GRAVITATIONAL LENSING

20 - SIS, NIS, ELLIPTICAL MODELS

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SOLUTIONS OF THE LENS EQUATION

$$y = x - \frac{x}{|x|}$$

$$x = \frac{\theta}{\theta_E}$$

If 0<y<1, the solution are two:

$$x_- = y - 1 \qquad x < 0 \qquad \theta_- = \beta - \theta_E$$

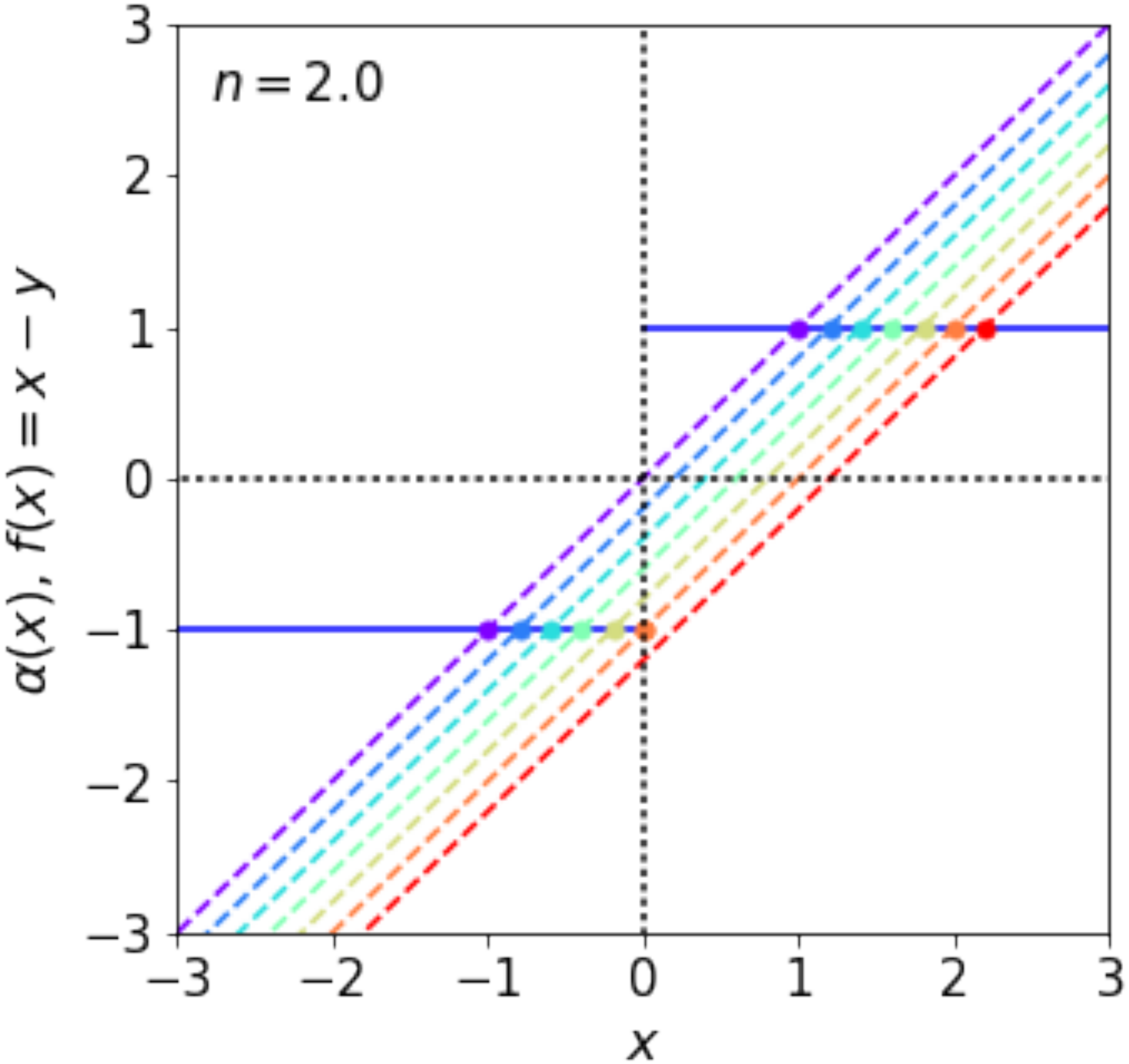
$$x_+ = y + 1 \qquad x > 0 \qquad \theta_+ = \beta + \theta_E$$

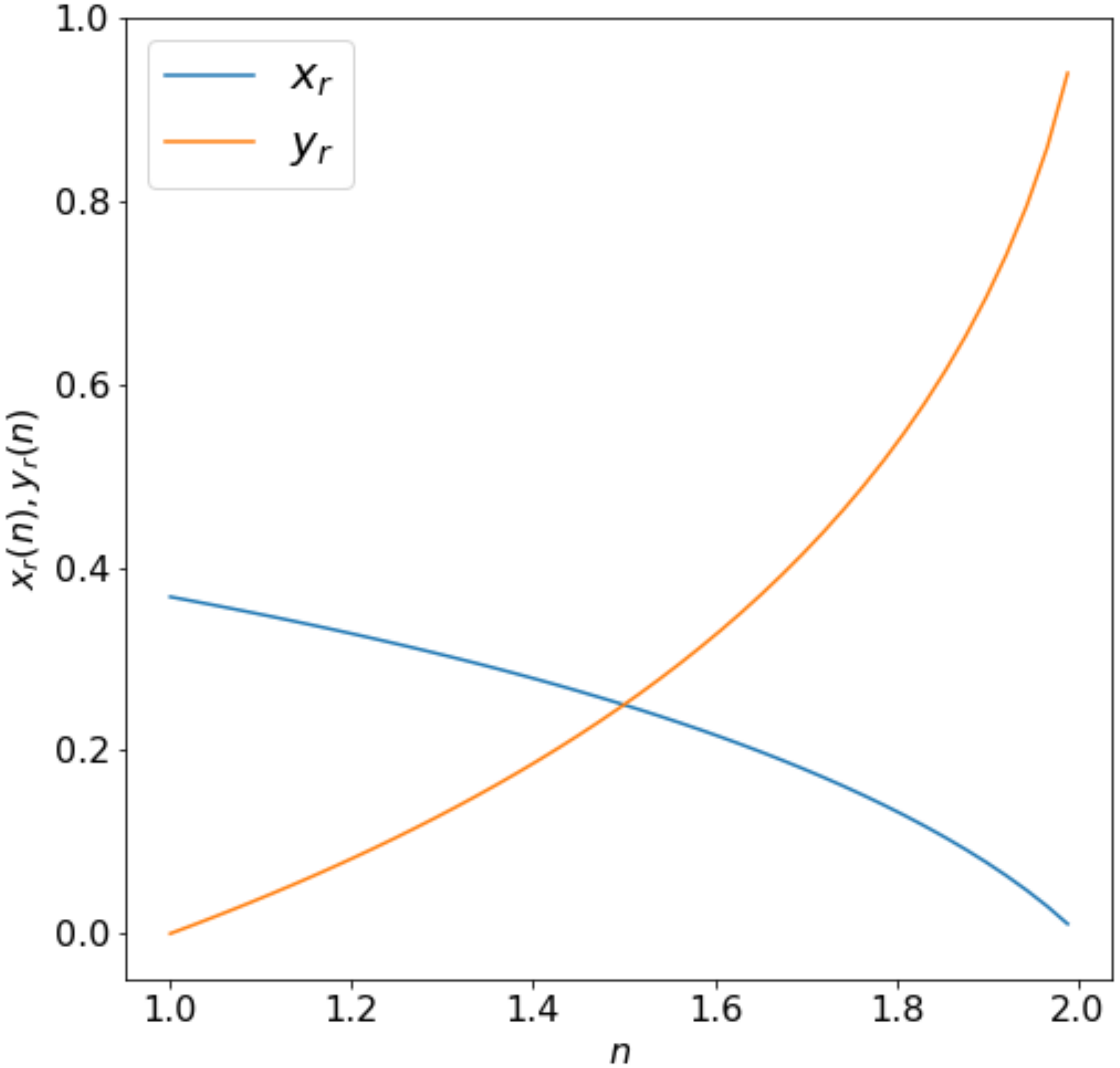
Otherwise, there is only one solution at

$$x_+ = y + 1$$

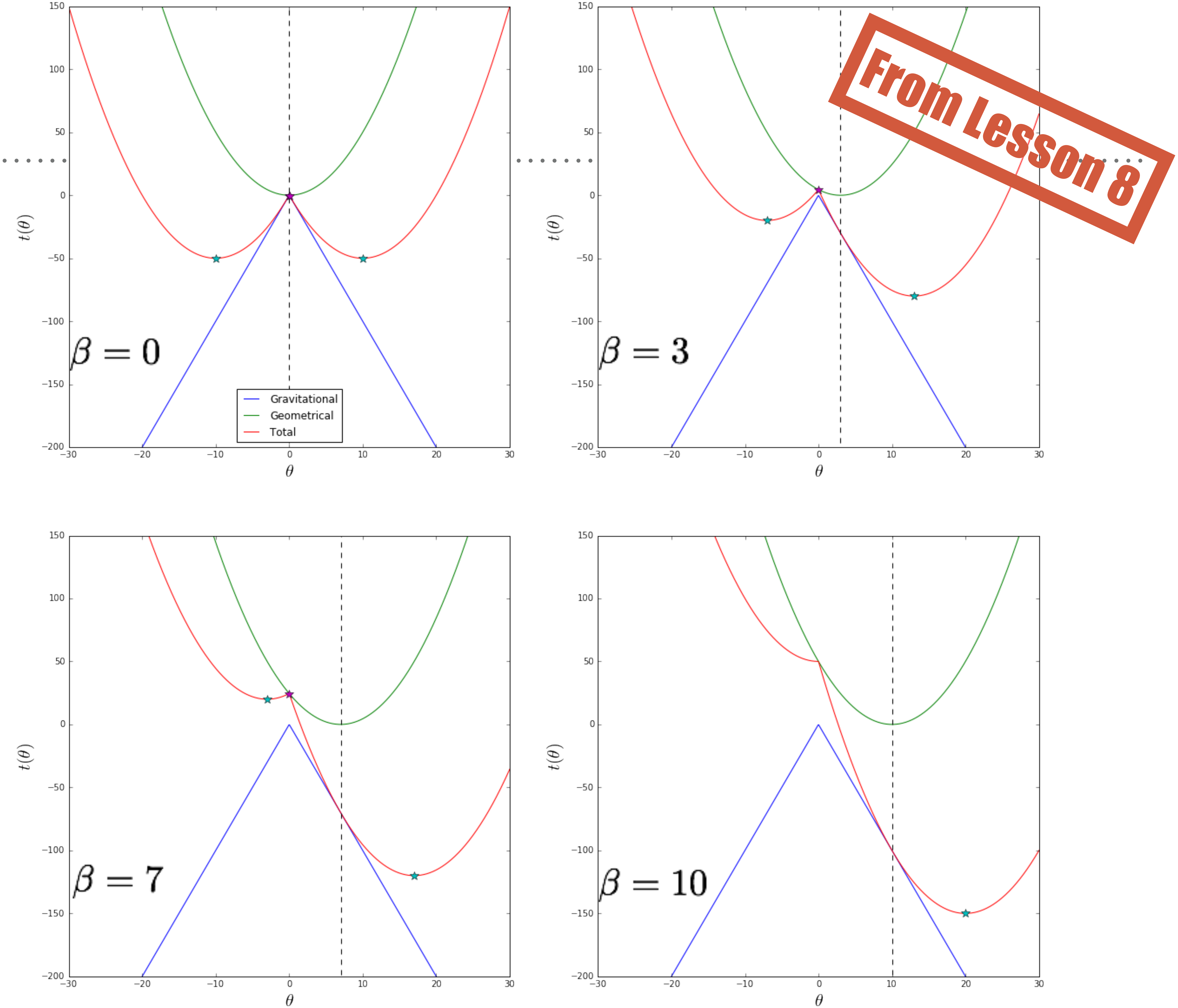
Thus, ***the circle of radius y=1 ("cut") plays the same role of the radial caustic*** *for the power-law lens with n<2, separating the source plane into regions with different image multiplicity.*

