GRAVITATIONAL LENSING

LECTURE 6

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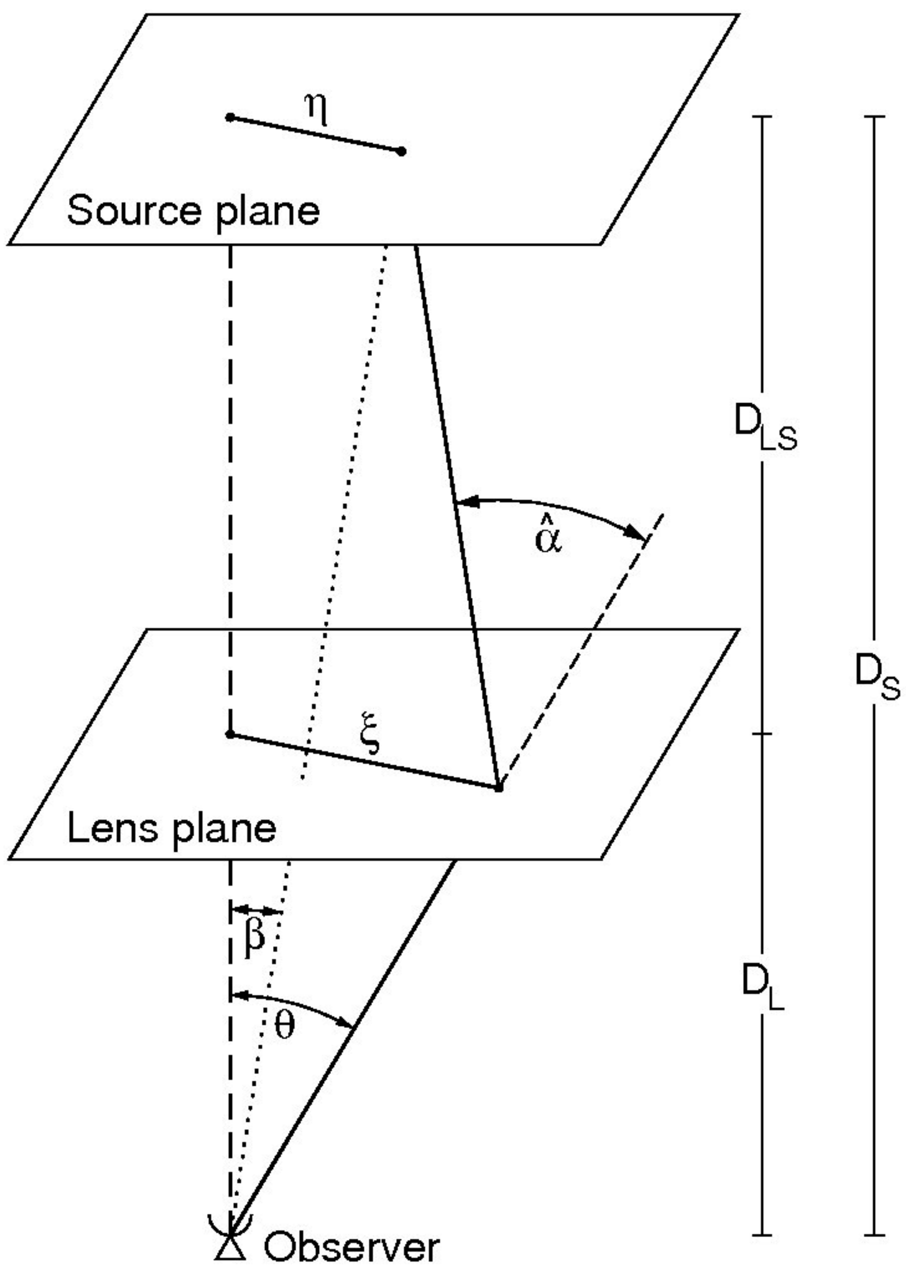
CONTENTS

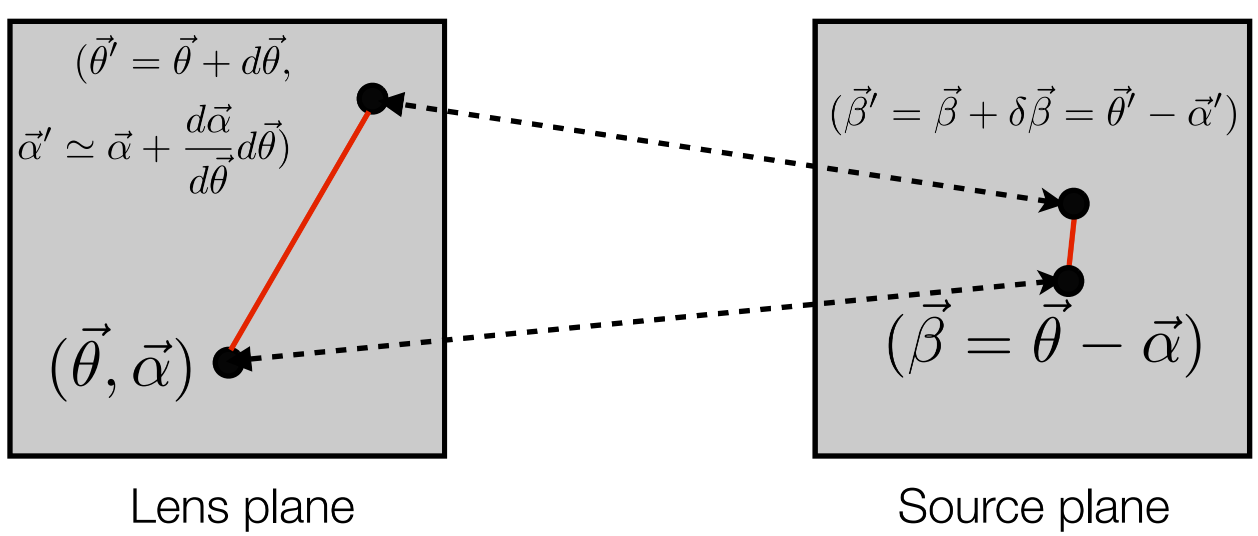
LENS MAPPING (FIRST ORDER)



