GRAVITATIONAL LENSING

3 - DEFLECTION ANGLE (CONTINUATION) - LENS EQUATION

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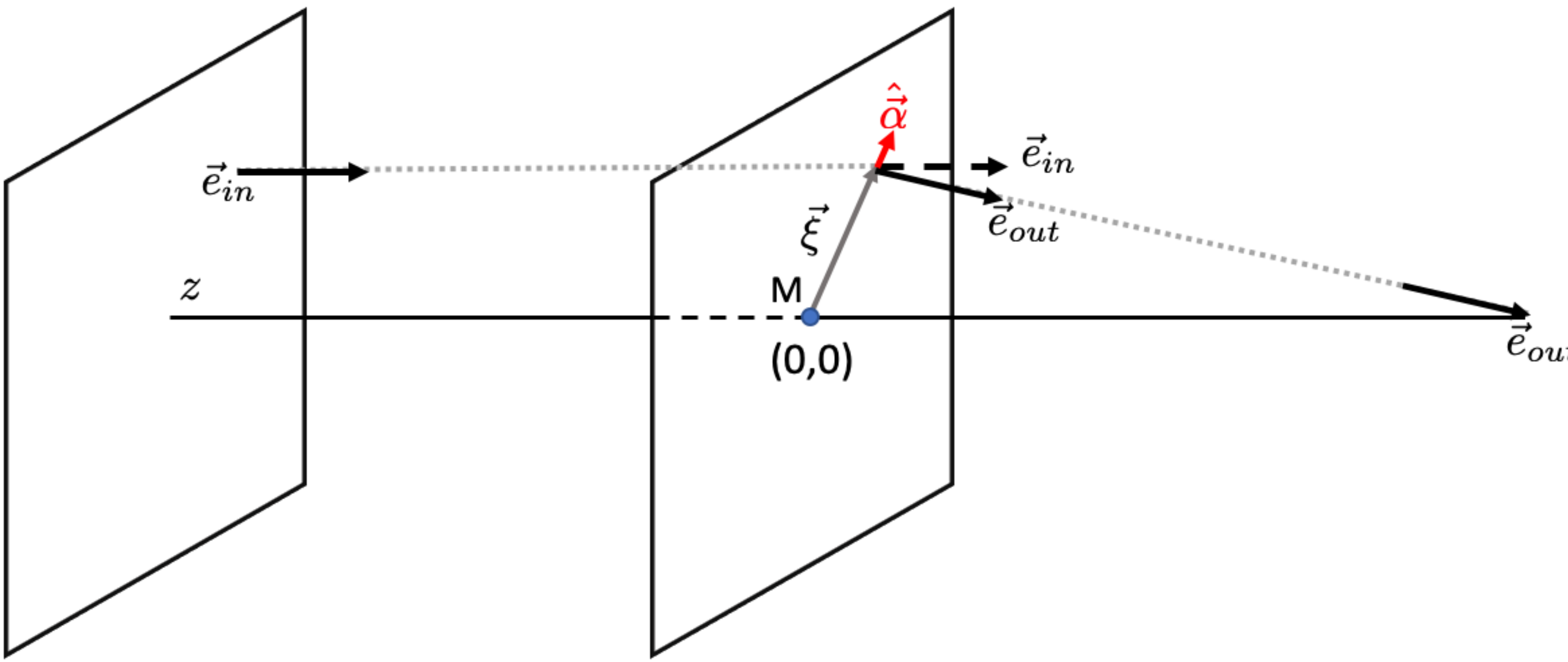
AA 2018-2019

GENERALISATION OF THE DEFLECTION ANGLE FORMULA

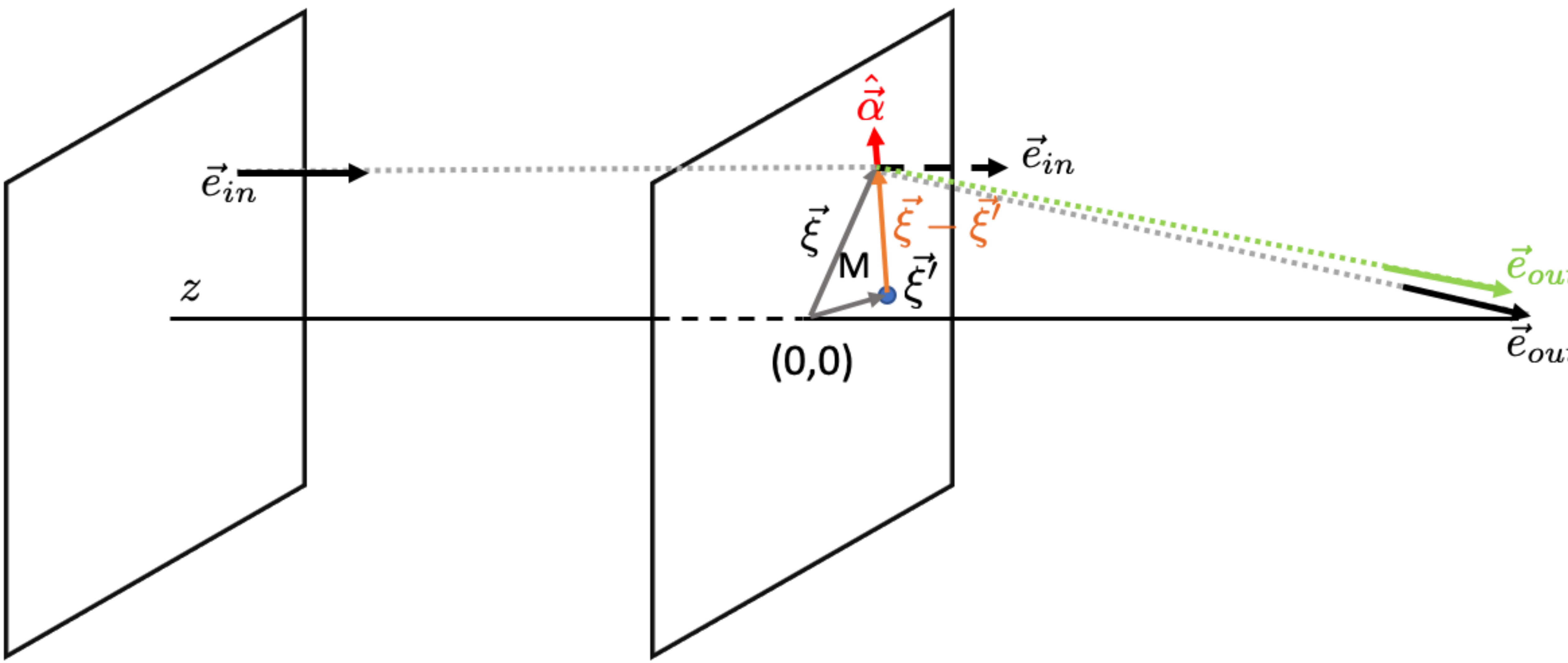
$$\hat{\overrightarrow{\alpha}}(\overrightarrow{\xi}) = \frac{4GM}{c^2\xi}\overrightarrow{e}_{\xi} = \frac{4GM}{c^2}\frac{\overrightarrow{\xi}}{|\xi|^2}$$



Using "Thin screen approximation"

GENERALISATION OF THE DEFLECTION ANGLE FORMULA

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2} \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2}$$



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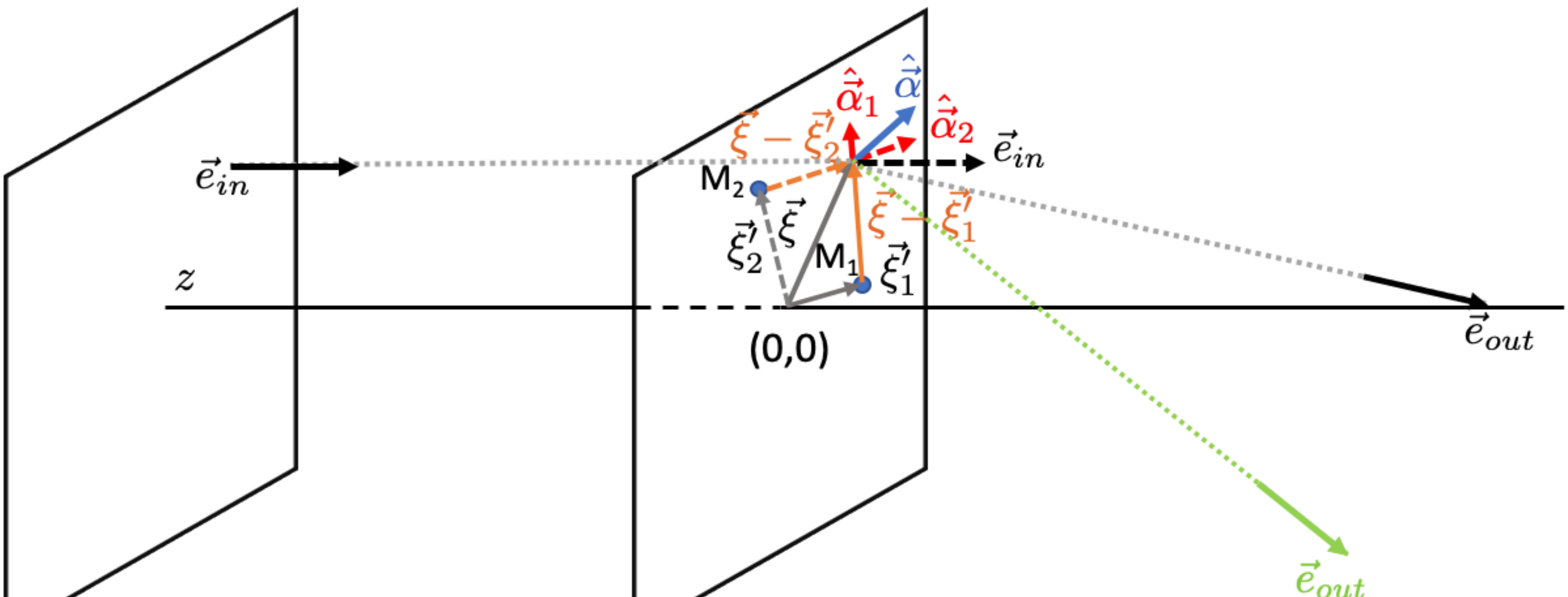
GENERALISATION OF THE DEFLECTION ANGLE FORMULA

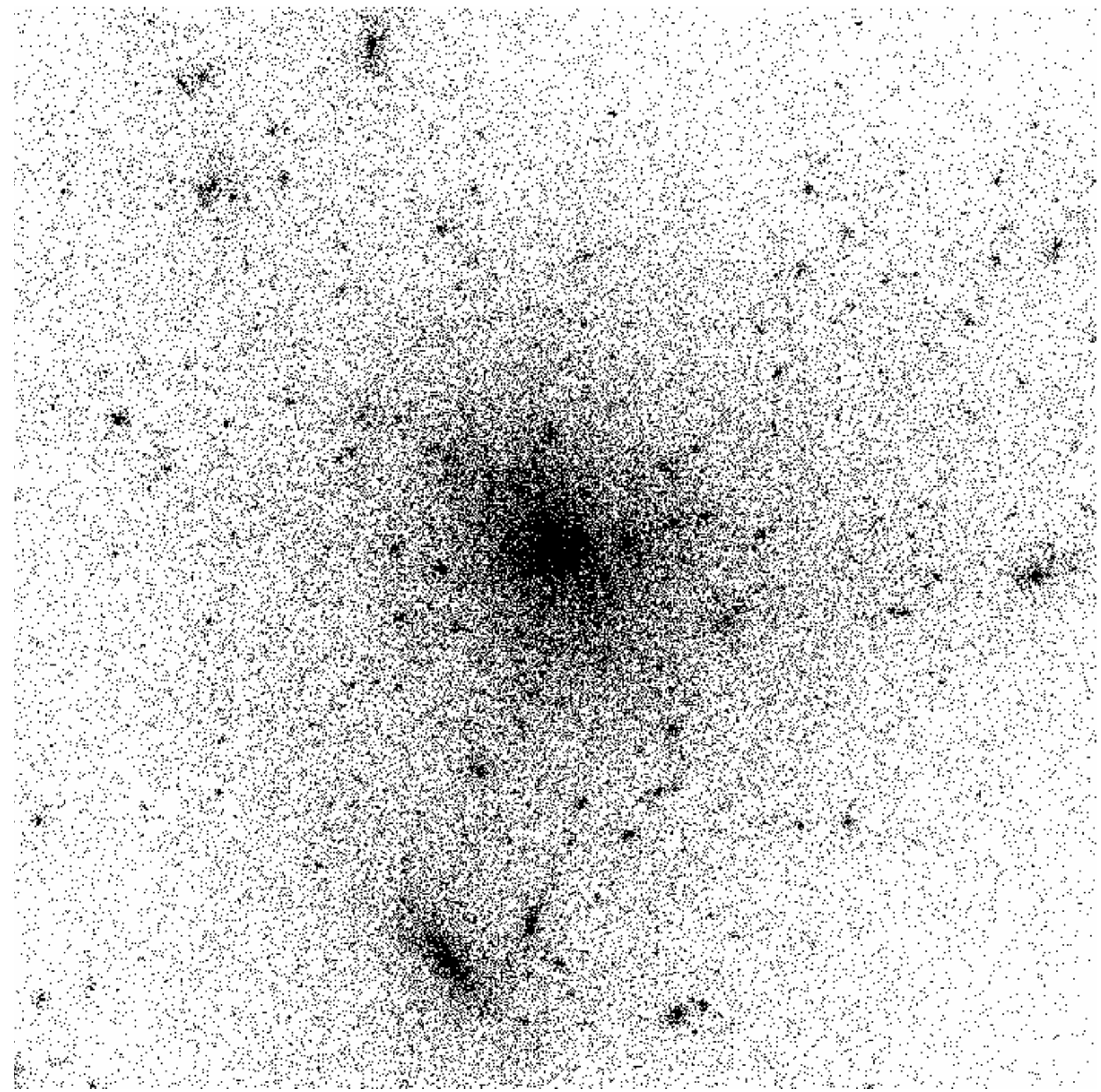
$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2}$$



Using "Thin screen approximation"

DEFLECTION BY AN ENSEMBLE OF POINT MASSES

- Structure formation is often studied using numerical simulations
- Galaxies, galaxy clusters, etc. are described by ensembles of particles
- The calculation of the deflection angle by direct summation of all contributions from each particle has a computational cost $O(N^2)$



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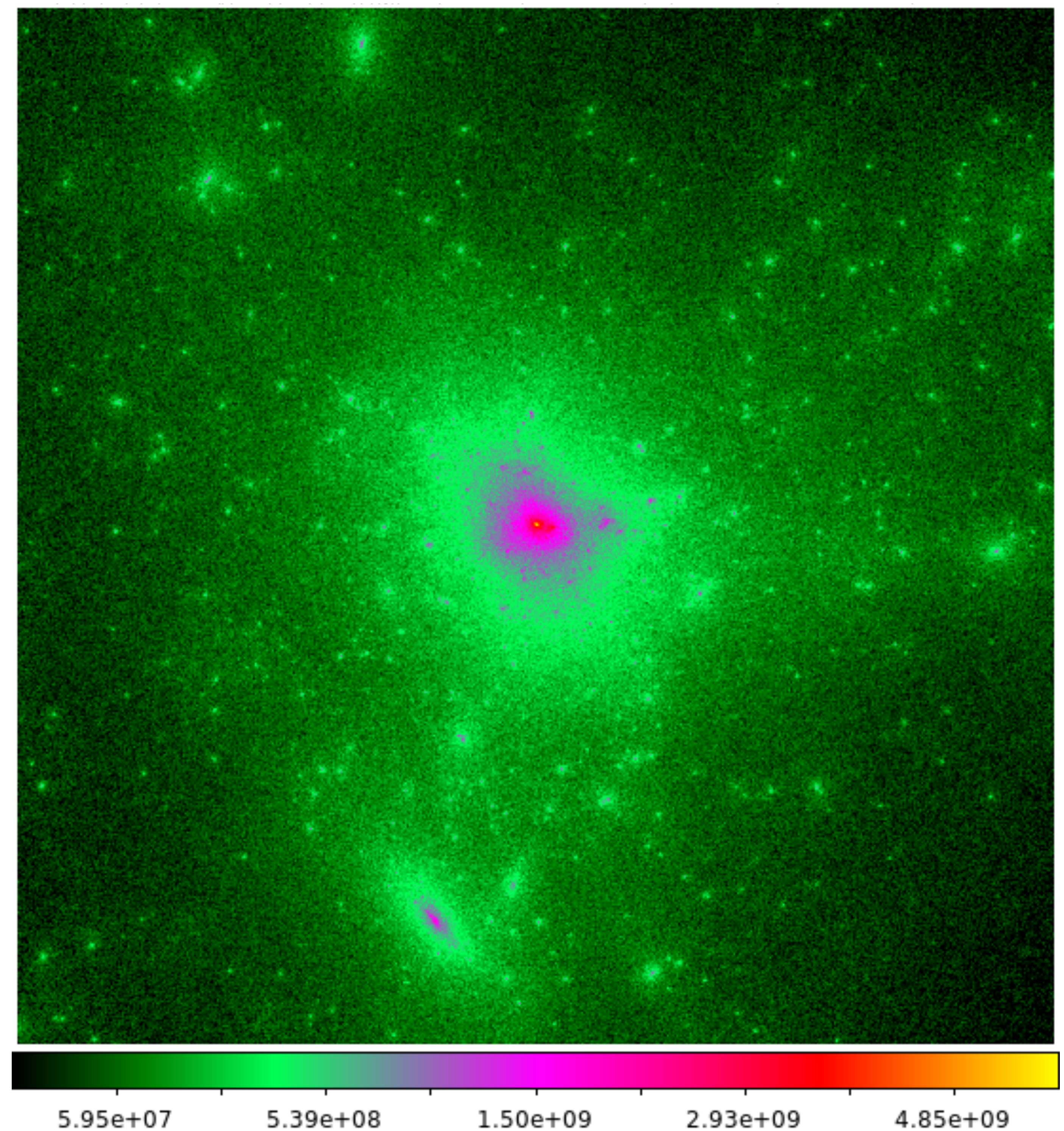