IMAGE DIAGRAM (SIS)





$$y_{cut} = \lim_{x \to 0} y(x) = \pm 1$$



From Lesson 8
$\beta = 0$
$\beta = 3$
$\beta = 7$
$\beta = 10$
Gravitational
Geometrical
Total
$t(\theta)$
θ

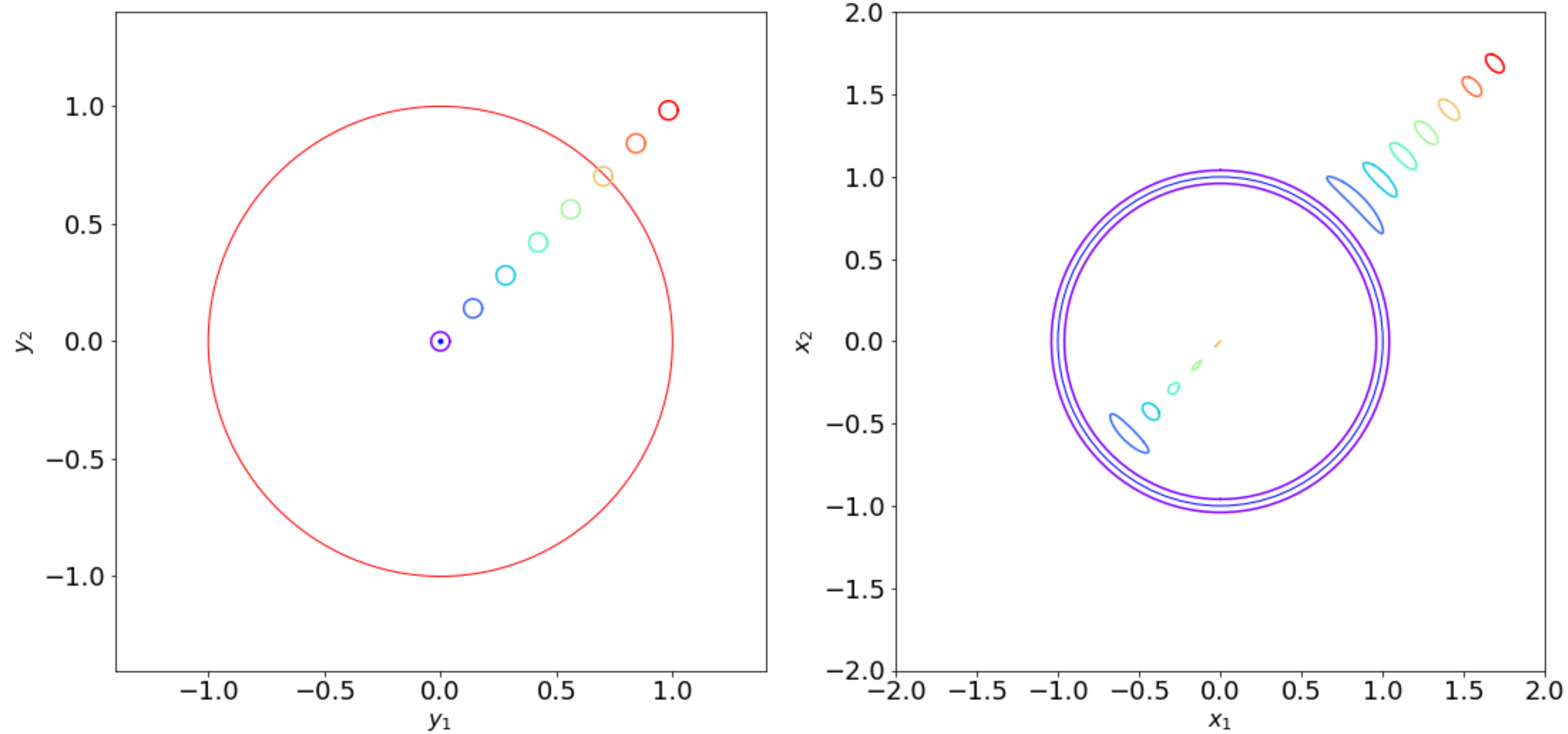
THE SINGULAR ISOTHERMAL SPHERE

$$\alpha(x) = \frac{x}{|x|} \Rightarrow \qquad \frac{d\alpha}{dx} = 0 \;\; \forall\, x \neq 0$$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1$

Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.

THE SINGULAR ISOTHERMAL SPHERE



THE SINGULAR ISOTHERMAL SPHERE

And the tangential magnification is easily computed:

$$\det A = \frac{y}{x}\frac{dy}{dx} = 1 - \frac{1}{|x|} = \frac{|x| - 1}{|x|}$$

$$\mu(x) = \frac{|x|}{|x| - 1}$$

$$x_+ = y + 1$$

$$x_- = y - 1 \quad (< 0)$$

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y}$$

$$\mu_- = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

THE SINGULAR ISOTHERMAL SPHERE

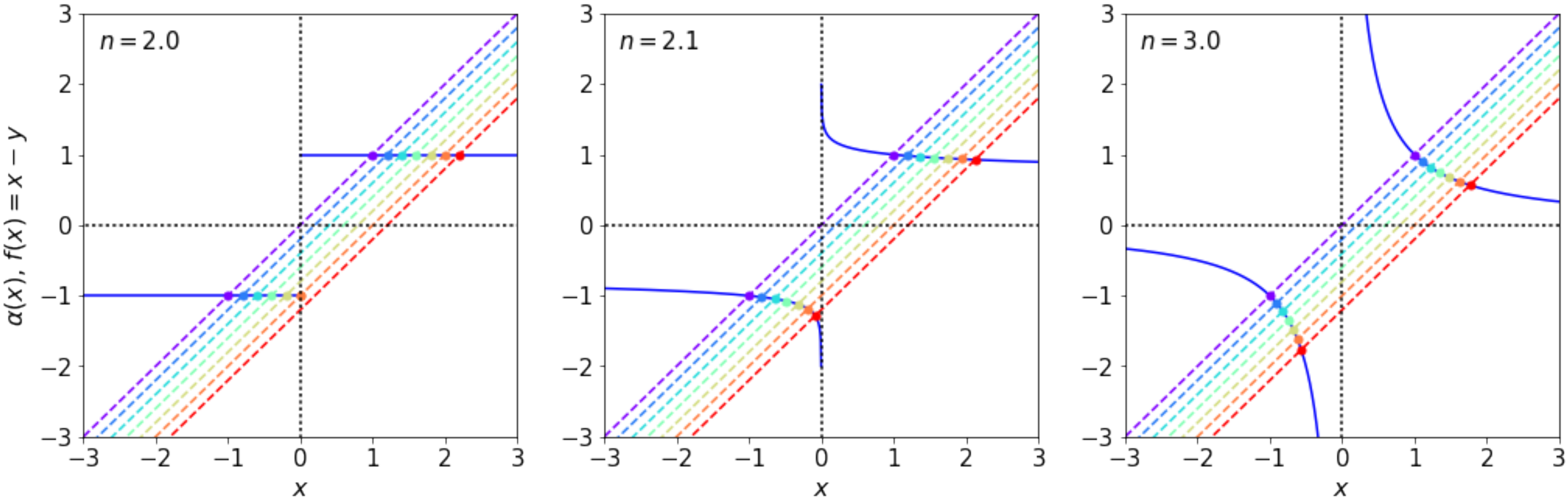
A peculiarity of the SIS is that convergence and shear care equal:

$$\gamma(x) = \bar{\kappa}(x) - \kappa(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = \frac{1}{x} - \frac{1}{2x} = \frac{1}{2x} = \kappa(x)$$