- *we derived the lens equation*

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

- *Assuming that the d.a. does not vary significantly over the scale dΘ:*



$$(\vec{\beta}' - \vec{\beta}) = \left(I - \frac{d\vec{\alpha}}{d\vec{\theta}}\right)(\vec{\theta}' - \vec{\theta}) = A(\vec{\theta}' - \vec{\theta})$$

LENS MAPPING (FIRST ORDER)

$$A = \frac{\partial \beta}{\partial \theta} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_i \theta_j} \right)$$

*A is called "**the lensing Jacobian**": it is a symmetric second rank tensor describing the first order mapping between lens and source planes.*

This tensor can be written as the sum of an isotropic part, proportional to its trace, and an anisotropic traceless part.

$$A_{iso,i,j} = \frac{1}{2} \mathrm{Tr} A \delta_{i,j}$$

$$A_{aniso,i,j} = A_{i,j} - \frac{1}{2} \mathrm{Tr} A \delta_{i,j}$$

ANISOTROPIC PART

$$A_{aniso,i,j} = A_{i,j} - \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

$$\begin{aligned}\left(A - \frac{1}{2}\mathrm{tr}A \cdot I\right)_{ij} &= \delta_{ij} - \Psi_{ij} - \frac{1}{2}(1 - \Psi_{11} + 1 - \Psi_{22})\delta_{ij} \\ &= -\Psi_{ij} + \frac{1}{2}(\Psi_{11} + \Psi_{22})\delta_{ij} \\ &= \begin{pmatrix} -\frac{1}{2}(\Psi_{11} - \Psi_{22}) & -\Psi_{12} \\ -\Psi_{12} & \frac{1}{2}(\Psi_{11} - \Psi_{22}) \end{pmatrix}\end{aligned}$$

*Introducing the **shear**:*

$$\gamma_1 = \frac{1}{2}\left(\Psi_{11} - \Psi_{22}\right)$$

$$\gamma_2 = -\Psi_{12} = -\Psi_{21}$$

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

Symmetric, trace-less tensor

ISOTROPIC PART

$$A_{iso,i,j} = \frac{1}{2}\mathrm{Tr}A\delta_{i,j}$$

$$\begin{aligned} \frac{1}{2}\mathrm{tr}A \cdot I &= \left[1 - \frac{1}{2}(\Psi_{11} + \Psi_{22})\right]\delta_{ij} \\ &= \left(1 - \frac{1}{2}\triangle\Psi\right)\delta_{ij} = (1-\kappa)\delta_{ij} \end{aligned}$$

Remember: $\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$

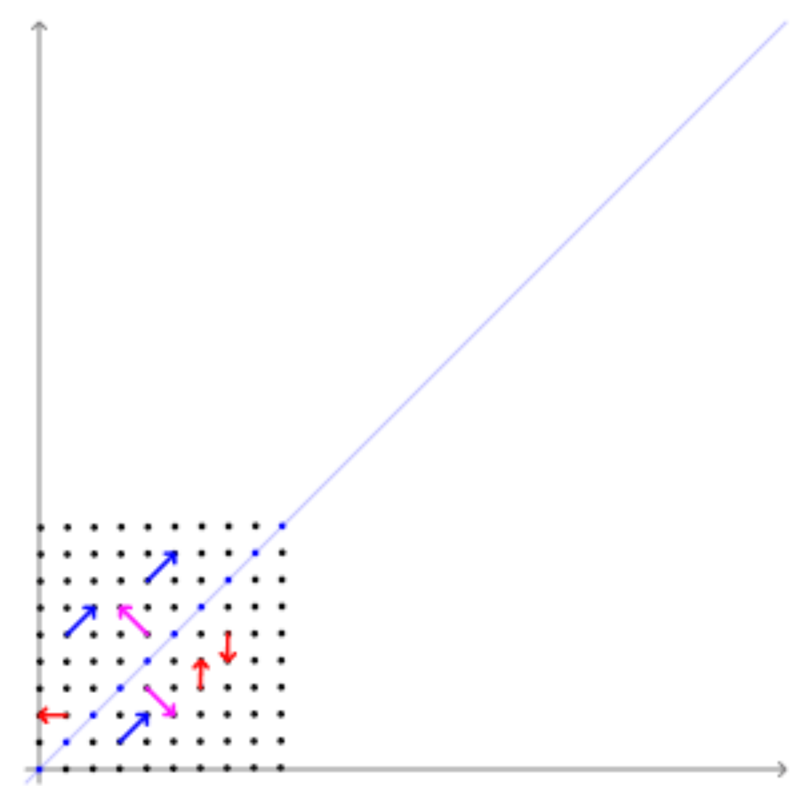
LENSING JACOBIAN

$$A = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

Lens mapping at first order is a linear application, distorting areas.