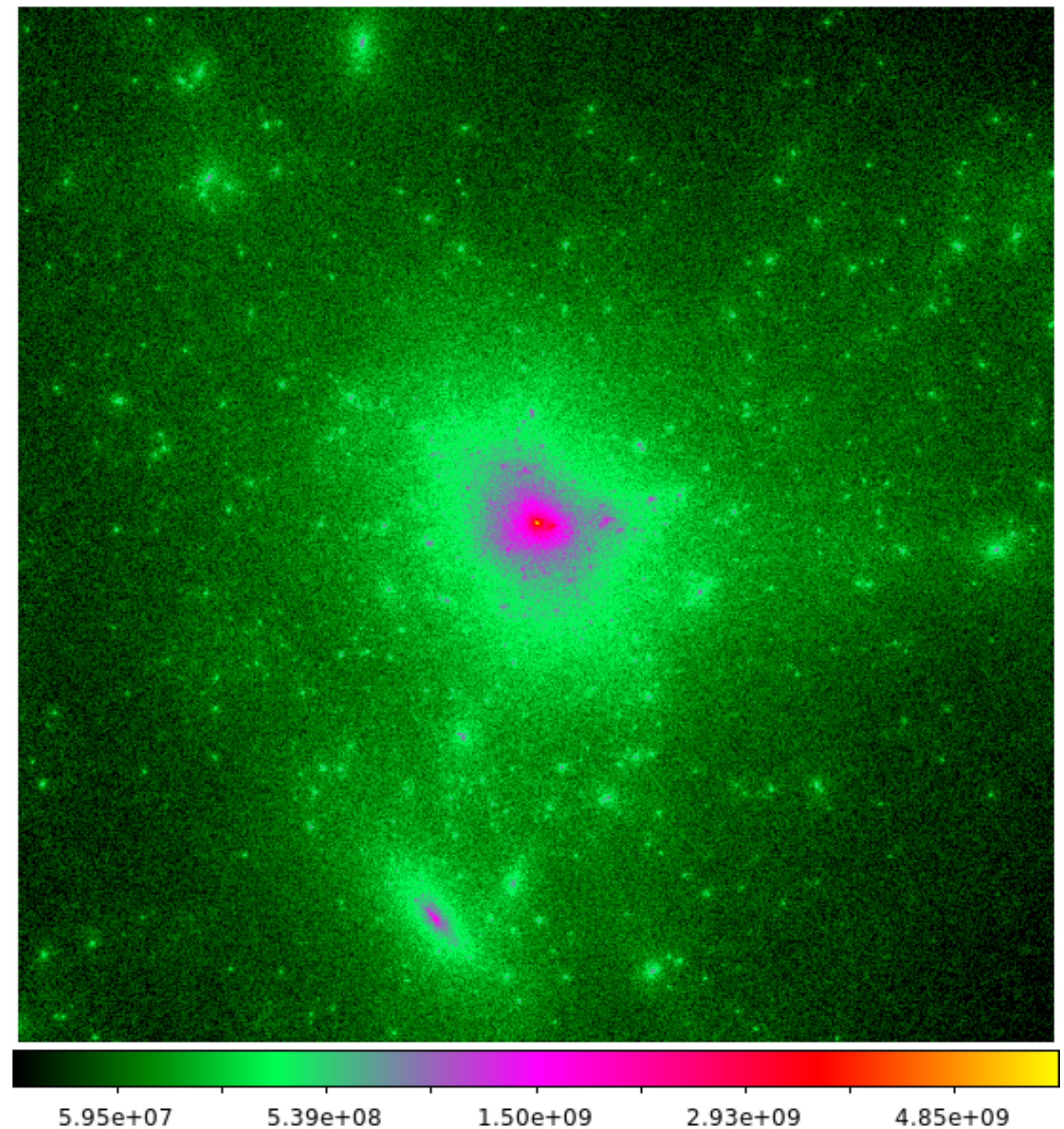
DEFLECTION BY AN EXTENDED MASS DISTRIBUTION

- This can be easily generalized to the case of a continuum distribution of mass

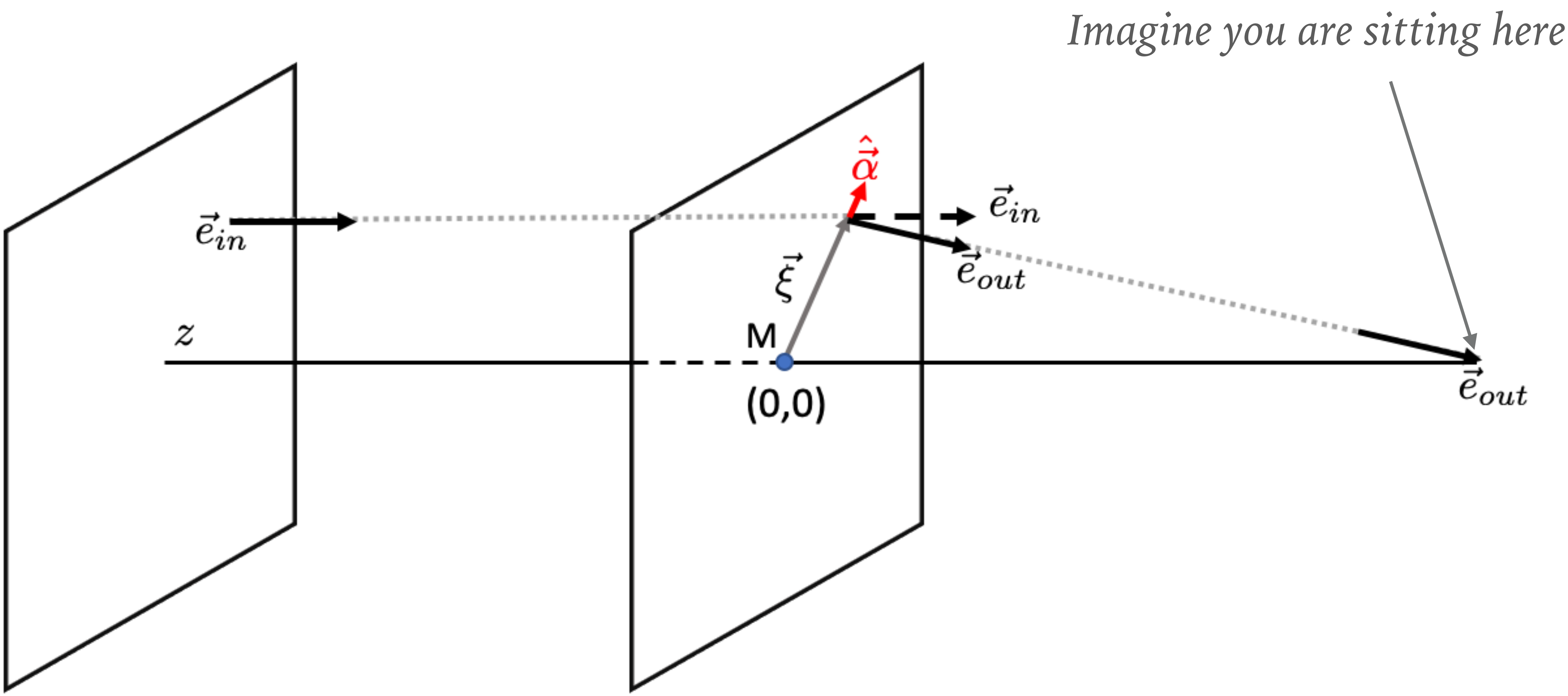
$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$$

$$d\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \Sigma(\vec{\xi}') d\xi'^2$$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \Sigma(\vec{\xi}') d\xi'^2$$



IS ANY DEFLECTION RIGHT?



Some rays are deflected in the right way, others are not!