$$\gamma_1 = \frac{1}{2x}\cos 2\phi$$

$$\gamma_2 = \frac{1}{2x}\sin 2\phi$$

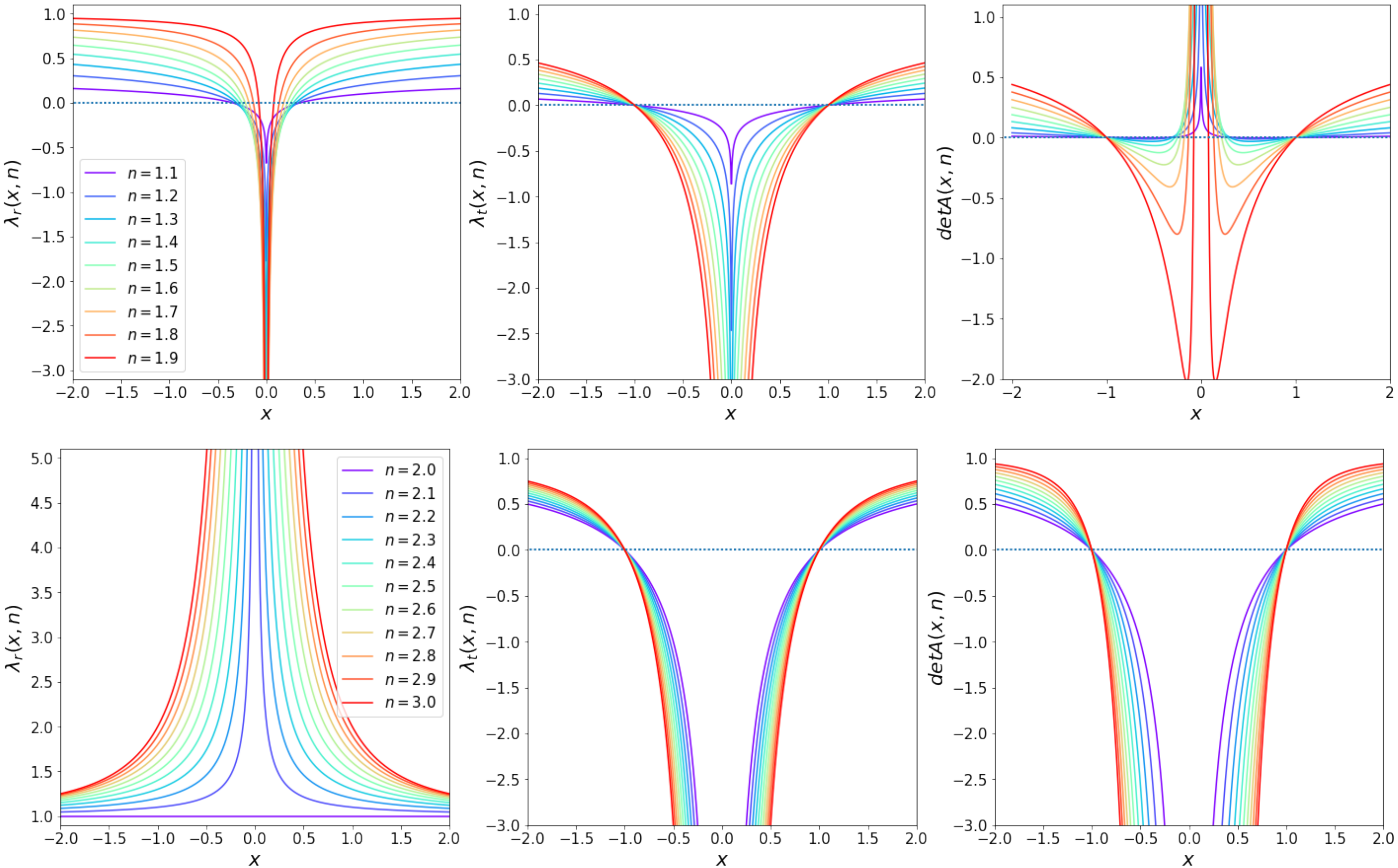
IMAGE DIAGRAM (N>=2)



PL lenses with n>2 always have 2 images, because the time delay surface is not continuously deformable.

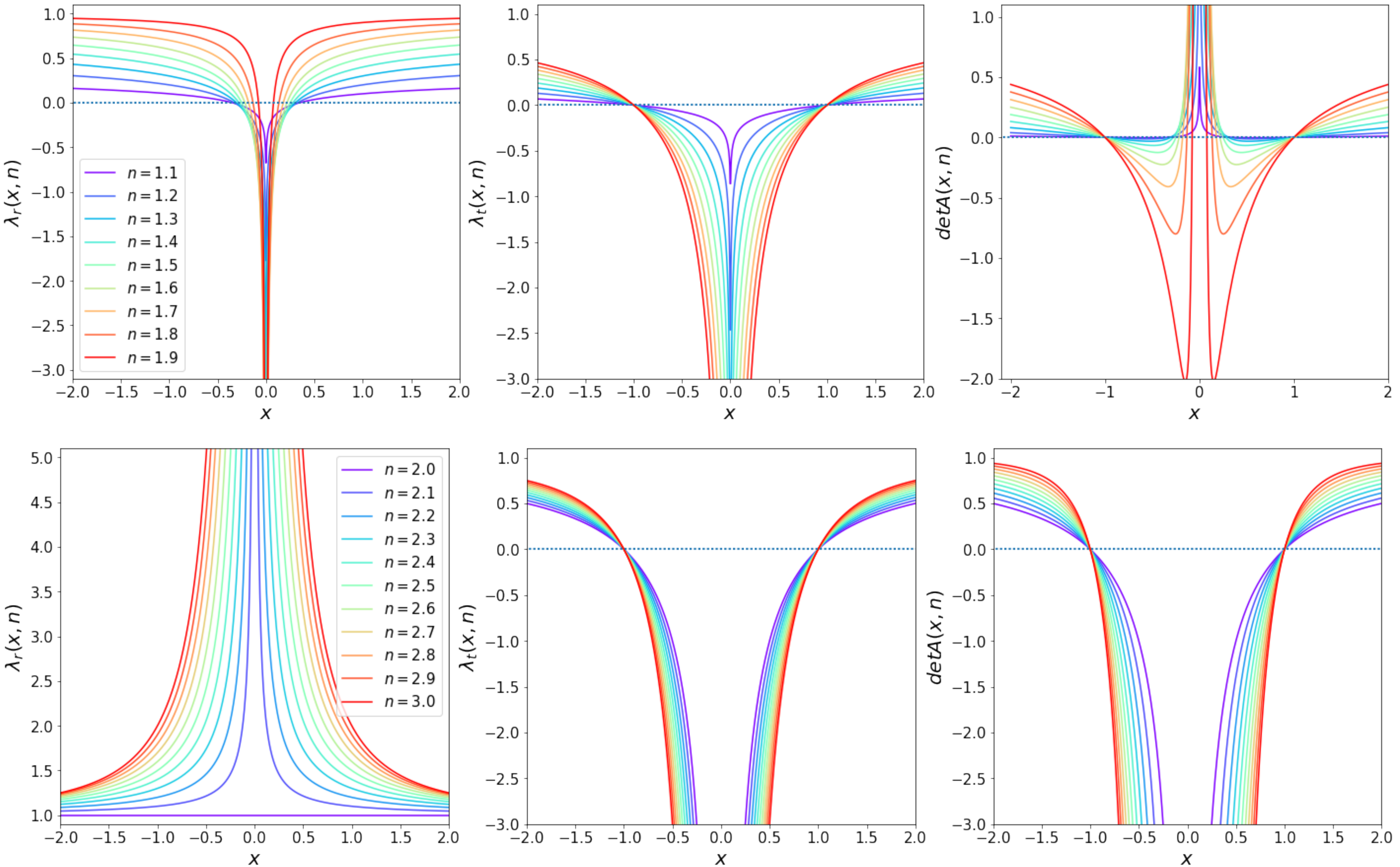
In addition, the images are radially de-magnified!

$$
\begin{aligned}
\lambda_t(x) &= 1 - x^{1-n} \\
\lambda_r(x) &= 1 - (2-n)x^{1-n}
\end{aligned}
$$



no radial critical line if n>=2!

$$\lambda_t(x) = 1 - x^{1-n}$$
$$\lambda_r(x) = 1 - (2-n)x^{1-n}$$



SOFTENED PROFILES: THE NON-SINGULAR ISOTHERMAL SPHERE

The profiles considered so far have surface density profiles with a singularity at x=0. We consider another class of lenses which have a flat core.

Given the simplicity of the model, we investigate the effects of the core by modifying the SIS lens:

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\frac{1}{\sqrt{\xi^2+\xi_c^2}} = \frac{\Sigma_0}{\sqrt{1+\xi^2/\xi_c^2}}$$

$$\Sigma_0 = \frac{\sigma_v^2}{2G\xi_c}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Choosing $$\xi_0 = 4\pi\left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\frac{1}{\sqrt{\xi^2+\xi_c^2}} = \frac{\Sigma_0}{\sqrt{1+\xi^2/\xi_c^2}}$$

$$\kappa(x) = \frac{1}{2\sqrt{x^2+x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The mass profile is computed as follows

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2+x_c^2} - x_c$$

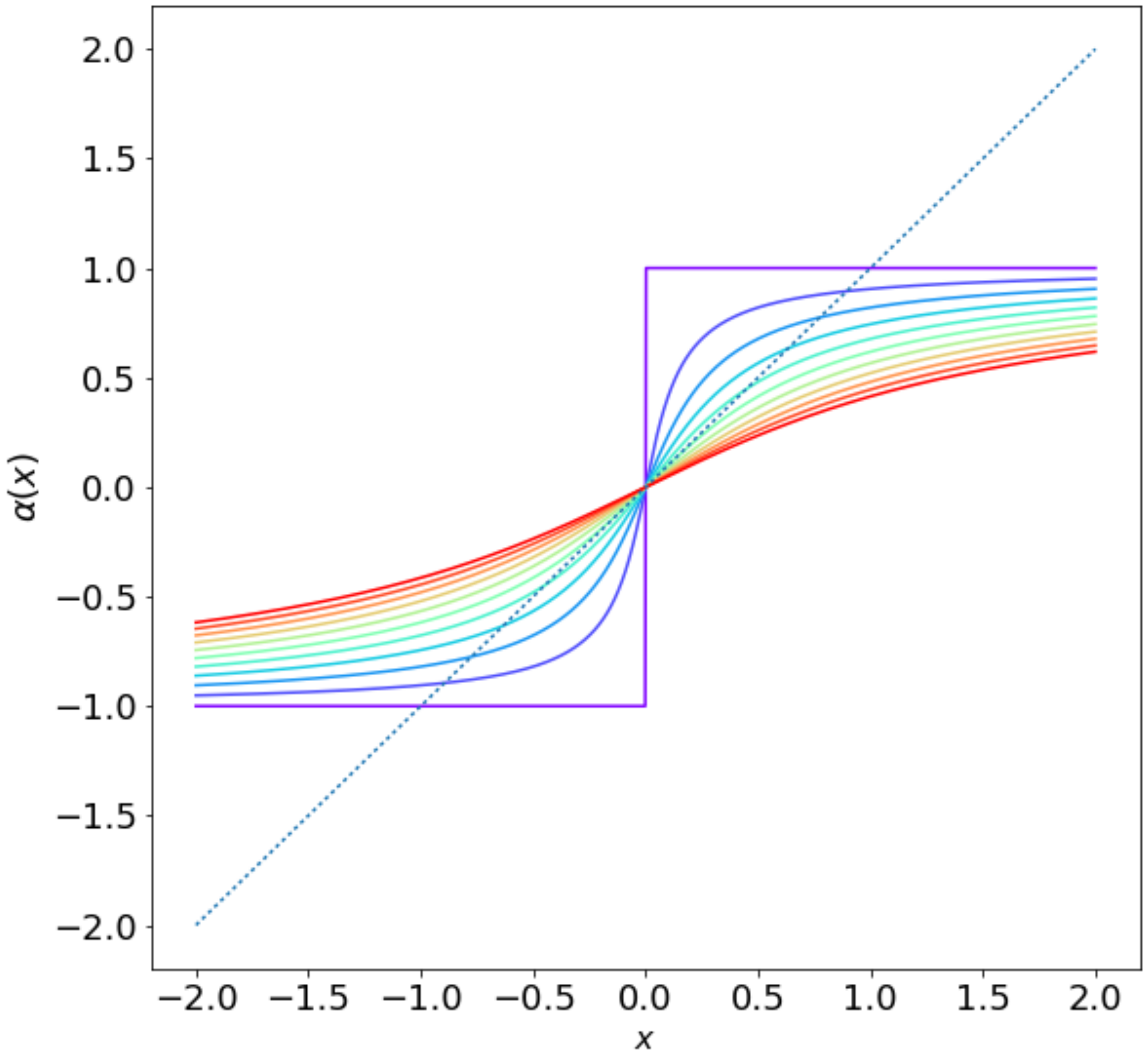
The deflection angle is

$$\alpha(x) = \frac{m(x)}{x} = \sqrt{1+\frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

The shear is

$$\gamma(x) = \frac{\sqrt{x^2+x_c^2}-x_c}{x^2} - \frac{1}{2\sqrt{x^2+x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)



*If the core is too large, **the derivative of the deflection angle is never larger than 1**…*

No multiple images!

NON SINGULAR ISOTHERMAL SPHERE (NIS)

We can search for the tangential critical line:

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2+x_c^2} - x_c \qquad m(x)/x^2 = 1$$

$$\sqrt{x^2+x_c^2} - x_c = x^2$$

$$x^2(x^2+2x_c-1) = 0$$

$$x_t = \sqrt{1-2x_c}$$

Note that the tangential critical line exists only if $x_c<1/2$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

and the radial critical line:

$$\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)=1+\frac{m(x)}{x^2}-2\kappa(x)=0$$

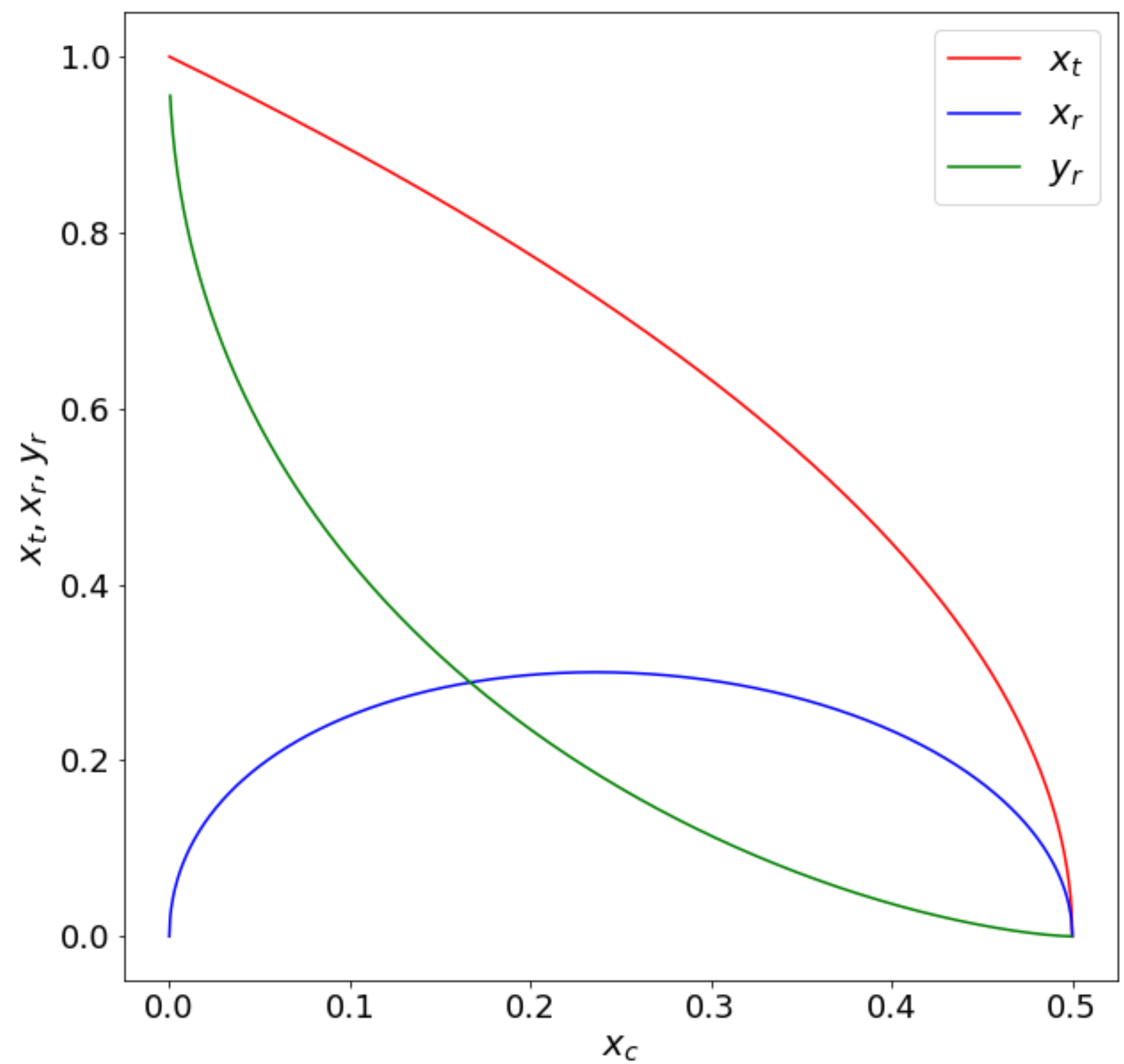
$$1+\frac{\sqrt{x^2+x_c^2}-x_c}{x^2}-\frac{1}{\sqrt{x^2+x_c^2}}=0$$

$$x_r^2=\frac{1}{2}\left(2x_c-x_c^2-x_c\sqrt{x_c^2+4x_c}\right)$$

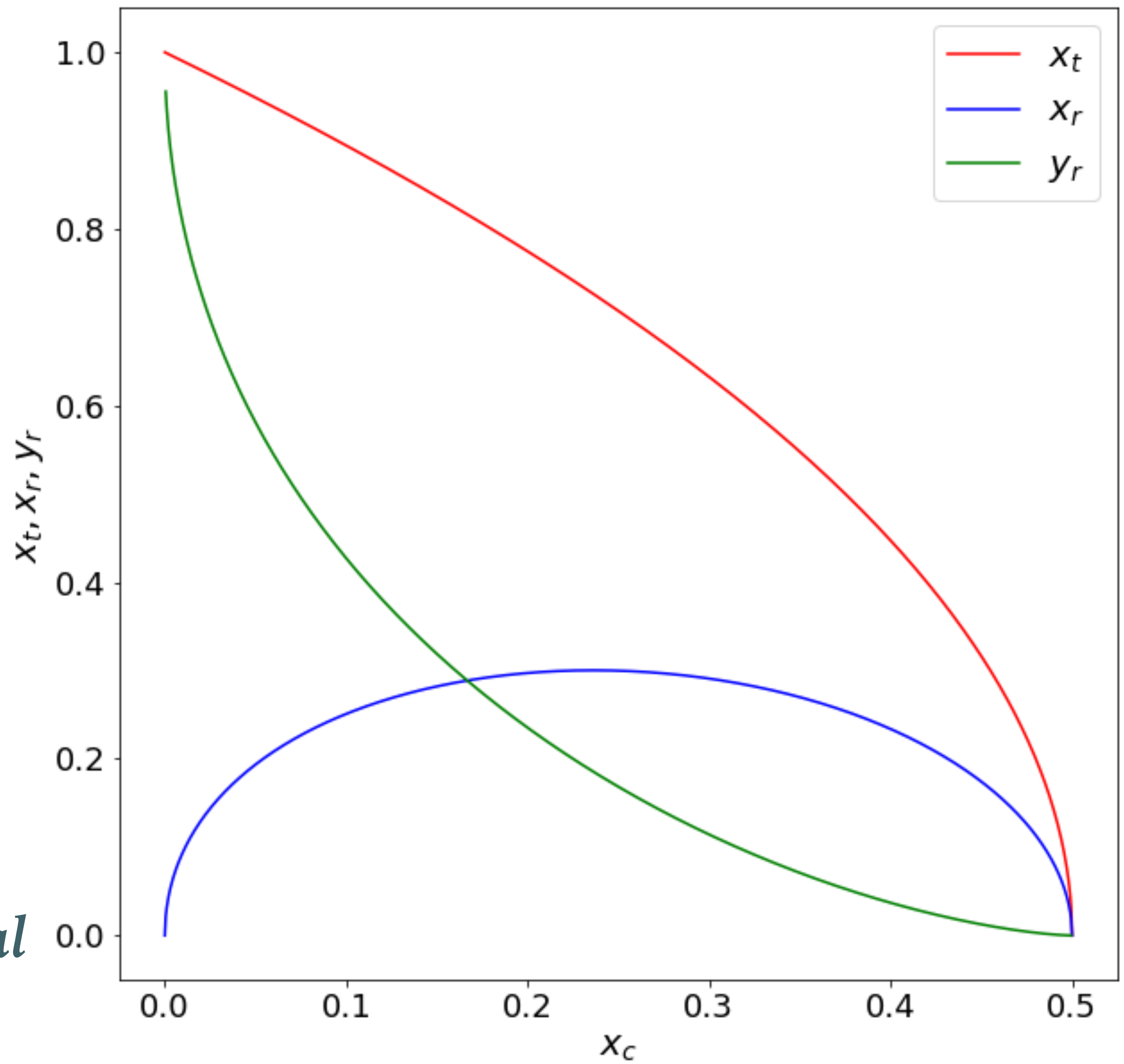
$$x_r^2\geq 0 \text{ for } x_c\leq 1/2.$$

