GRAVITATIONAL LENSING

18 - SOFTENED LENSES

Massimo Meneghetti

AA 2017-2018

SOFTENED PROFILES: THE NON-SINGULAR ISOTHERMAL SPHERE

The profiles considered so far have surface density profiles with a singularity at x=0. We consider another class of lenses which have a flat core.

Given the simplicity of the model, we investigate the effects of the core by modifying the SIS lens:

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\Sigma_0 = \frac{\sigma_v^2}{2G\xi_c}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Choosing $\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_\mathrm{L} D_\mathrm{LS}}{D_\mathrm{S}}$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\kappa(x) = \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The mass profile is computed as follows

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2 + x_c^2} - x_c$$

The deflection angle is

$$\alpha(x) = \frac{m(x)}{x} = \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

The shear is

$$\gamma(x) = \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

We can search for the tangential critical line:

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2+x_c^2} - x_c \qquad m(x)/x^2 = 1$$

$$\sqrt{x^2+x_c^2} - x_c = x^2$$

$$x^2(x^2+2x_c-1) = 0$$

$$x_t = \sqrt{1-2x_c}$$

Note that the tangential critical line exists only if $x_c < 1/2$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

and the radial critical line:

$$\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)=1+\frac{m(x)}{x^2}-2\kappa(x)=0$$

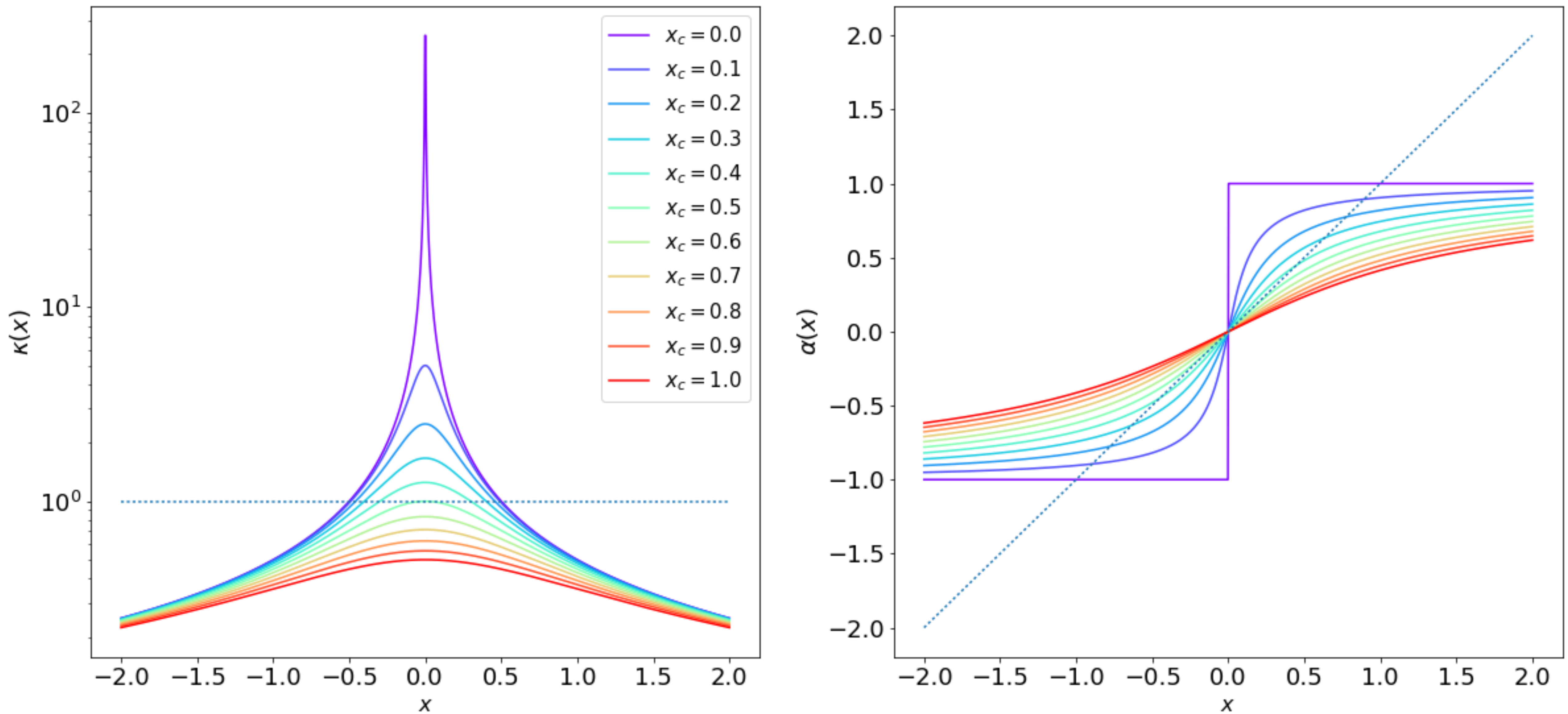
$$1+\frac{\sqrt{x^2+x_c^2}-x_c}{x^2}-\frac{1}{\sqrt{x^2+x_c^2}}=0$$

$$x_r^2=\frac{1}{2}\left(2x_c-x_c^2-x_c\sqrt{x_c^2+4x_c}\right)$$

$$x_r^2\geq 0 \text{ for } x_c\leq 1/2.$$

Thus, the existence condition for the radial critical is the same as for the tangential critical line

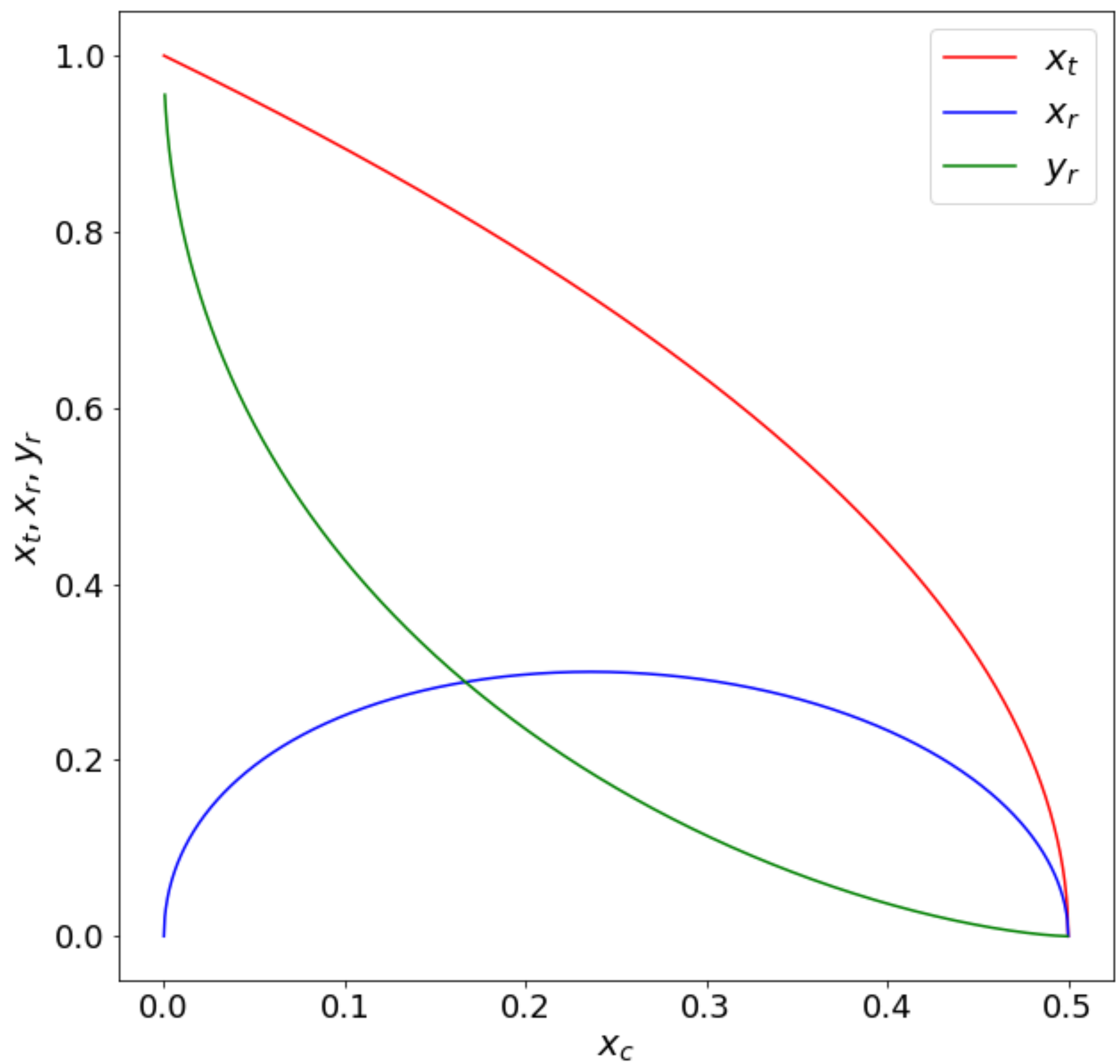
NON SINGULAR ISOTHERMAL SPHERE (NIS)



