GRAVITATIONAL LENSING

LECTURE 21

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AA 2016-2017

CONTENTS

THE SINGULAR ISOTHERMAL SPHERE

The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is in isothermal and hydrostatic equilibrium, its density profile is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

velocity dispersion of the gas particles

The profile is "unphysical"

- *singularity near the center*
- *mass is infinite*

THE SINGULAR ISOTHERMAL SPHERE

For lensing purposes, we are interested in the projection of this profile:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

$$\begin{aligned}
\Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\
&= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\
&= \frac{\sigma_v^2}{2G\xi} \,.
\end{aligned}$$

THE SINGULAR ISOTHERMAL SPHERE

As usual, we can switch to dimensionless units.

Let's take $$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_L D_{LS}}{D_S}$$

Then: $$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}} = \frac{1}{2x}\Sigma_{cr}\,.$$

$$\kappa(x) = \frac{1}{2x}$$

Thus, the SIS lens is a power-law lens with n=2!

THE SINGULAR ISOTHERMAL SPHERE

The mass profile is readily computed:

$$m(x) = |x|$$

as well as the deflection angle:

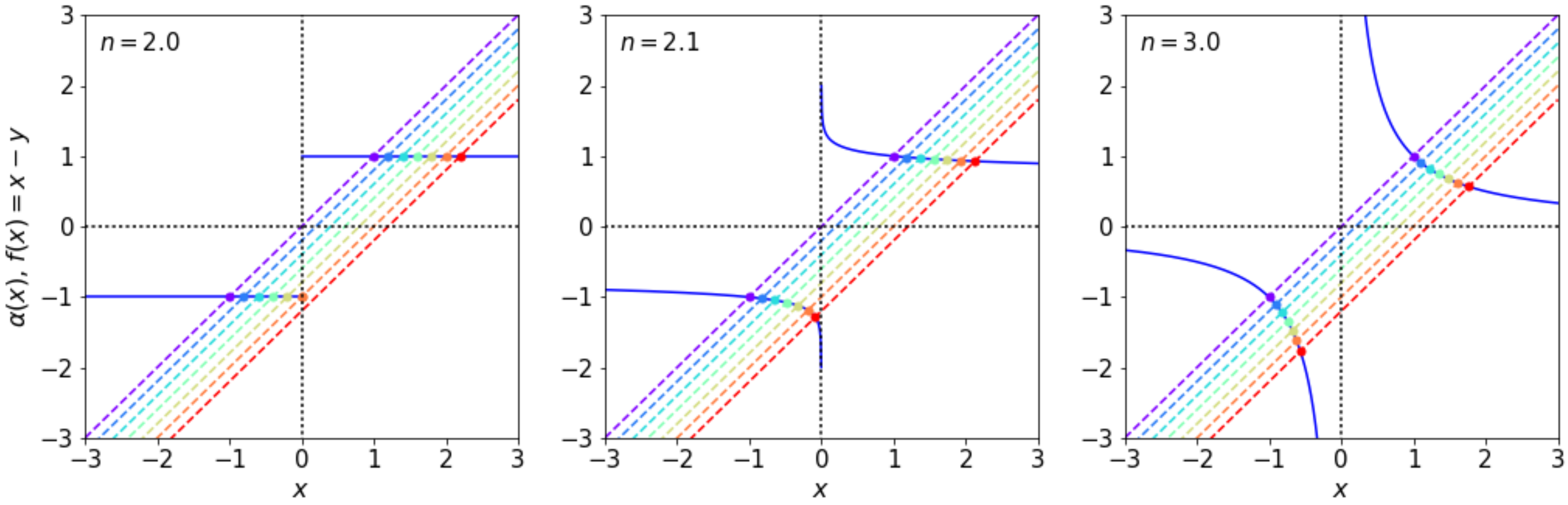
$$\alpha(x) = \frac{x}{|x|}$$

The lens equation reads

$$y = x - \frac{x}{|x|}$$

How many solutions does this equation have?

IMAGE DIAGRAM (N>2)



THE SINGULAR ISOTHERMAL SPHERE

If $0<y<1$, the solution are two:

