Multi-variable Quantification of BDDs in External Memory using Nested Sweeping (Extended Version)

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Abstract. Previous research on the Adiar BDD package has been successful at designing algorithms capable of handling large Binary Decision Diagrams (BDDs) stored in external memory. To do so, it uses consecutive sweeps through the BDDs to resolve computations. Yet, this approach has kept algorithms for multi-variable quantification, the relational product, and variable reordering out of its scope.

In this work, we address this by introducing the *nested sweeping* framework. Here, multiple concurrent sweeps pass information between each other to compute the result. We have implemented the framework in Adiar and used it to create a new external memory multi-variable quantification algorithm. Compared to conventional depth-first implementations, Adiar with nested sweeping is able to solve more benchmark instances and/or solve them faster.

1 Introduction

The ability of Binary Decision Diagrams (BDDs) to represent Boolean formulae as small directed acyclic graphs (DAGs) have made them an invaluable tool to solve many complex problems. For example, recently they have been used to check type-and-effect systems [35,36], to generate proofs for SAT and QBF solvers [14–16], for circuit synthesis [22,32], to solve games [37,45,53], and for symbolic model checking [3,19,20,23,26,28,34].

Implementations of decision diagrams conventionally make use of recursive depth-first algorithms and a unique node table [10,21,29,33,40,52]. Both of these introduce random access, which pauses the entire computation while missing data is fetched [30,39,44]. For large enough instances, data has to reside on disk and the resulting I/O-operations that ensue become the bottle-neck.

Adiar [49] is a BDD package written in C++ based on the ideas of Lars Arge [5]: the depth-first recursive algorithms are replaced with iterative algorithms. Here, one or more priority queues reorder the execution of recursive calls such that they are synchronised with a level-by-level traversal of the inputs. This makes Adiar's algorithms, unlike the conventional recursive implementations, optimal in the I/O-model [1] of Aggarwal and Vitter [5,6]. In turn, this enables it to manipulate BDDs beyond the reach of conventional BDD packages at a negligible cost to its running time [49].

Yet, the ideas in [5,6,49] only provide a translation of the simplest BDD algorithms. Specifically, it only provides a translation for the operations that recurse-once-and-return for each node in the BDD's graph. This does not provide a way to translate the more complex BDD algorithms that recurse more than once per BDD node, e.g. multi-variable quantification. Hence, until this work, Adiar could not easily be used for solving Quantified Boolean formulæ (QBF). Furthermore, game solving and symbolic model checking has until now been out of reach for Adiar.

1.1 Contributions

In Section 3, we introduce the notion of nested sweeping to provide a framework on which these more complex BDD operations can be implemented. Here, an outer bottom-up sweep accumulates the results from multiple nested inner sweeps. With this framework in hand, we implement an I/O-efficient multivariable quantification that simulates the one in conventional BDD packages. Furthermore, we identify in Section 3.2 optimisations for the nested sweeping framework in general and in Section 3.3 for the quantification operation in particular. Section 4 provides an overview of the implementation while Section 5 investigates the impact of nested sweeping in practice to solve QBF-encodings of two-player games and for reasoning about the transition system in Conway's Game of Life [24]. We compare in Section 6 our approach to related work and finally provide our conclusions and sketch out future work in Section 7.

2 Preliminaries

2.1 The I/O-Model

Aggarwal and Vitter introduced the I/O-model [1] to analyse the cost of transferring data to and from a slow storage device. Here, computations can only operate on data that resides in *internal* memory, e.g. the RAM, with a finite size of M. Hence, if the input of size N (or some intermediate result) exceeds M then it needs to be transferred to and from *external* memory, e.g. the disk. Yet, each such data transfer (I/O) moves an entire consecutive block of B elements; an algorithm's I/O-complexity is the number of I/Os it uses.

One needs $\operatorname{scan}(N) \triangleq N/B$ I/Os to linearly scan through a consecutive list of N elements in external memory [1]. Assuming N > M, one needs to use $\Theta(\operatorname{sort}(N))$ I/Os to sort N elements, where $\operatorname{sort}(N) \triangleq N/B \cdot \log_{M/B}(N/B)$ [1]. Furthermore, one can design an I/O-efficient priority queue capable of doing N insertions and deletions in $\Theta(\operatorname{sort}(N))$ I/Os [4]. For simplicity, we overload $\operatorname{scan}(N)$ to be N and $\operatorname{sort}(N)$ to be $N\log N$ when referring to an algorithm's time complexity rather than its I/O complexity.