Distortion directions are given by the ***eigenvectors*** *of A.*

Distortion amplitudes in these directions are given by the ***eigenvalues****.*



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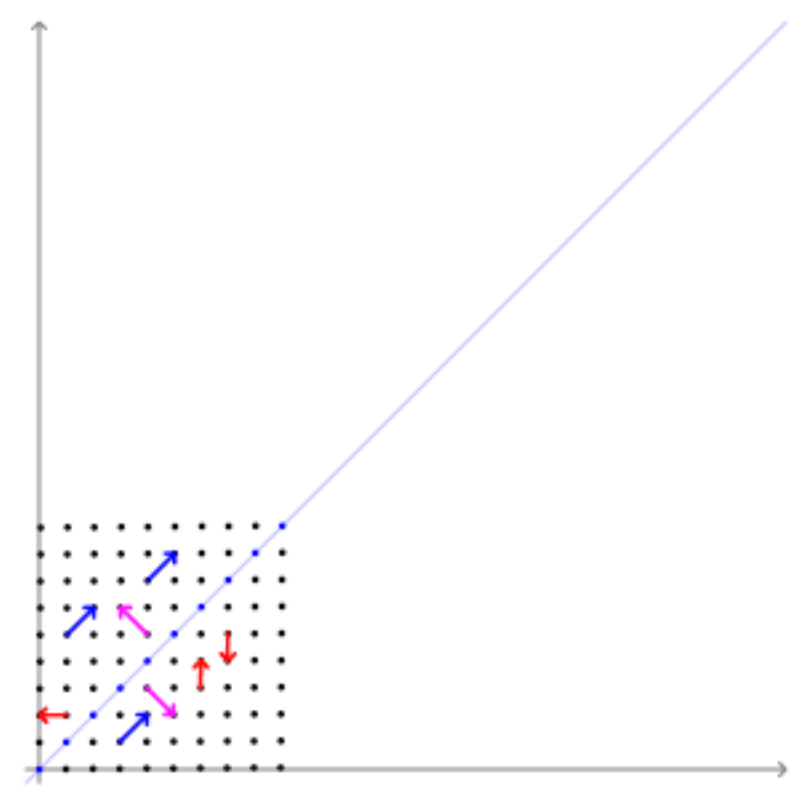


EIGENVALUES

$$\det A = (1-\kappa-\gamma)(1-\kappa+\gamma)$$
$$\gamma = \sqrt{\gamma_1^2+\gamma_2^2}$$

There is thus an orthogonal coordinate transformation $R(\varphi)$, a rotation by an angle φ, which brings the Jacobian matrix into diagonal form.

Generally, the Jacobian matrix transforms as

$$A \to A' = R(\varphi)^T A R(\varphi)$$

This shows that the shear components transform under coordinate rotations as

$$\gamma_1 \to \gamma_1' = \gamma_1 \cos(2\varphi) + \gamma_2 \sin(2\varphi)$$
$$\gamma_2 \to \gamma_2' = -\gamma_1 \sin(2\varphi) + \gamma_2 \cos(2\varphi)$$

i.e. unlike a vector! Since the shear components are mapped onto each other after rotations of $\varphi = \pi$ rather than $\varphi = 2\pi$, they form a so-called spin-2 field.

EXAMPLE: FIRST ORDER DISTORTION OF A CIRCULAR SOURCE

$$\beta_1^2 + \beta_2^2 = \beta^2$$

In the reference frame where A is diagonal:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma & 0 \\ 0 & 1-\kappa+\gamma \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

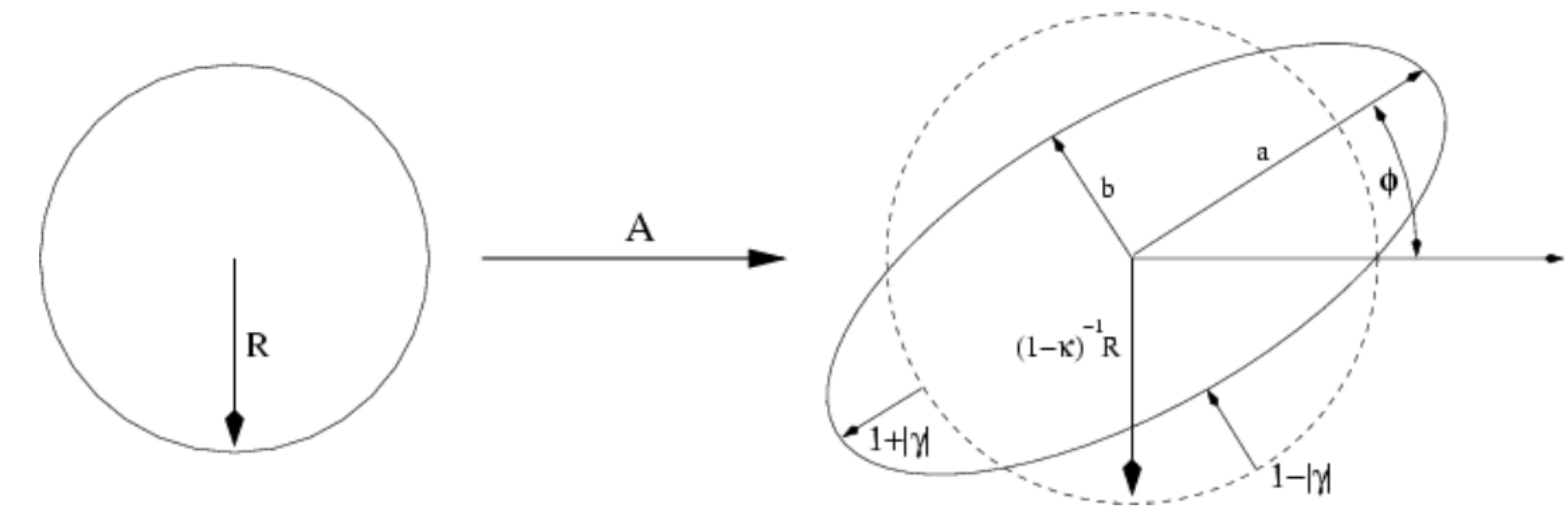
$$\begin{aligned} \beta_1 &= (1-\kappa-\gamma)\theta_1 \\ \beta_2 &= (1-\kappa+\gamma)\theta_2 \end{aligned}$$

