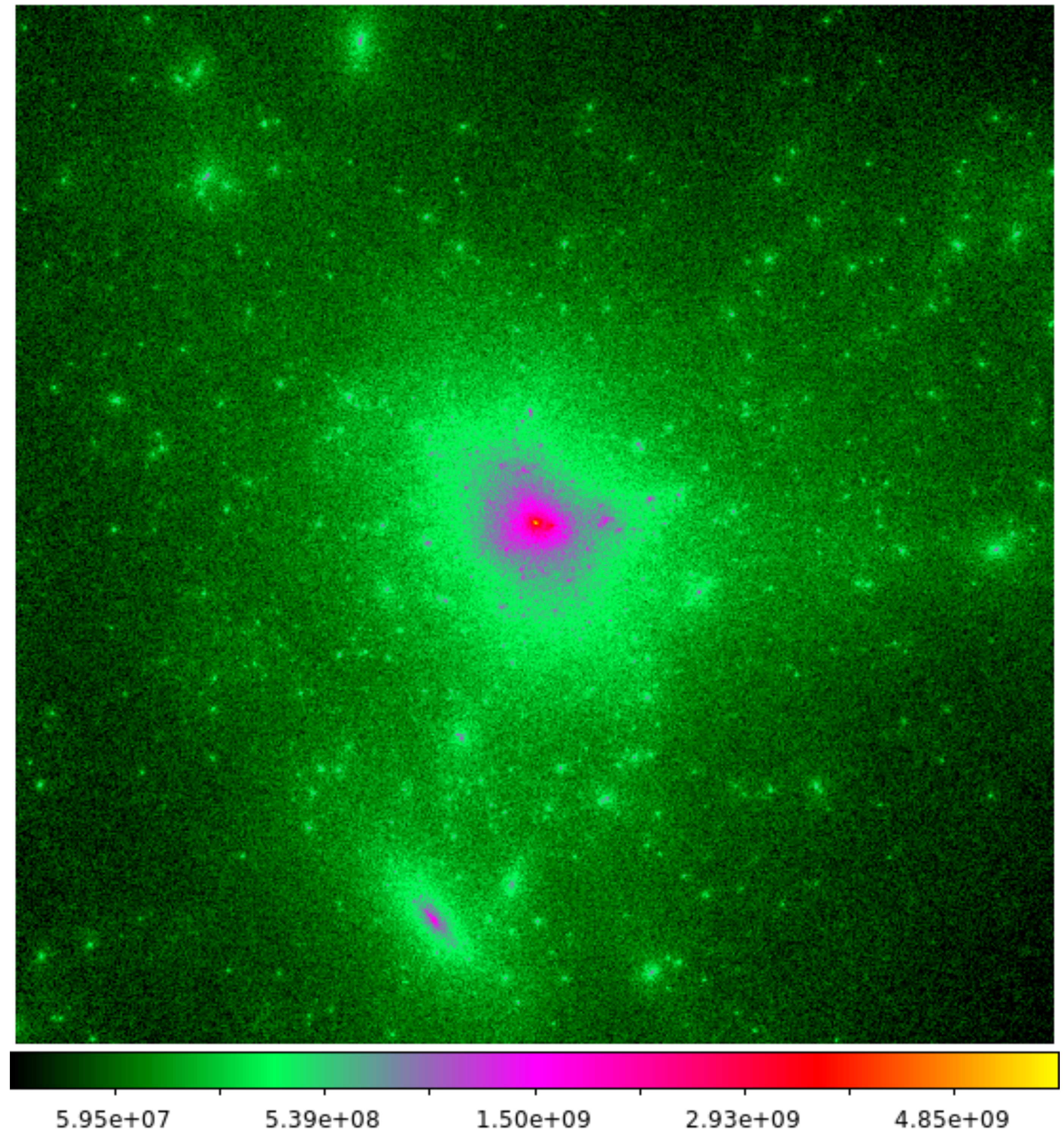
# DEFLECTION BY AN ENSEMBLE OF POINT MASSES

- Structure formation is often studied using numerical simulations
- Galaxies, galaxy clusters, etc. are described by ensembles of particles
- The calculation of the deflection angle by direct summation of all contributions from each particle has a computational cost $O(N^2)$



# DEFLECTION BY AN EXTENDED MASS DISTRIBUTION

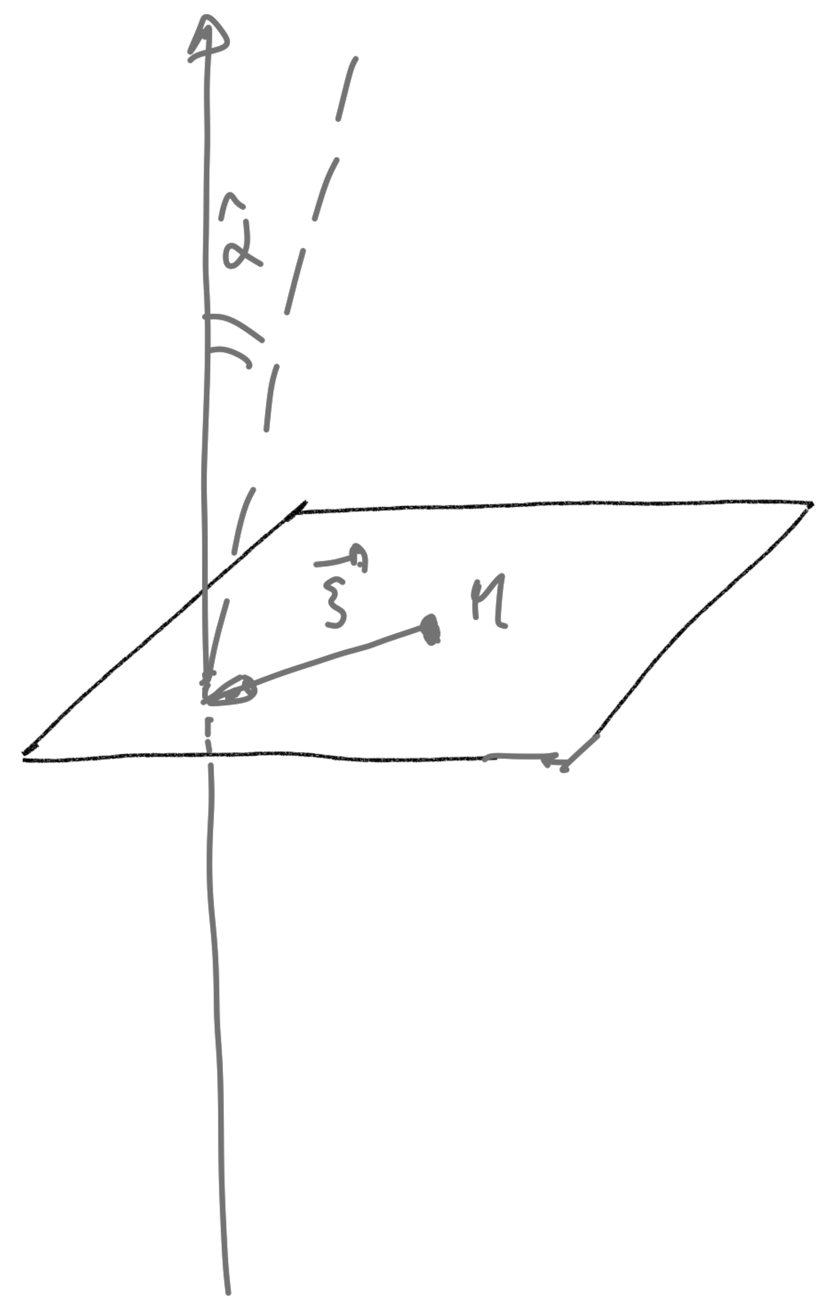
- This can be easily generalized to the case of a continuum distribution of mass




$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\, \mathrm{d}z$$

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\, \mathrm{d}^2\xi'$$

# DEFLECTION BY A POINT MASS



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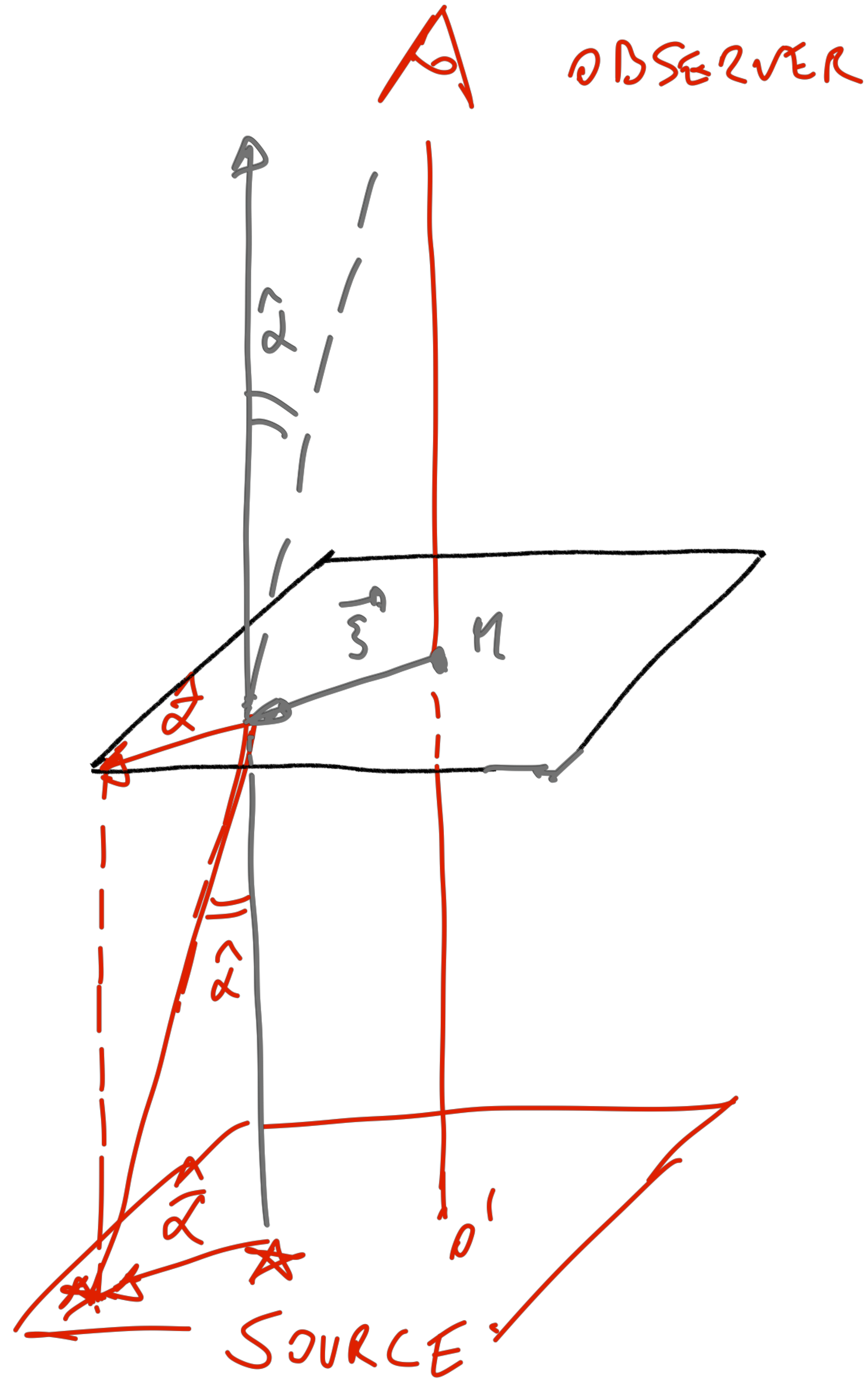
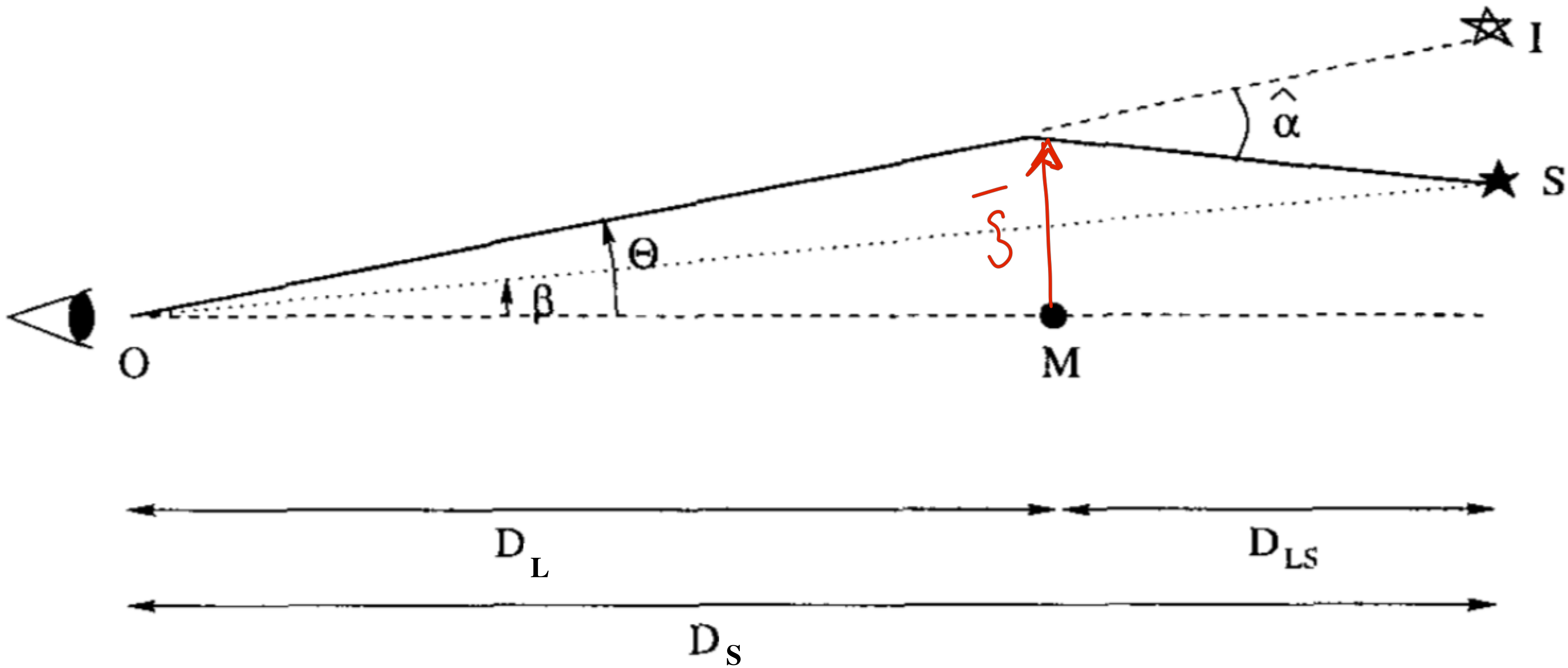


# LENS EQUATION



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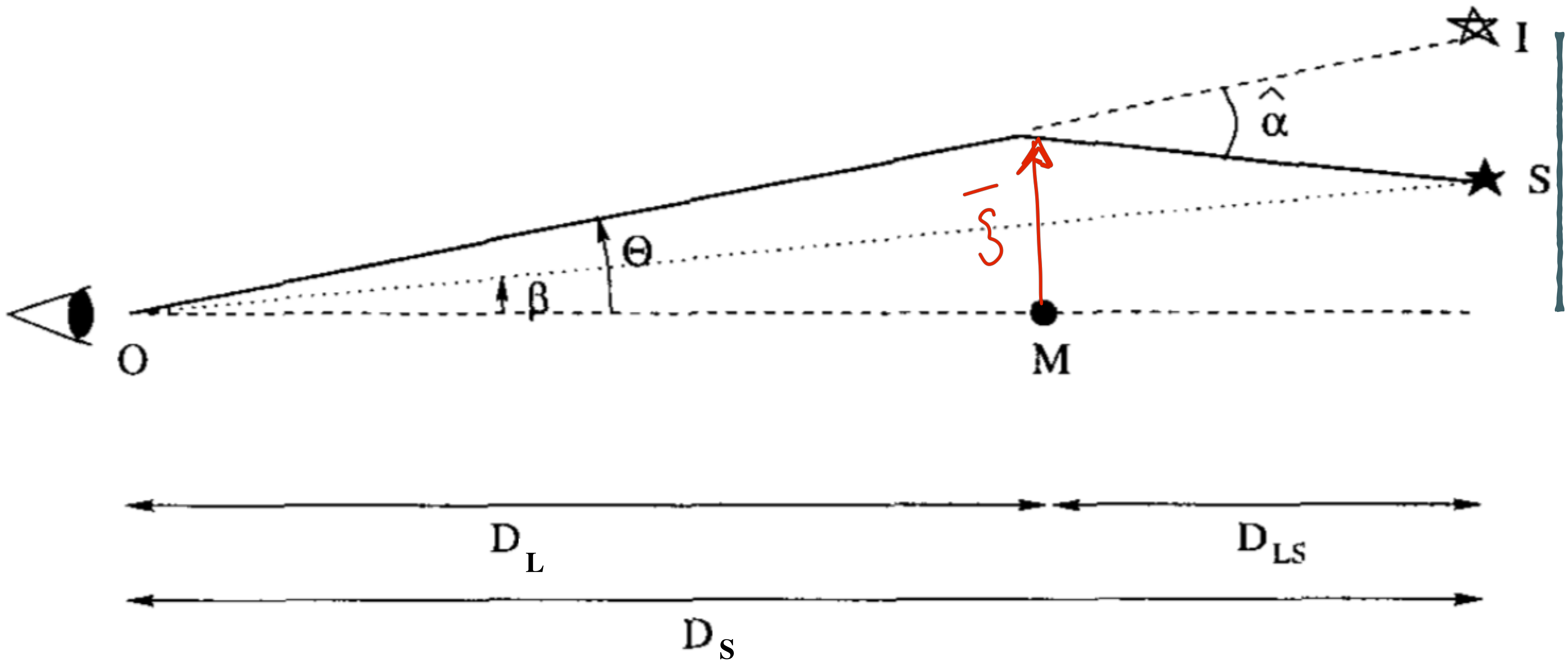

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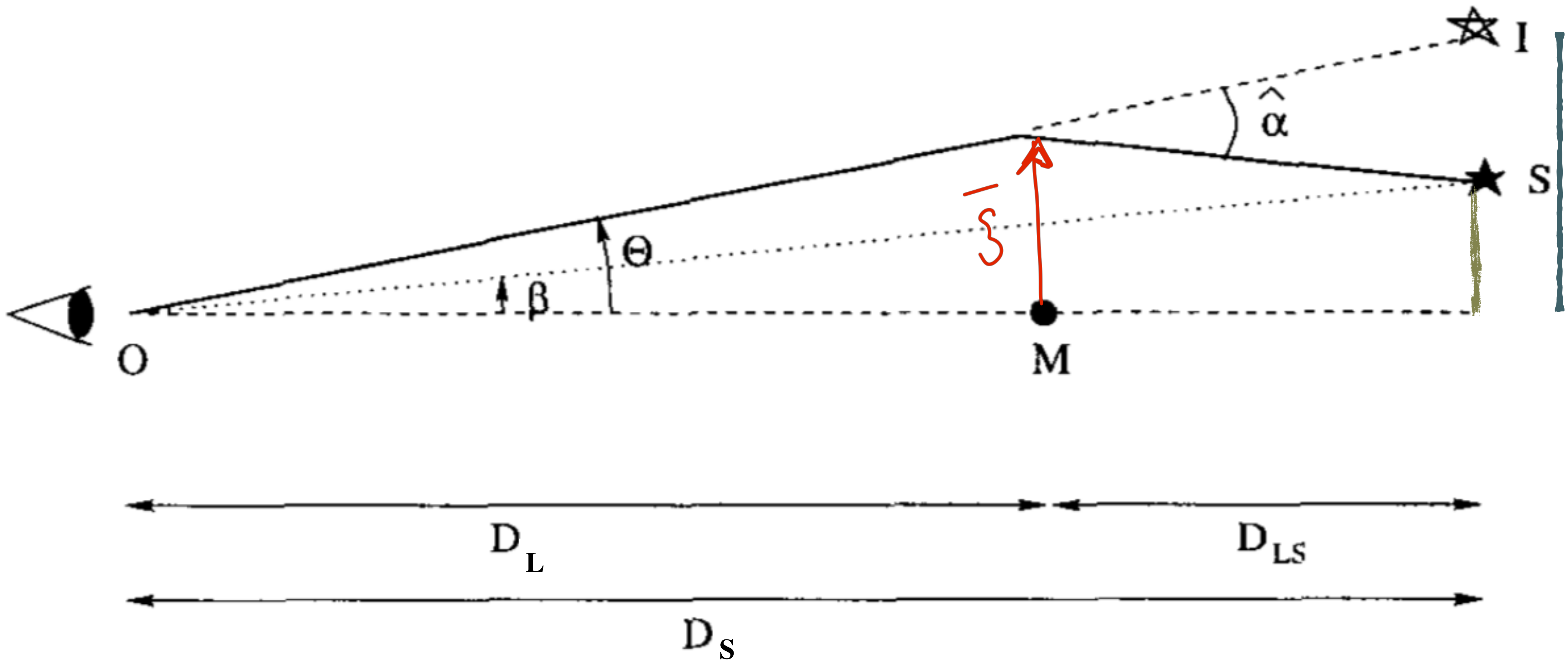

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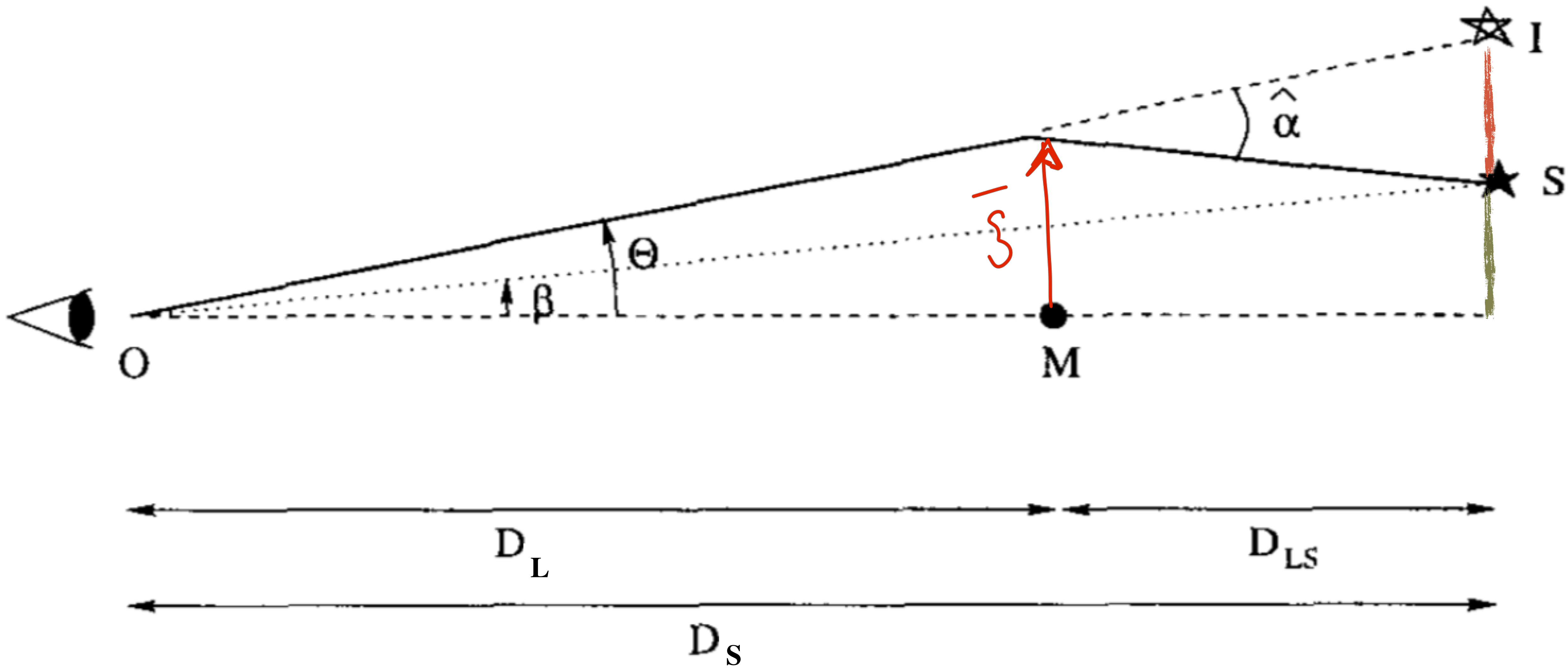

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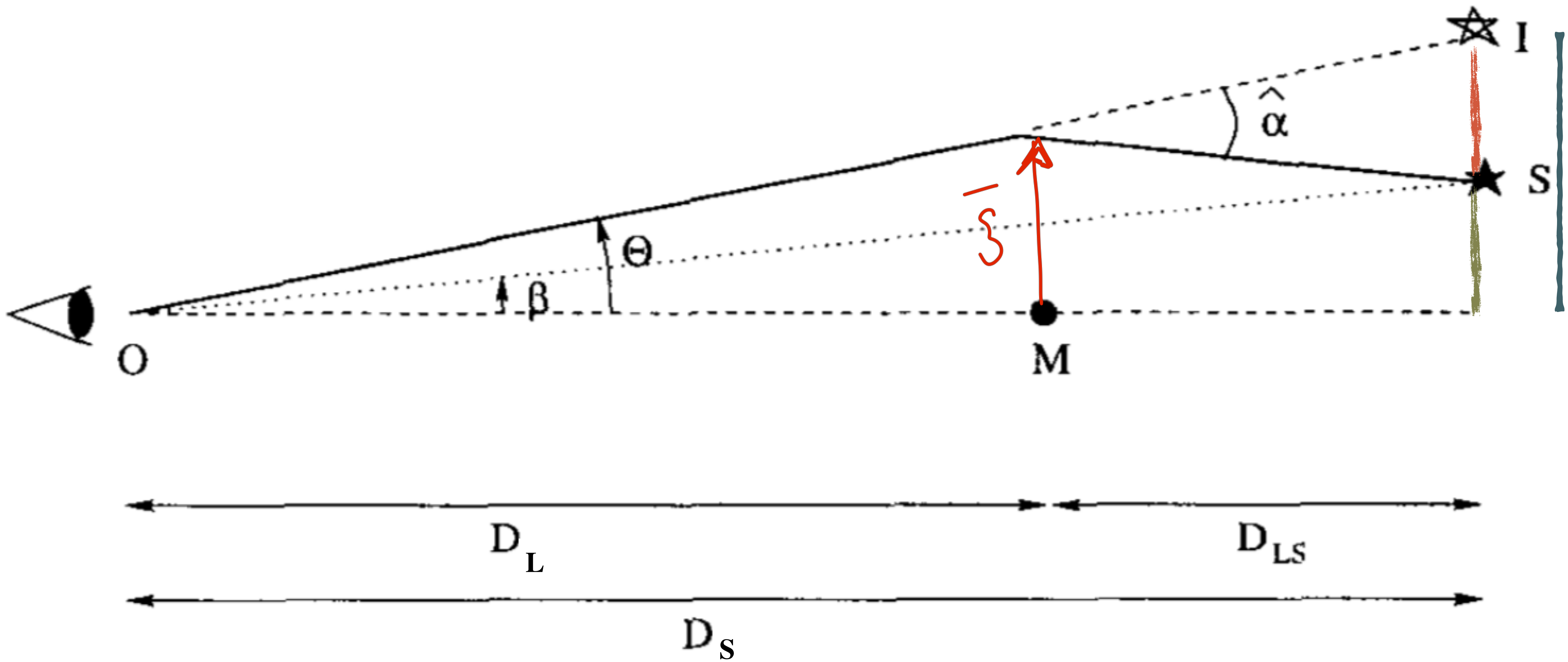
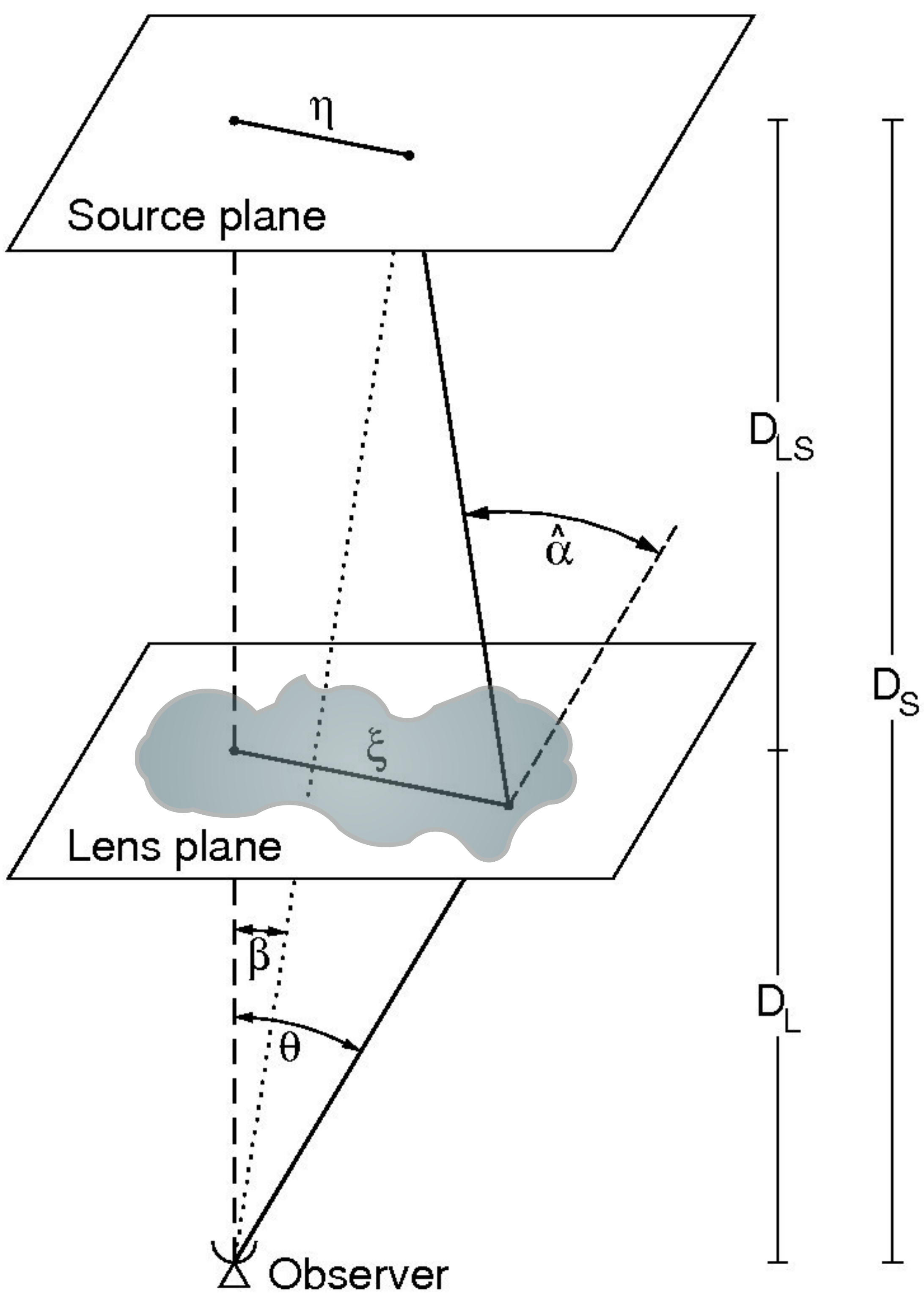



$$D_S\theta = D_S\beta + D_{LS}\hat{\alpha} \Rightarrow \beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}(\theta)$$

# LENS EQUATION




*Remember that:*

1) *we are using the "Thin Screen Approximation"*
2) *positions on the lens and source planes are defined by vectors*
3) *the deflection angle itself is a vector*

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$$

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} \qquad \vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$

# OTHER NOTATIONS

*Quite often, an alternative way is chosen to write the lens equation: the so called “dimension-less” notation.*

*This implies the choice of a reference angle (or length) to scale the source and image positions and the deflection angle:*

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S} \qquad \vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) \qquad \vec{\beta} = \vec{\theta} - \vec{\alpha}$$

$$\theta_0 = \frac{\xi_0}{D_L} = \frac{\eta_0}{D_S}$$

*the reference angle subtends the reference scales on the lens and on the source planes*

*dividing both members of the lens equation by the reference angle…*

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x})$$

$$\vec{\alpha}(\vec{x}) = \frac{\vec{\alpha}(\theta)}{\theta_0} = \frac{D_L}{\xi_0}\frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta})$$