1. Consider the function $f(x) = \sqrt{5x}$ on $[4, 6]$. It is clear that $p = 5$ is a fixed point of f.

   (a) (10 points) Check all of the conditions of Theorem 2.8 and justify a conclusion that the iterative sequence $\{p_n\}$ (for $p_0 \in [4, 6]$ and for $p_0 \neq 5$) converges linearly to $p = 5$.

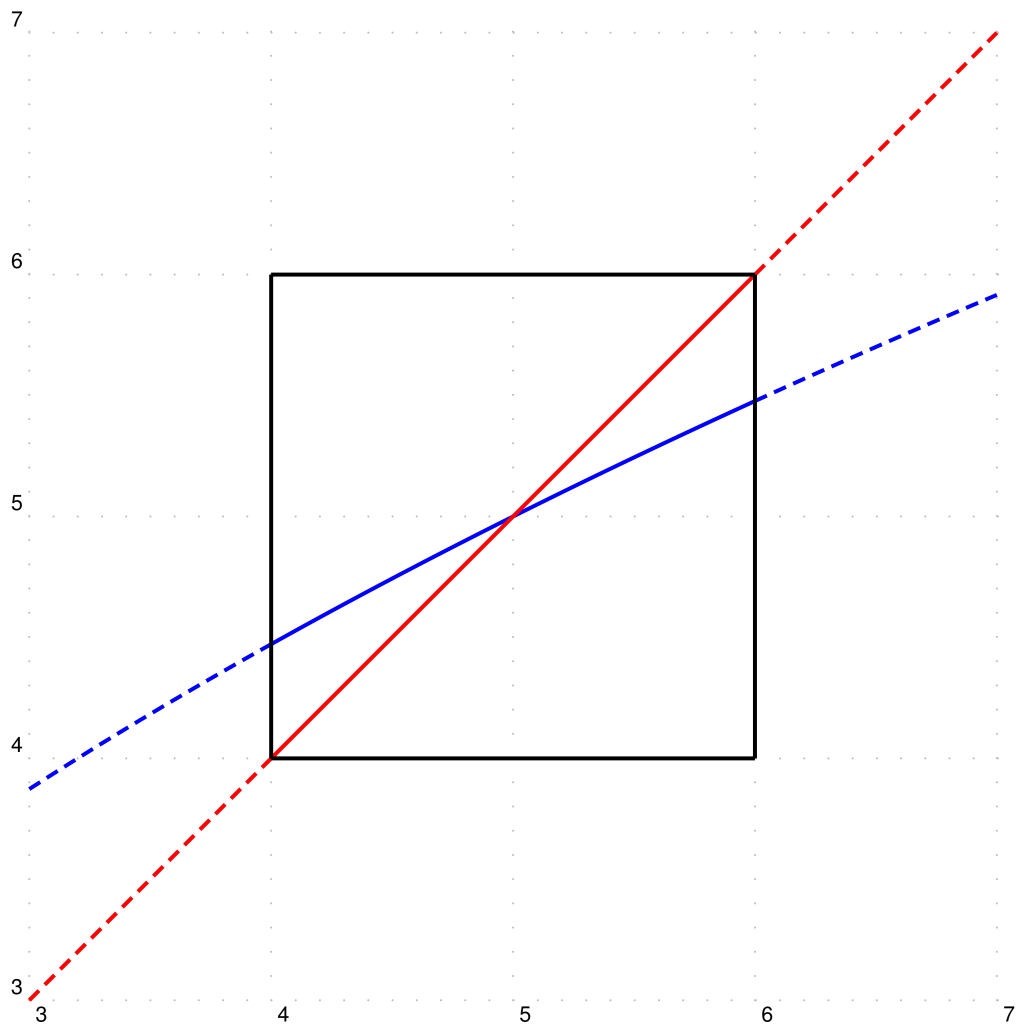
   First we must check that $f(x)$ is continuous on $[4, 6]$, such that $\forall x \in [4, 6], f(x) \in [4, 6]$.

$$f(x) = \sqrt{5x}$$

$$\begin{aligned} f(4) &= \sqrt{5 \times 4} \\ &= 2\sqrt{5} \end{aligned} \qquad\qquad \begin{aligned} f(6) &= \sqrt{5 \times 6} \\ &= \sqrt{30} \end{aligned}$$

$$f(x) \in [2\sqrt{5}, \sqrt{30}]$$

$$\boxed{\therefore \forall x \in [4, 6],\ f(x) \in [4, 6]}$$



The blue line is the function $f(x) = \sqrt{5x}$ and the red line is the function $f(x) = x$

A we will see later, the derivative of thus function is never less than zero on [4,6], the function is continous and increasing. Thus the first condition of Theorem 2.8 is met.

