GRAVITATIONAL LENSING

14 - BINARY LENSES

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COMPLEX LENS EQUATION (WITT, 1990)

- Thus:

$$z_s = z - \sum_{i=1}^{N} \frac{m_i}{z^* - z_i^*}$$

- Taking the conjugate:

$$z^* = z_s^* + \sum_{i=1}^{N} \frac{m_i}{z - z_i}$$

- We obtain z* and substitute it back into the original equation, which results in a (N^2+1)th order complex polynomial in the unknown z, $\mathbf{p^{N^2+1}(z)=0}$

- This equation can be solved only numerically, even in the case of a binary lens

COMPLEX LENS EQUATION (WITT, 1990)

- Note that the solutions are not necessarily solutions of the lens equations (spurious solutions)
- One has to check if the solutions are solutions of the lens equation
- Rhie (2001,2003): maximum number of images is $5(N-1)$ for $N>2$

JACOBIAN DETERMINANT

The Jacobian determinant is (on the real plane):

$$\det A = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \left(\frac{\partial y_1}{\partial x_2}\right)^2$$

How do we write it in complex notation?

JACOBIAN DETERMINANT

The complex derivatives (Wirtinger derivatives) of z_s are:

$$\begin{aligned}
\frac{\partial z_s}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)(y_1 + iy_2) = \frac{1}{2}\left(\frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}\right) \\
\frac{\partial z_s}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)(y_1 + iy_2) = \frac{1}{2}\left(\frac{\partial y_1}{\partial x_1} - \frac{\partial y_2}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\right)
\end{aligned}$$

JACOBIAN DETERMINANT

Note that in lensing these two derivatives are equal!

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\end{aligned}$$

Thus:

$$\left(\frac{\partial z_s}{\partial z}\right)^2 = \frac{1}{4}\left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 + 2\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\right]$$

$$\left(\frac{\partial z_s}{\partial z^*}\right)\left(\frac{\partial z_s}{\partial z^*}\right)^* = \frac{1}{4}\left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 - 2\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\right] + \left(\frac{\partial y_1}{\partial x_2}\right)^2$$

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$$

By taking the difference of these two equations:

$$
\left(\frac{\partial z_s}{\partial z}\right)^2 - \left(\frac{\partial z_s}{\partial z^*}\right)\left(\frac{\partial z_s}{\partial z^*}\right)^* = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \left(\frac{\partial y_1}{\partial x_2}\right)^2 = \det A
$$

JACOBIAN DETERMINANT (OR INVERSE MAGNIFICATION)

Now, we can use the lens equation:

$$z_s = z - \sum_{i=1}^{N} \frac{m_i}{z^* - z_i^*}$$

To obtain:

$$\frac{\partial z_s}{\partial z} = 1 \qquad \frac{\partial z_s}{\partial z^*} = \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2}$$

so that

$$\det A = 1 - \left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2$$

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so that

$$\left(\frac{\partial z_s}{\partial z}\right)^2 - \left(\frac{\partial z_s}{\partial z^*}\right)\left(\frac{\partial z_s}{\partial z^*}\right)^* = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \left(\frac{\partial y_1}{\partial x_2}\right)^2 = \det A$$

$$\det A = 1 - \left|\sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2}\right|^2$$

CRITICAL LINES

From this equation:

$$\det A = 1 - \left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2$$

We see that on the critical lines (detA=0)

