GRAVITATIONAL LENSING

LECTURE 8

Docente: Massimo Meneghetti

AA 2016-2017

CONTENTS

- second order lensing effects

SECOND ORDER LENS EQUATION

$$\beta_i \simeq \underbrace{\frac{\partial \beta_i}{\partial \theta_j}}_{A_{ij}} \theta_j$$

Thus, second order lens mapping is described by a rank-3 tensor, which can be written in terms of the derivatives of the shear.

SECOND ORDER LENS EQUATION

$$\beta_i \simeq \frac{\partial \beta_i}{\partial \theta_j}\theta_j + \frac{1}{2}\frac{\partial^2 \beta_i}{\partial \theta_j \partial \theta_k}\theta_j\theta_k$$

$$A_{ij}$$

$$\frac{\partial A_{ij}}{\partial \theta_k} = D_{ijk}$$

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$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} \qquad D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}$$

Thus, second order lens mapping is described by a rank-3 tensor, which can be written in terms of the derivatives of the shear.

COMPLEX NOTATION

It is sometimes very convenient to use a complex notation to describe quantities on the lens and on the source planes:

$$v = (v_1, v_2) \longrightarrow v = v_1 + iv_2$$

With this notation,

$$\alpha = \alpha_1 + i\alpha_2$$

$$\gamma = \gamma_1 + i\gamma_2.$$

We can also define complex differential operators. They turn out to be "spin ladder" operators:

$$\partial = \partial_1 + i\partial_2$$

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We can also define complex differential operators. They turn out to be "spin ladder" operators:

$$\partial = \partial_1 + i\partial_2$$ *Spin raising operator*

$$\partial^\dagger = \partial_1 - i\partial_2$$ *Spin lowering operator*

COMPLEX NOTATION

$$\partial\hat{\Psi} = \partial_1\hat{\Psi} + i\partial_2\hat{\Psi} = \alpha_1 + i\alpha_2 = \alpha$$

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From spin-0 scalar field to spin-1 vector field (deflection angle)

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$$\partial^\dagger\partial = \partial_1^2 + \partial_2^2 = \triangle$$

$$\partial^\dagger\partial\hat{\Psi} = \triangle\hat{\Psi} = 2\kappa$$

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From spin-1 vector field to spin-0 scalar field

$$\frac{1}{2}\partial\partial\hat{\Psi} = \frac{1}{2}\partial\alpha = \gamma$$

The shear is a spin-2 tensor field

$$\partial^{-1}\partial^\dagger\gamma = \frac{1}{2}\partial^{-1}\partial^\dagger\partial\partial\hat{\Psi} = \partial^\dagger\partial\hat{\Psi} = \kappa$$

COMPLEX NOTATION

$$\begin{aligned} F &= \frac{1}{2}\partial\partial^{\dagger}\partial\hat{\Psi} = \partial\kappa \\ G &= \frac{1}{2}\partial\partial\partial\hat{\Psi} = \partial\gamma \end{aligned}$$

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\end{aligned}
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COMPLEX NOTATION

$$F = \frac{1}{2}\partial\partial^\dagger\partial\hat{\Psi} = \partial\kappa$$

Spin-1

$$G = \frac{1}{2}\partial\partial\partial\hat{\Psi} = \partial\gamma$$

Spin-3

$$F = F_1 + iF_2 = (\gamma_{1,1} + \gamma_{2,2}) + i(\gamma_{2,1} - \gamma_{1,2})$$

$$G = G_1 + iG_2 = (\gamma_{1,1} - \gamma_{2,2}) + i(\gamma_{2,1} + \gamma_{1,2})$$

$$D_{ij1} = \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix} \qquad D_{ij2} = \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & 2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}$$

COMPLEX NOTATION

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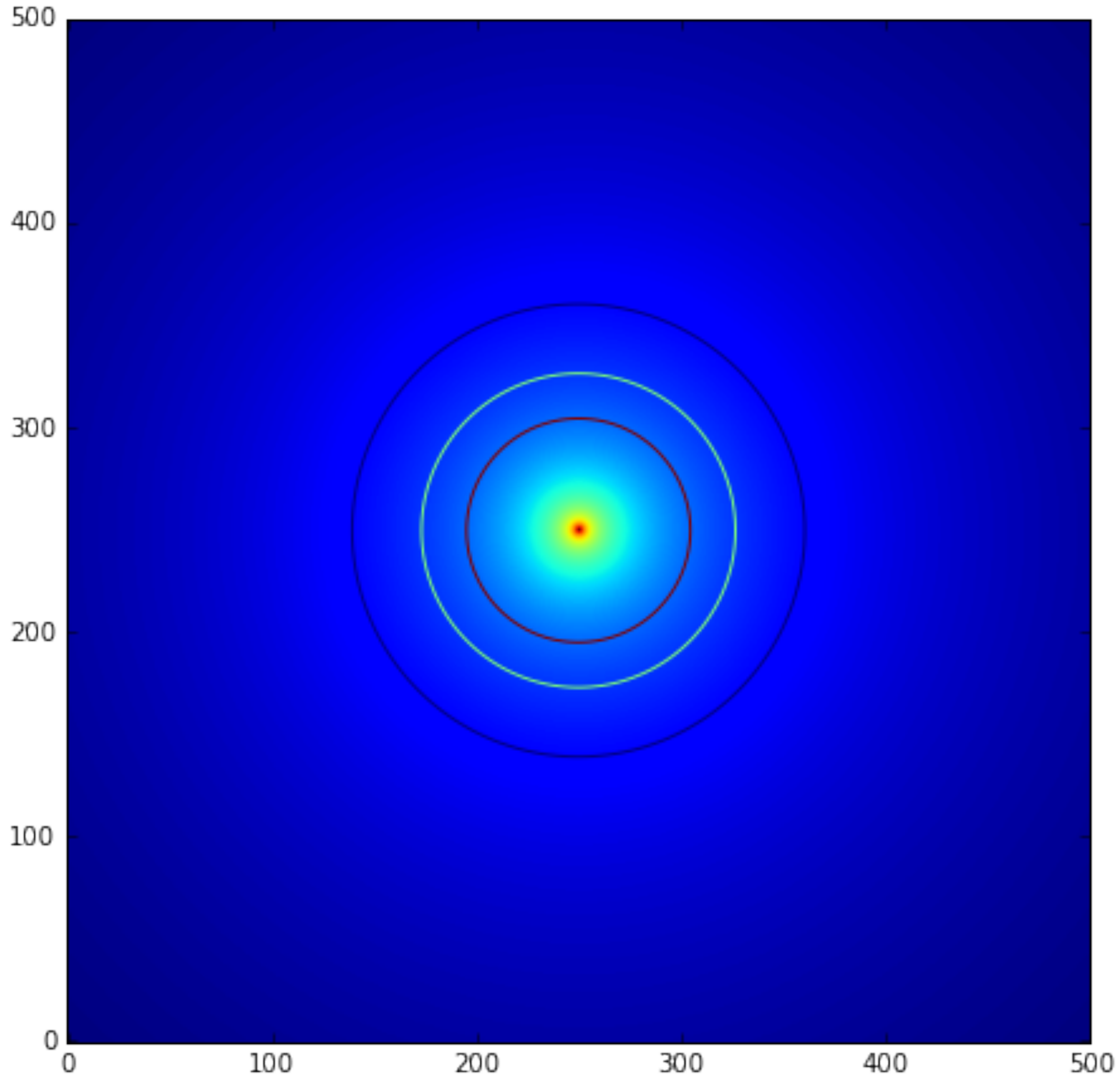
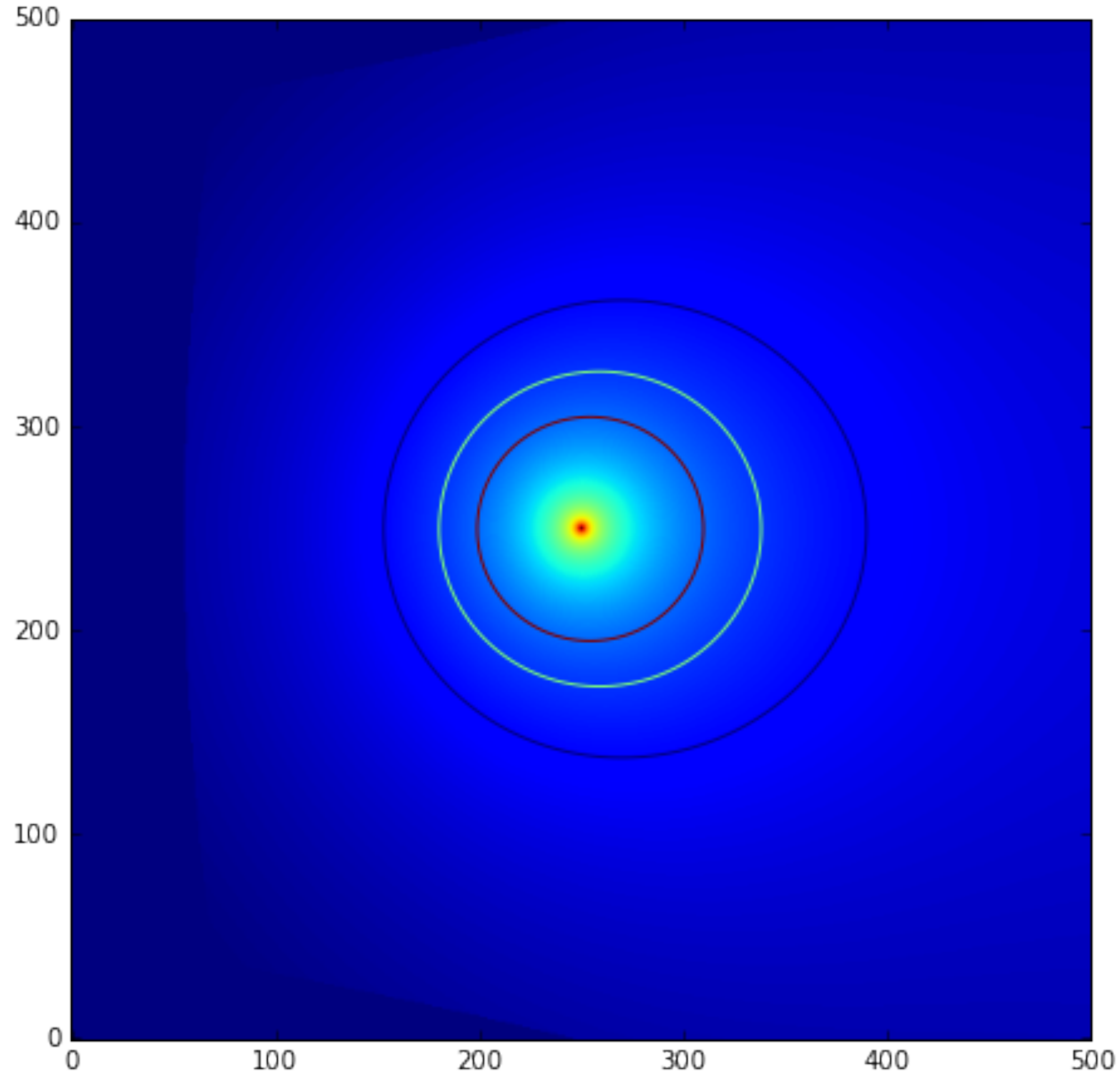
Thus, the second order term in the lens equation can be written in terms of F and G.

