GRAVITATIONAL LENSING

LECTURE 15

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AA 2016-2017

MULTIPLE POINT MASSES

- In the case of multiple point masses, we can use the superposition principle to compute the total deflection angle: total deflection angle = sum of individual deflections
- compared to an individual point mass, the spatial symmetry is broken
- The mass scale of the system is the total mass=sum of the individual masses
- We may use this mass to define an equivalent Einstein radius and use it to scale all angles

COMPLEX LENS EQUATION (WITT, 1990)

- For a system of N-lenses we obtained:

$$z_s = z - \sum_{i=1}^{N} \frac{m_i}{z^* - z_i^*}$$

- Taking the conjugate:

$$z_s^* = z^* - \sum_{i=1}^{N} \frac{m_i}{z - z_i}$$

- We obtain z* and substitute it back into the original equation, which results in a (N^2+1)th order complex polynomial equation

$$p(z) = \sum_{i=0}^{N} c_i z^i$$

- This equation can be solved only numerically, even in the case of a binary lens

COMPLEX LENS EQUATION (WITT, 1990)

- Note that the solutions are not necessarily solutions of the lens equations (spurious solutions)
- One has to check if the solutions are solutions of the lens equation
- Rhie (2001,2003): maximum number of images is 5(N-1) for N>2

JACOBIAN DETERMINANT

The Jacobian determinant is (on the real plane):

$$\det A = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \left(\frac{\partial y_1}{\partial x_2}\right)^2$$

How do we write it in complex notation?

JACOBIAN DETERMINANT

The complex derivatives of zs are:

$$\begin{aligned}
\frac{\partial z_s}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)(y_1 + iy_2) = \frac{1}{2}\left(\frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}\right) \\
\frac{\partial z_s}{\partial z^*} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)(y_1 + iy_2) = \frac{1}{2}\left(\frac{\partial y_1}{\partial x_1} - \frac{\partial y_2}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\right)
\end{aligned}$$

JACOBIAN DETERMINANT

Note that in lensing these two derivatives are equal!

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\end{aligned}
$$

Thus:

$$
\left(\frac{\partial z_s}{\partial z}\right)^2 = \frac{1}{4}\left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 + 2\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\right]
$$

$$
\left(\frac{\partial z_s}{\partial z^*}\right)\left(\frac{\partial z_s}{\partial z^*}\right)^* = \frac{1}{4}\left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 - 2\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\right] + \left(\frac{\partial y_1}{\partial x_2}\right)^2
$$

JACOBIAN DETERMINANT

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$$

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$$

By taking the difference of these two equations:

$$
\left(\frac{\partial z_s}{\partial z}\right)^2 - \left(\frac{\partial z_s}{\partial z^*}\right)\left(\frac{\partial z_s}{\partial z^*}\right)^* = \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \left(\frac{\partial y_1}{\partial x_2}\right)^2 = \det A
$$

JACOBIAN DETERMINANT

Now, we can use the lens equation:

$$z_s = z - \sum_{i=1}^{N} \frac{m_i}{z^* - z_i^*}$$

To obtain:

$$\frac{\partial z_s}{\partial z} = 1 \qquad \frac{\partial z_s}{\partial z^*} = \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2}$$

so that

$$\det A = 1 - \left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2$$

CRITICAL LINES

From this equation:

$$\det A = 1 - \left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2$$

We see that on the critical lines (detA=0)

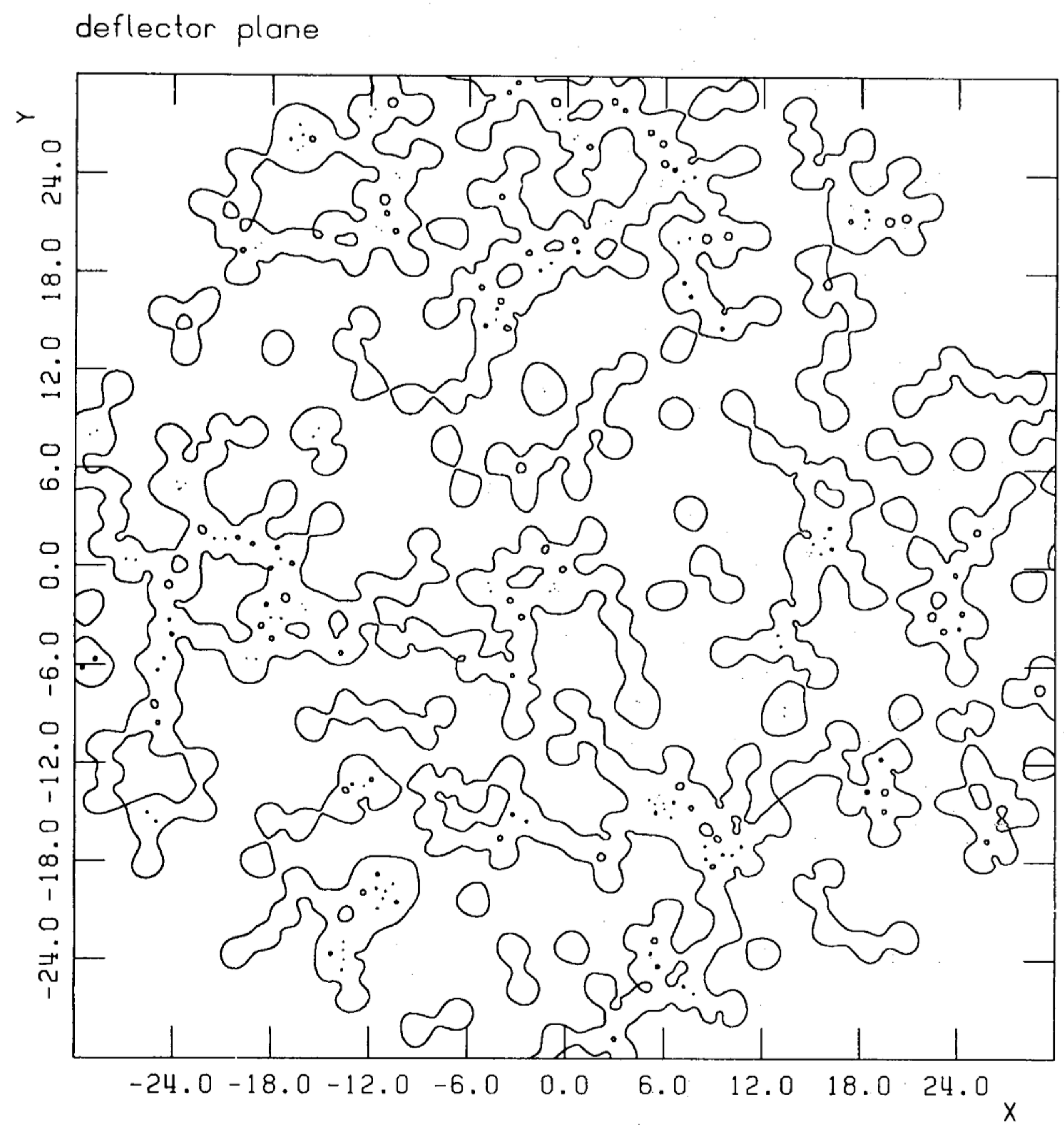
$$\left| \sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} \right|^2 = 1$$

This sum has to be satisfied on the unit circle:

$$\sum_{i=1}^{N} \frac{m_i}{(z^* - z_i^*)^2} = e^{i\phi} \qquad \phi \in [0, 2\pi)$$

Getting rid of the fraction, this equation can be turned into a polynomial of degree 2N: for each phase, there are <=2N critical points. Solving for all phases, we find up to 2N critical lines.

CRITICAL LINES



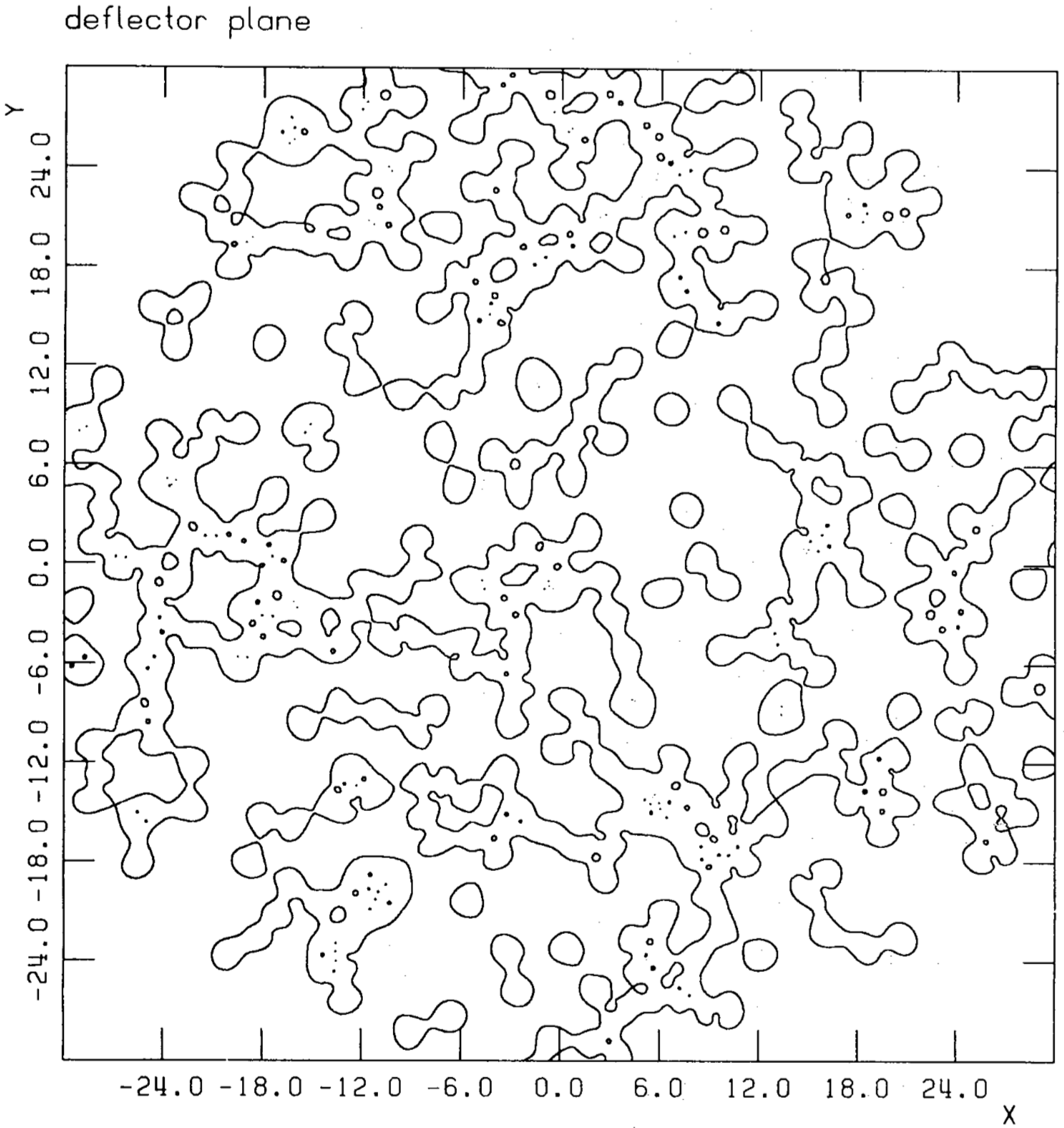
critical lines originated by 400 stars

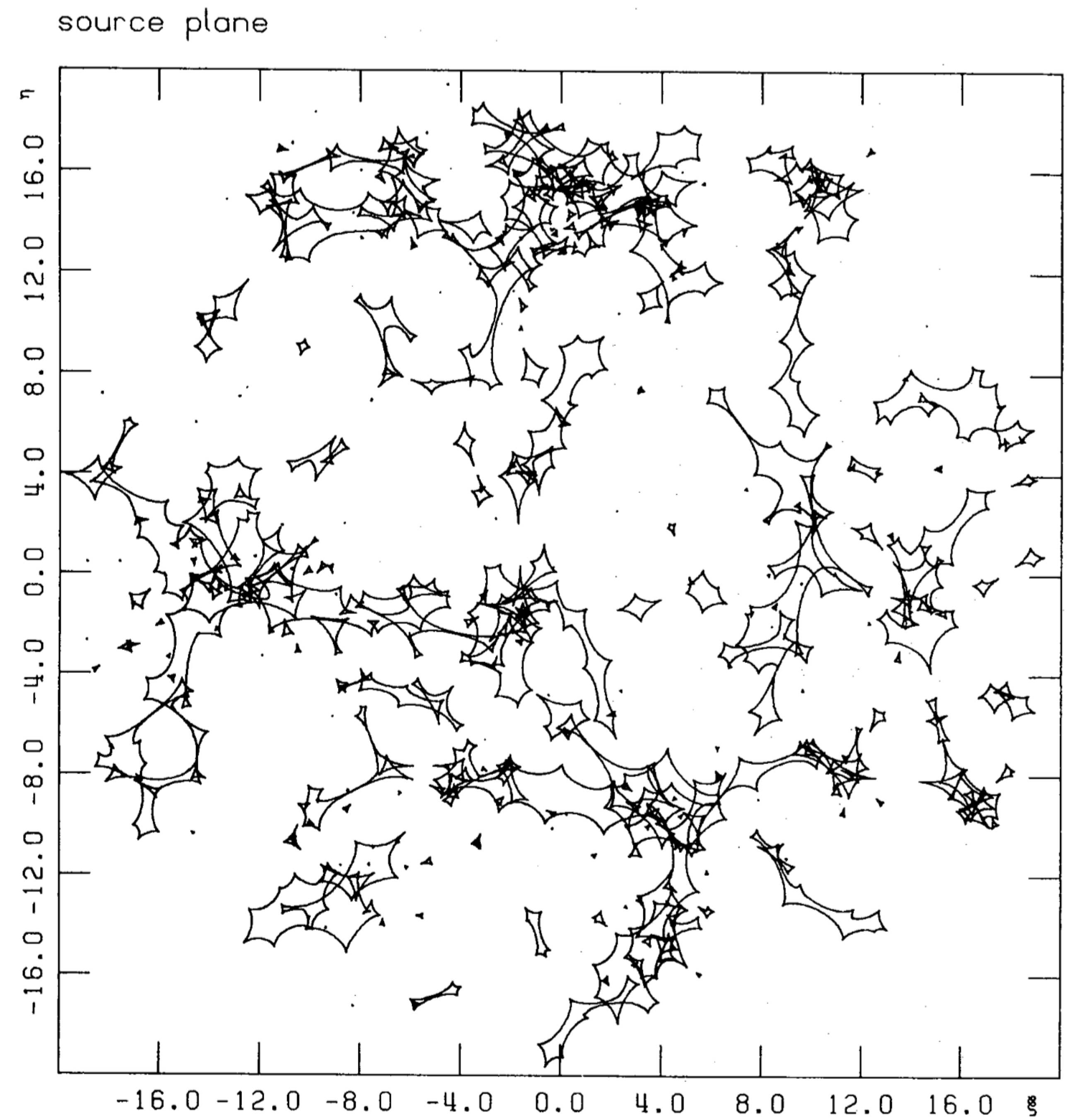
Witt, 1990, A&A, 236, 311

MAGNIFICATION

- obviously, the magnification is the inverse of the Jacobian determinant
- can you imagine what would happen to a source moving in the background of such a network of critical lines?

CRITICAL LINES AND CAUSTICS





critical lines and caustics originated by 400 stars

Witt, 1990, A&A, 236, 311

BINARY LENSES

- Lens equation:
$$z_s = z - \frac{m_1}{z^* - z_1^*} - \frac{m_2}{z^* - z_2^*}$$

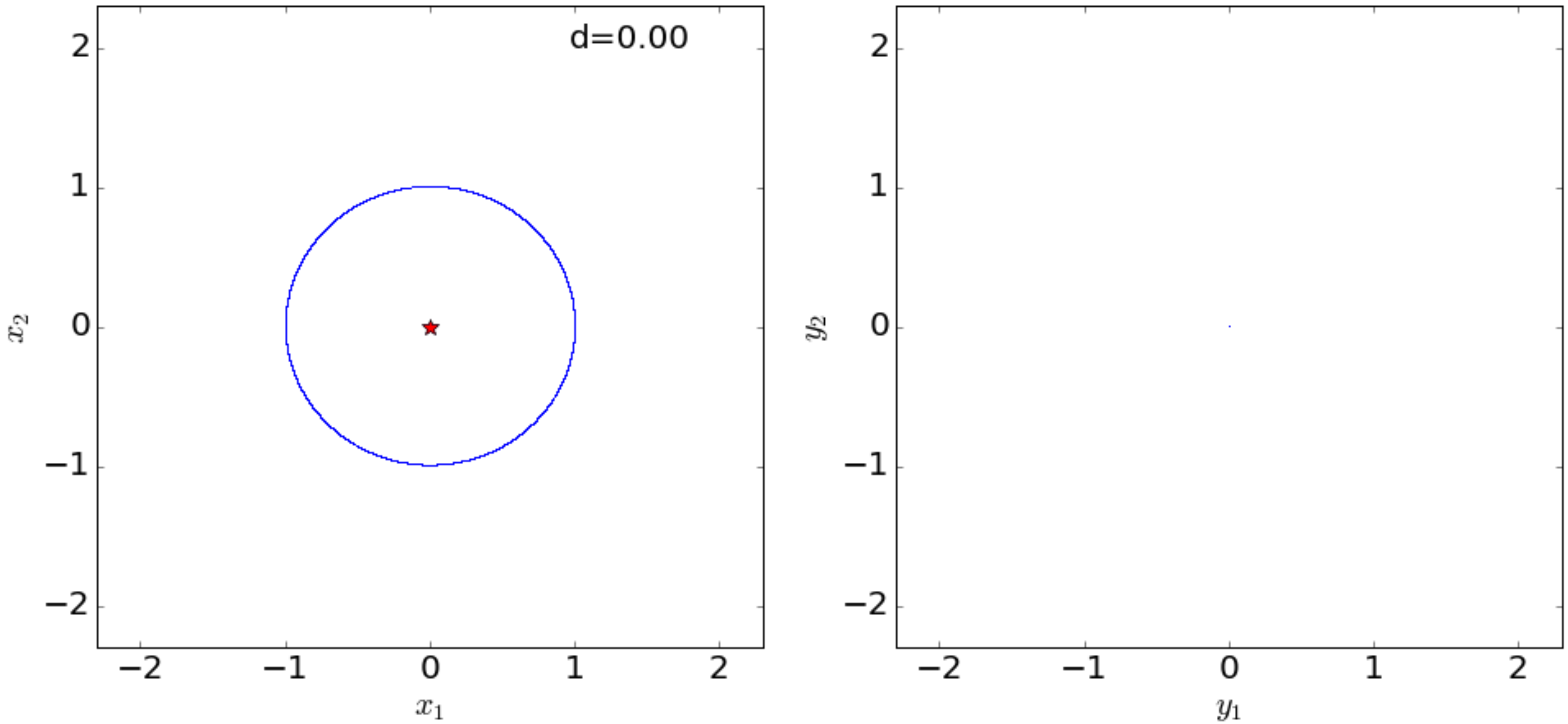
- determinant of the Jacobian:
$$\det A = 1 - \left|\frac{\partial z_s}{\partial z^*}\right|^2$$
$$\frac{\partial z_s}{\partial z^*} = \frac{m_1}{(z^* - z_1^*)^2} + \frac{m_2}{(z^* - z_2^*)^2}$$

- condition for critical points:
$$\frac{\partial z_s}{\partial z^*} = e^{i\phi}$$

- resulting fourth grade polynomial:

$$z^4 - z^2(2z_1^{*2} + e^{i\phi}) - zz_1^* 2(m_1 - m_2)e^{i\phi} + z_1^{*2}(z_1^{*2} - e^{i\phi}) = 0$$

BINARY LENSES:
TWO LENSES WITH THE SAME MASS (Q=1) AND VARYING DISTANCE



critical lines *caustics*

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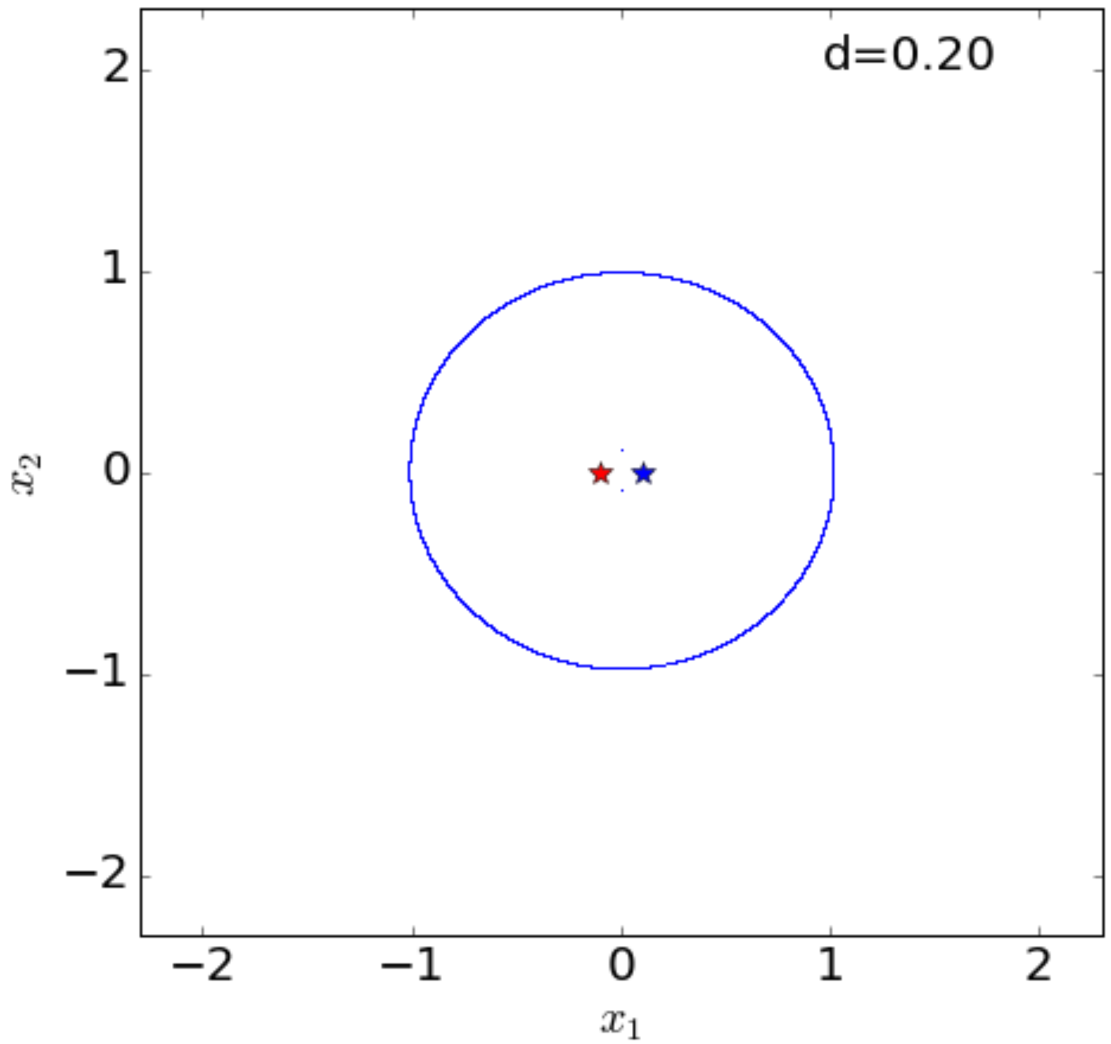
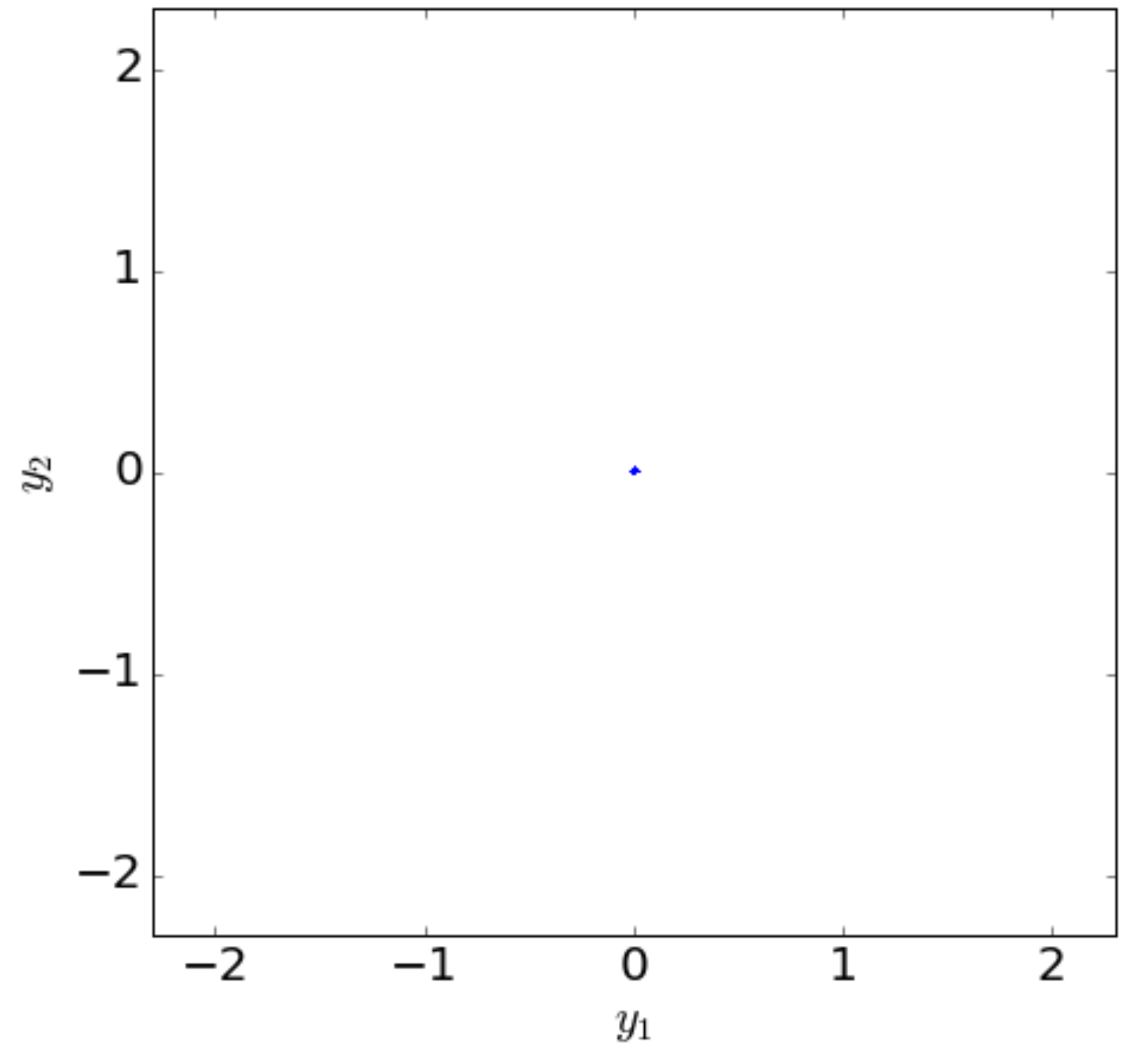


critical lines



caustics

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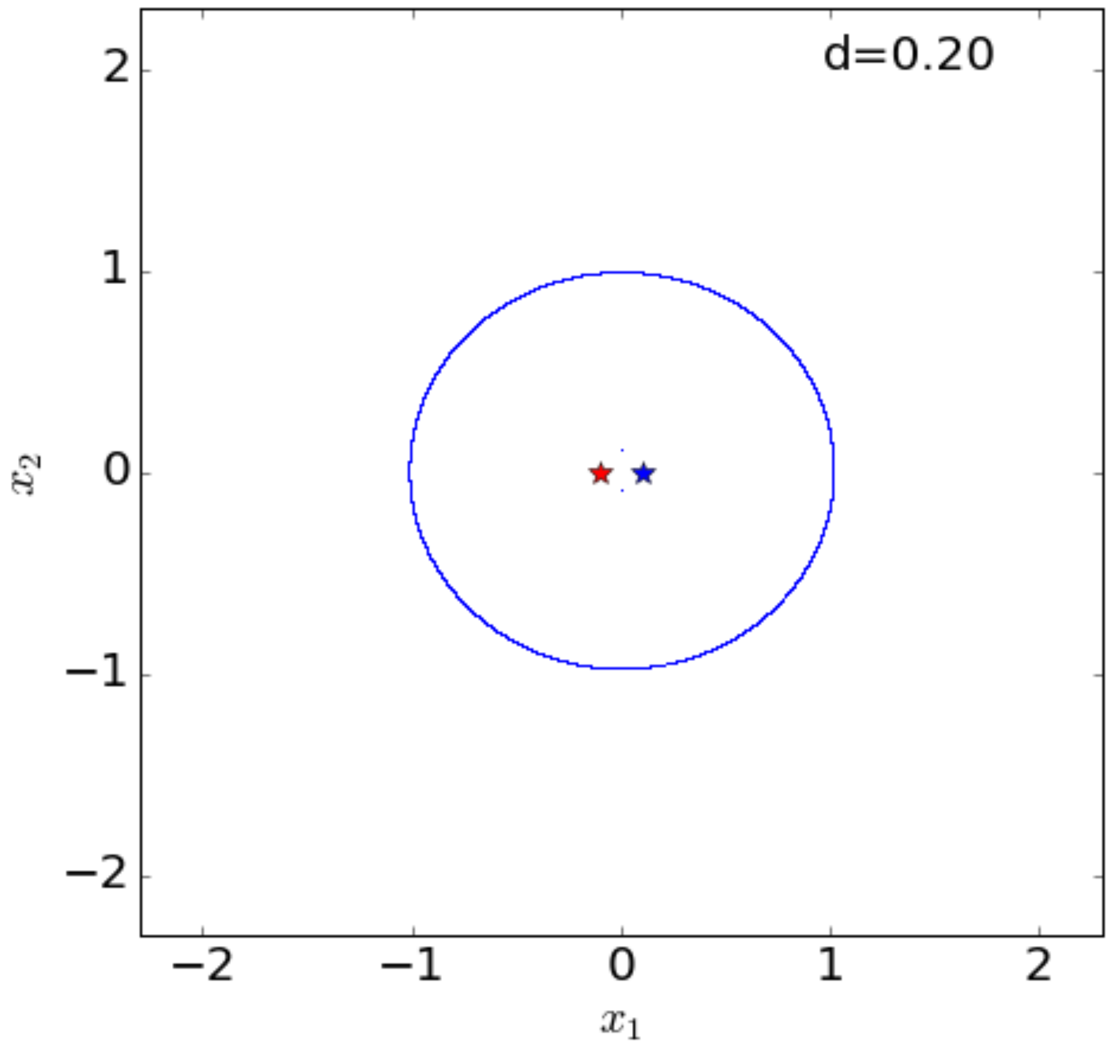