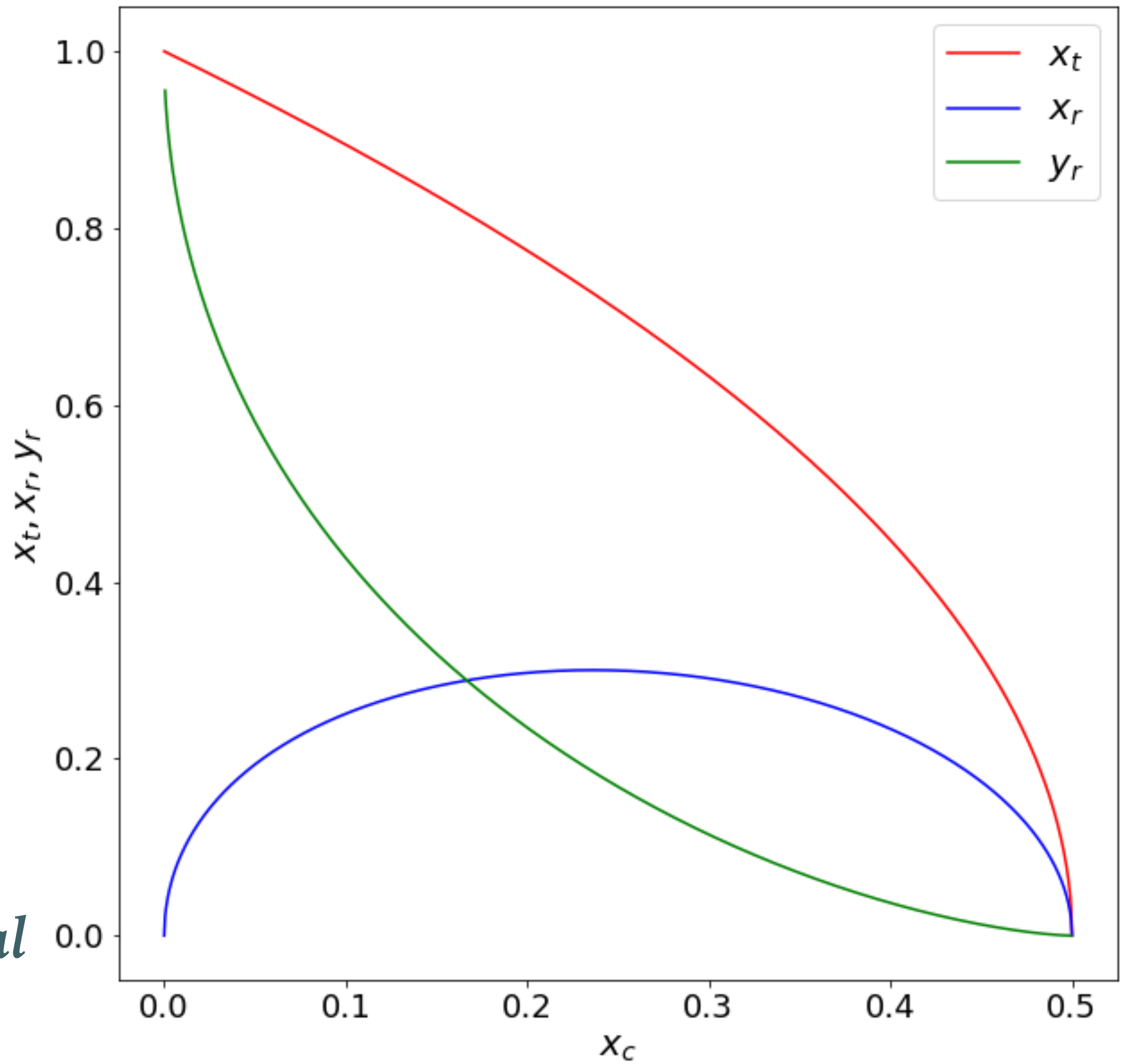
If the core is too large, the convergence does not exceed 1 and the derivative of the deflection angle is never larger than 1…

No critical lines! No multiple images!

NON SINGULAR ISOTHERMAL SPHERE (NIS)

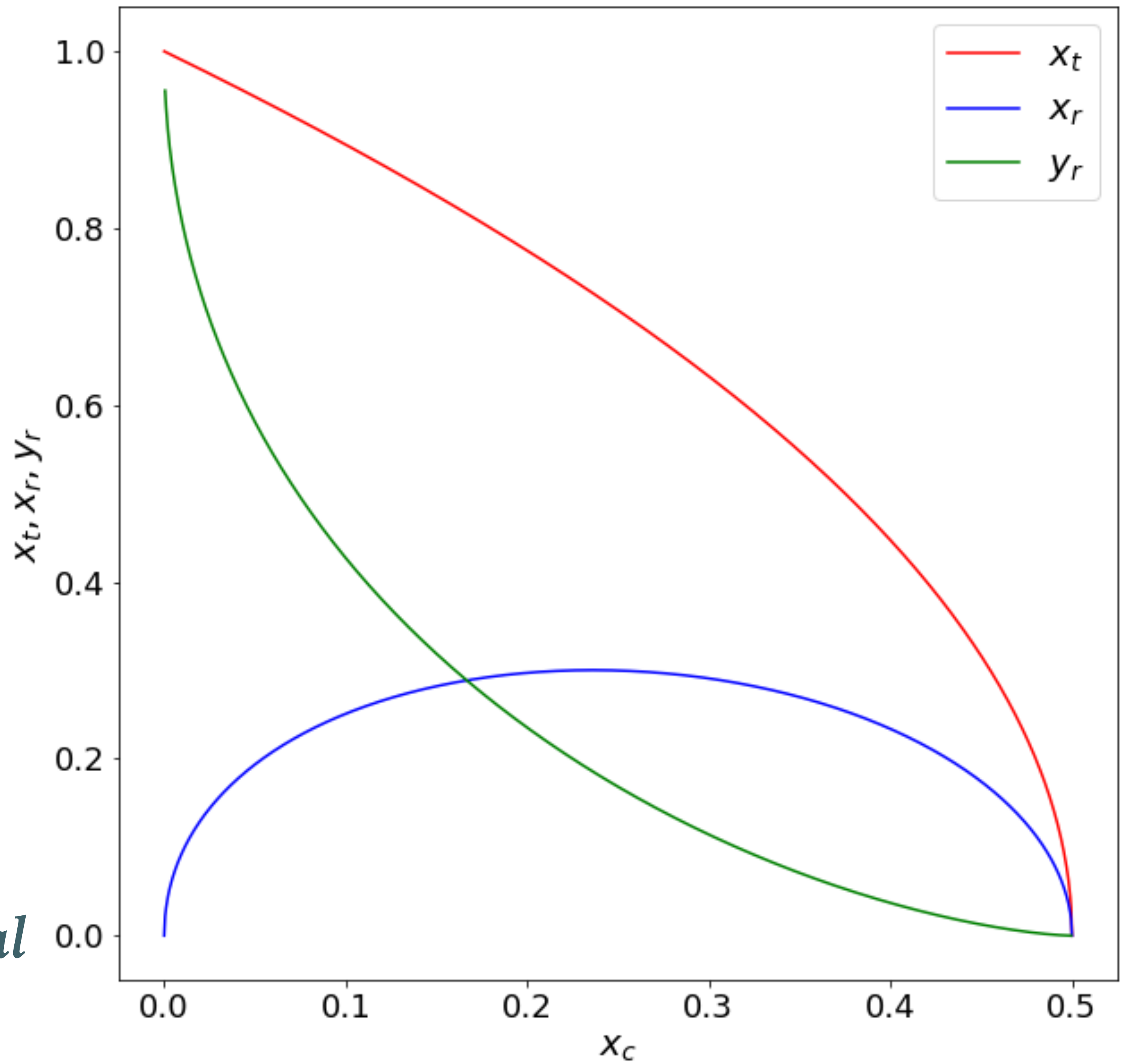


NON SINGULAR ISOTHERMAL SPHERE (NIS)



Small core, small radial critical line, large tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)



Small core, small radial critical line, large tangential critical line

large core, small radial critical line, small tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The lens equation can be reduced to the form:

$$y = x - \frac{m(x)}{x} = x - \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

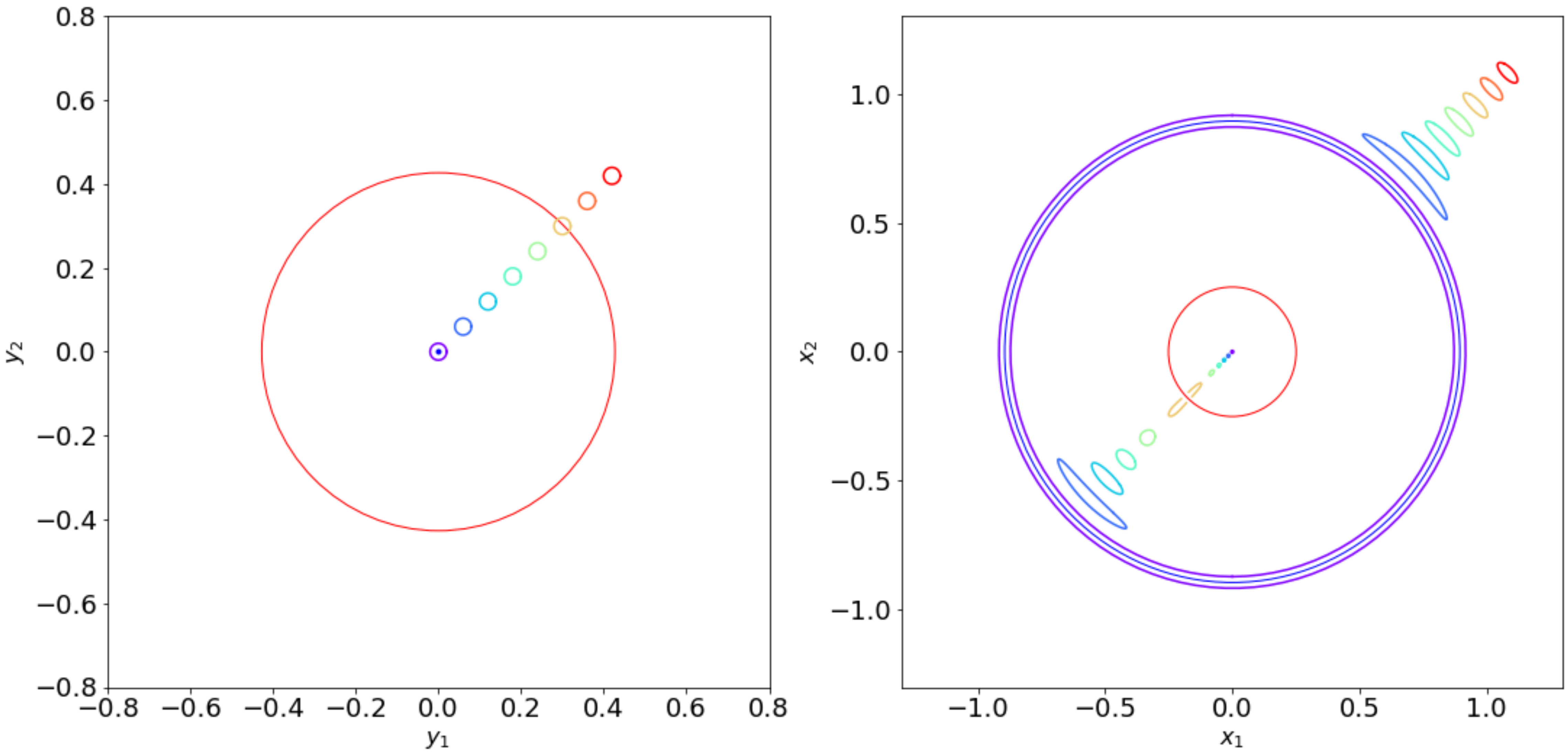
$$x^3 - 2yx^2 + (y^2 + 2x_c - 1)x - 2yx_c = 0\,.$$