$$\beta_1 = A_{11}\theta_1 + A_{12}\theta_2 + \frac{1}{2}(D_{111}\theta_1^2 + 2D_{112}\theta_1\theta_2 + D_{221}\theta_2^2)$$
$$\beta_2 = A_{22}\theta_2 + A_{12}\theta_1 + \frac{1}{2}(D_{222}\theta_2^2 + 2D_{221}\theta_1\theta_2 + D_{112}\theta_1^2)$$

SECOND ORDER DISTORTIONS OF A CIRCULAR SOURCE





$F_1 = 0.5 \quad G_1 = 0$

$F_2 = 0 \quad G_2 = 0$

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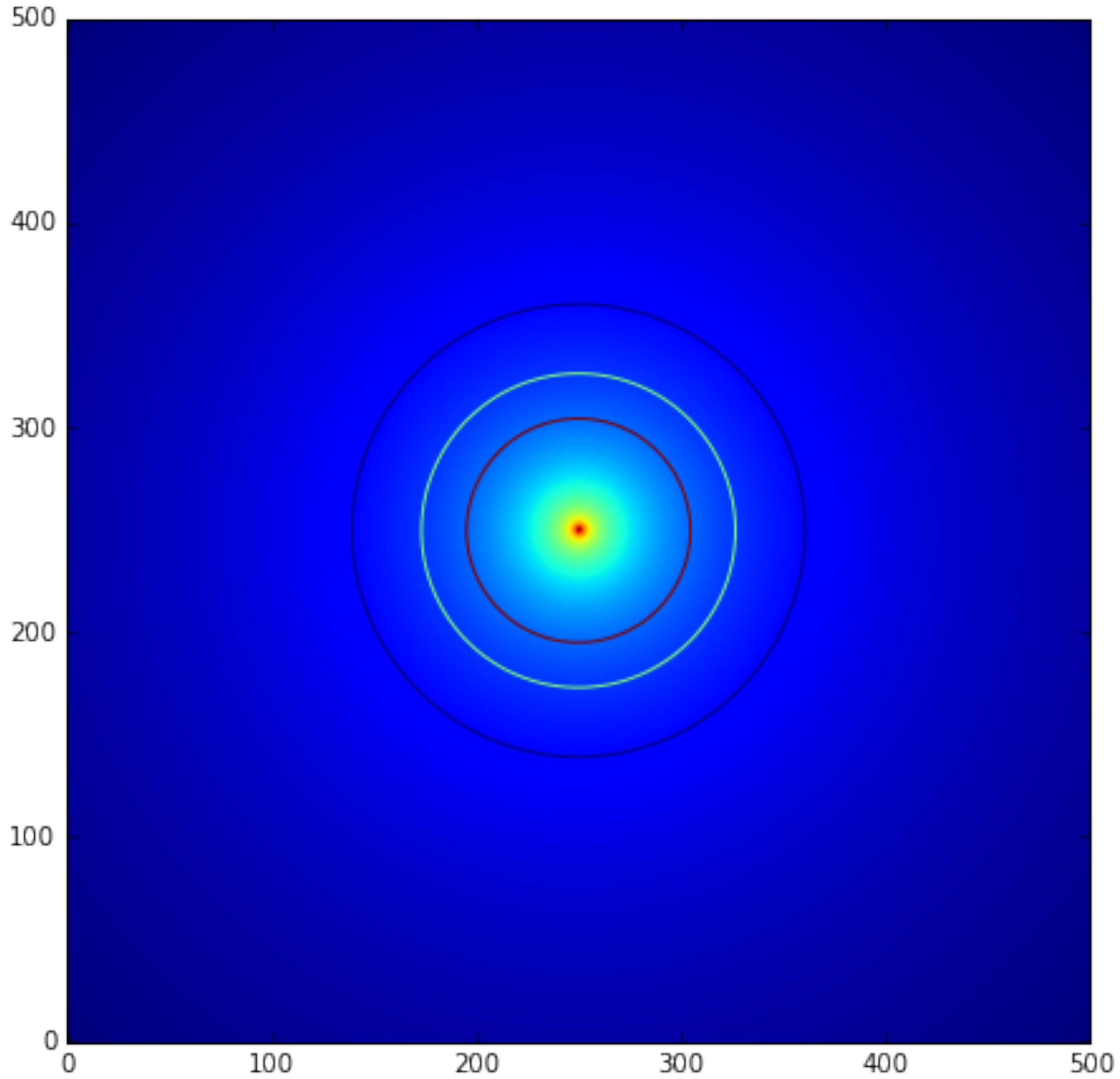