$$\beta^2 = \beta_1^2 + \beta_2^2 = (1-\kappa-\gamma)^2\theta_1^2 + (1-\kappa+\gamma)^2\theta_2^2$$

This is the equation of an ellipse with semi-axes:

$$a = \frac{\beta}{1-\kappa-\gamma} \qquad b = \frac{\beta}{1-\kappa+\gamma}$$

EXAMPLE: FIRST ORDER DISTORTION OF A CIRCULAR SOURCE

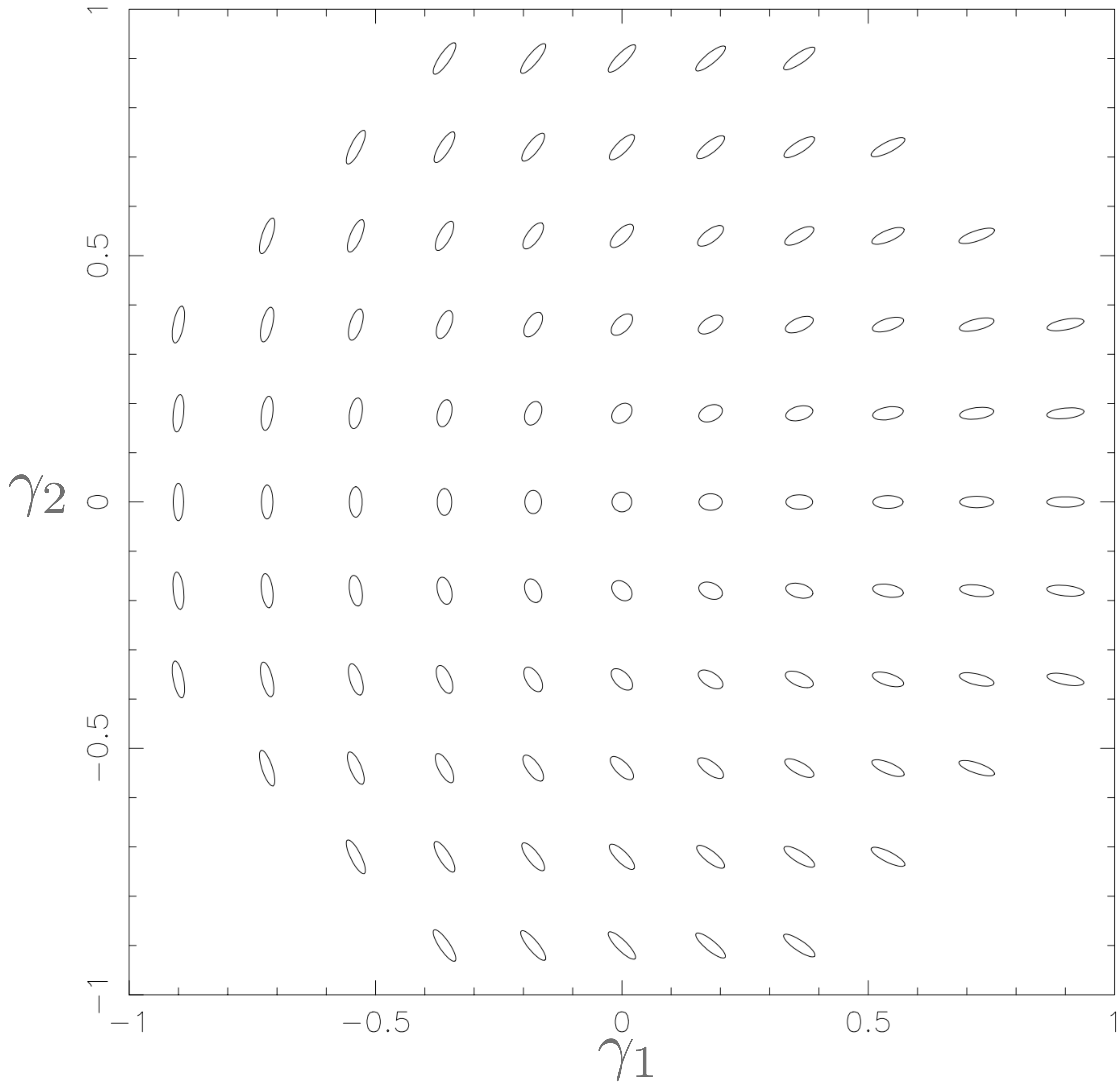


convergence: *responsible for isotropic expansion or contraction*

shear: *responsible for anisotropic distortion*

Ellipticity: $$e = \frac{a-b}{a+b} = \frac{\gamma}{1-\kappa} = g$$

SHEAR DISTORTIONS



CONSERVATION OF SURFACE BRIGHTNESS

The source ***surface brightness*** *is*

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

In phase space, the radiation emitted is characterized by the density

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p}$$

In absence of photon creations or absorptions, f is conserved (Liouville theorem)