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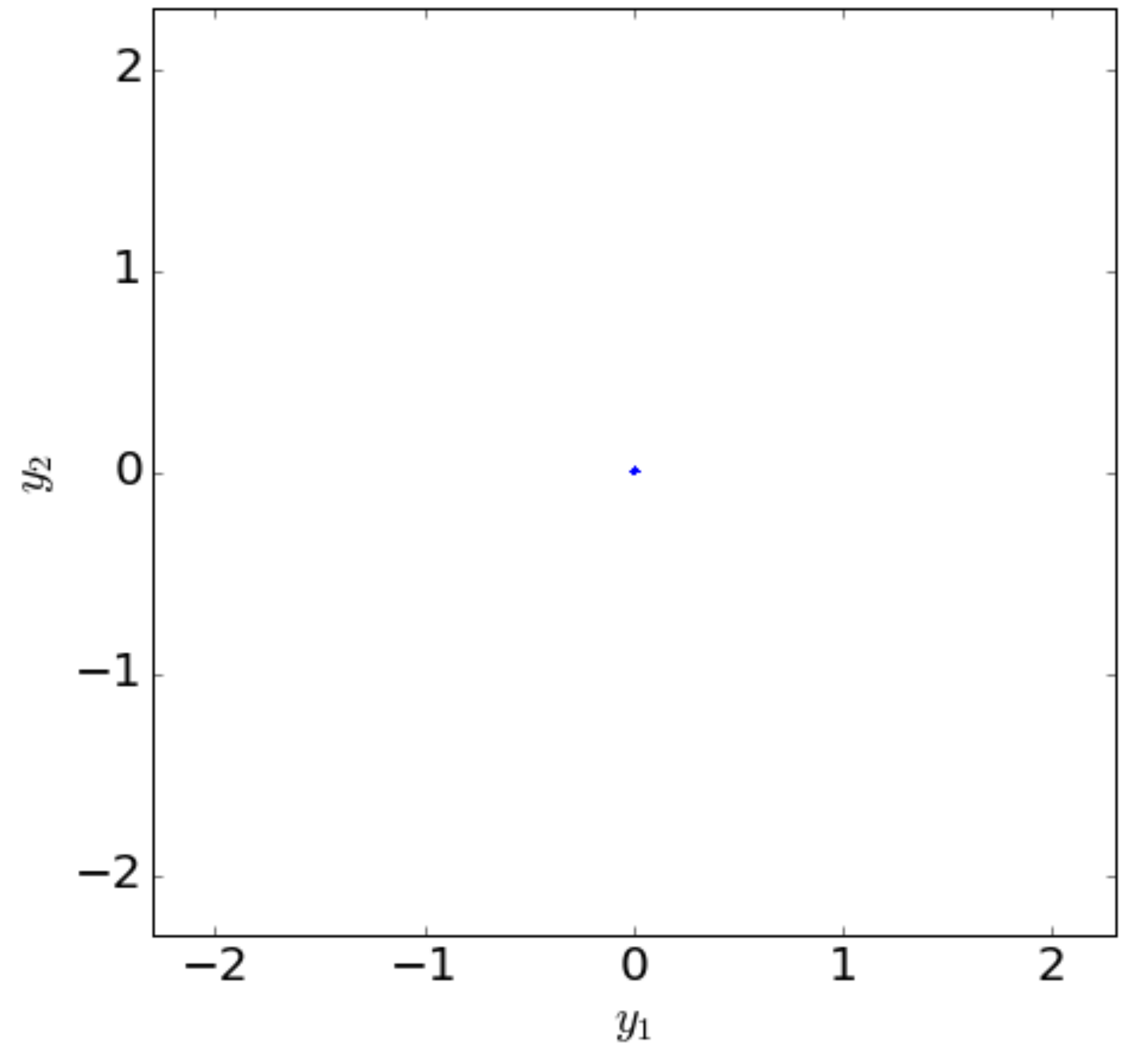

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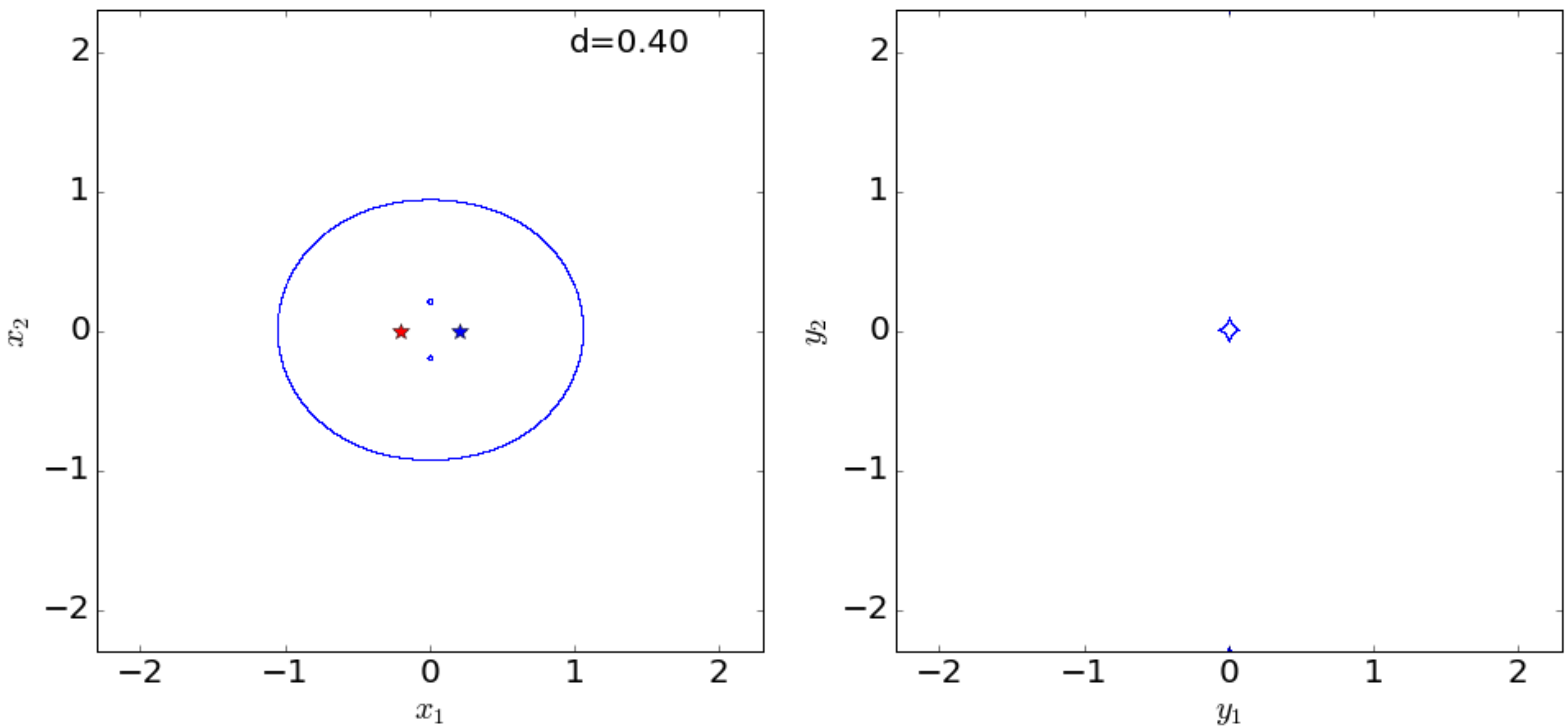

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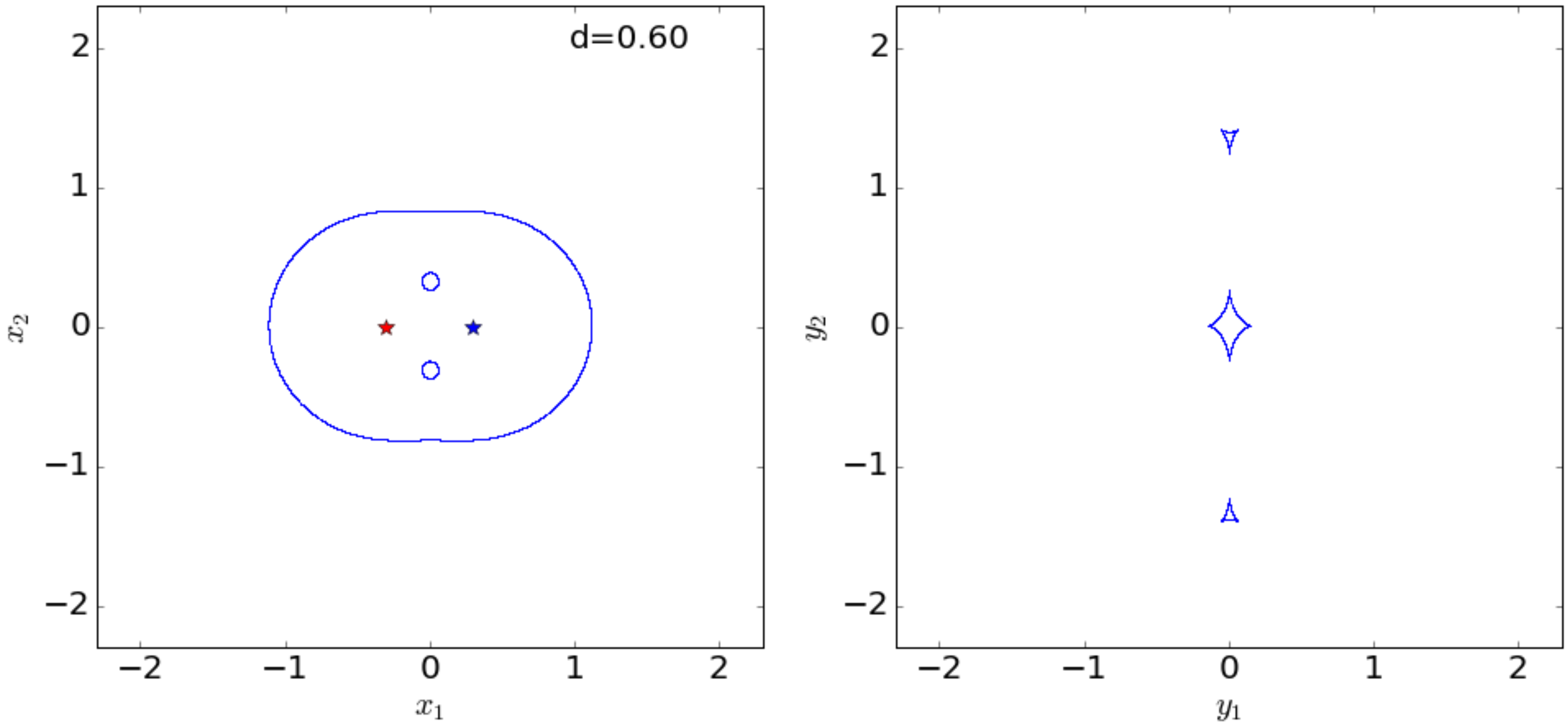

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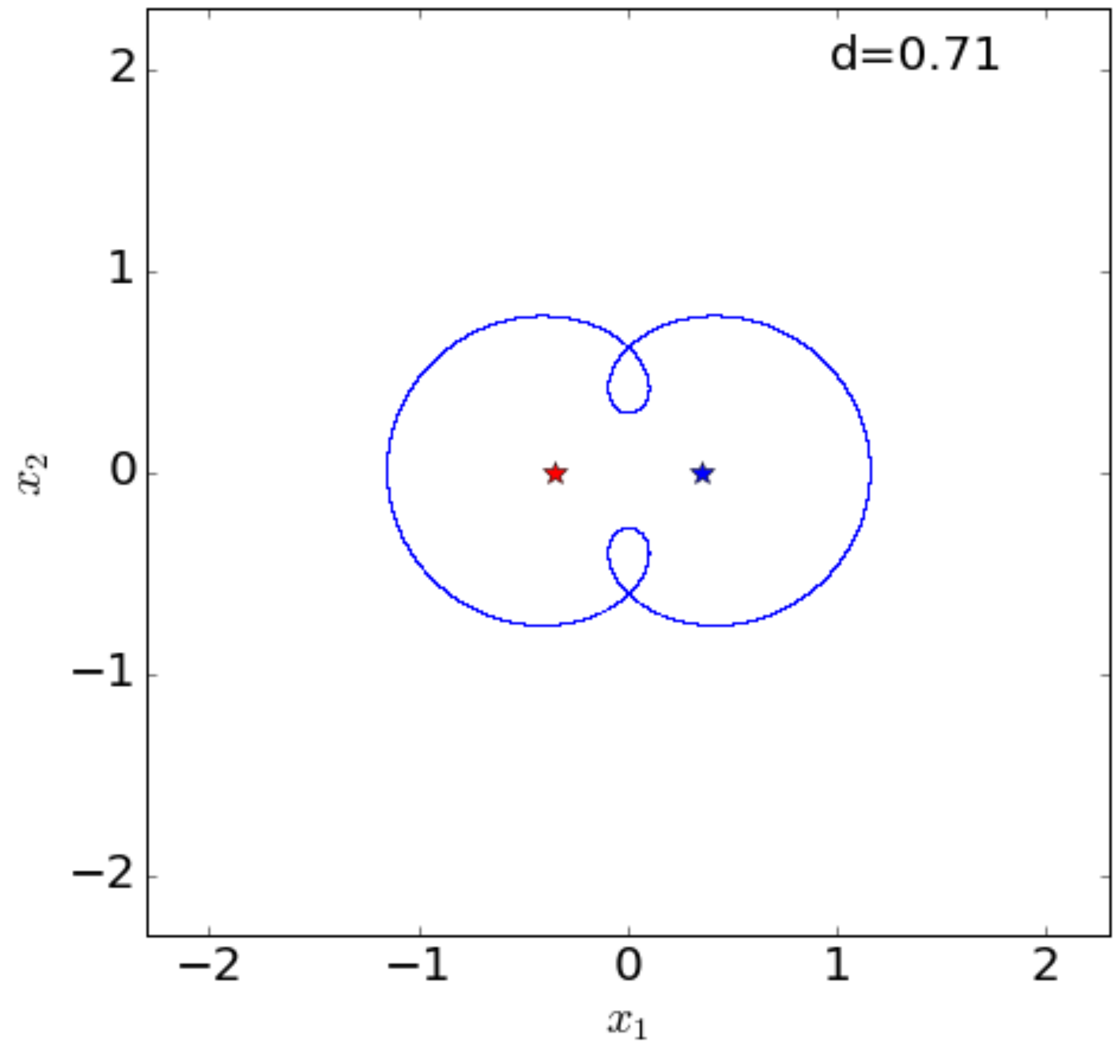

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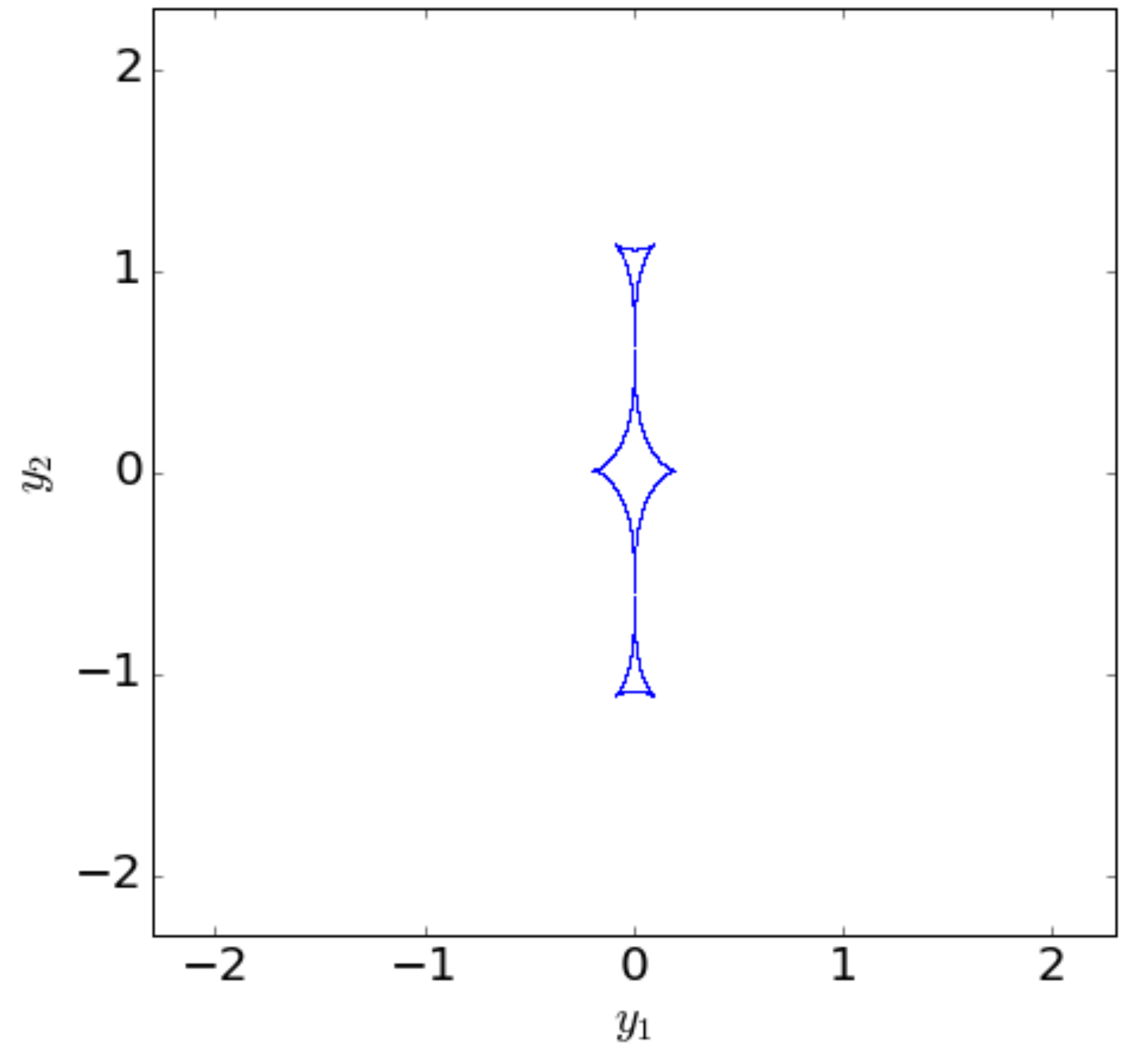

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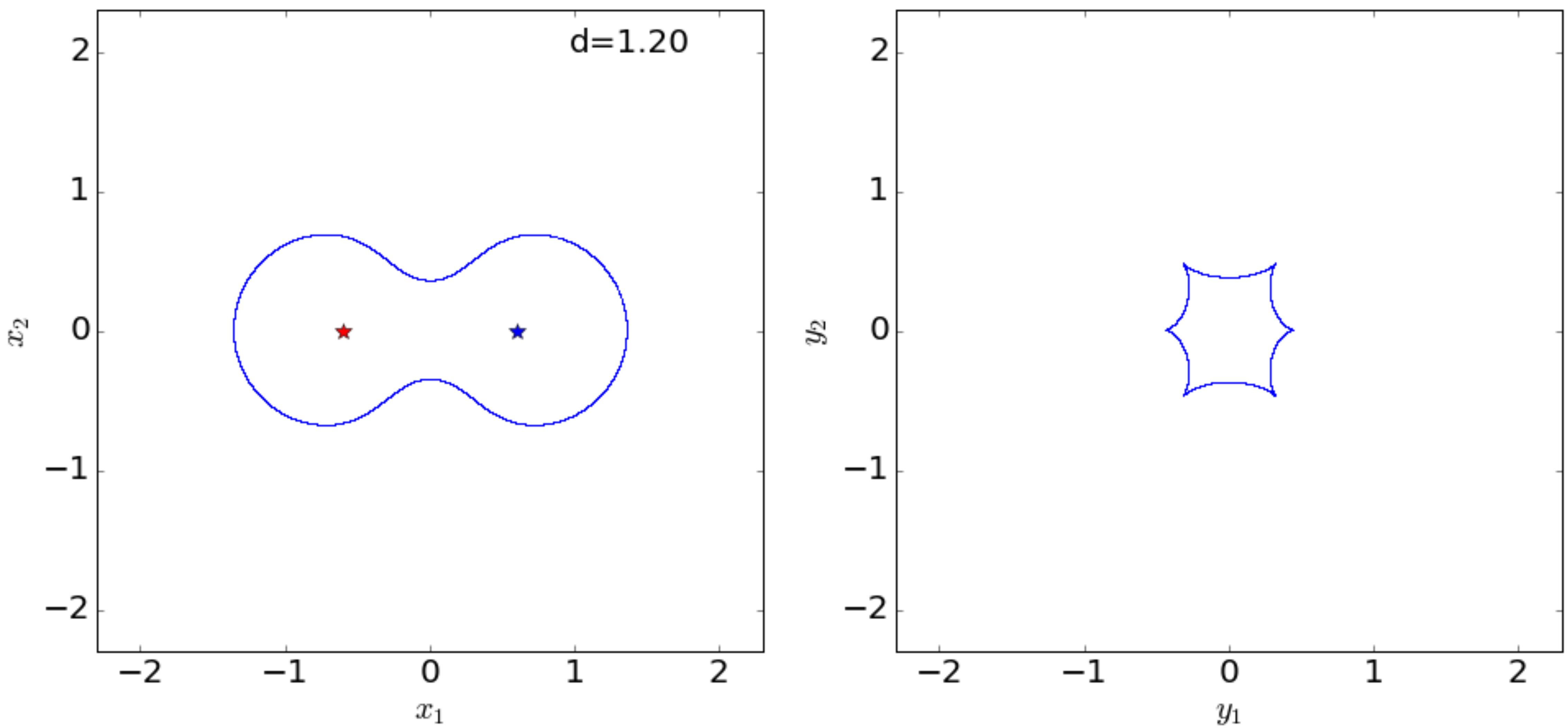

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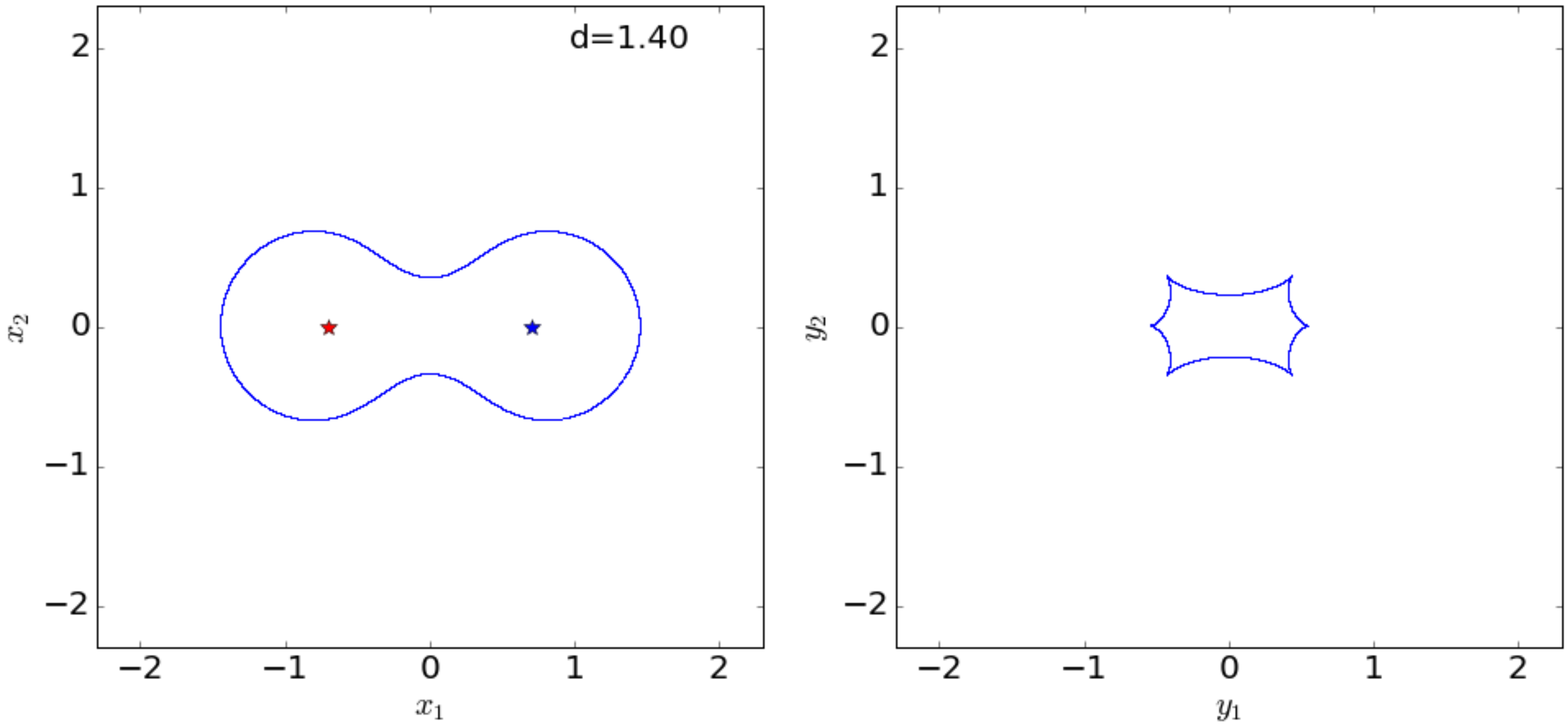

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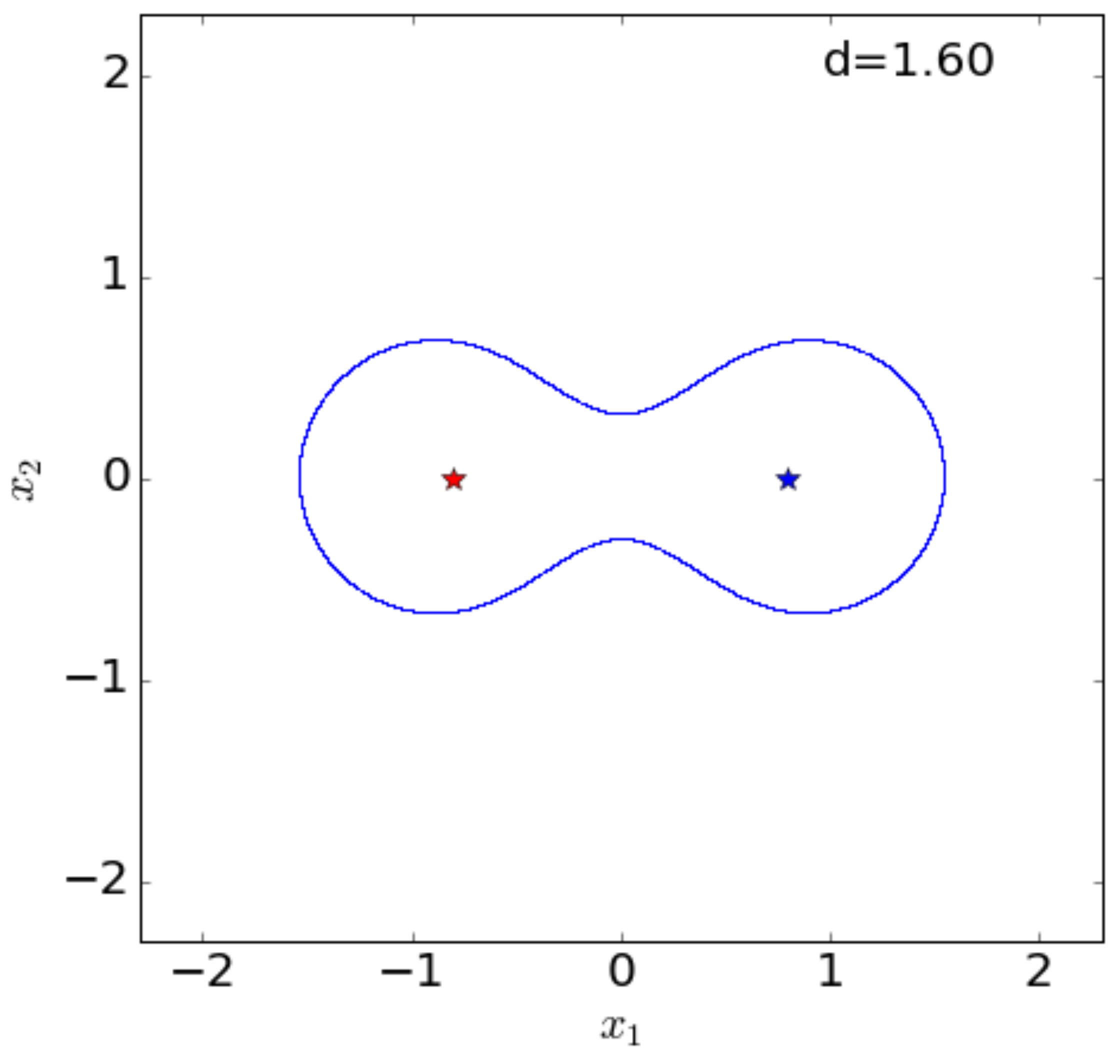