Thus, the existence condition for the radial critical is the same as for the tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)



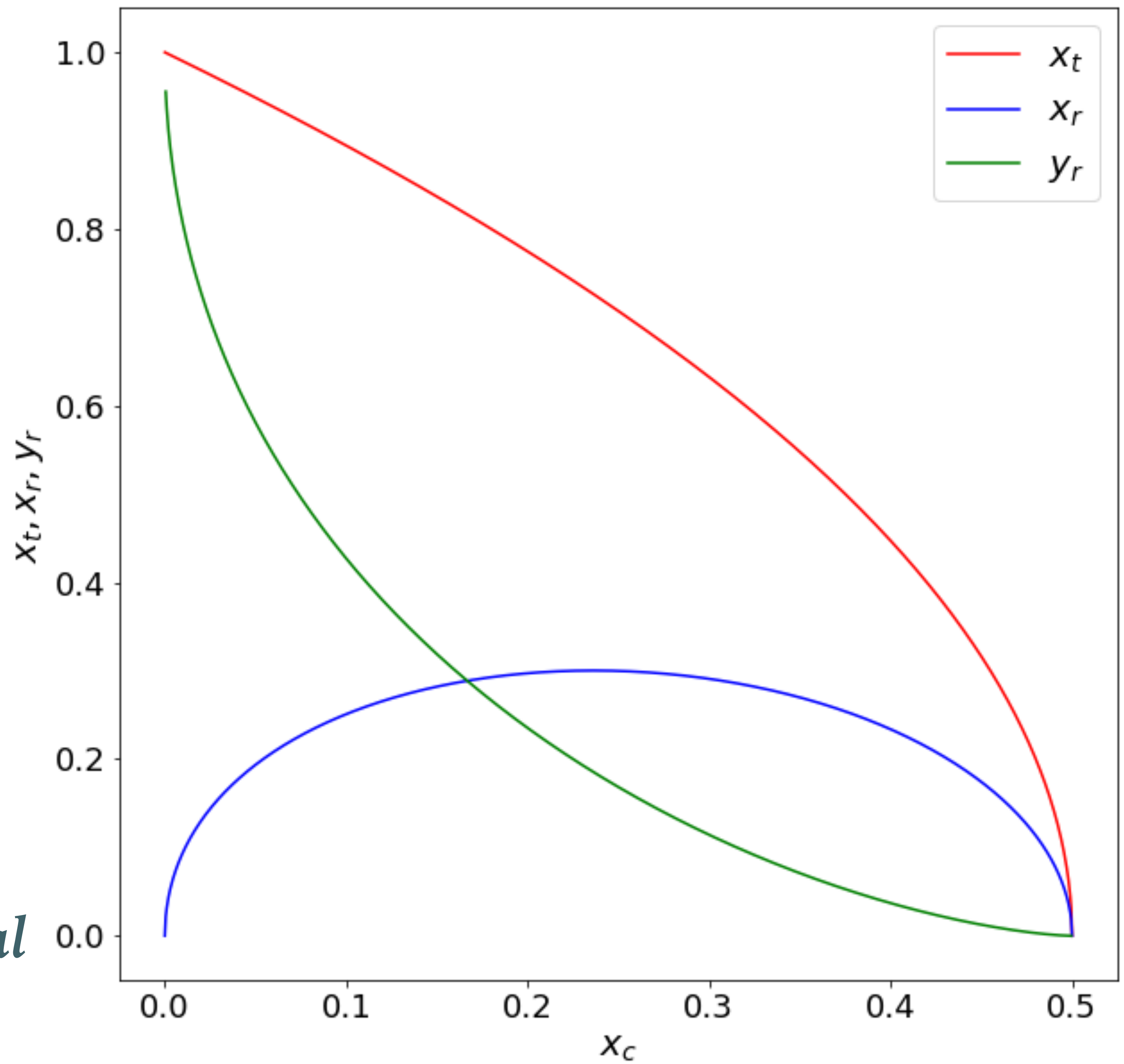
NON SINGULAR ISOTHERMAL SPHERE (NIS)



Small core, small radial critical line, large tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Small core, small radial critical line, large tangential critical line



large core, small radial critical line, small tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The lens equation can be reduced to the form:

$$y = x - \frac{m(x)}{x} = x - \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

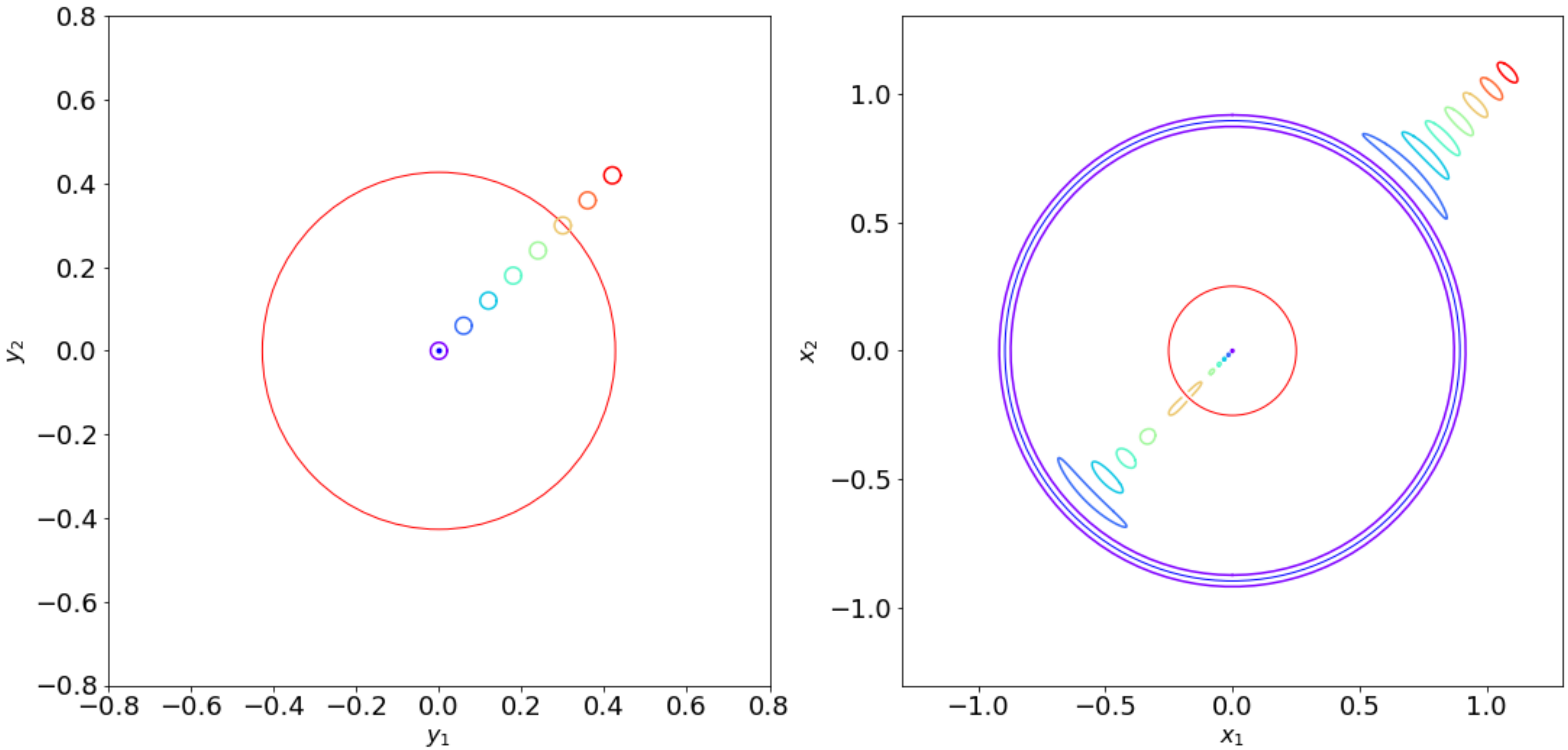
$$x^3 - 2yx^2 + (y^2 + 2x_c - 1)x - 2yx_c = 0 \,.$$

There are up to three solutions, but, again the existence of multiple images depends on y and x_c*…*

In particular on whether:

- *the radial caustic exist*
- *the source is inside or outside the radial caustic*

NON SINGULAR ISOTHERMAL SPHERE (NIS)



Three images if the source is inside the radial caustic; One image otherwise.

