GRAVITATIONAL LENSING

19 - AXIALLY SYMMETRIC LENSES

Massimo Meneghetti

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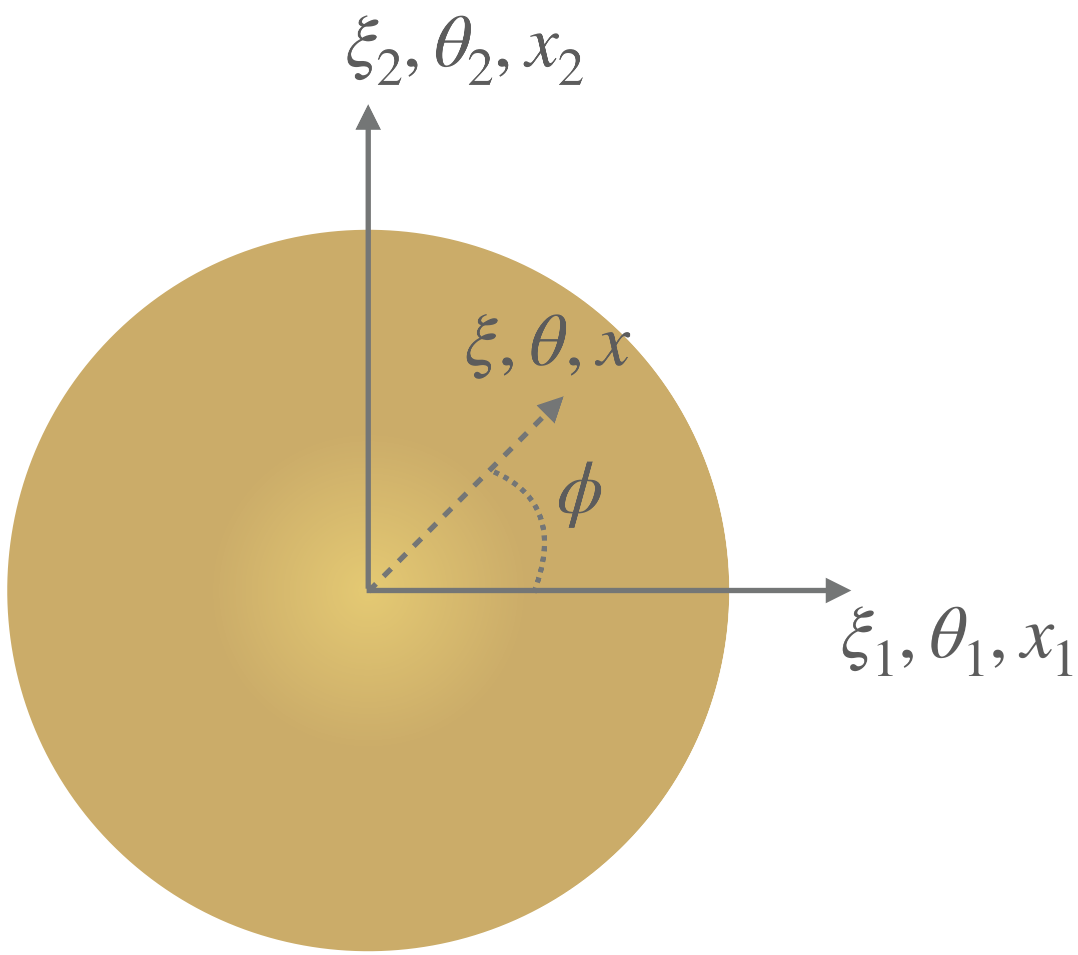
RECAP



$$\xi_0 = D_L \theta_0 = \text{scala di rif. arbitraria}$$

$$m(x) = \frac{2\pi \int_0^{\xi} \xi' \Sigma(\xi') d\xi'}{\pi \xi_0^2 \Sigma_{crit}} = \frac{M(\xi)}{M_{crit}(\xi_0)}$$

