GRAVITATIONAL LENSING

8 - NUMERICAL STUDY OF A COMPLEX LENS
MICROLENSING AND POINT-MASS LENS (1)

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AA 2018-2019

*Please refer to Jupyter Notebook 7**b_2020***

for the first part of this lecture or click here

MICROLENSING

- Microlensing is a lensing regime which include effects produced by a broad range of masses: from planets to ensembles of stars
- given the small sizes of the lens, these are (to first-order) assimilated to point masses.
- microlensing effects are mostly detectable and searched within our own galaxy, in particular by monitoring huge amounts of stars in the bulge of the MW or in the Magellanic Clouds
- nevertheless, microlensing effects are important also in extragalactic lenses. Small masses in distant galaxies, for example, introduce perturbations to the lensing signal of their hosts

THE POINT MASS LENS MODEL

- The deflection angle of the point mass lens was derived in the first lecture
- the lensing potential can be readily derived

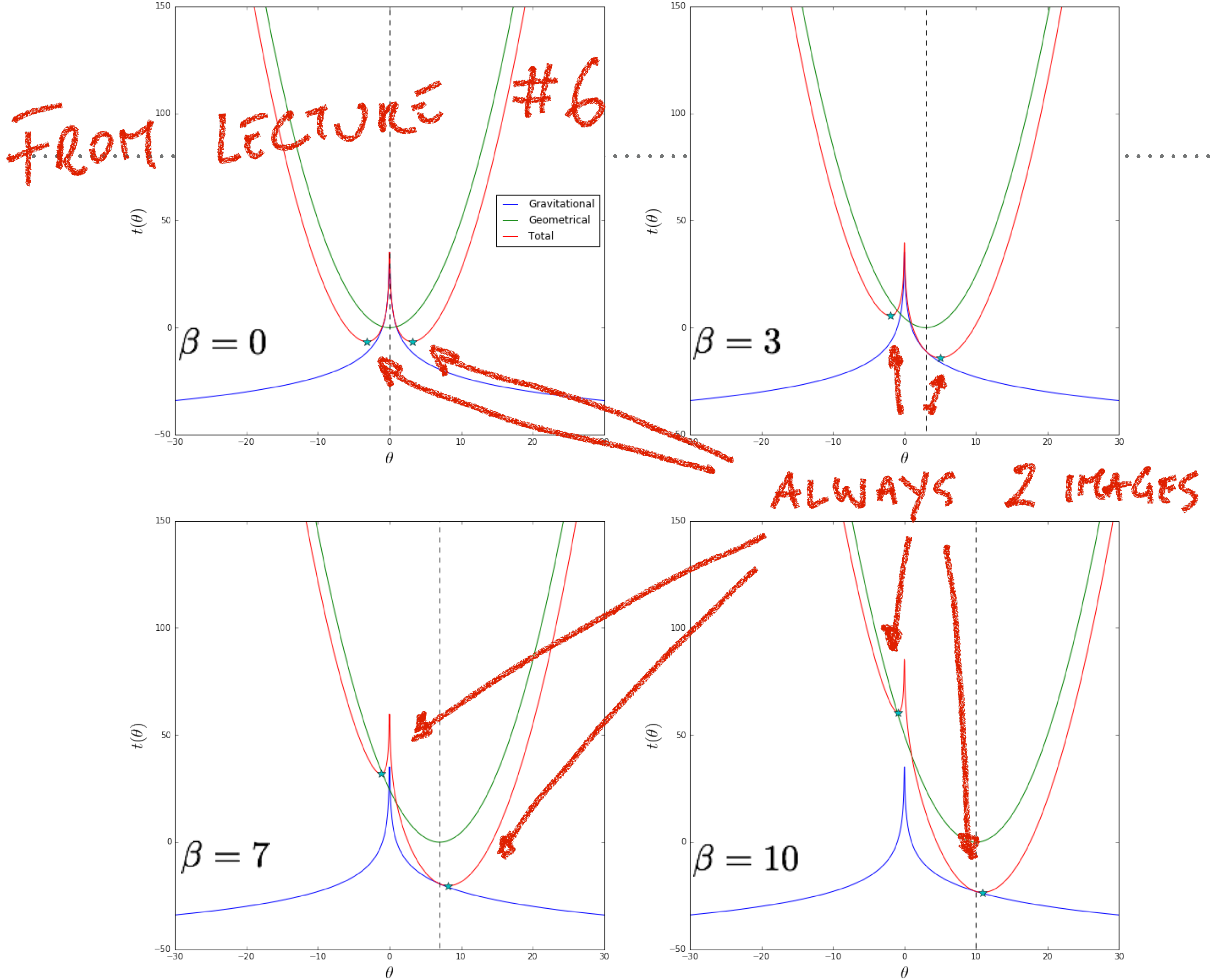
$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2}\frac{\vec{\xi}}{|\vec{\xi}|^2}; \quad \hat{\vec{\alpha}}(\vec{\theta}) = \frac{4GM}{c^2 D_L}\frac{\vec{\theta}}{|\vec{\theta}|^2}$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\nabla}_\theta \hat{\Psi}(\vec{\theta})$$