Intuitively, an algorithm is I/O-inefficient if it uses an entire I/O to retrieve a block but does not make use of a significant portion of the B elements within. That is, random access can result in N I/Os. For all realistic values of N, M, and B, this is several magnitudes larger than both $\operatorname{scan}(N)$ and $\operatorname{sort}(N)$.

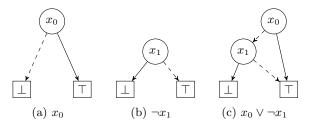


Fig. 1: Examples of Reduced Ordered Binary Decision Diagrams. Terminals are drawn as boxes surrounding their Boolean value. Internal nodes are drawn as circles and contain their decision variable. Arcs to the *high* and *low* child are respectively drawn solid and dashed.

2.2 Binary Decision Diagrams

As shown in Fig. 1, a Binary Decision Diagram [13] (BDD) (based on [2, 31]) represents an n-ary Boolean function as a singly-rooted directed acyclic graph (DAG). Each of its two sinks, referred to as terminals, contain one of the two Boolean values, $\mathbb{B} = \{\top, \bot\}$. These represent the function's output values. An internal BDD node, v, is associated in v-var with a Boolean input variable x_i . Furthermore, it has two BDD nodes as children, v-low and v-high. These three values in f encode the ternary if-then-else

$$v.var ? v.high : v.low$$
.

What are colloquially referred to as BDDs are in fact Reduced Ordered Binary Decision Diagrams (ROBDDs). An Ordered BDD (OBDD) restricts each variable to occur at most once on each path from the root to a terminal and to occur according to a certain order, π . This gives rise to a levelisation of the OBDD where each level, ℓ , is associated with an input variable, x_i . For sake of simplicity, we assume that π is the identity order. A Reduced OBDD further restricts the DAG such that (1) no nodes are duplicates of another and (2) no node is redundant, i.e. v.high = v.low. Assuming the variable ordering, π , is fixed, ROBDDs are a unique canonical form of the Boolean function it represents.

Quantification Algorithm The levelisation of OBDDs allows the recursive BDD algorithms to both be efficient and elegant. For example, the or operation works by a product construction of the two input BDDs. Here, each node of the output BDD simulates, according to π , the decision(s) taken on the shallowest BDD node(s) in the product of nodes from the input.

Since $(\exists x : \phi) \equiv \phi[\top/x] \lor \phi[\bot/x]$, the or operation can be used as the basis for an existential quantification (\exists) for a set of input variables, $X = \{x_i, x_j, \dots, x_k\}$. As shown in Fig. 2, if v is a terminal then this (sub)BDD depends on none of the to-be quantified variables. Otherwise, both its children are resolved recursively into intermediate results, exi0 and exi1. If the decision variable of the

```
1 exists (v, X)

2 if v = \bot \lor v = \top

3 return v

4 exi0 \leftarrow exists (v.low, X)

5 exi1 \leftarrow exists (v.high, X)

6 if v.var \not\in X

7 return Node \{v.var, exi0, exi1\}

8 return or (exi0, exi1)
```

Fig. 2: A recursive multi-variable exists operation.

root, v.var, should not be quantified, a new node with variable v.var is created from the two recursive results. Otherwise, exi0 and exi1 are instead combined (recursively once more) with a nested or operation.

Similarly, one can implement a universal quantification (\forall) by use of a nested and operation. For clarity, our contributions in Section 3 are only phrased with respect to the exists operation. But, everything that follows also applies to forall by replacing or with and.

Relational Product The relational product computes the set of states after taking a step in a transition system with the formula $\exists \vec{x} : S(\vec{x}) \land R(\vec{x}, \vec{x'})$. Hence, the support for a multi-variable quantification operation is key for the application of BDDs in the context of symbolic model checking.

2.3 I/O-efficient BDD Manipulation

The Adiar [49] BDD package builds on top of Lars Arge's ideas [5,6] on how to improve the I/O complexity of BDD manipulation. To not introduce random access, