# GRAVITATIONAL LENSING

## LECTURE 19

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*AA 2016-2017*

# EXTENDED LENSES

- Cosmic structures like galaxies and galaxy clusters are characterized by bound mass distributions, which cannot be approximated by point lenses
- Indeed these are *extended lenses*, and their lensing properties are determined by e.g. their surface mass density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\ \mathrm{d}z$$

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\ \mathrm{d}^2\xi'$$

# EXTENDED LENSES

- Recall that the surface density is related to the lensing potential by

$$\triangle_\theta \Psi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G} \frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

# WHAT ARE THE RELEVANT PROPERTIES OF THE LENSES?

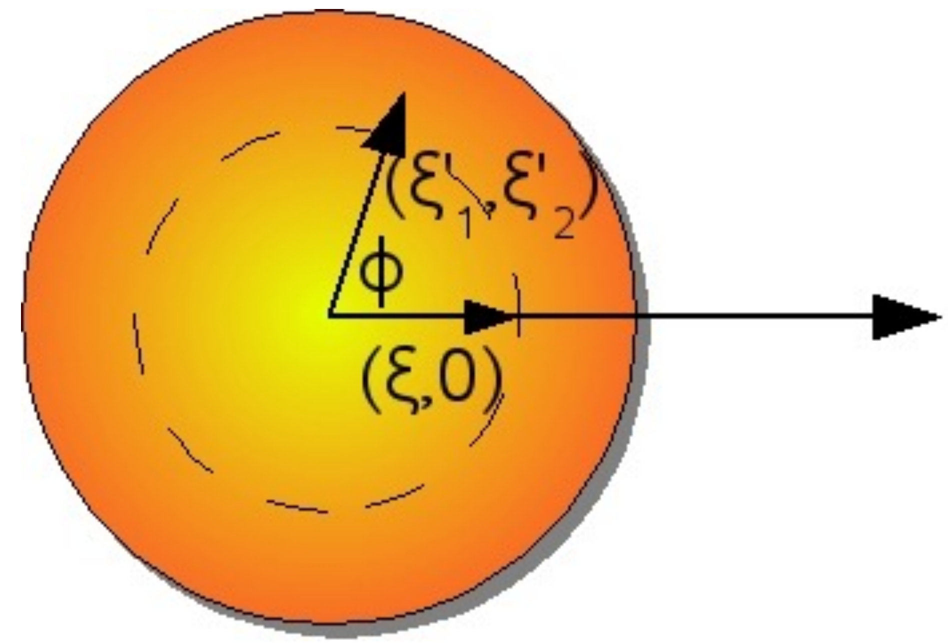
- The surface density distribution of a lens (and its potential) can be characterized by means of
    - the profile
    - the shape of the iso-density (iso-potential) contours
    - the smoothness
    - the environment where the lens resides
- In this and in the following lessons, we will study how these features determine the ability of a mass distribution to produce lensing effects.
- We will do that by building analytical models with increasing level of complexity.

# AXIALLY SYMMETRIC, CIRCULAR LENSES

- Axially symmetric, circular models are the simplest lens models for describing extended mass distributions
- For these lenses $\Sigma(\vec{\xi}) = \Sigma(|\vec{\xi}|)$
- Several quantities relevant for lensing can be derived in a simple manner by using the symmetry properties of the lens.
- One example is the deflection angle…

# DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} \, \mathrm{d}^2\xi'$$




$$\begin{aligned}
\vec{\xi} - \vec{\xi}' &= (\xi - \xi' \cos\phi, -\xi' \sin\phi) \\
|\vec{\xi} - \vec{\xi}'|^2 &= \xi^2 + \xi'^2 \cos^2\phi - 2\xi\xi' \cos\phi + \xi'^2 \sin^2\phi \\
&= \xi^2 + \xi'^2 - 2\xi\xi' \cos\phi
\end{aligned}$$