$$m(x) = 2 \int_0^x x' \kappa(x') dx'$$

$$\kappa(x) = \frac{m'(x)}{2x}$$

POWER-LAW LENS

$$\kappa(x) = \frac{3-n}{2} x^{1-n}$$ n=parameter

$$m(x) = 2\int_0^x x'\kappa(x')dx' = (3-n)\int_0^x x'^{2-n}dx' = x^{3-n}$$

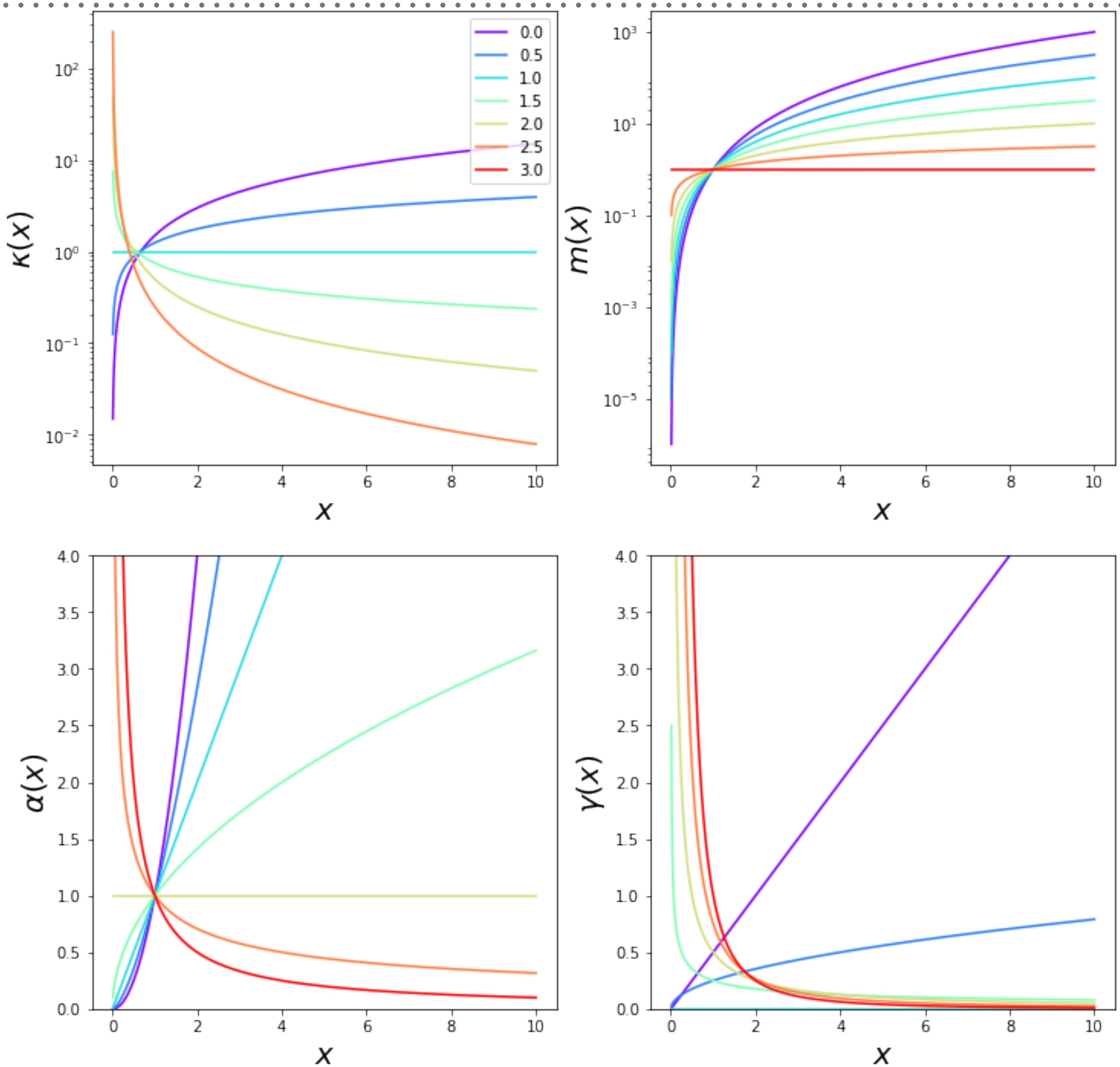
$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$

$$\gamma(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = \bar{\kappa}(x) - \kappa(x) = x^{1-n} - \frac{3-n}{2}x^{1-n} = \frac{n-1}{2}x^{n-1}$$

$$y = x - \alpha(x) = x - x^{2-n}$$

$$\det A = \frac{y}{x}\frac{dy}{dx} = (1 - x^{1-n})[1 - (2-n)x^{1-n}]$$

POWER-LAW LENS



POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

The tangential critical line has equation $x=1$ for any value of the slope parameter n.

The caustic is the point $y=0$

$$y_{cau} = x_{crit} - \alpha(x_{crit}) \ \Rightarrow y_{cau,t} = 0 \ \ \forall \ n$$

Actually, this is true for any axially symmetric lens!

$$\lambda_t = 1 - \frac{m(x)}{x^2} = 0 \ \Rightarrow y_{cau,t} = x_{crit,t} \left[1 - \frac{m(x_{crit,t})}{x^2_{crit,t}} \right] = 0$$

POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

Instead, the size of the radial critical line depends on n:

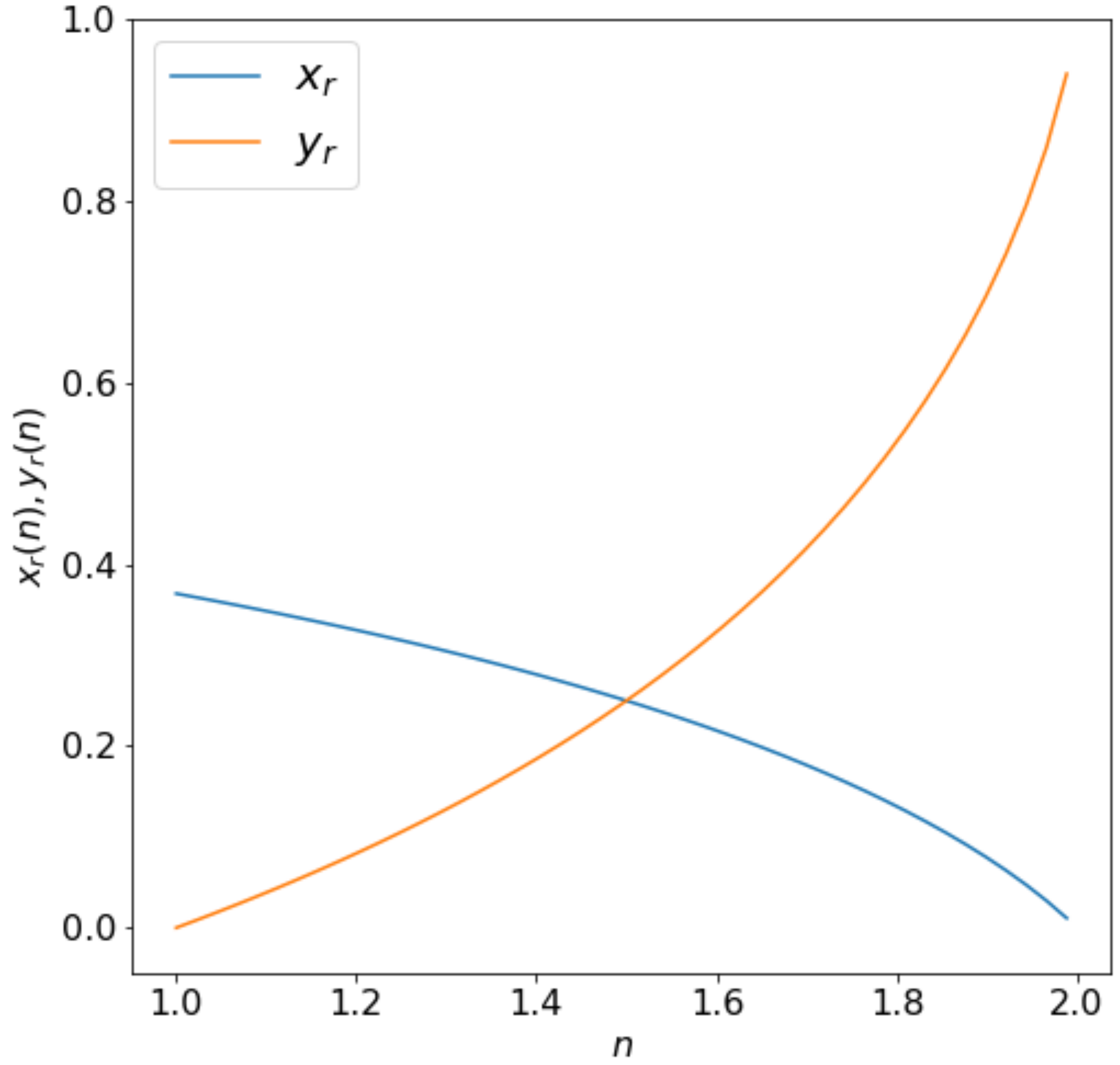
$$\lambda_r = 1 + \frac{m(x)}{x^2} - \frac{m'(x)}{x} = 1 + x^{1-n} - (3-n)x^{1-n} = 1 - x^{1-n}(2-n) = 0$$