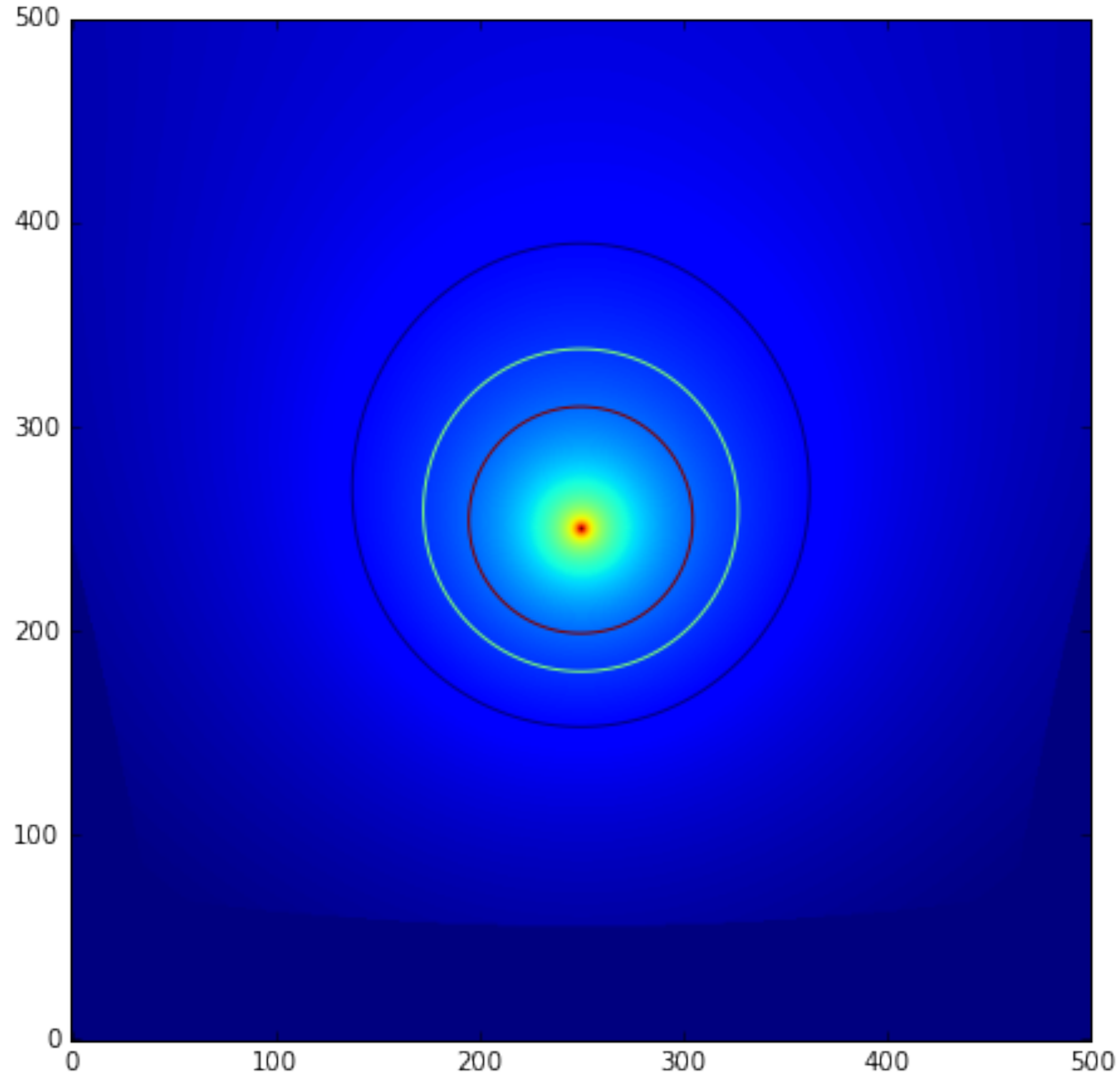

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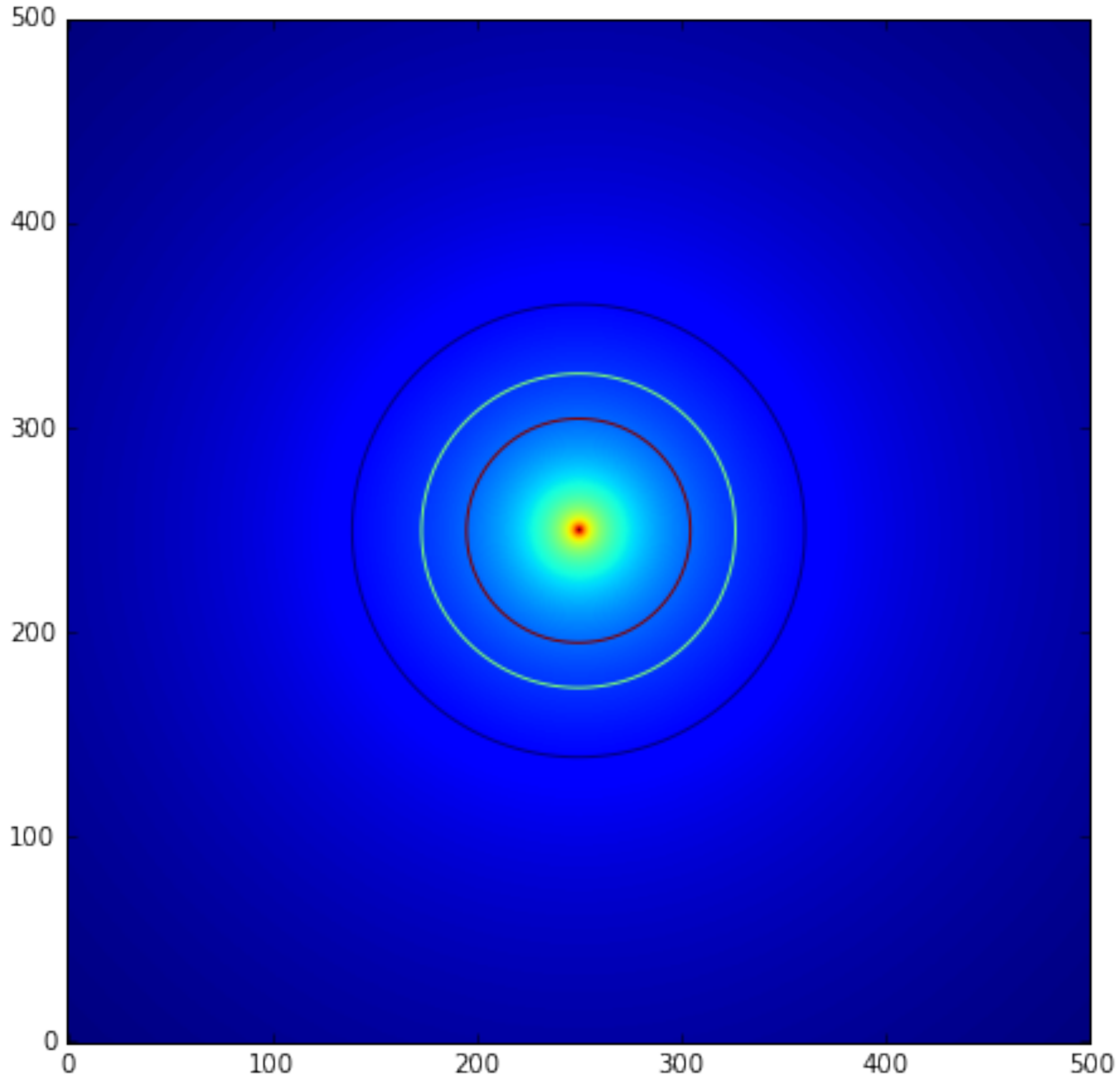

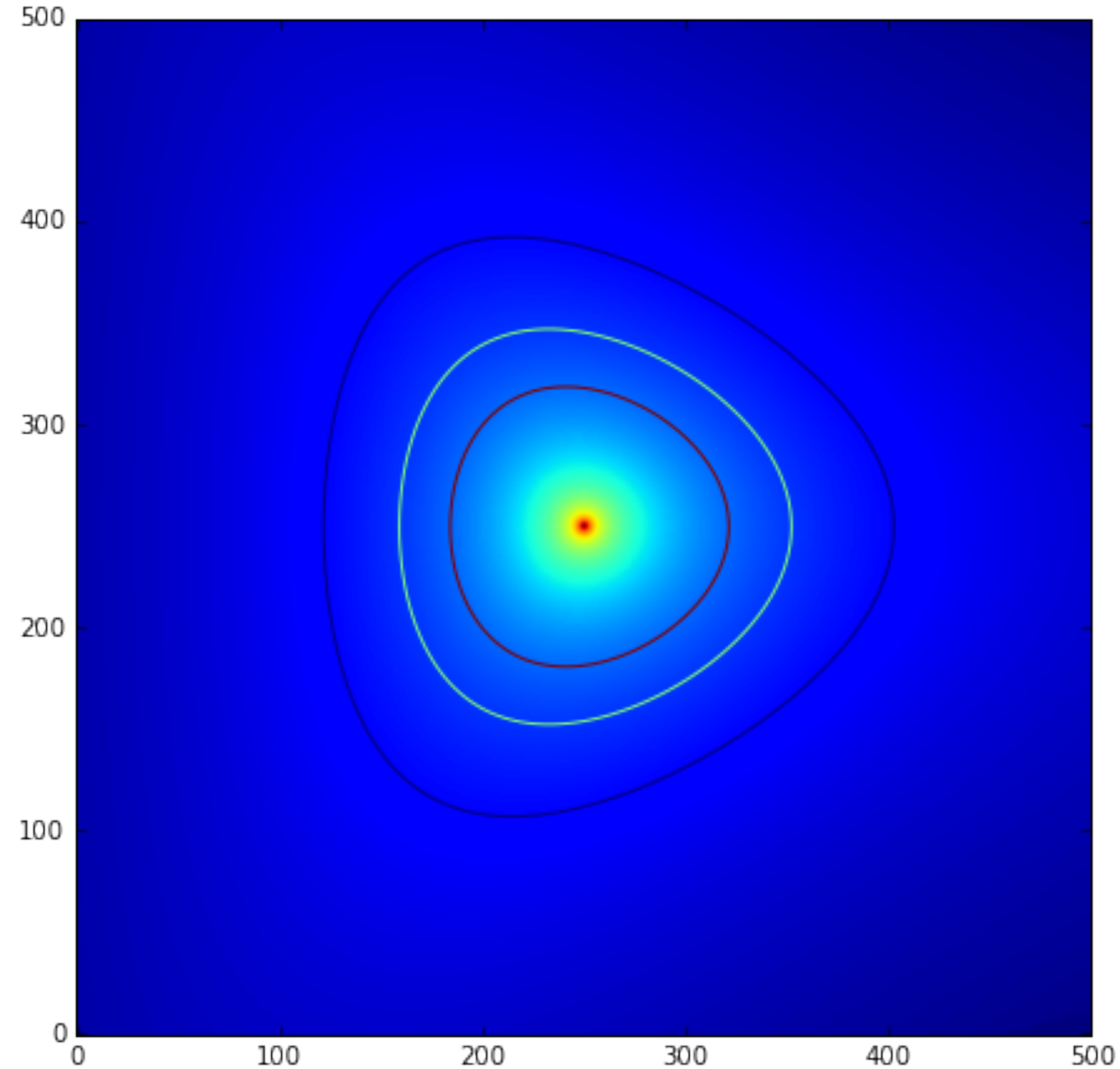

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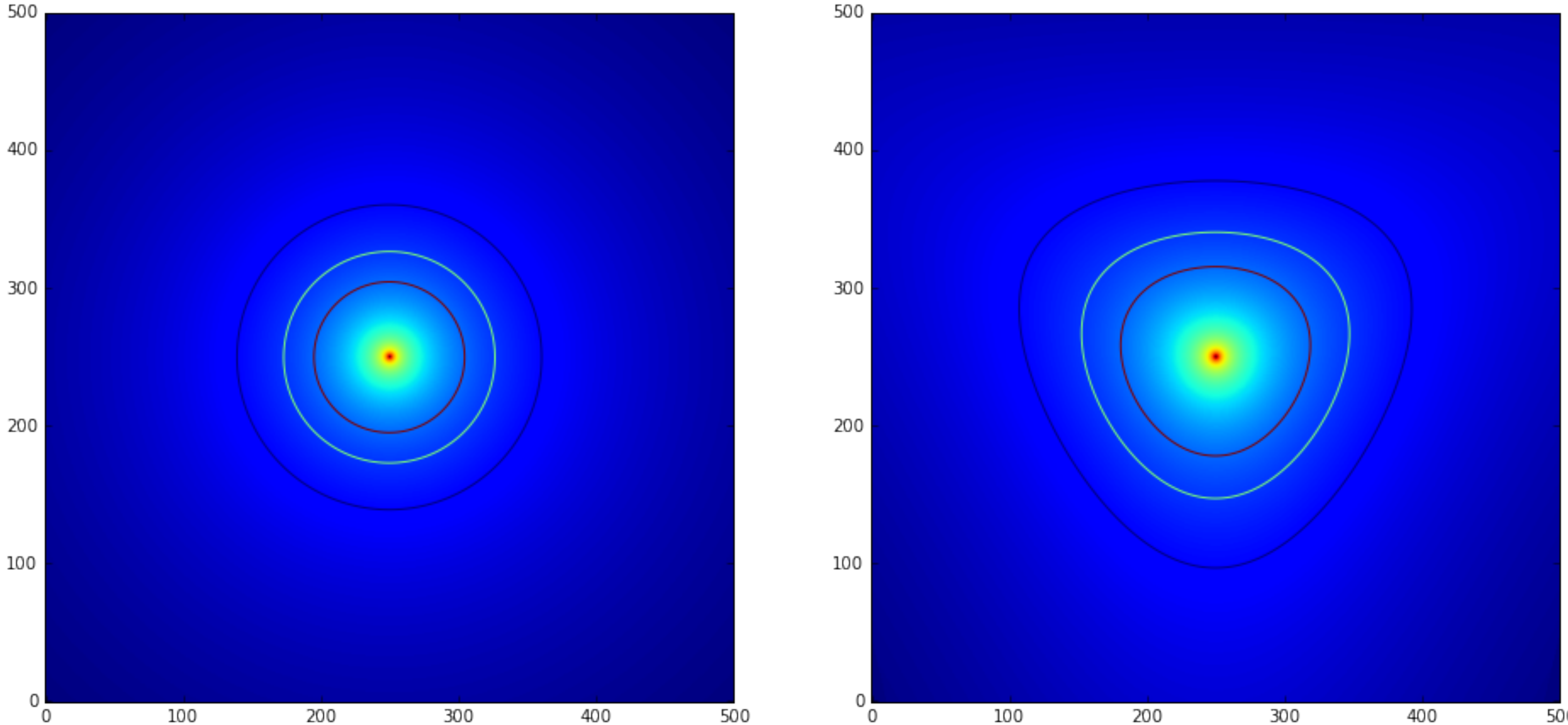