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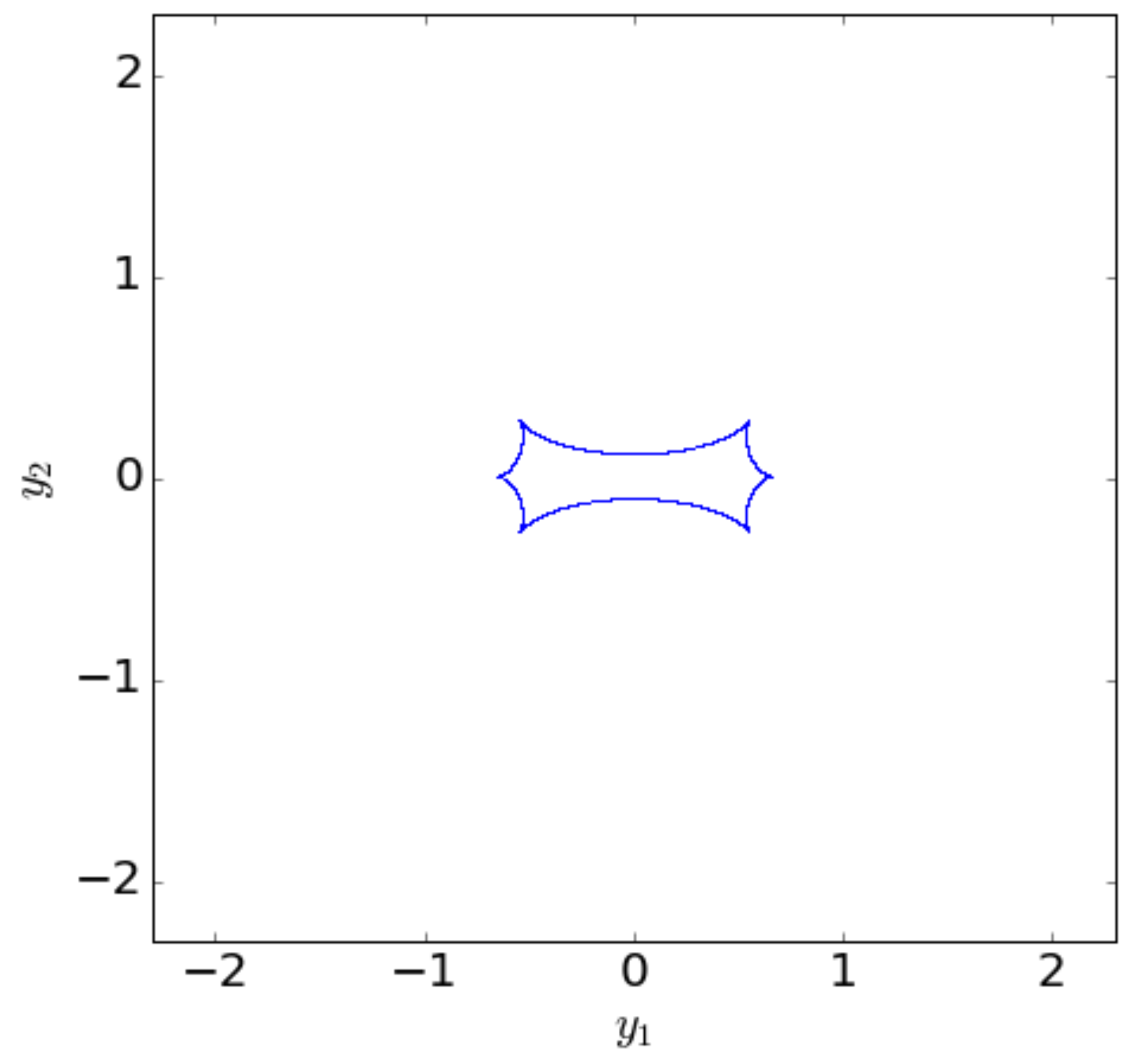

caustics

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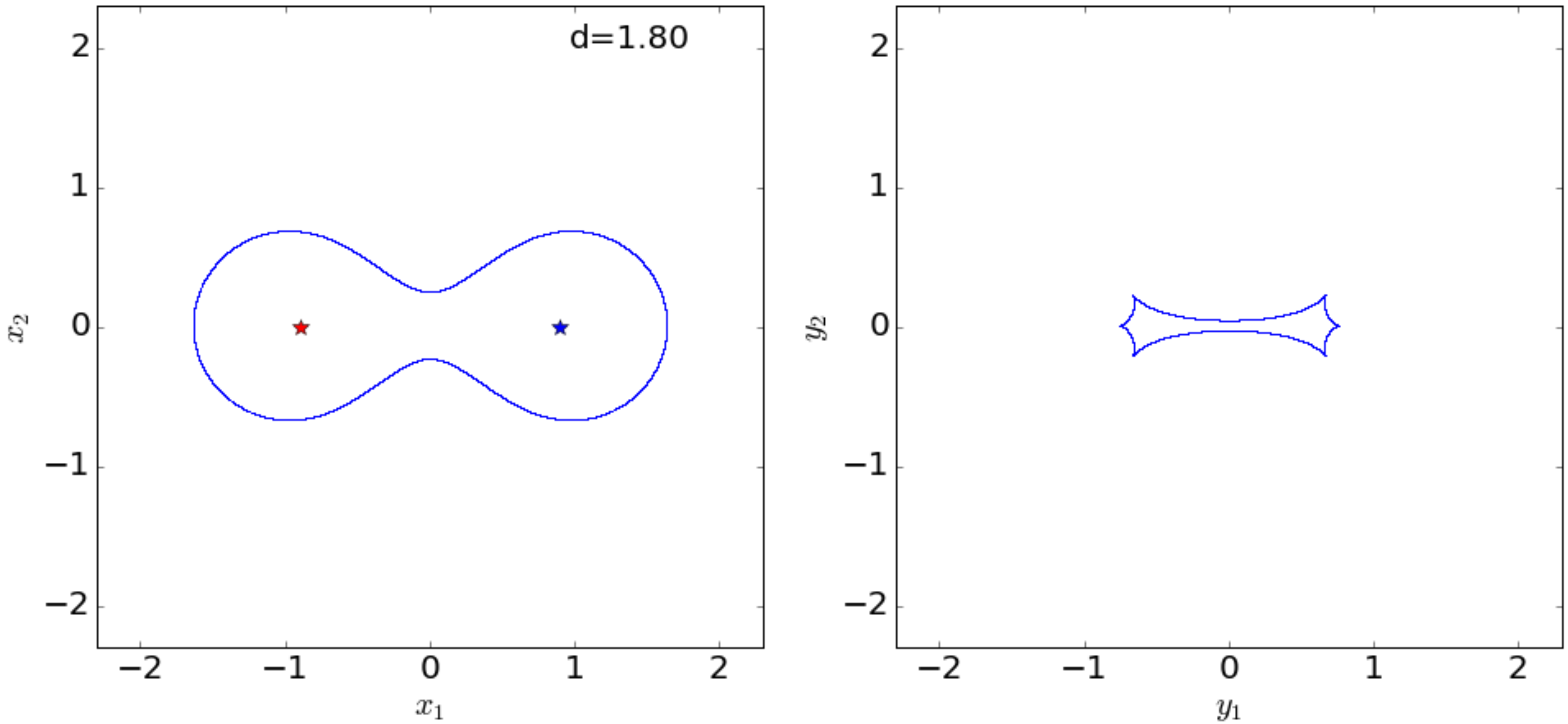

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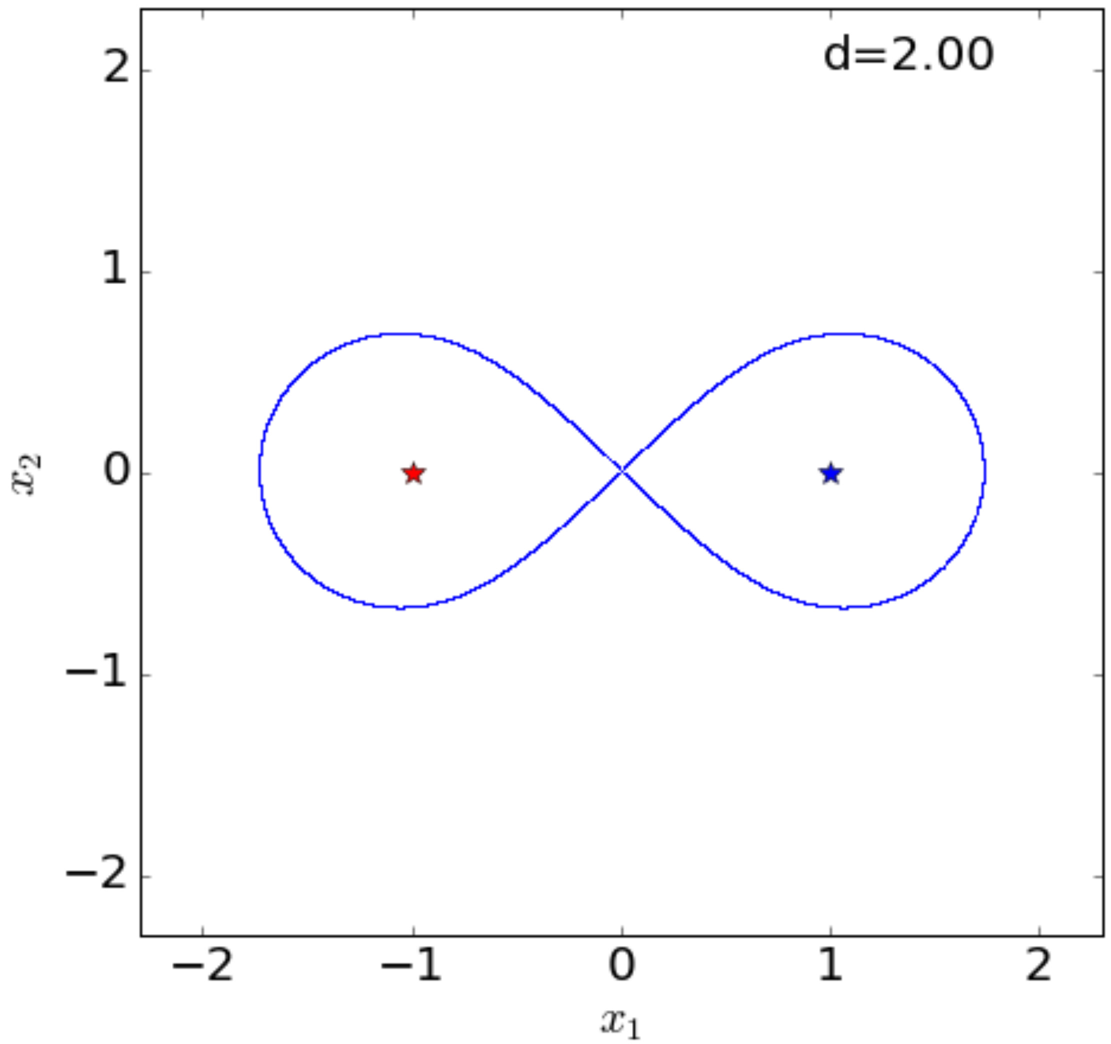

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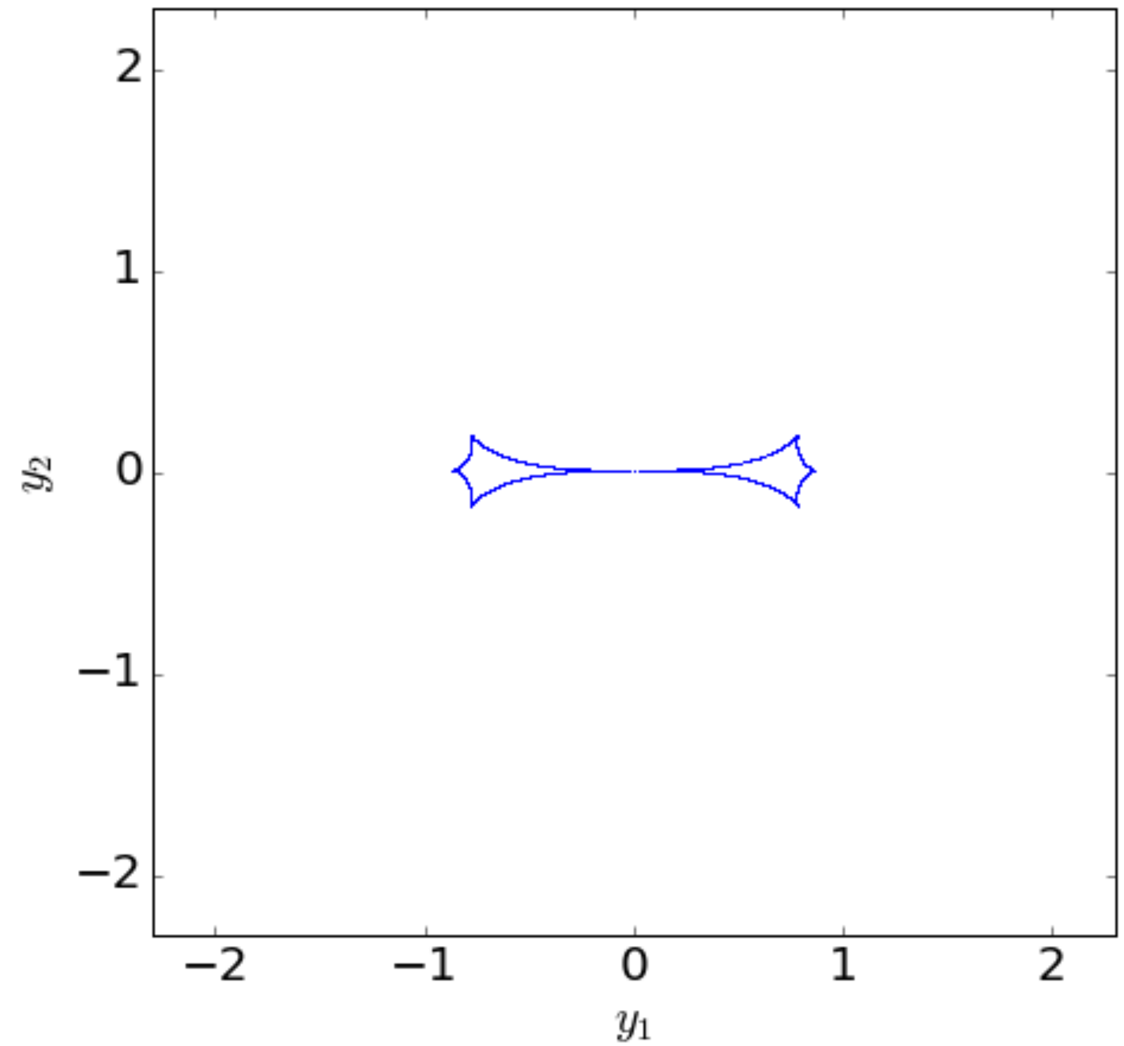

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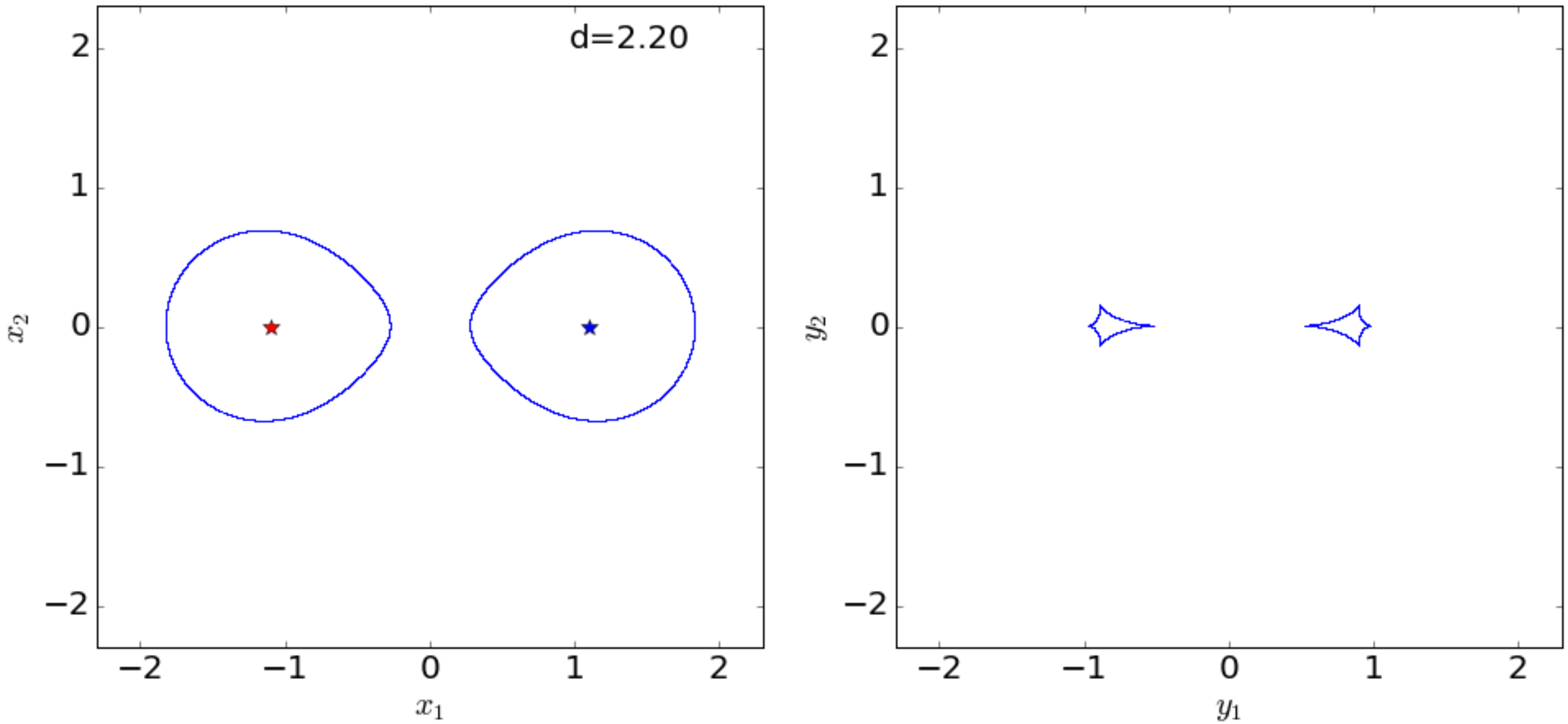

critical lines *caustics*

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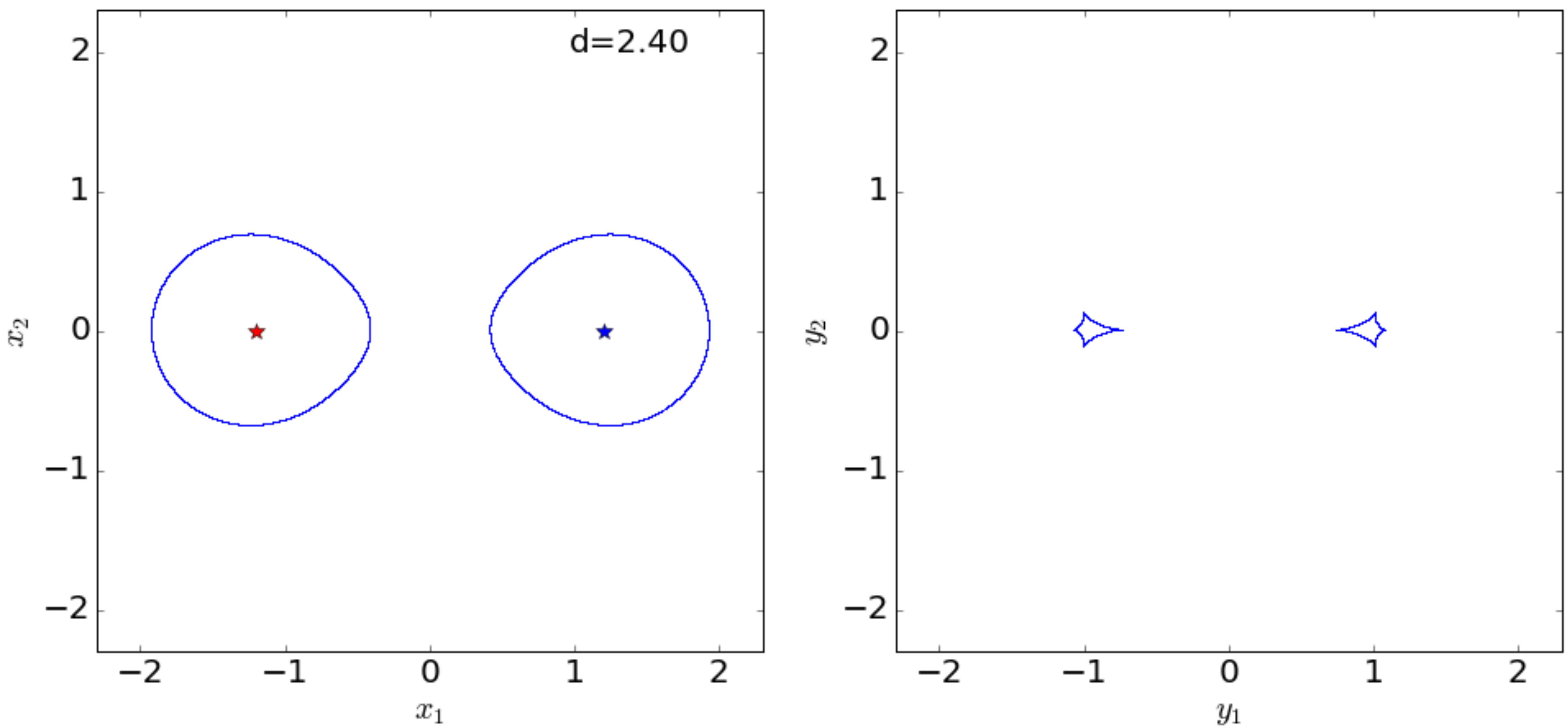
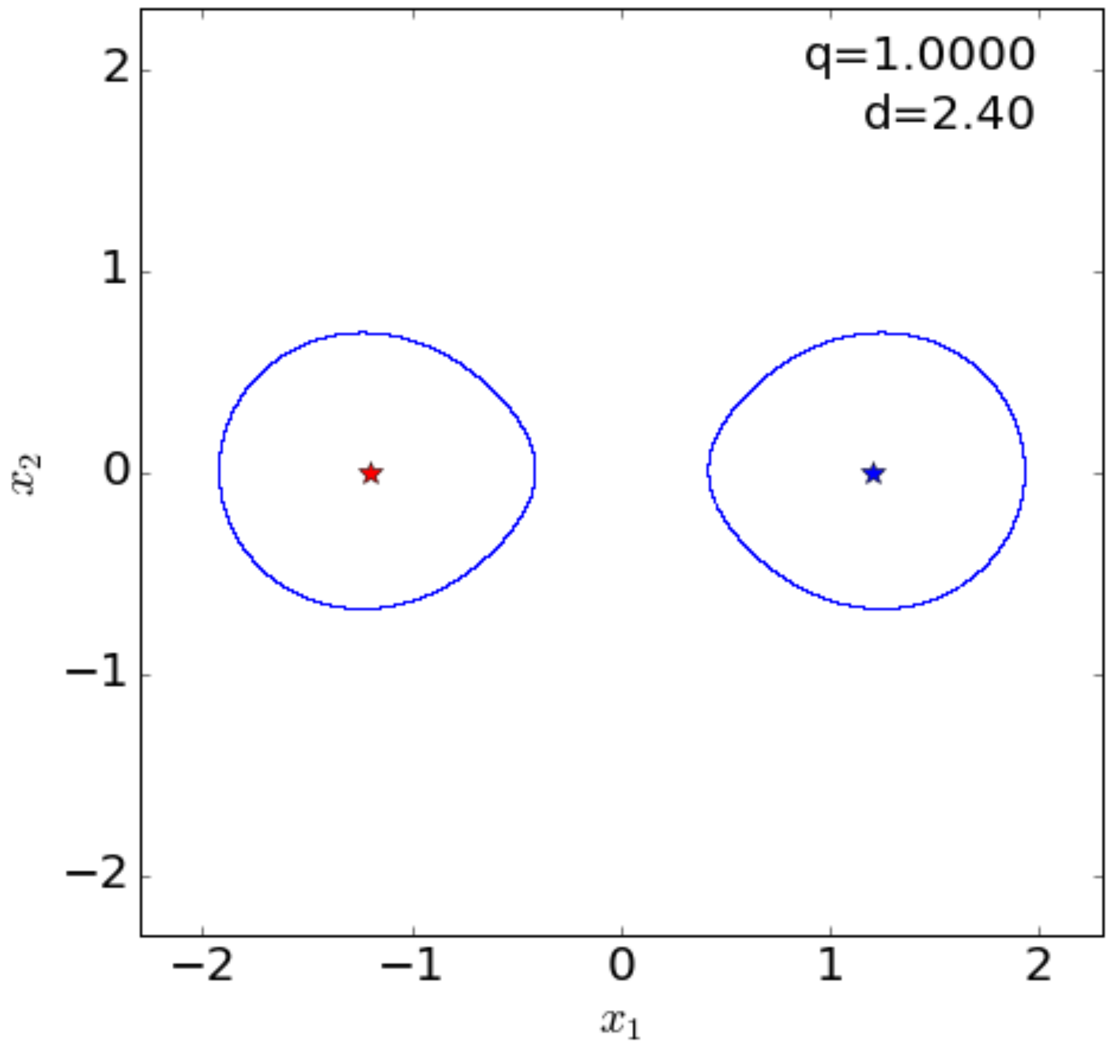


critical lines *caustics*

BINARY LENSES:
TWO LENSES WITH THE VARYING MASS AND FIXED DISTANCE



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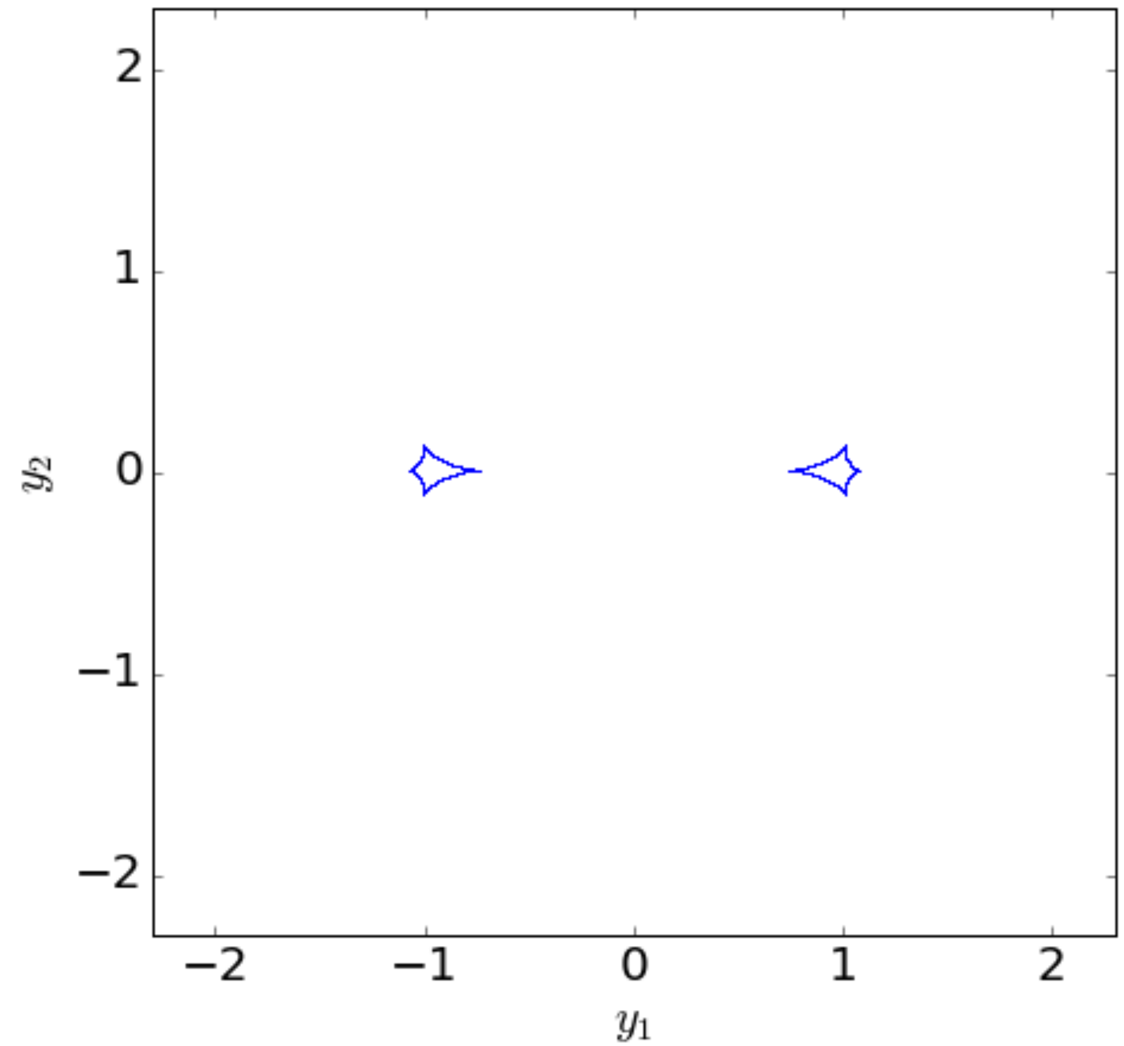

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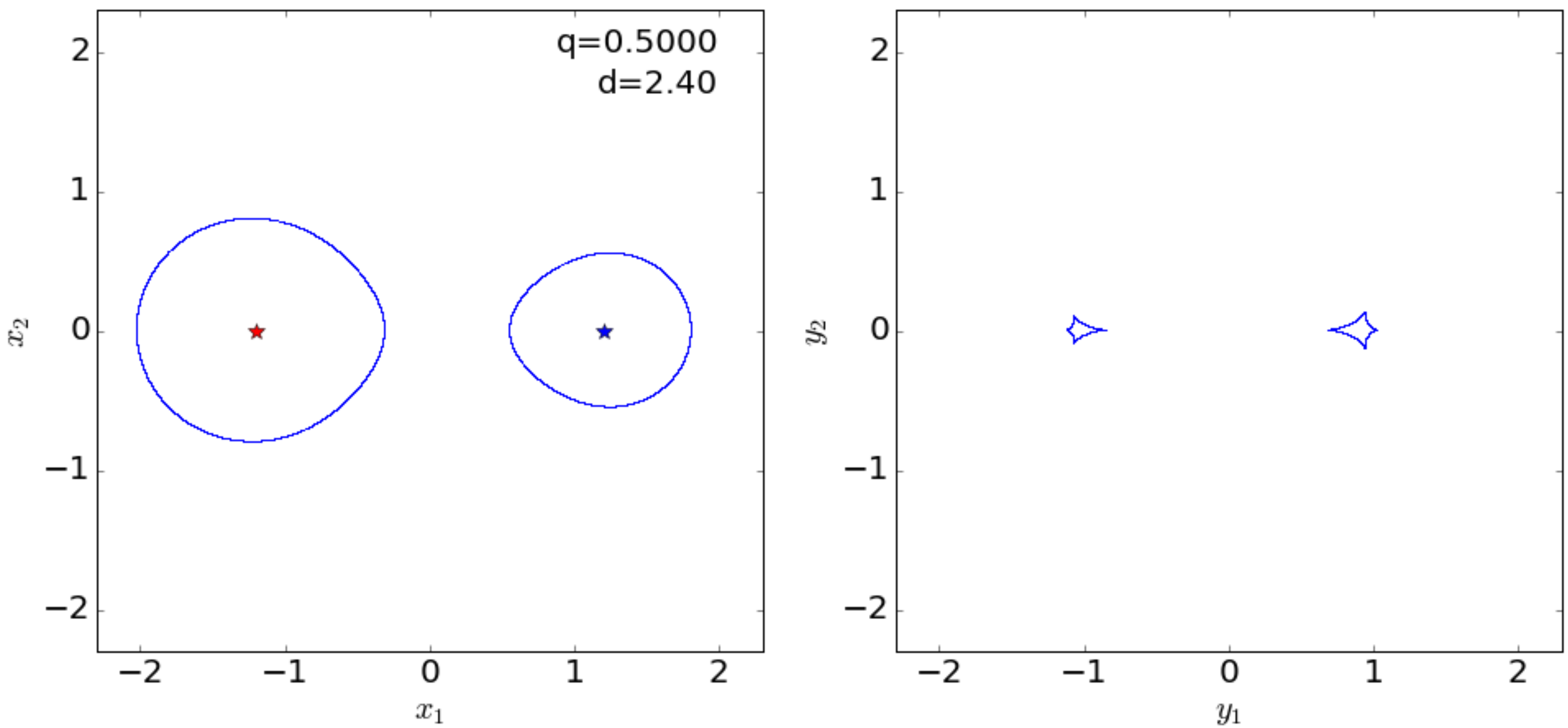

critical lines

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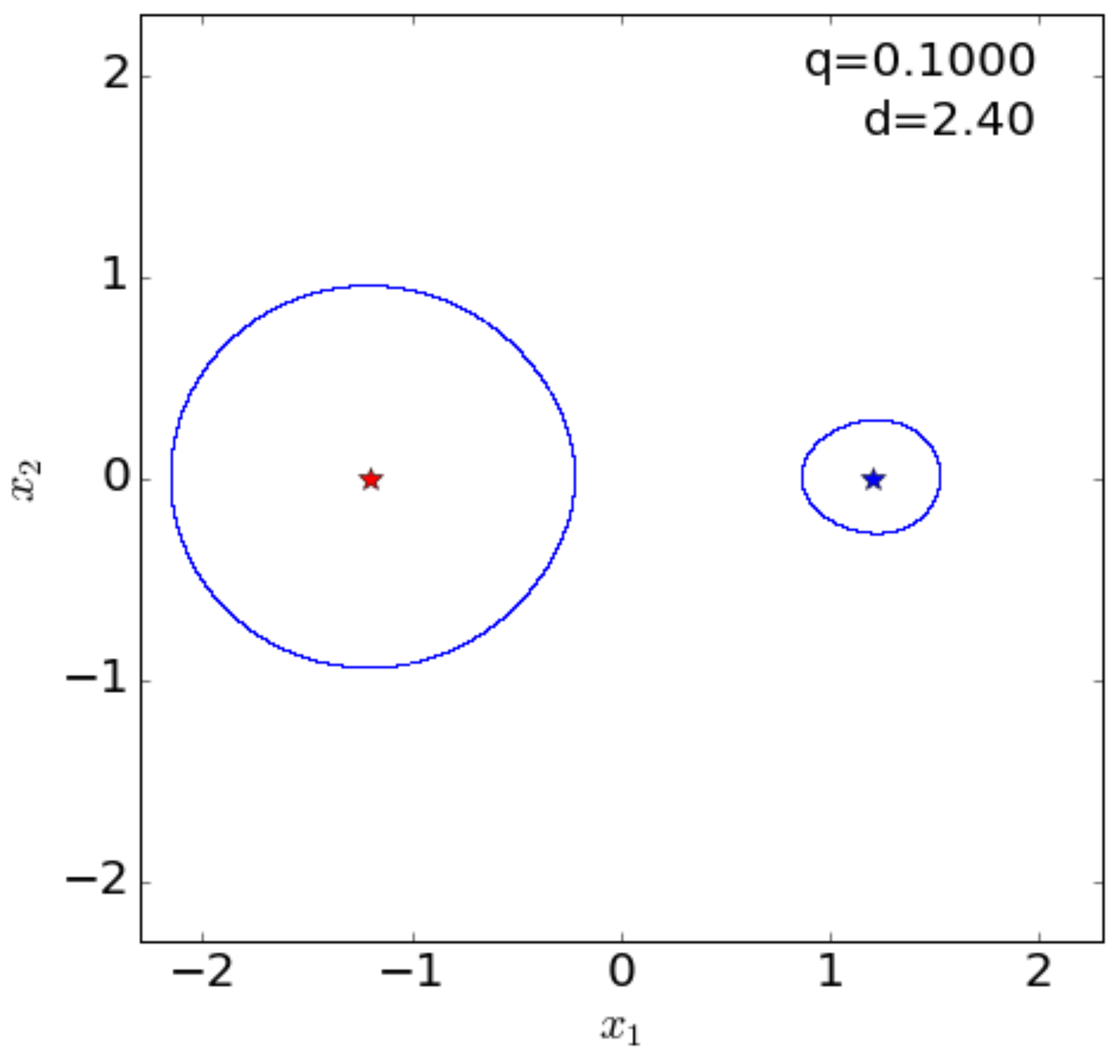

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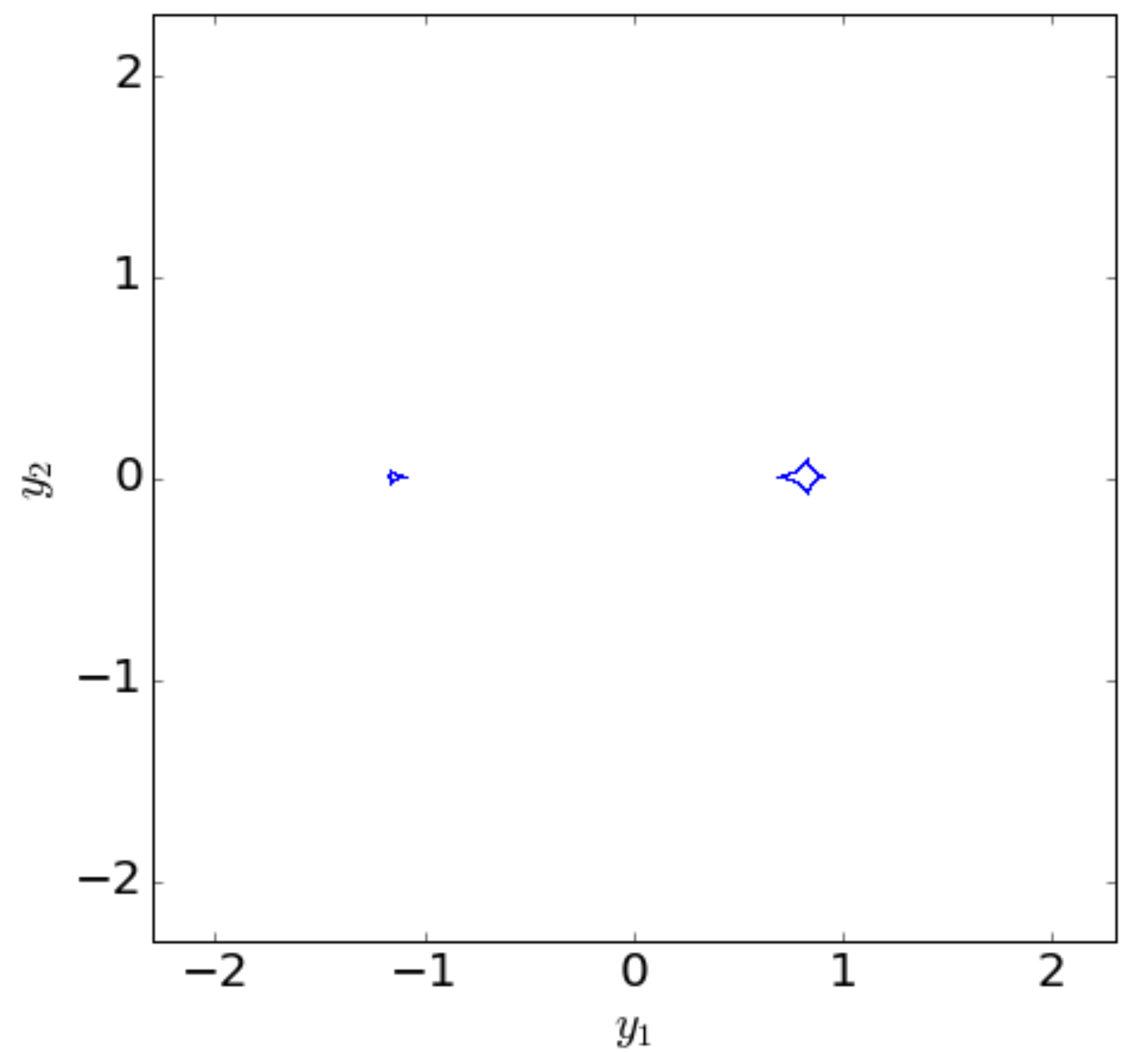

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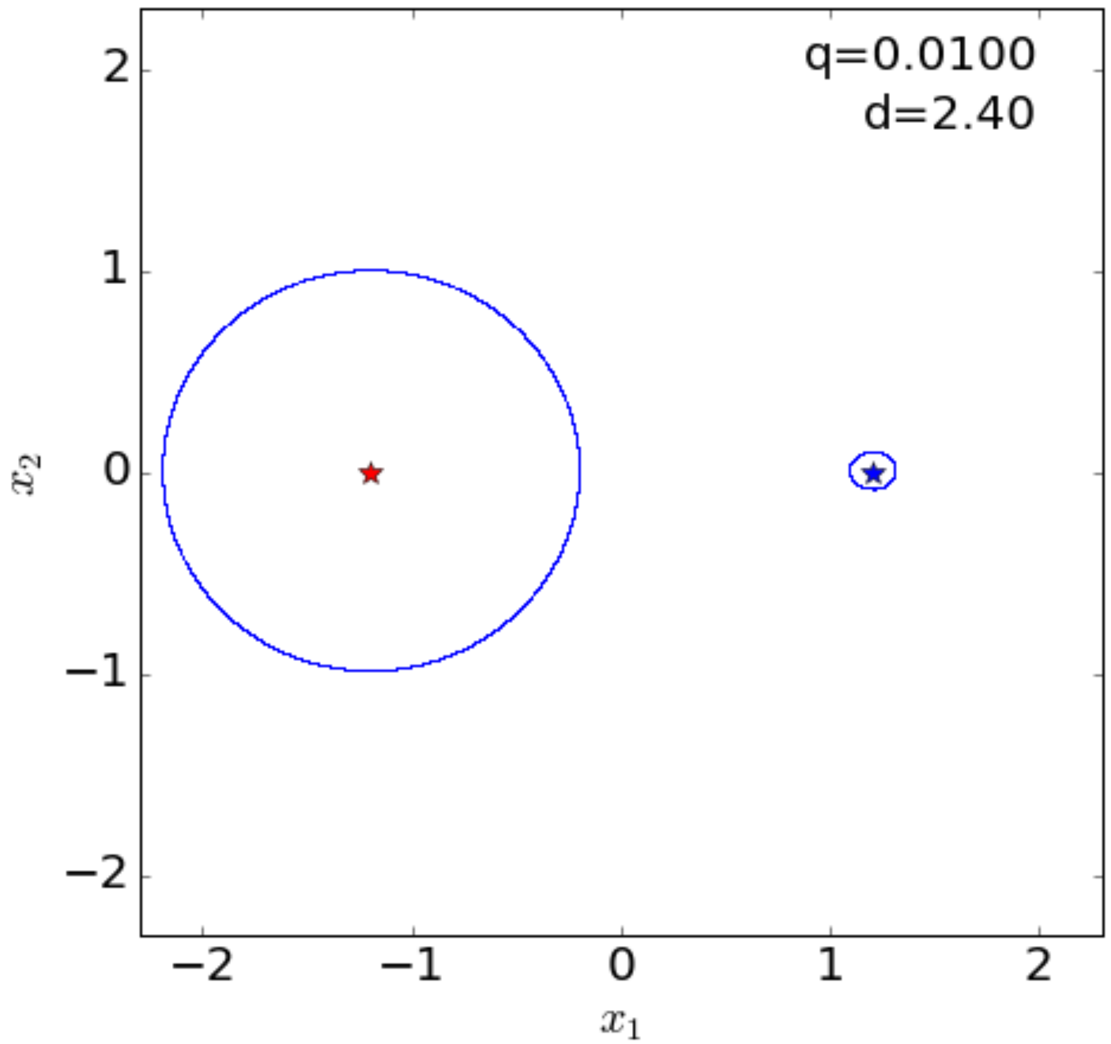

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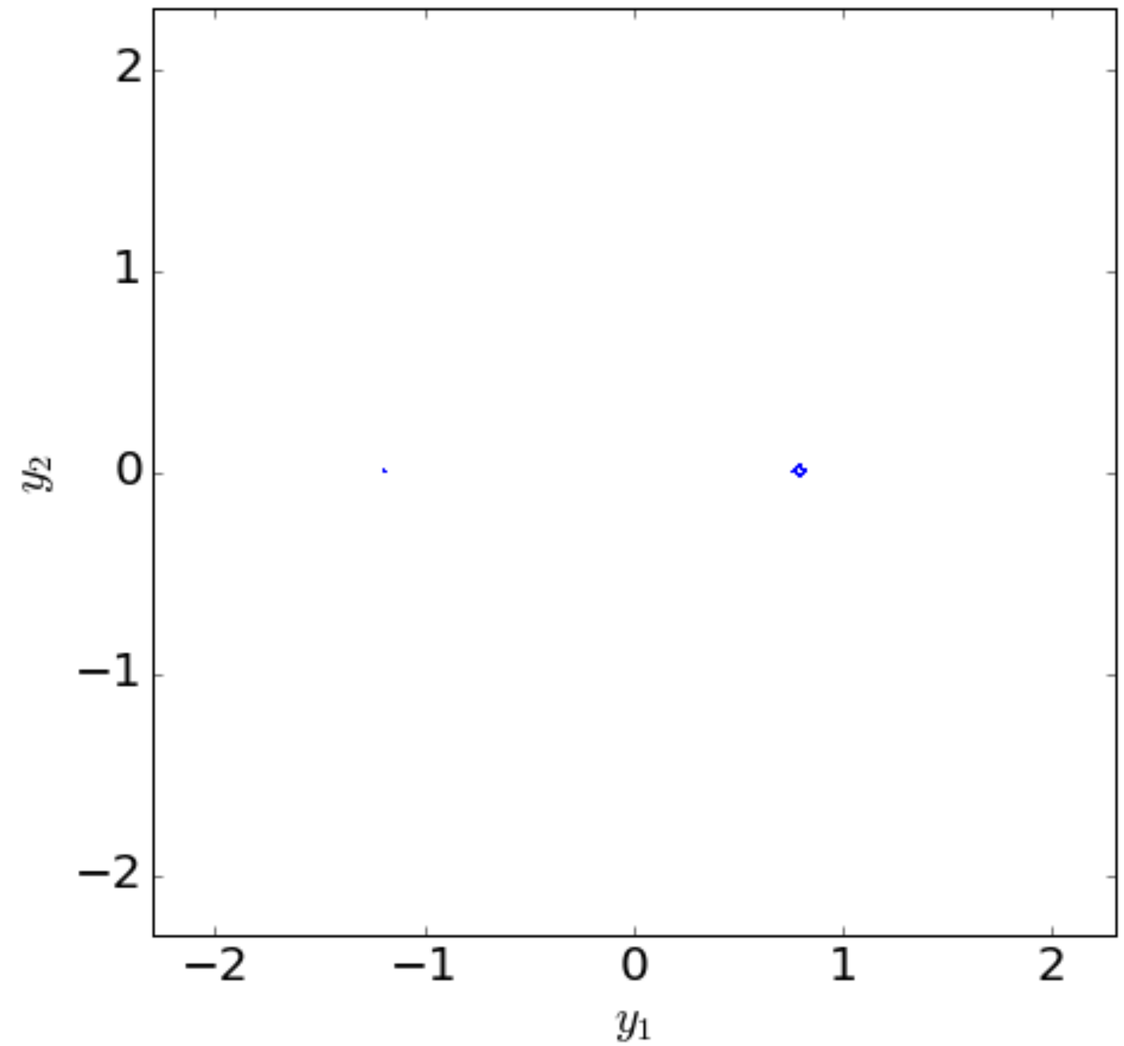

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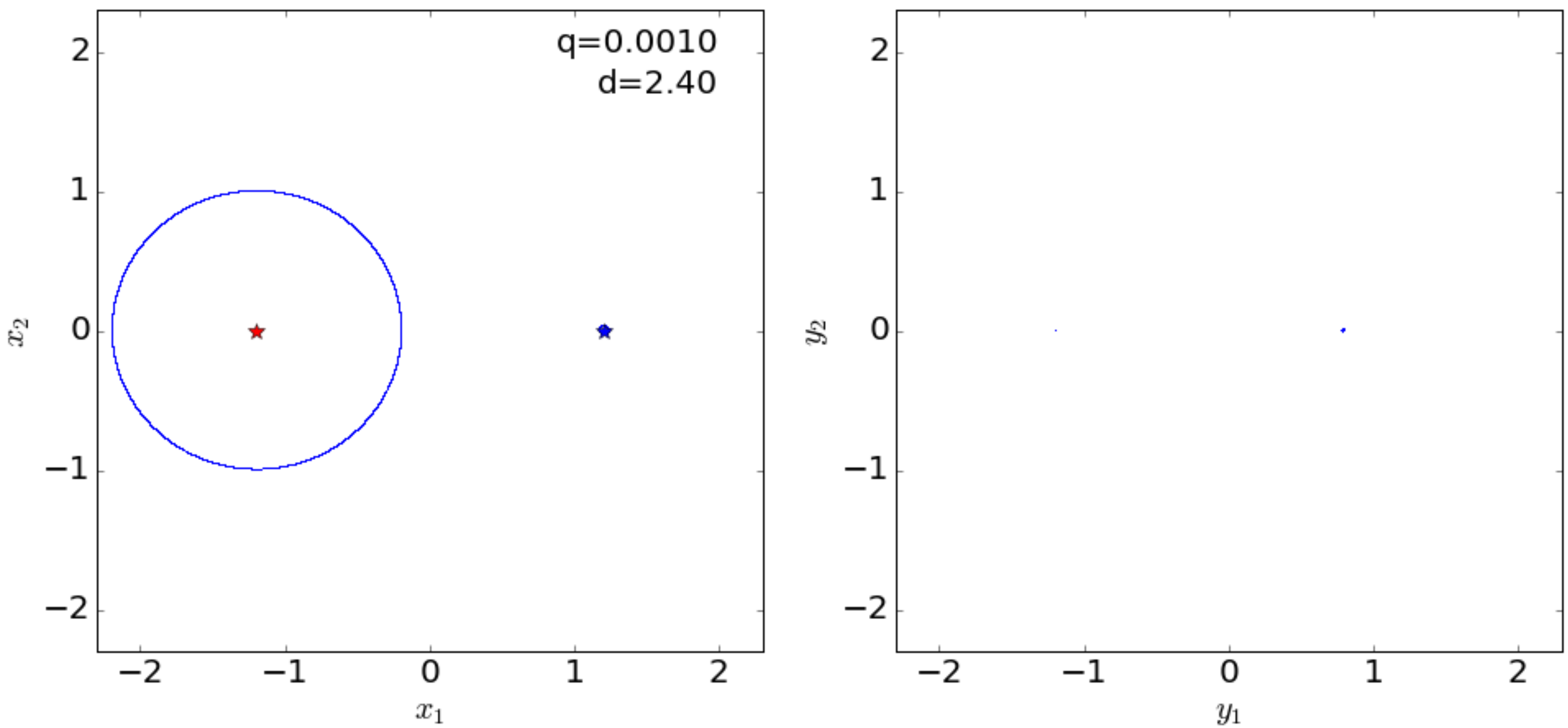

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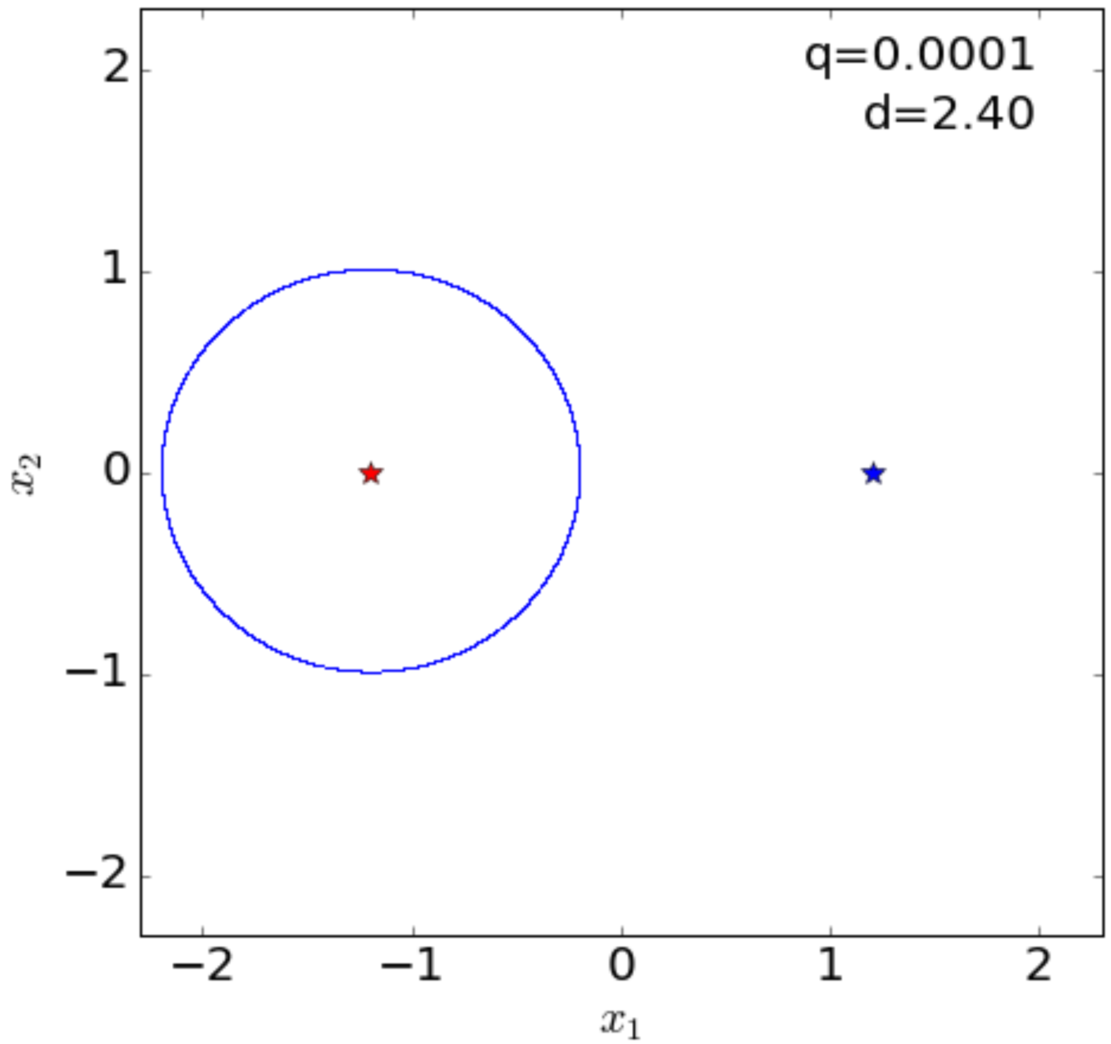

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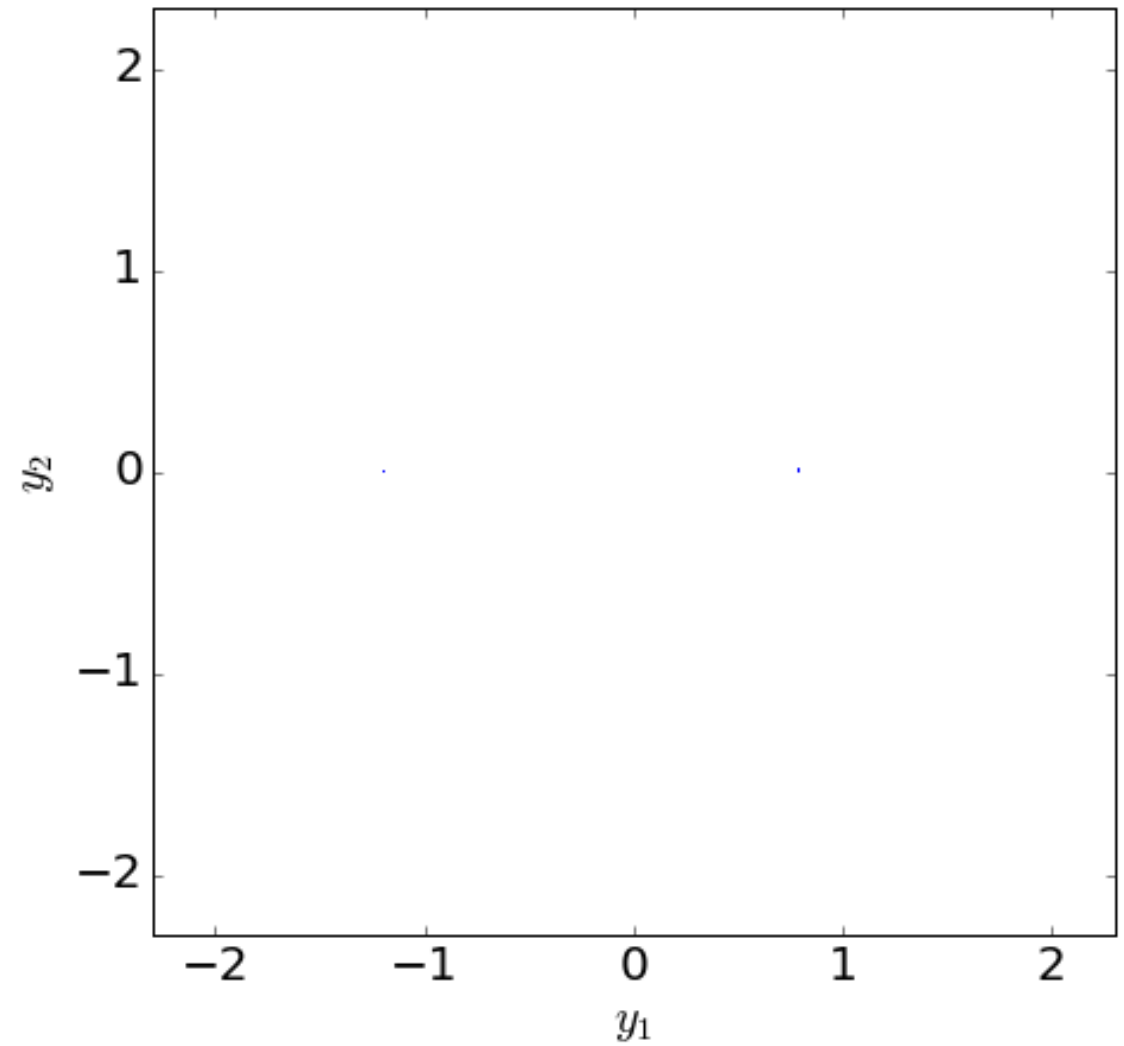

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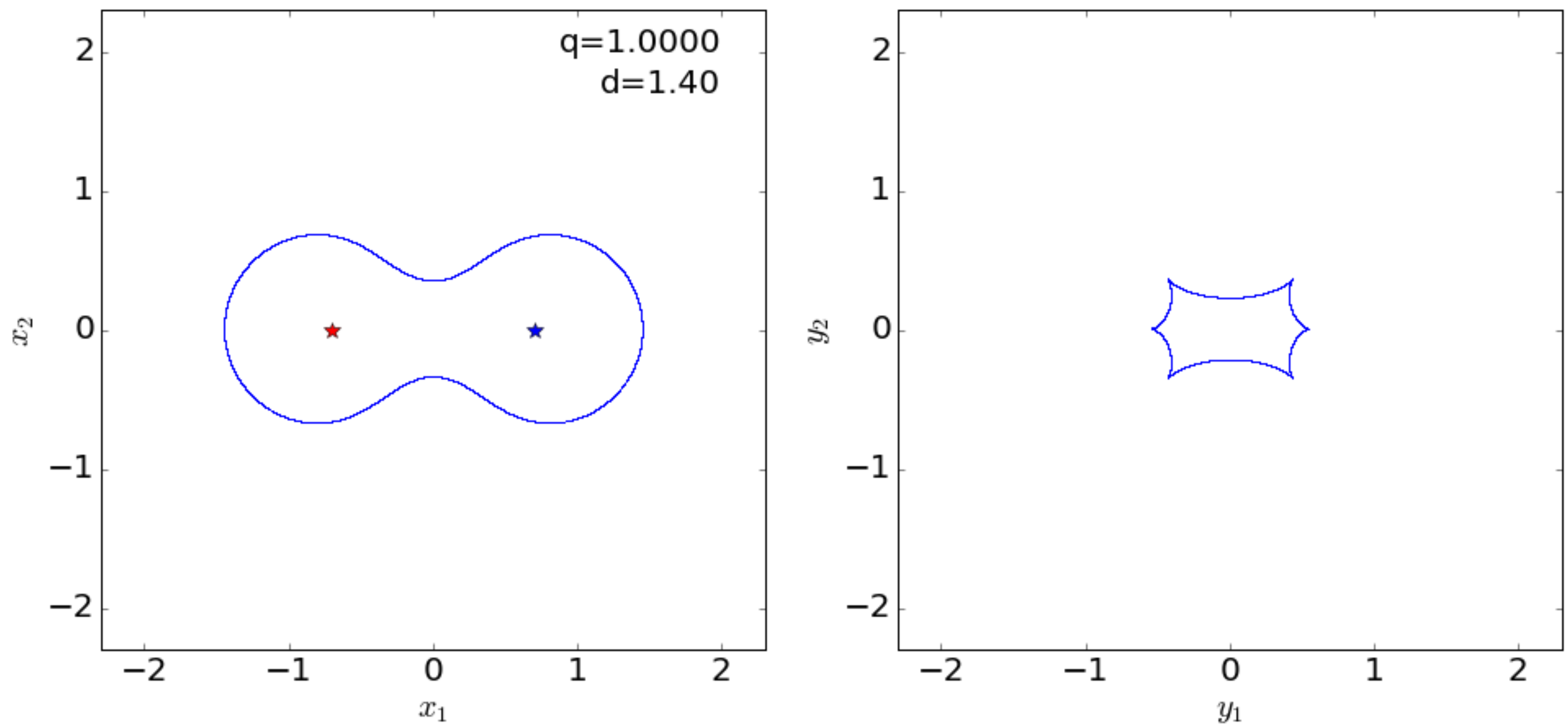

critical lines *caustics*

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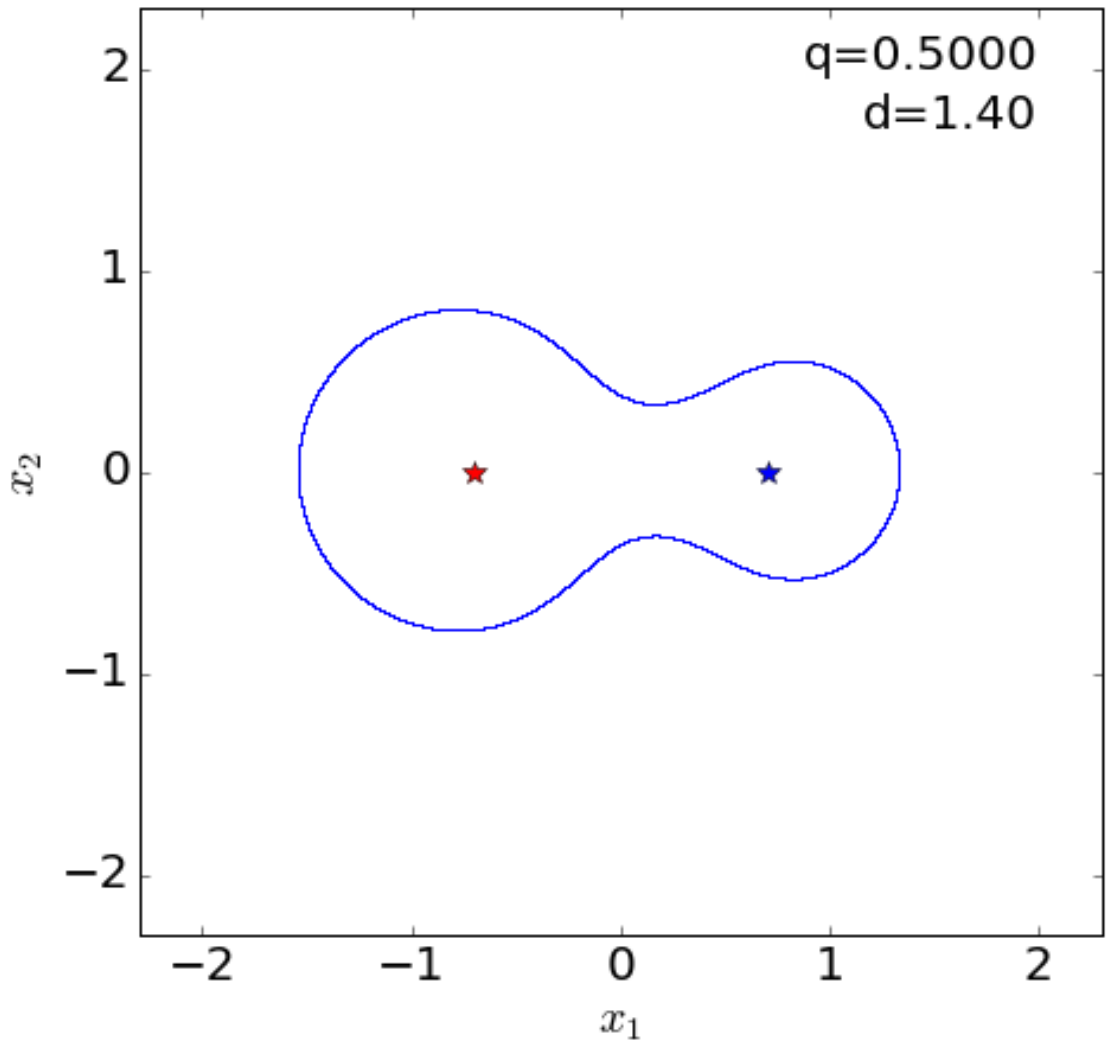

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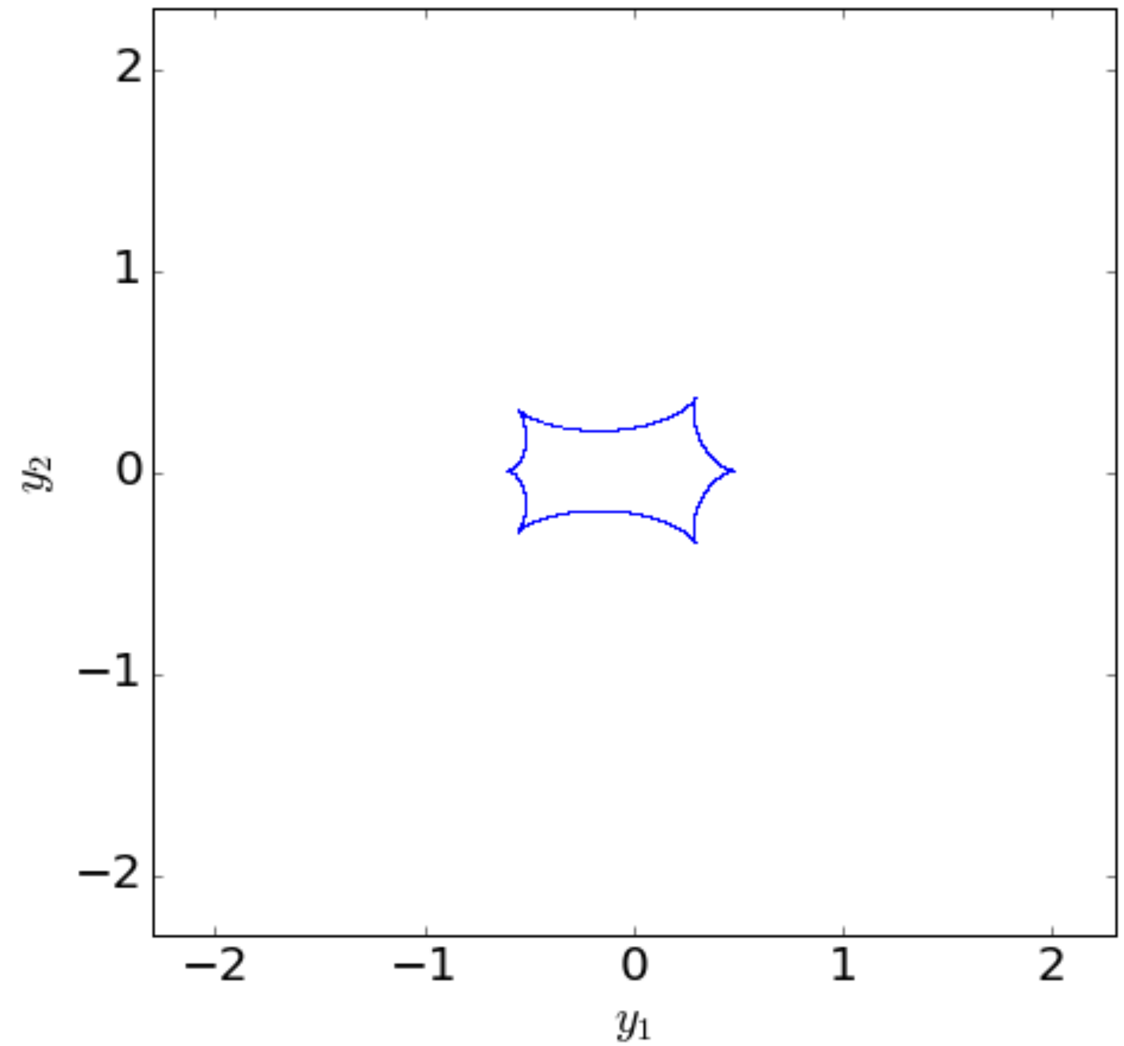

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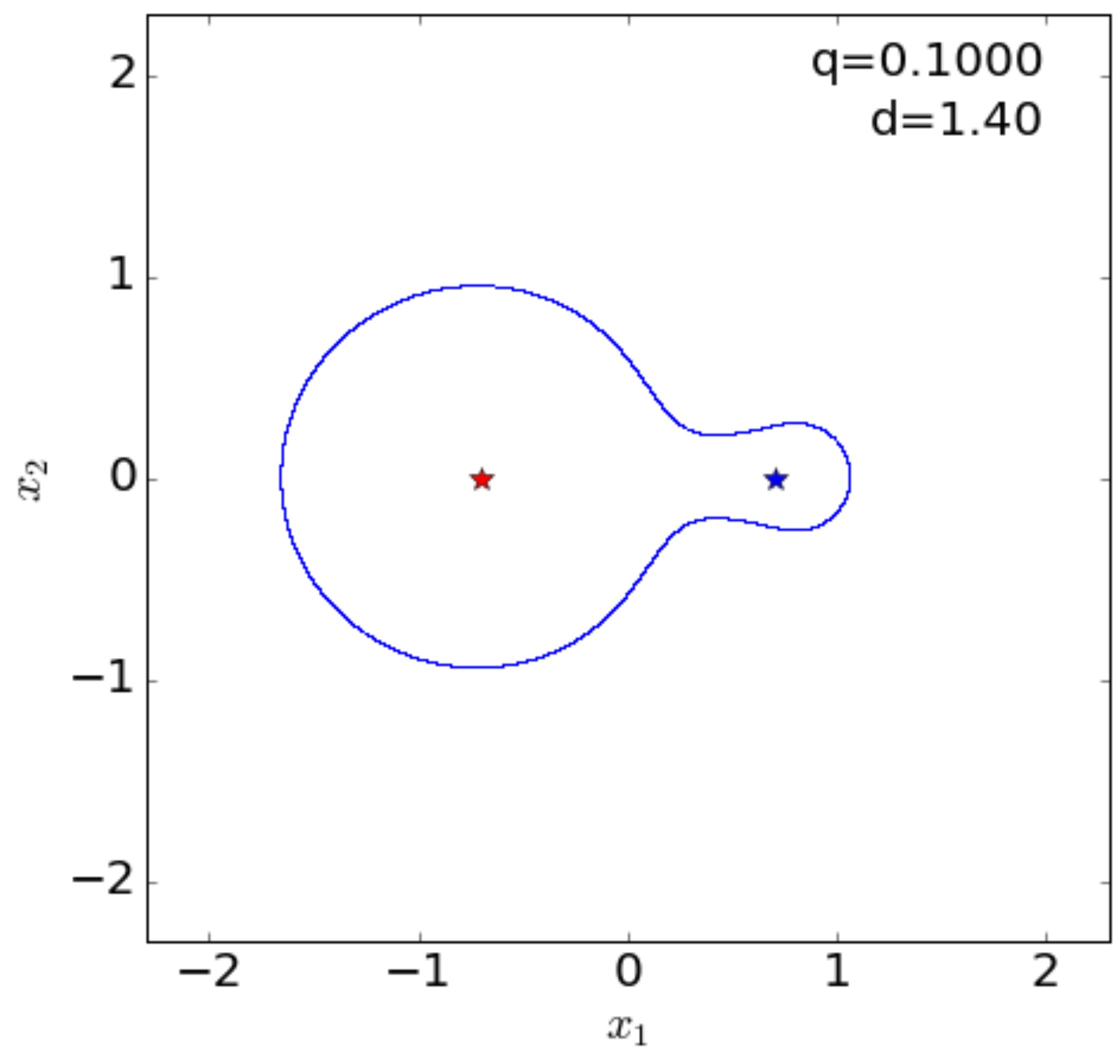

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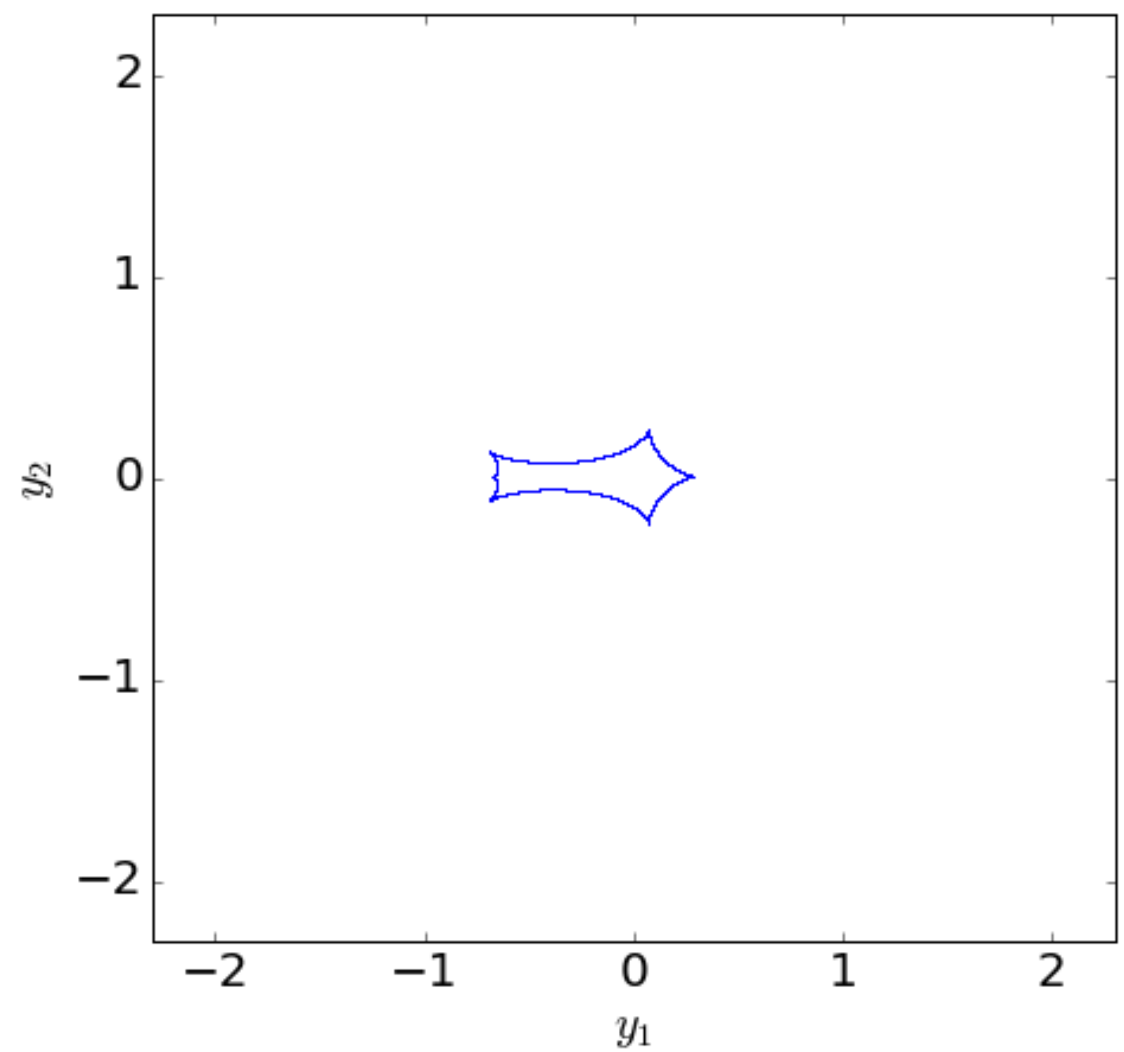

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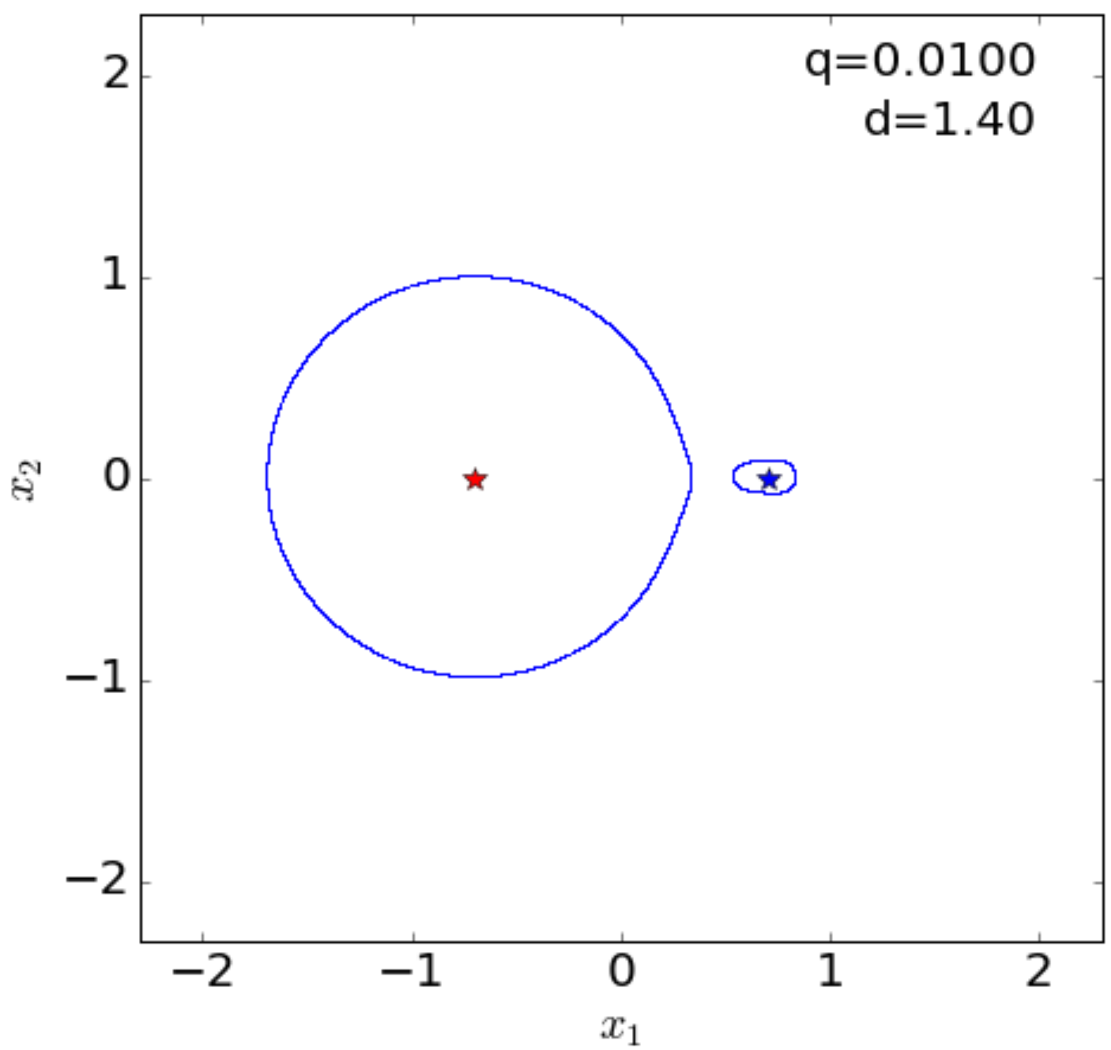

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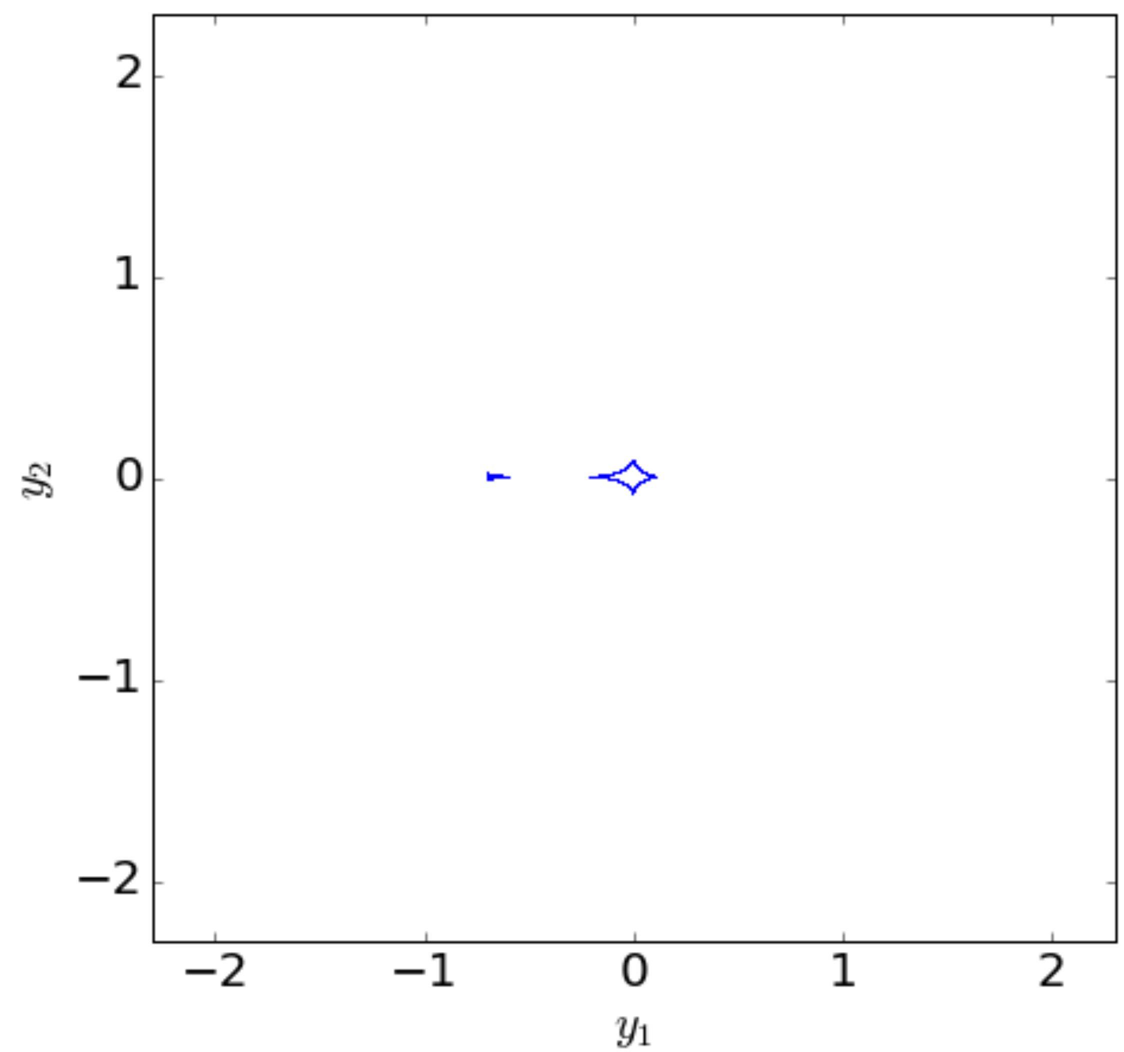

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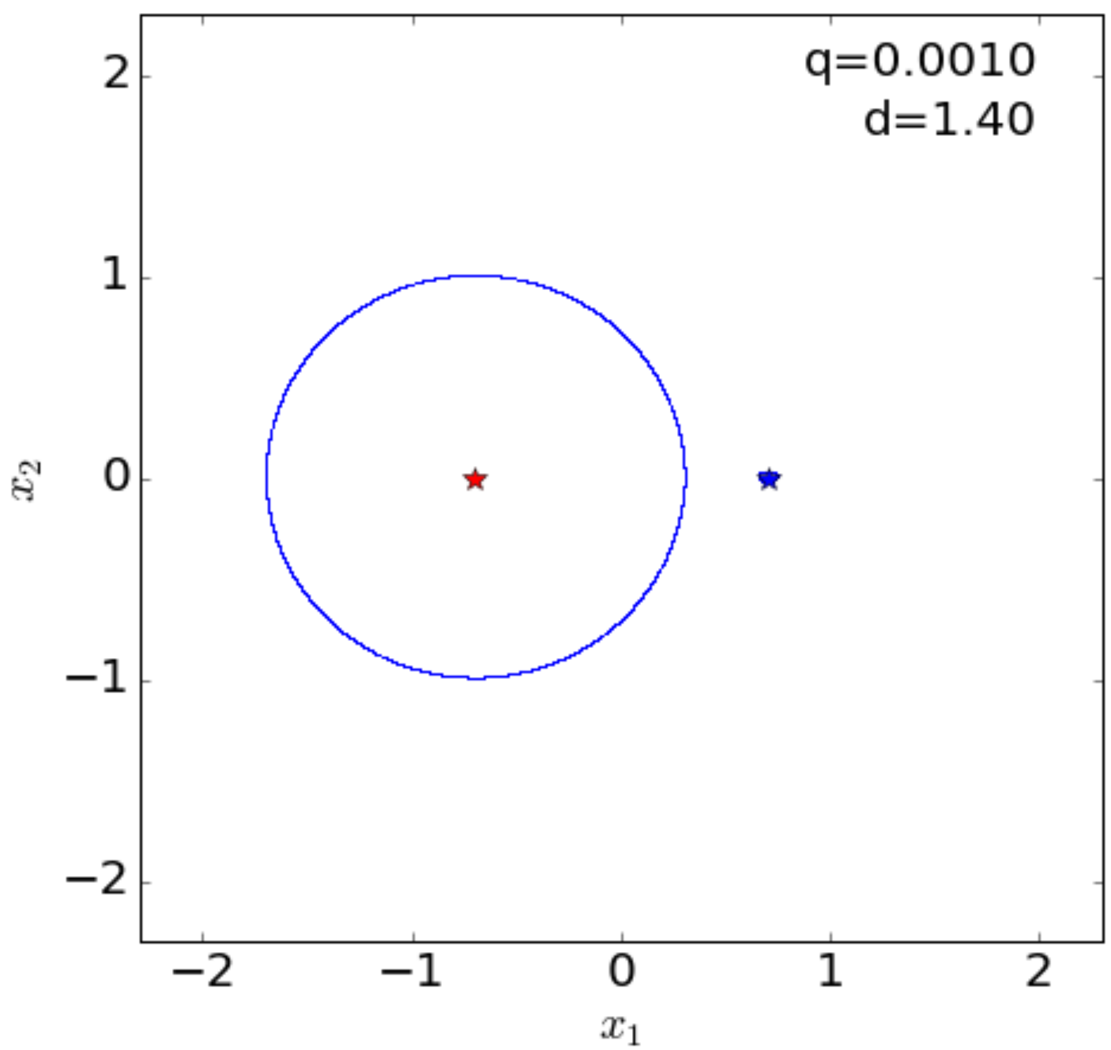

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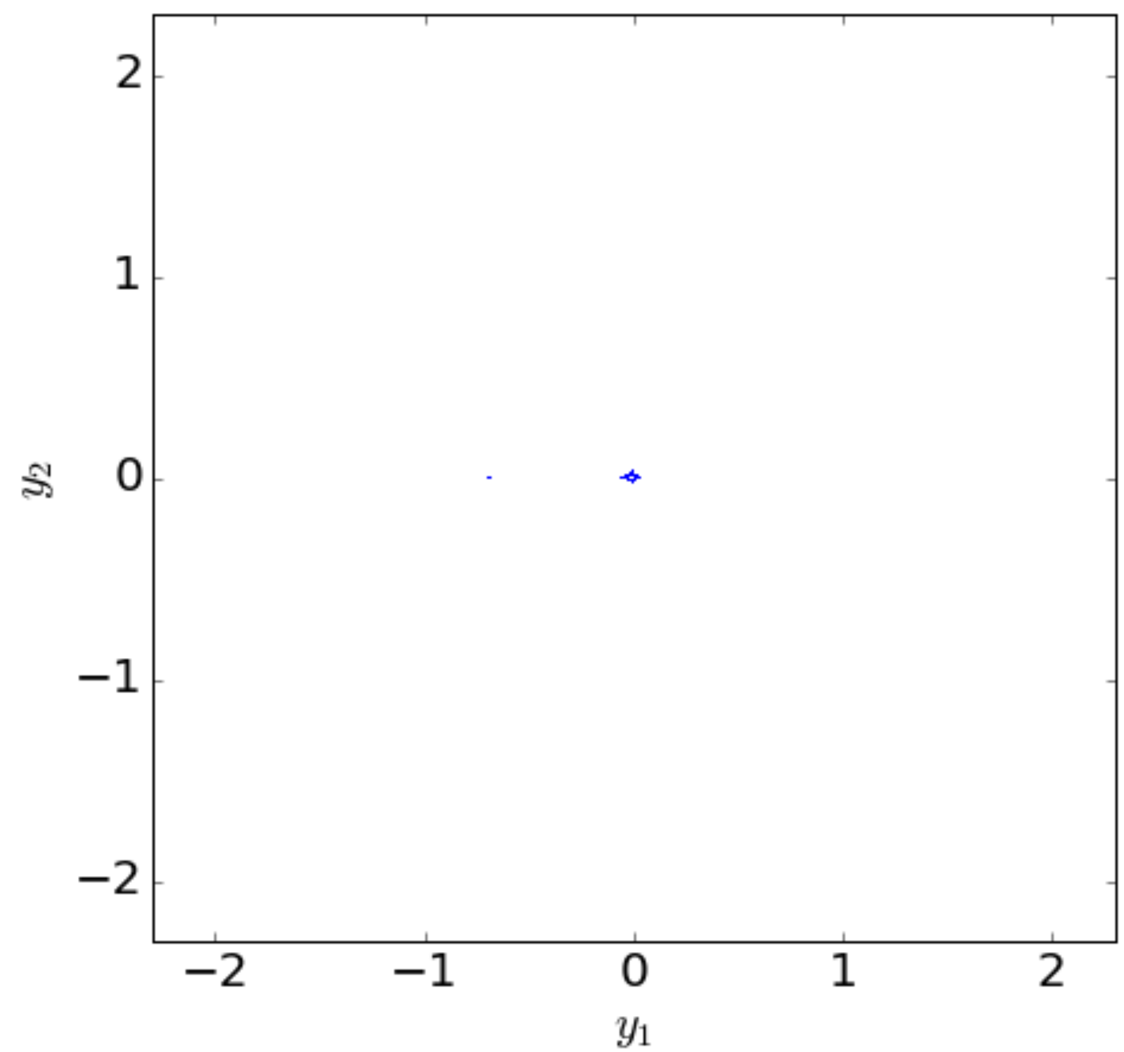
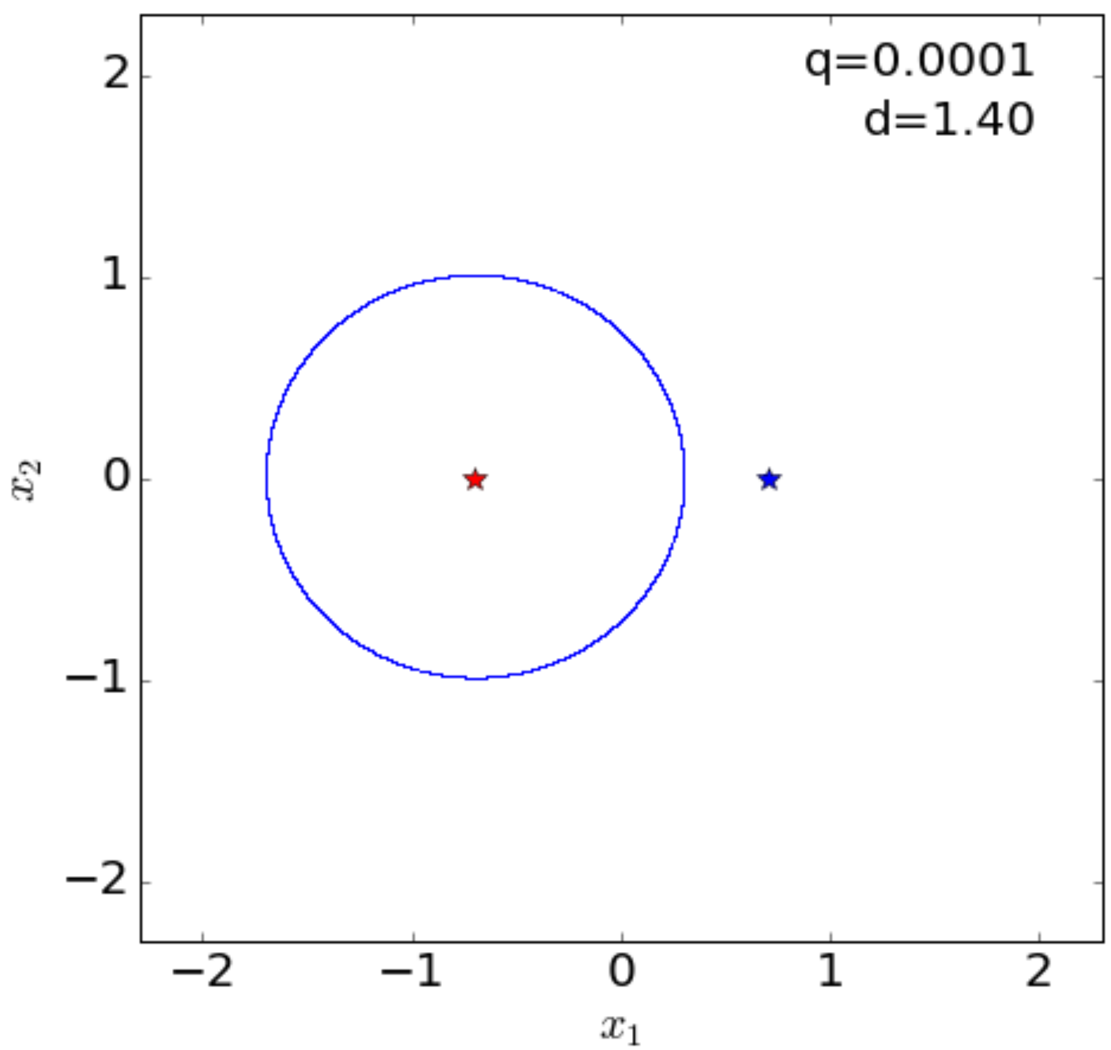
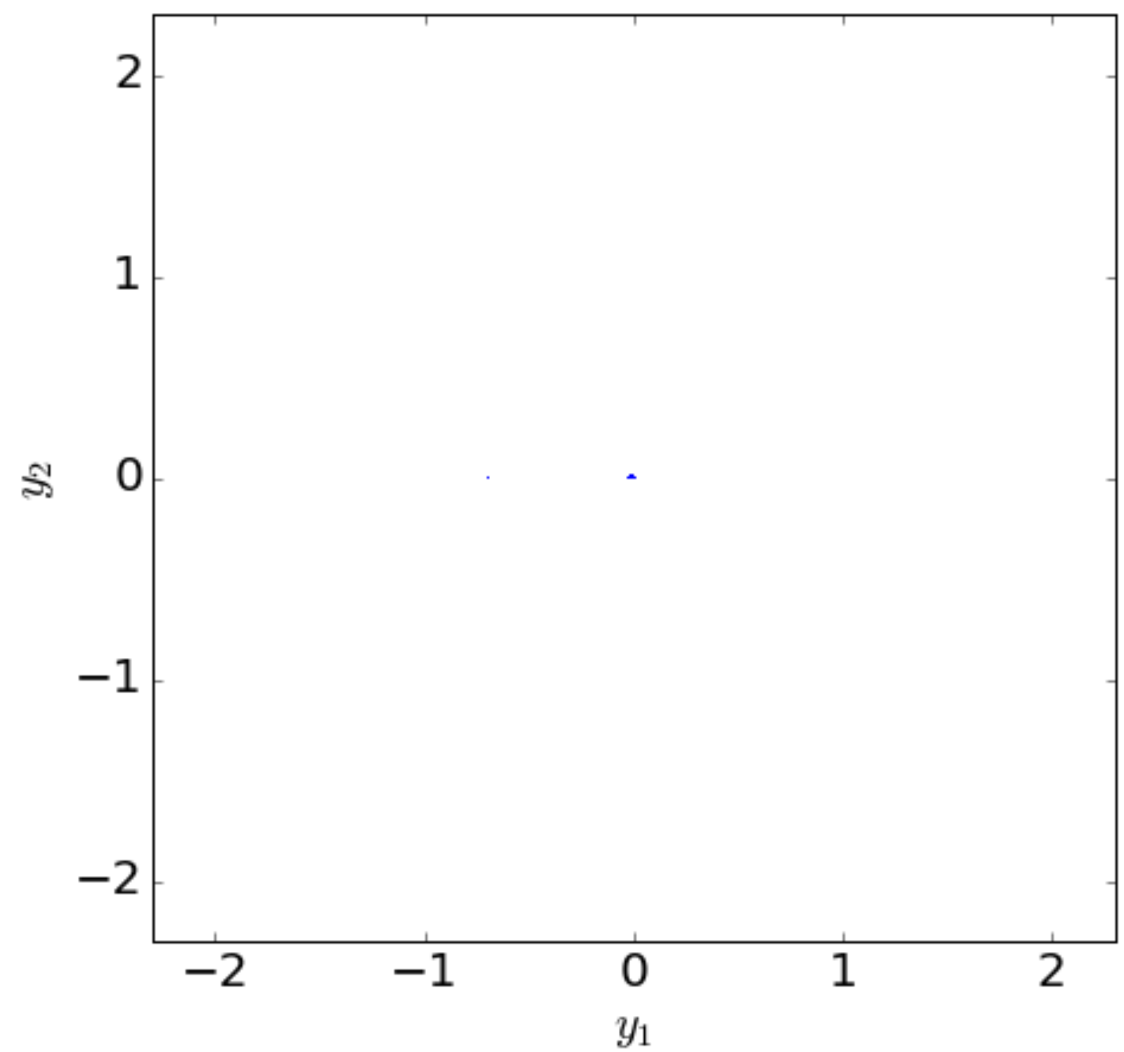


caustics

BINARY LENSES:
TWO LENSES WITH THE VARYING MASS AND FIXED DISTANCE



critical lines



caustics

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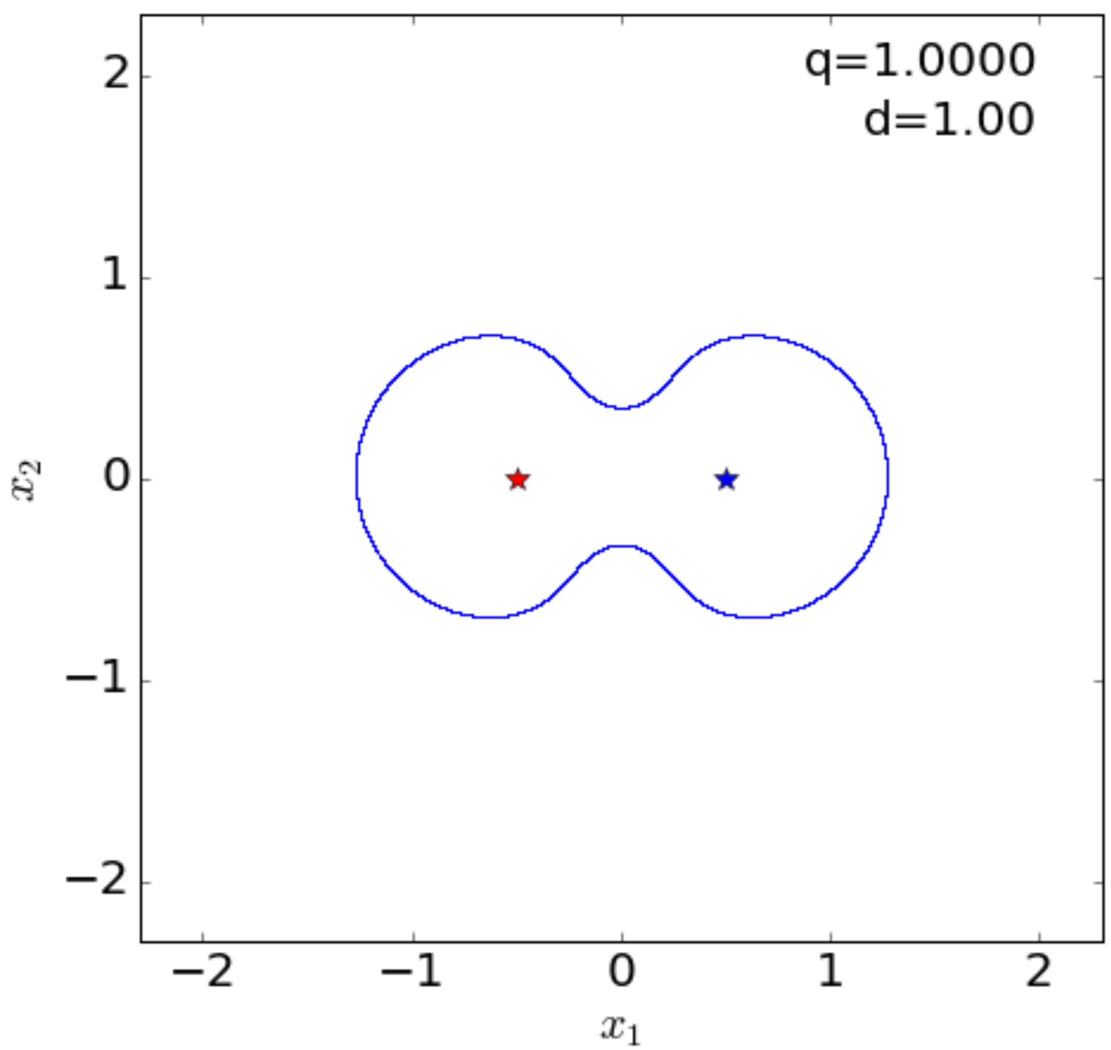

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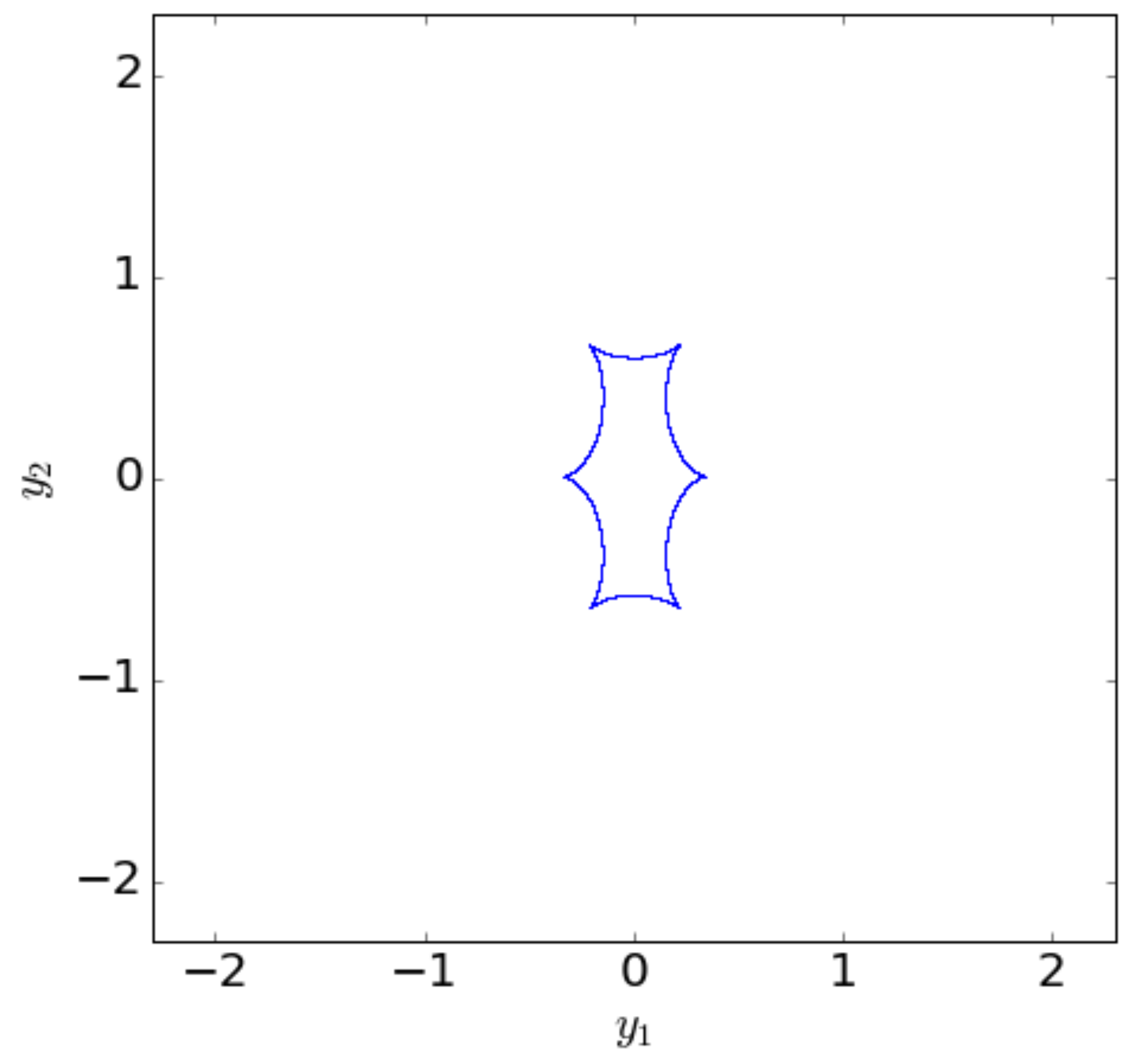

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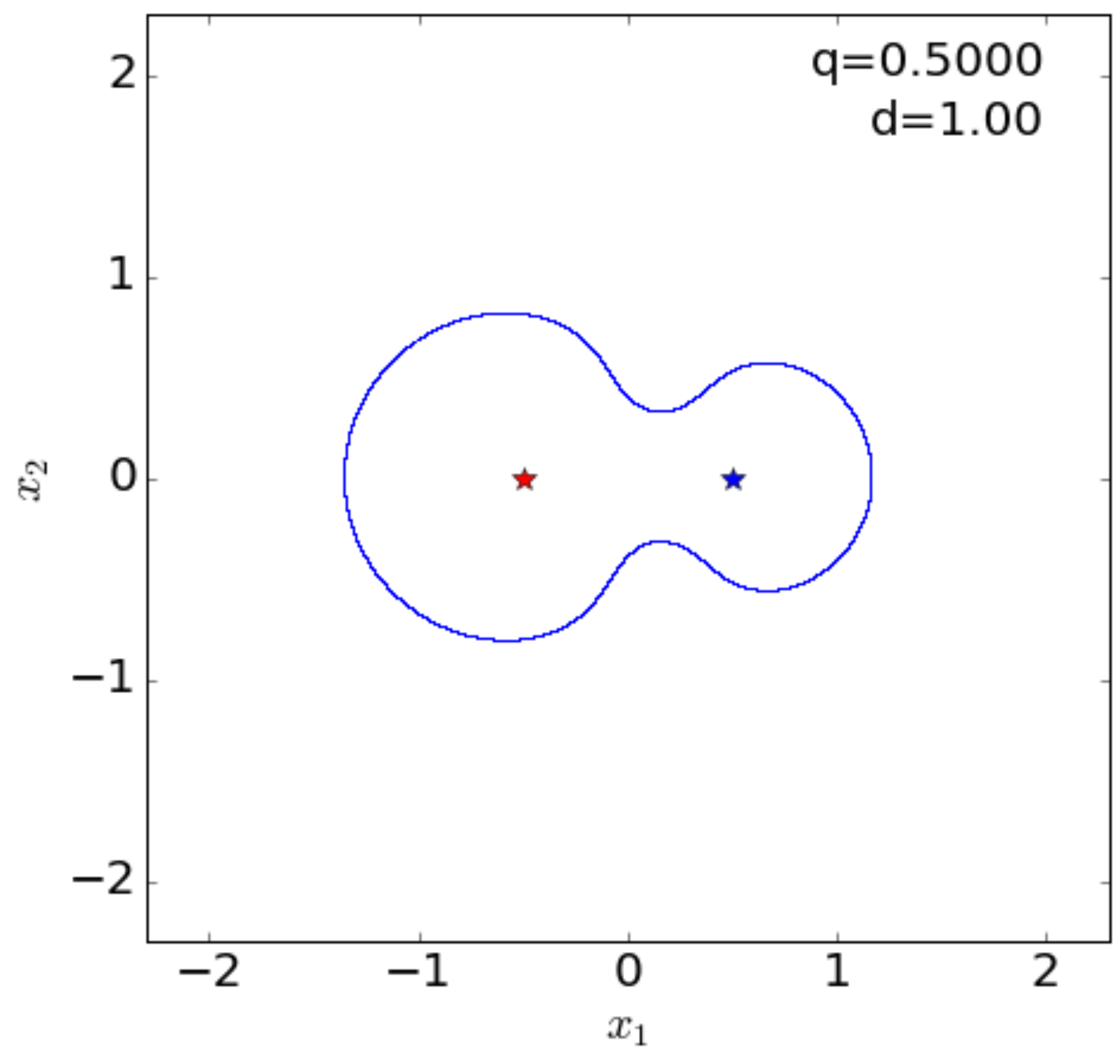
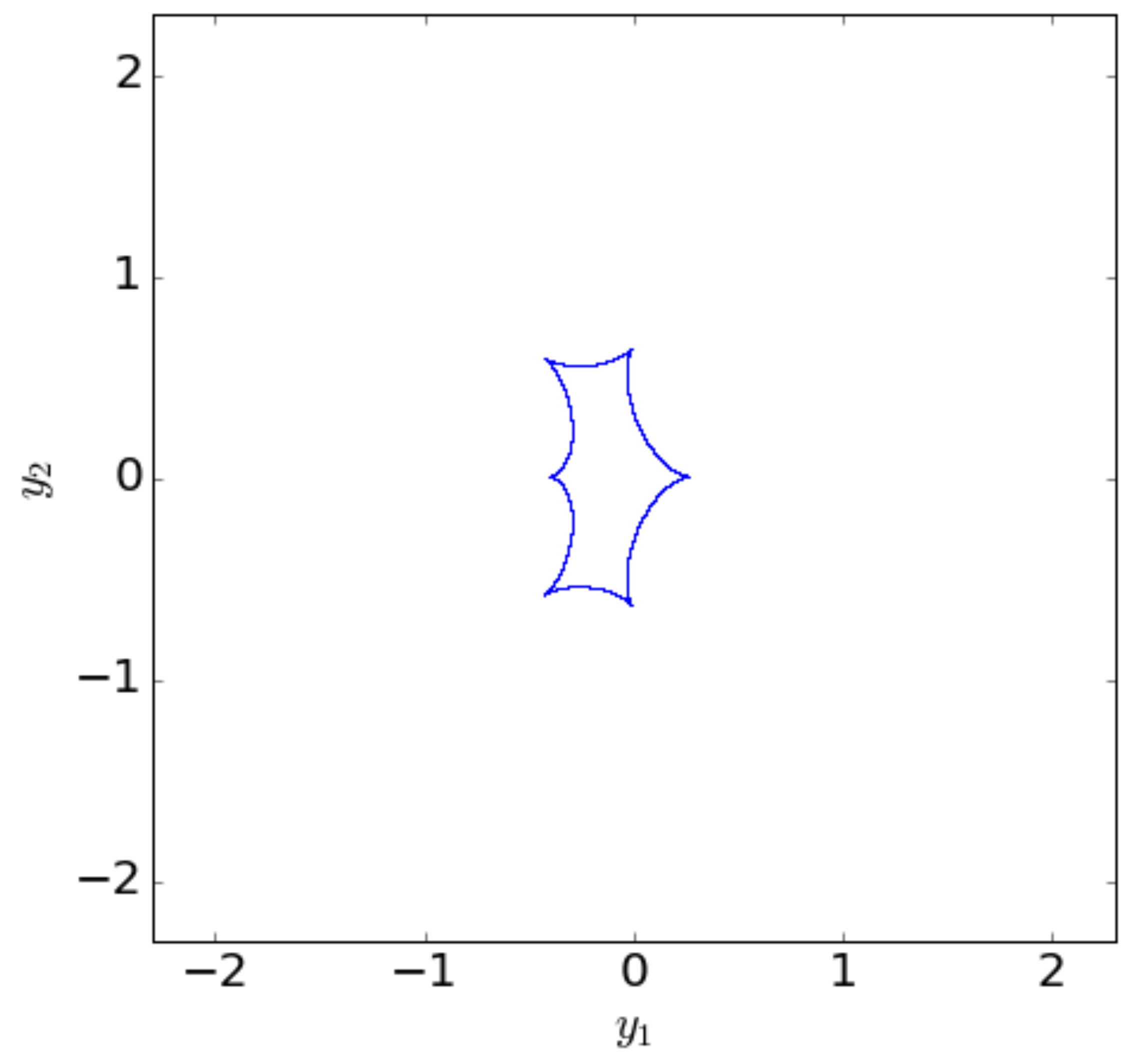


critical lines



caustics

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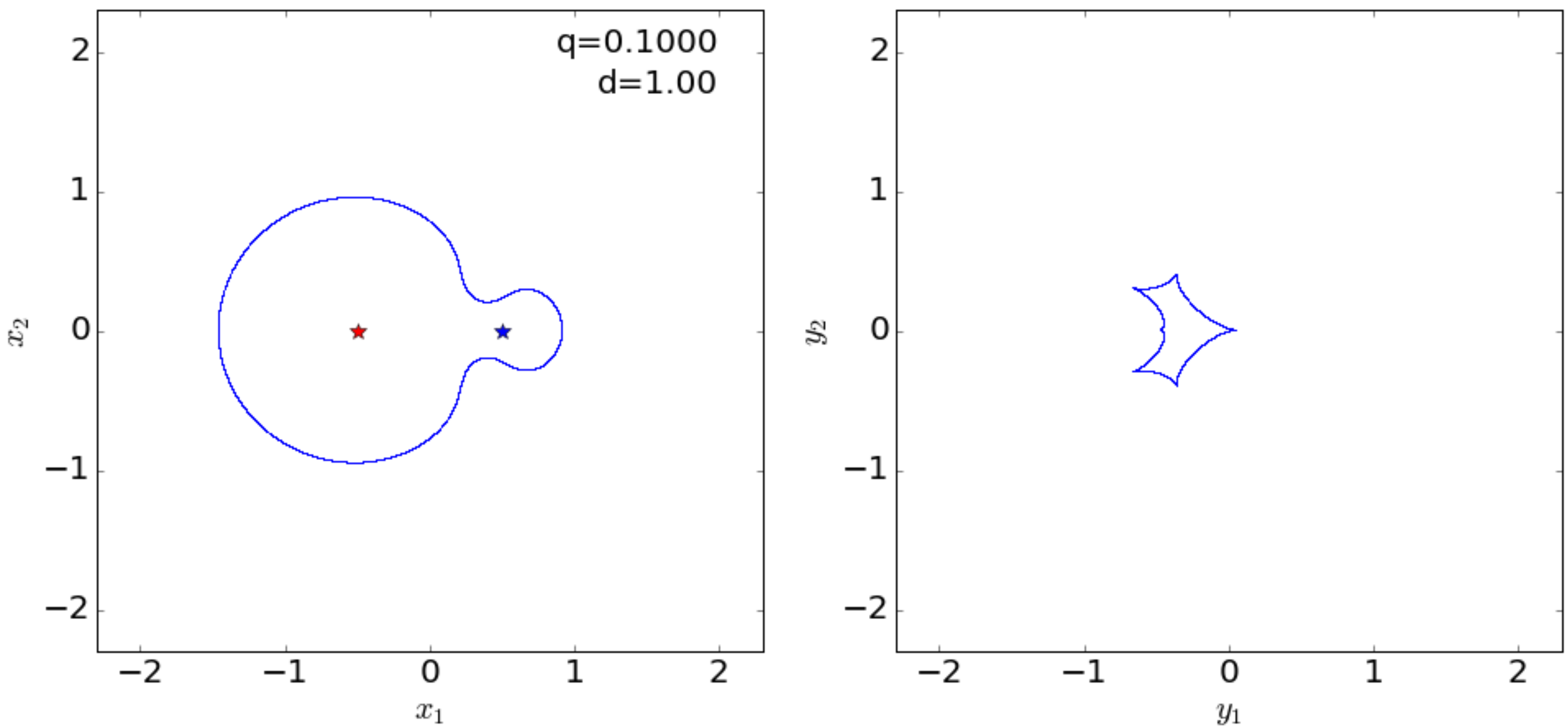

critical lines *caustics*

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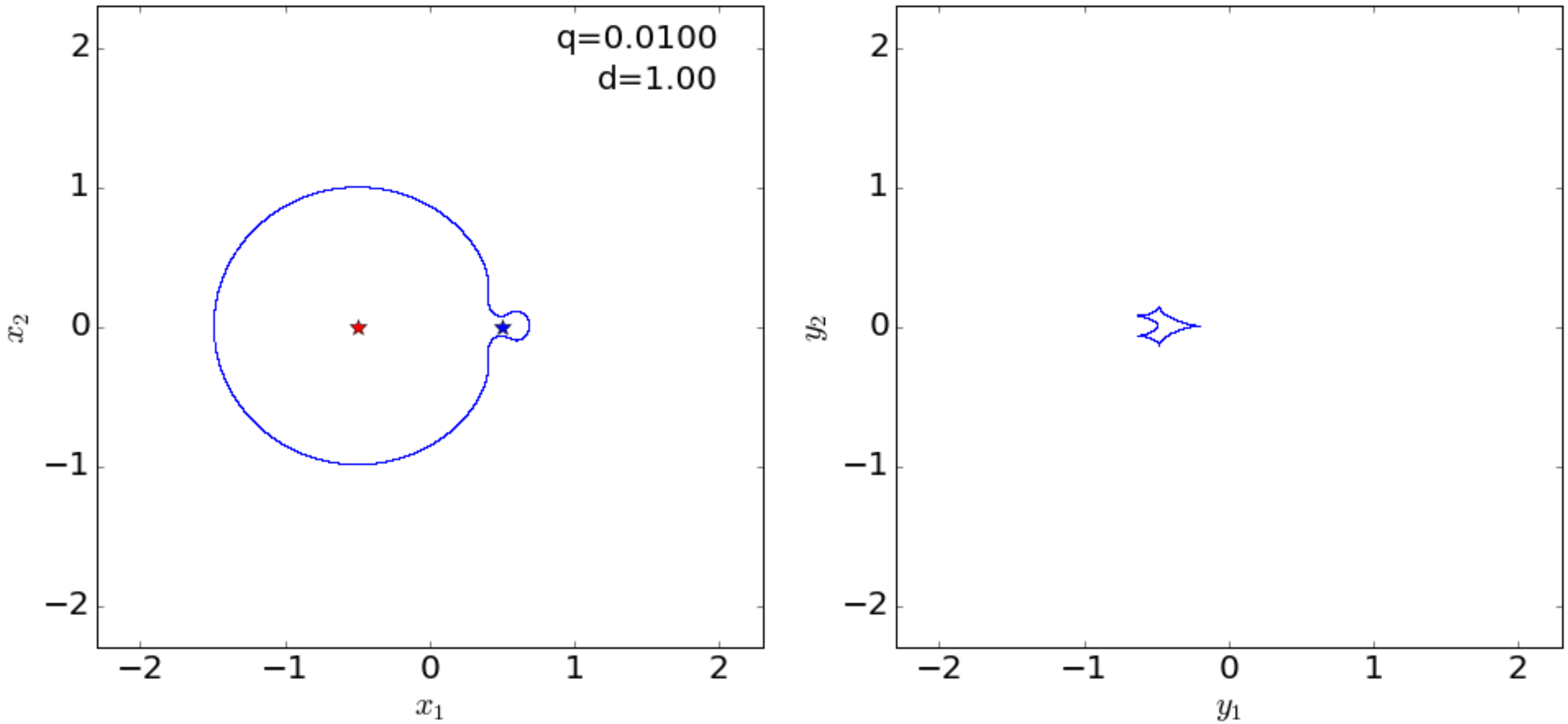

critical lines *caustics*

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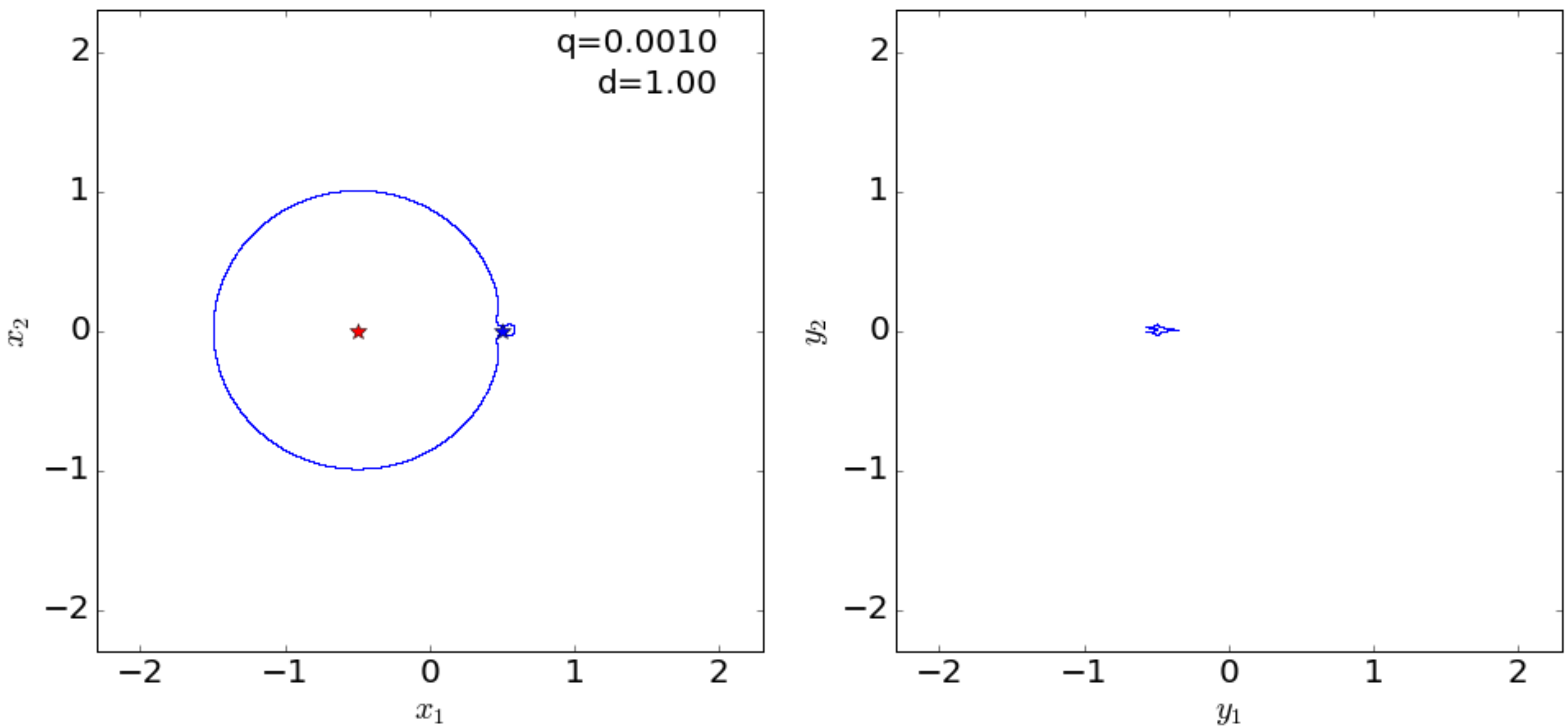