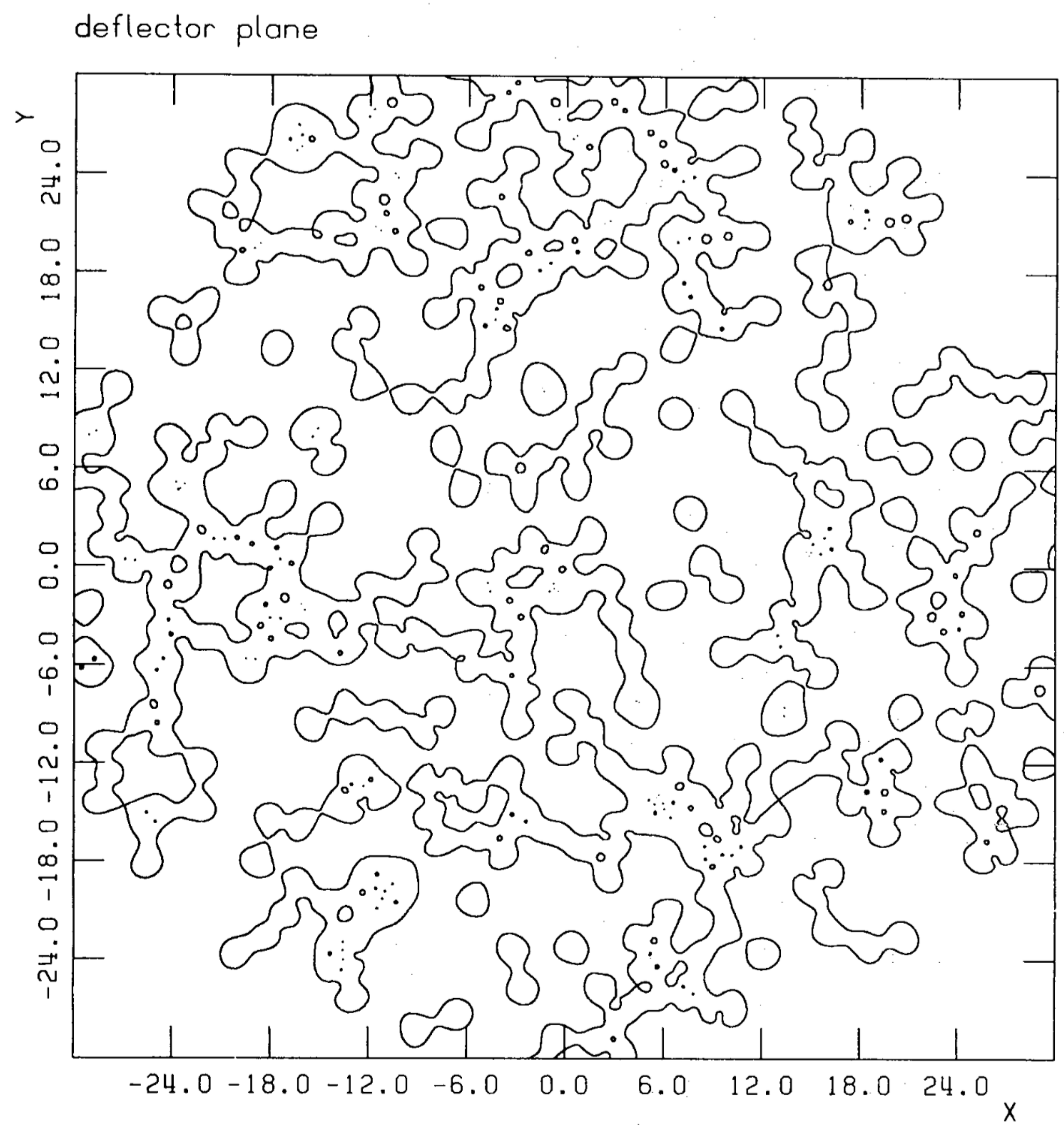
$$\left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2 = 1$$

This sum has to be satisfied on the unit circle:

$$\sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} = e^{i\phi} \qquad \phi \in [0, 2\pi)$$

Getting rid of the fraction, this equation can be turned into a polynomial of degree 2N: for each phase, there are <=2N critical points. Solving for all phases, we find up to 2N critical lines.