$$x_- = y - 1$$

$$x_+ = y + 1$$

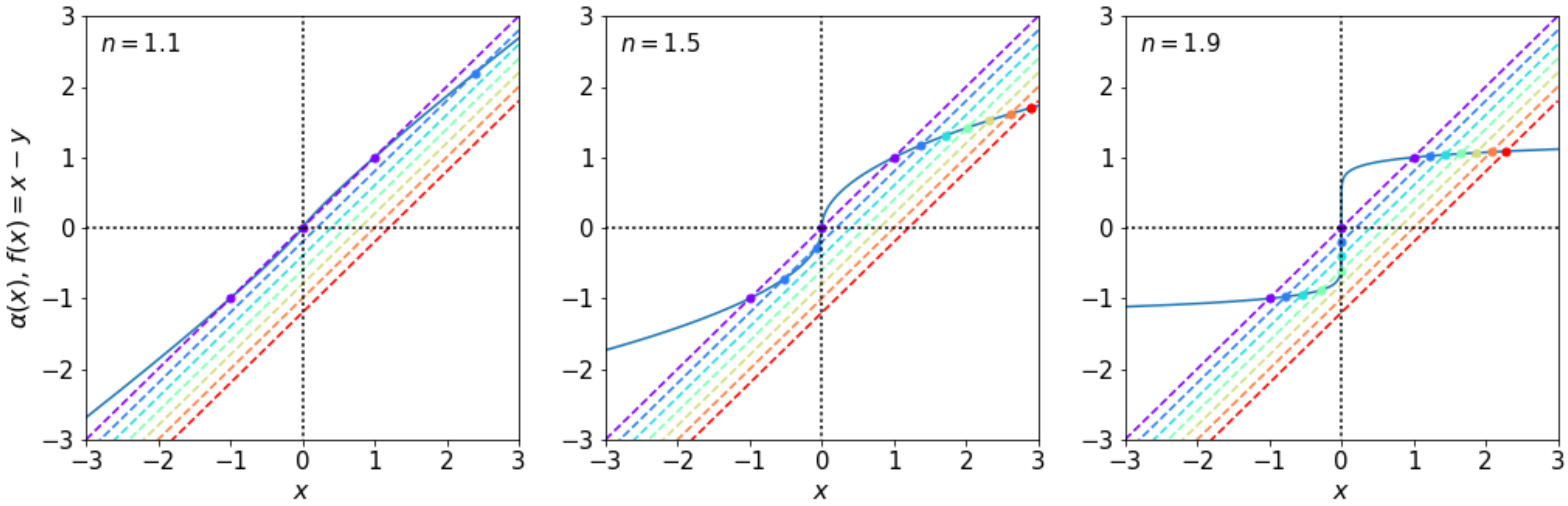
$$\theta_\pm = \beta \pm \theta_E$$

Otherwise, there is only one solution at

$$x_+ = y + 1$$

Thus, the circle of radius $y=1$ plays the same role of the radial caustic for the power-law lens with $n<2$, separating the source plane into regions with different image multiplicity.

IMAGE DIAGRAM



radial critical line:

$$\left.\frac{d\alpha(x)}{dx}\right|_{x_r} = 1$$

THE SINGULAR ISOTHERMAL SPHERE

On the other hand, for the SIS: $d\bar{\alpha}/dx = 0$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1.$

Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.

THE SINGULAR ISOTHERMAL SPHERE

The shear can be computed easily:

$$\gamma(x) = \frac{m(x)}{x} - \kappa(x) = \frac{1}{2x}$$

$$\begin{aligned} \gamma_1 &= -\frac{1}{2}\frac{\cos 2\phi}{x} \\ \gamma_2 &= -\frac{1}{2}\frac{\sin 2\phi}{x} \end{aligned}$$

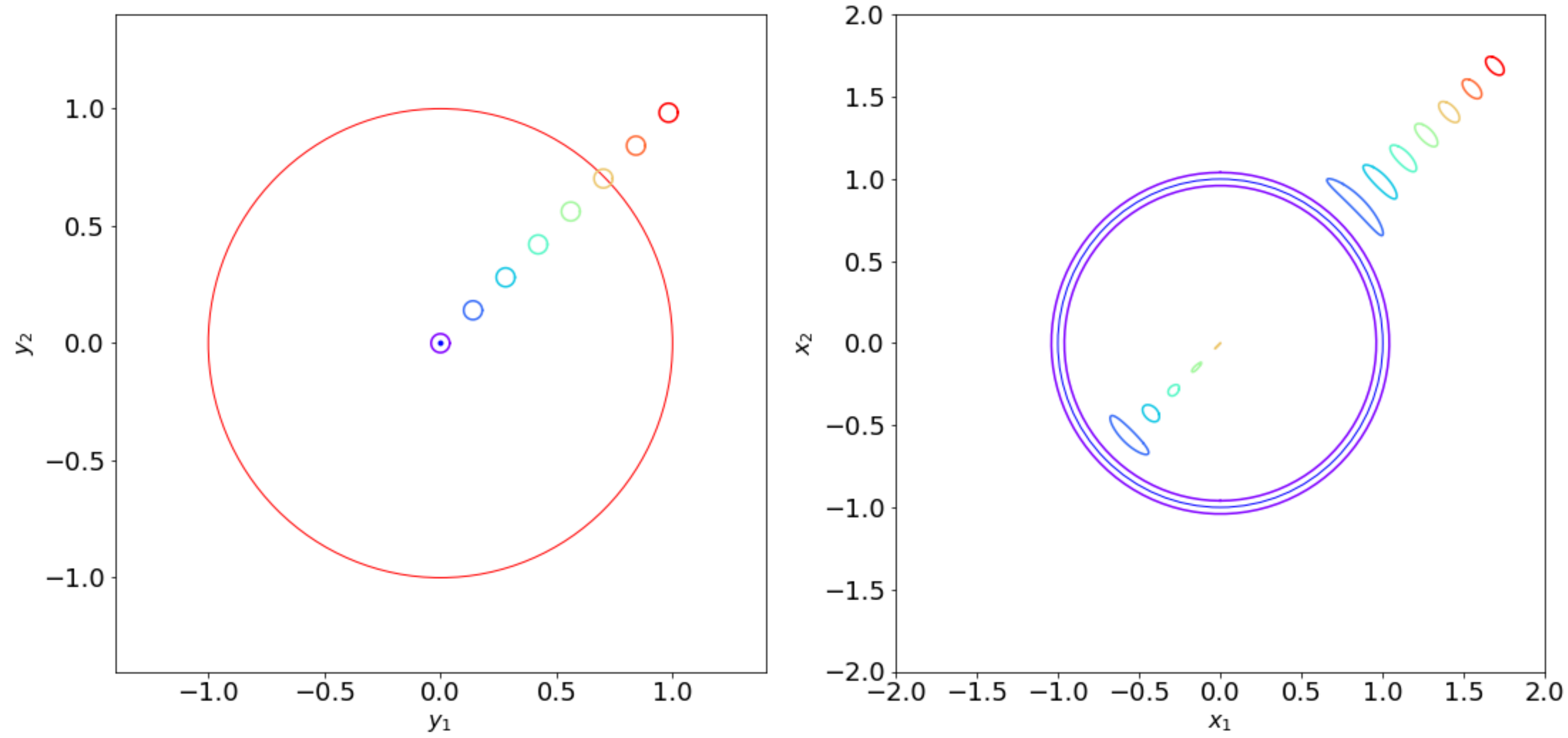
THE SINGULAR ISOTHERMAL SPHERE

as well as the magnification

$$\mu = \frac{|x|}{|x|-1}$$

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \quad ; \quad \mu_- = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