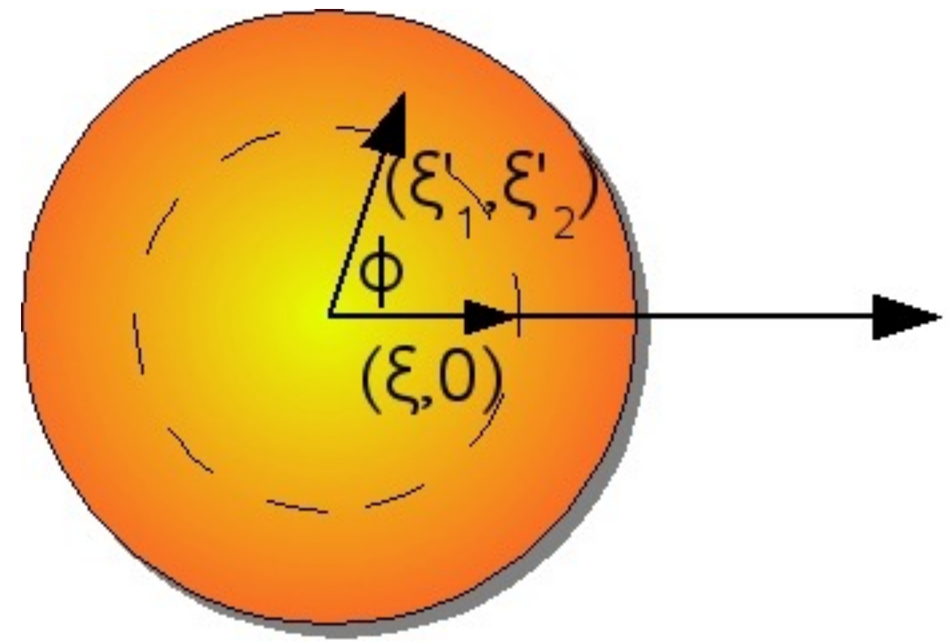
$$\begin{aligned}
\hat{\alpha}_1(\vec{\xi}) &= \frac{4G}{c^2} \int_0^\infty \mathrm{d}\xi' \xi' \Sigma(\xi') \int_0^{2\pi} \mathrm{d}\phi \frac{\xi - \xi' \cos\phi}{\xi^2 + \xi'^2 - 2\xi\xi' \cos\phi} \\
\hat{\alpha}_2(\vec{\xi}) &= \frac{4G}{c^2} \int_0^\infty \mathrm{d}\xi' \xi' \Sigma(\xi') \int_0^{2\pi} \mathrm{d}\phi \frac{-\xi' \sin\phi}{\xi^2 + \xi'^2 - 2\xi\xi' \cos\phi}
\end{aligned}$$

$$\boxed{\hat{\alpha}(\xi) = \frac{4G}{c^2} \frac{2\pi \int_0^\xi \Sigma(\xi')\xi' \, \mathrm{d}\xi'}{\xi} = \frac{4GM(\xi)}{c^2\xi}}$$

*For an axially symmetric lens, the deflection is "radial": it depends only on the distance from the lens center.*

# DEFLECTION ANGLE OF AN AXIALLY SYMMETRIC LENS

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{(\vec{\xi}-\vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi}-\vec{\xi'}|^2}\,\mathrm{d}^2\xi'$$




$$\begin{aligned}
\vec{\xi}-\vec{\xi'} &= (\xi-\xi'\cos\phi, -\xi'\sin\phi) \\
|\vec{\xi}-\vec{\xi'}|^2 &= \xi^2+\xi'^2\cos^2\phi-2\xi\xi'\cos\phi+\xi'^2\sin^2\phi \\
&= \xi^2+\xi'^2-2\xi\xi'\cos\phi
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\hat{\alpha}_2(\vec{\xi}) &= \frac{4G}{c^2}\int_0^\infty \mathrm{d}\xi'\xi'\Sigma(\xi')\int_0^{2\pi}\mathrm{d}\phi\frac{-\xi'\sin\phi}{\xi^2+\xi'^2-2\xi\xi'\cos\phi}
\end{aligned}$$

$\frac{2\pi}{\xi}$ $\qquad$ $\xi' < \xi$

$$\boxed{\hat{\alpha}(\xi) = \frac{4G}{c^2}\frac{2\pi\int_0^\xi \Sigma(\xi')\xi'\,\mathrm{d}\xi'}{\xi} = \frac{4GM(\xi)}{c^2\xi}}$$

*For an axially symmetric lens, the deflection is “radial”: it depends only on the distance from the lens center.*