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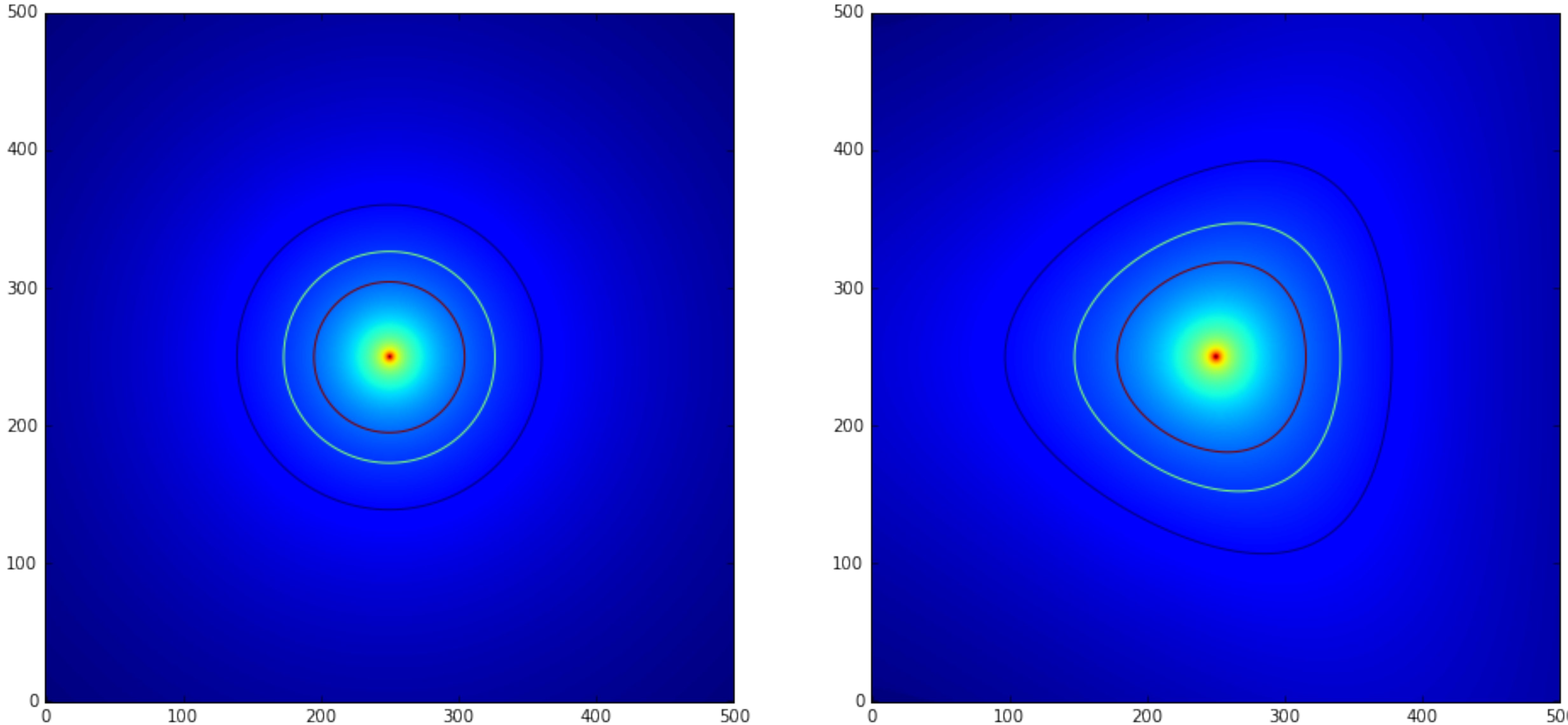
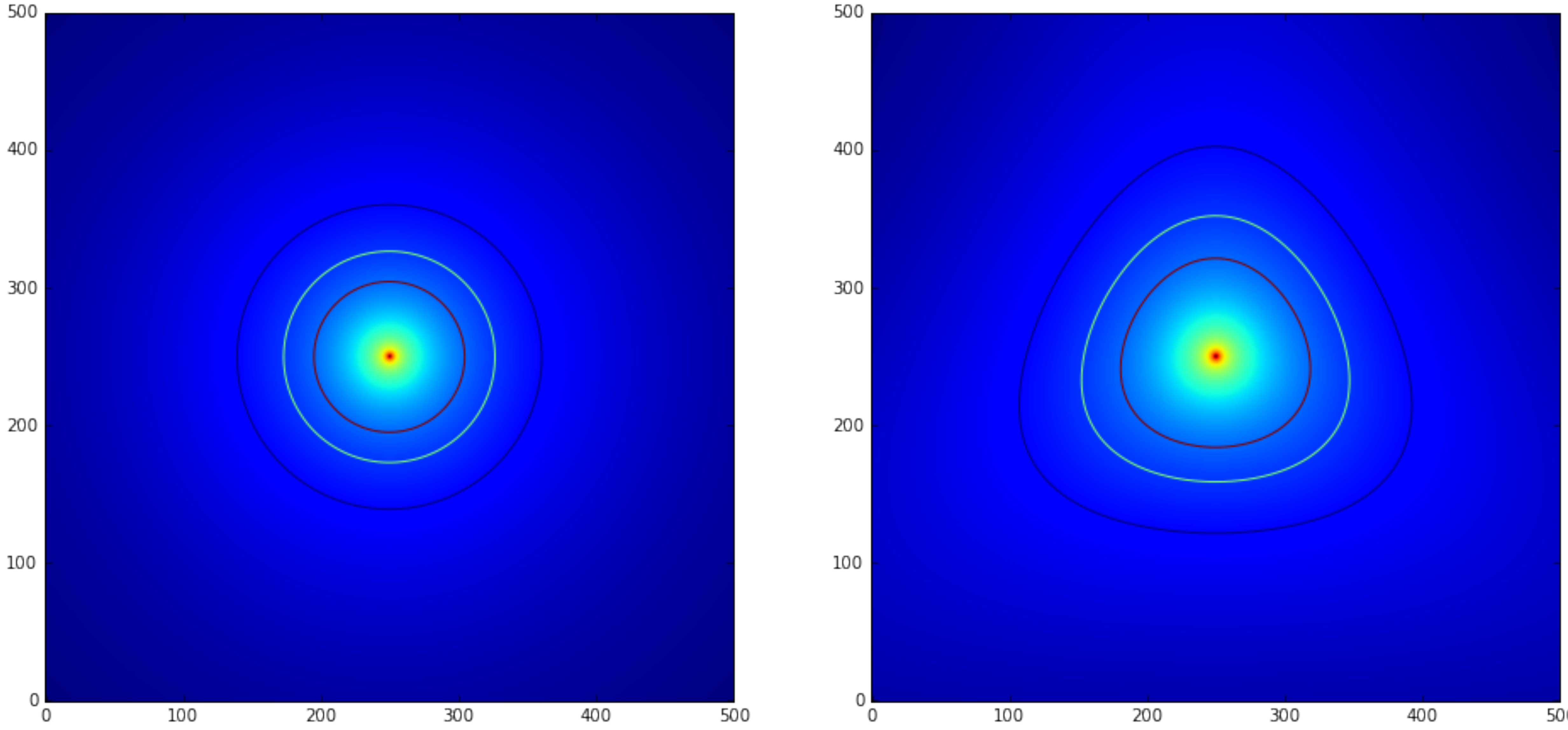
SECOND ORDER DISTORTIONS OF A CIRCULAR SOURCE