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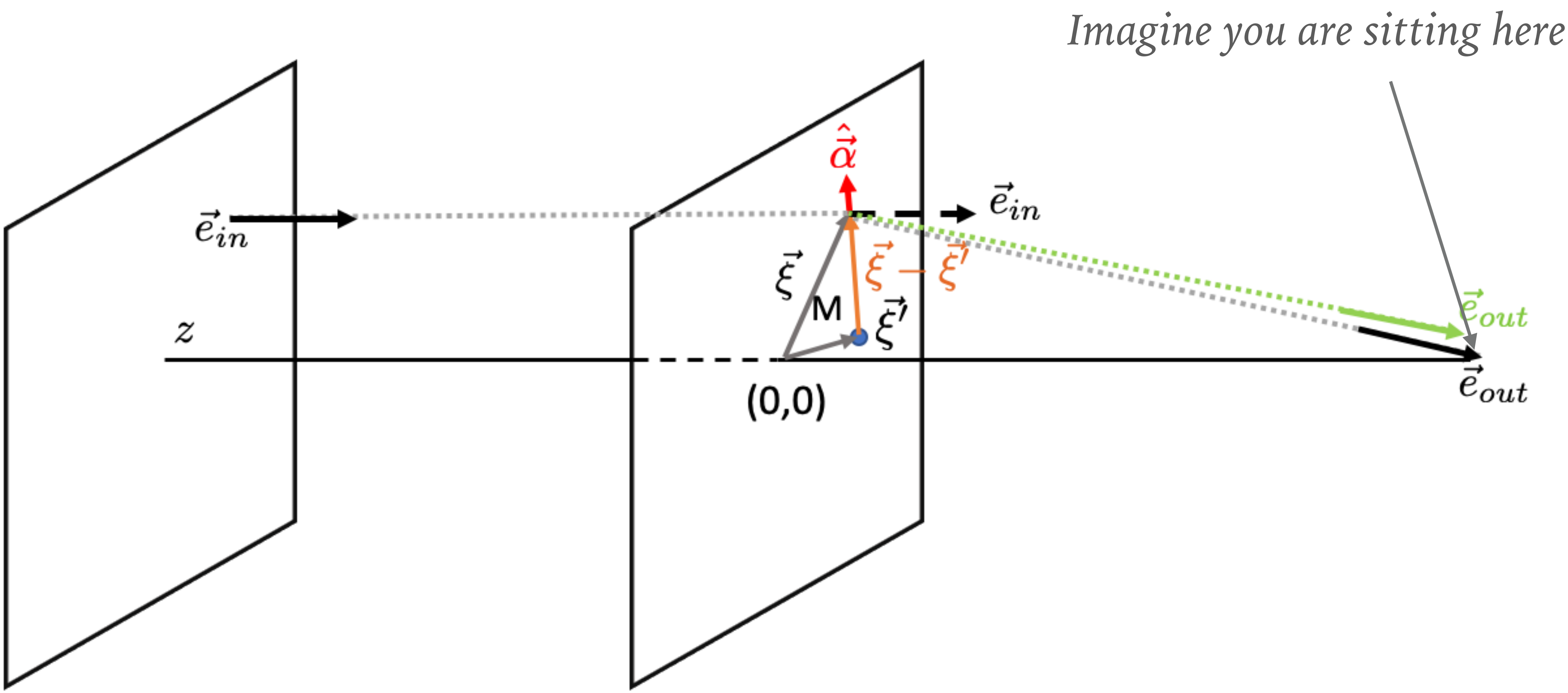
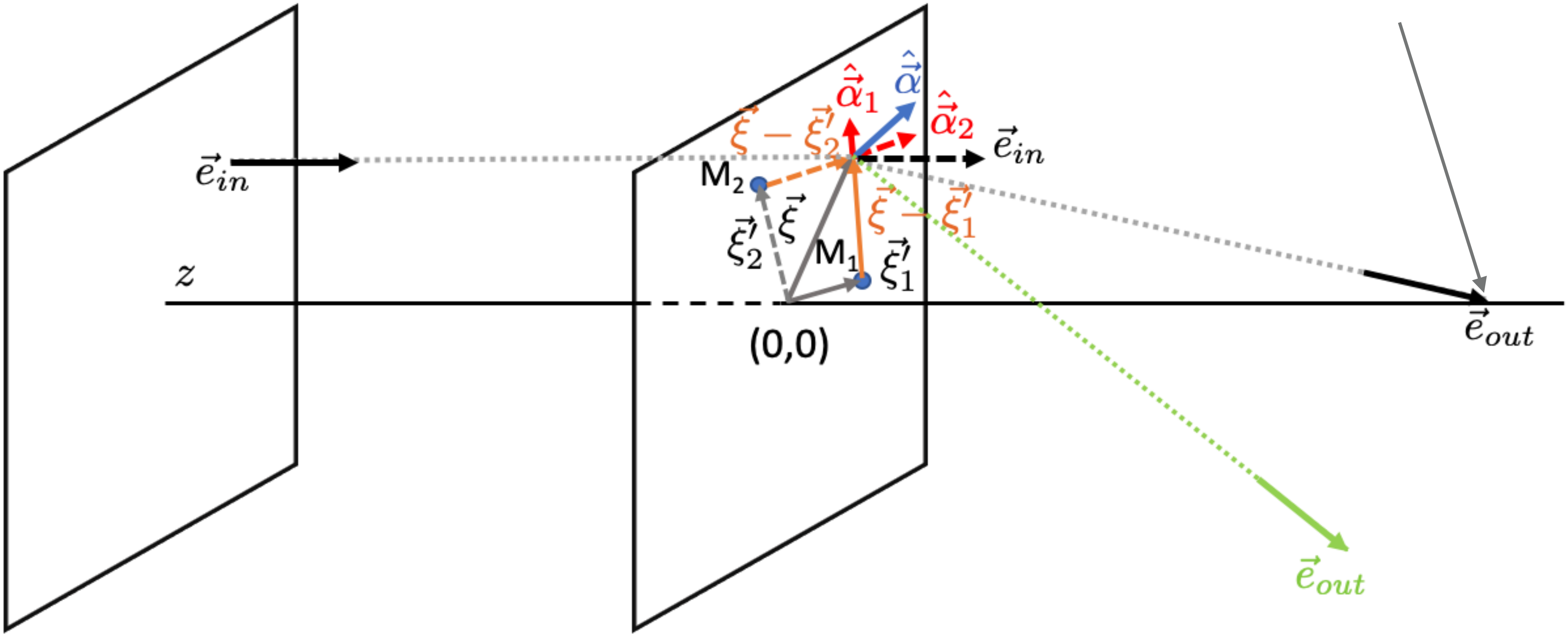


Some rays are deflected in the right way, others are not!

IS ANY DEFLECTION RIGHT?

Imagine you are sitting here



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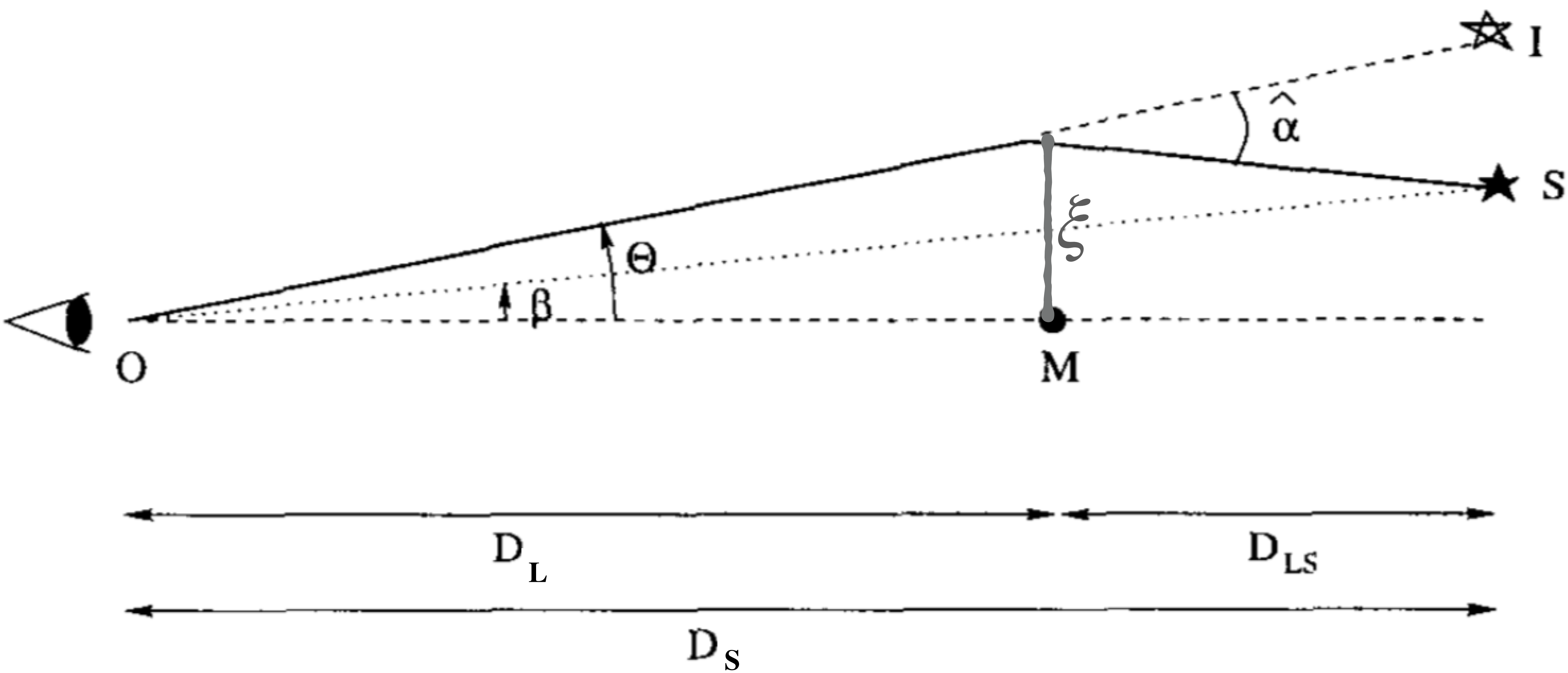
LENS EQUATION



