PROBLEMS FOR THE 1^{st} FRENCH TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

Keywords: 1. additive number theory, diophantine equations – 2. number theory, congruence – 3. combinatorics, discrete dynamics – 4. rearrangements – 5. analytic geometry, convergence – 6. inequalities – 7. games, graph theory – 8. metric geometry, inequalities – 9. algebra, rational fractions – 10. algebra, polynomials.

Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}, \mathbb{Q}, \mathbb{C}$	sets of integer, rational and complex numbers
$\mathbb{R}, \mathbb{R}^2$	real line and real plane
$\mathbb{F}_p = \{0, 1, \ldots, p-1\}$	field of residue classes modulo a prime p
$K[x], K(x)$	sets of polynomials and rational fractions with coefficients in K
φ	Euler's totient function
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$

1. Arithmetic Functions

1. Let $n > 2$ be a natural number, and let $1 = a_1 < a_2 < \ldots < a_{\varphi(n)} = n-1$ be the positive integers less than n which are coprime to n. Find the following sums

$$\sum_{i=1}^{\varphi(n)} a_i \quad \text{and} \quad \sum_{i=1}^{\varphi(n)/2} a_i.$$

2. Denote by $\psi(n)$ the number of triples $(x, y, z) \in \mathbb{Z}^3$ such that

$$0 \leq x < y, \quad 0 < z, \quad \gcd(x, y, z) = 1 \quad \text{and} \quad yz = n.$$

Find a formula for $\psi(n)$ depending on n and its prime divisors.

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3. Let a, b be positive integers. Denote by $f(a, b, n)$ the number of solutions of the equation

$$ax + by = n$$

in natural numbers x and y. Find a formula for $f(a, b, n)$.

4. Denote by $g(n)$ the number of quadruples (a, b, x, y) of natural numbers such that

$$a < b, \quad \gcd(a, b) = \gcd(x, y) = 1 \quad \text{and} \quad ax + by = n.$$

Give a formula for $g(n)$ or estimate it.

5. Suggest and study analogous questions for linear equations $a_1x_1 + a_2x_2 + \ldots + a_kx_k = n$.

2. Fibonacci and Beyond

Let a and b be integers. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by the recurrence relation

$$x_n = ax_{n-1} + bx_{n-2} \quad \text{for any } n \geq 3, \text{ where } x_2 = x_1 = 1.$$

Let p be a natural number. Denote by $(\overline{x}_n)_{n\in\mathbb{N}}$ the sequence of remainders of $(x_n)_{n\in\mathbb{N}}$ after the division by p: for all $n \in \mathbb{N}$, we have $\overline{x}_n \in \{0, 1, \ldots, p-1\}$ such that $\overline{x}_n \equiv x_n \bmod p$.

1. Take $a = b = 1$, then $(x_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence. Suppose that p is prime.

a) Show that the sequence $(\overline{x}_n)_{n\in\mathbb{N}}$ is periodic, *i.e.* there exists $k \in \mathbb{N}$ such that $\overline{x}_{n+k} = \overline{x}_n$ for any $n \in \mathbb{N}$. The minimal such k is called the *period* of the sequence.

b) Show that if $p \equiv 1, 4 \bmod 5$, then $\overline{x}_{p-1} = 0$ and $\overline{x}_p = \overline{x}_{p+1} = 1$.

c) Show that if $p \equiv 2, 3 \bmod 5$, then $\overline{x}_{p-1} = 1$, $\overline{x}_p = p - 1$ and $\overline{x}_{p+1} = 0$.

d) Show that if $p \equiv 3 \bmod 4$, then $\overline{x}_n \neq 0$ for any odd n.

e) Find lower and upper bounds for the period of the sequence $(\overline{x}_n)_{n\in\mathbb{N}}$.