$$dN = \frac{dE}{h\nu} = \frac{dE}{cp}$$

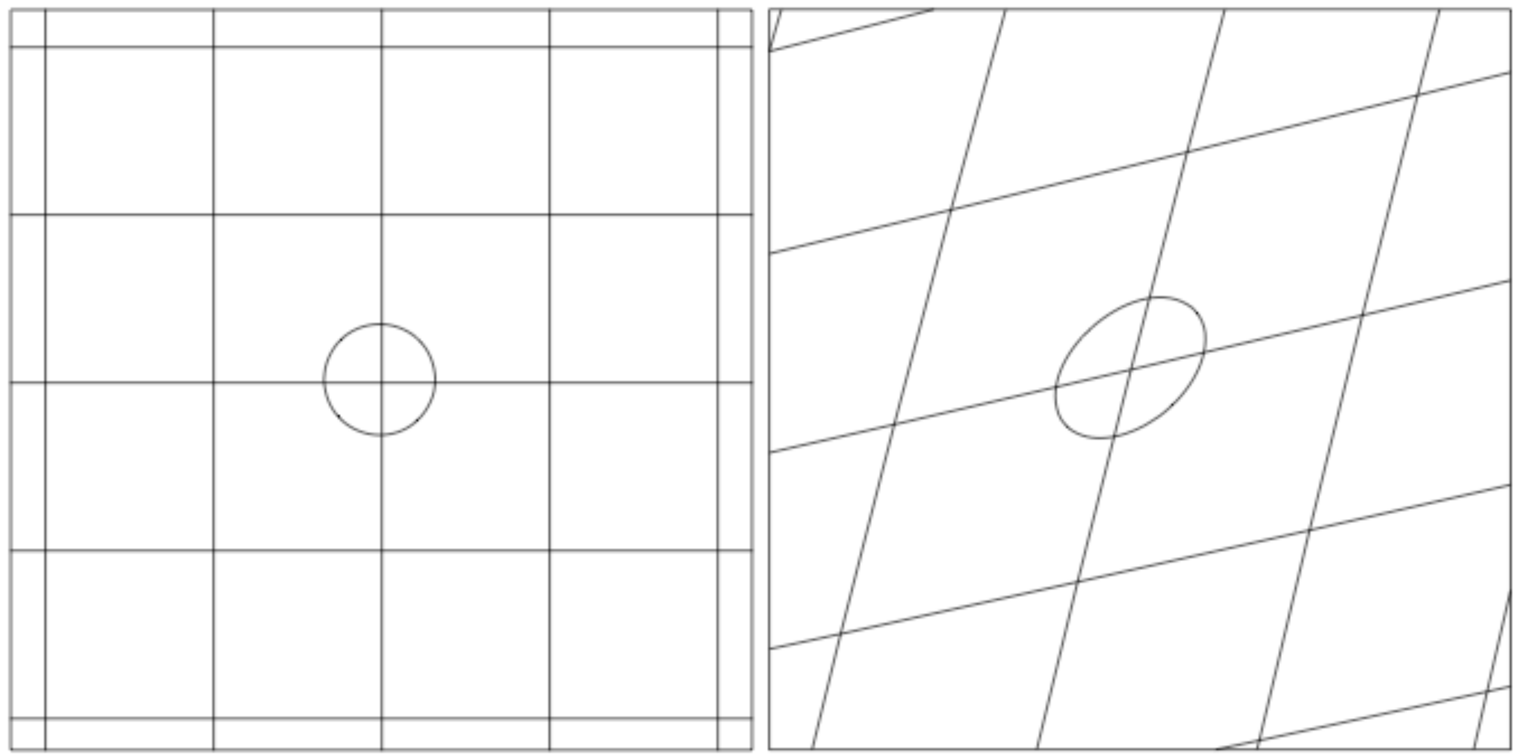
$$d^3x = c dt dA$$

$$d^3\vec{p} = p^2 dp d\Omega$$

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p} = \frac{dE}{hcp^3 dA dt d\nu d\Omega} = \frac{I_\nu}{hcp^3}$$

Since GL does not involve creation or absorption of photons, neither it changes the photon momenta (achromatic!), surface brightness is conserved!