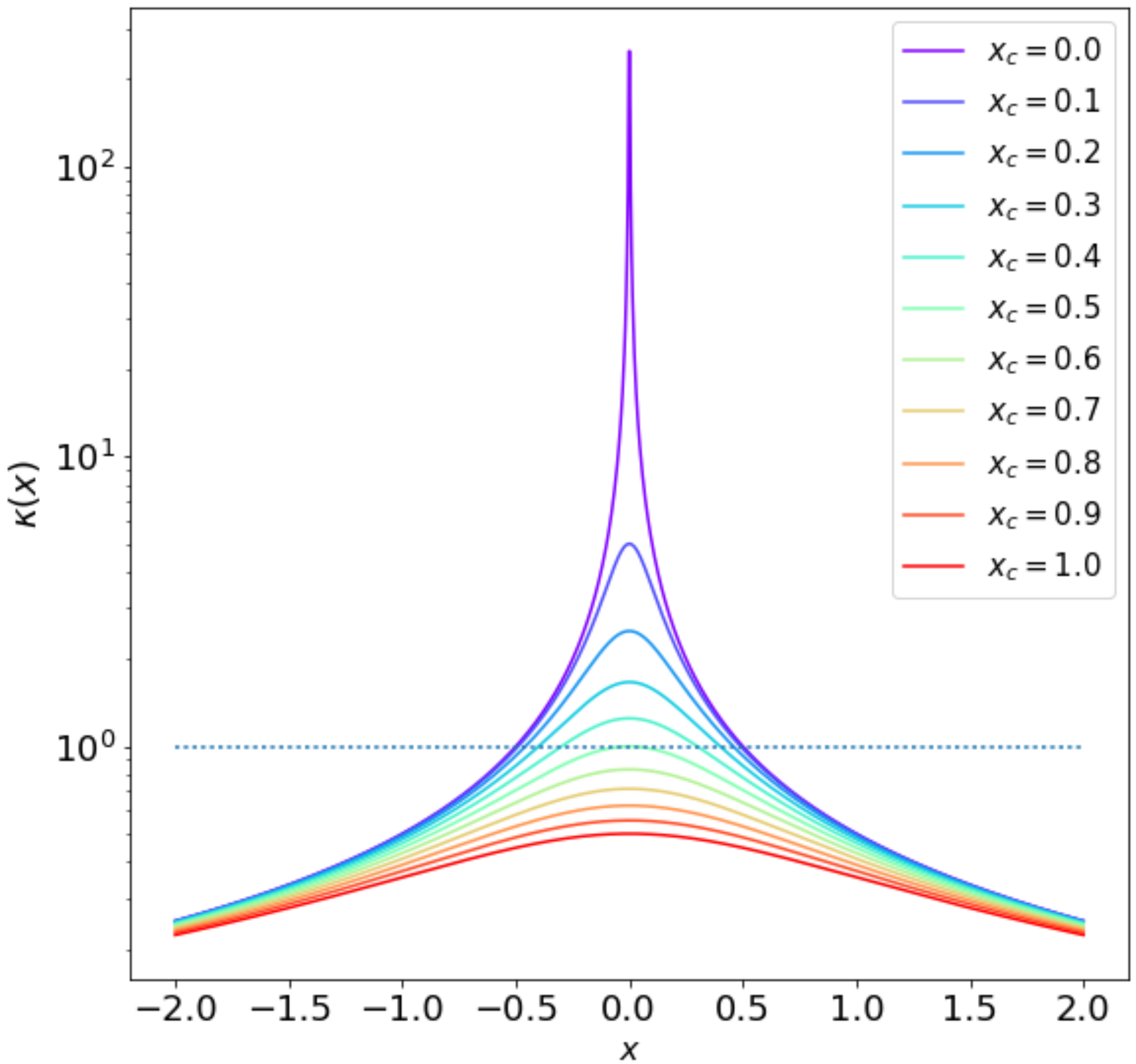
Parity: changes at each critical line (remember: maxima, minima, saddle points of TDS.

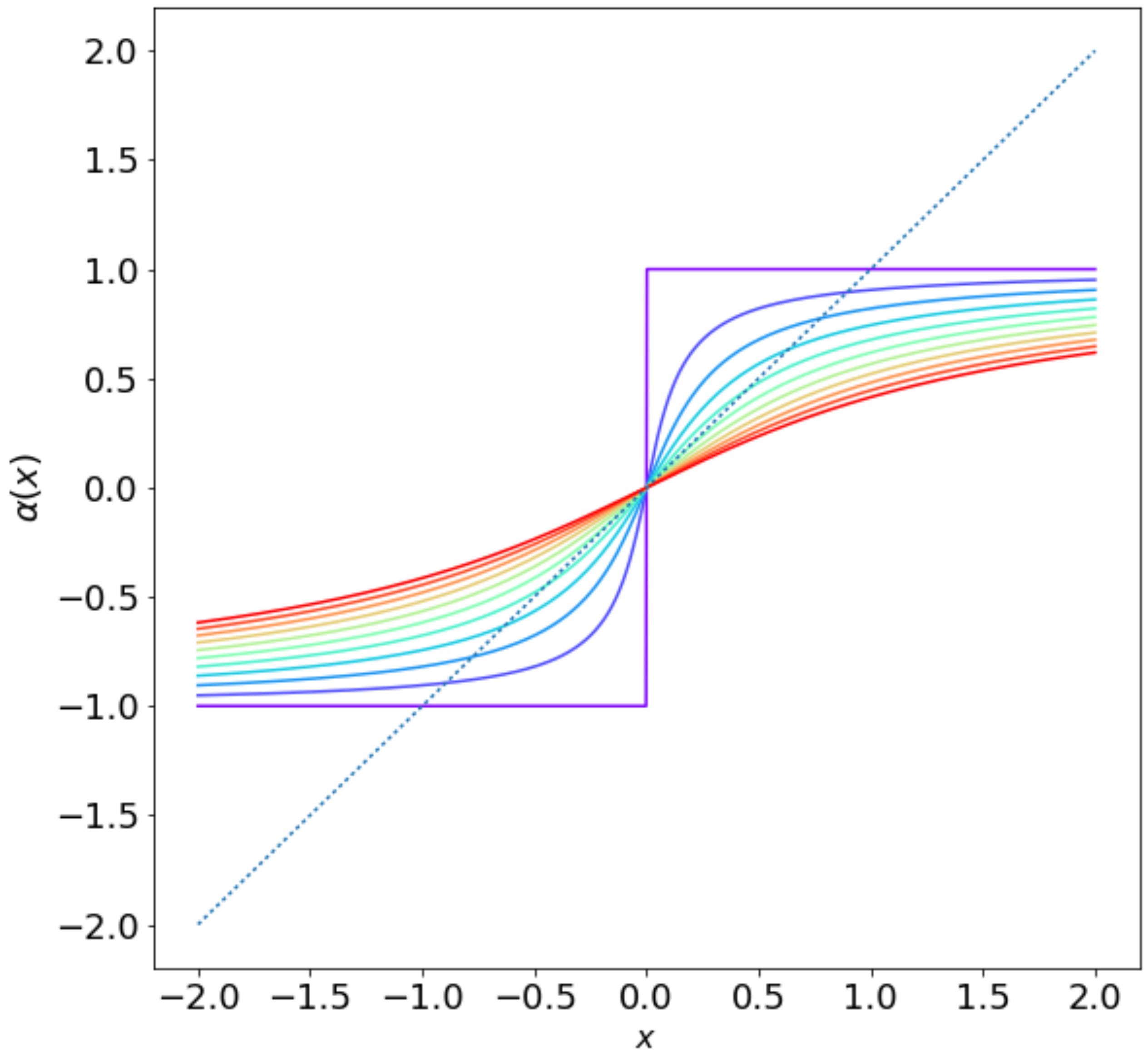
IMPORTANT!

We can see for the NIS that, when multiple images exist, $\kappa(0) > 1$…

In this case the lens is called ***supercritical****.*

Supercriticality is a <u>sufficient and necessary condition</u> for strong lensing by axisymmetric lenses (Sect. 2.5, "Principles of gravitational lensing", Congdon & Keeton)





SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

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Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

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Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

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Ensures that the mass inside elliptical iso-contours is independent on f

Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

Let's derive the convergence in dimensionless units:

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

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$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

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In polar coordinates:

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2}$$

$$\kappa(x, \varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

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With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