Now we must find a positive constant $k \,\Big|\, k < 1 \wedge \forall x \in (4, 6), \quad | f'(x) | \leq k.$

$$f'(x) = \frac{\sqrt{5}}{2\sqrt{x}}$$

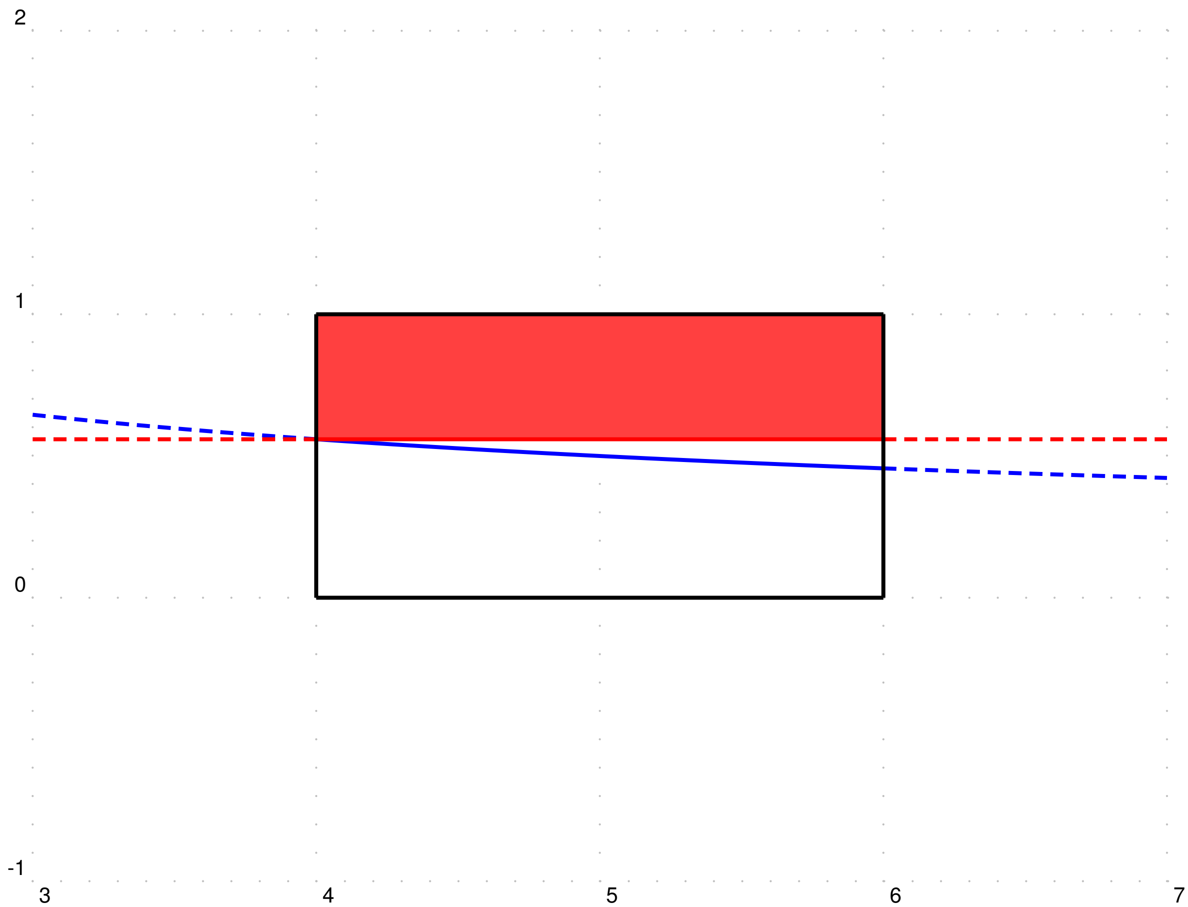
$$f'(4) = \frac{\sqrt{5}}{2\sqrt{4}} \qquad\qquad f'(6) = \frac{\sqrt{5}}{2\sqrt{6}}$$
$$= \frac{\sqrt{5}}{4} \qquad\qquad = \frac{\sqrt{30}}{12}$$

$$f'(x) \in (\frac{\sqrt{30}}{12}, \frac{\sqrt{5}}{4})$$

Thus,

$$\frac{\sqrt{30}}{12} < f'(x) < \frac{\sqrt{5}}{4} \leq k < 1$$

$$\boxed{\therefore k \in [\frac{\sqrt{5}}{4}, 1)}$$



The blue line is the function $f'(x)$ and the red shaded area is $| f'(x) | \leq k < 1$.

