

# Symbolic Model Checking in External Memory

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**Abstract.** We extend the external memory BDD package Adiar with support for monotone variable substitution. Doing so, it now supports the relational product operation at the heart of symbolic model checking. We also identify additional avenues for merging variable substitution fully and the conjunction operation partially inside the relational product’s existential quantification step. For smaller BDDs, these additional ideas improve the running of Adiar for model checking tasks up to 47%. For larger instances, the computation time is mostly unaffected as it is dominated by the existential quantification.

Adiar’s relational product is about one order of magnitude slower than conventional depth-first BDD implementations. Yet, its I/O-efficiency allows its running time to be virtually independent of the amount of internal memory. This allows it to compute on BDDs with much less internal memory and potentially to solve model checking tasks beyond the reach of conventional implementations.

Compared to the only other external memory BDD package, CAL, Adiar is several orders of magnitude faster when computing on larger instances.

**Keywords:** Time-forward Processing · External Memory Algorithms · Binary Decision Diagrams · Symbolic Model Checking

## 1 Introduction

Binary Decision Diagrams [13] (BDDs) are a concise and canonical representation of  $n$ -ary Boolean functions as directed acyclic graphs. Starting with [11, 17, 21], BDDs have become a popular tool for symbolic model checking. To this day, they are still used for model checking probabilistic [25, 27, 40] and multi-agent systems [25, 26, 44, 60] and CTL\* formulas [5]. Furthermore, they are also still used for verifying network configurations [3, 4, 12, 45], circuits [30–32], and feature models [22, 24]. They also have found recent use for the generation of extended resolution proofs for SAT and QBF problems [14–16]. Furthermore, recent research efforts have made progress on fundamental BDD-based procedures, e.g. [29, 41], and the very implementation of BDDs, e.g. [10, 28, 54]

The conciseness of BDDs mitigate the state space explosion problem of model checking. Yet, there is an inherent lower bound to how much space (or computation time) is needed to meaningfully encapsulate the state space. Hence, the size of the BDDs grows together with the size and the complexity of the

model under verification. Inevitably, BDDs must outgrow the machine’s RAM for some models. Yet, most implementations use recursion and hash tables for memoisation [10, 23, 28, 43, 59]. Both recursion and the hash tables introduce cache misses [34, 48]. As the BDDs outgrow the RAM, these cache misses turn into memory swaps which slows computation down by several orders of magnitude [54]. This puts an upper limit on what BDDs can solve in practice.

Unlike conventional BDD implementations, Adiar [54] is designed to handle BDDs that are too large for the RAM. To this end, it replaces the conventional approach of memoised recursion with time-forward processing [6, 20] algorithms. Unlike depth-first recursion [7], this technique is efficient in the I/O-model [1] of Aggarwal and Vitter. This I/O-efficiency, in turn, allows Adiar in practice to process large BDDs without being affected by the disk’s speed. Using this technique is only at the cost of a small overhead to its running time [54].

### 1.1 Contributions

In [57], we extended the external memory BDD package Adiar with efficient multi-variable quantification, based on the notion of nested sweeps. The framework in [57] was evaluated on quantifiers that occur in Quantified Boolean Formulas. In this paper, we evaluate the effectiveness of nested sweeps in the context of symbolic model checking. This requires some extensions to Adiar in order to improve the *relational product*. This operation is needed for the computation of symbolic successors and predecessors of a set of states. We show in Section 3.1 how monotone variable substitutions can be piggy-backed “for free” onto Adiar’s other algorithms. Furthermore, we show in Section 3.2 how to combine the conjunction and quantification operations to further improve the running time of the relational product. Finally, Section 5 identifies future work and provides recommendations based on our experimental results in Section 4.

## 2 Preliminaries

### 2.1 I/O Model

To make algorithmic analysis tractable, Aggarwal and Vitter’s I/O-model [1] is an abstraction of the machine’s complex memory hierarchy. This model consists of two levels of memory: the *internal memory*, e.g. the RAM, and the *external memory*, e.g. the disk. To compute on some data, it has to reside in internal memory. Yet, whereas the external memory is unlimited, the internal memory has only space for  $M$  elements. Hence, if the input of size  $N$  or some auxiliary data structure exceeds  $M$  then data has to be offloaded to external memory. Yet, each data transfer between the two levels, i.e. each read and write, consists of  $B$  sized *blocks* of consecutive data. Intuitively, an algorithm is I/O-efficient if it makes sure to use a substantial portion of the  $B$  elements in each block that has been read.

An algorithm’s I/O-complexity is the number of data transfers, I/Os, it uses. For example, reading an input sequentially requires  $N/B$  I/Os whereas random

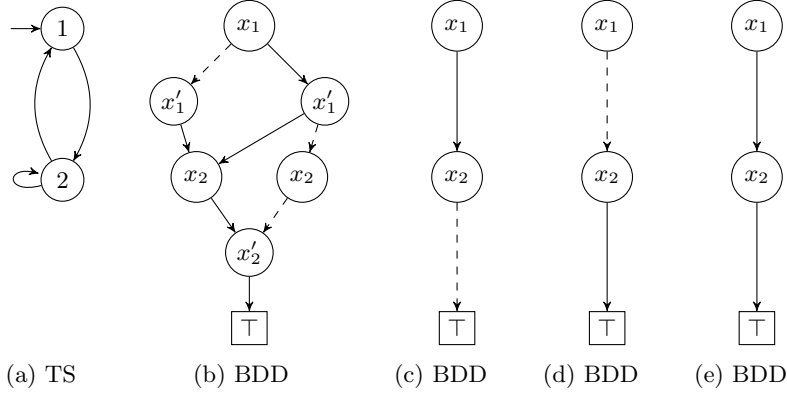


Fig. 1: The transition system in 1a can be represented as the BDD for its relation 1b and the BDD for its initial state in 1c via a *unary* encoding and an *interleaved* variable ordering. One (More) relational product of these two BDDs creates the one in 1d (1e).