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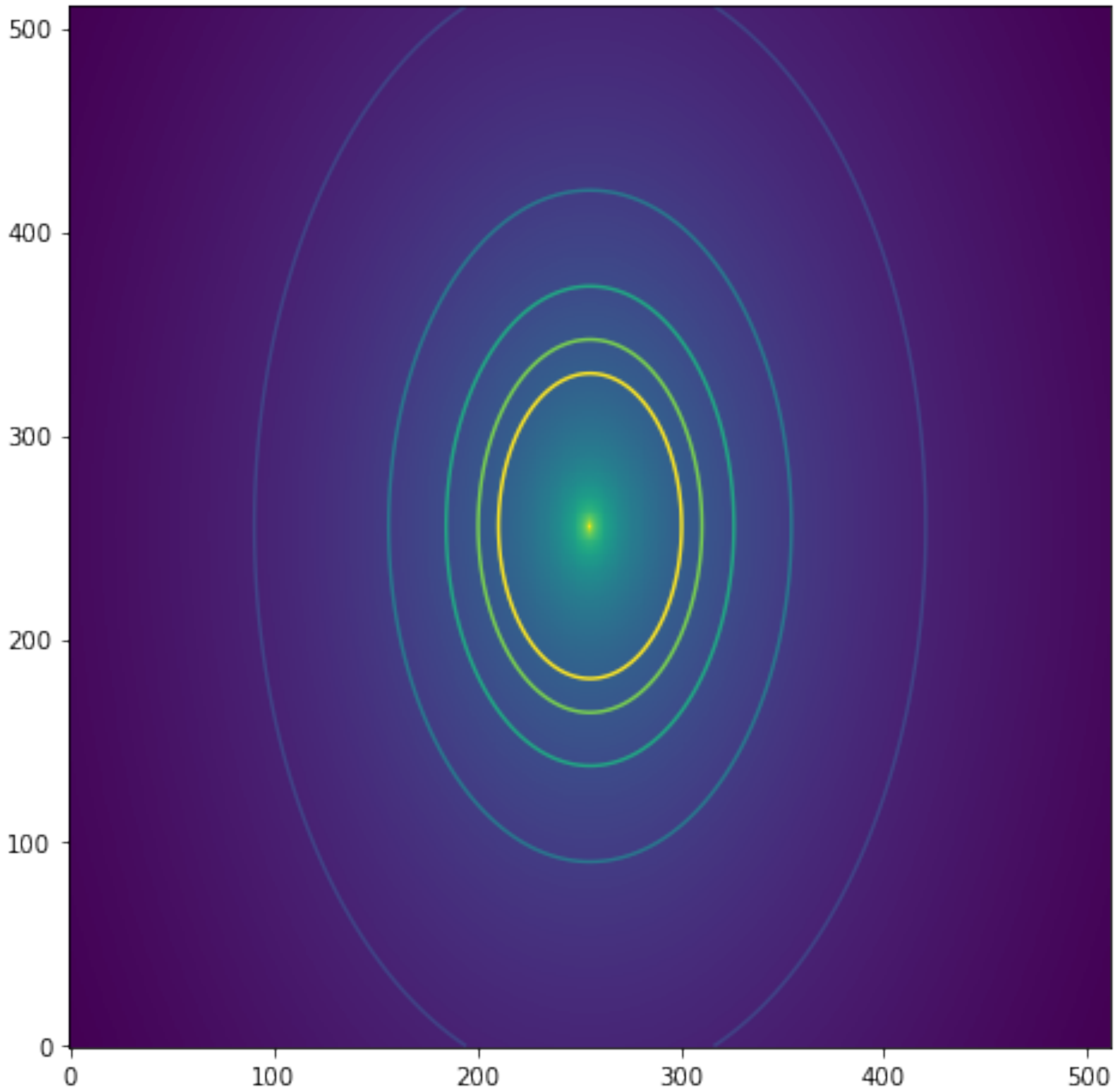
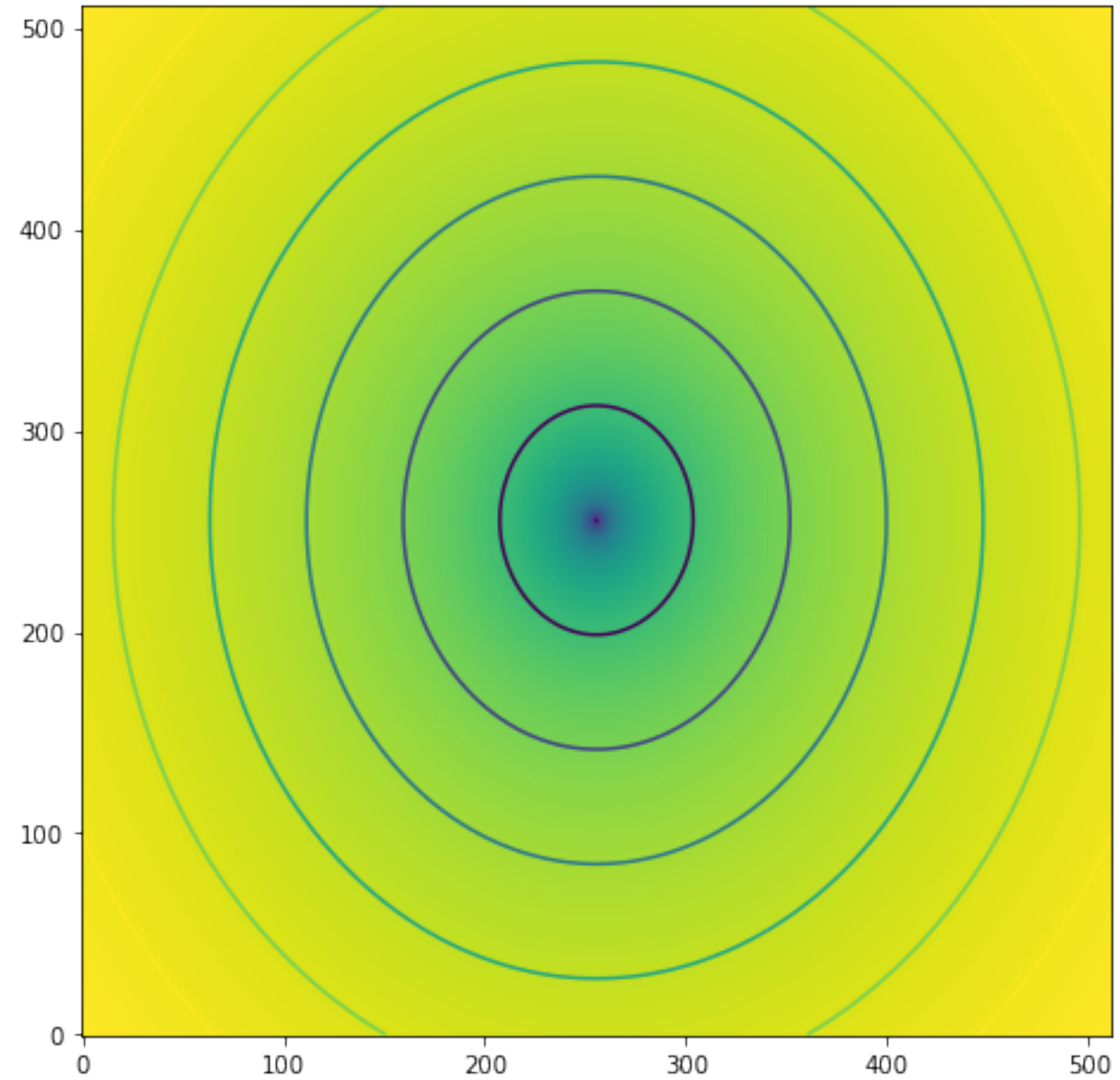
$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

Solved with Green's method (Kormann et al. 1994):

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

CONVERGENCE AND POTENTIAL





SINGULAR ISOTHERMAL ELLIPSOID

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Let's compute the deflection angle:

SINGULAR ISOTHERMAL ELLIPSOID

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Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

SINGULAR ISOTHERMAL ELLIPSOID

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$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
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SINGULAR ISOTHERMAL ELLIPSOID

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$$\begin{aligned}\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)\\ \alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\end{aligned}$$

Analogy with the SIS: the deflection angle does not depend on x!

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) = \frac{\sqrt{f}}{f'} \arcsin(f'\sin\varphi)$$

The component of the shear:

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)\\ \alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\end{aligned}$$

The component of the shear:

$$\begin{aligned}\gamma_1 &= \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1}-\frac{\partial\alpha_2}{\partial x_2}\right)\\ \gamma_2 &= \frac{\partial\alpha_1}{\partial x_2}\end{aligned}$$

$$\begin{aligned}\gamma_1 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi\\ \gamma_2 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

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The component of the shear:

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Similarly to the SIS: $\gamma = \kappa$

SINGULAR ISOTHERMAL ELLIPSOID

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

SINGULAR ISOTHERMAL ELLIPSOID

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whose eigenvalues are:

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,.\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}
\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\
\lambda_r &= 1-\kappa+\gamma = 1\,.
\end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

SINGULAR ISOTHERMAL ELLIPSOID

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As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

SINGULAR ISOTHERMAL ELLIPSOID

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\begin{aligned}
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$$

As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

$$\kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)} \quad \Rightarrow \quad \vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\left[\cos\varphi, \sin\varphi\right]$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}[\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}[\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned} y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ . \end{aligned}$$

There is no radial caustic, but there is the cut, which can be computed as

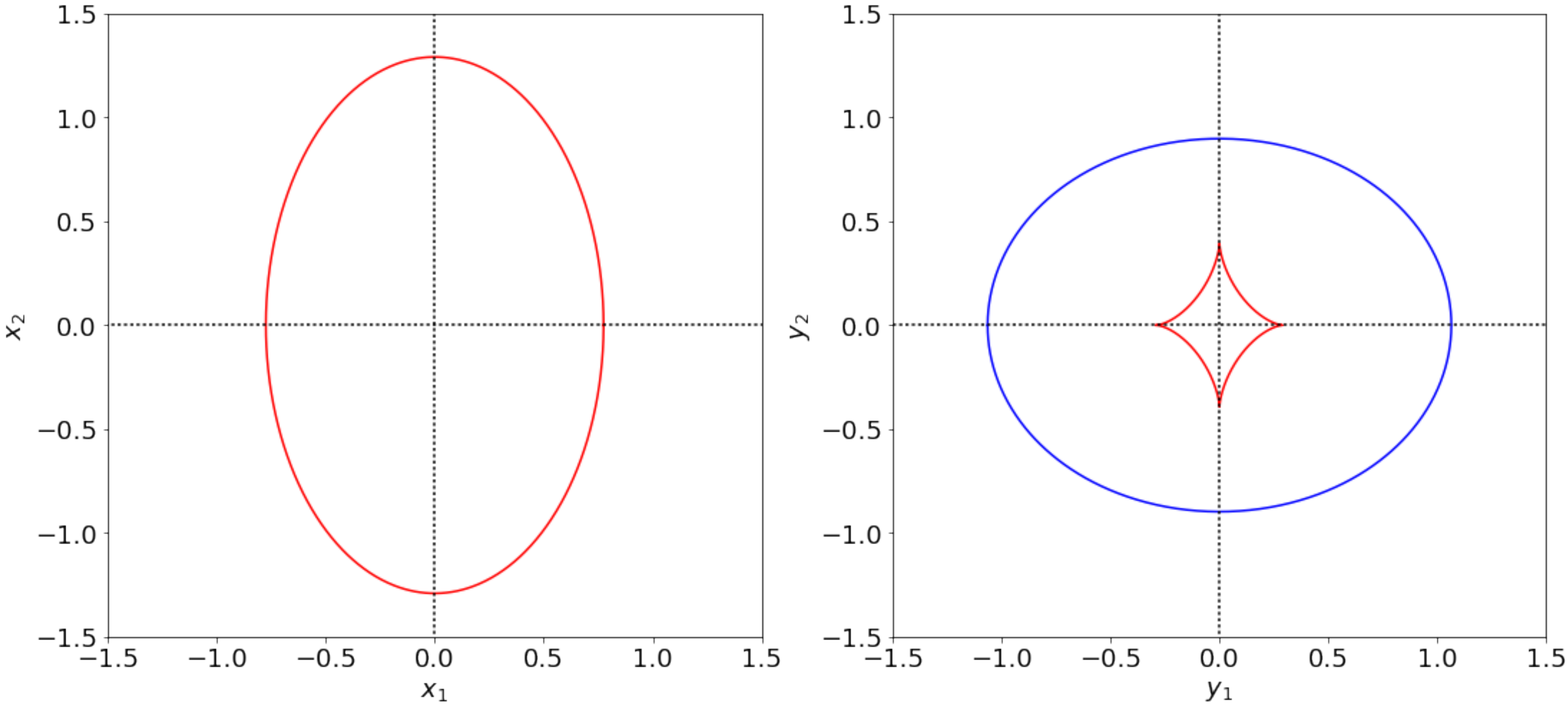
$$\vec{y}_c = \lim_{x\to 0}\vec{y}(x,\varphi) = -\vec{\alpha}(\varphi)$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{y}_c = \lim_{x\to 0} \vec{y}(x,\varphi) = -\vec{\alpha}(\varphi)$$

$$\begin{aligned} y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ . \end{aligned}$$