$$\vec{\nabla}\ln|\vec{x}| = \frac{\vec{x}}{|\vec{x}|^2}$$

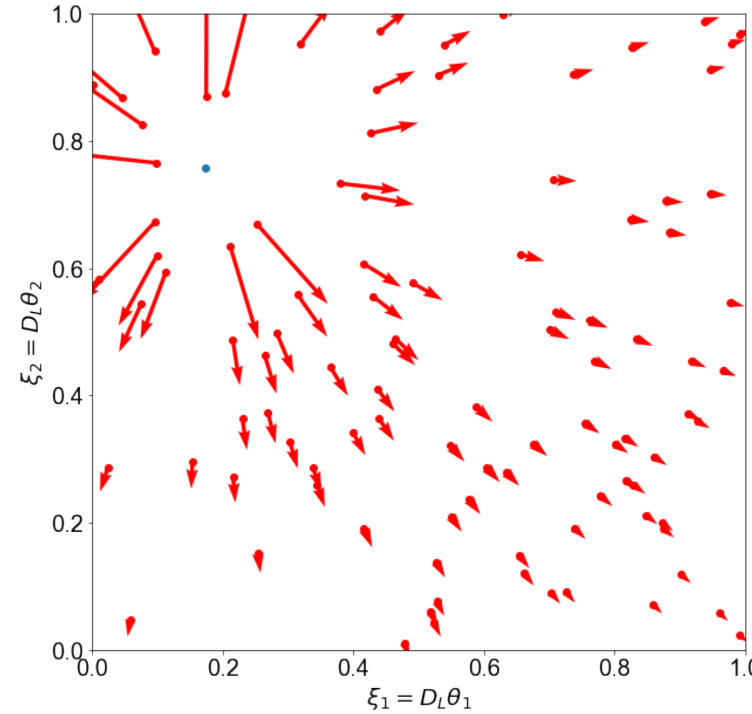
$$\Downarrow$$

$$\hat{\Psi}(\vec{\theta}) = \frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}\ln|\vec{\theta}|$$


FROM LECTURE #6
Gravitational
Geometrical
Total
$\beta = 0$
$\beta = 3$
$\beta = 7$
$\beta = 10$
$t(\theta)$
θ
ALWAYS 2 IMAGES

LENS EQUATION

- the deflection angle always points away from the lens



$$\alpha(\theta) = \frac{4GM}{c^2\theta}\frac{D_{LS}}{D_L D_S}$$

- given the symmetry of the lens, we can omit the vector notation in most equations (e.g. in the lens equation)
- the lens equation reads:

$$\beta = \theta - \alpha(\theta) = \theta - \frac{4GM}{c^2\theta}\frac{D_{LS}}{D_L D_S}$$