For readability, we suppress the  $\perp$  terminal. The  $\top$  terminal is drawn as a box while BDD nodes are drawn as circles surrounding their decision variable. The *then*, resp. *else*, child is drawn solid, resp. dashed.

access costs up to  $N$  I/Os. Furthermore, one can sort  $N > M$  elements in  $\Theta(\text{sort}(N)) \triangleq \Theta(N/B \cdot \log_{M/B}(N/B))$  I/Os [1]. For most realistic values of  $N$ ,  $M$ , and  $B$ ,  $N/B < \text{sort}(N) \ll N$ .

## 2.2 Binary Decision Diagrams

Binary Decision Diagrams [13] (BDDs), based on [2, 42], represent Boolean formulae as singly-rooted binary directed acyclic graphs. These consist of two sink nodes (*terminals*) with the Boolean values  $\top$  and  $\perp$ . Each non-sink node (*BDD node*) contains a decision variable,  $x_i$ , together with two children,  $v_\top$  and  $v_\perp$ . Together, these three values represent the if-then-else decision  $x_i ? v_\top : v_\perp$ . Hence, the BDD represents an  $n$ -ary Boolean formula. In particular, each path from the root to the  $\top$  terminal represents one (or more) assignments for which the function outputs  $\top$ . For example, Fig. 1c represents the formula  $x_1 \wedge \neg x_2$ .

What are colloquially called BDDs are in particular *ordered* and *reduced* BDDs. A BDD is ordered, if the decision variables occur only once on each path and always according to the same ordering [13]. An ordered BDD is furthermore reduced if it neither contains duplicate subgraphs nor any BDD nodes have their two children being the same [13]. Assuming the variable ordering is fixed, reduced and ordered BDDs are a canonical representation of a Boolean formulae [13].

**Relational Product** In the case of symbolic model checking, one or more BDDs,  $R_{\vec{x}, \vec{x}'}$ , represent relations between unprimed Boolean variables,  $\vec{x}$ , that

encode the *current* and primed variables,  $\vec{x}'$ , that encode the *next* state. These relations are applied to sets of states,  $S_{\vec{x}}$ , which is identified using only the unprimed variables. For example, the transition system in Fig. 1a, can be represented via a unary encoding with the two variables  $x_1$  and  $x_2$  which represent the states 1 and 2, respectively. The transitions would then be the relation in Fig. 1b. In particular, Fig. 1b represents each of the three transitions in Fig. 1a as the disjunction of the following three formulas.

$$x_1 \wedge \neg x'_1 \wedge \neg x_2 \wedge x'_2 \quad \neg x_1 \wedge x'_1 \wedge x_2 \wedge \neg x'_2 \quad \neg x_1 \wedge \neg x'_1 \wedge x_2 \wedge x'_2$$

As Fig. 1b is a disjunction of all three transitions, it is a joint [18] relation<sup>1</sup>. Instead, one could also keep Fig. 1b as three separate BDDs for a disjoint [18] relation. This has the benefit of representing the transition system symbolically via smaller BDDs at the cost of having to apply the transitions one-by-one [18]. The initial state of the transition system would be the BDD in Fig. 1c, i.e. the formula  $x_1 \wedge \neg x_2$ .

These BDDs are manipulated using the following two operations (the *relational product*) to obtain the next or the previous set of states.

$$Next(S_{\vec{x}}, R_{\vec{x}, \vec{x}'}) \triangleq (\exists \vec{x} : S_{\vec{x}} \wedge R_{\vec{x}, \vec{x}'})[\vec{x}' / \vec{x}] \quad (1)$$

$$Prev(S_{\vec{x}}, R_{\vec{x}, \vec{x}'}) \triangleq \exists \vec{x}' : S_{\vec{x}}[\vec{x} / \vec{x}'] \wedge R_{\vec{x}, \vec{x}'} \quad (2)$$

For example, *Next* of Fig. 1c and Fig. 1b is the BDD shown in Fig. 1d. The transitive closure of applying *Next*, i.e. the result of  $Next^*$  of Fig. 1c and Fig. 1b, is shown in Fig. 1e.

### 2.3 I/O-efficient BDD Manipulation

The Adiar [54] BDD package (based on [7]) aims to compute efficiently on BDDs that are so large they have to be stored on the disk. To do so, rather than using depth-first recursion, it processes the BDDs level by level with time-forward processing [6, 20]. This makes it optimal in the I/O-model [7]. As shown in Fig. 2a, the basic BDD operations, such as the **And** ( $\wedge$ ), are computed in two phases. First, the resulting, but not necessarily reduced, BDD is computed in a top-down **Apply** sweep [54]. Secondly, this BDD is made canonical in a bottom-up **Reduce** sweep [54]. As shown in Fig. 2b, more complex BDD operations, such as **Exists** ( $\exists$ ), are computed by accumulating the result of multiple nested **Apply-Reduce** sweeps bottom-up in an outer **Reduce** sweep [57]. Here, each set of nested **Apply-Reduce** sweeps computes the **Or** ( $\vee$ ) of a to-be quantified variable's cofactors [57].

<sup>1</sup> This entire example only pertains to a unary encoding of a transition system with asynchronous semantics. If the transition system is synchronous, then the conjunction would be used instead and one has to join all transitions together into a single relation.

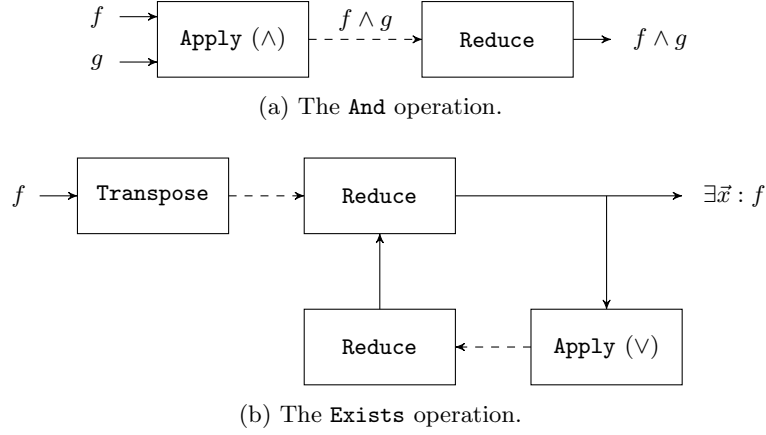


Fig. 2: The Apply-Reduce pipelines in Adiar.

### 3 I/O-efficient Relational Product

#### 3.1 I/O-efficient Variable Substitution

The work in [54, 57] covers all BDD operations in Eqs. (1) and (2) but the variable substitution ( $[\vec{x}'/\vec{x}]$ ) between primed and unprimed variables. Hence, this operation's design is the last step towards an I/O-efficient relational product.

In general, the substitution algorithm is equivalent to changing the variable ordering. Yet, this operation is notorious for being hard to compute since it greatly affects the structure and the size of the BDD. But, in the context of symbolic model checking, it merely suffices to consider variable substitutions,  $\pi$ , that are *monotone* with respect to the variable ordering. That is, if a variable  $x_i$  is prior to  $x_j$  in the given BDD then  $\pi(x_i)$  is also prior to  $\pi(x_j)$  in the substituted BDD. For example, the BDDs in Fig. 1 use a variable ordering where primed and unprimed variables are interleaved and so the variable substitutions in Eqs. (1) and (2) are monotone. This monotonicity is useful, since the BDDs before and after such a substitution are isomorphic.

**Proposition 1.** *A monotone variable substitution can be applied to a BDD of  $N$  nodes in  $\mathcal{O}(N)$  time and using  $2 \cdot \frac{N}{B}$  I/Os.*

*Proof.* Using  $\frac{N}{B}$  I/Os, one can stream through the  $N$  BDD nodes in external memory and apply  $\pi$  onto all BDD nodes requiring  $\mathcal{O}(N)$  time. The result is simultaneously streamed back to external memory using another  $\frac{N}{B}$  I/Os.  $\square$

This is optimal if the input is reduced and the output needs to exist separately in external memory. Yet, if the input still needs to be reduced, then substitution can be integrated into the **Reduce** algorithm from [54] as follows: when reducing the level for variable  $x_i$ , output its new nodes with variable  $\pi(x_i)$ .

**Proposition 2.** *A monotone variable substitution can be applied during the **Reduce** sweep to an unreduced BDD with  $n$  levels in  $\mathcal{O}(n)$  time, using no additional space in internal memory, and not using any additional I/Os.*

Particular to Eq. (1), variable substitution can be integrated into the quantification operation’s outer **Reduce** sweep which precedes it.

Similarly, substitution in Eq. (2) can become part of the conjunction operation that succeeds it. Here, each BDD node of  $S_{\bar{x}}$  would be mapped on-the-fly as they are streamed from external memory. The implementation of such an idea can be greatly simplified with one additional restriction on  $\pi$ . Note, in the context of model checking, variable substitutions are not only monotone but also *affine*, i.e.  $\pi(x_i) = x_{\alpha \cdot i + \beta}$  for some  $\alpha, \beta \in \mathbb{N}$  and  $\alpha \geq 1$ . Hence, by storing  $\alpha$  and  $\beta$  in internal memory, one can defer applying  $\pi$  until they are read as part of the succeeding BDD operation. This makes variable substitution a constant-time operation by adding an overhead to all other BDD operations.

**Proposition 3.** *A monotone and affine variable substitution can be applied to a BDD in  $\mathcal{O}(1)$  time, using  $\mathcal{O}(1)$  additional space in internal memory, and using no additional I/Os.*

In practice,  $\alpha$  is always 1. Hence, only  $\beta$  needs to be stored.

### 3.2 An I/O-efficient AndExists

Just as Propositions 2 and 3 move the variable substitution inside another operation, the relational product’s performance can be further improved by merging the conjunction and existential quantification into a single **AndExists** [62].

The quantification operation’s outer **Reduce** sweep requires the input to be transposed [57]. Hence, Fig. 2b includes an initial transposition step. The **Apply** step of the **And** in Fig. 2a produces an unreduced output which already is transposed [54]. This means that naively combining Figs. 2a and 2b into an **AndExists** will result in some computational steps having no effect: the reduced BDD from the **And** is immediately transposed and reduced once more as part of the **Exists**. Hence, the **Reduce** step of the **And** and the **Transpose** step of the **Exists** can be skipped.

**Optimisation 1.** *Use the unreduced result of the conjunction operation as the input to the quantification operation’s outer **Reduce** sweep.*

This makes the quantification operation’s outer **Reduce** sweep also do double duty as the conjunction operation’s **Reduce** sweep. This saves the linearithmic time and I/Os otherwise needed to first reduce the result of the conjunction and to then transpose it.

Furthermore, experiments in [57] indicate, the quantification algorithm can become up to  $\sim 21\%$  faster by pruning a BDD node (and possibly its subtrees) where quantification is trivial because one of or both its children are terminals.

**Optimisation 2.** *During the conjunction’s **Apply** sweep, prune the resulting BDD nodes that have to-be quantified variables.*

## 4 Experimental Evaluation

We have added the `bdd_replace` function to Adiar to provide support for variable substitution. For now, it only supports monotone substitutions with the three propositions presented in Section 3.1. Building on this, we added the `bdd_relprod`, `bdd_relnext`, and `bdd_relprev` operations for model checking applications. This includes the ideas in Section 3.2. Optimisation 1 was added in its general form to both the `bdd_exists` and the `bdd_forall` functions: transposition is skipped if the input BDD is unreduced.

With this in hand, we have conducted multiple experiments to evaluate the impact and utility of this work. In particular, we have sought to answer the following research questions:

1. What is the impact of the proposed optimisations in Section 3?
2. How does Adiar’s Relational Product operation compare to conventional depth-first implementations?
3. How does Adiar’s Relational Product operation compare to the breadth-first algorithms of CAL?