$$x_{crit,r} = (2-n)^{1/(n-1)}$$

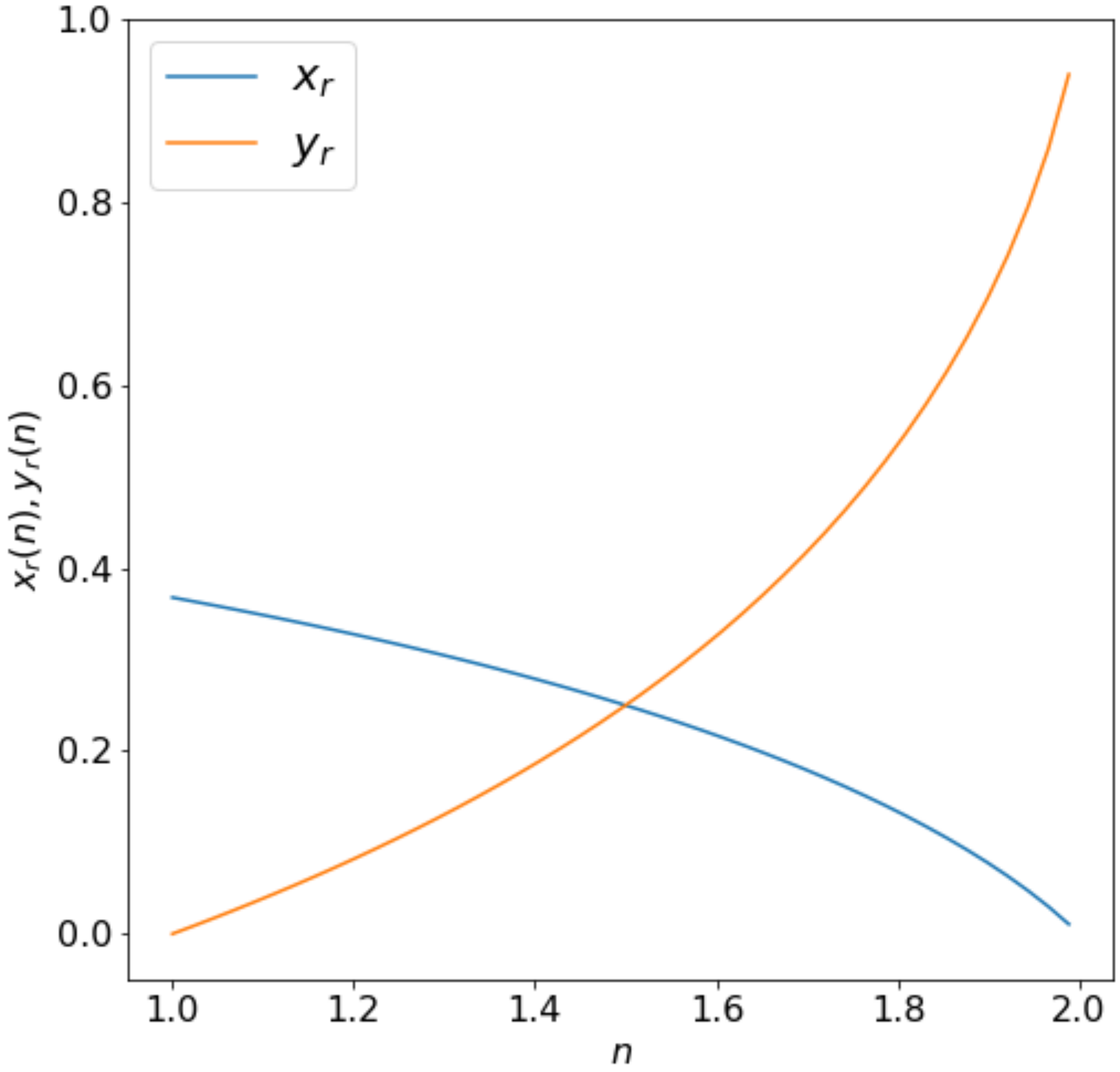
And the corresponding caustic is:

$$y_{cau,r} = x_{crit,r} - x_{crit,r}^{2-n}$$

RADIAL CRITICAL LINE

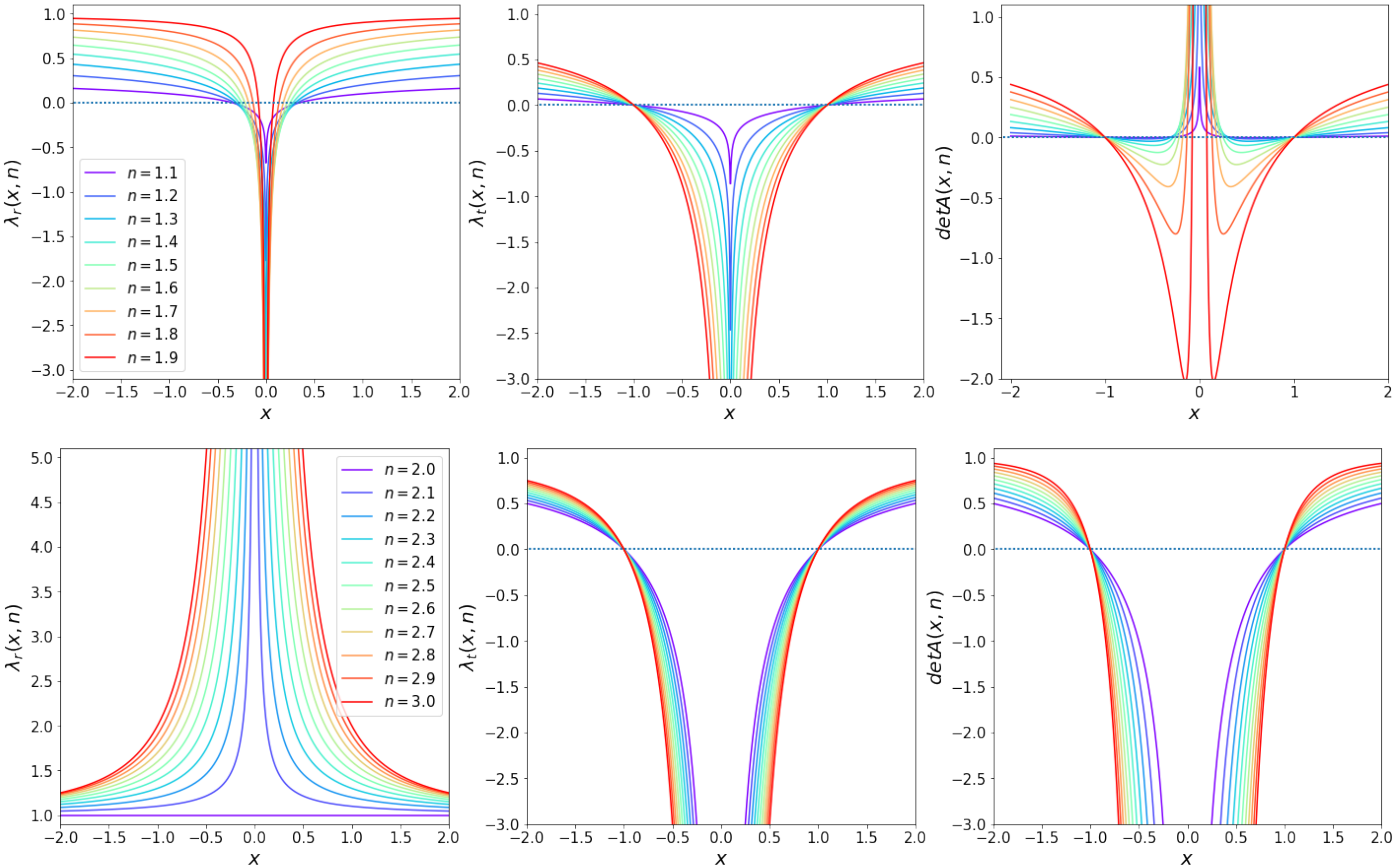


RADIAL CRITICAL LINE



Large n, small radial critical line

$$
\begin{aligned}
\lambda_t(x) &= 1 - x^{1-n} \\
\lambda_r(x) &= 1 - (2-n)x^{1-n}
\end{aligned}
$$



no radial critical line if n>=2!