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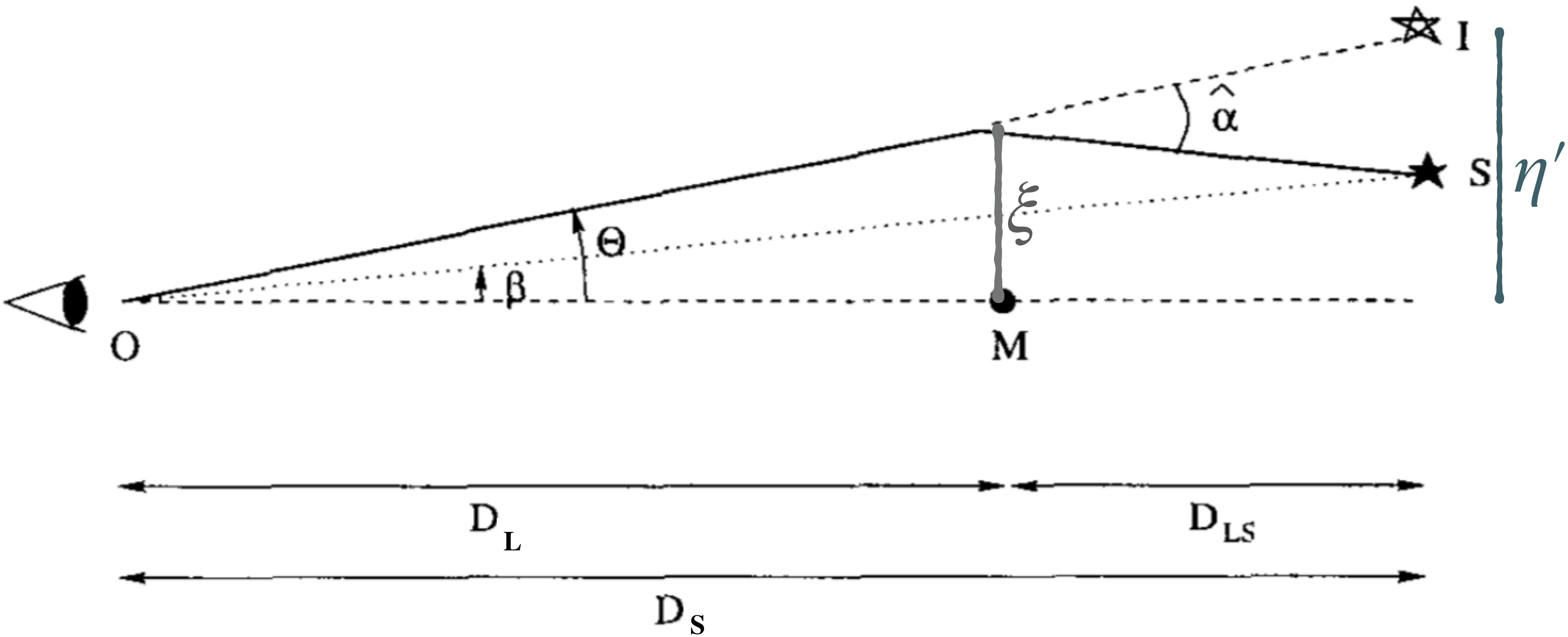

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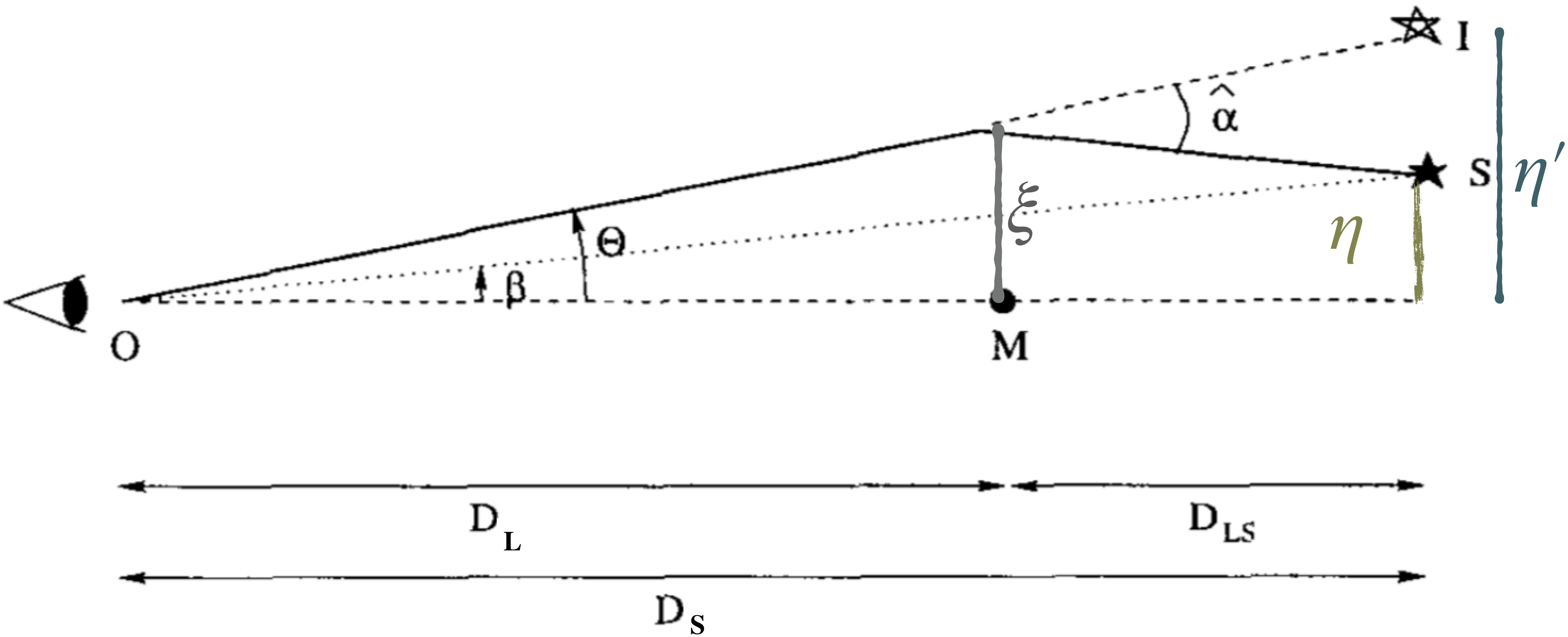
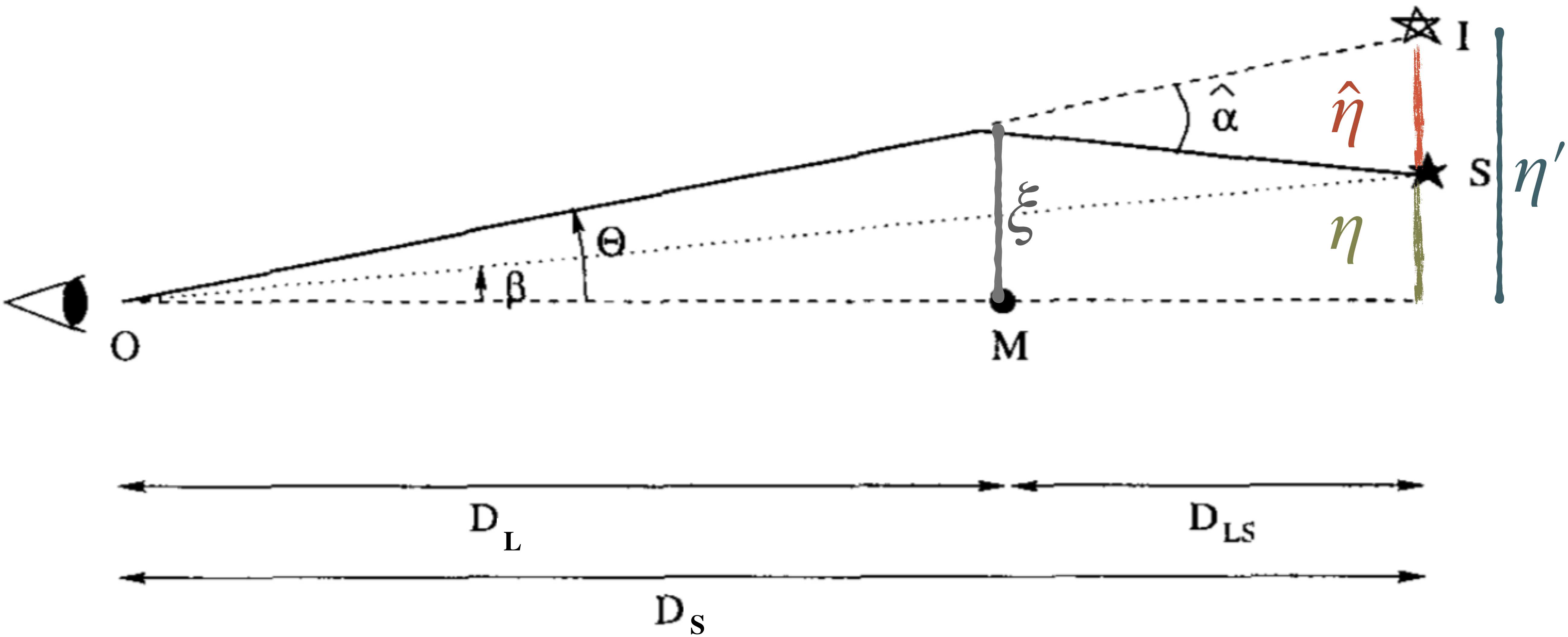