### 4.1 Benchmarks

We have extended our BDD benchmarking suite<sup>2</sup> [54] with the foundations for a symbolic model checker for Petri Nets [49] and Asynchronous Boolean Networks [33, 61]. This benchmark explores the given model symbolically as follows:

– *Reachability:*

The set of reachable states,  $S_{\text{reach}}$ , is computed via the transitive closure  $\text{Next}^*(S_I, R)$  on the initial state,  $S_I$ , and transition relation  $R$ . This uses `bdd_relnext` up to a polynomial number of times with respect to the model and its state space.

– *Deadlock:*

The set of deadlocked states are identified via  $S_{\text{reach}} \setminus \text{Prev}(S_{\text{reach}}, R)$ . This requires a single use of `bdd_relprev` if a joint partitioning [18] is used. If a disjoint partitioning [18] is used then this operation is called once for each transition in the model.

Based on [46], the variable ordering is predetermined by analysing the given model<sup>3</sup> with Sloan’s algorithm [51]. For each model, we have run them with both a joint and a disjoint [18] partitioning of the transition relation.

Based on preliminary experiments, we identified 75 model instances that were solvable with Adiar. In particular, these are 16 Petri nets from the 2021–2023

<sup>2</sup> [github.com/SSoelvsten/bdd-benchmark](https://github.com/SSoelvsten/bdd-benchmark)

<sup>3</sup> Even though [46] suggests one runs the algorithm on a bipartite read/write graph derived from the model, we instead run it on an incidence graph derived from the model. Our preliminary experiments indicate this further decreases the size of the BDDs.

Model Checking Competitions [36–38], 41 Boolean networks distributed with AEON [9], and 18 Boolean networks distributed with PyBoolNet [35].

For all 150 of these instances, none of the BDDs generated during either benchmark grew larger than  $2 \cdot 10^6$  BDD nodes (48 MiB). Furthermore, the BDDs encoding the respective initial states required only 294 or fewer BDD nodes (6.9 KiB). While computing the transition relation’s transitive closure, the BDDs grew slowly. That is, most, if not all, BDD computations in either benchmark are too small to be within the current scope of Adiar [56]. Hence, inspired by the recent work of Pastva and Henzinger [48], we have also created the following two additional benchmarks.

– *Next*:

Given a BDD with a set of states,  $S_{\vec{x}}$ , and another one with a relation,  $R_{\vec{x}, \vec{x}'}$ , the next set of states,  $Next(S_{\vec{x}}, R_{\vec{x}, \vec{x}'})$ , is computed with a `bdd_relnext`.

– *Prev*:

Given a BDD with a set of states,  $S_{\vec{x}}$ , and another one with a relation,  $R_{\vec{x}, \vec{x}'}$ , the previous set of states,  $Prev(S_{\vec{x}}, R_{\vec{x}, \vec{x}'})$ , is computed with a `bdd_relprev`.

For inputs, we have followed the approach in [48]. The above-described reachability analysis has been extended to save the (joint) transition relation BDD,  $R_{\vec{x}, \vec{x}'}$ , together with the first state BDD,  $S_{\vec{x}}$ , constructed of each order of magnitude. Using this, we have generated BDDs by running reachability analysis on all models from the 2020–2023 Model Checking Competitions [36–39]. This was done using LibBDD [9] as the BDD backend and a time limit of 1 h on a Ubuntu 24.4 machine with a 12-core 3.6 GhZ Intel i7-12700 processor and 64 GiB of memory. This has resulted in serialised BDDs from 124 model instances. These BDDs have been made publically available at the following DOI:

<https://doi.org/10.5281/zenodo.13928216>

For our evaluation, we focus on the three models **GPUForwardProgress** 20a (2021), **SmartHome** 16 (2020), and **ShieldPPPs** 10a (2020) where we could generate a set of states with a magnitude of  $2^{25}$  BDD nodes<sup>4</sup>. The relation sizes are 500 MiB, 270 MiB, and 0.1 MiB, respectively. For state size, we focus on the four largest orders of magnitude constructed, i.e. from  $2^{22}$  (40 MiB) to  $2^{25}$  BDD nodes (320 MiB).

## 4.2 Hardware, Settings, and Measurements

Similar to [52, 54–57], we ran our experiments on machines at the Centre for Scientific Computing in Aarhus. These machines run Rocky Linux (Kernel 4.18.0-513) with 48-core 3.0 GHz Intel Xeon Gold 6248R processors, 384 GiB of RAM,

<sup>4</sup> A fourth model, **SmartHome** 17 (2020), also created a set of states this large. Yet, the serialized BDD for the relation turned out to be corrupted. The published set of BDDs above has fixed this and other data corruptions. Furthermore, the repository also includes large BDDs that required more than 1 h to be generated.



3.5 TiB of SSD disk (with 48 GiB of swap memory). The benchmark and BDD packages were compiled with GCC 10.1.0 and Rust 1.72.1. Due to an update to the cluster’s machines, LibBDD [9] was compiled with GCC 13.2.0 and Rust 1.77.1 for the *Next* and *Prev* benchmarks.

Each BDD package was given 9/10th of these 384 GiB of internal memory<sup>5</sup>; this leaves the remaining space for the OS and the benchmark’s other data structures. Otherwise, each BDD package was given a single CPU core and initialised with their respective default and/or recommended settings.

For *Reachability* and *Deadlock* (Table 2 and Fig. 5), we have measured the running time 3 times with a timeout of 48 h. For *Next*, and *Prev* (Table 2 and Fig. 6) we measured the running time of each instance 5 times and bounded the running time to 12 h. For RQ 1, we tried to minimise the noise due to hardware and the OS in the reported numbers. Similar to [54, 56, 57] (based on [19]), we intended to do so by reporting the smallest measured running time. But, some measurements seem to have been taken while the machines were in a particularly good state. Using the minimum in this case makes RQ 1 much harder to investigate. Hence, we instead resort to reporting the median.

For the data shown in Fig. 7 for RQ 2, we deemed that a single measurement of each data point would suffice.