$$
\begin{aligned}
\lambda_t(x) &= 1 - x^{1-n} \\
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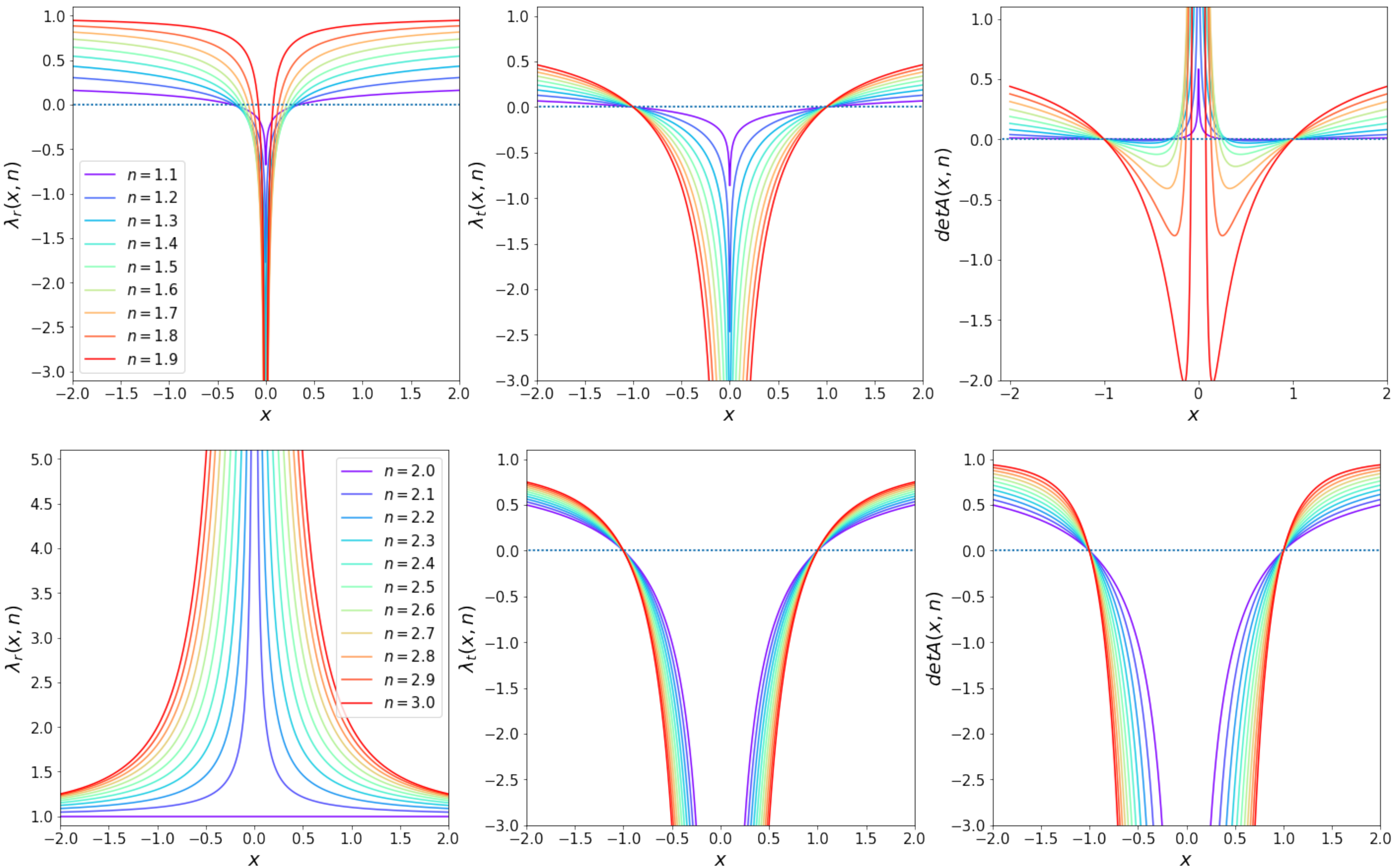
\end{aligned}
$$



THE SINGULAR ISOTHERMAL SPHERE

The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is in isothermal and hydrostatic equilibrium, its density profile is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

velocity dispersion of the gas particles

The profile is "unphysical"

- *singularity near the center*
- *mass is infinite*

THE SINGULAR ISOTHERMAL SPHERE

For lensing purposes, we are interested in the projection of this profile:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \qquad r^2 = \xi^2 + z^2$$

$$\begin{aligned} \Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\ &= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\ &= \frac{\sigma_v^2}{2G\xi}\,. \end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

As usual, we can switch to dimensionless units.

Let's take

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_L D_{LS}}{D_S}$$

Then:

$$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}} = \frac{1}{2x}\Sigma_{cr}\,.$$

$$\kappa(x) = \frac{1}{2x}$$

Thus, the SIS lens is a power-law lens with n=2!

THE SINGULAR ISOTHERMAL SPHERE

The mass profile is readily computed:

$$m(x) = |x|$$

as well as the deflection angle:

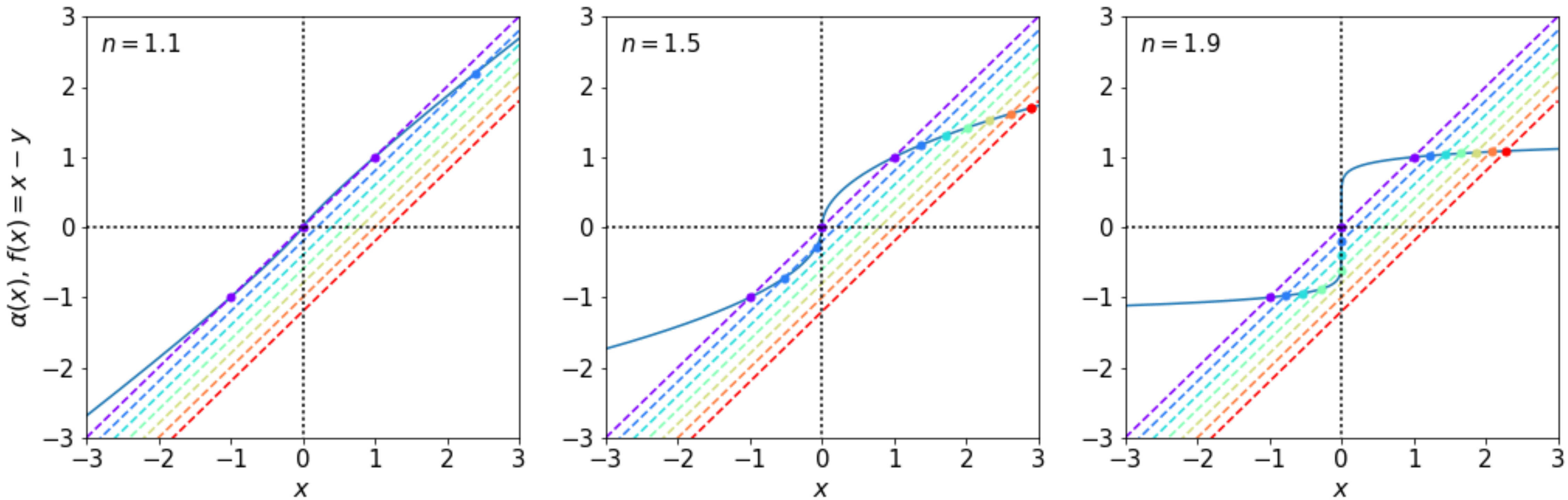
$$\alpha(x) = \frac{x}{|x|}$$

The lens equation reads

$$y = x - \frac{x}{|x|}$$

How many solutions does this equation have?

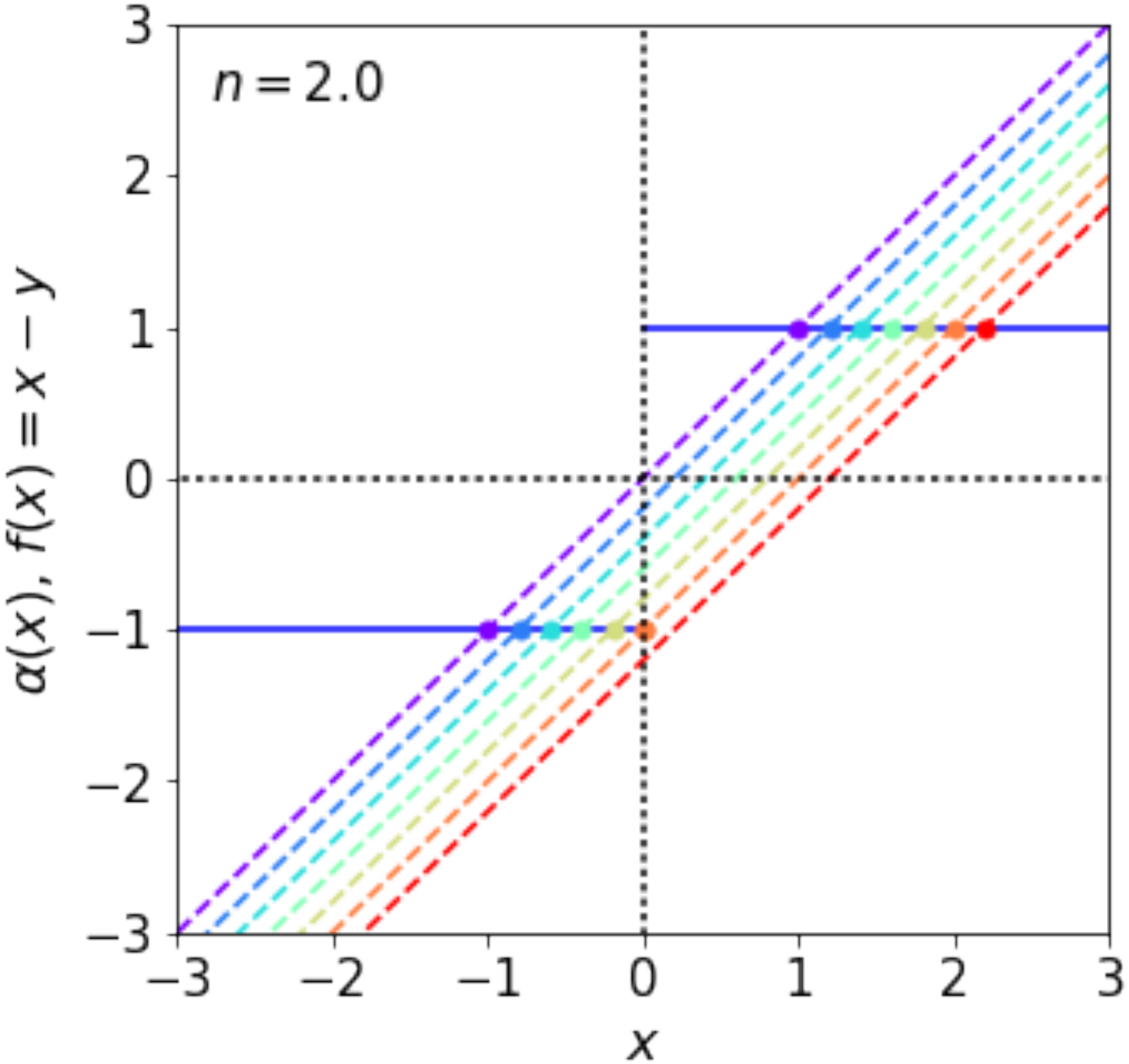
IMAGE DIAGRAM



radial critical line:

$$\left.\frac{d\alpha(x)}{dx}\right|_{x_r} = 1$$

IMAGE DIAGRAM

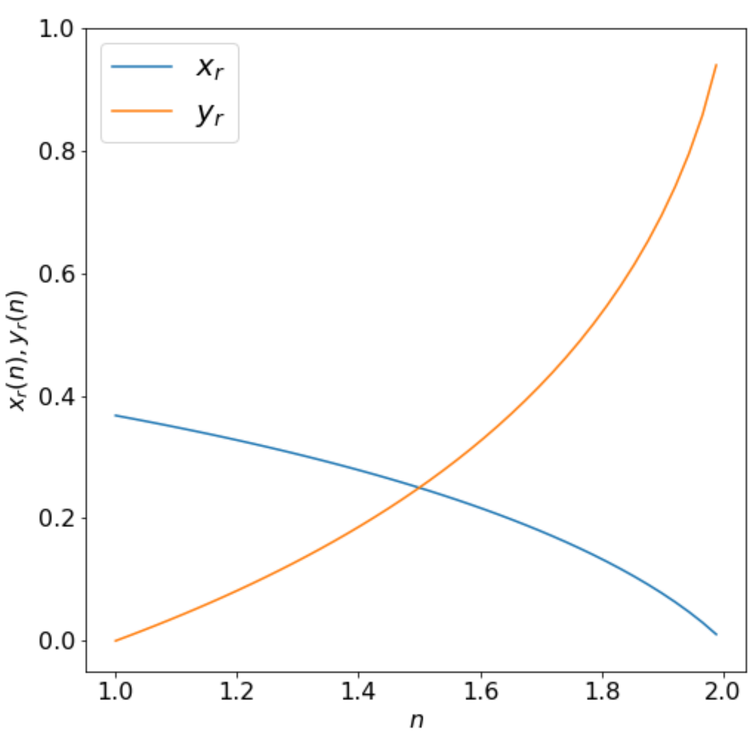


$$y_{cut} = \lim_{x \to 0} y(x) = \pm 1$$

$$\frac{d\alpha}{dx} = 0 \;\; \forall \, x$$

There is no radial critical line!