There are up to three solutions, but, again the existence of multiple images depends on y and x_c*…*

In particular on whether:

- *the radial caustic exist*
- *the source is inside or outside the radial caustic*

NON SINGULAR ISOTHERMAL SPHERE (NIS)



Three images if the source is inside the radial caustic; One image otherwise.

Parity: changes at each critical line (remember: maxima, minima, saddle points of TDS.

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

Ensures that the mass inside elliptical iso-contours is independent on f

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Let's derive the convergence in dimensionless units:

SINGULAR ISOTHERMAL ELLIPSOID

Let's derive the convergence in dimensionless units:

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2\xi_2^2}} \frac{\xi_0}{\xi_0}$$

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

Let's derive the convergence in dimensionless units:

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}} \frac{\xi_0}{\xi_0}$$

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{D_{\mathrm{S}}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

In polar coordinates:

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2}$$

$$\kappa(x, \varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad\qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

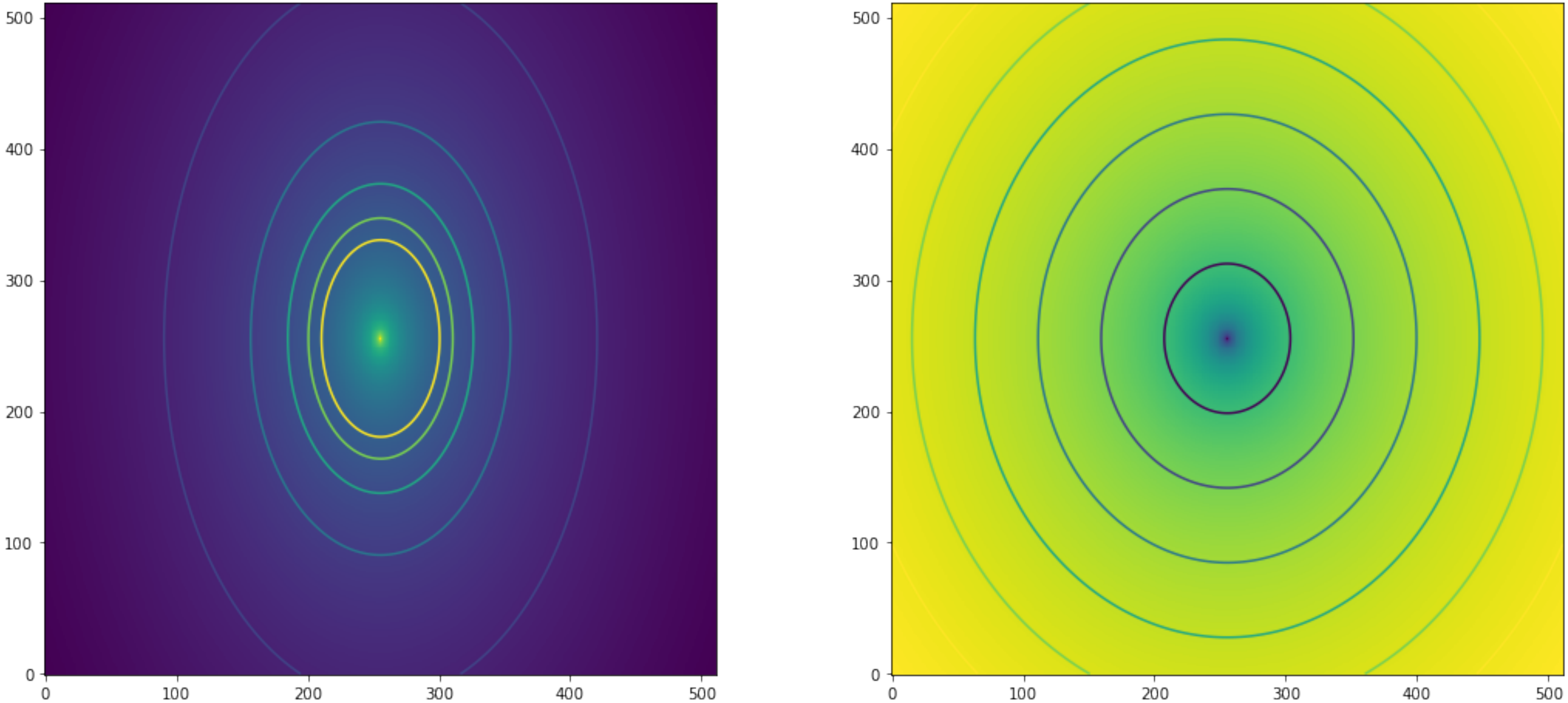
With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

Solved with Green's method (Kormann et al. 1994):

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

CONVERGENCE AND POTENTIAL



SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}\left(f'/f\cos\varphi\right)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

$$\begin{aligned} \alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ \alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi) \end{aligned}$$

Analogy with the SIS: the deflection angle does not depend on x!

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) = \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)$$

The component of the shear:

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) = \frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi)$$

The component of the shear:

$$\gamma_1 = \frac{1}{2}\left(\frac{\partial \alpha_1}{\partial x_1} - \frac{\partial \alpha_2}{\partial x_2}\right)$$

$$\gamma_2 = \frac{\partial \alpha_1}{\partial x_2}$$

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
\end{aligned}$$

The component of the shear:

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_2}{\partial x_2}\right) \\
\gamma_2 &= \frac{\partial\alpha_1}{\partial x_2}
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi \\
\gamma_2 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi
\end{aligned}$$

Similarly to the SIS: $\gamma = \kappa$

SINGULAR ISOTHERMAL ELLIPSOID

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}
\gamma_1 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi \\
\gamma_2 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi
\end{aligned}$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

whose eigenvalues are:

$$\begin{aligned}
\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\
\lambda_r &= 1-\kappa+\gamma = 1\,.
\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned} \lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,. \end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned} \lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,. \end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\ .\end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

$$\kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)} \qquad \Rightarrow \qquad \vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\left[\cos\varphi, \sin\varphi\right]$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\,[\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned} y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ . \end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)} [\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned} y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)} \cos\varphi - \frac{\sqrt{f}}{f'} \mathrm{arcsinh}\left(\frac{f'}{f} \cos\varphi\right) \\ y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)} \sin\varphi - \frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi) \ . \end{aligned}$$

There is no radial caustic, but there is the cut, which can be computed as

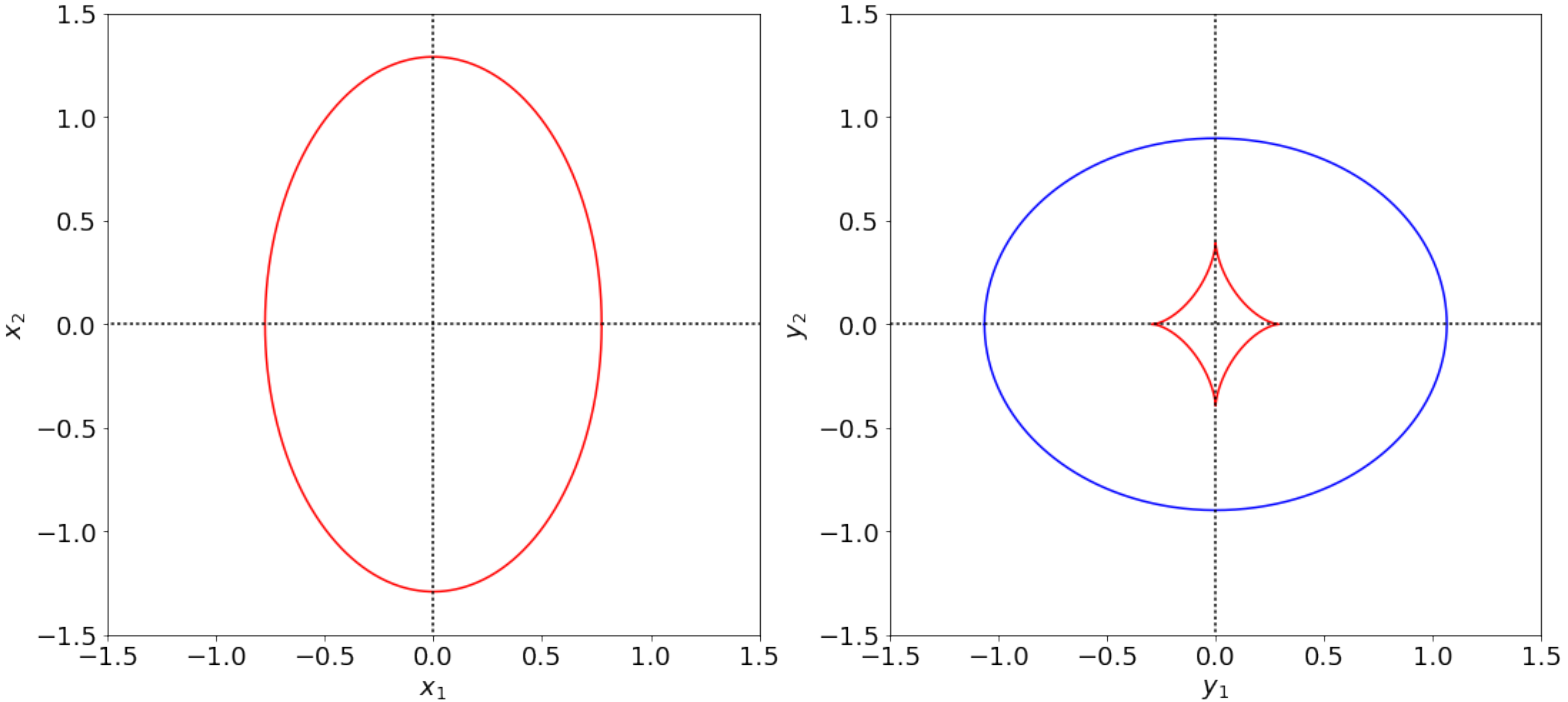
$$\vec{y}_c = \lim_{x \to 0} \vec{y}(x, \varphi) = -\vec{\alpha}(\varphi)$$