MAGNIFICATION



Kneib & Natarajan (2012)

$$F_\nu = \int_I I_\nu(\vec{\theta}) d^2\theta = \int_S I_\nu^S[\vec{\beta}(\vec{\theta})]\mu d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

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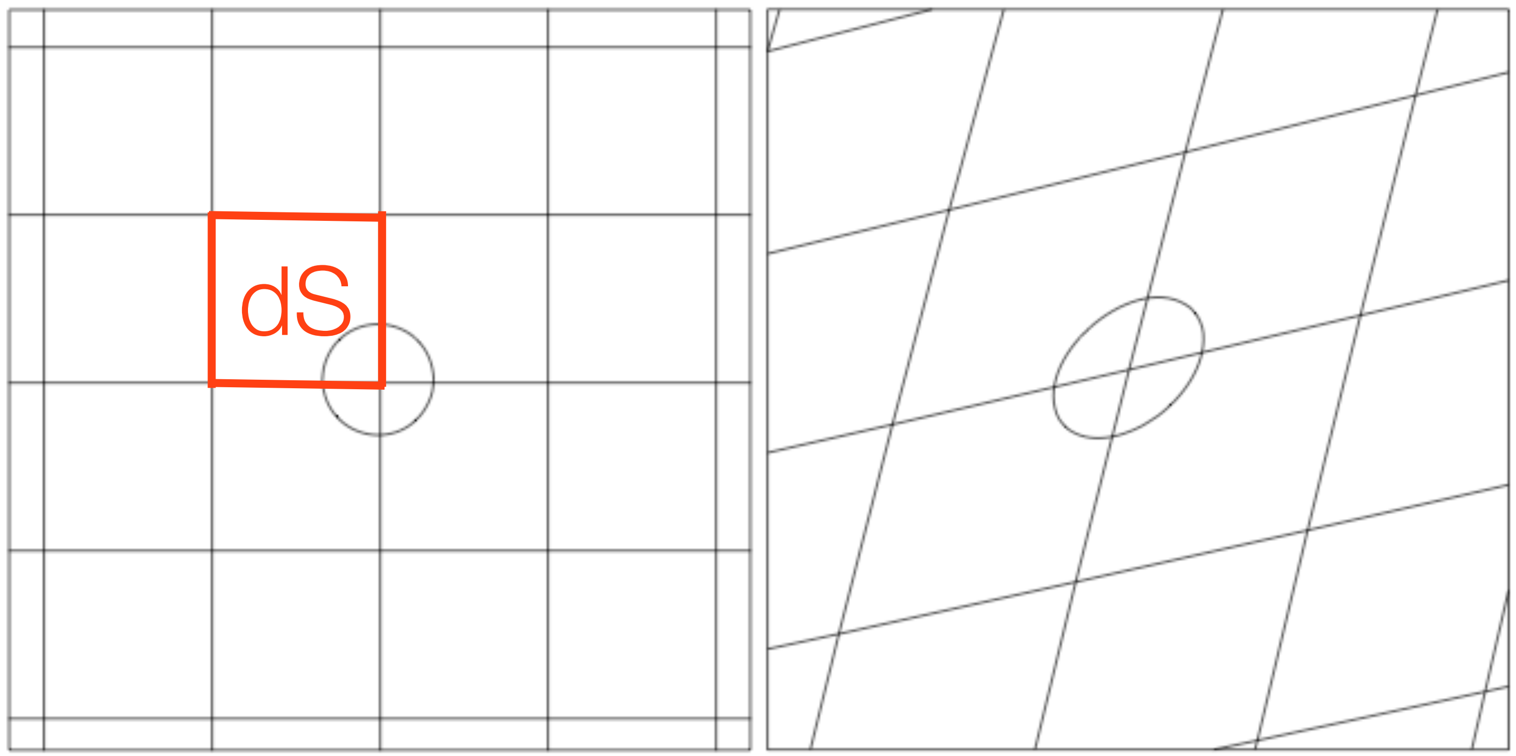

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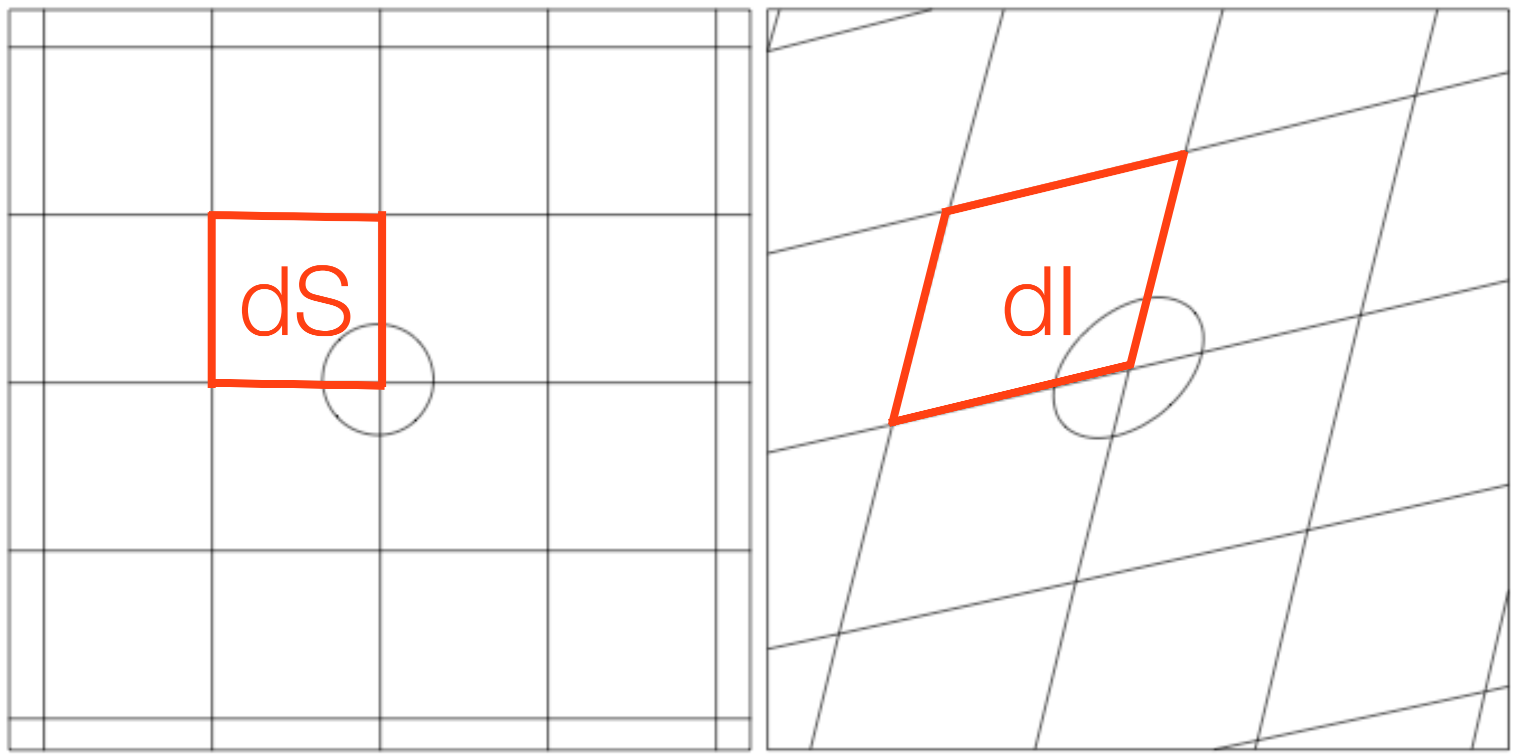

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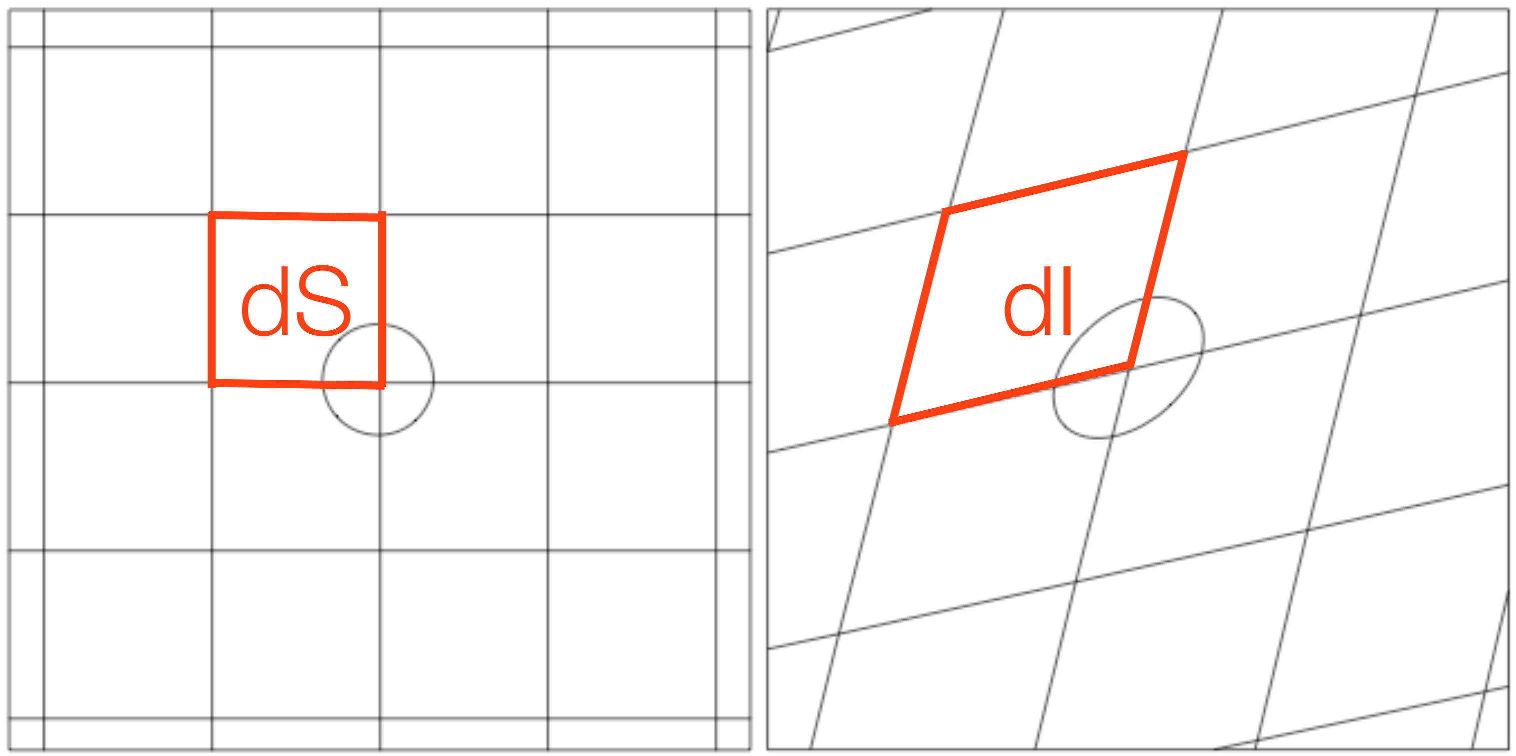


Kneib & Natarajan (2012)

$$\boxed{\mu = \frac{dI}{dS} = \frac{\delta\theta^2}{\delta\beta^2} = \det A^{-1}}$$

$$F_\nu = \int_I I_\nu(\vec{\theta})d^2\theta = \int_S I_\nu^S[\vec{\beta}(\vec{\theta})]\mu d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

CRITICAL LINES AND CAUSTICS

Both convergence and shear are functions of position on the lens plane:

$\kappa = \kappa(\vec{\theta})$

$\gamma = \gamma(\vec{\theta})$

The determinant of the lensing Jacobian is

$$\det A = (1 - \kappa - \gamma)(1 - \kappa + \gamma) = \mu^{-1}$$

*The **critical lines** are the lines where the eigenvalues of the Jacobian are zero:*

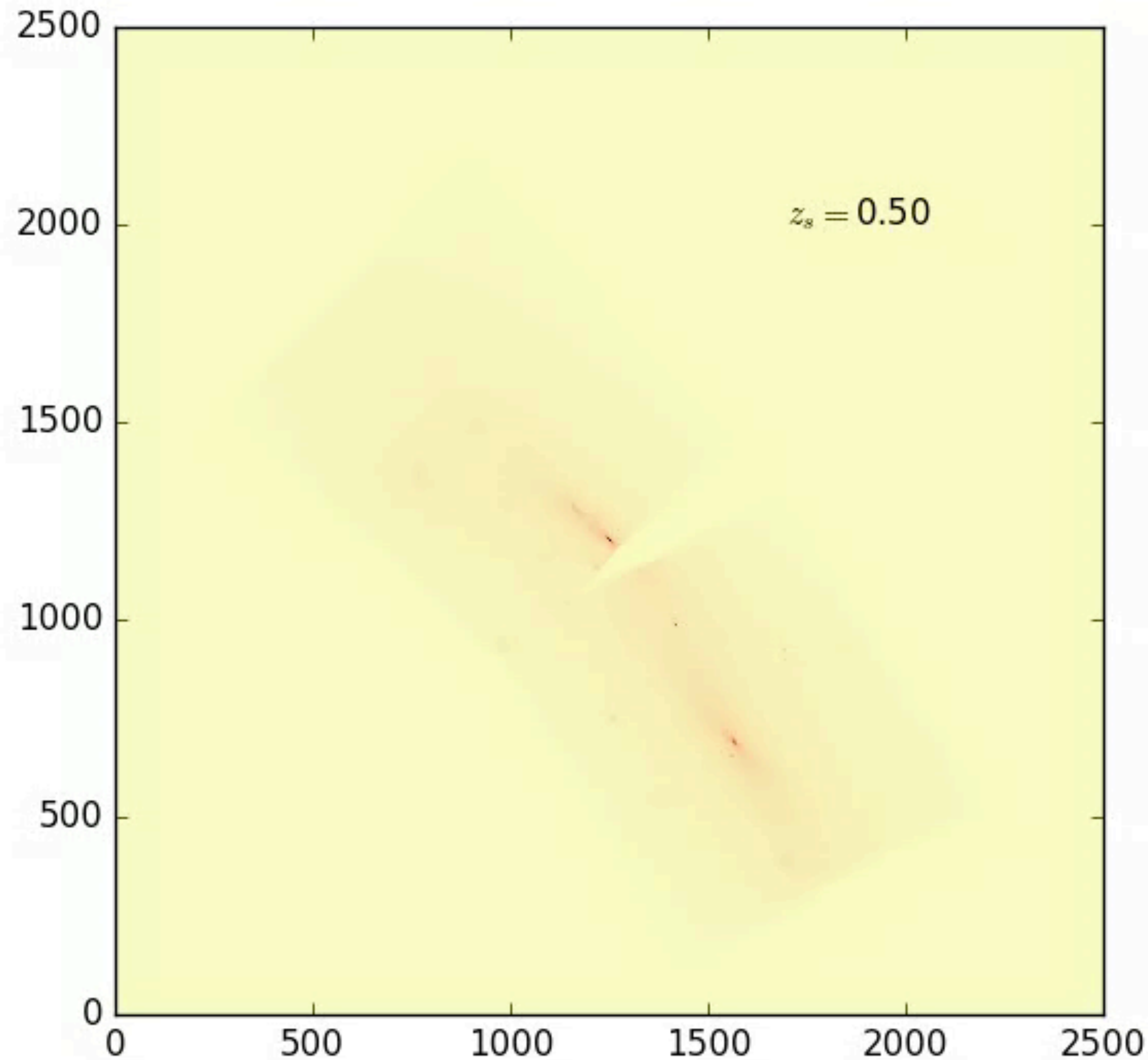
$(1 - \kappa - \gamma) = 0$ *tangential critical line*

$(1 - \kappa + \gamma) = 0$ *radial critical line*

Along these lines the magnification diverges!

*Via the lens equations, they are mapped into the **caustics**…*

VISUALIZING THE CAUSTICS



VISUALIZING THE CAUSTICS