THE SINGULAR ISOTHERMAL SPHERE



SOFTENED PROFILES: THE NON-SINGULAR ISOTHERMAL SPHERE

The profiles considered so far have surface density profiles with a singularity at x=0. We consider another class of lenses which have a flat core.

Given the simplicity of the model, we investigate the effects of the core by modifying the SIS lens:

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\Sigma_0 = \frac{\sigma_v^2}{2G\xi_c}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Choosing $\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\kappa(x) = \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The mass profile is computed as follows

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2 + x_c^2} - x_c$$

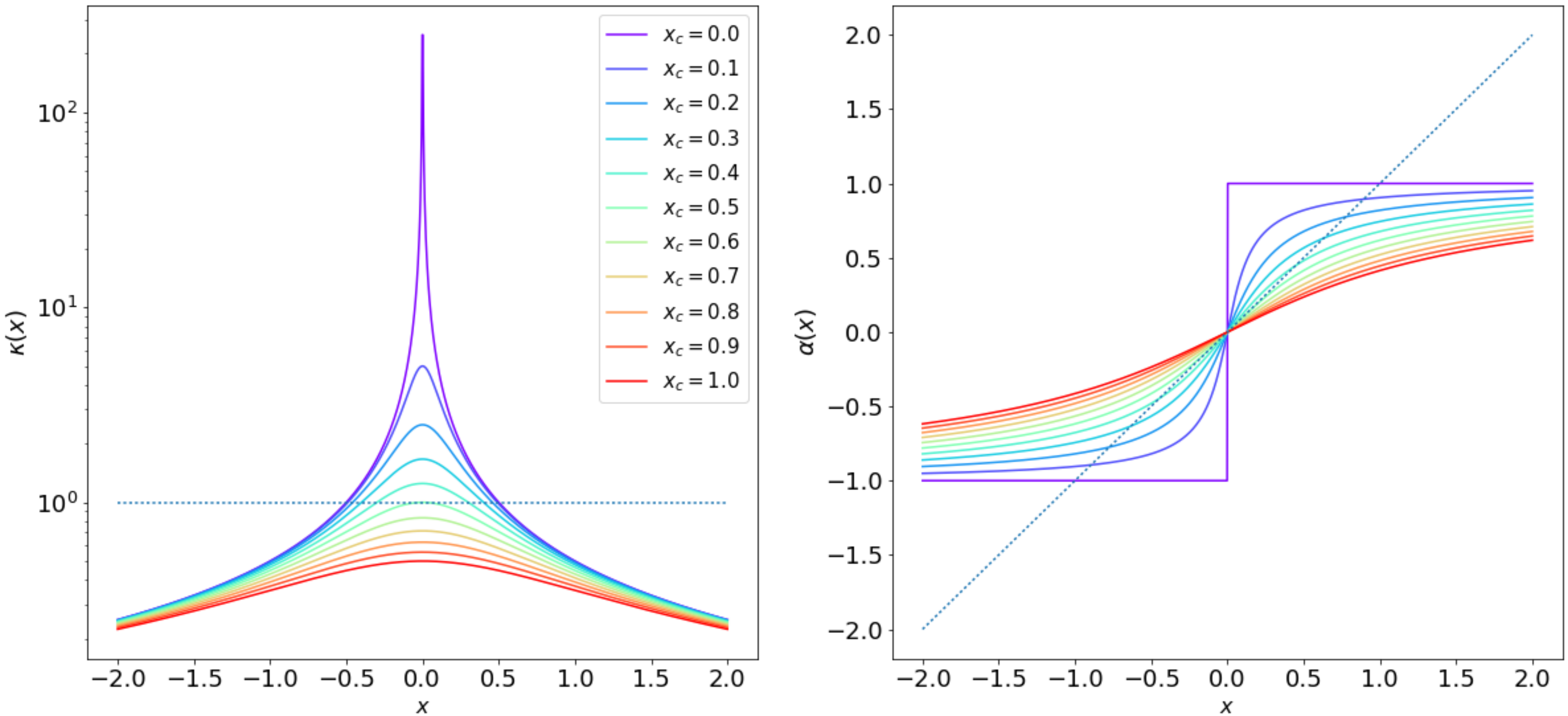
The deflection angle is

$$\alpha(x) = \frac{m(x)}{x} = \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

The shear is

$$\gamma(x) = \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)



Note that if the core is too large, the convergence does not exceed 1 and the derivative of the deflection angle decreases…

NON SINGULAR ISOTHERMAL SPHERE (NIS)

We can search for the tangential critical line:

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2+x_c^2} - x_c \qquad m(x)/x^2 = 1$$

$$\sqrt{x^2+x_c^2} - x_c = x^2$$

$$x^2(x^2+2x_c-1) = 0$$

$$x_t = \sqrt{1-2x_c}$$

Note that the tangential critical line exists only if $x_c<1/2$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

and the radial critical line:

$$\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)=1+\frac{m(x)}{x^2}-2\kappa(x)=0$$

