Due:

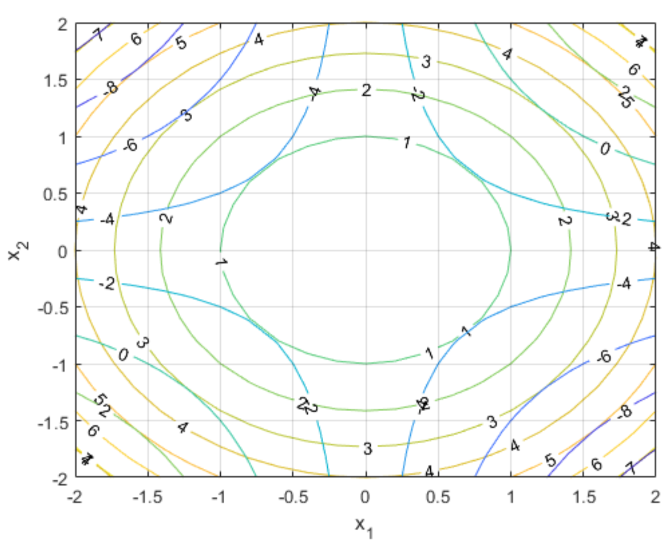
1. Consider the following nonlinear programming problem

$$\begin{aligned} &\text{minimize} \quad x_1^2 + x_2^2 \\ &\text{subject to} \quad 2x_1x_2 = 3 \end{aligned}$$

Do a **contour map** of the function $f(x_1, x_2) = x_1^2 + x_2^2$ and superimpose the relation given by equality constraint $h(x_1, x_2) = 0$, where $h(x_1, x_2) = 2x_1x_2 - 3$

You can use MATLAB to do this. For example, modify the code below to plot both the level curves of f and $h(x_1, x_2) = 0$

```
[X,Y] = meshgrid(-2.0:.2:2.0, -2.0:.2:2.0);
Z = X.^2+Y.^2;
P = 2.*X.*Y-3;
[c,h]=contour(X,Y,Z);
clabel(c,h);
hold on;
[v,j]=contour(X,Y,P);
clabel(v,j);
grid on;
xlabel('x_1');
ylabel('x_2');
syms X Y Lambda
F = X^2+Y^2;
H = 2*X*Y-3;
L = F + Lambda * H;
g = gradient(L, [X, Y, Lambda]);
disp(g);
```



(a) Define the Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda h(x_1, x_2)$$

$$\boxed{\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(2x_1x_2 - 3) = x_1^2 + x_2^2 + 2\lambda x_1 x_2 - 3\lambda}$$

Also defined in line 12.

(b) Calculate the **gradient** of $\mathcal{L}$

$$\nabla \mathcal{L} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \frac{\partial \mathcal{L}}{\partial x_2} \\ \frac{\partial \mathcal{L}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2x_2\lambda \\ 2x_2 + 2x_1\lambda \\ 2x_1x_2 - 3 \end{bmatrix}$$

(c) Find $x_1^\star, x_2^\star$, and $\lambda^\star$ such that

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

and compute the value of f at $(x_1^\star, x_2^\star)$

We can rewrite the question as a system of equations. The partial derivatives are calculated above.

$$\begin{aligned} 2x_1 + 2x_2\lambda &= 0 \\ 2x_2 + 2x_1\lambda &= 0 \\ 2x_1x_2 - 3 &= 0 \end{aligned}$$

Using wolfram to solve gives us four solutions.

$$\nabla \mathcal{L} = 0 \begin{cases} x_1 = \pm\sqrt{\frac{3}{2}},\ x_2 = \pm\sqrt{\frac{3}{2}},\ \lambda = -1 \\ x_1 = \pm i\sqrt{\frac{3}{2}},\ x_2 = \pm i\sqrt{\frac{3}{2}},\ \lambda = 1 \end{cases}$$

Both of the real solutions, when evaluated on the objective function give 3 as a result.

(d) Is the point $(x_1^\star, x_2^\star)$ found in part (c) the **optimal solution** for problem (1)? Justify your answer.

By definition, the solution corresponding to the original constrained optimization is always a saddle point of the Lagrangian function. The Hessianis test applied on this problem is inconclusive, so I'm not able to confidently say this solution is the best, however my intuition says it is.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$$

(a) Show that the function $f(x_1, x_2)$ has only one **stationary point** (i.e. a point where $\nabla f(\mathbf{x}^\star) = (0, 0)$).

We can start by completing the requisite partial derivatives

$$\frac{\partial f}{\partial x_1} = 2x_1 + 8 \text{ and } \frac{\partial f}{\partial x_2} = -4x_2 + 12 \qquad \therefore \nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + 8 \\ -4x_2 + 12 \end{bmatrix}$$

Stationary points occur at points where the gradient is zero, so we can just set each expression to zero and solve.

$$2x_1 + 8 = 0 \to x_1 = 4 \text{ and } -4x_2 + 12 = 0 \to x_2 = -3$$

This is the only solution to these equations, thus it is also the only stationary point for the equation.

(b) Show that the stationary point is neither a maximum nor a minimum.

We can show this by computing the Hessian matrix and using its determinate to conclude the properties of the point. We will start by computing the requisite partial derivatives.

$$\frac{\partial^2 f}{\partial x_1^2} = 2, \ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \ \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0, \text{ and } \frac{\partial^2 f}{\partial x_2^2} = -4$$

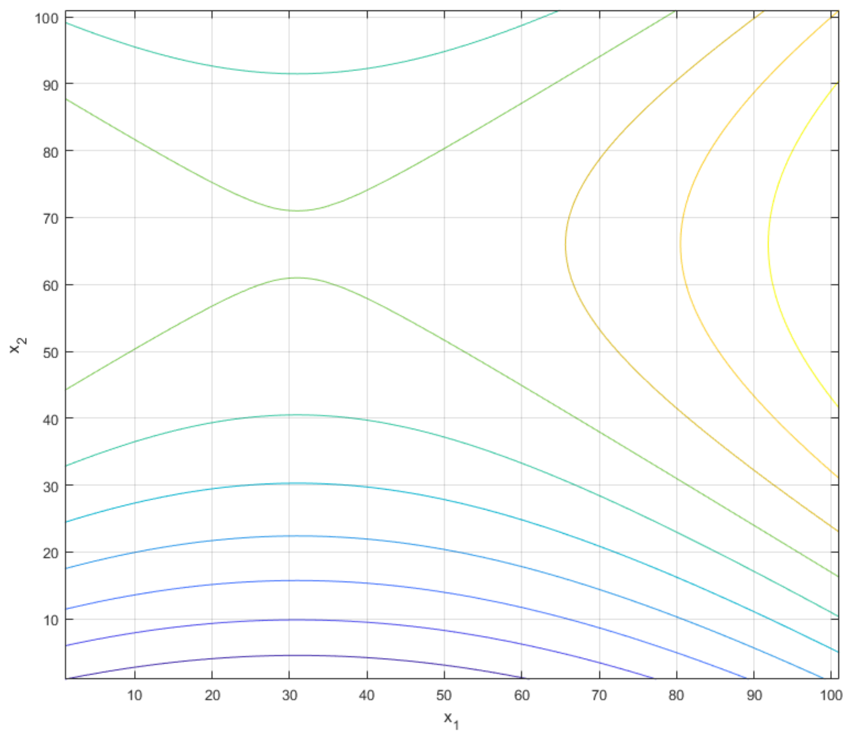
Therefore our Hessian matrix (H) and its determinant are

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \qquad \text{and} \qquad det(H) = 2 \times -4 - 0 \times 0 = -8$$

Therefore, by the second partial derivative test, we know that this point is a saddle point, and by definition is neither a maximum or a minimum.

(c) Show that the contour lines (level curves) of f are hyperbolas. We can do this using MATLAB

```
[X,Y] = meshgrid(-10.0:.2:10.0,-10.0:.2:10.0);
Z = 8.*X+12.*Y+X.^2-2.*Y.^2;
contour(Z);
grid on;
xlabel('x_1');
ylabel('x_2');
```



- For $f : \mathbb{R}^2 \to \mathbb{R}$ the **gradient** of f is given by

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix},$$

and the **Hessian** is the 2×2 matrix given by

$$H = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Contour lines, are found by studying the set of all *level sets*, defined as

$$\mathcal{L}_c(f) = \{(x_1, x_2) \mid f(x_1, x_2) = c\},$$

where $c \in \mathbb{R}$