- this is clearly quadratic in θ, with positive discriminant

$$\theta^2 - \beta\theta - \frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S} = 0$$

- so, for each source there are two images, whose positions can be determined by solving the lens equation

SOLUTIONS OF THE LENS EQUATION

- we introduce the Einstein radius:

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$$

- by inserting into the lens equation:

$$\beta = \theta - \frac{4GM}{c^2\theta}\frac{D_{LS}}{D_L D_S} = \theta - \frac{\theta_E^2}{\theta}$$

- if we divide by θ_E, we obtain an a-dimensional form of the lens equation

$$\frac{\beta}{\theta_E} = \frac{\theta}{\theta_E} - \frac{\theta_E}{\theta} \Rightarrow y = x - \frac{1}{x}$$

- this is a very convenient way of writing the lens equation, because we get rid of all constants.

$$x = \frac{\theta}{\theta_E} \qquad y = \frac{\beta}{\theta_E}$$

SOLUTIONS OF THE LENS EQUATION

$$y = x - \frac{1}{x}$$

$$\Downarrow$$

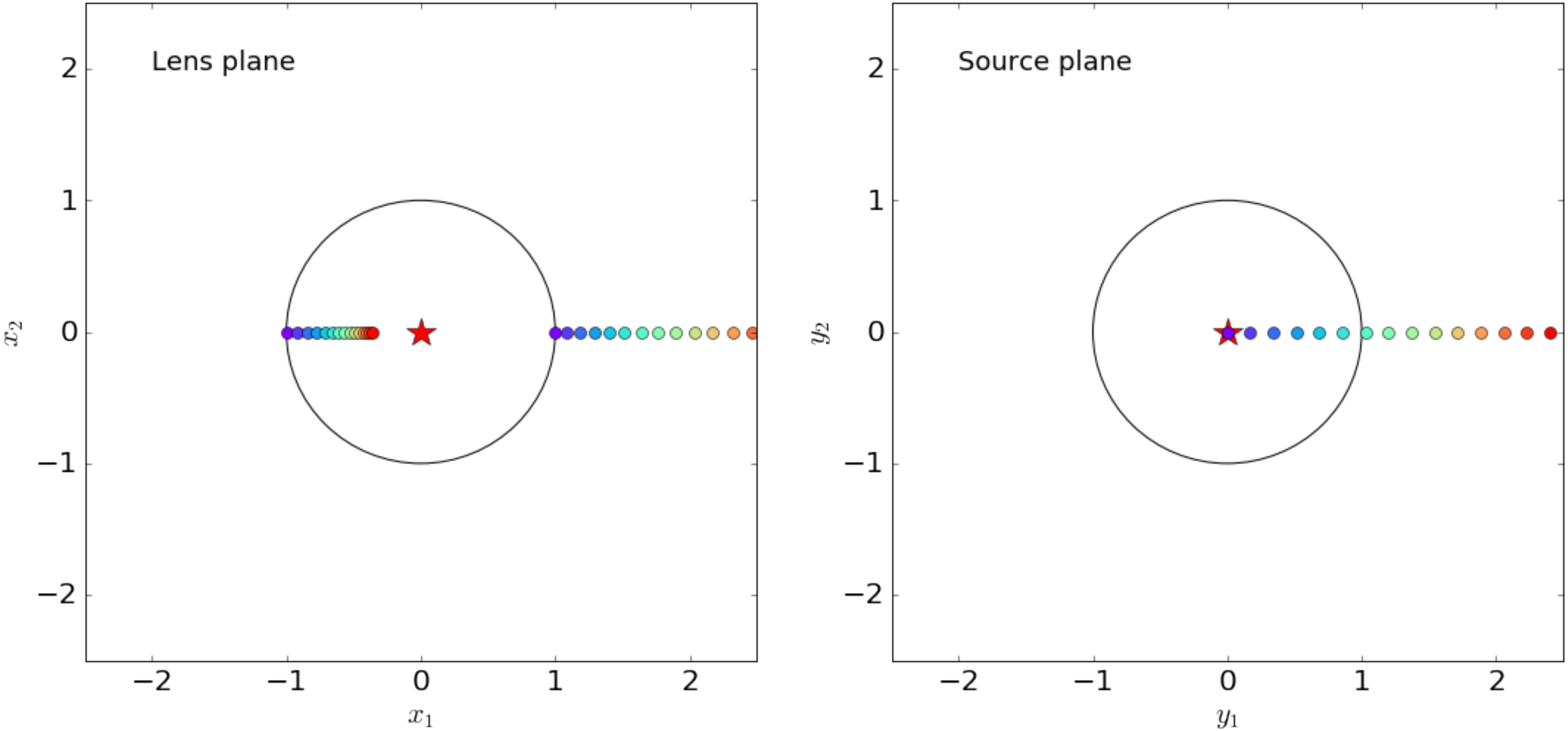
$$x^2 - xy - 1 = 0$$

$$\Downarrow$$

$$x_{\pm} = \frac{1}{2}\left[y \pm \sqrt{y^2 + 4}\right]$$

or $$\theta_{\pm} = \frac{1}{2}\left[\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right]$$ *(in angular units)*