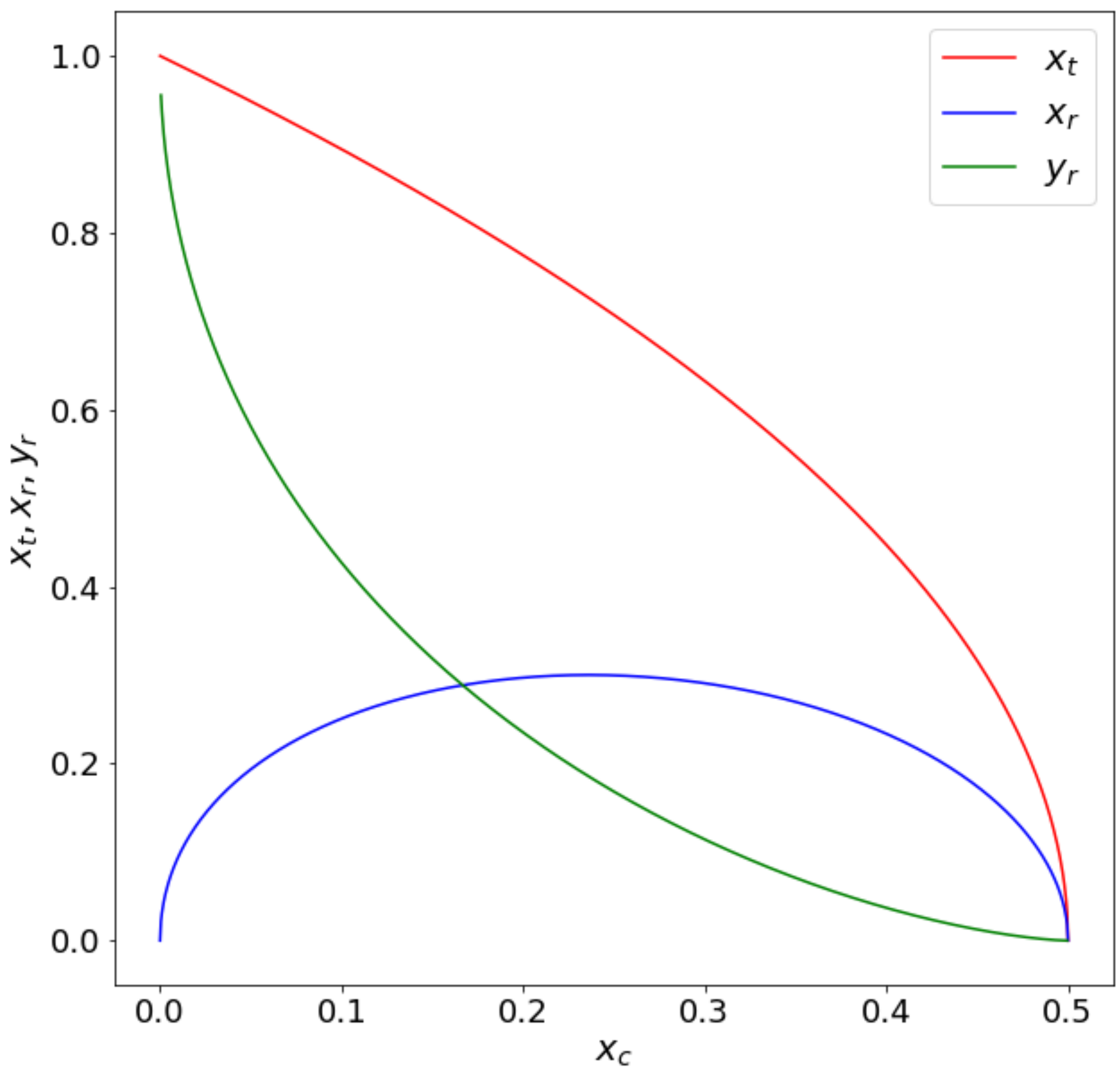
$$1+\frac{\sqrt{x^2+x_c^2}-x_c}{x^2}-\frac{1}{\sqrt{x^2+x_c^2}}=0$$