2. Investigate the previous questions in the case that p is not prime.

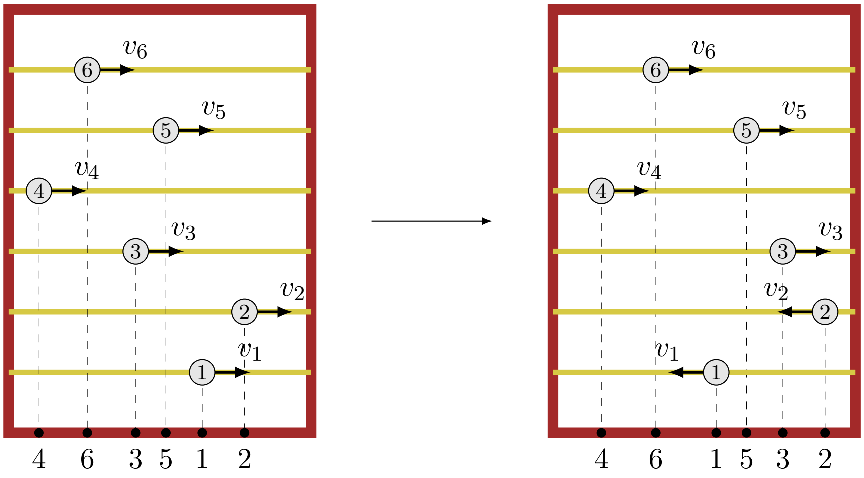
3. Study the problem for other values of a and b, for instance when $a = 2012$ and $b = 2011$.

3. An Abacus

Consider a simple abacus: a solid frame with n horizontal wires and one bead on each wire. We assume that the beads are numbered from 1 to n and slide on wires with constant nonzero speed $v_1, v_2, \ldots, v_n$ respectively. If a bead hits the frame, then it changes its direction to the opposite. At any moment when no two beads lie on the same vertical line we project the numbers of the beads to the lower edge of the frame and obtain a permutation of the first n natural numbers (see the figure).

1. For which initial position and velocities of the beads all permutations of the natural numbers $1, 2, \ldots, n$ will occur in such a way? First, consider the cases $n = 2, 3, 4, 5$ and suppose that the beads are simultaneously leaving the left edge of the frame and slide to the right with some distinct velocities of your choice.

2. Do there exist an initial position and velocities such that any permutation of the first n natural numbers will occur exactly once before reiteration?



3. In general, describe the set of permutations that will occur, given an initial position and velocities of the beads.

4. Study abacuses having more than one bead on some wires. For instance, investigate the case that one of the wires contains several beads that move with the same velocity, and if two beads collide then they switch their directions to opposite.

5. Study the analogous problem when there are accelerations.

4. **Rearrangements**

Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be positive integers. Denote by $D(a_1, a_2, \ldots, a_n)$ the set of all pairs (a, b) of positive integers such that there exists a rearrangement $x_1, x_2, \ldots, x_n$ of the sequence $a_1, a_2, \ldots, a_n$ with the following property:

$$|x_i - x_{i+1}| = a \text{ or } b \quad \text{for any } 1 \leq i < n.$$

1. Given $a, b \in \mathbb{N}$, find all positive integers n such that $(a, b) \in D(1, 2, \ldots, n)$.

2. Given $a, b, K \in \mathbb{N}$, find the largest integer n such that $(a, b) \in D(a_1, a_2, \ldots, a_n)$ for some distinct natural numbers $a_1, a_2, \ldots, a_n$ not exceeding K.

3. Given $a, b, n \in \mathbb{N}$, find the minimal possible value of the difference $a_n - a_1$ such that the pair (a, b) belongs to the set $D(a_1, a_2, \ldots, a_n)$, where $a_1 < a_2 < \ldots < a_n$.

4. Denote by $C(a_1, a_2, \ldots, a_n)$ the set of all pairs (a, b) of positive integers such that there exists a rearrangement $x_1, x_2, \ldots, x_n$ of the sequence $a_1, a_2, \ldots, a_n$ with the property:

$$|x_i - x_{i+1}| = a \text{ or } b \quad \text{for any } 1 \leq i \leq n, \text{ where } x_{n+1} = x_1.$$

Study the questions 1–3 for the set $C(a_1, a_2, \ldots, a_n)$.

5. Investigate analogous problems for triples (a, b, c), quadruples (a, b, c, d) and so on.

5. **Convergent Triangles**

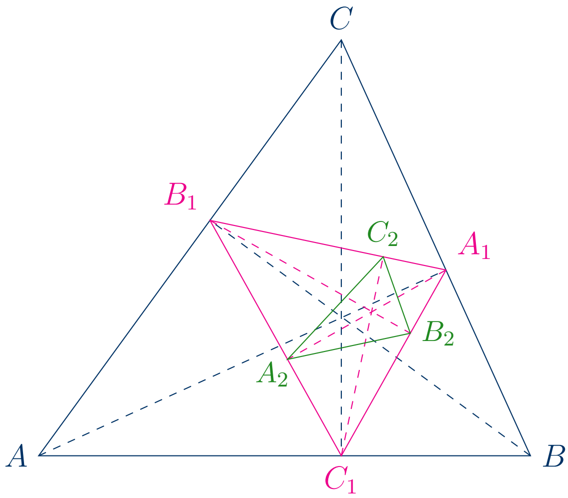
Let A, B, C be the vertices of a triangle, and K_1, L_1, M_1 the midpoints of the sides BC, AC and AB respectively. Denote by f the operation of transition from a triangle ΔABC to the triangle whose vertices are the midpoints of the sides of ΔABC. By iterating this operation, we define the sequence of triangles $\Delta K_n L_n M_n$ such that