SOLUTIONS OF THE LENS EQUATION

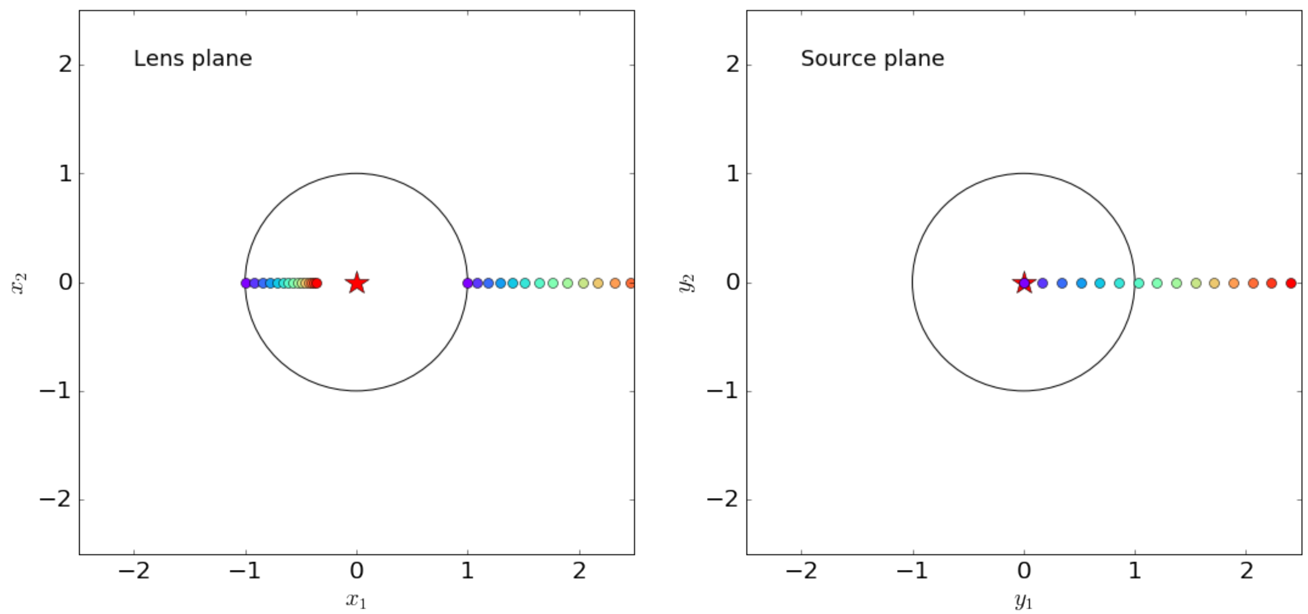


If y>0 $\quad x_+ > 1 \quad -1 < x_- < 0$

$$x_\pm = \frac{1}{2}\left[y \pm \sqrt{y^2+4}\right]$$

As $y \to \infty$, *we see that* $x_+ \to \infty$, *while* $x_- \to 0$

PROPERTIES OF THE IMAGES



$$x_\pm = \frac{1}{2}\left[y \pm \sqrt{y^2+4}\right]$$

For y=0, the image is a full ring: $x_\pm = \pm 1$

