Examples of research problems for high school students

1. Fitting one triangle inside another. Let triangles P and Q have edge lengths p_1, p_2, p_3 and q_1, q_2, q_3 respectively. What are necessary and sufficient conditions on the p_i and q_i for P to contain a congruent copy of Q?

2. Longest chords of polygons. Let P be a plane convex n-gon with side lengths $a_1, \ldots, a_n$. Let b_i be the length of the longest chord of P parallel to the ith side.

a) Prove that

$$\sqrt{8} < \sum_{i=1}^{n} \frac{a_i}{b_i} \leqslant 4.$$

When does the equality on the right hold?

b) Is it true that

$$3 \leqslant \sum_{i=1}^{n} \frac{a_i}{b_i} \ ?$$

c) What are 3-dimensional analogs of these inequalities?

3. Reptiles. An n-reptile is a two dimensional region that can be tiled by n congruent tiles, each similar to the whole region.

a) Describe all n-reptiles that are convex polygons.
b) Give examples of non-convex n-reptiles.
c) Give examples of reptiles with "holes".
d) Does there exist a 2-reptile that is also a 3-reptile?

4. Integer parts of powers. Consider the sequences $[(\frac{3}{2})^n]$ and $[(\frac{4}{3})^n]$ where n is a positive integer.

a) Prove that there are infinitely many composite numbers in each of these sequences.
b) Do the sequences also contain infinitely many primes?

5. Collection of disks with no three in a line. What is the radius of the smallest disk in which one can place n unit disks so that no straight line intersects more than two of them?

6. Difference of two gives one. Solve the following equation

$$x^m - 2y^n = 1$$

in integers x, y, m, $n > 1$.

7. Omnipotent Queens. What is the minimal number $Q(n)$ of Queens one has to place on the $n \times n$ chessboard to attack all its cells?

8. Irreducible compositions. Let $g(x) = (x - a_1) \ldots (x - a_n)$ be a polynomial with distinct integer roots $a_1, \ldots, a_n$.

a) Denote $f(x) = ax^2 + bx + c$. Find all integers a, b, c such that the polynomial $f \circ g$ is irreducible over $\mathbb{Q}$.
b) Investigate the same question for polynomials f of degree greater than 2.

9. A problem on neighbors. The *diameter* of a polygon (not necessarily convex) is the greatest distance between any two of its points.

a) A unit square is divided into convex polygons. Suppose that the diameter of each of these polygons does not exceed $\frac{1}{30}$. Prove that there is a polygon P with six or more *neighbors*, that is, polygons touching P in at least one point.

b) The same problem in the general case that the polygons are not necessarily convex.

c) Replace in a) the constant $\frac{1}{30}$ by a positive real number $\epsilon > 0$. Find the greatest positive integer $N(\epsilon)$ such that, for any partition of the unit square into convex polygons of diameter $\leq \epsilon$, at least one of these polygons would have $N(\epsilon)$ or more neighbors.

d) The problem c) in the case that the polygons are not necessarily convex.