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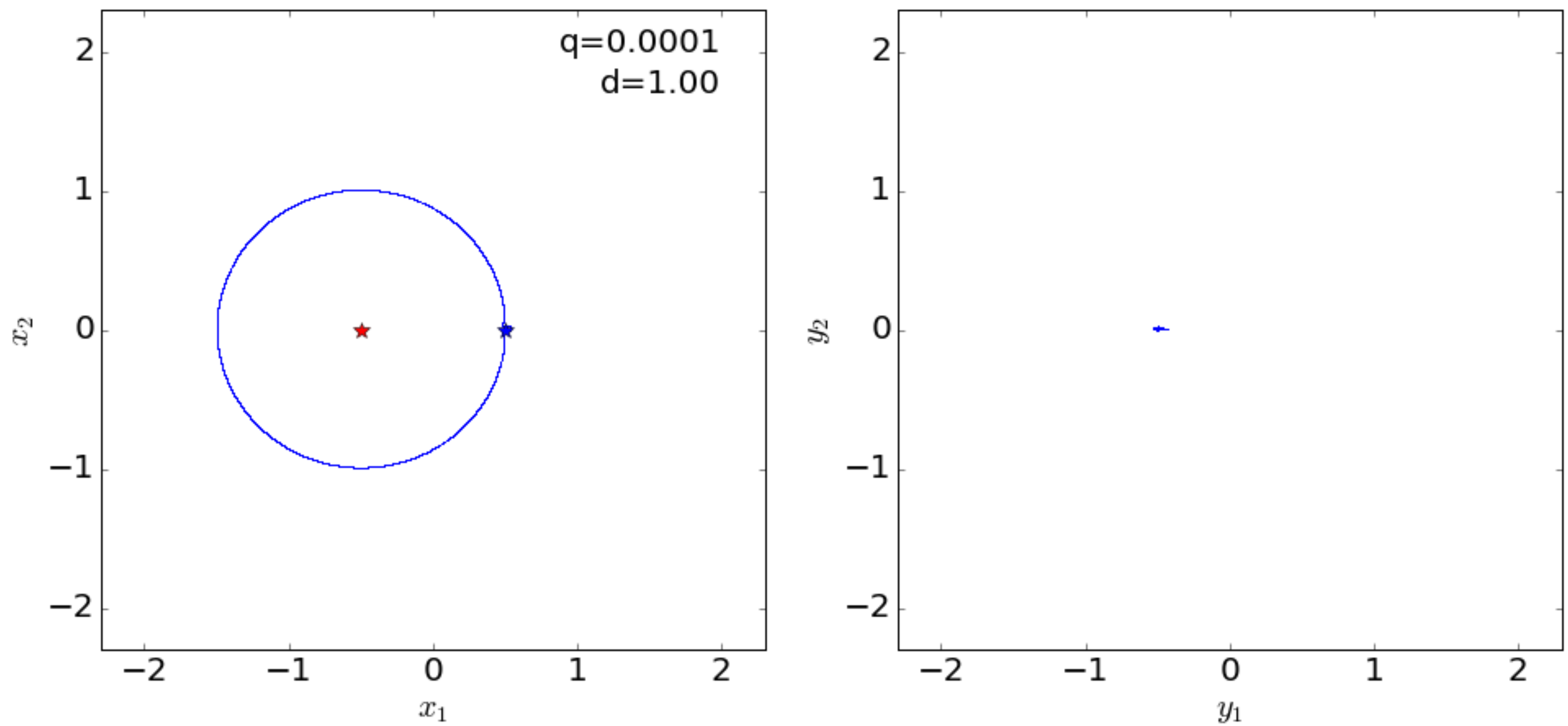

critical lines *caustics*

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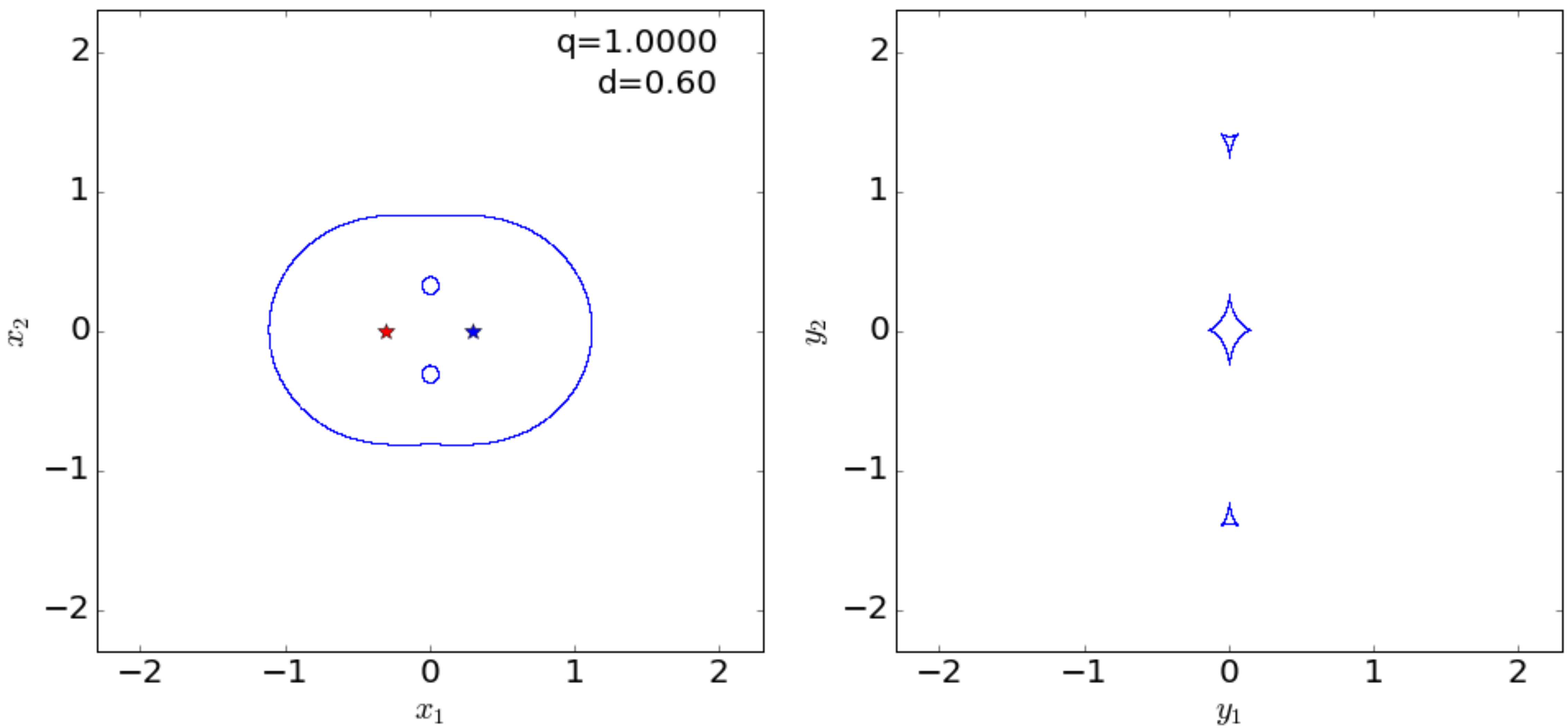

critical lines *caustics*

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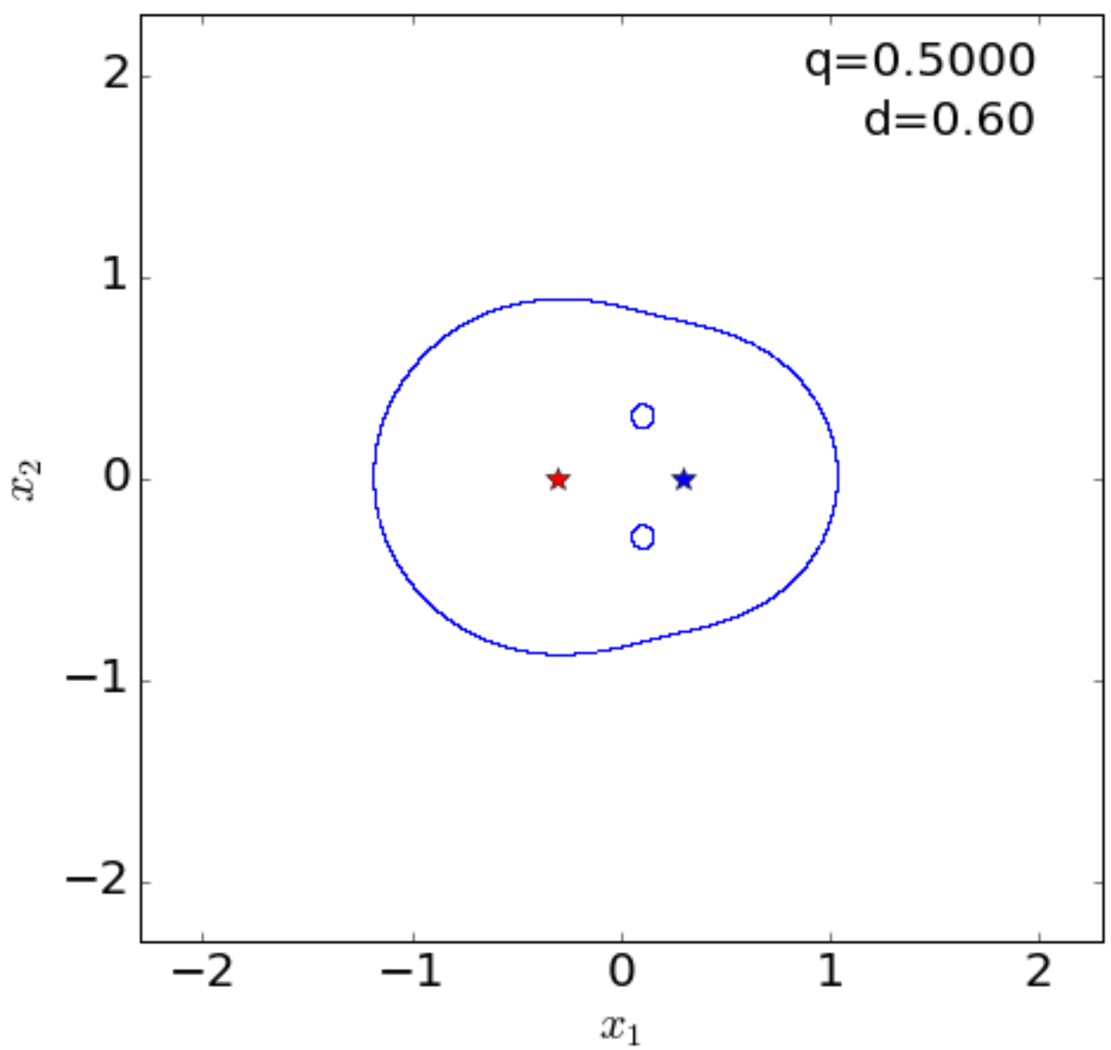
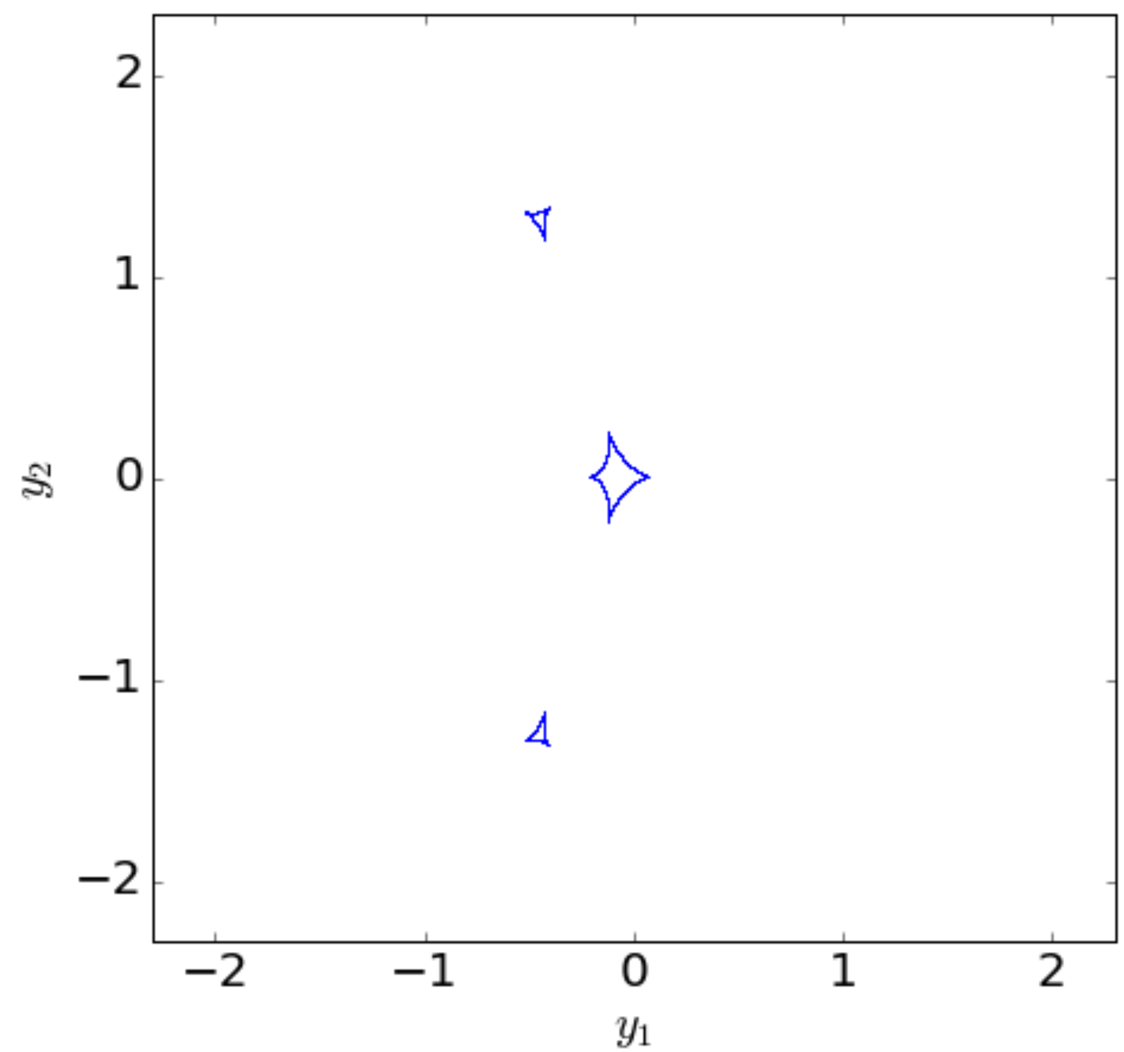


critical lines



caustics

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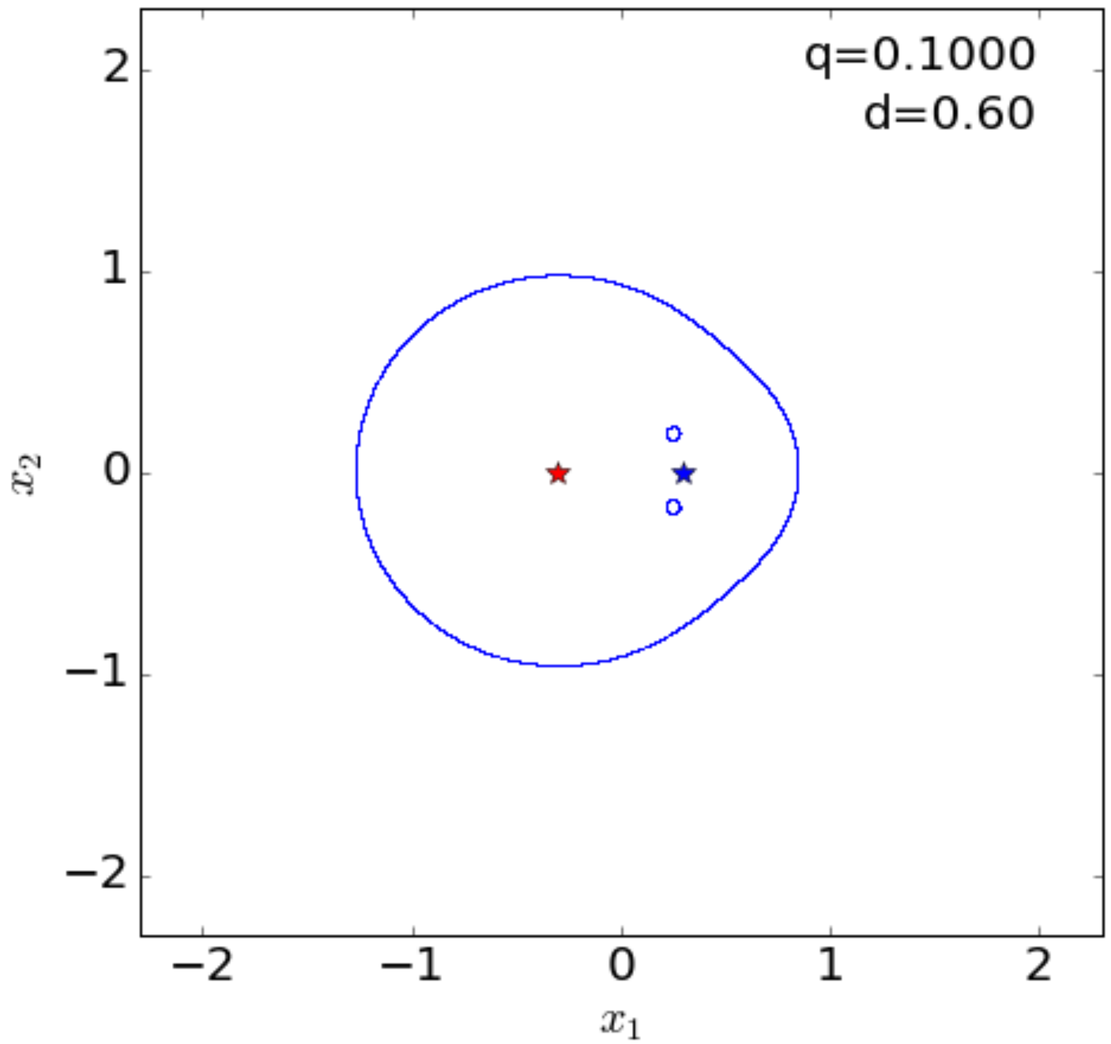
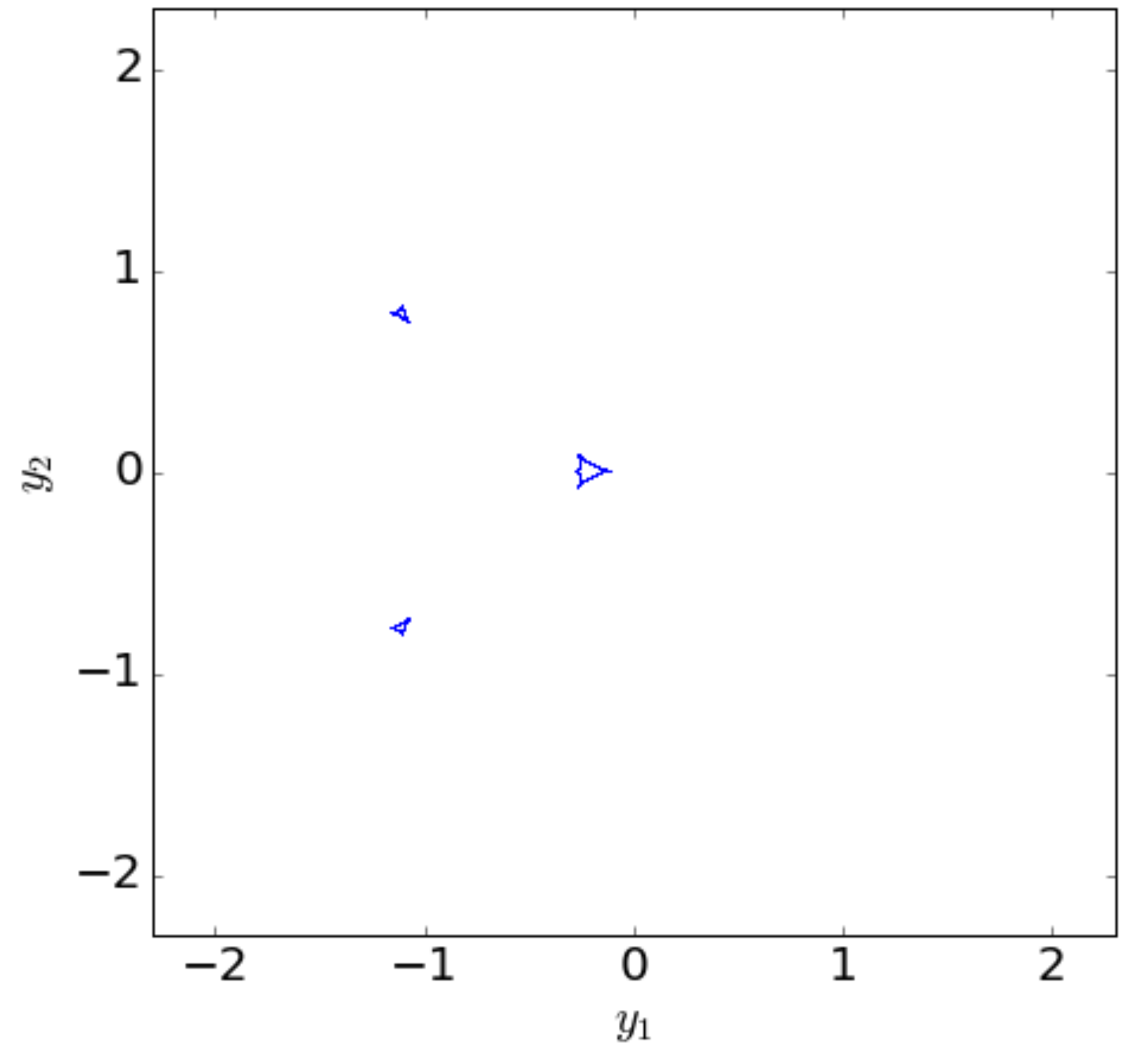


critical lines



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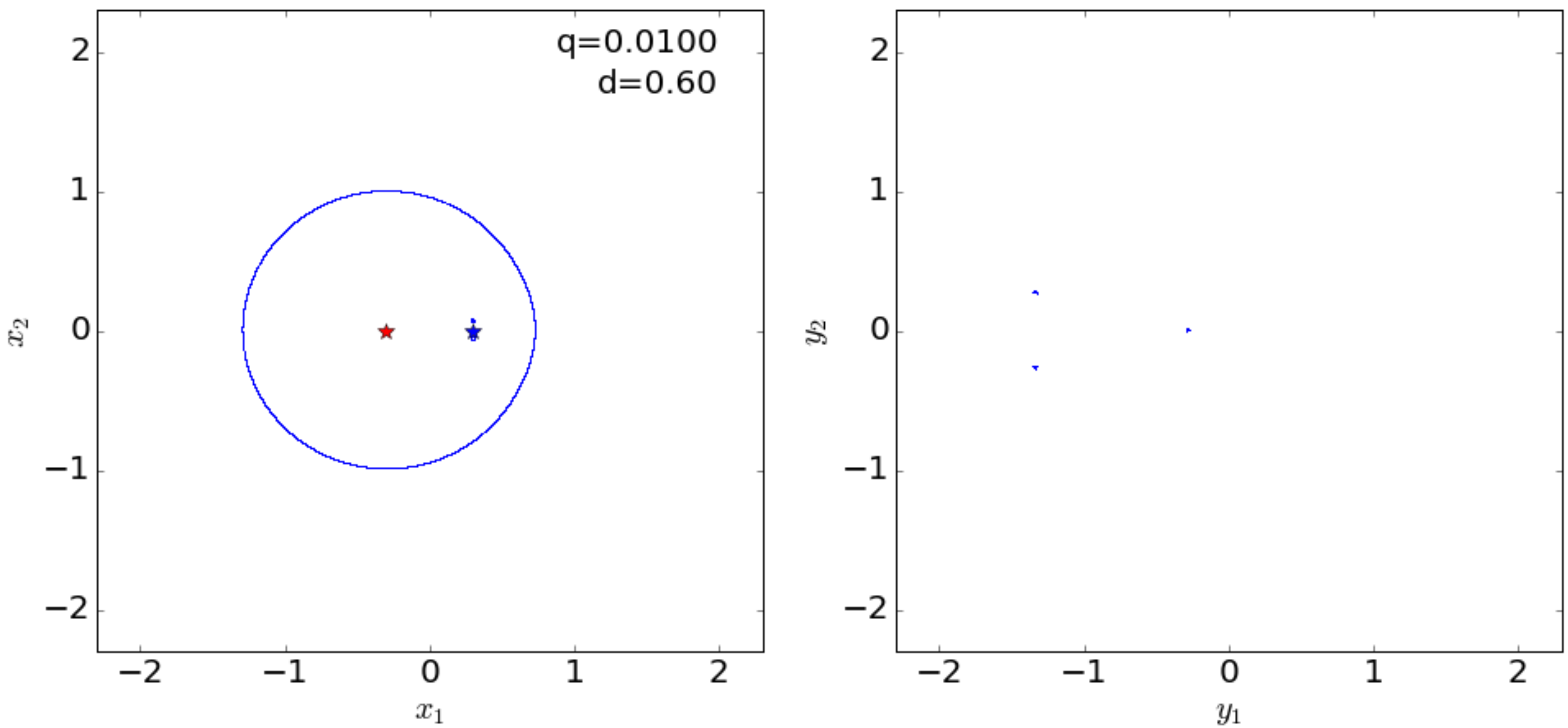

critical lines

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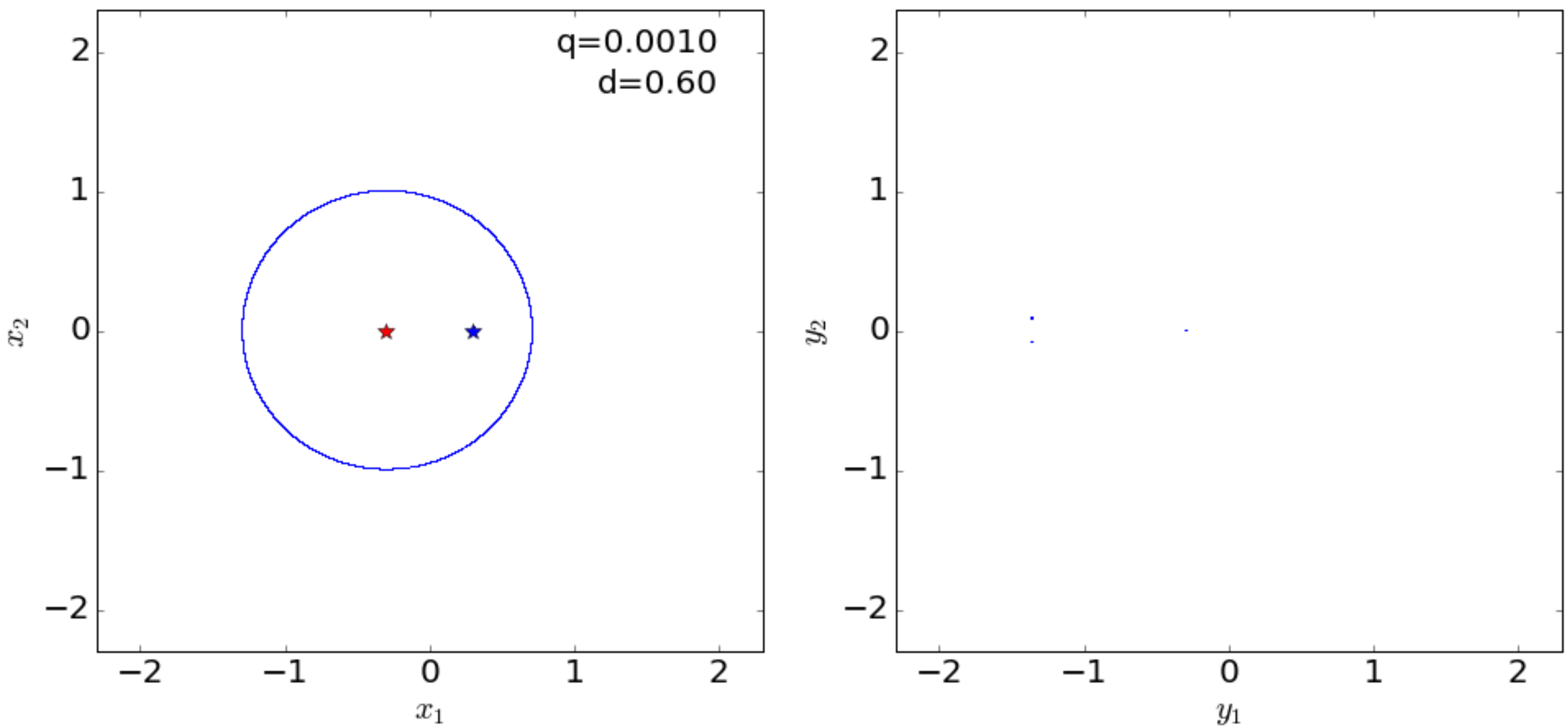