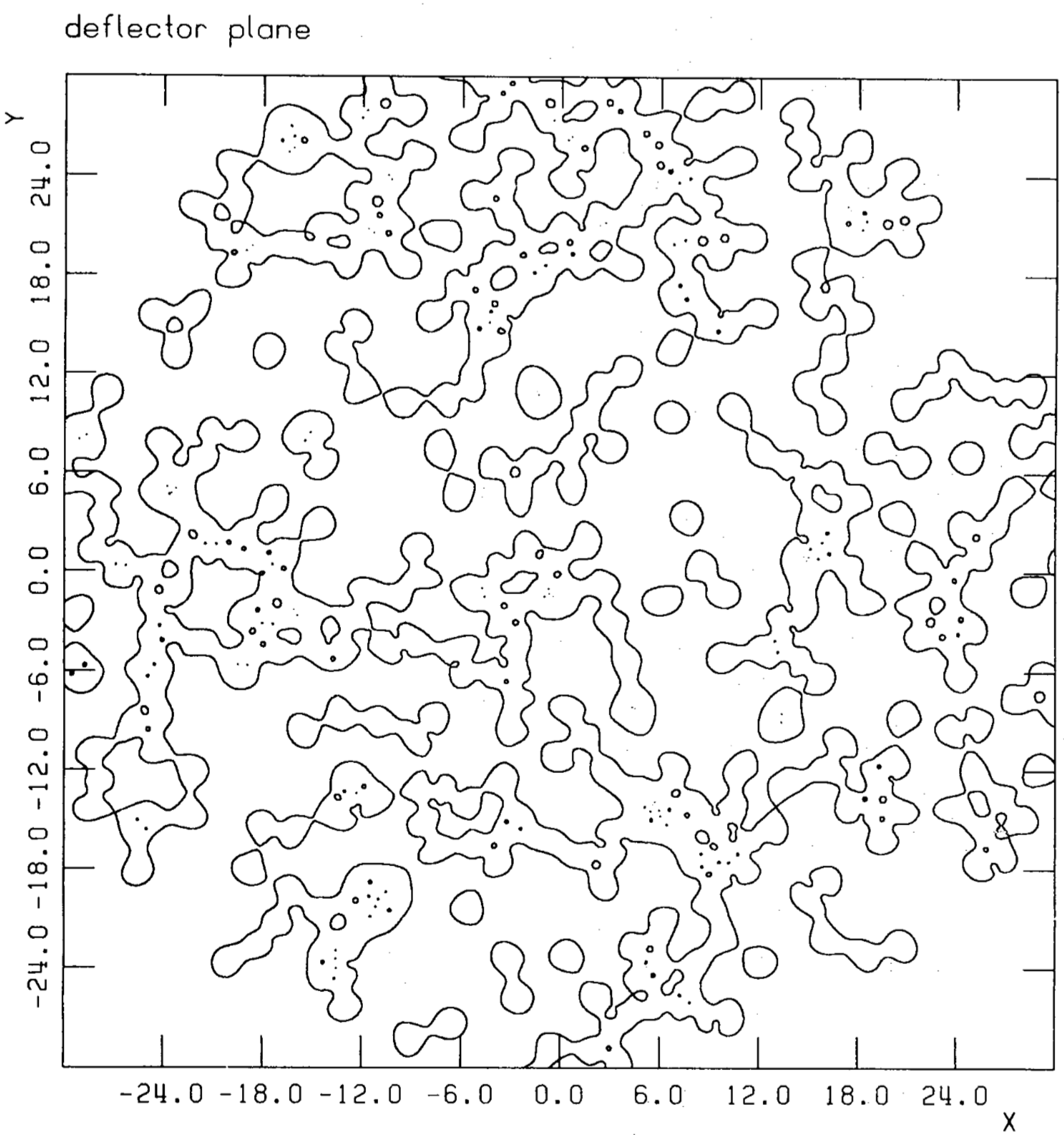
CRITICAL LINES

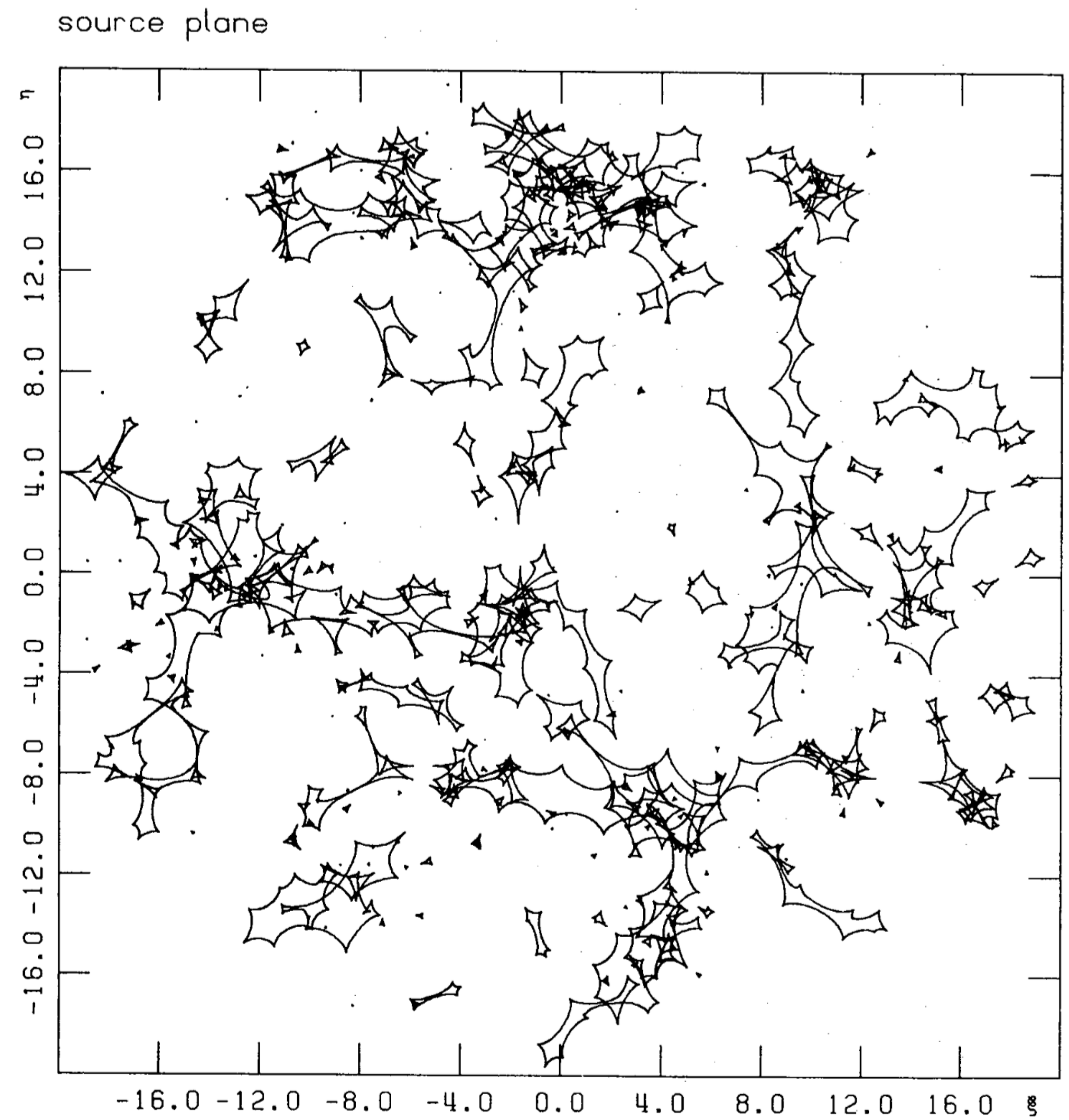


critical lines originated by 400 stars

Witt, 1990, A&A, 236, 311

CRITICAL LINES AND CAUSTICS





critical lines and caustics originated by 400 stars

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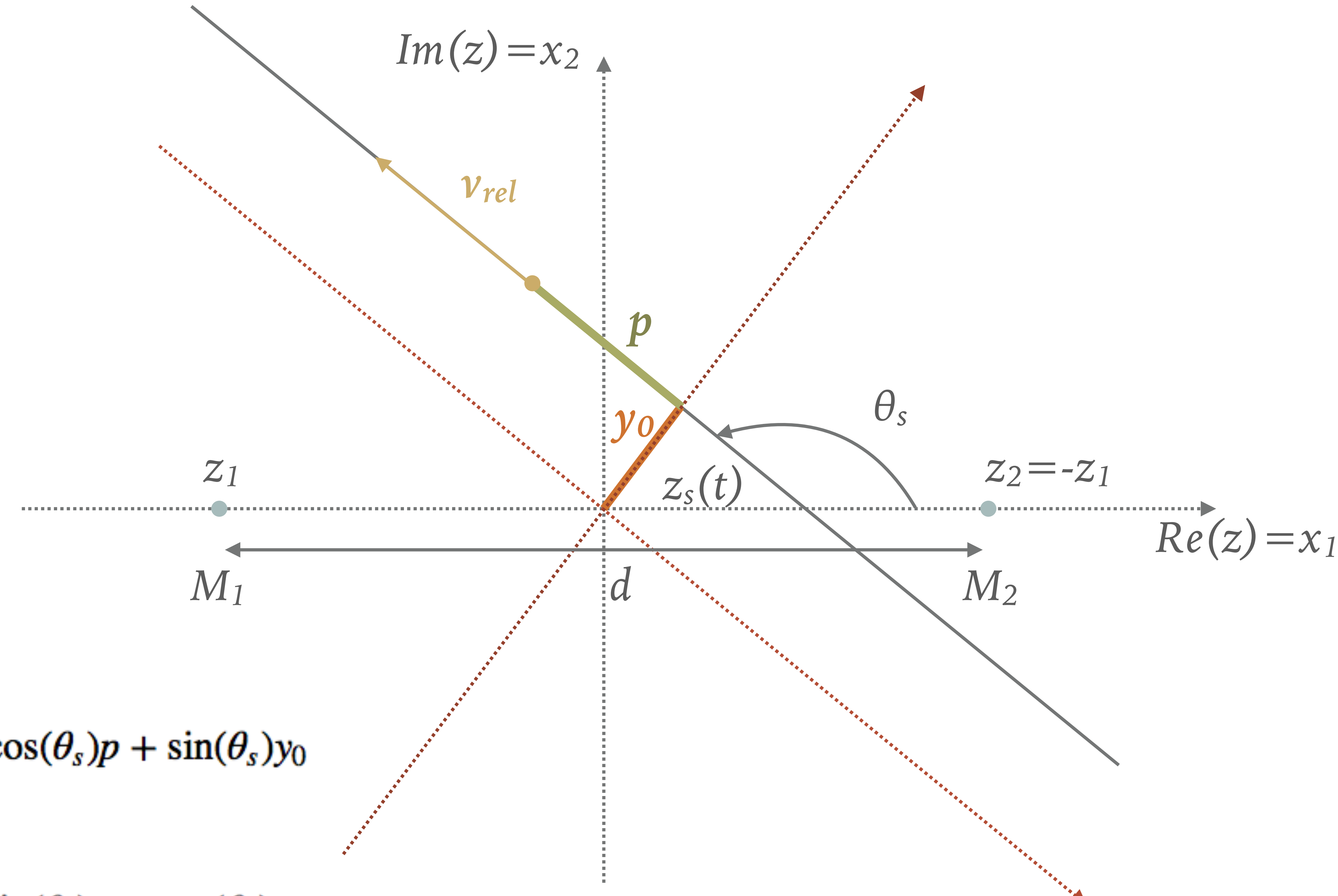
BINARY LENSES



$$\Re(z_s) = \cos(\theta_s)p + \sin(\theta_s)y_0$$

$$\Im(z_s) = \sin(\theta_s)p - \cos(\theta_s)y_0$$

BINARY LENSES

- Lens equation:

$$z_s = z - \frac{m_1}{z^* - z_1^*} - \frac{m_2}{z^* - z_2^*}$$

- determinant of the Jacobian:

$$\det A = 1 - \left|\frac{\partial z_s}{\partial z^*}\right|^2$$

$$\frac{\partial z_s}{\partial z^*} = \frac{m_1}{(z^* - z_1^*)^2} + \frac{m_2}{(z^* - z_2^*)^2}$$

- condition for critical points:

$$\frac{\partial z_s}{\partial z^*} = e^{i\phi}$$

- resulting fourth grade polynomial (z_2=-z_1):

$$z^4 - z^2(2z_1^{*2} + e^{i\phi}) - zz_1^* 2(m_1 - m_2)e^{i\phi} + z_1^{*2}(z_1^{*2} - e^{i\phi}) = 0$$