# DEFLECTION ANGLE IN ADIMENSIONAL FORM

$$\boxed{\hat{\alpha}(\xi) = \frac{4G}{c^2}\frac{2\pi\int_0^{\xi}\Sigma(\xi')\xi'\,\mathrm{d}\xi'}{\xi} = \frac{4GM(\xi)}{c^2\xi}}$$

*The adimensional, reduced deflection angle is:*

$$\begin{aligned}
\alpha(x) &= \frac{D_{\mathrm{L}}D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}}\hat{\alpha}(\xi_0 x) \\
&= \frac{D_{\mathrm{L}}D_{\mathrm{LS}}}{\xi_0 D_{\mathrm{S}}}\frac{4GM(\xi_0 x)}{c^2\xi}\frac{\pi\xi_0}{\pi\xi_0} \\
&= \frac{M(\xi_0 x)}{\pi\xi_0^2\Sigma_{\mathrm{cr}}}\frac{1}{x} \equiv \frac{m(x)}{x},
\end{aligned}$$

*We have introduced the dimensionless mass:*

$$\alpha(x) = \frac{2}{x}\int_0^x x'\kappa(x')\mathrm{d}x' \Rightarrow m(x) = 2\int_0^x x'\kappa(x')\mathrm{d}x'.$$

# LENS EQUATION

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \qquad\qquad \vec{\alpha}(\vec{x}) = \frac{m(\vec{x})}{x^2}\vec{x}$$

*Given that the deflection angle and x are parallel, so will be y!*

$$y = x - \frac{m(x)}{x}$$

# CONVERGENCE AND SHEAR

*To compute convergence and shear, we need to compute the partial derivatives of the deflection angle:*

$$\begin{aligned}
\frac{\partial \alpha_1}{\partial x_1} &= \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_1^2}{x^3} + m\frac{x_2^2 - x_1^2}{x^4}\,, \\
\frac{\partial \alpha_2}{\partial x_2} &= \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_2^2}{x^3} + m\frac{x_1^2 - x_2^2}{x^4}\,, \\
\frac{\partial \alpha_1}{\partial x_2} &= \frac{\partial \alpha_2}{\partial x_1} = \frac{\mathrm{d}m}{\mathrm{d}x}\frac{x_1 x_2}{x^3} - 2m\frac{x_1 x_2}{x^4}
\end{aligned}$$

# CONVERGENCE AND SHEAR

*From the partial derivatives of the deflection angle we obtain:*

$$\begin{aligned}
\kappa(x) &= \frac{1}{2x}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\,, \\
\gamma_1(x) &= \frac{1}{2}(x_2^2 - x_1^2)\left(\frac{2m(x)}{x^4} - \frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}\right) \\
\gamma_2(x) &= x_1x_2\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3} - \frac{2m(x)}{x^4}\right)\,.
\end{aligned}$$

# CONVERGENCE AND SHEAR

*For an axially symmetric lens the following relation exists between convergence and shear:*

$$\gamma(x) = \frac{m(x)}{x^2} - \kappa(x)$$

$$\frac{m(x)}{x^2} = 2\pi \frac{\int_0^x x' \kappa(x') \mathrm{d}x'}{\pi x^2} = \overline{\kappa}(x)$$

$$\gamma(x) = \overline{\kappa}(x) - \kappa(x)$$

# LENSING JACOBIAN

$$A = I - \frac{m(x)}{x^4}\begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} - \frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{1}{x^3}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}.$$

# DETERMINANT OF THE LENSING JACOBIN

$$
\begin{aligned}
\det A &= \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1 - \frac{m(x)}{x^2}\right)\left[1 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{m(x)}{x}\right)\right] \\
&= \left(1 - \frac{m(x)}{x^2}\right)\left(1 + \frac{m(x)}{x^2} - 2\kappa(x)\right) \\
&= \left(1 - \frac{\alpha(x)}{x}\right)\left(1 - \frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right) .
\end{aligned}
$$

# CRITICAL LINES

*The critical lines can be found by setting* $\det A = 0$*:*

$$m(x)/x^2 = 1$$

$$\mathrm{d}(m(x)/x)/\mathrm{d}x = 1$$

*The first defines the "tangential" critical line. The second the "radial" critical line.*

*If they exist, what is the shape of the critical lines for this kind of lenses? What are the caustics?*