$$x_r^2=\frac{1}{2}\left(2x_c-x_c^2-x_c\sqrt{x_c^2+4x_c}\right)$$

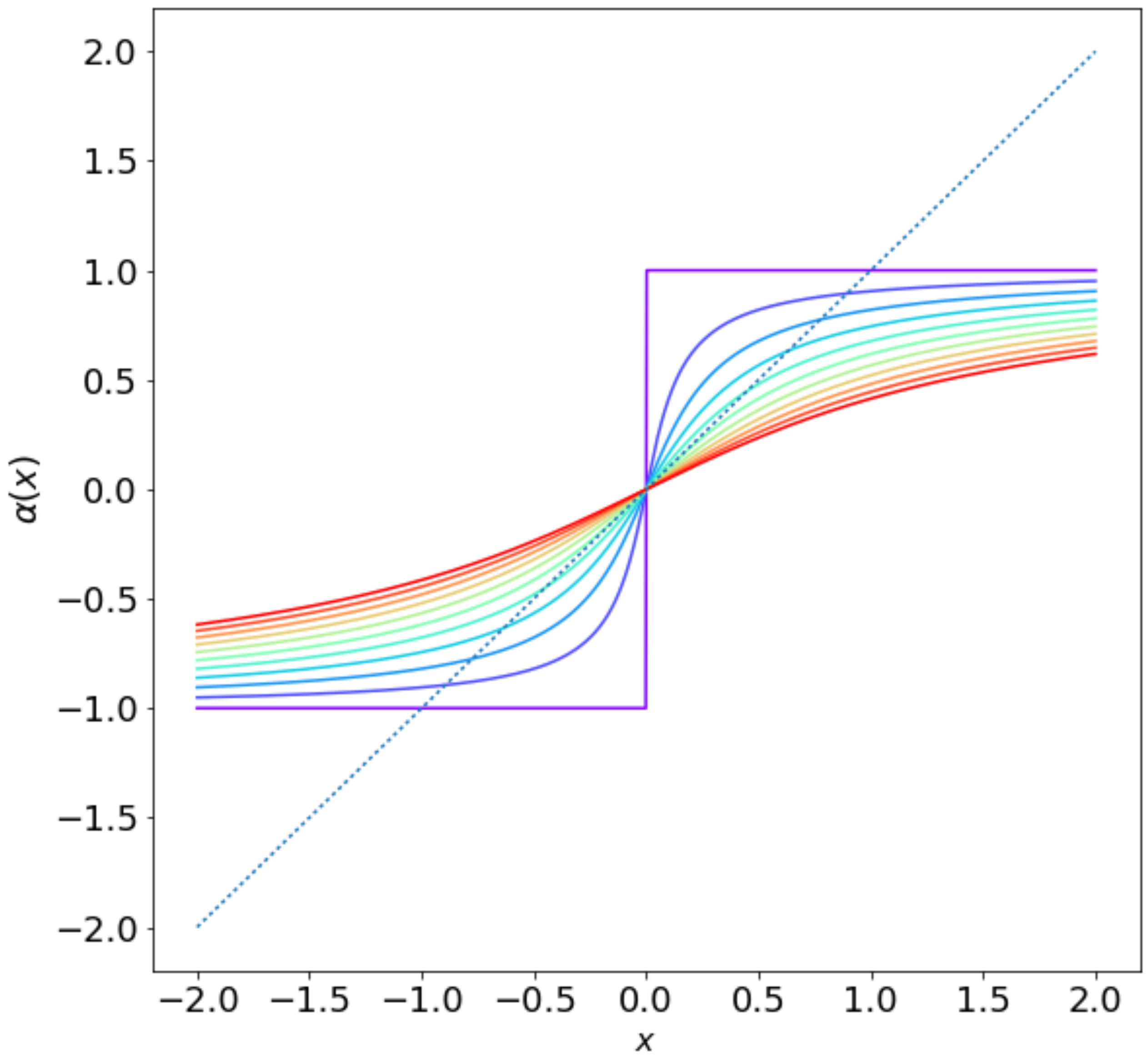
$$x_r^2\geq 0 \text{ for } x_c\leq 1/2.$$

Thus, the existence condition for the radial critical is the same as for the tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)



NON SINGULAR ISOTHERMAL SPHERE (NIS)



As you can see, this has implications also for the existence of multiple images…

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The lens equation can be reduced to the form:

$$y = x - \frac{m(x)}{x} = x - \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

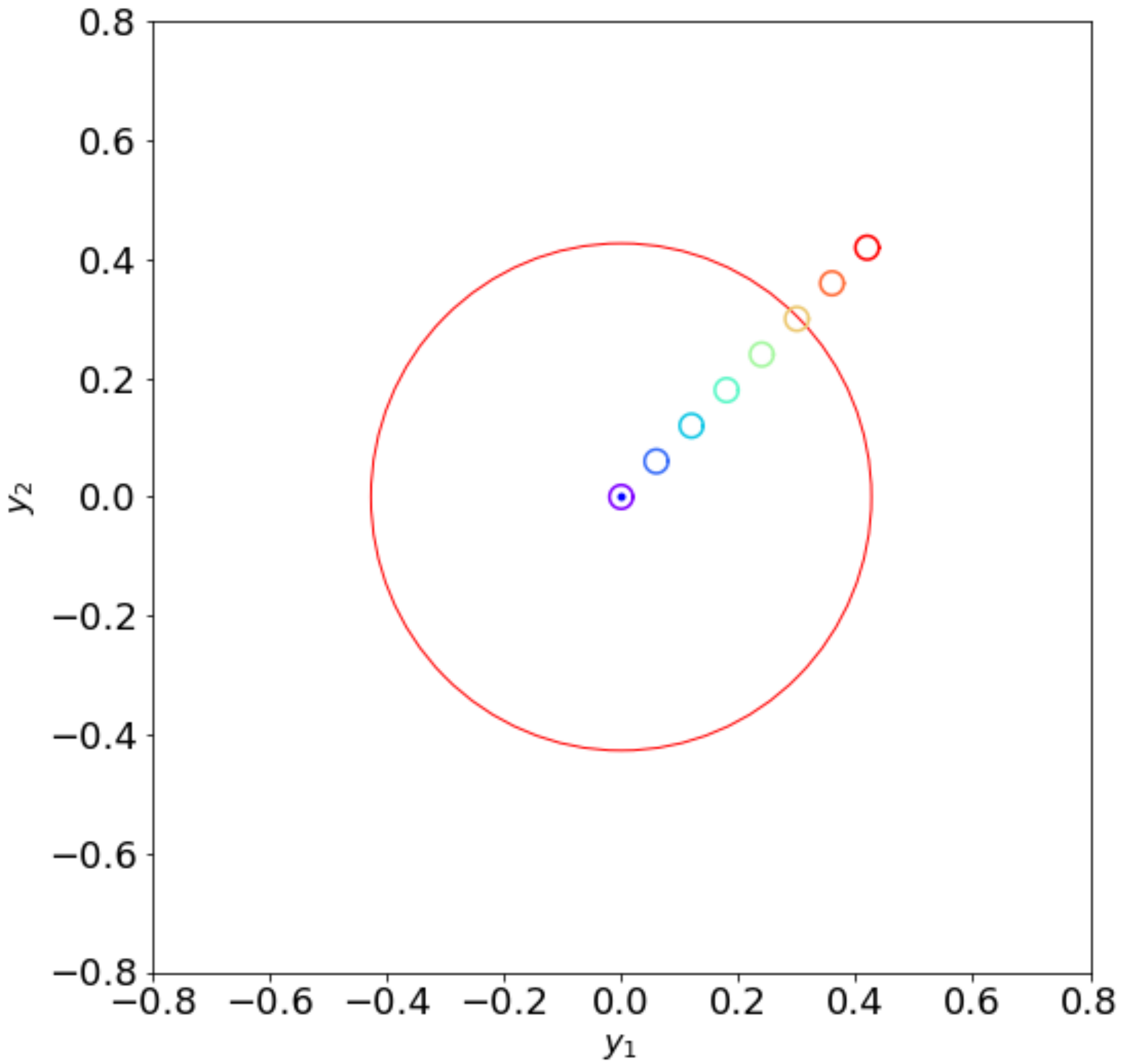
$$x^3 - 2yx^2 + (y^2 + 2x_c - 1)x - 2yx_c = 0 \, .$$