However, there is a line that almost plays the role of the caustic (at least to determine the image multiplicity…)



THE SINGULAR ISOTHERMAL SPHERE

If $0<y<1$, the solution are two:

$$x_- = y - 1$$

$$x_+ = y + 1$$

$$\theta_\pm = \beta \pm \theta_E$$

Otherwise, there is only one solution at

$$x_+ = y + 1$$

Thus, the circle of radius $y=1$ plays the same role of the radial caustic for the power-law lens with $n<2$, separating the source plane into regions with different image multiplicity.

THE SINGULAR ISOTHERMAL SPHERE

$$\frac{d\alpha}{dx} = 0 \ \ \forall \ x$$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1$

Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.

THE SINGULAR ISOTHERMAL SPHERE

The shear can be computed easily:

$$\gamma(x) = \bar{\kappa}(x) - \kappa(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = \frac{1}{x} - \frac{1}{2x} = \frac{1}{2x} = \kappa(x)$$

$$\gamma_1 = \frac{1}{2x}\cos 2\phi$$

$$\gamma_2 = \frac{1}{2x}\sin 2\phi$$

THE SINGULAR ISOTHERMAL SPHERE

as well as the magnification

$$\mu = \frac{|x|}{|x|-1}$$

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \quad ; \quad \mu_- = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

THE SINGULAR ISOTHERMAL SPHERE

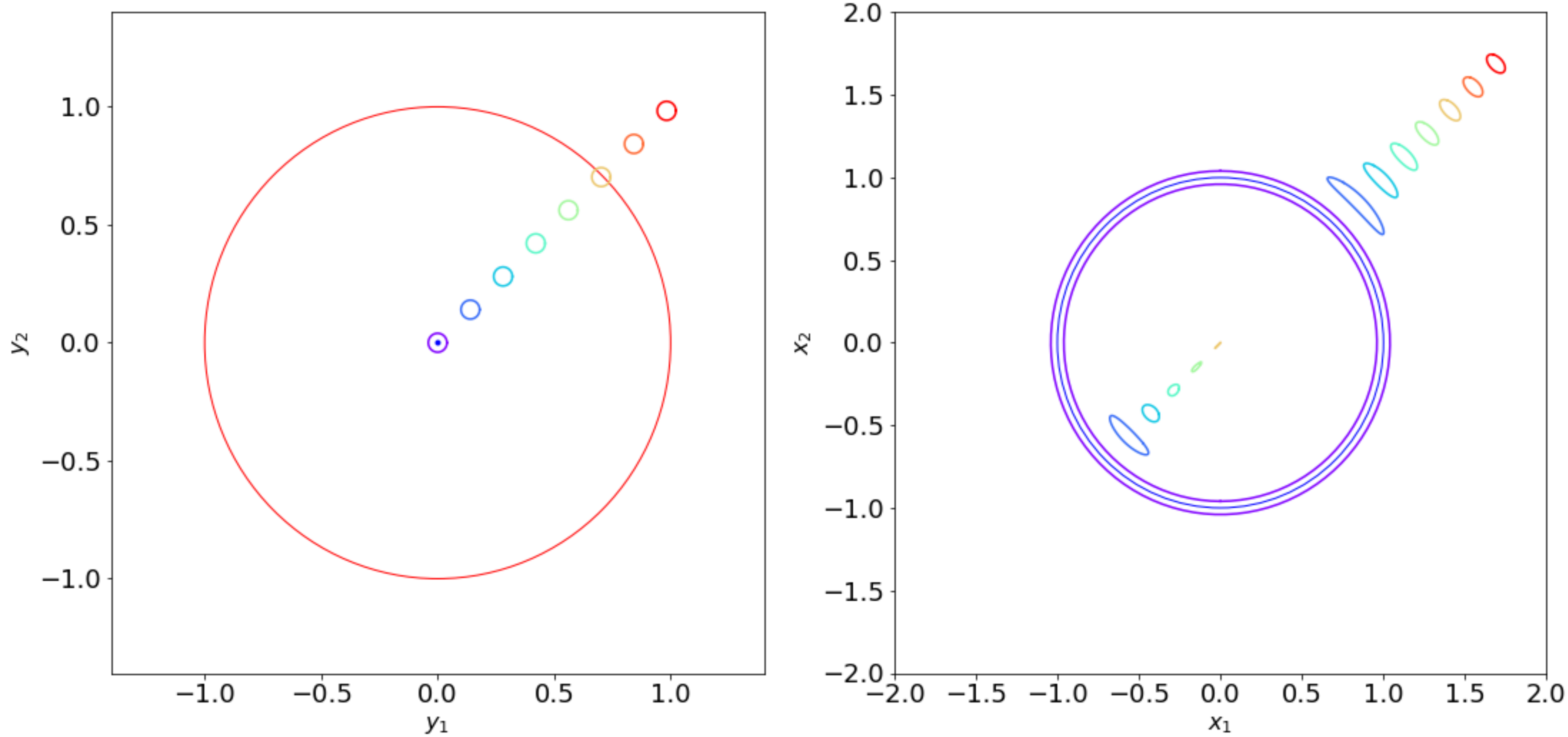
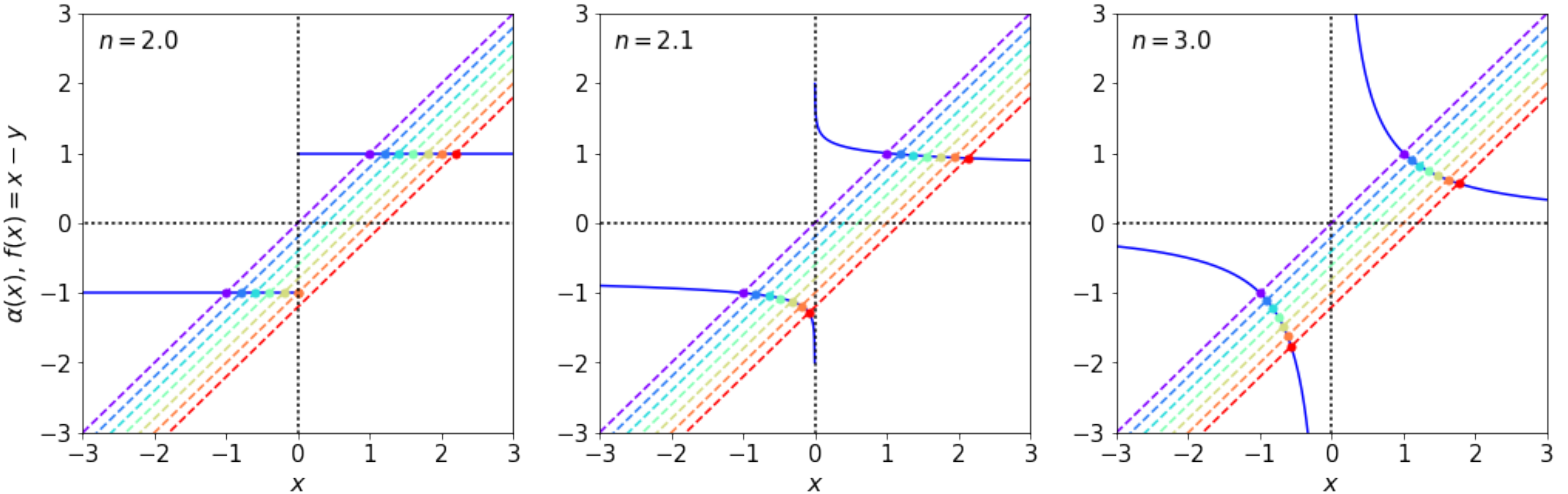