LENS EQUATION

$$\eta' = \eta + \hat{\eta}$$



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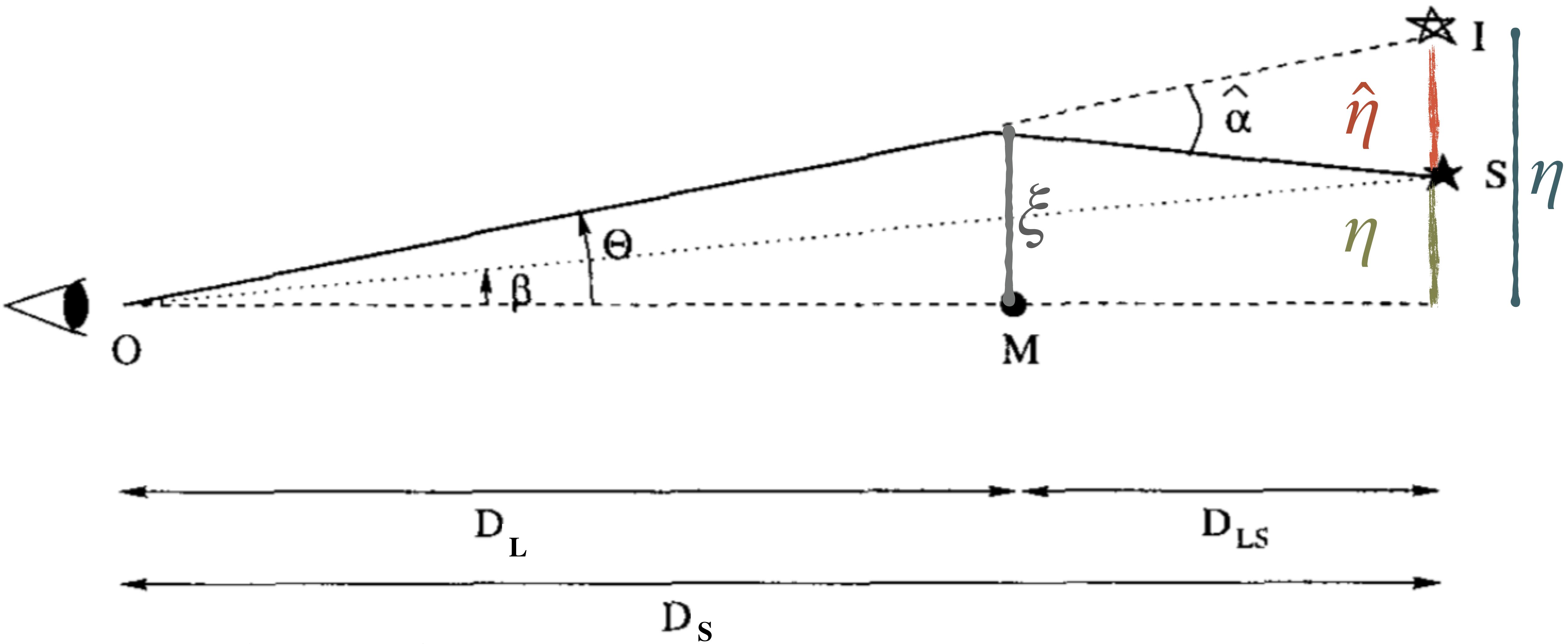

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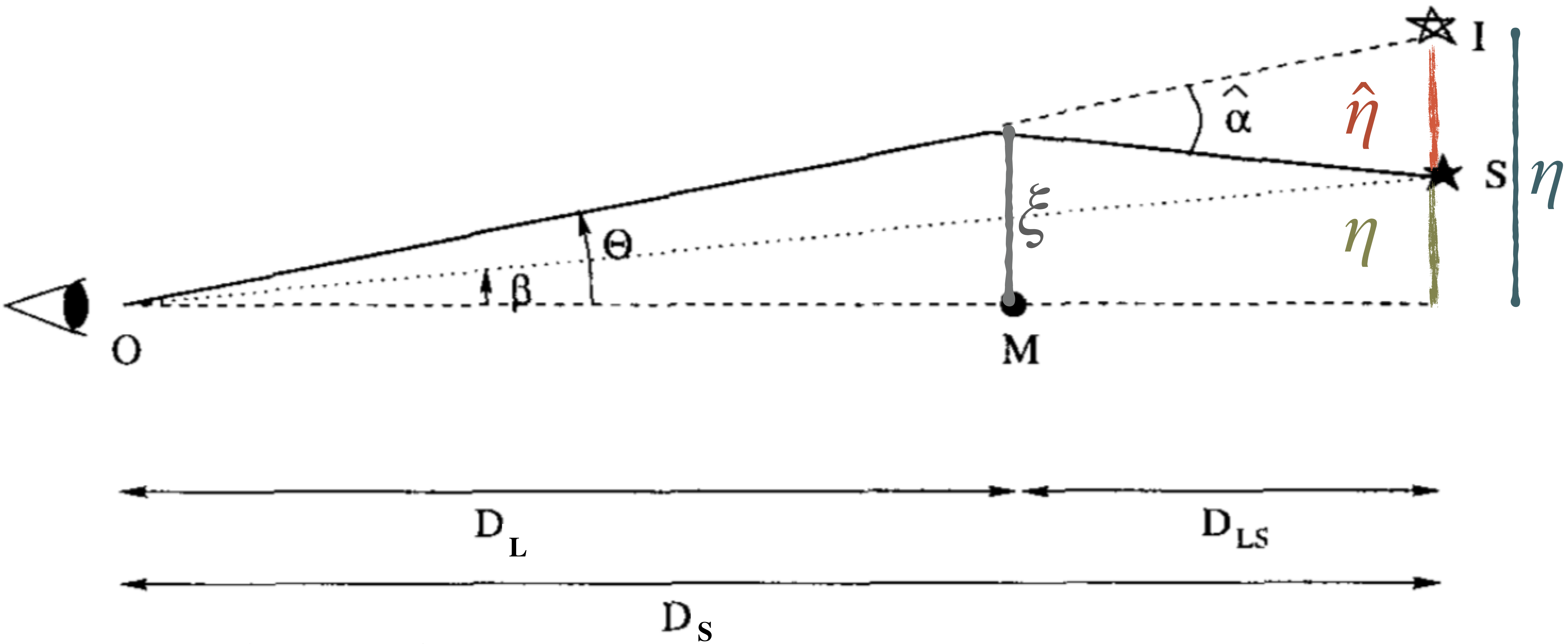

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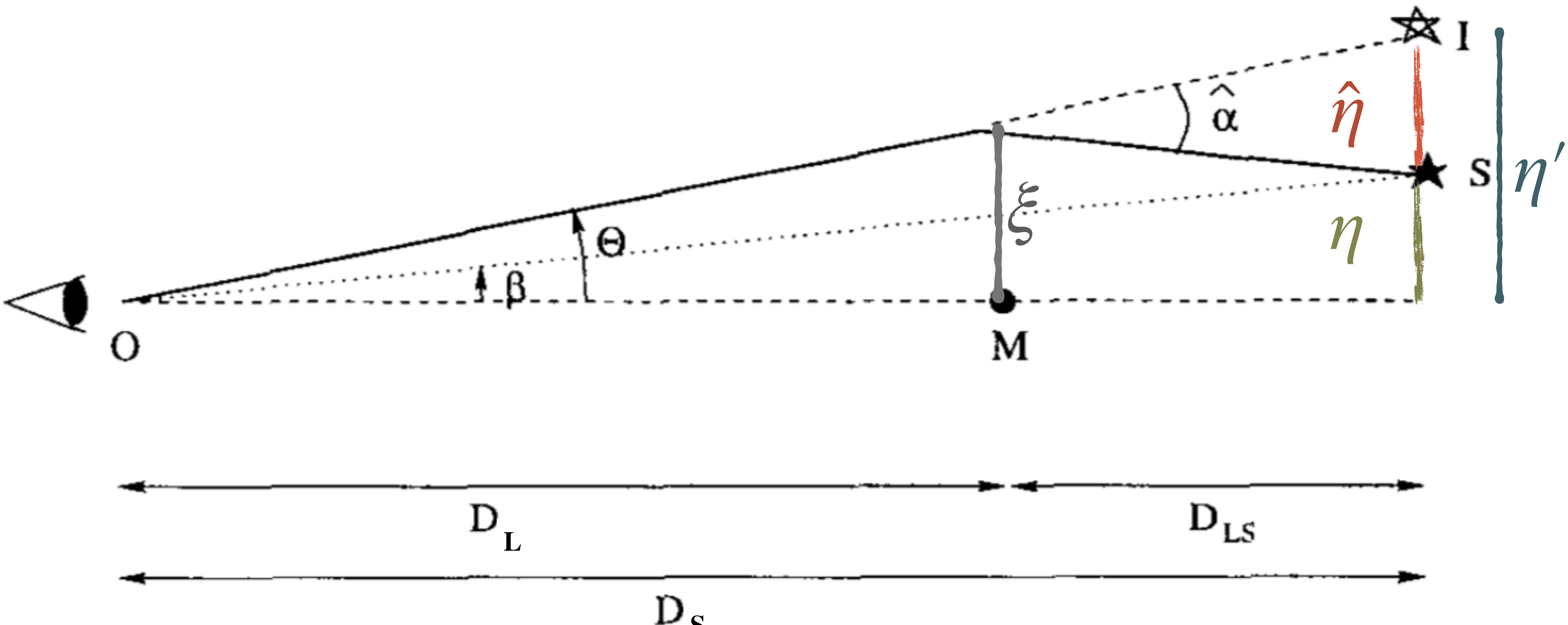

$$\eta' = D_S\theta \qquad \eta = D_S\beta \qquad \hat{\eta} = D_{LS}\hat{\alpha}(\theta)$$