SINGULAR ISOTHERMAL ELLIPSOID

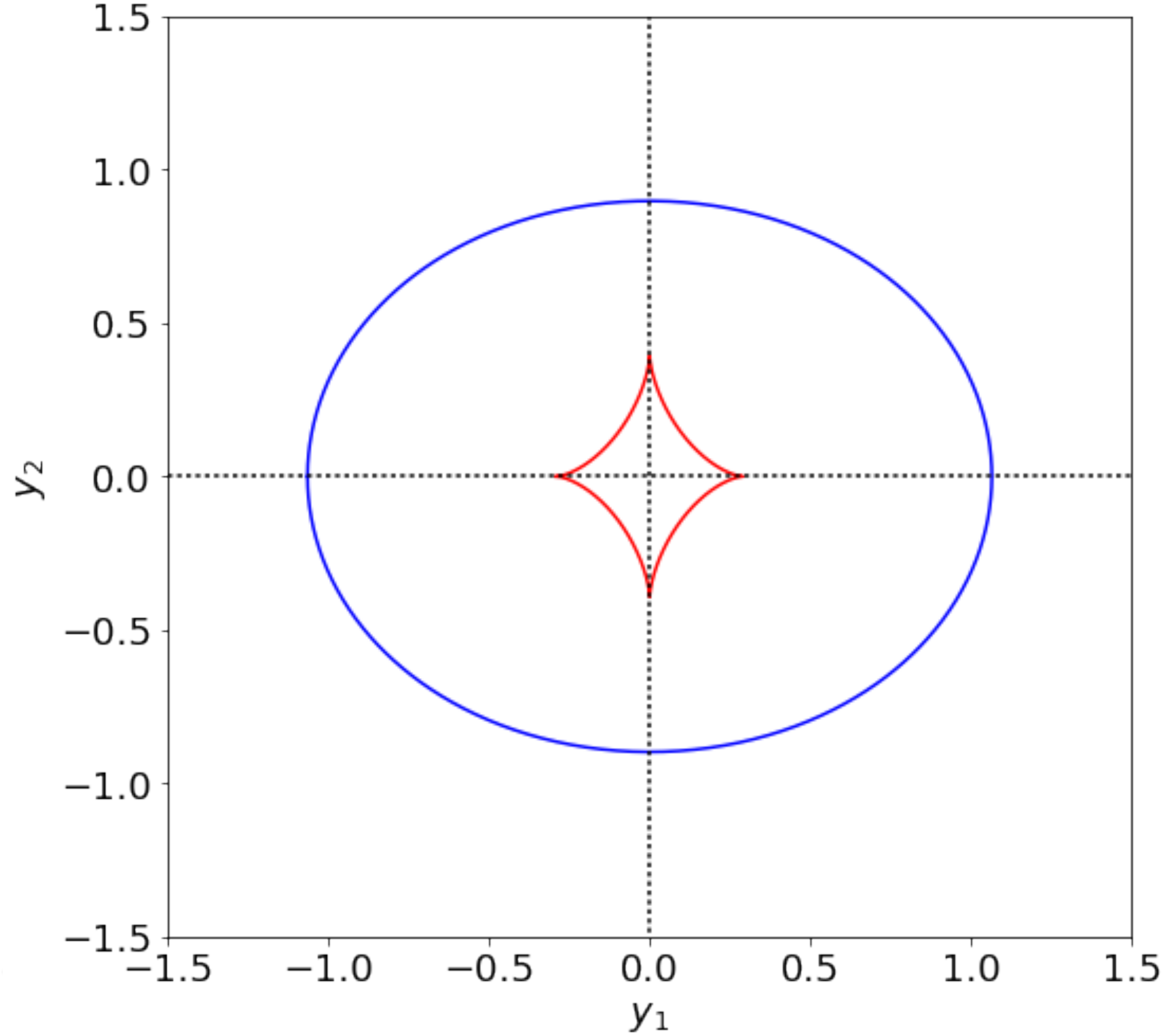
$$\vec{y}_c = \lim_{x \to 0} \vec{y}(x, \varphi) = -\vec{\alpha}(\varphi)$$

$$\begin{aligned} y_{c,1} &= -\frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ y_{c,2} &= -\frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi) \, . \end{aligned}$$

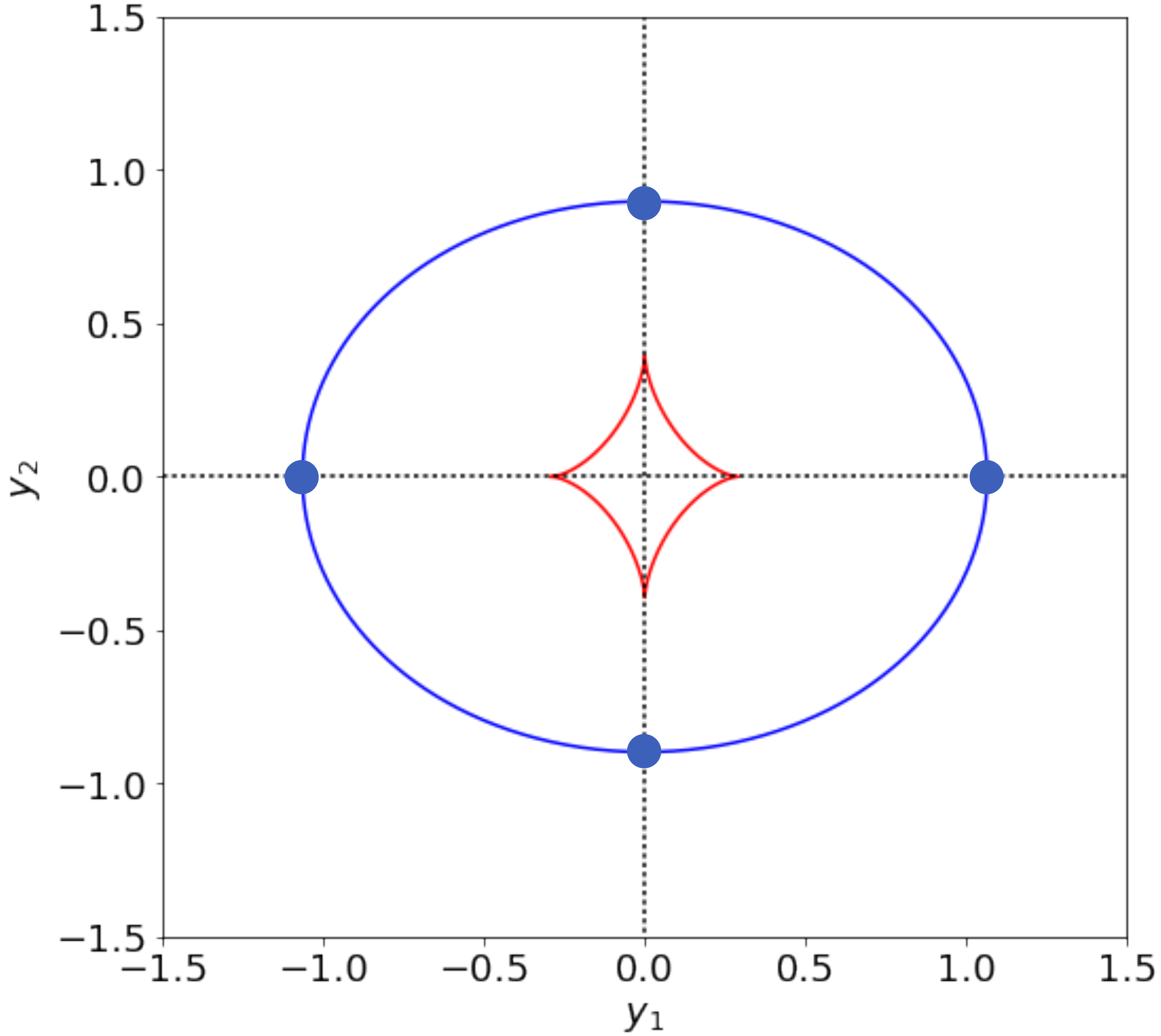
CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC

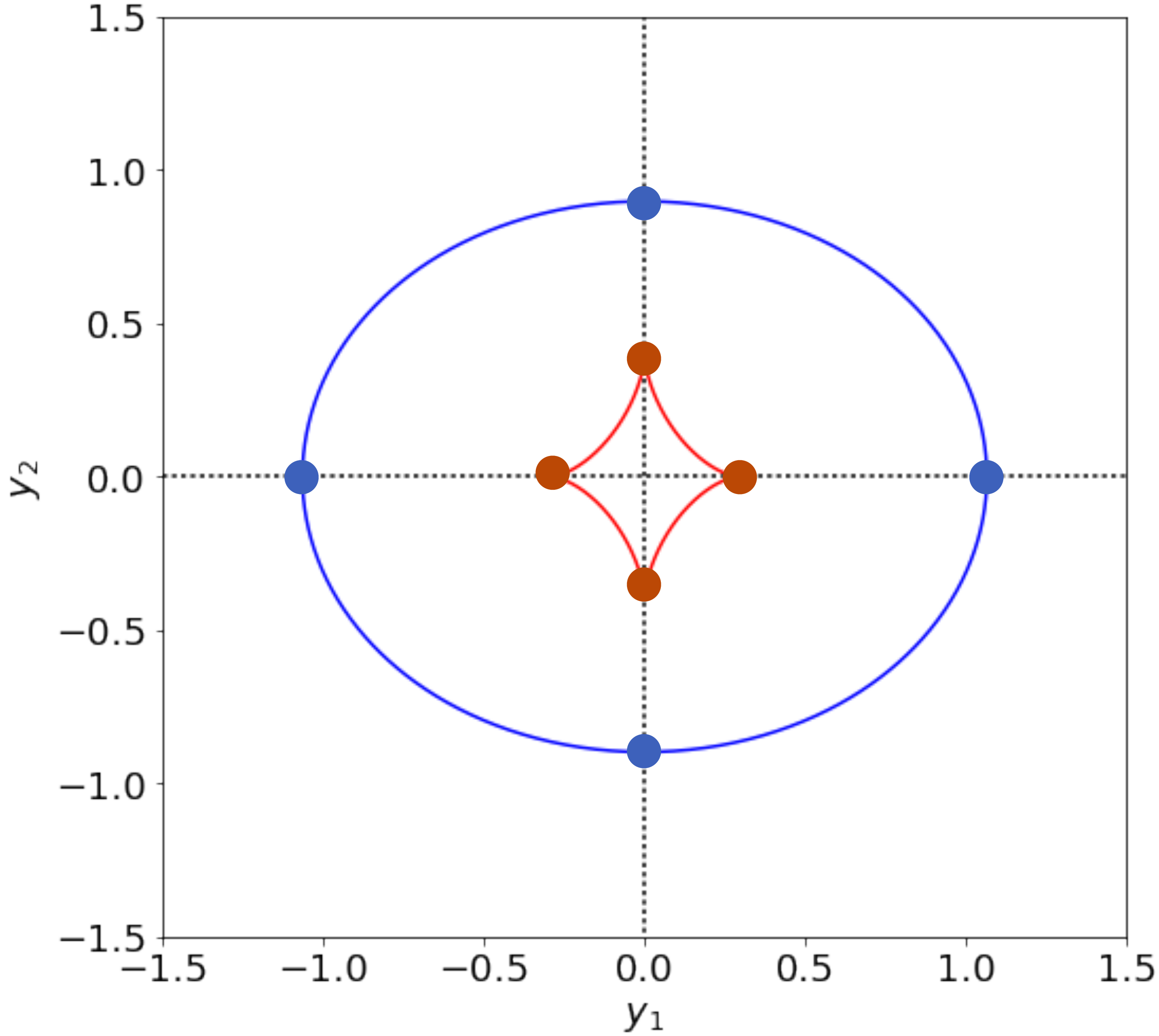


CRITICAL LINE, CUT, CAUSTIC



$$
\begin{aligned}
s_{1,\pm,c} &= [y_{c,1}(\varphi = 0, \pi), 0] \, , \\
s_{2,\pm,c} &= [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

CRITICAL LINE, CUT, CAUSTIC



$$s_{1,\pm,c} = [y_{c,1}(\varphi = 0, \pi), 0] \,,$$
$$s_{2,\pm,c} = [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]$$

$$s_{1,\pm,t} = [y_{t,1}(\varphi = 0, \pi), 0] \,,$$
$$s_{1,\pm,t} = [0, y_{t,2}(\varphi = \pi/2, -\pi/2)]$$

CRITICAL LINE, CUT, CAUSTIC

$$s_{1,\pm,c} = [y_{c,1}(\varphi = 0, \pi), 0] \,,$$
$$s_{2,\pm,c} = [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]$$

$$s_{1,\pm,t} = [y_{t,1}(\varphi = 0, \pi), 0] \,,$$
$$s_{1,\pm,t} = [0, y_{t,2}(\varphi = \pi/2, -\pi/2)]$$

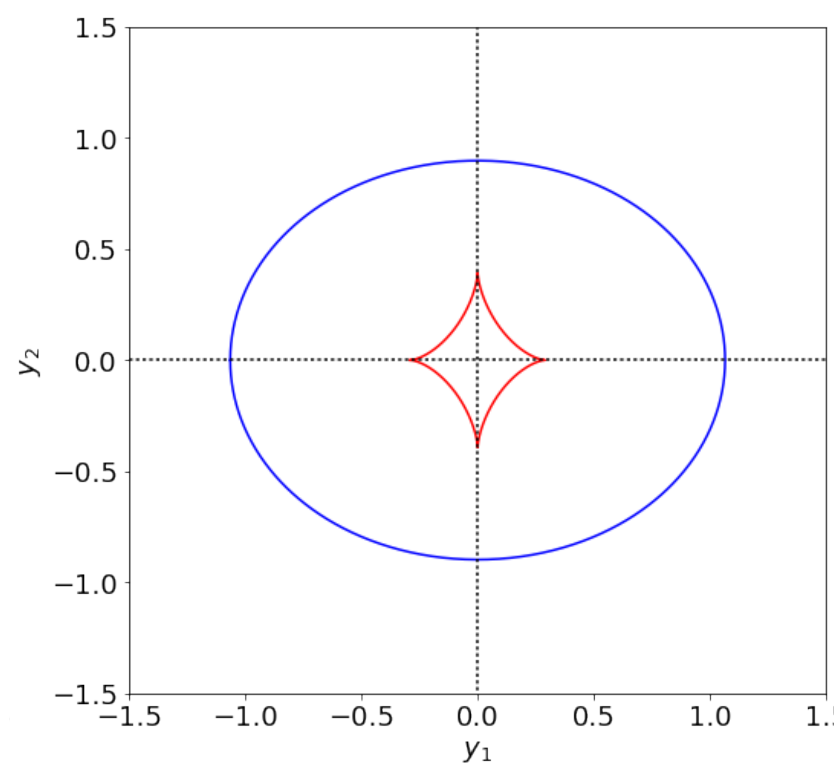
$$y_{c,1} = -\frac{\sqrt{f}}{f'} \text{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$
$$y_{c,2} = -\frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi) \,.$$

$$y_{t,1} = \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'} \text{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$
$$y_{t,2} = \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi) \,.$$