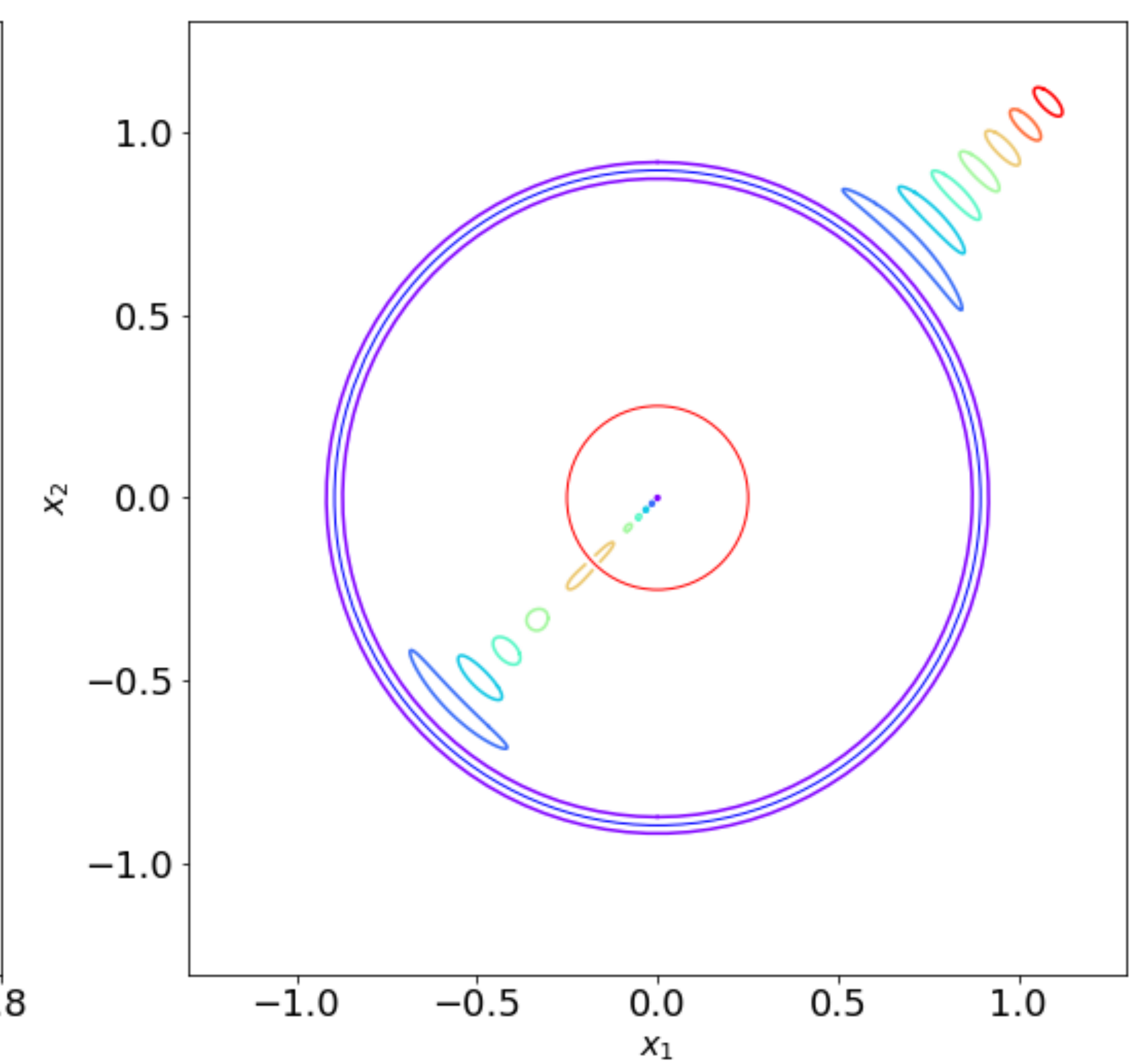
There are up to three solutions, but, again the existence of multiple images depends on y and x_c…