### 4.3 RQ 1: Effect of the Optimisations

We have implemented the ideas from Section 3.1 and Section 3.2 in order of their complexity and expected benefit in practice: Proposition 1 (—), Optimisation 1 (◊), Optimisation 2 (◈), Proposition 2 (◈), and finally Proposition 3 (◈). For evaluation, we have measured the running time of these five accumulated set of features. Figures 3 and 4 show the relative performance increase compared to only using Proposition 1.

Table 1 shows for each benchmark the total running time of each optimisation. The running time of the two model checking tasks, *Reachability* and *Deadlock*, are mainly affected by the optimisations. On the other hand, the running time of *Next* and *Prev* are comparatively unaffected. This suggest that, as the size of BDDs increases, the computational cost of variable substitution and

<sup>5</sup> CUDD and LibBDD ignore any given memory limit and hence may have used more.

Table 1: Total running time (seconds) of each version of Adiar. The # column indicates the number of instances that were solved by all five versions.

	#	Prop. 1	Prop. 1 Opt. 1	Prop. 1 Opt. 1+2	Prop. 1+2 Opt. 1+2	Prop. 1+2+3 Opt. 1+2
<i>Reachability</i>	147	4134.2	3918.1	3738.6	3627.5	3659.7
<i>Deadlock</i>	147	416.4	269.3	246.5	248.1	223.8
<i>Next</i>	12	24921.8	24378.0	23994.6	23257.7	23628.1
<i>Prev</i>	10	77541.9	78068.2	76862.0	75693.8	76353.8

the conjunction, which the improvements pertain to, decreases in comparison to the cost of quantifying variables.

Optimisation 1 (◇) improves the total running time between  $-0.7\%$  and  $35.3\%$  depending on the benchmark. In particular, it mainly improves the running for the smaller BDDs in *Reachability* and *Deadlock*.

Both the relation,  $R_{\vec{x}, \vec{x}'}$ , and states,  $S_{\vec{x}}$ , have many edges to  $\perp$  that shortcut the  $\wedge$  operator and gives Optimisation 2 (◆) ample opportunity to prune subtrees. Hence, it improves the total running time between  $4.6\%$  and  $15.7\%$  depending on the benchmark. In particular, for the *Next* benchmark with *ShieldPPPs* of a magnitude  $2^{24}$ , this optimisation decreases the intermediate BDD size by  $30\%$ . Yet, the number of nodes processed during the quantification operation's nested inner **Apply-Reduce** sweeps is unaffected. Hence, this optimisation may only improve performance of the conjunction's **Apply** and **Reduce** sweeps (the latter of which is merged with the quantification operation's outer **Reduce** sweep due to Optimisation 1 (◇)) rather than save work during existential quantification.

Proposition 2 (◆) further improves the total running time for *Reachability* and *Next* by  $3.0\%$  and  $3.1\%$ , respectively.

Finally, Proposition 3 (◆) has no effect on the *Reachability* and the *Next* benchmarks. Table 1 shows it adds an overhead of up to  $1.6\%$ . This slowdown is due to mapping all BDD nodes on-the-fly with an  $\alpha = 1$  and  $\beta = 0$ , i.e. without any actual changes. This overhead can be circumvented by only incorporating this remapping into the  $\wedge$ -operation in *Prev*. Yet, doing so would require a substantially higher implementation effort and it would also remove the ability to use this optimisation elsewhere. On the other hand, Table 1 also shows Proposition 3 (◆) improves *Deadlock* by  $9.8\%$  when compared to the version only including up to Proposition 2 (◆). For *Prev*, there is a slowdown of  $0.8\%$ . Yet, since Proposition 2 (◆) does not apply to this benchmark and the version in-

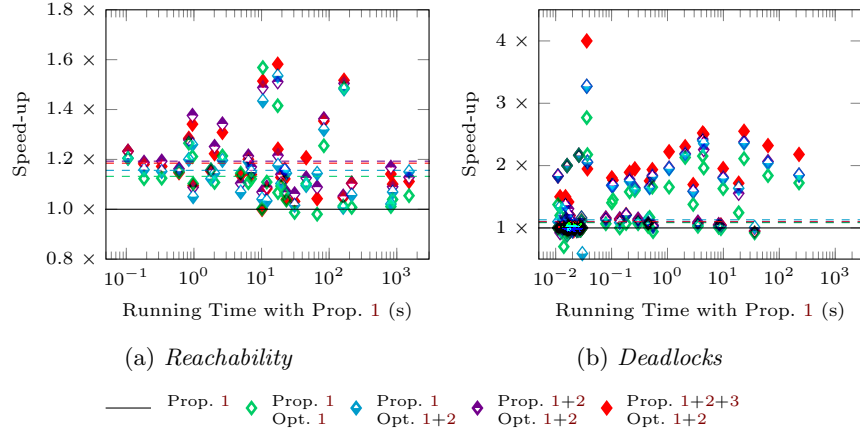


Fig. 3: Impact of optimisations on model checking tasks running time. Averages are drawn as dashed lines.

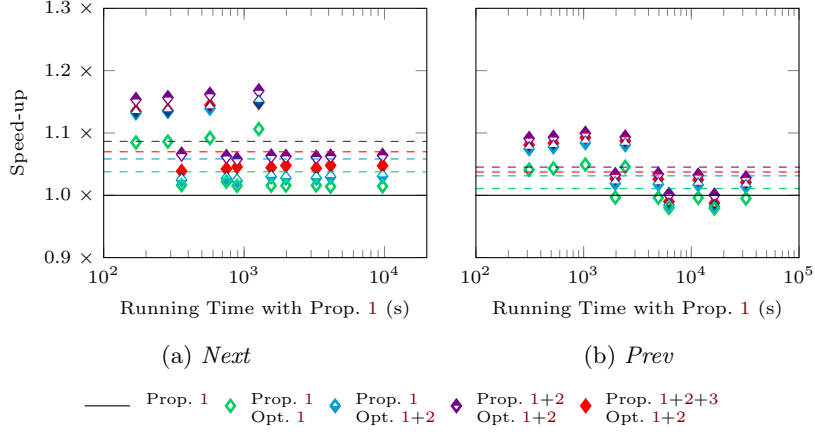


Fig. 4: Impact of optimisations on running time of a single relational product. Averages are drawn as dashed lines.

cluding Proposition 3 (◆) is 0.7% faster than the one only with Optimisations 1 and 2 (◆), this can be attributed to noise and the whims of the compiler.