CRITICAL LINE, CUT, CAUSTIC



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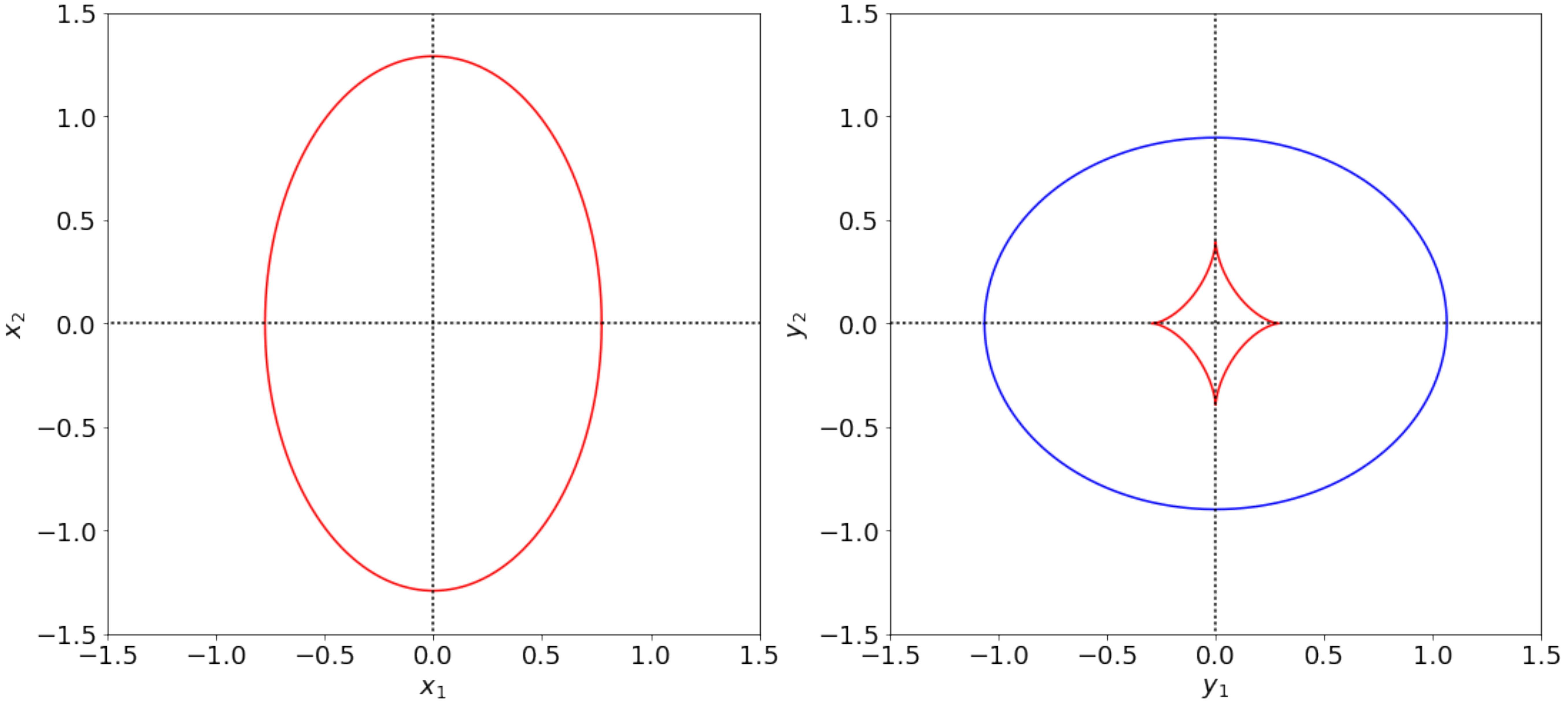

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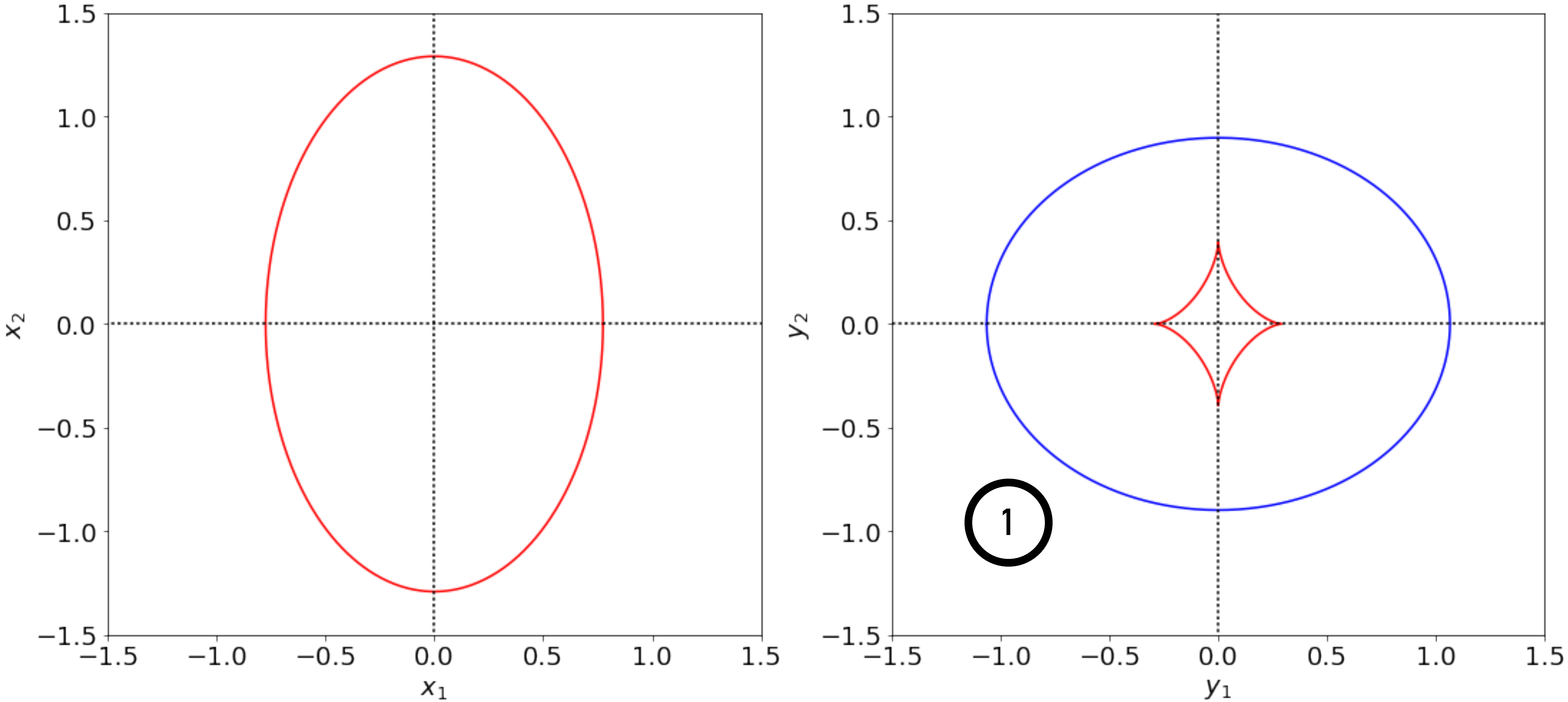

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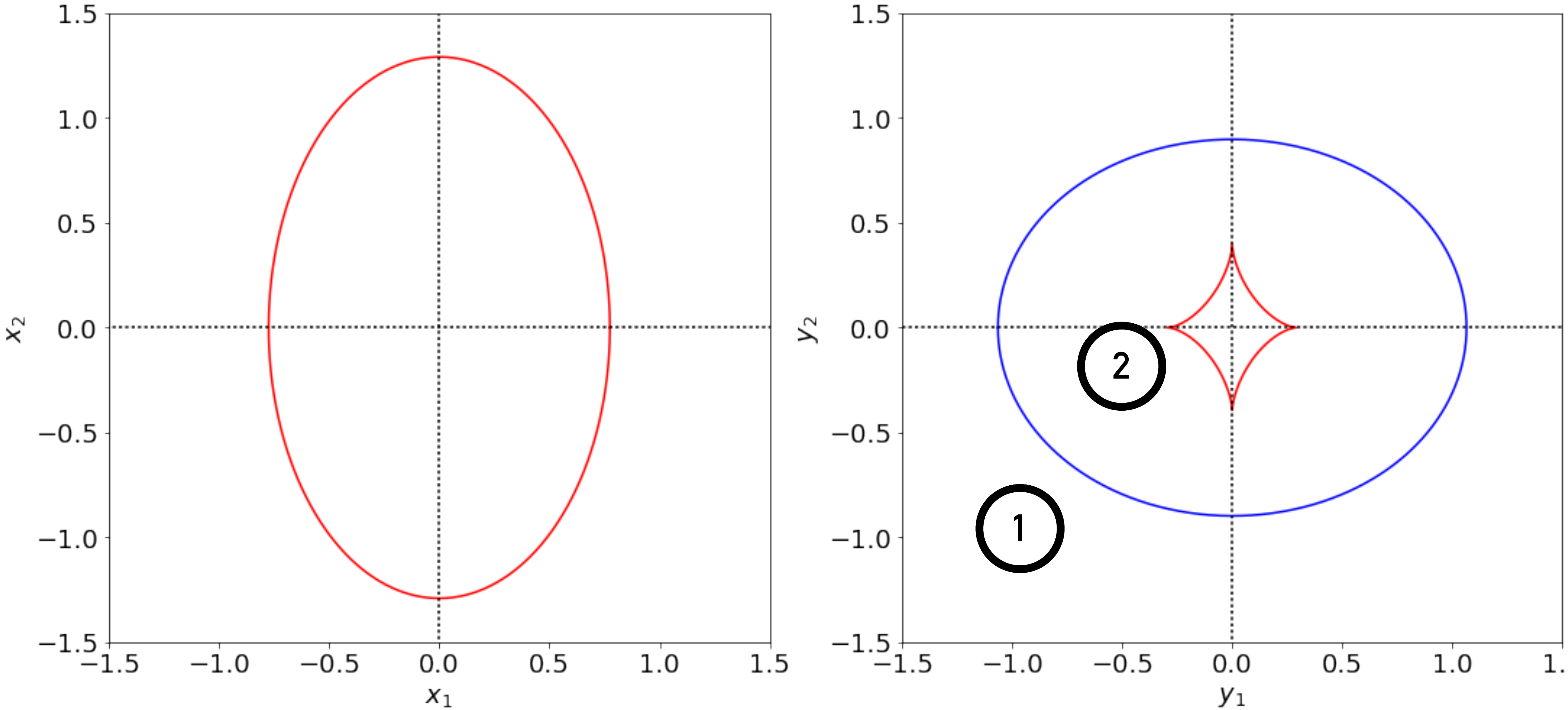

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