$$\Delta K_1 L_1 M_1 = f(\Delta ABC) \quad \text{and} \quad \Delta K_{n+1} L_{n+1} M_{n+1} = f(\Delta K_n L_n M_n) \quad \text{for all } n \in \mathbb{N}.$$

1. Prove that the sequence of triangles $\Delta K_n L_n M_n$ converges to a point P, determine that point and estimate the speed of convergence.

Now, we consider the operation g of transition from a triangle ΔABC to the triangle whose vertices are the feet of the altitudes of ΔABC. Let

$$\Delta A_1B_1C_1 = g(\Delta ABC) \quad \text{and} \quad \Delta A_{n+1}B_{n+1}C_{n+1} = g(\Delta A_nB_nC_n) \quad \text{for all } n \in \mathbb{N}.$$



2. Find a formula for the point A_1 in terms of the points A, B and C.

3. Is it true that if the limit of the sequence of triangles $\Delta A_nB_nC_n$ exists, then it necessarily falls into the circumscribed circle of ΔABC? Consider also the circumscribed circle of the triangle $\Delta A_1B_1C_1$.

4. Investigate the convergence of the sequence $(\Delta A_nB_nC_n)_{n\in\mathbb{N}}$. What happens if one of the triangles $\Delta A_nB_nC_n$ is right?

5. Denote by H the limit of the sequence $(\Delta A_nB_nC_n)_{n\in\mathbb{N}}$ if that limit exists. Show that $H = H(\widehat{A}, \widehat{B}, \widehat{C})$ is a continuous function of the angles $\widehat{A}, \widehat{B}, \widehat{C}$, but that it is nowhere differentiable. Estimate the difference $H(\widehat{A}+\epsilon, \widehat{B}-\epsilon, \widehat{C}) - H(\widehat{A}, \widehat{B}, \widehat{C})$ in terms of $\epsilon > 0$.

6. Study the analogous problem for the feet of the angle bisectors.

6. **Inequalities for Sides**

Let T be the set of triples (a, b, c) of positive real numbers such that there exists a non-degenerate triangle with side length a, b, c.

1. Let $n \geq 3$ be an integer, and a_1, a_2, …, a_n positive real numbers satisfying the inequality

$$(a_1^2 + a_2^2 + \ldots + a_n^2)^2 > (n-1)(a_1^4 + a_2^4 + \ldots + a_n^4).$$

Show that every triple (a_i, a_j, a_k) with distinct indices $1 \leq i, j, k \leq n$ belongs to T.

2. For a given positive integer n, find the infimum and supremum of the expression

$$\frac{\sqrt[n]{a^n+b^n} + \sqrt[n]{b^n+c^n} + \sqrt[n]{c^n+a^n}}{a+b+c} \quad \text{over all } (a,b,c) \in T.$$

3. Describe the set of positive real numbers x such that the inequality $a^x < b^x + c^x$ holds for any triple $(a, b, c) \in T$ with $a \geq b \geq c$.

4. For a given positive integer n, find the largest real $k = k(n)$ such that the inequality

$$a^n + b^n + c^n \geq k(a + b + c)^n$$

holds for all triples $(a, b, c) \in T$.

5. In the questions $2 - 4$, you may also replace the set T by

a) the set T_{ac} of triples (a, b, c) for which there exists an *acute-angled* triangle with side length a, b, c;
b) the set T_{ob} of triples (a, b, c) for which there exists an *obtuse-angled* triangle with side length a, b, c;
c) the set T_{re} of triples (a, b, c) of positive real numbers such that $a^2 = b^2 + c^2$.