#### 4.4 RQ 2: Comparison to Depth-first Implementations

**Unlimited Memory** We have measured the running time for the same benchmarks with the depth-first BDD packages BuDDy 2.4 [43], CUDD 3.0 [59], LibBDD 0.5.22 [9], and Sylvan 1.8.1 [23]. Sylvan is not included for the *Next* and *Prev* measurements since the manual BDD reconstruction results in a stack overflow<sup>6</sup>. Figures 5 and 6 show scatter plots of the running time of Adiar (with all features from Section 3) in comparison to the other implementations.

That the BDD size in *Reachability* and *Deadlock* are small is clearly evident in Table 2 and Fig. 5. Earlier, the disk-based algorithms of Adiar have proven to be several orders of magnitude slower than the conventional depth-first algorithms when computing on small BDDs [54, 56, 57]. This is also evident in Fig. 5 where the gap between Adiar and depth-first implementations becomes smaller as the computation time, and with it the BDDs, grow larger. Furthermore, the gap in Fig. 5b is smaller than in Fig. 5a since *Deadlock* only includes computation on the larger BDD that represents the entire state space.

As the BDD sizes are larger in *Next* and *Prev*, the gap between Adiar and depth-first implementations is expected to be smaller. Indeed, as Table 2 and Fig. 6 show, the gap is only of a single order of magnitude or less.

<sup>6</sup> More precisely, its garbage collection algorithm starts out by creating a task for each BDD root. This is to mark all nodes that are still alive. Yet, if the input BDD is big enough then the number of BDD nodes at a single level may outgrow the worker queues in Lace [63].

Table 2: Total Running time of Adiar (with Prop. 1, 2, and 3 and Opt. 1 and 2) and other implementations of BDDs to solve all benchmarks. The # column indicates the number of instances that were solved by all BDD packages.

	#	Adiar	BuDDy	CAL	CUDD	LibBDD	Sylvan
<i>Reachability</i>	149	3659.7	44.2	2989.7	118.4	797.9	62.4
<i>Deadlock</i>	149	223.8	11.3	12935.69	73.6	67.9	9.6
<i>Next</i>	12	23628.1	1827.43	TO	10021.34	2958.89	—
<i>Prev</i>	10	76353.8	4252.37	TO	6890.05	6815.761	—

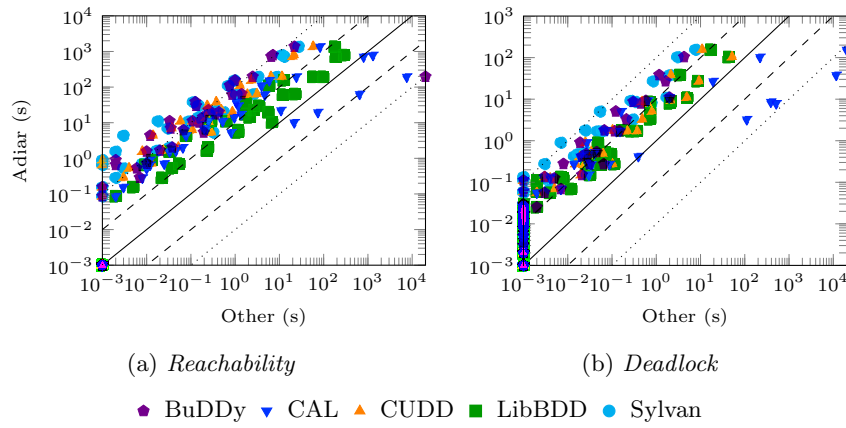


Fig. 5: Running time of Adiar on model checking tasks compared to other implementations. Timeouts are shown as markers at the top and the right.

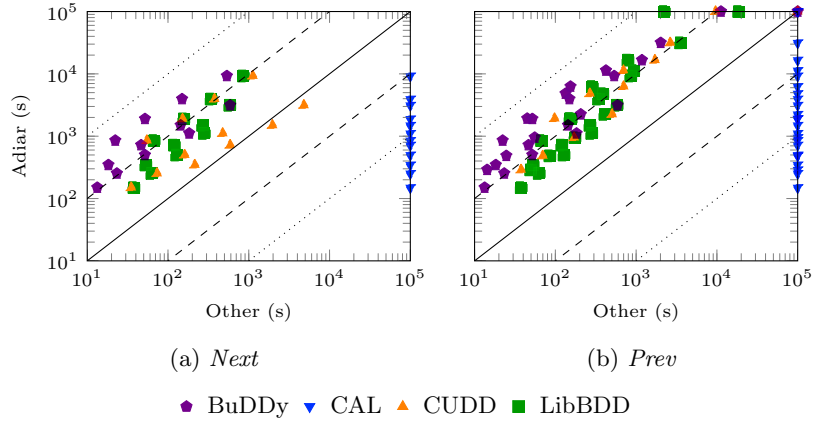


Fig. 6: Running time of Adiar on a single relational product compared to other implementations. Timeouts are shown as markers at the top and the right.

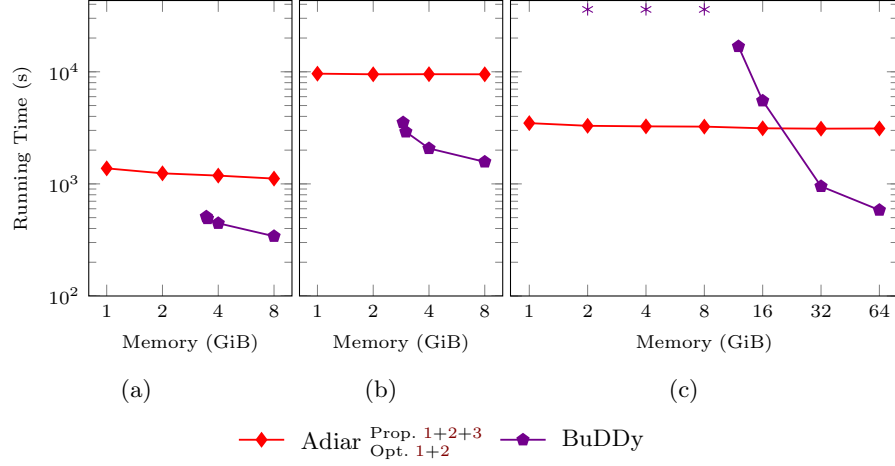


Fig. 7: Running time for *Next* depending on available internal memory. Timeouts are marked as stars. The models are `GPUForwardProgress 20a` (7a), `SmartHome 16` (7b), and `ShieldPPPs 10a` (7c) with a state space BDD of magnitude  $2^{25}$ .

**Limited Memory** At time of writing, Adiar still mainly focuses on handling BDDs that are too large to fit into main memory. That is, the results from our experiments shown in Figs. 5 and 6 are heavily biased against Adiar. The benefit of doing so is for the experiments to provide a worst-case comparison between Adiar and other implementations.

To turn the tables, Fig. 7 shows for each of the three models of the *Next* benchmark the running time of Adiar and BuDDy when given different amounts of internal memory. Each of these data points were only measured once. In `GPUForwardProgress 20a` (Fig. 7a) and `SmartHome 16` (Fig. 7b), BuDDy is unable to complete its computation with less than 3.45 and 2.9 GiB of memory, respectively. As the memory increases beyond this point, its performance quickly increases. For the `ShieldPPPs 10a` model (Fig. 7c), BuDDy needs more than 12 h to finish its computation when given less than 12 GiB of memory. This is due to repeated need to do garbage collection which in turn clears the computation cache and makes it repeat previous computations. Adiar, on the other hand, does not use any memoisation; its priority queues are implicitly doing double-duty as a computation cache [54]. It neither suffers from garbage collection: BDD nodes are stored in files on disk where the entire file is deleted when it is not needed anymore [54]. Yet, Adiar’s use of I/O-efficient files and priority queues allows it to only slow down by 23% as its internal memory is decreased far below what BuDDy needs.

#### 4.5 RQ 3: Comparison to CAL (Breadth-first Implementation)

Table 2 and Figs. 5 and 6 also include measurements of the breadth-first BDD package, CAL [50]. Similar to our previous results in [57], CAL’s running time

is on-par with the depth-first implementations for smaller instances but then quickly deteriorates as the BDDs grow larger. This is due to its use of conventional depth-first recursion for the smaller instances [56].

The overhead of its breadth-first algorithms, on the other hand, makes it much slower than Adiar for the larger instances in Figs. 5a and 5b. This overhead is especially apparent in the *Next* and *Prev*. Here, CAL times out after 12 h for all instances.

## 5 Conclusions and Future Work

In this work, we have successfully designed an I/O-efficient relational product in Adiar which enables BDD-based symbolic model checking beyond the limits of the machine’s memory. These algorithms, as they exist today, are much faster at processing large BDDs than a conventional BDD package that either needs to repeatedly run garbage collection to stay within its memory limits or that needs to offload its BDDs to the disk by means of swap memory. In fact, Adiar’s running time is virtually independent of its memory limits.

### Optimisations for Relational Product

Towards designing this I/O-efficient relational product, we have identified multiple optimisations particular to the design of Adiar’s algorithms. Based on our results in Section 4, we recommend the following with respect to the optimisations we have proposed in Section 3.

**Recommendation.** *Propositions 1 and 2 and Optimisations 1 and 2 should be included in an I/O-efficient relational product. A safer alternative to Proposition 3 may be worth the additional implementation effort.*

The gap in running time between Adiar and depth-first implementations is for larger instances about one order of magnitude. This gap is about twice the size than our previous results in [52, 54, 56, 57]: in those benchmarks, Adiar is only up to a factor 4 slower than the conventional approach. In [62], a combined **AndExists** BDD operation roughly halves the running time for depth-first implementations. The optimisations in Section 3.2 aim to also create a combined **AndExists** within the time-forward processing paradigm of Adiar. The gap in performance indicates more ideas are needed.

**Recommendation.** *Future work on an I/O-efficient relational product should further integrate the **Exists** within the **And**. To this end, it may be worth investigating alternatives to Optimisation 2; for example, partial quantification in [57] may prove performant in the context of symbolic model checking with BDDs.*

Additional measurements have indicated that for larger instances, the **And** (even without the optimisations in Section 3.2) is responsible for less than 1/10th of the total computation time.

**Recommendation.** *Future work on an I/O-efficient relational product should especially focus on improving the I/O-efficient **Exists** operation’s performance.*

### Small-scale BDD Computation

The BDD library CAL [50] (based on [8, 47]) is to the best of our knowledge the only other implementation aiming at efficiently computing on BDDs stored on the disk. In practice, it switches from the conventional depth-first to the breadth-first algorithms described in [50] when the size of the BDDs exceed  $2^{19}$  BDD nodes [56]. Yet, these breadth-first algorithms are in practice one or more orders of magnitude slower than Adiar’s algorithms. This makes Adiar vastly outperform CAL at solving our benchmarks as the BDDs involved have grown large enough.

But, similar to our previous results in [54, 56, 57], the gap in performance between Adiar and depth-first implementations is still several orders of magnitude for smaller instances. As the BDDs in model checking tasks start out small and only grow slowly, this overhead bars the current version of Adiar from being applicable for model checking in practice.

**Recommendation.** *Similar to CAL, one should combine Adiar’s time-forward processing algorithms with the conventional depth-first approach. Both [56] and [55] have paved the way for getting Adiar’s algorithms to work in tandem with a unique node table.*

Yet, doing such a vast engineering task is outside the scope of this paper. We leave the task of combining the strengths of depth-first recursion and time-forward processing as future work.

### Generic Variable Substitution

We have in this work only focused on monotone variable substitution. This still leaves a variable substitution that is I/O-efficient for the general case as an open problem, i.e. an algorithm capable of handling substitutions that change the order of the BDD’s levels. If it is possible to design such an algorithm which is efficient with respect to time, space, and I/Os then it can be used as the backbone for novel external memory variable reordering algorithms.

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## Data Availability Statement

Our experiments are created based on the BDD Benchmarking Suite [53, 54]. The obtained data and its analysis are available in our accompanying artifact [58]. This artifact also includes the inputs, pre-compiled binaries, and scripts to recreate our experiments. Finally, it also provides the source code to read, change, and recompile said binaries.

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## A SCC Decomposition

We also implemented the following third foundational model checking operation for our experiments in Section 4.

– *SCC Decomposition*:

The reachable states are decomposed into their strongly connected components (SCCs) via the CHAIN algorithm [41]. This uses both `bdd_relnext` and `bdd_relnprev` up to a polynomial number of times with respect to the model and its state space. As per [29], each deadlock state identified during the *Deadlock* stage is an SCC which can be skipped.

The 75 model instances were in particular the ones where Adiar seemed able to consecutively solve *Reachability*, *Deadlock*, and *SCC Decomposition* within 2 days. Likewise, the results in measurements in Table 2 and Fig. 5 were run with a timeout of 48 h.

This benchmark was left out of the above experiments, since they do not add new knowledge. In particular, the BDD size stayed too small to be within Adiar’s current scope. If anything, the results shown below further cements the need for incorporating the conventional depth-first algorithms with Adiar’s I/O-efficient time-forward processing.

### A.1 RQ 1: Effect of the Optimisations

Table 3 and Fig. 8 shows the effect of each optimisation on this particular benchmark. Similar to *Reachability* and *Deadlock*, the *SCC Decomposition* is positively affected by the optimisations due to the small BDD size.

Optimisation 1 (◇) improves the total running time for *SCC Decomposition* with 20.7%. Optimisation 2 (◆) further improves the running time by 14.0% and Proposition 2 (◆) by 4.8%. Finally, Proposition 3 (◆) improves the total running time by 1.5%

### A.2 RQ 2: Comparison to Depth-first Implementations

Table 4 and Fig. 9 shows the running time of Adiar and the depth-first BDD packages described in Section 4. Here, the gap between Adiar and the depth-first BDD packages clearly show that the size of the BDDs stay small. This is due to the fact that the Chain algorithm in [41] repeatedly explores the (remaining) set of states from single pivot states.

Table 3: Total running time (seconds) of each version of Adiar on SCC Decomposition. The # column indicates the number of instances that were solved by all five versions.

	#	—Prop. 1	◇Prop. 1 Opt. 1	◆Prop. 1 Opt. 1+2	◆Prop. 1+2 Opt. 1+2	◆Prop. 1+2+3 Opt. 1+2
SCC	144	734040.5	562074.5	483553.0	460388.5	453712.4

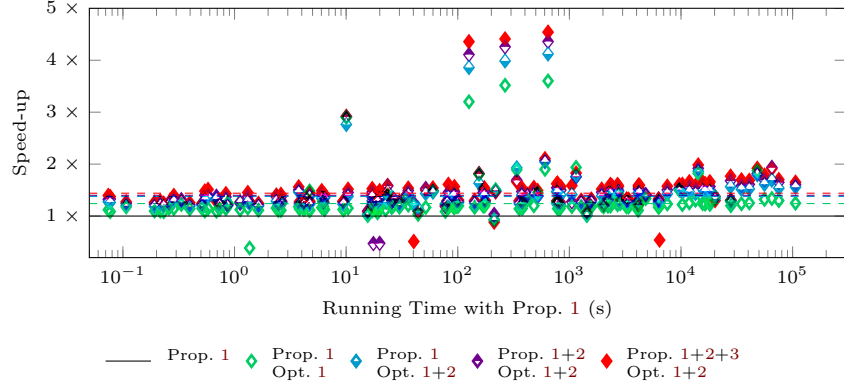


Fig. 8: Impact of optimisations on SCC Decomposition running time. Averages are drawn as dashed lines.

### A.3 RQ 3: Comparison to CAL (Breadth-first Implementation)

Table 4 and Fig. 9 also shows the running time of CAL for solving the SCC decomposition tasks. Whereas CAL becomes slower than Adiar for the largest instances in Figs. 5a and 5b, the same is not evident in Fig. 9. This is further testament to the small size of the BDDs in this benchmark.

Table 4: Total Running time of Adiar (with Prop. 1, 2, and 3 and Opt. 1 and 2) and other implementations of BDDs for SCC decomposition. The # column indicates the number of instances that were solved by all BDD packages.

	#	Adiar	BuDDy	CAL	CUDD	LibBDD	Sylvan
<i>SCC</i>	147	567188.5	680.6	22679.93	1840.0	10201.9	862.2

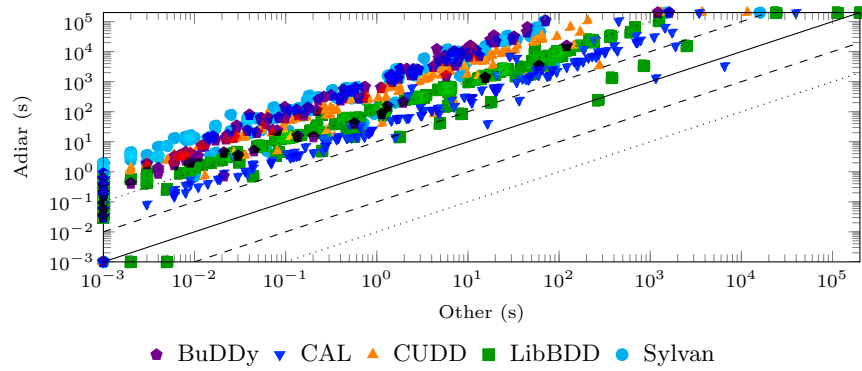


Fig. 9: Running time of Adiar on SCC decomposition compared to other implementations. Timeouts are shown as markers at the top and the right.