In particular on whether:

- *the radial caustic exist*
- *the source is inside or outside the radial caustic*

NON SINGULAR ISOTHERMAL SPHERE (NIS)





TIME DELAYS

As seen earlier, lensing introduces a time delay:

$$t(x) = \frac{(1+z_{\mathrm{L}})}{c} \frac{D_{\mathrm{L}} D_{\mathrm{S}}}{D_{\mathrm{LS}}} \frac{\xi_0^2}{D_{\mathrm{L}}^2} \left[\frac{1}{2}(x-y)^2 - \Psi(x) \right] = \frac{(1+z_{\mathrm{L}})}{c} \frac{D_{\mathrm{L}} D_{\mathrm{S}}}{D_{\mathrm{LS}}} \tau(x)$$

If there are multiple images, each of them is probing a different line of sight…

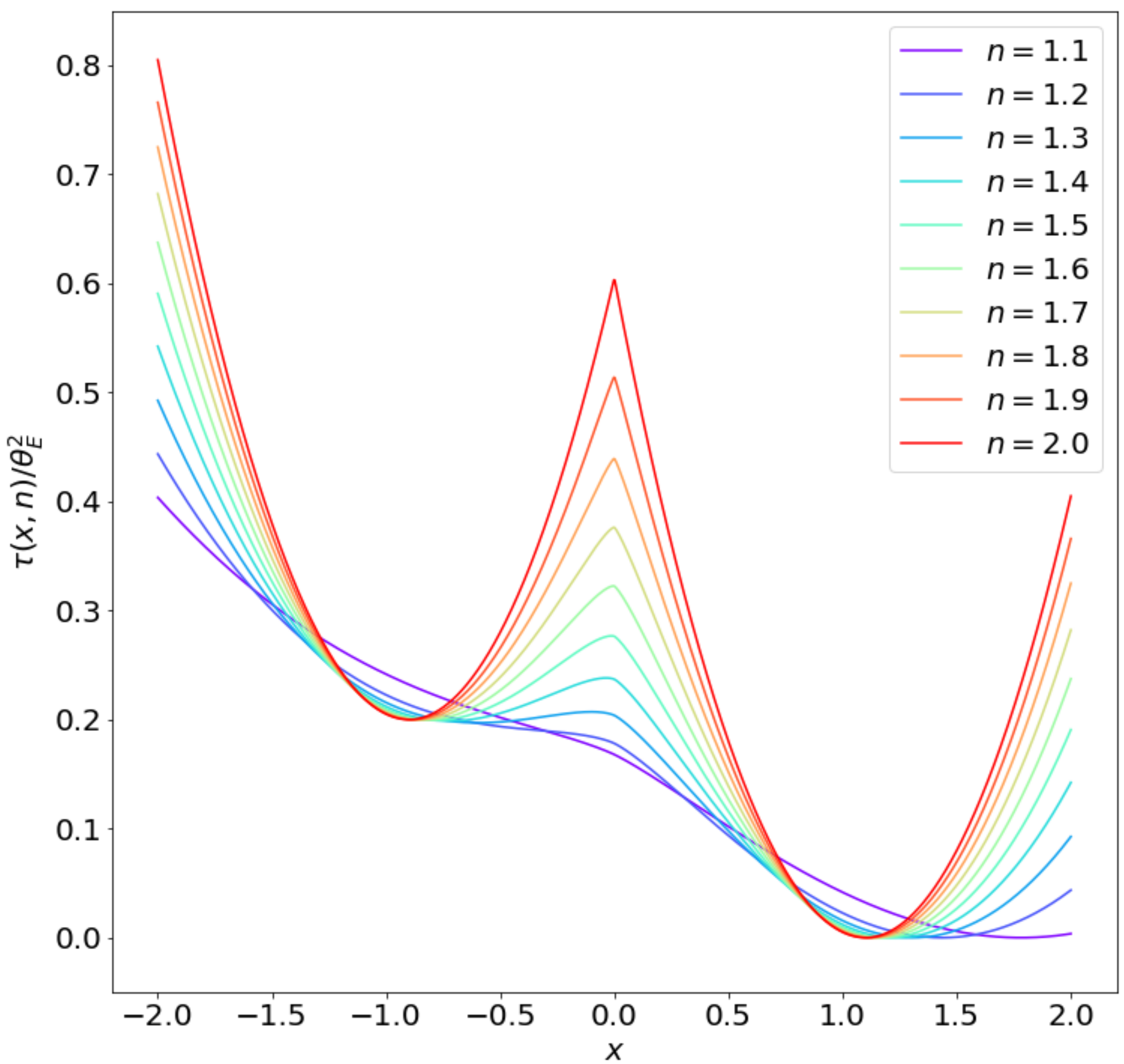
If the source is intrinsically variable, we may be able to measure a delay between the images.