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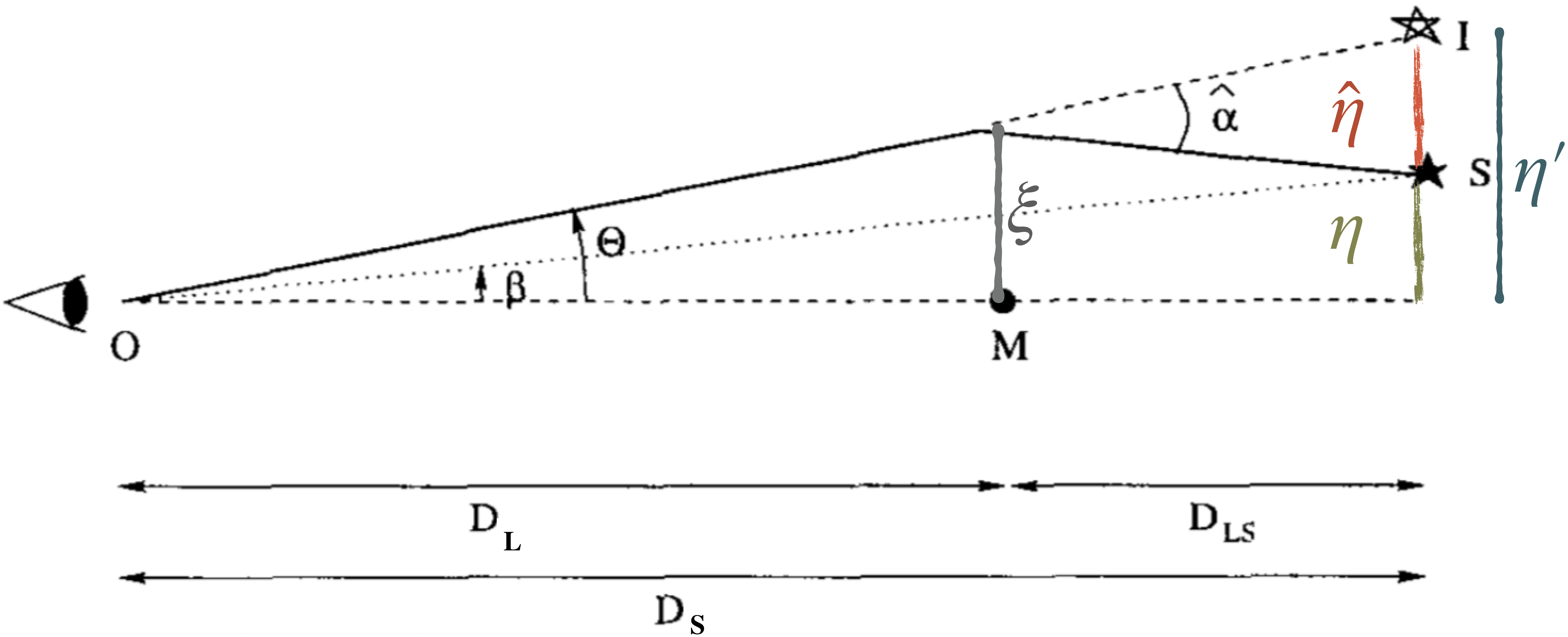


$$\eta' = D_S\theta \qquad \eta = D_S\beta \qquad \hat{\eta} = D_{LS}\hat{\alpha}(\theta)$$

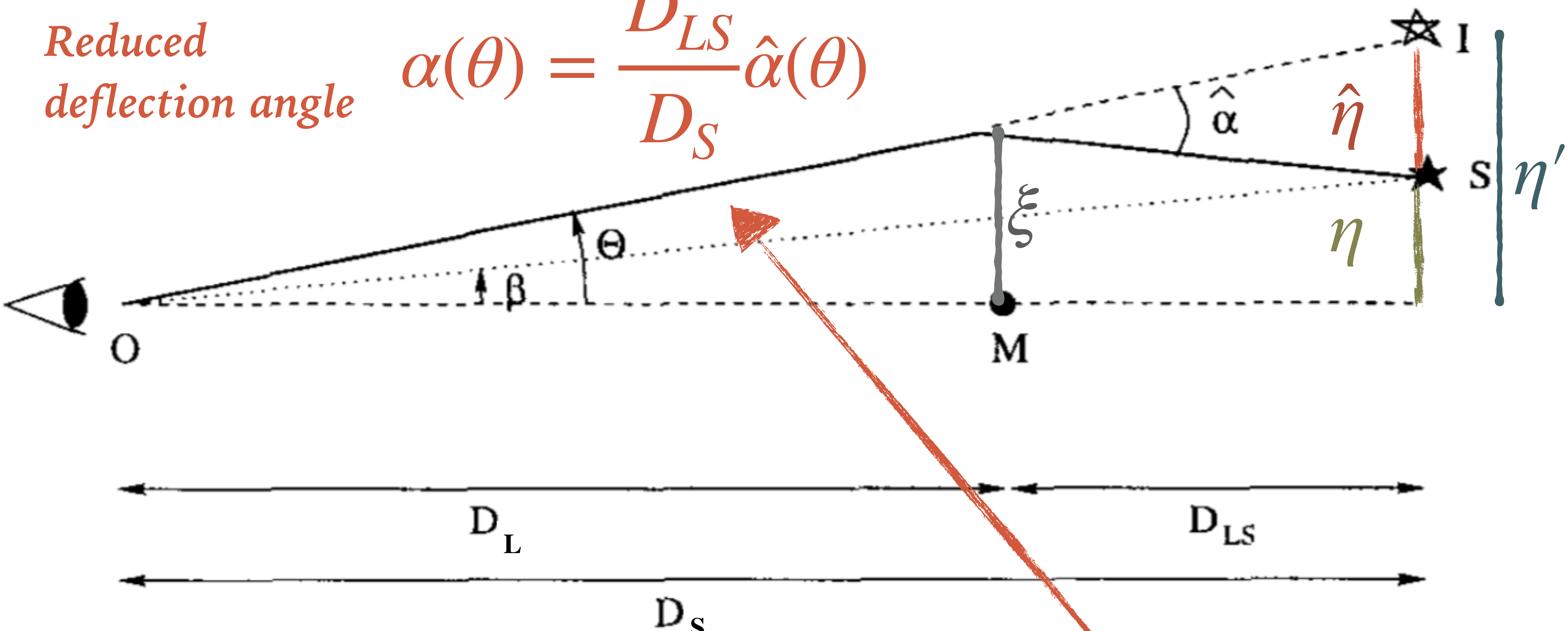
$$D_S\theta = D_S\beta + D_{LS}\hat{\alpha}(\theta) \Rightarrow \beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}(\theta)$$