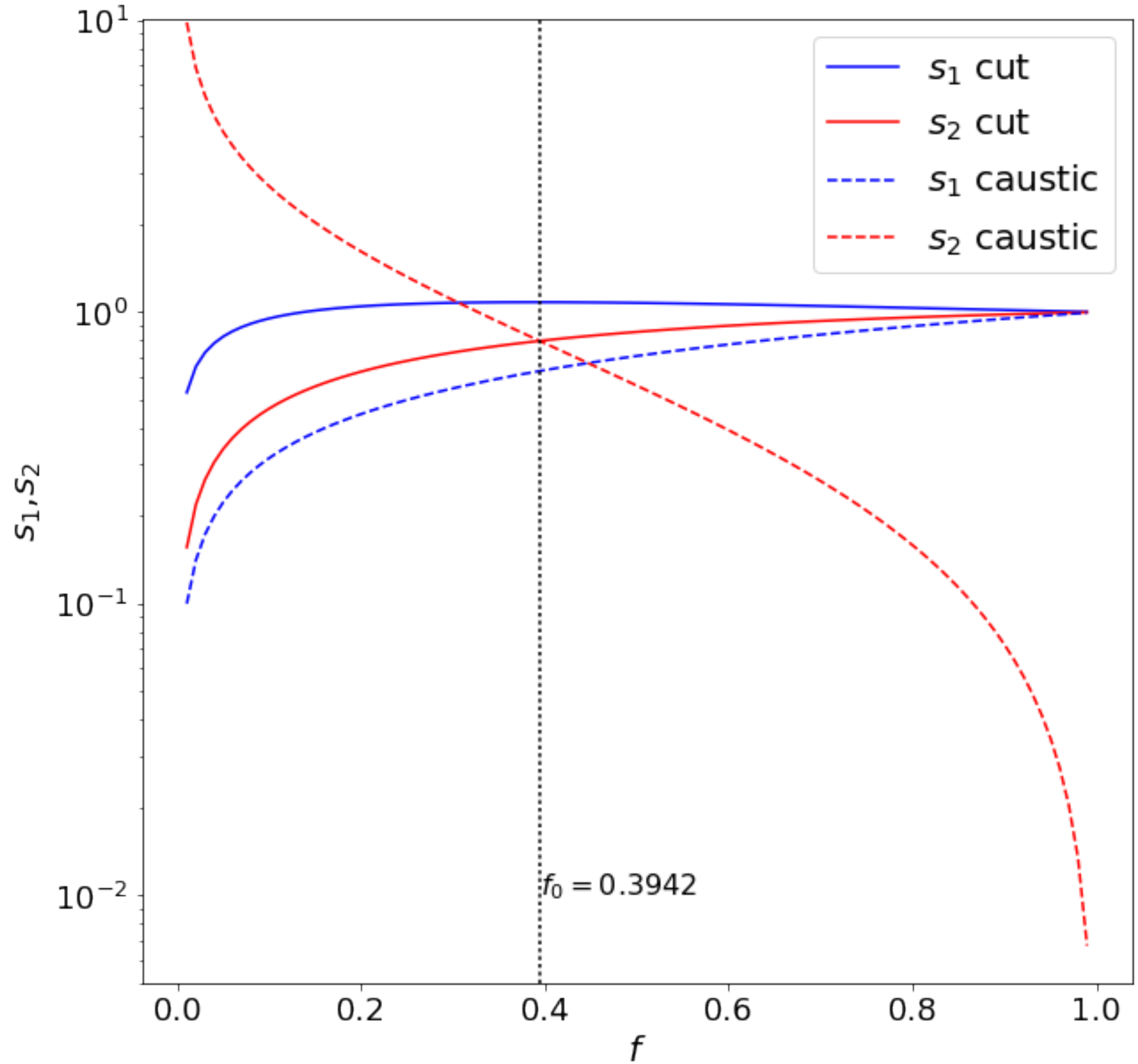
We can easily see that

$$s_{1,c} > s_{1,t}$$

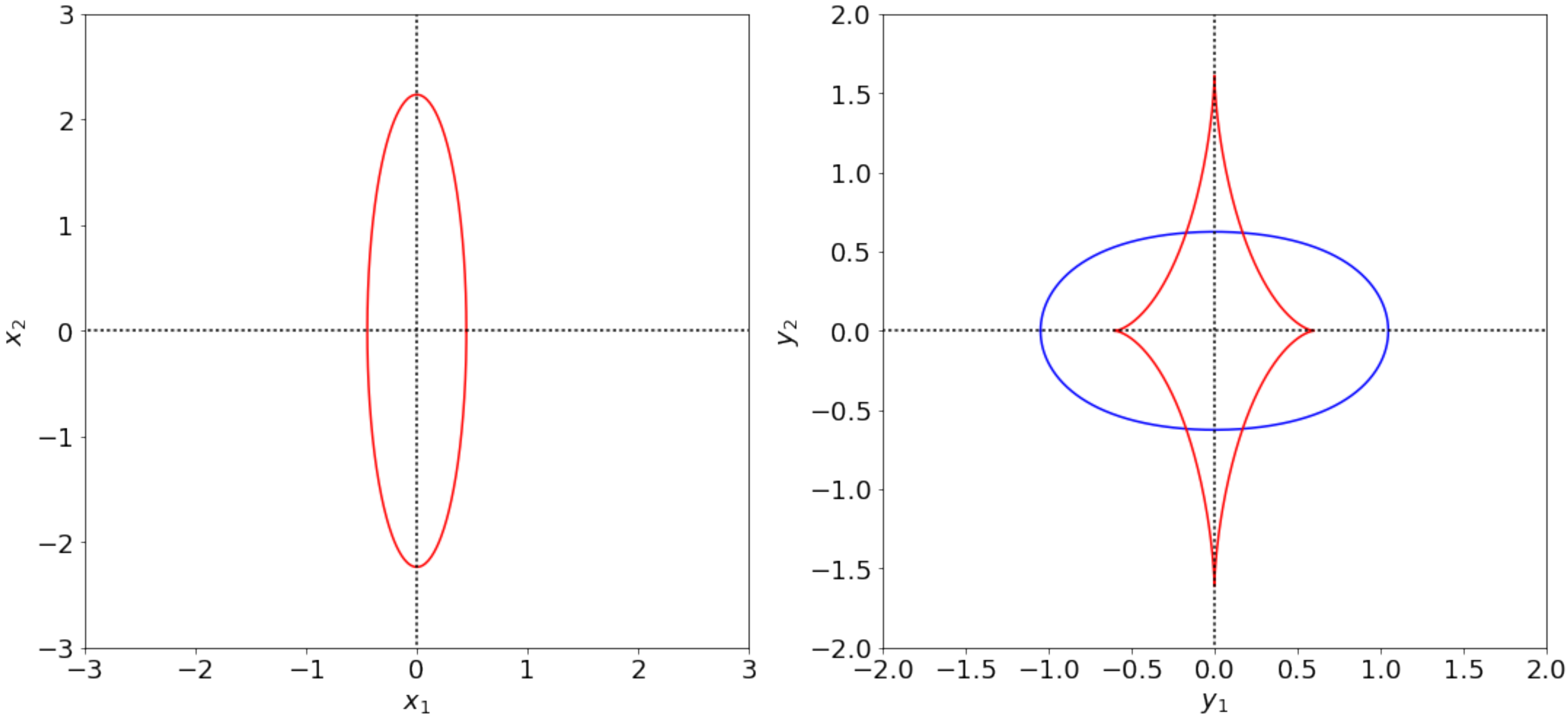
independent on f.



CRITICAL LINE, CUT, CAUSTIC

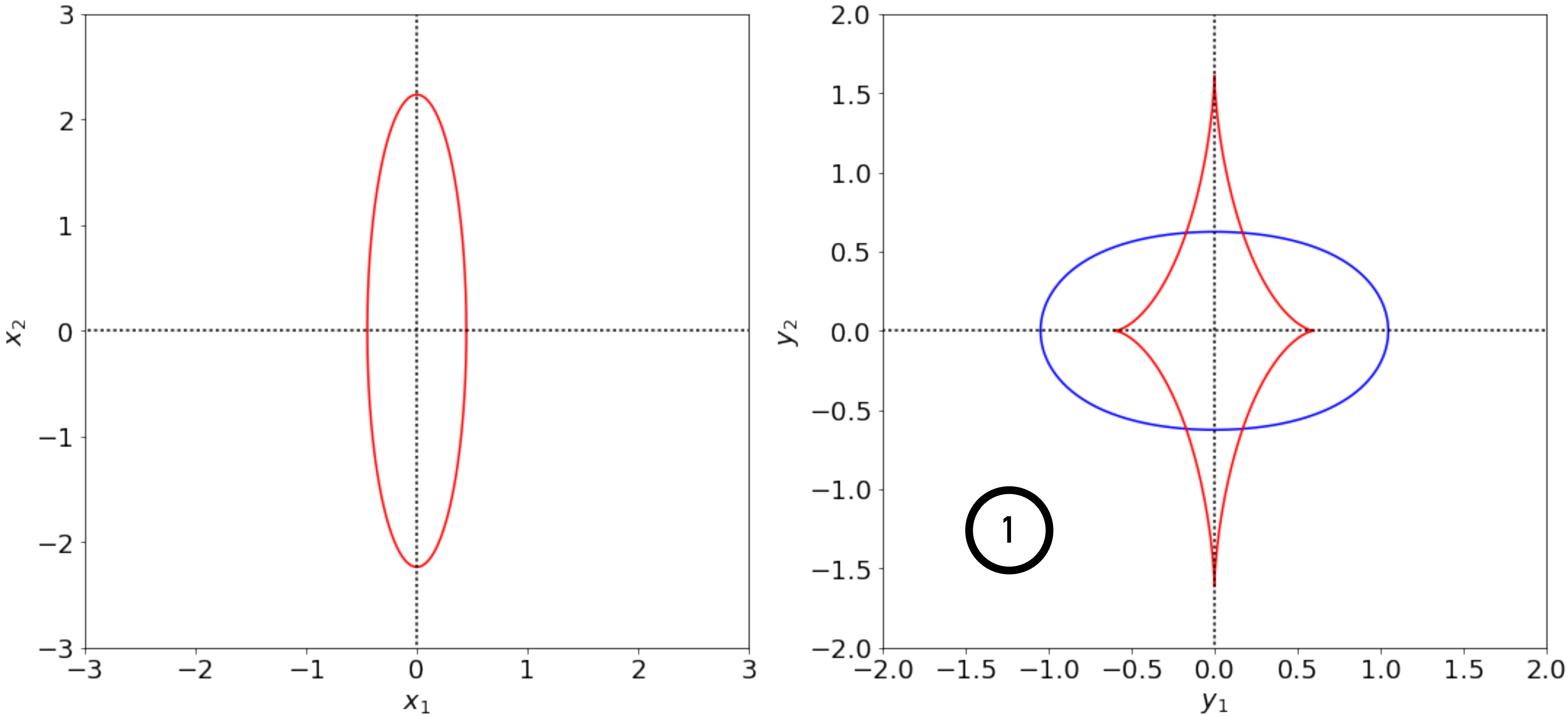


CRITICAL LINE, CUT, CAUSTIC



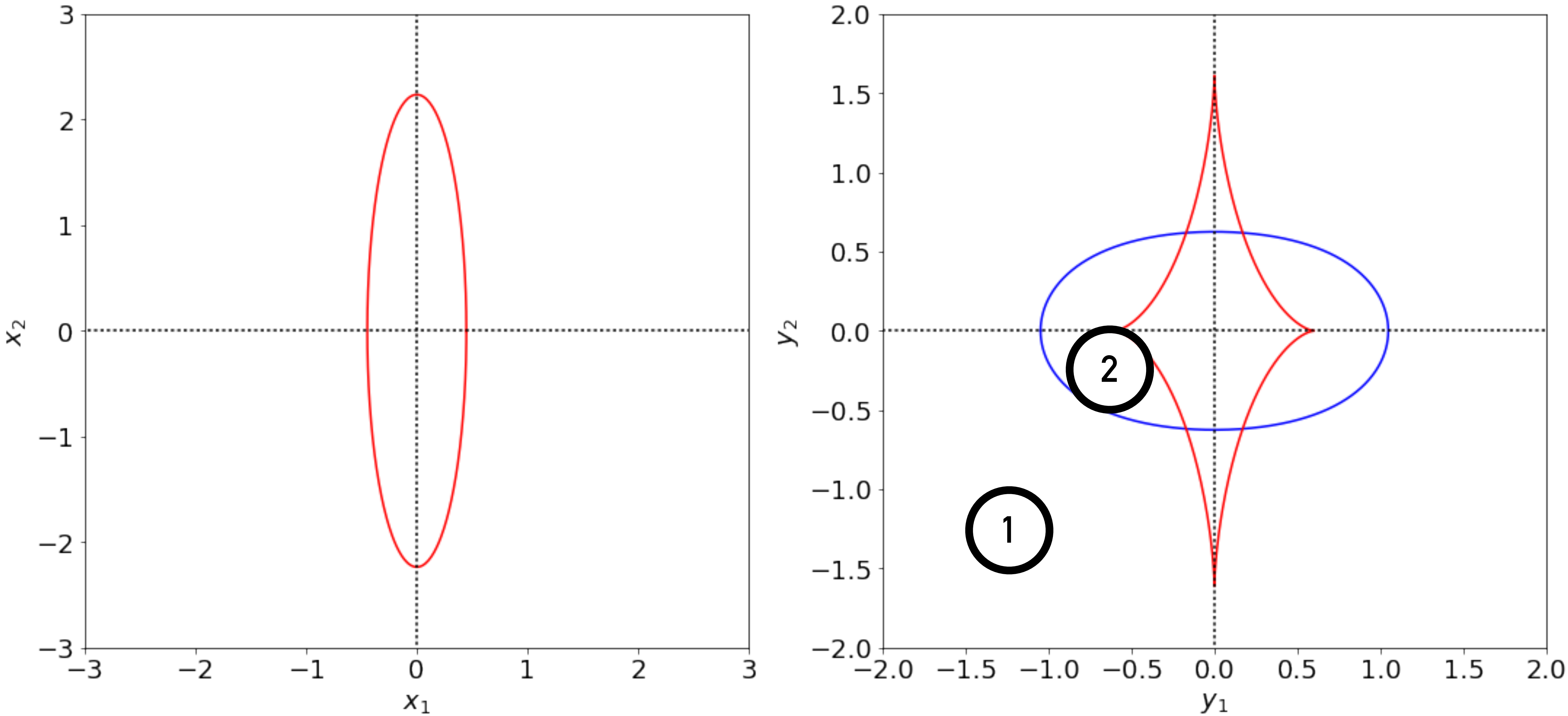
This is important for the image multiplicity…