F_1=0 G_1=-0.5

F_2=0 G_2=0

SECOND ORDER DISTORTIONS OF A CIRCULAR SOURCE



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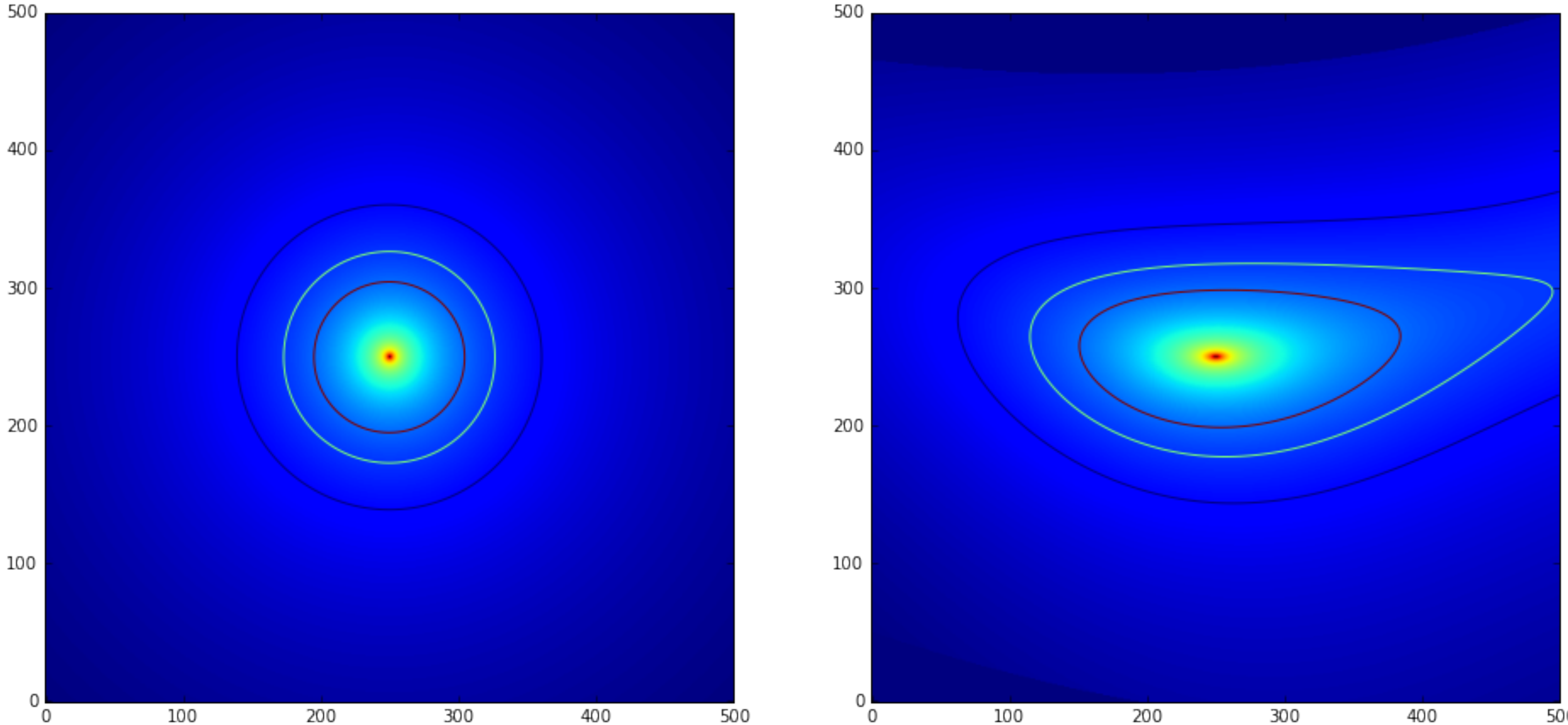
SECOND ORDER DISTORTIONS OF A CIRCULAR SOURCE



$\gamma_1 = 0.3 \quad F_1 = 0.2 \quad G_1 = 0$

$\gamma_2 = 0 \quad F_2 = 0 \quad G_2 = 0.5$

GRAVITATIONAL TIME DELAY

In Lecture 1: $n = 1 - \frac{2\Phi}{c^2}$

$$t_{grav} = \int \frac{dz}{c'} - \int \frac{dz}{c} = \frac{1}{c}\int (n-1)dz = -\frac{2}{c^3}\int \Phi dz$$

Remember that $\hat{\Psi}(\vec{\theta}) = \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z)\mathrm{d}z$

Therefore, $t_{\mathrm{grav}} = -\frac{D_{\mathsf{L}} D_{\mathsf{LS}}}{D_{\mathsf{S}}} \frac{1}{c} \hat{\Psi}$

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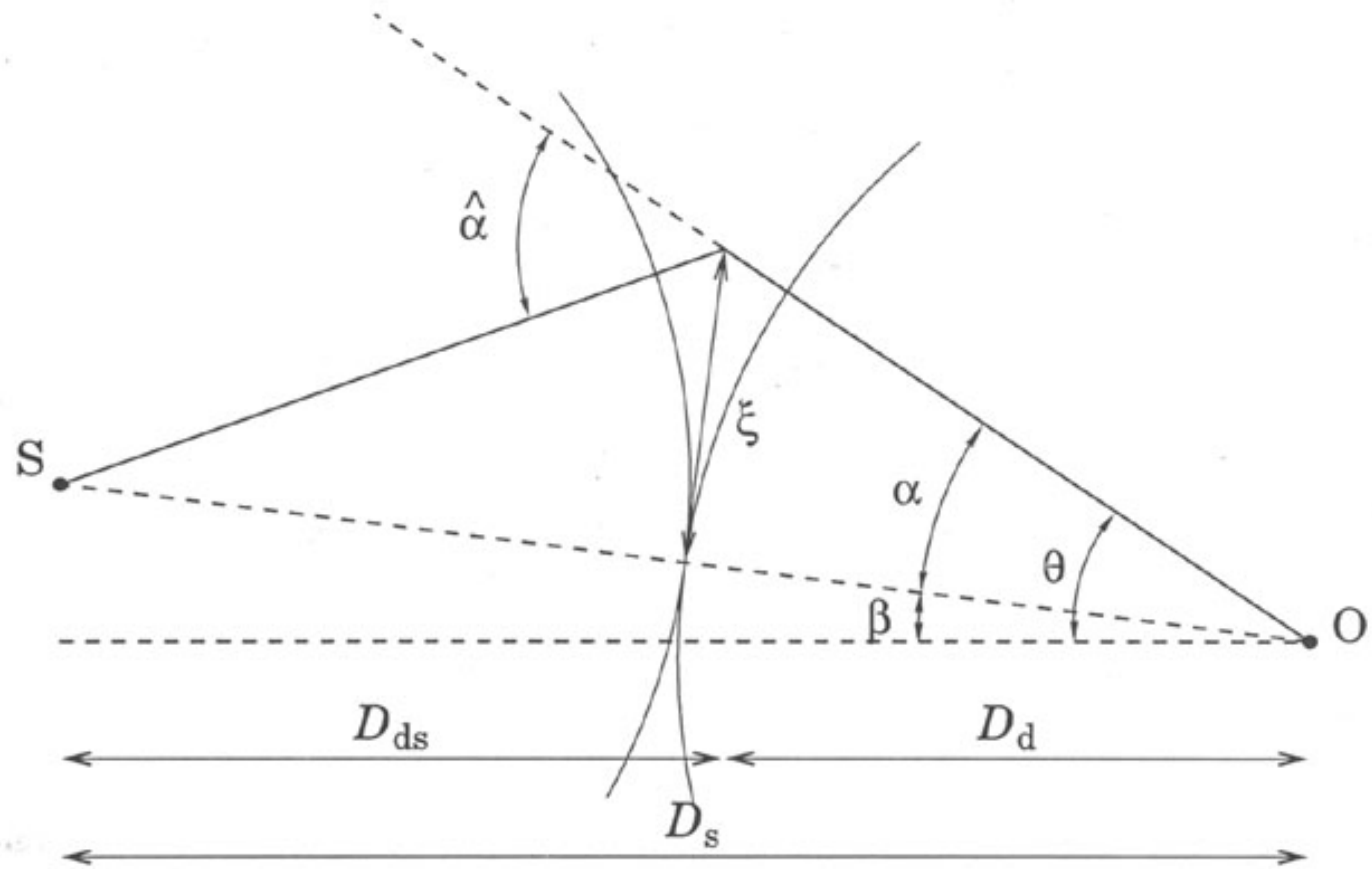
Integrating along the line of sight!

GEOMETRICAL TIME DELAY



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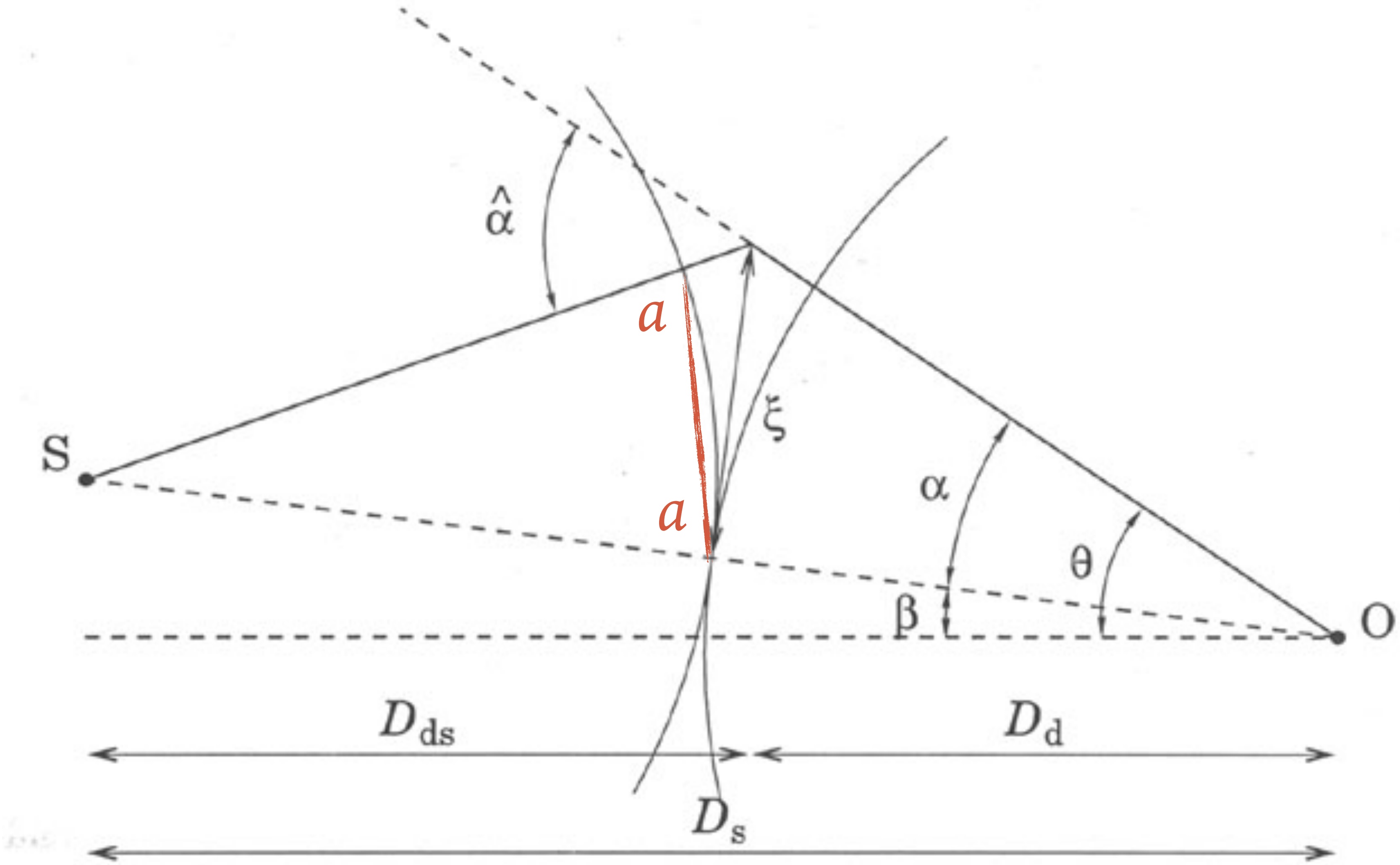