IMAGE DIAGRAM (N>=2)



PL lenses with n>2 always have 2 images, because the time delay surface is not continuously deformable.

In addition, the images are radially de-magnified!

$$
\begin{aligned}
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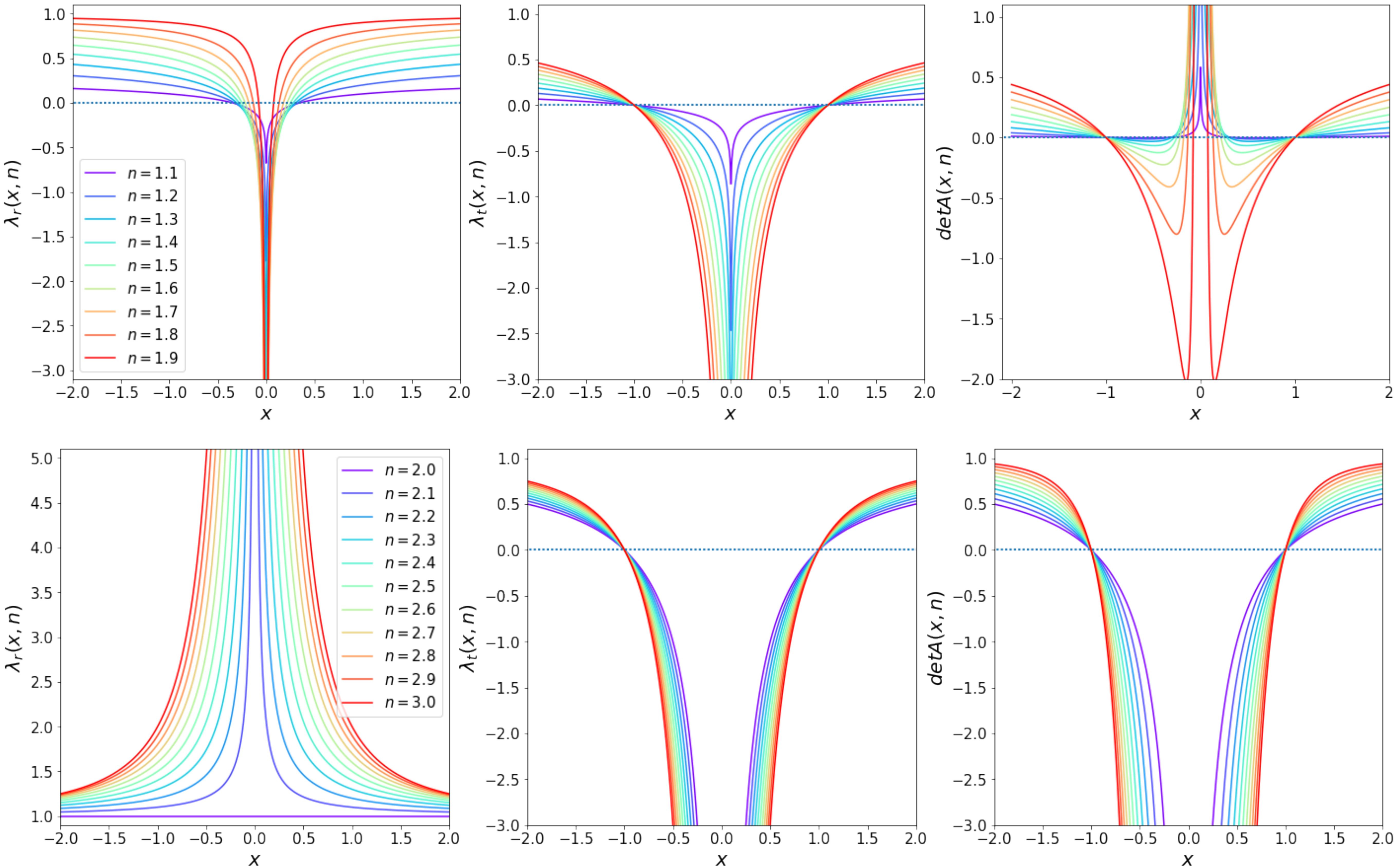
\end{aligned}
$$



no radial critical line if n>=2!

$$\lambda_t(x) = 1 - x^{1-n}$$
$$\lambda_r(x) = 1 - (2-n)x^{1-n}$$