critical lines *caustics*

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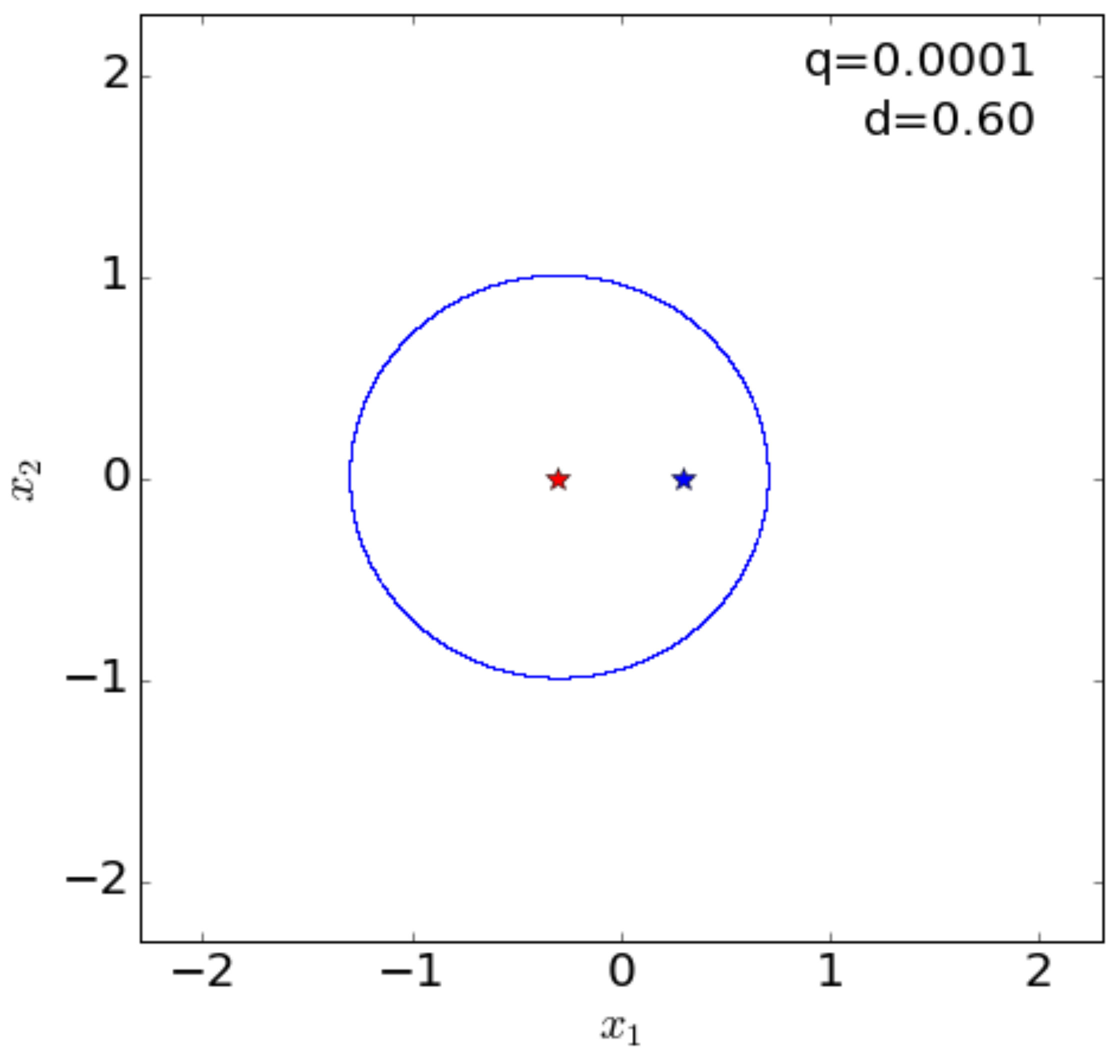
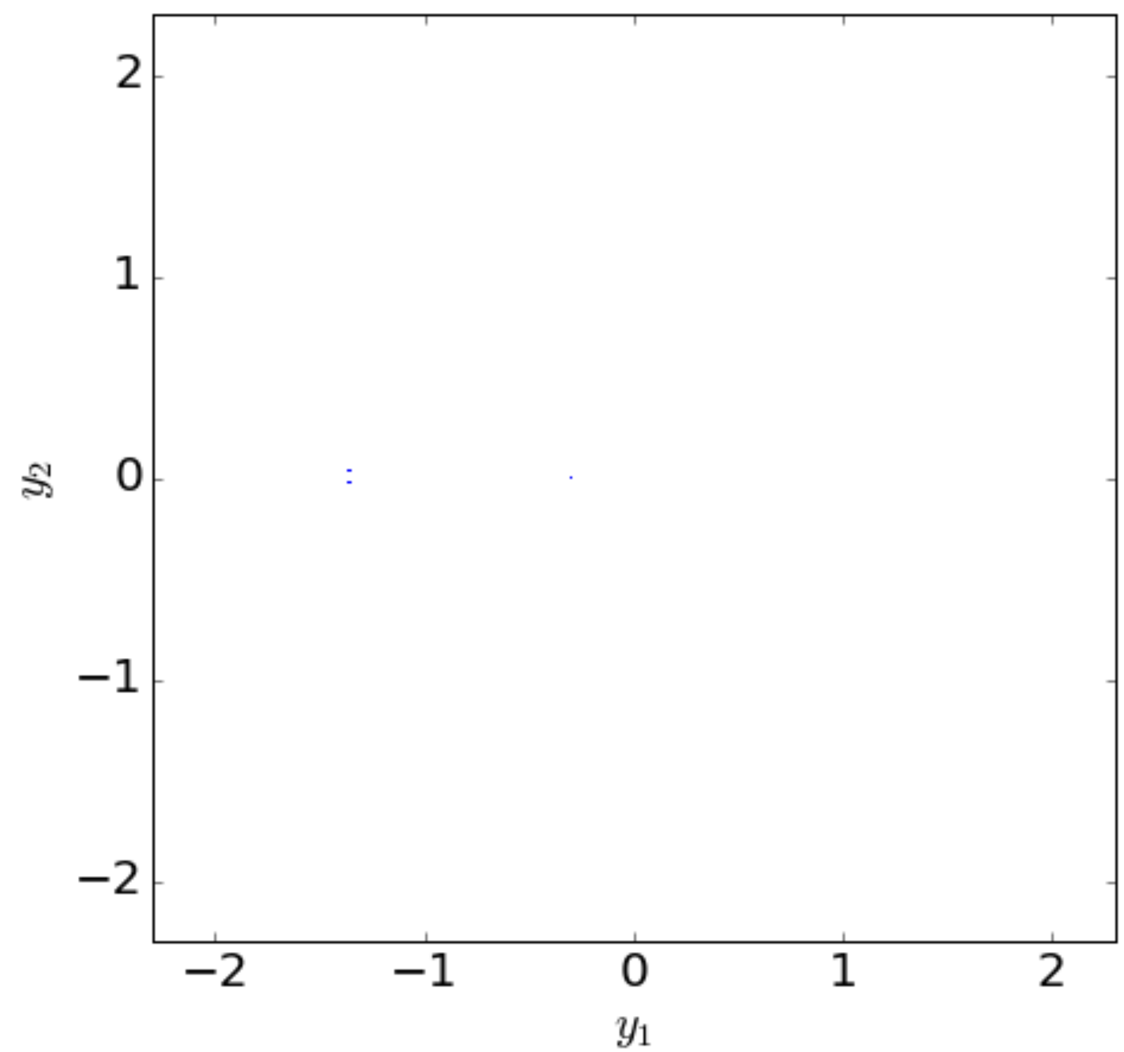
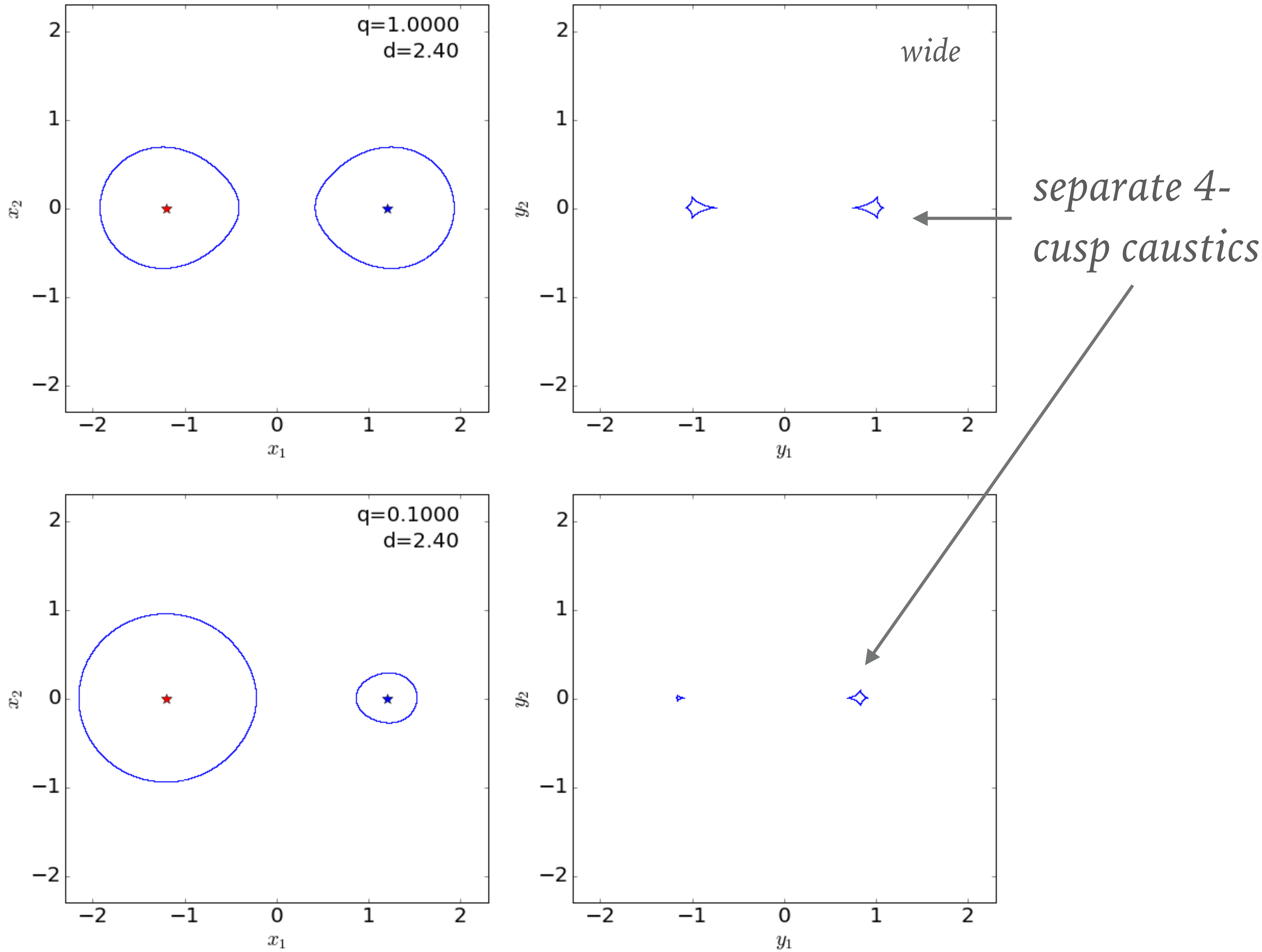


critical lines



caustics

BINARY LENSES: TOPOLOGY CLASSIFICATION



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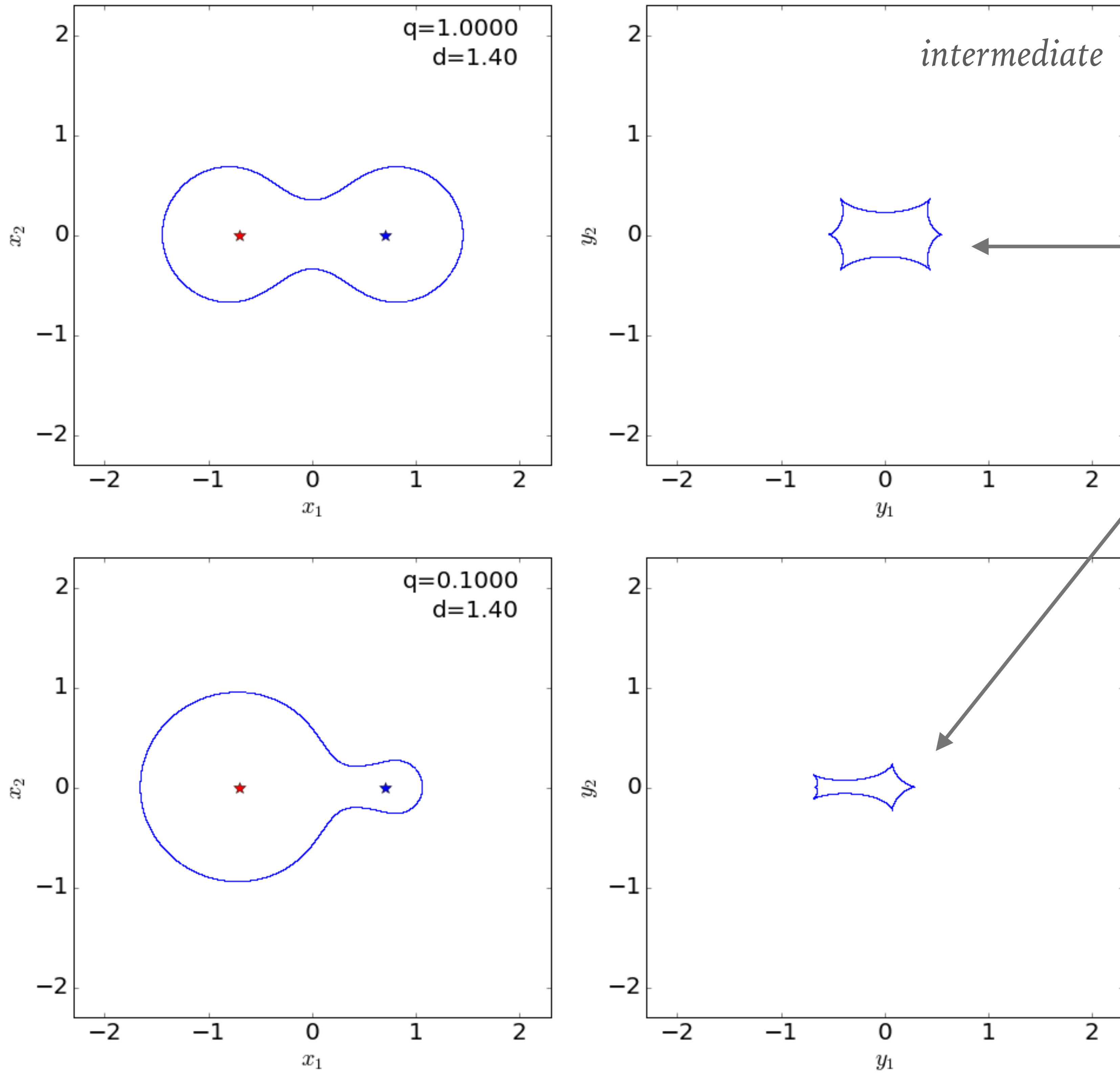
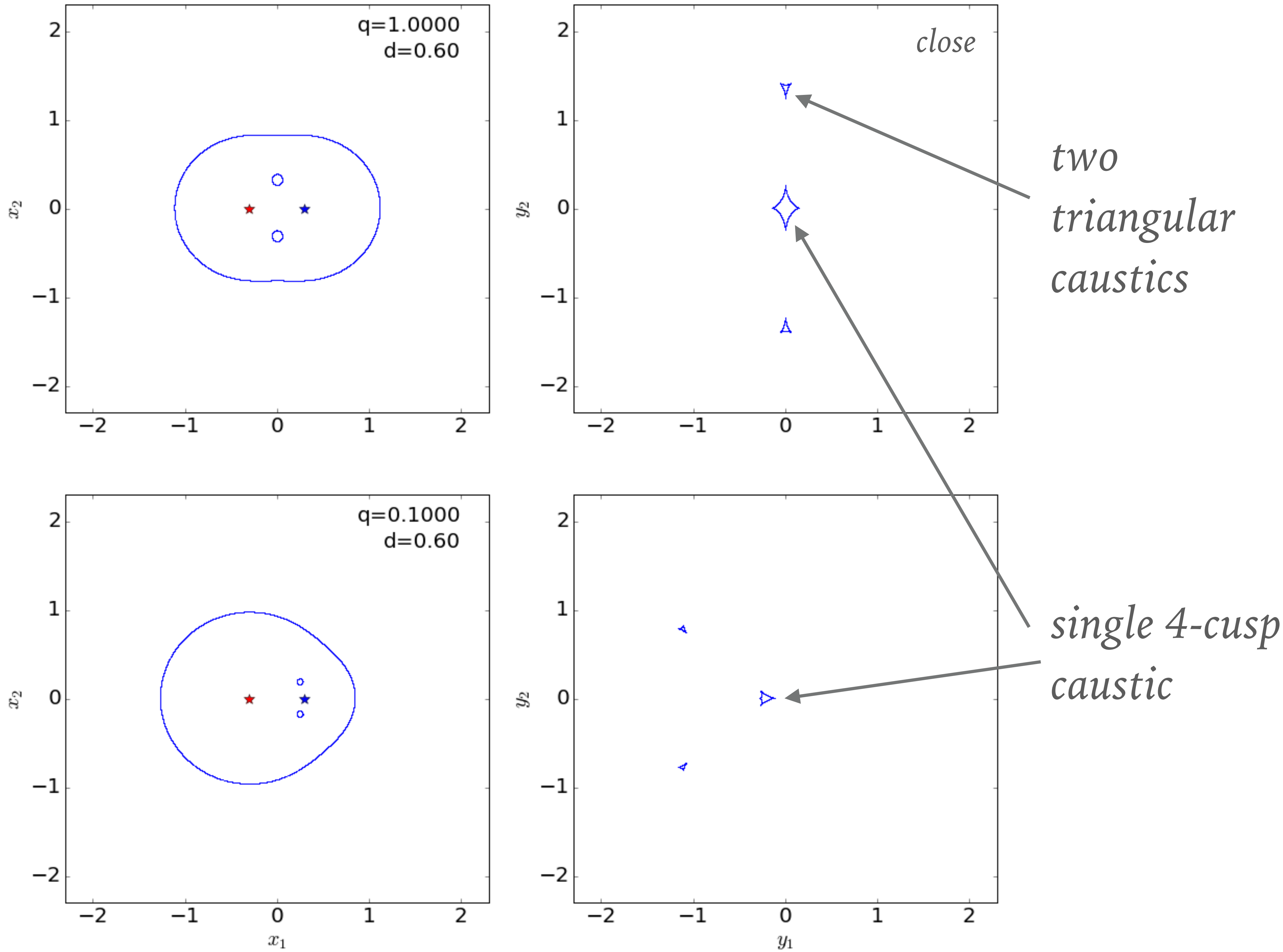


single 6-cusp caustic

BINARY LENSES: TOPOLOGY CLASSIFICATION

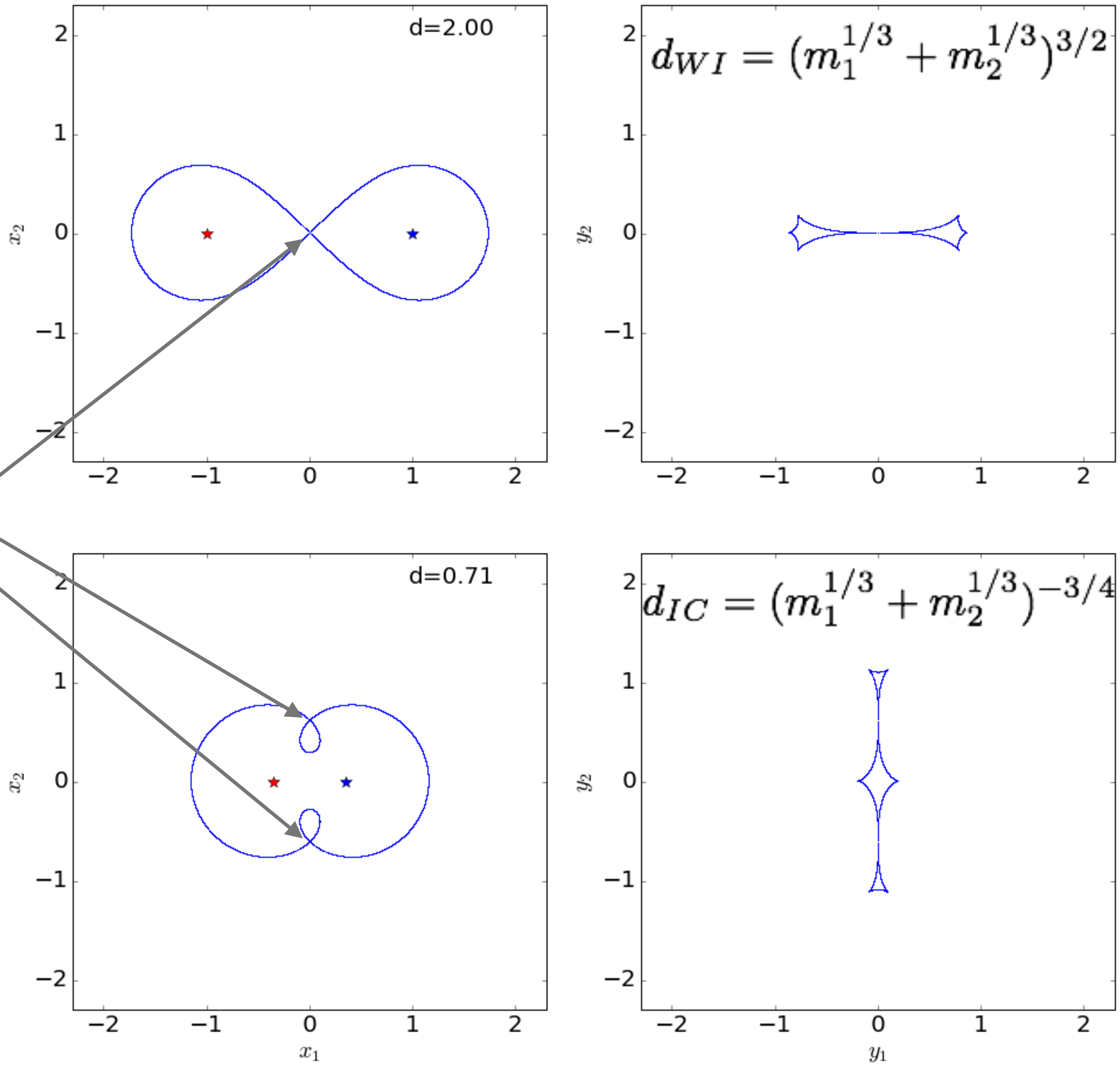


TRANSITIONS

Touching critical lines

$$\det A = 0$$

$$\frac{\partial \det A}{\partial z^*} = 0$$



MULTIPLE IMAGES

- Lens equation: $$z_s = z - \frac{m_1}{z^* - z_1^*} - \frac{m_2}{z^* - z_2^*}$$
- complex polynomial: $$p_5(z) = \sum_{i=0}^{5} c_i z^i$$

$$\Delta m = \frac{m_1 - m_2}{2} \qquad m = \frac{m_1 + m_2}{2} \qquad z_2 = -z_1 \qquad z_1 = z_1^*$$

$$c_0 = z_1^2[4(\Delta m)^2 z_s + 4m\Delta m z_1 + 4\Delta m z_s z_s^* z_1 + 2m z_s^* z_1^2 + z_s z_s^{*2} z_1^2 - 2\Delta m z_1^3 - z_s z_1^4]$$

$$c_1 = -8m\Delta m z_s z_1 - 4(\Delta m)^2 z_1^2 - 4m^2 z_1^2 - 4m z_s z_s^* z_1^2 - 4\Delta m z_s^* z_1^3 - z_s^{*2} z_1^4 + z_1^6$$

$$c_2 = 4m^2 z_s + 4m\Delta m z_1 - 4\Delta m z_s z_s^* z_1 - 2z_s z_s^{*2} z_1^2 + 4\Delta m z_1^3 + 2z_s z_1^4$$

$$c_3 = 4m z_s z_s^* + 4\Delta m z_s^* z_1 + 2z_s^{*2} z_1^2 - 2z_1^4$$

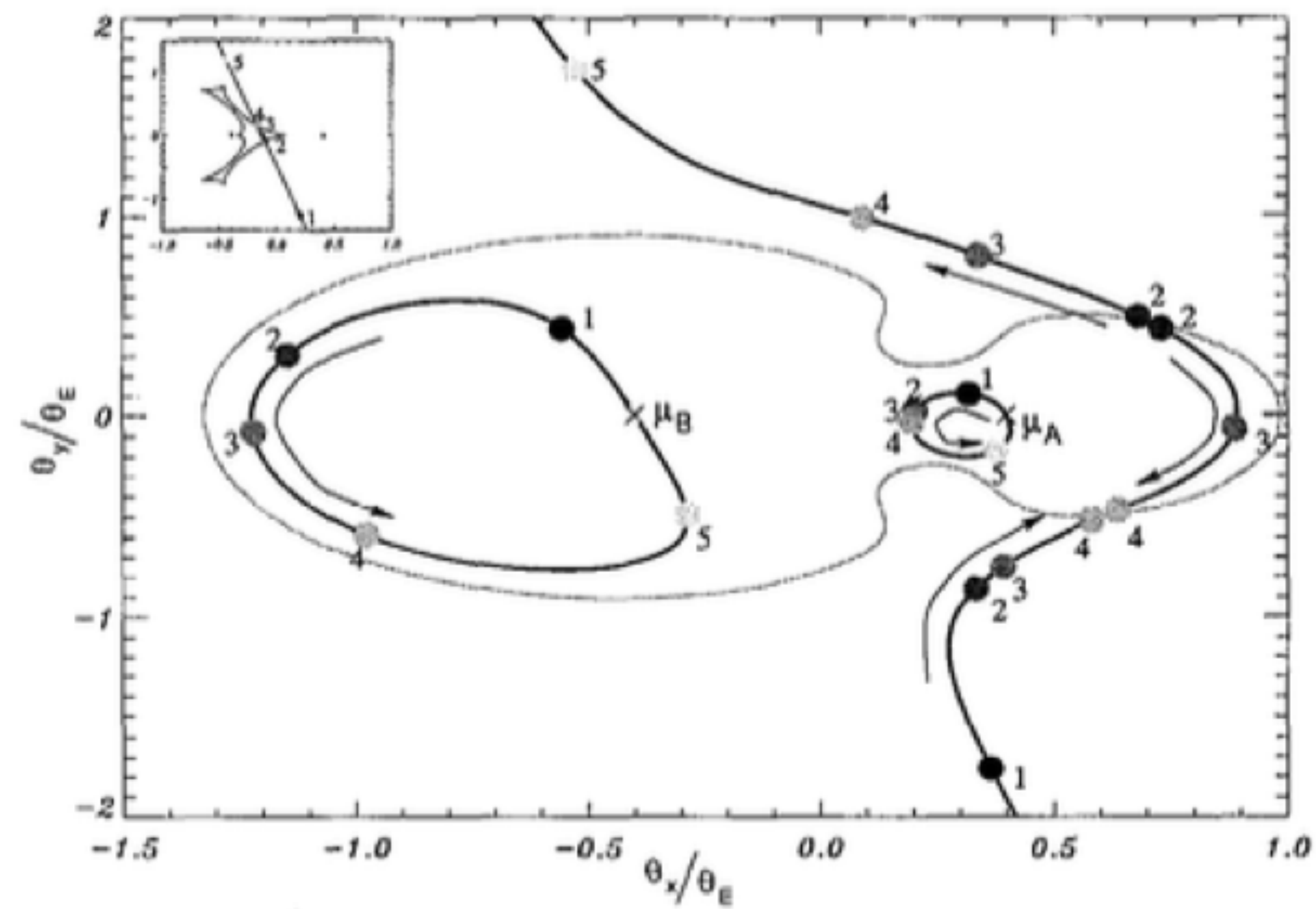
$$c_4 = -2m z_s^* + z_s z_s^{*2} - 2\Delta m z_1 - z_s z_1^2$$

$$c_5 = z_1^2 - z_s^{*2}$$

Witt & Mao, 1995, ApJ, 447, L105

- 3 or 5 images

MULTIPLE IMAGES



Mollerach & Roulet, "Gravitational Lensing and Microlensing"

IMAGE MAGNIFICATION

- magnification at the image position:

$$\mu = \det A^{-1} = \left[1 - \left|\frac{m_1}{(z^* - z_1^*)^2} + \frac{m_2}{(z^* - z_2^*)^2}\right|\right]^{-1}$$

- total magnification:

$$\mu_{tot} = \sum_{i=1}^{n_i} |\mu_i|$$

- of course, the magnification varies as a function of *z*…