CRITICAL LINE, CUT, CAUSTIC



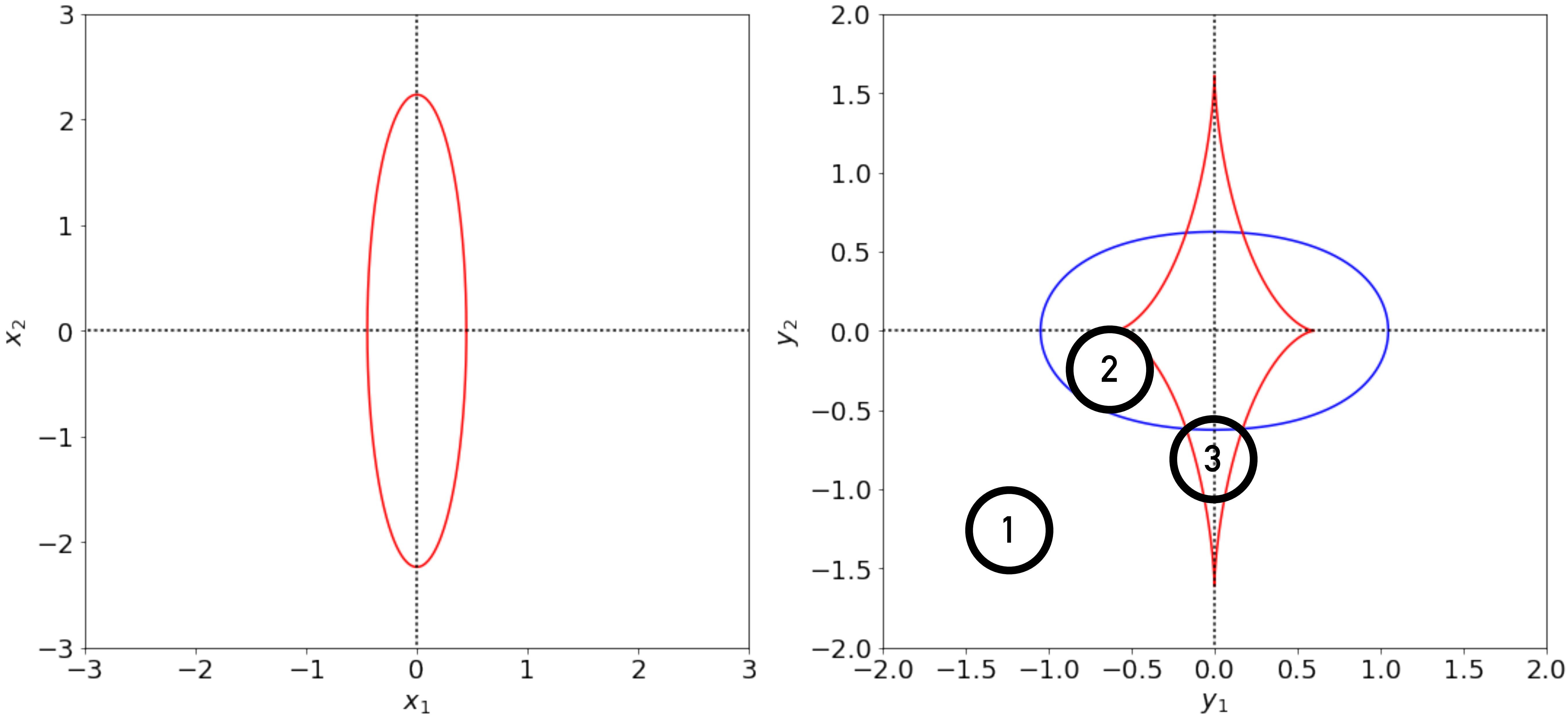
This is important for the image multiplicity…

CRITICAL LINE, CUT, CAUSTIC



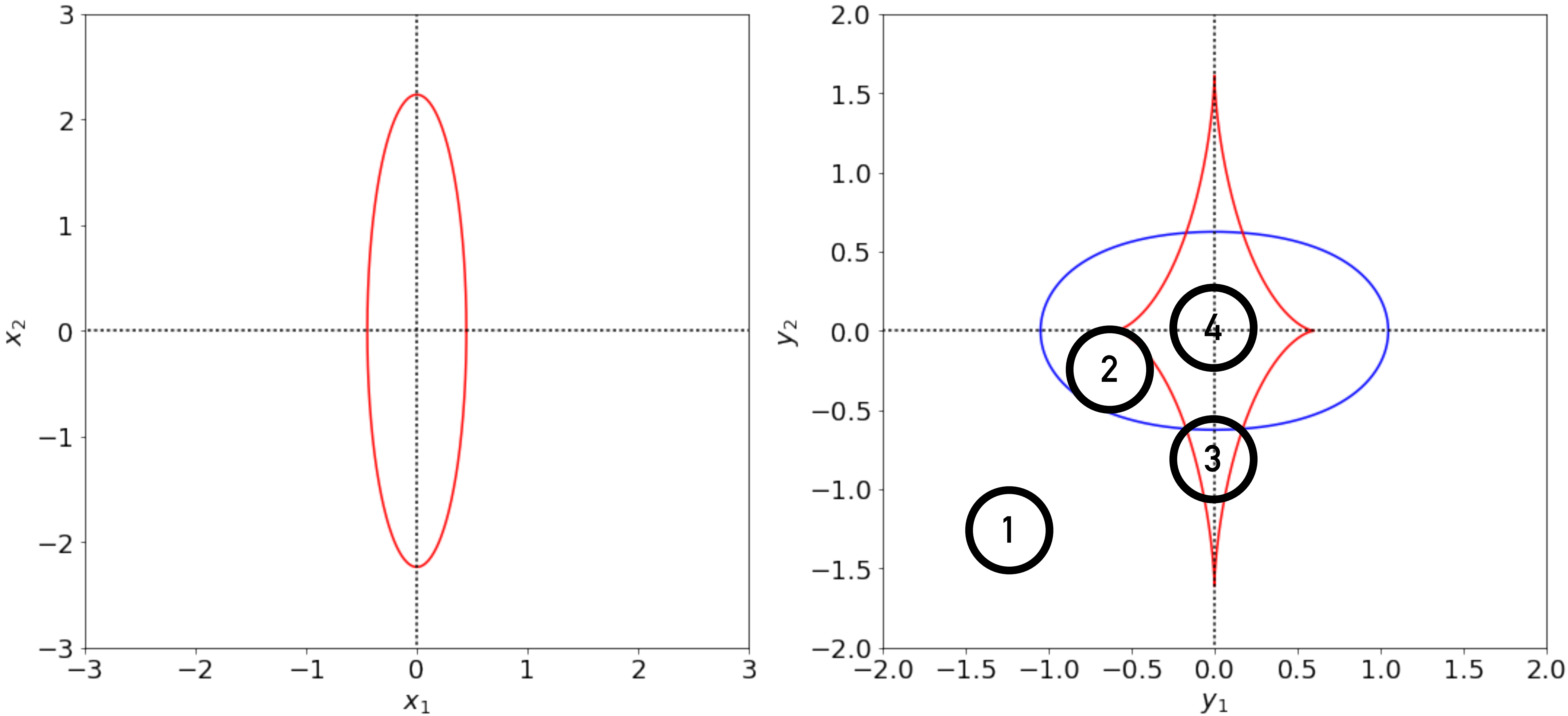
This is important for the image multiplicity…