7. **A Game on Cycles**

Consider an undirected graph G in the plane possibly with loops but with no multiple edges. Let V be the set of its vertices, and E the set of its edges. A *path of length* k is a sequence of distinct edges (v_0, v_1), (v_1, v_2), …, $(v_{k-1}, v_k) \in E$. We call $v_0, v_k \in V$ respectively the *start* and *end vertex* of the path, and $v_1, \dots, v_{k-1} \in V$ are the *internal vertices*. We say that a path is *simple* if its vertices are distinct, except possibly for the start and end vertex. A *cycle of length* k is a **simple** path of length k such that $v_0 = v_k$. (So, the cycles of length 1 are exactly the loops of G, and there is no cycle of length 2.)

In directed graphs, an oriented edge is called an *arrow*. One defines analogously the notions of a *directed path*, a *simple directed path* and a *directed cycle*. Note that in a directed graph, for any two vertices a, b, there exists at most one arrow from a to b and at most one arrow from b to a, which are denoted by (a, b) and (b, a) respectively. (So, there can be directed cycles of length 2.)

Let $n \geq 5$ and $k \geq 1$ be positive integers. Suppose that n points are given in the plane.

1. The high school students Clara and Carl play the following game: they alternately draw an edge between two points that are not connected by an edge yet. Clara starts the game. In each of the following cases, find a winning strategy for one of the students:

a) The loser is the one after whose move the obtained undirected graph will contain a cycle of length at least k.
b) The loser is the one after whose move the obtained undirected graph will contain a cycle of length exactly k.
c) The winner is the one after whose move the obtained undirected graph will contain a cycle of length at least k.
d) The winner is the one after whose move the obtained undirected graph will contain a cycle of length exactly k.

Consider two different problems: when loops are allowed and when loops are not allowed.

2. Formulate and study similar questions in the directed case.

3. Investigate variations of the game when Clara and Carl draw edges/arrows of two distinct colours, say red and blue respectively. The loser/winner is the one after whose move the obtained undirected/directed graph will contain a one-color cycle of length at least/exactly k.

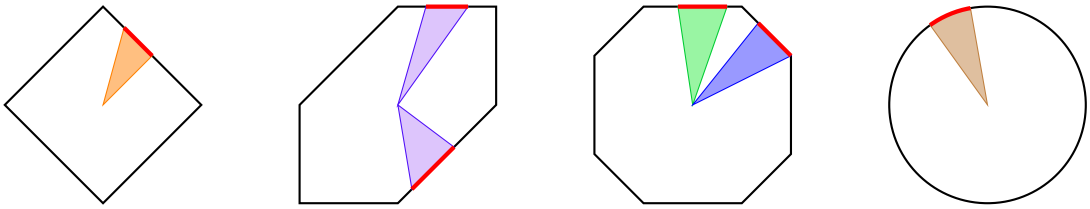
4. Suggest and study the analogous game for more than two players.

8. **Average Chord Length**

Let L be a closed curve symmetric about the origin O and bounding a convex shape in the plane $\mathbb{C} = \mathbb{R}^2$. Define the L-*distance* $d_L(M, N)$ between two distinct points M, N in $\mathbb{R}^2$ as the ratio $d(M, N)/d(O, P)$, where d is the usual Euclidean distance and P is an intersection point of L and the parallel to (MN) through O.

A *chord* is a segment joining two points of L. We are interested in the *average chord length of* L, in the following sense:

- consider a parametrization of L by $M(\theta) = r(\theta)e^{i\theta}$, $\theta \in [0, 2\pi]$,
- integrate the quantity $d_L(M(u), M(v))$ times the areas of the elementary triangles $\Delta OM(u)M(u + du)$ and $\Delta OM(v)M(v + dv)$ for $u, v \in [0, 2\pi]$,
- divide the result by the square of the area of the convex shape bounded by L.



1. Show that the elementary triangle $\Delta OM(u)M(u + du)$ has area $\frac{1}{2}r(u)^2\, du$.

2. Show that the average chord length is

$$E(L) = \frac{\int_{u=0}^{2\pi} \int_{v=0}^{2\pi} d_L(M(u), M(v))r(u)^2 r(v)^2\, du\, dv}{\left(\int_{u=0}^{2\pi} r(u)^2\, du\right)^2}$$

3. Show that $1/2 < E(L) < 2$. Improve these bounds the best you can.

4. Compute $E(L)$ when L is a) a square; b) a parallelogram; c) a circle.

5. How does $E(L)$ vary when L is changed by a linear transformation?

6. Suppose L is a polygon with vertices $v_0, v_1, \ldots, v_{2n-1}$. Denote by s_i the side $[v_i v_{i+1}]$, where $v_{2n} = v_0$.

a) For $t \in [0, 1]$, denote by $s_i(t)$ the point $(1-t)v_i + tv_{i+1}$ on s_i. Show that $d_L(s_i(s), s_j(t))$ is piecewise linear in s and t.

b) Relate $E(L)$ to the quantities

$$a_{ij} = \int_{s=0}^{1} \int_{t=0}^{1} d_L(s_i(s), s_j(t))\, ds\, dt$$

c) Find an algorithm to compute $E(L)$ when L is a polygon.