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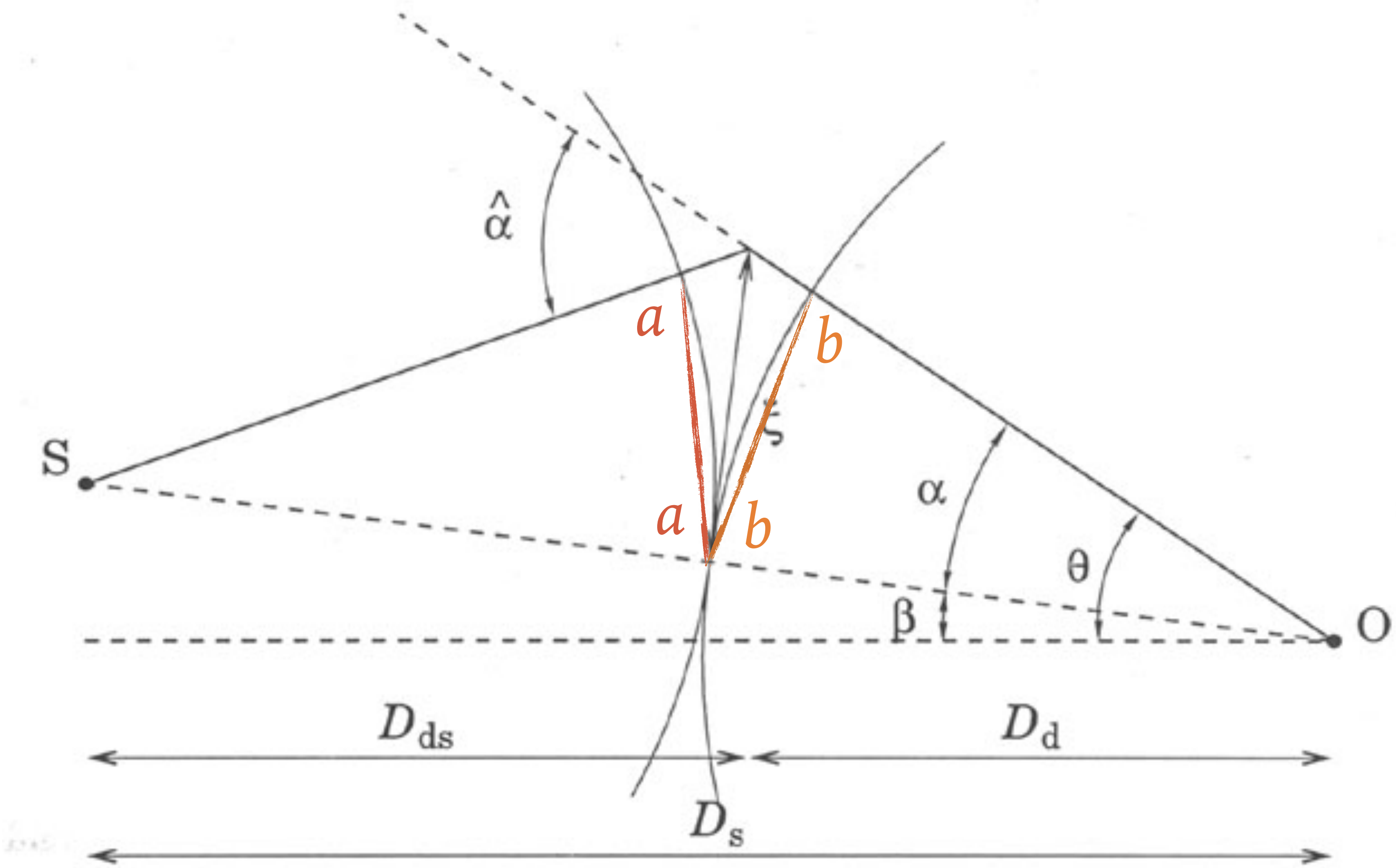

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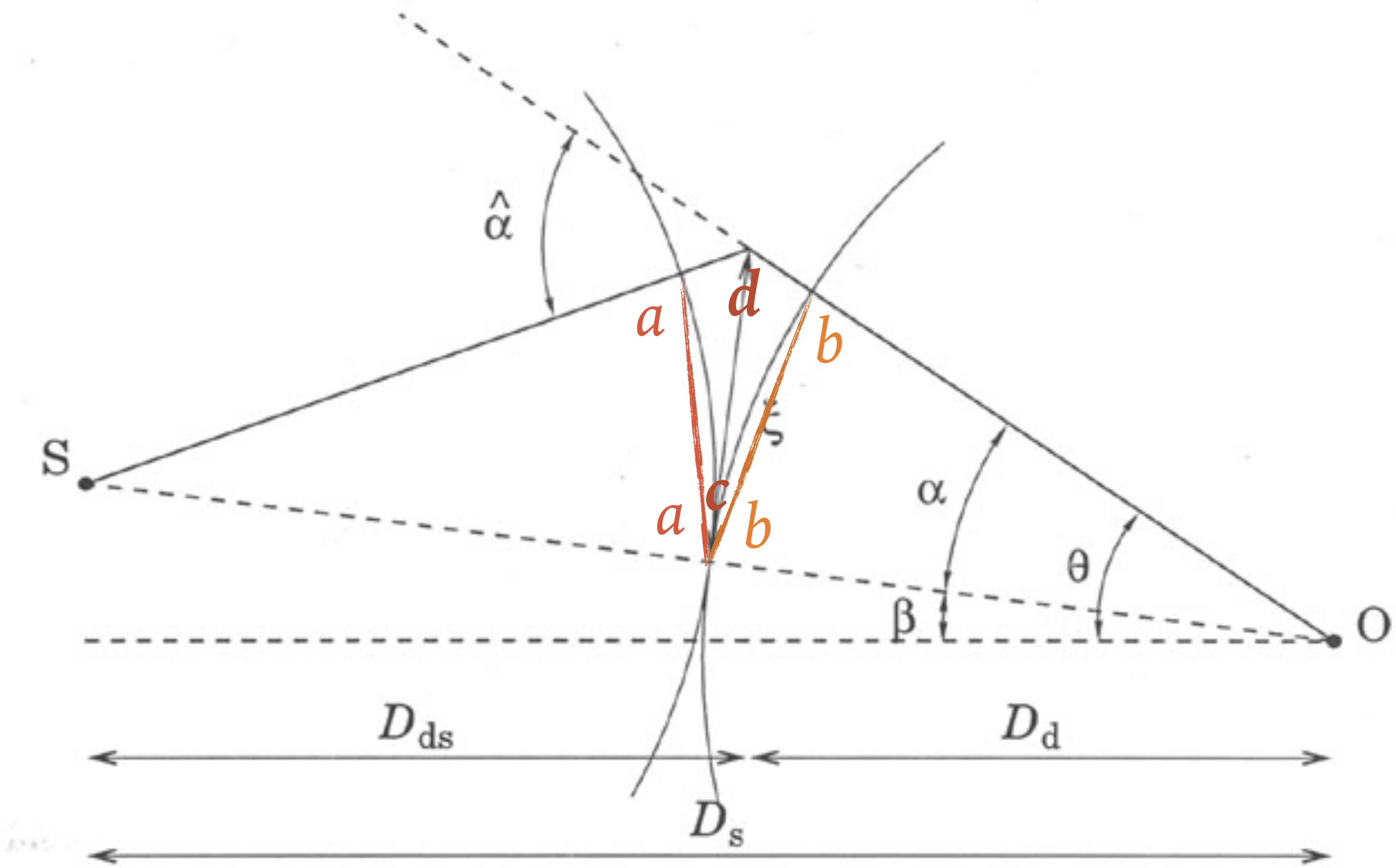