LENS EQUATION

$$\eta' = \eta + \hat{\eta}$$

Reduced deflection angle

$$\alpha(\theta) = \frac{D_{LS}}{D_S}\hat{\alpha}(\theta)$$



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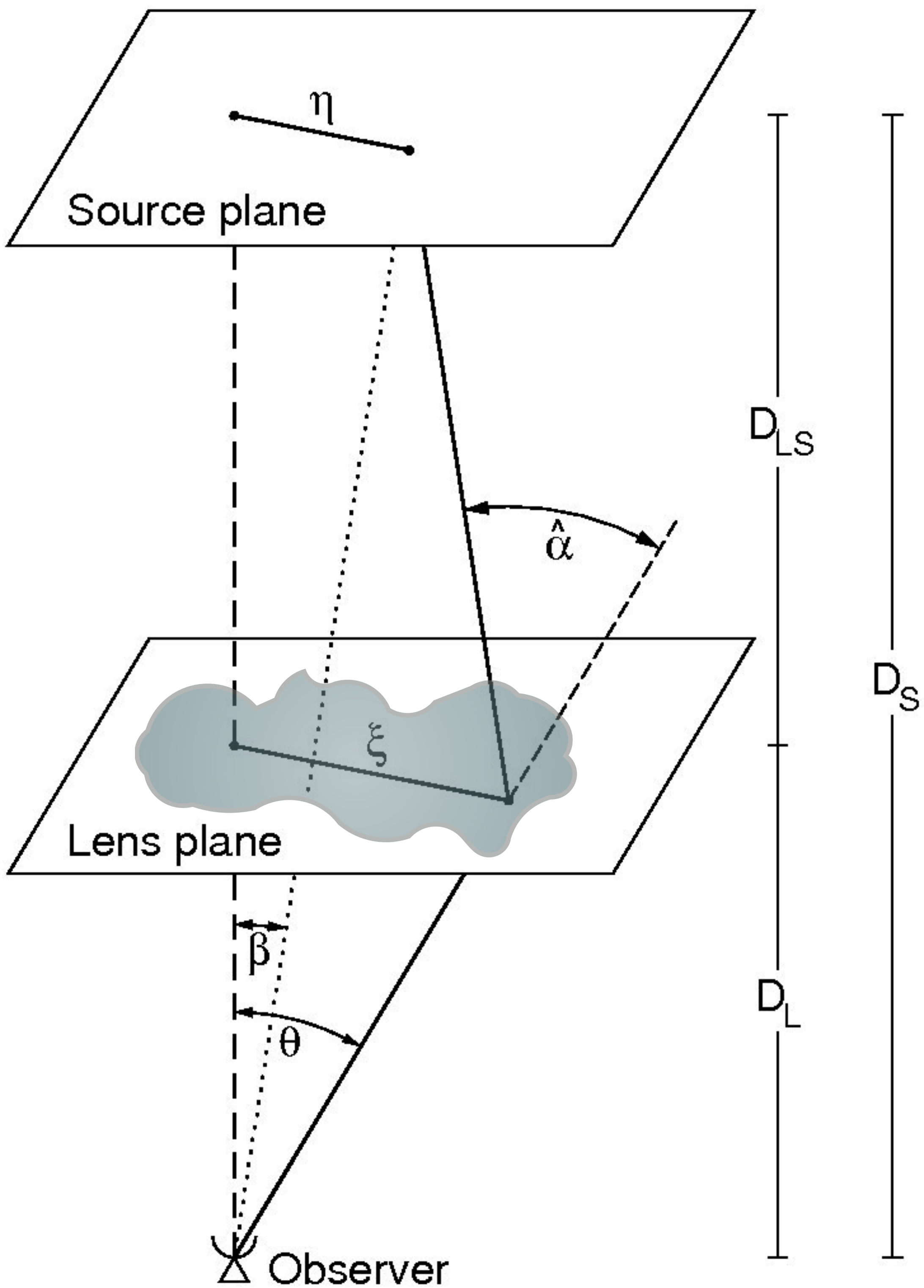

Remember that:

1) *positions on the lens and source planes are defined by vectors*
2) *the deflection angle itself is a vector*

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$$

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\vec{\hat{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

DIMENSIONLESS NOTATION

*Quite often, an alternative way is chosen to write the lens equation: the so called "**dimension-less**" notation.*

*This implies the choice of a **reference angle (or length)** to scale the source and image positions and the deflection angle:*

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \quad \vec{\beta} = \frac{\vec{\eta}}{D_S} \quad \vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) \quad \vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\theta_0 = \frac{\xi_0}{D_L} = \frac{\eta_0}{D_S}$$

the reference angle subtends the reference scales on the lens and on the source planes

$$\frac{\vec{\theta}}{\theta_0} = \frac{\vec{\beta}}{\theta_0} - \frac{\vec{\alpha}(\vec{\theta})}{\theta_0}$$

dividing both members of the lens equation by the reference angle…

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x})$$

$$\vec{\alpha}(\vec{x}) = \frac{\vec{\alpha}(\vec{\theta})}{\theta_0} = \frac{D_L}{\xi_0}\vec{\alpha}(\vec{\theta})$$

LENSING POTENTIAL

$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \Phi dz$$

This formula tells us that the deflection is caused by the projection of the Newtonian gravitational potential on the lens plane.

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z)\mathrm{d}z$$

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We introduce the ***effective lensing potential***

1 *the lensing potential is the projection of the 3D potential*

2 *the lensing potential scales with distances*

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) = \vec{\alpha}(\vec{\theta})$$

The reduced deflection angle is the gradient of the lensing potential

$$\begin{aligned}
\vec{\nabla}_\theta \hat{\Psi}(\vec{\theta}) &= D_\mathrm{L}\vec{\nabla}_\perp \hat{\Psi} = \vec{\nabla}_\perp \left(\frac{D_\mathrm{LS}}{D_\mathrm{S}} \frac{2}{c^2} \int \hat{\Phi}(\vec{\theta}, z) \mathrm{d}z \right) \\
&= \frac{D_\mathrm{LS}}{D_\mathrm{S}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{\theta}, z) \mathrm{d}z \\
&= \vec{\alpha}(\vec{\theta})
\end{aligned}$$

NOTE THAT…

… the same result holds if we use the dimension-less notation:

$$\vec{\nabla}_x = \frac{\xi_0}{D_\mathrm{L}} \vec{\nabla}_\theta$$

$$\vec{\nabla}_x \hat{\Psi} = \frac{\xi_0}{D_\mathrm{L}} \vec{\nabla}_\theta \hat{\Psi} = \frac{\xi_0}{D_\mathrm{L}} \vec{\alpha}$$

By multiplying both sides of the equation by D_L^2/ξ_0^2 *we obtain:*

$$\frac{D_\mathrm{L}^2}{\xi_0^2} \vec{\nabla}_x \hat{\Psi} = \frac{D_\mathrm{L}}{\xi_0} \vec{\alpha} \quad \Rightarrow \quad \Psi = \frac{D_\mathrm{L}^2}{\xi_0^2} \hat{\Psi} \quad \Rightarrow \quad \vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

We have introduced the ***dimensionless counter-part of the lensing potential!***

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\triangle_\theta \hat{\Psi}(\vec{\theta}) = 2\kappa(\vec{\theta})$$

The laplacian of the lensing potential is twice the ***convergence:***

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G}\frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

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$[G]=L^3/M/T^2$

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$[D_X]=L$

The critical surface density is a characteristic density to distinguish between strong and weak gravitational lenses!

OTHER PROPERTIES OF THE LENSING POTENTIAL

$$\triangle_\theta \hat{\Psi}(\vec{\theta}) = 2\kappa(\vec{\theta})$$

The laplacian of the lensing potential is twice the ***convergence:***

We start from the poisson equation

$$\triangle\Phi = 4\pi G\rho$$

The surface mass density is then:

$$\Sigma(\vec{\theta}) = \frac{1}{4\pi G}\int_{-\infty}^{+\infty}\triangle\Phi dz$$

$$\kappa(\vec{\theta}) = \frac{1}{c^2}\frac{D_L D_{LS}}{D_S}\int_{-\infty}^{+\infty}\triangle\Phi dz$$

Let's introduce the Laplacian operator on the lens plane:

$$\triangle_\theta = \frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2} = D_L^2\left(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2}\right) = D_L^2\left(\triangle - \frac{\partial^2}{\partial z^2}\right)$$

Then:

$$\triangle\Phi = \frac{1}{D_L^2}\triangle_\theta\Phi + \frac{\partial^2\Phi}{\partial z^2}$$

OTHER PROPERTIES OF THE LENSING POTENTIAL

With this substitution:

$$\kappa(\vec{\theta}) = \frac{1}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}\left[\triangle_\theta \int_{-\infty}^{+\infty}\Phi \mathrm{d}z + D_{\mathrm{L}}^2 \int_{-\infty}^{+\infty}\frac{\partial^2\Phi}{\partial z^2}\mathrm{d}z\right]$$

where the second term in the sum is zero, if the lens if gravitationally bound!

Given the definition of lensing potential:

$$\kappa(\theta) = \frac{1}{2}\triangle_\theta\hat{\Psi}$$

Note that:

$$\triangle_\theta = D_{\mathrm{L}}^2\triangle_\xi = \frac{D_{\mathrm{L}}^2}{\xi_0^2}\triangle_x \qquad \kappa(\theta) = \frac{1}{2}\triangle_\theta\hat{\Psi} = \frac{1}{2}\frac{\xi_0^2}{D_{\mathrm{L}}^2}\triangle_\theta\Psi$$

$$\Rightarrow \quad \kappa(\vec{x}) = \frac{1}{2}\triangle_x\Psi(\vec{x})$$

DIMENSIONLESS NOTATION

From

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} \, \mathrm{d}^2\xi'$$

we obtain

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \mathrm{d}^2x' \kappa(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

Using

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

$$\Psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \kappa(\vec{x}') \ln |\vec{x} - \vec{x}'| \mathrm{d}^2x'$$

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Using

$$\vec{\nabla}_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x})$$

Convolution kernels

$$\Psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbf{R}^2} \kappa(\vec{x}') \ln|\vec{x} - \vec{x}'| d^2x'$$