Thus the second condition of Theorem 2.8 is met. Since $\forall x \in [4, 6], \;\; f'(x) \neq 0$ any $x \in [4, 6], \;\; x \neq 5$ will converge linearly to unique fixed point $x = 5$.

(b) (10 points) Generate a table including values of p_i and $\frac{|p_i - 5|}{|p_{i-1} - 5|^1}$ for $0 \leq i \leq 20$.

i	p_i	Asymptotic Error
0	4.58257569495584	0.5217803813052
1	4.7867398586908	0.510895361703103
2	4.89220801821161	0.50544832769337
3	4.94581035736896	0.502724244716311
4	4.97283136521286	0.501362132467313
5	4.98639717893233	0.500681067497319
6	4.99319395724437	0.500340533906638
7	4.99659581977788	0.500170266973093
8	4.9982976200792	0.500085133489079
9	4.99914873757483	0.500042566744727
10	4.99957435066968	0.500021283373047
11	4.99978717080521	0.500010641685841
12	4.99989358427018	0.50000532084444
13	4.99994679185198	0.500002660418948
14	4.99997339585521	0.500001330210742
15	4.99998669790991	0.500000665095256
16	4.99999334895053	0.50000033258057
17	4.99999667447416	0.500000166323559
18	4.9999983372368	0.500000083061597
19	4.99999916861833	0.500000041664331
20	4.99999958430915	0.500000021366322

(c) (5 points) Estimate the asymptotic error constant λ.

$$\boxed{\lambda = \frac{1}{2}}$$

2. Show that the sequences below converge linearly to $p = 0$. How many terms are required before $|p_n - p| < 5 \times 10^{-2}$?

(a) (10 points) $p_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{|\ p_{n+1}\ |}{|\ p_n\ |^{\alpha}} = \lim_{n\to\infty} \frac{|\ \frac{1}{n+1}\ |}{|\ \frac{1}{n}\ |^{\alpha}} = \lim_{n\to\infty} \frac{n^{\alpha}}{n+1}$$

We will have convergence for $\alpha = 1$, but not $\alpha = 2$ so this sequence will converge linearly to 0.

Terms to be within 5×10^{-2}

$$|\ p_n - p\ | < 5 \times 10^{-2} \rightarrow \frac{1}{n} < 5 \times 10^{-2} \rightarrow \boxed{n > 20}$$

(b) (10 points) $p_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\mid p_{n+1} \mid}{\mid p_n \mid^{\alpha}} = \lim_{n\to\infty} \frac{\mid \frac{1}{(n+1)^2} \mid}{\mid \frac{1}{n^2} \mid^{\alpha}} = \lim_{n\to\infty} (\frac{n^{\alpha}}{n+1})^2$$

Similarly to part a, will have convergence for $\alpha = 1$, but not $\alpha = 2$ so this sequence will also converge linearly to 0.

Terms to be within 5×10^{-2}

$$\mid p_n - p \mid < 5 \times 10^{-2} \to \frac{1}{n^2} < 5 \times 10^{-2} \to 20 < n^2 \to \boxed{n > \sqrt{20} \approx 4.47214}$$

3. (10 points) Show that $p_n = \left(\frac{1}{10}\right)^{2^n}$ converges to 0 quadratically.

$$\lim_{n\to\infty} \frac{\mid p_{n+1} \mid}{\mid p_n \mid^{\alpha}} = \lim_{n\to\infty} \frac{\mid 10^{-2^{n+1}} \mid}{\mid 10^{-2^n} \mid^{\alpha}} = \lim_{n\to\infty} \frac{10^{-2^n 2}}{10^{-2^n \alpha}}$$

We will have convergence for $\alpha = 1$ as well as $\alpha = 2$, but not $\alpha = 3$, so this this sequence will quadratically to 0.

4. (a) (10 points) Write down the formula for a sequence p_n that converges to $p = 0$ with order $\alpha = 3$.

$$p_n = 100^{-3^n}$$

(b) (5 points) Generate the first 5 terms of this sequence.

n	p_n
0	1×10^{-2}
1	1×10^{-6}
2	1×10^{-18}
3	1×10^{-54}
4	1×10^{-162}