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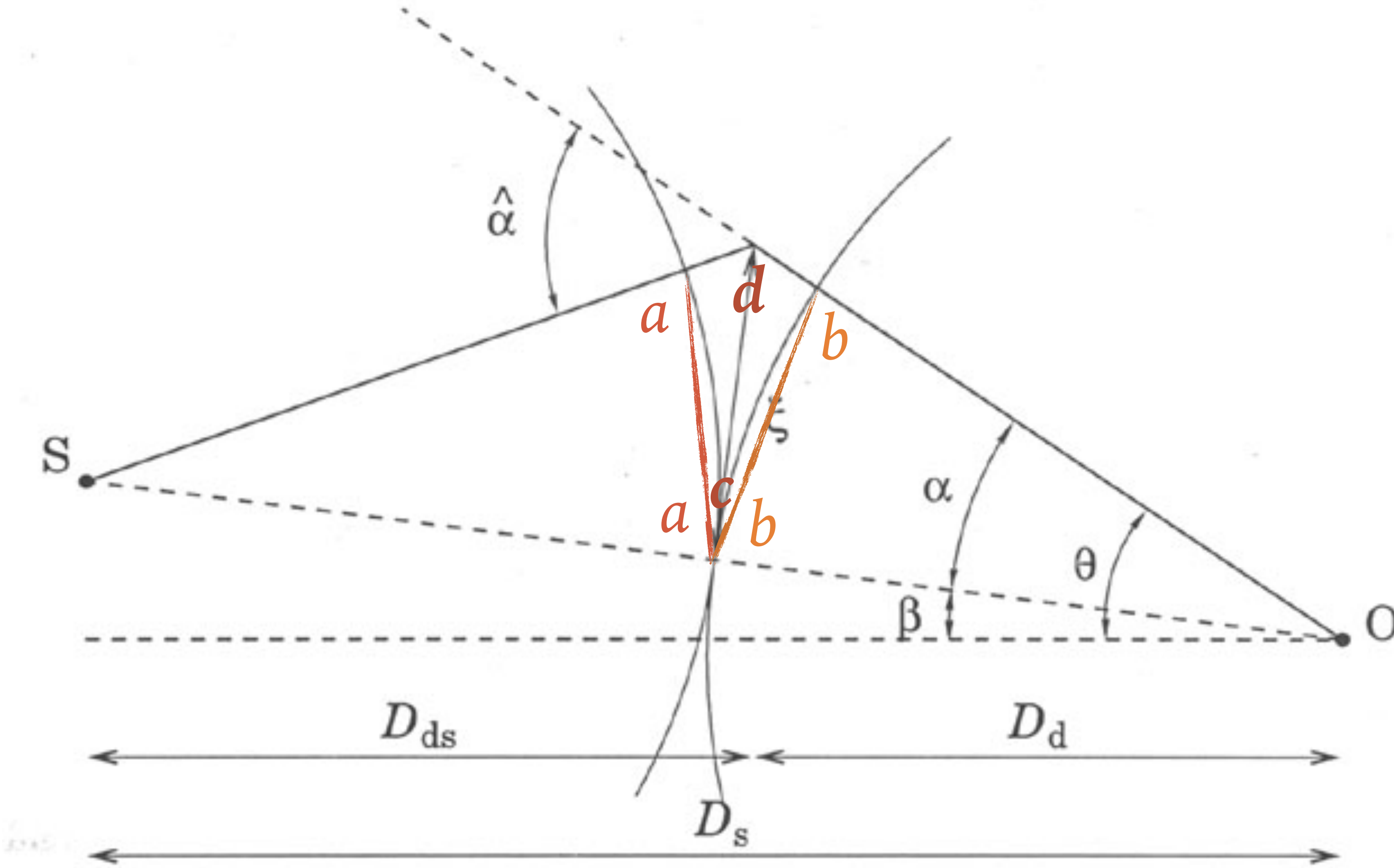
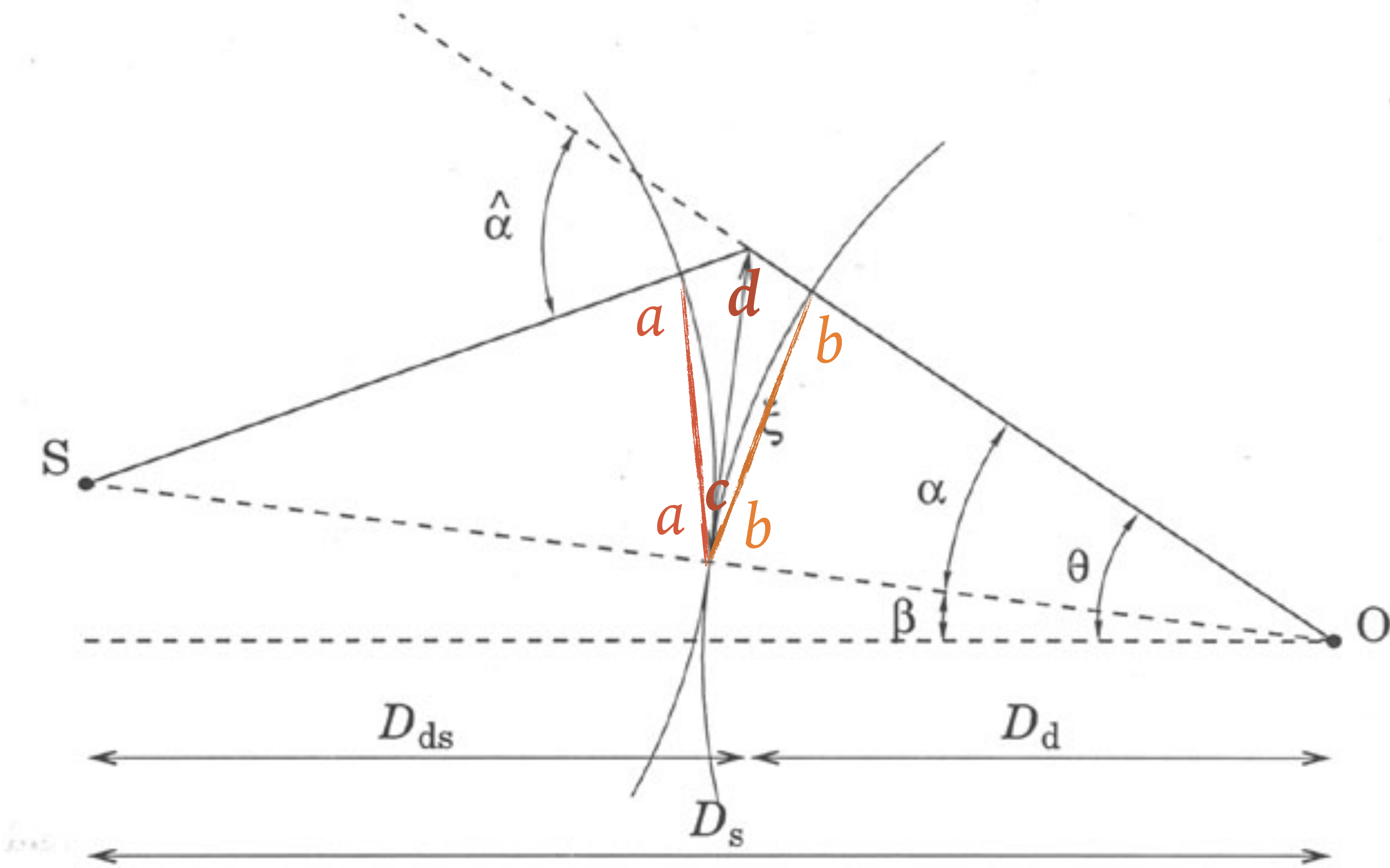


$$c + d + \pi - a + \pi - b = 2\pi$$
$$\Rightarrow c + d = a + b$$

GEOMETRICAL TIME DELAY



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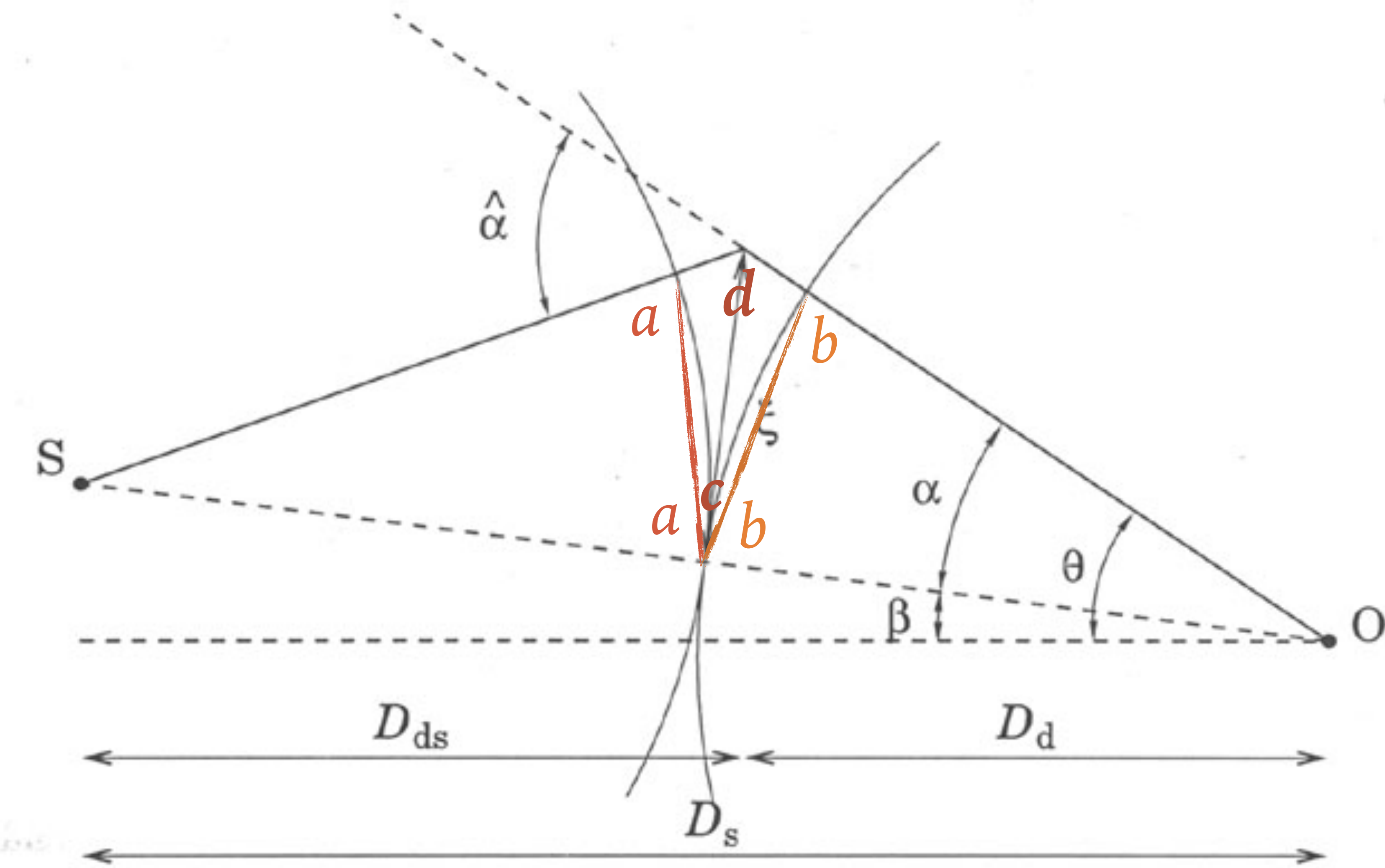