7. Find a formula for $E(L)$ when L is a regular n-gon, where n is even and $n \geqslant 4$.

8. Show that any convex hexagon symmetric about O can be sent by a linear transform to a hexagon $H_{x,y}$ with vertices (x, y), $(1, 1)$, $(-1, 1)$, $(-x, -y)$, $(-1, -1)$, $(1, -1)$, where $x \geqslant 1$, $y \geqslant 0$ and $x + y \leqslant 2$.

9. Find a formula for $E(H_{x,y})$ for such (x, y).

10. Find the range of $E(H)$ for all hexagons.

11. Compute $E(L)$ for examples of L of your choice.

12. Find or conjecture the range of $E(L)$ for all possible L.

13. Formulate and study the analogous problem in dimension 3. For instance, compute $E(L)$ when L is a) a sphere; b) a regular octahedron; c) a cube.

9. **Roots of Unity**

Let n be a natural number. There are exactly n distinct complex numbers satisfying the equation $z^n = 1$, they are called n^{th} *roots of unity.*

Let K be a field. We denote by $K(x)$ the set of rational fractions with coefficients in K, that is,

$$\frac{p(x)}{q(x)} = \frac{a_k x^k + \ldots + a_1 x + a_0}{b_l x^l + \ldots + b_1 x + b_0},$$

where $p(x)$, $q(x)$ are polynomials with coefficients in K and $q(x) \not\equiv 0$.

1. Find all rational fractions $f \in \mathbb{Q}(x)$ such that, for any $n \in \mathbb{N}$ and each n^{th} root of unity $s \in \mathbb{C}$, the number $f(s)$ is an n^{th} root of unity as well.

2. Find all rational fractions $f \in \mathbb{Q}(x)$ with the following property:

> there exists $n_0 \in \mathbb{N}$ such that, for any natural $n \geq n_0$ and each n^{th} root of unity $s \in \mathbb{C}$, the number $f(s)$ is an n^{th} root of unity as well.

3. Study the same questions for finite extensions of $\mathbb{Q}$. For instance when $K = \mathbb{Q}[\sqrt{5}]$, find all rational fractions $f \in K(x)$ preserving the set of n^{th} roots of unity for any natural number n large enough.

10. **Multiple Roots**

Let K be some field, and $K[x]$ the set of polynomials with coefficients from K. Given a polynomial $Q \in K[x]$ and one of its roots $a \in K$, the *multiplicity* of a is the largest integer m such that $(x - a)^m$ divides Q. A root a is called *multiple* if its multiplicity is at least 2.

1. Consider the field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ of residue classes modulo a prime p. For a given $n \in \mathbb{N}$, denote by α_n the probability that a random polynomial $Q \in \mathbb{F}_p[x]$ of degree n has a multiple root in $\mathbb{F}_p$. Find α_n or estimate it. What is the limit of α_n as $n \to \infty$?

2. The same questions for a finite extension of $\mathbb{F}_p$.

3. The sequence of complex polynomials $P_n(x) \in \mathbb{C}[x]$ is such that

$$P_1(x) = ax + b \quad \text{and} \quad P_n(x) = x\Big(P_{n-1}(x) + P_{n-1}(cx + d)\Big) \quad \text{for any } n \geq 2,$$

where a, b, c, d are given complex numbers.

a) Find all $n \in \mathbb{N}$ such that the polynomial $P_n(x)$ has a multiple root.

b) Find all $a, b, c, d \in \mathbb{C}$ such that **none** of the polynomials $P_n(x)$ has a multiple root.

c) Study the analogous problem when the polynomial $P_1(x)$ is nonlinear. You can also consider the sequence

$$P_n(x) = Q(x) \cdot \Big(P_{n-1}(x) + P_{n-1}(cx + d)\Big), \quad n \geq 2,$$

for a given polynomial $Q(x) \in \mathbb{C}[x]$.

4. Suggest and study additional directions of research.

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