This is the ***Einstein ring*** $x = \frac{\theta}{\theta_E}$

One of the images is internal to the Einstein radius, the other is external

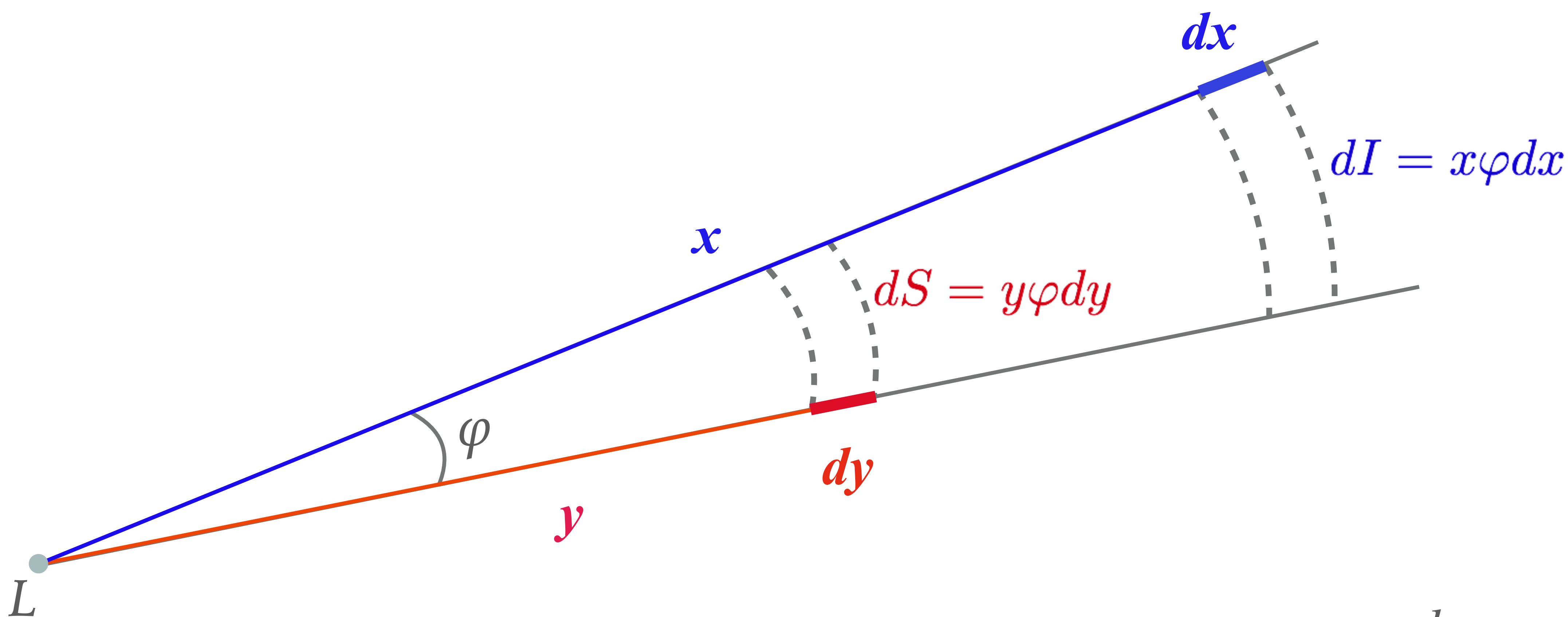
SIZE OF THE EINSTEIN RADIUS

$$\theta_E \equiv \sqrt{\frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm L}D_{\rm S}}} \qquad\qquad D \equiv \frac{D_{\rm L}D_{\rm S}}{D_{\rm LS}}$$

$$\begin{aligned}\theta_E &\approx (10^{-3})'' \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{D}{10\mathrm{kpc}}\right)^{-1/2}, \\ &\approx 1'' \left(\frac{M}{10^{12} M_\odot}\right)^{1/2} \left(\frac{D}{\mathrm{Gpc}}\right)^{-1/2},\end{aligned}$$

For a star like the sun within the MW, the Einstein radius is of the order of milli-arcseconds!

MAGNIFICATION



$$\det A(x) = \frac{y}{x}\frac{dy}{dx}$$

CRITICAL LINES AND CAUSTICS

$$y = x - \frac{1}{x}$$

From the lens equation, it follows that:

$$\lambda_t(x) = \frac{y}{x} = 1 - \frac{1}{x^2}$$

$$\lambda_r(x) = \frac{dy}{dx} = 1 + \frac{1}{x^2}$$

The second eigenvalue is always positive (no critical line). The first is zero on the circle

$$x^2 = 1$$

Thus, the Einstein ring is the tangential critical line! The corresponding caustic is a point at $y=0$

IMAGE MAGNIFICATION

Clearly,

$$\det A(x) = \frac{y}{x}\frac{dy}{dx}$$

$$\lambda_t(x) = \frac{y}{x} = 1 - \frac{1}{x^2}$$

$$\lambda_r(x) = \frac{dy}{dx} = 1 + \frac{1}{x^2}$$

$$\det A(x) = \frac{y}{x}\frac{dy}{dx} = \left(1 - \frac{1}{x^4}\right)$$

$$\Rightarrow$$

$$\mu(x) = \det A^{-1}$$

$$\mu(x) = \left(1 - \frac{1}{x^4}\right)^{-1}$$

IMAGE PARITY

Note that:

$$y > 0 \quad \Rightarrow \quad x_+ > 0, \quad x_- < 0$$

$$\mu_t = \frac{x}{y} \quad \Rightarrow \quad \mu_t(x_+) > 0, \quad \mu_t(x_-) < 0$$

$$\mu_r = \frac{dx}{dy} > 0$$

⇒ *Thus the parity of the images is different!*

Not surprising given that the two images are separated by the critical line

SOURCE MAGNIFICATION

Let's compute now the source magnification. This is the sum of the magnifications of the two images

$$x_{\pm} = \frac{1}{2}\left[y \pm \sqrt{y^2+4}\right]$$

$$\begin{aligned} \frac{x}{y} &= \frac{1}{2}\left(1 \pm \frac{\sqrt{y^2+4}}{y}\right) \\ \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{2}\left(1 \pm \frac{y}{\sqrt{y^2+4}}\right) . \end{aligned}$$

Thus the magnifications at the two image positions are

$$\begin{aligned} \mu_{\pm}(y) &= \frac{1}{4}\left(1 \pm \frac{\sqrt{y^2+4}}{y}\right)\left(1 \pm \frac{y}{\sqrt{y^2+4}}\right) \\ &= \frac{1}{4}\left(1 \pm \frac{\sqrt{y^2+4}}{y} \pm \frac{y}{\sqrt{y^2+4}} + 1\right) \\ &= \frac{1}{4}\left(2 \pm \frac{2y^2+4}{y\sqrt{y^2+4}}\right) = \frac{1}{2}\left(1 \pm \frac{y^2+2}{y\sqrt{y^2+4}}\right) \end{aligned}$$