The models we have studied can be used to predict the time delay between the images. The fundamental ingredient is the lensing potential:

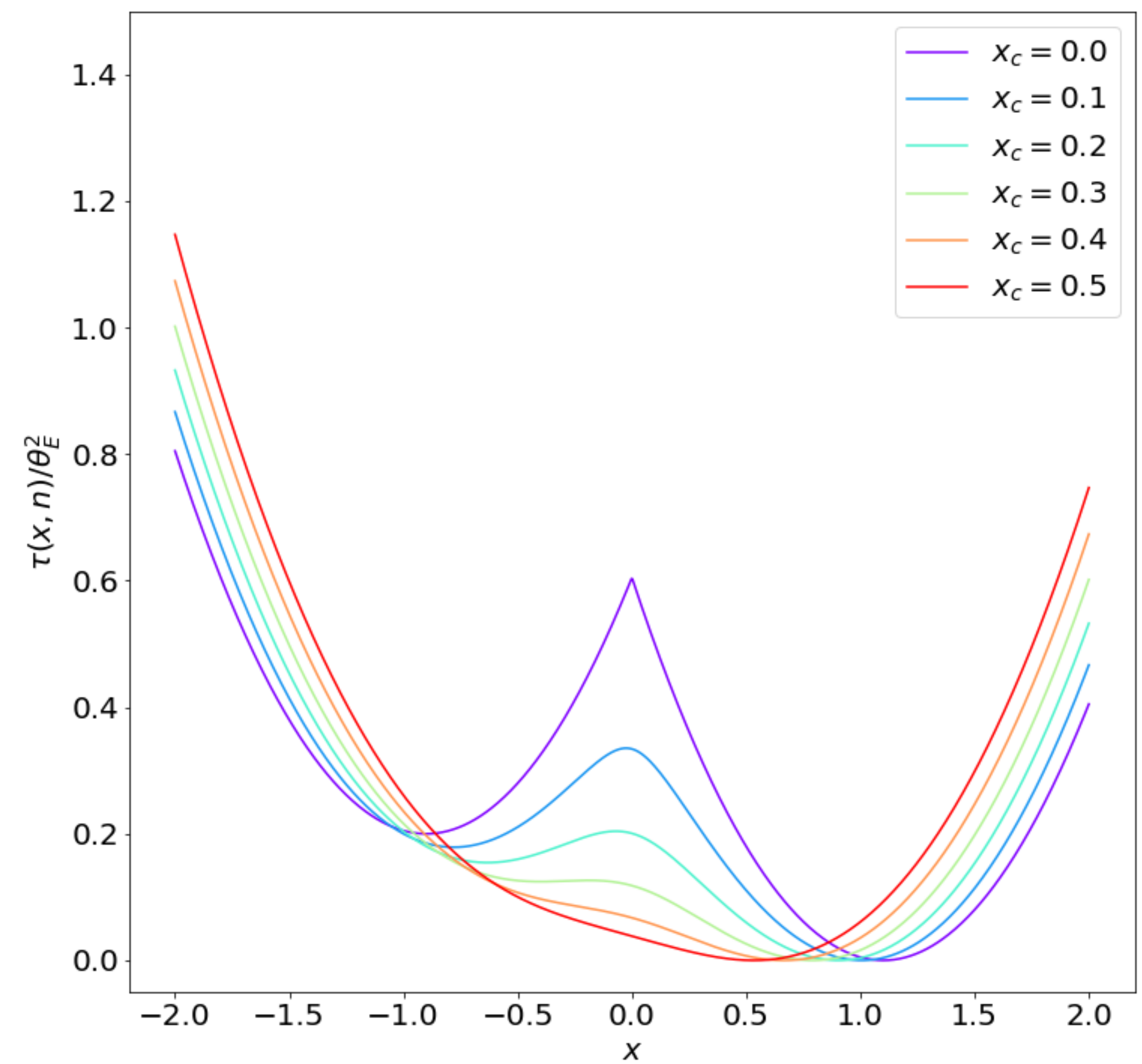
$$\Psi(x) = \frac{1}{3-n} x^{3-n}$$ *power-law*

$$\Psi(x, x_c) = \sqrt{x^2 + x_c^2} - x_c \ln\left(x_c + \sqrt{x^2 + x_c^2} \right)$$ *NIS*

TIME DELAYS



$n = 1.1$
$n = 1.2$
$n = 1.3$
$n = 1.4$
$n = 1.5$
$n = 1.6$
$n = 1.7$
$n = 1.8$
$n = 1.9$
$n = 2.0$
$\tau(x, n)/\theta_E^2$
x



$x_c = 0.0$
$x_c = 0.1$
$x_c = 0.2$
$x_c = 0.3$
$x_c = 0.4$
$x_c = 0.5$
$\tau(x, n)/\theta_E^2$
x

TIME DELAYS

$$\tau(x) = \frac{\xi_0^2}{D_\mathrm{L}^2}\left[\frac{1}{2}(x-y)^2 - \frac{1}{3-n}x^{3-n}\right]$$

$$x - y = \alpha(x) = x^{2-n}$$

$$\tau(x_i) = \frac{\xi_0^2}{D_\mathrm{L}^2}\left[\frac{1}{2}x_i^{2(2-n)} - \frac{1}{3-n}x_i^{3-n}\right]$$

$$\Delta t_{ij} \propto \Delta\tau_{ij} = \frac{\xi_0^2}{D_\mathrm{L}^2}\left[\frac{1}{2}\left(x_j^{2(2-n)} - x_i^{2(2-n)}\right) - \frac{1}{3-n}\left(x_j^{3-n} - x_i^{3-n}\right)\right]$$

For $n = 2$, this formula gives:

$$\Delta\tau_{ij} = \frac{\xi_0^2}{D_\mathrm{L}^2}(x_i - x_j) = \theta_E^2\left(\frac{\theta_i}{\theta_E} - \frac{\theta_j}{\theta_E}\right) = \frac{1}{2}\left(\theta_i^2 - \theta_j^2\right) = \Delta\tau_{SIS}$$