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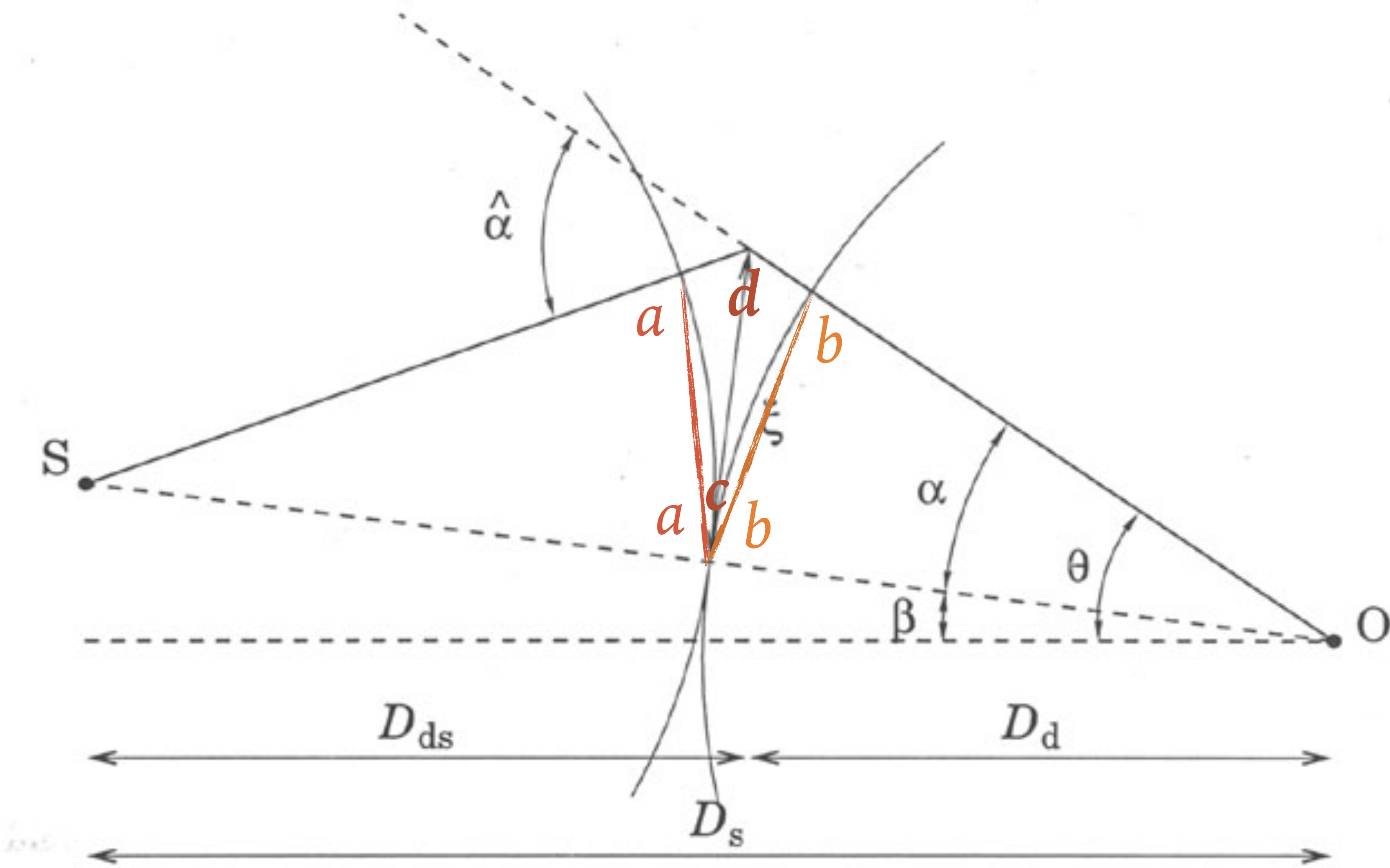

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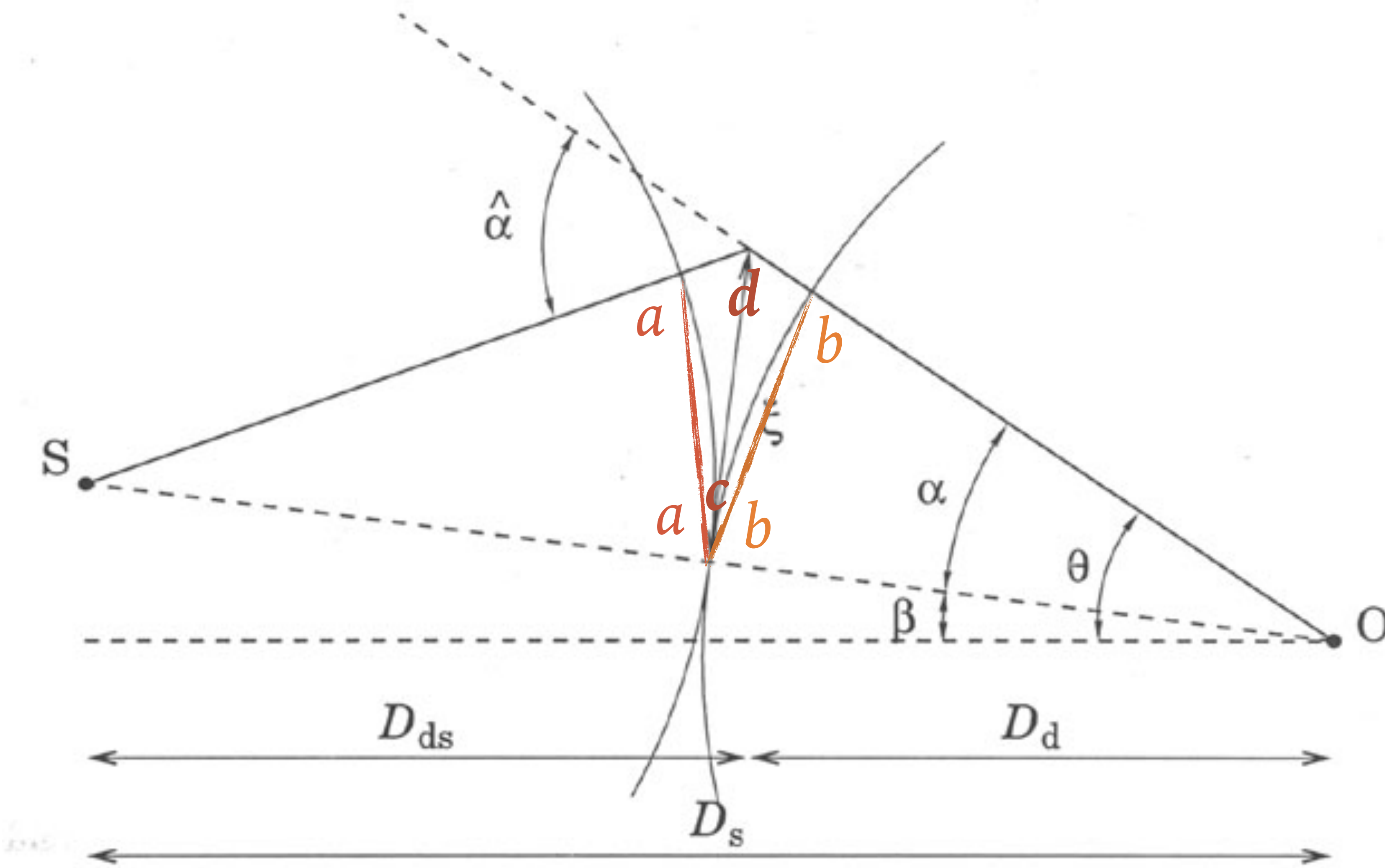

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$$\Rightarrow c = \frac{\hat{\alpha}}{2}$$

$$\Delta l = \xi \frac{\hat{\vec{\alpha}}}{2} = (\vec{\theta} - \vec{\beta}) \frac{D_{\mathsf{L}} D_{\mathsf{LS}}}{D_{\mathsf{S}}} \frac{\vec{\alpha}}{2} = (\vec{\theta} - \vec{\beta})^2 \frac{D_{\mathsf{L}} D_{\mathsf{LS}}}{2 D_{\mathsf{S}}}$$

TIME DELAY SURFACE

$$t(\vec{\theta}) = t_{geom} + t_{grav} \propto \left(\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \hat{\Psi}\right)$$

$$\vec{\nabla} t(\vec{\theta}) \propto \left(\vec{\theta} - \vec{\beta} - \vec{\nabla}\hat{\Psi}\right)$$

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Lens equation!

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Images form at the stationary points of t!

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Lens equation!

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$$T_{ij} = \frac{\partial^2 t(\vec{\theta})}{\partial \theta_i \partial \theta_j} \propto (\delta_{ij} - \Psi_{ij})$$

HESSIAN OF THE TIME DELAY SURFACE

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