CRITICAL LINE, CUT, CAUSTIC



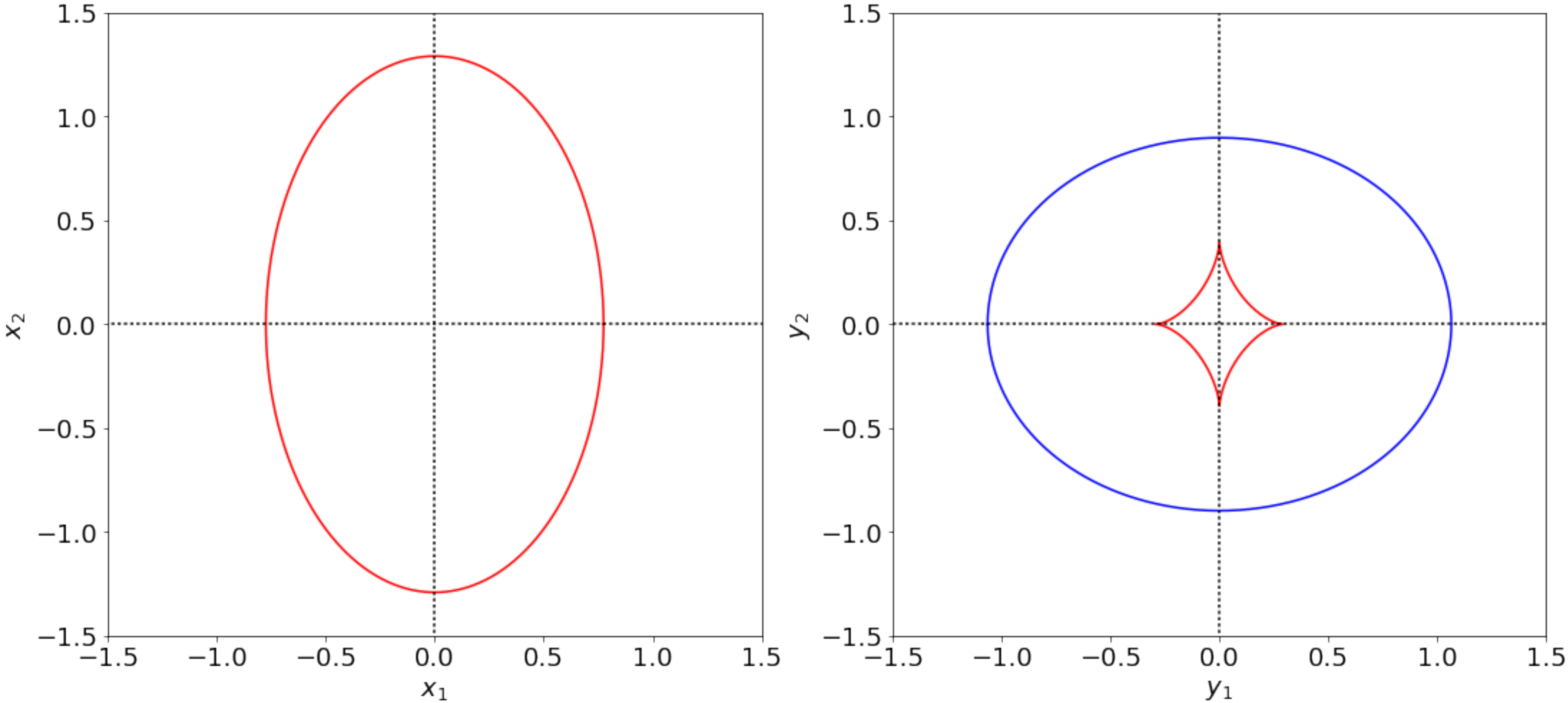
This is important for the image multiplicity…

CRITICAL LINE, CUT, CAUSTIC

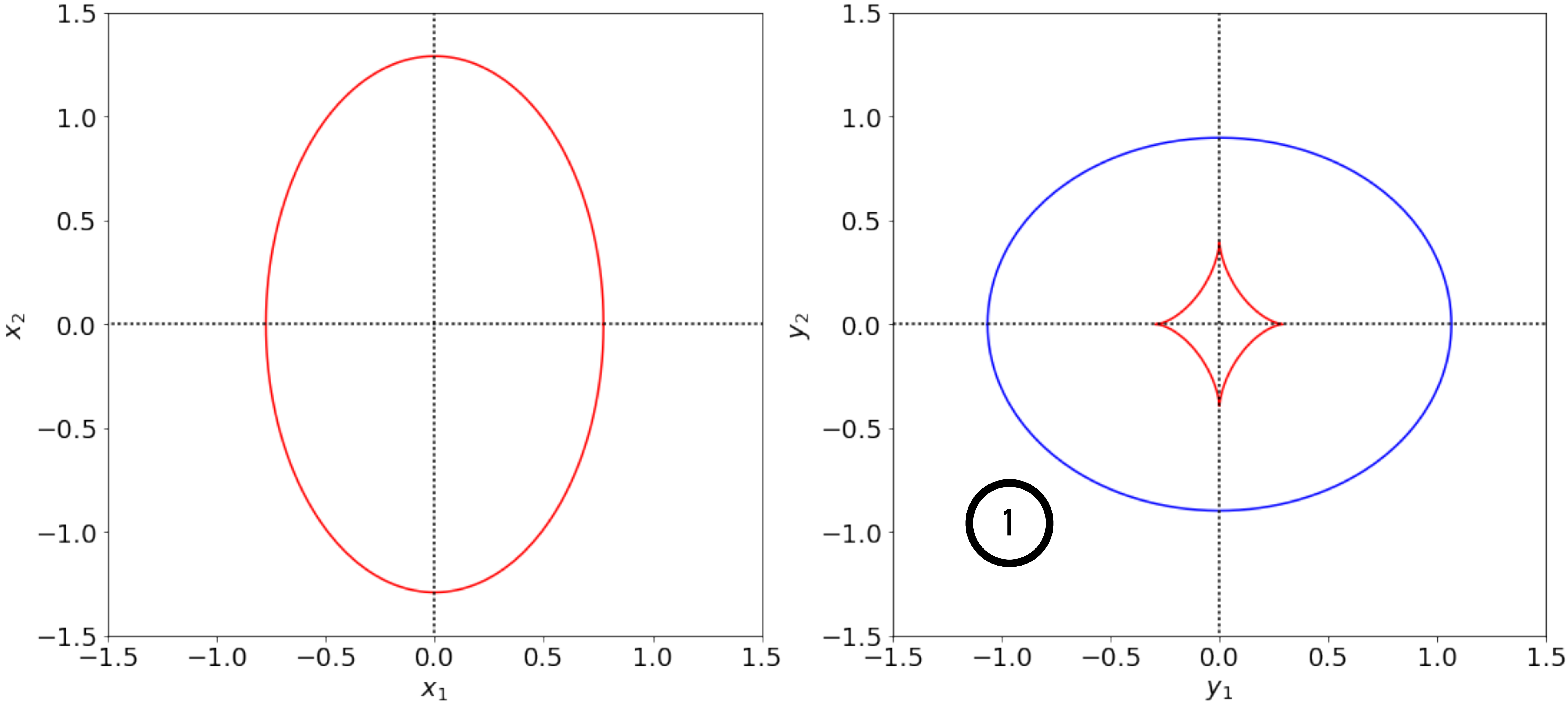


This is important for the image multiplicity…

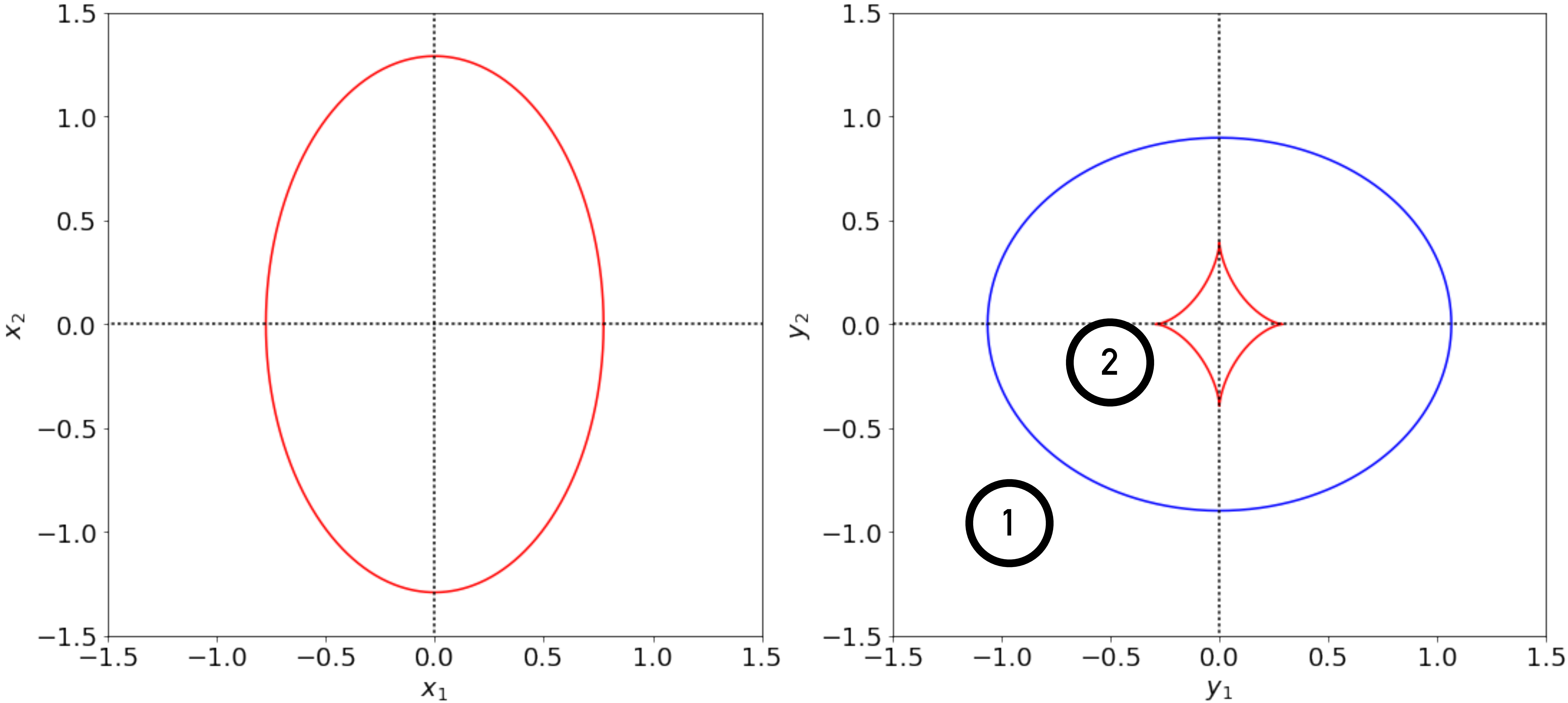
CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC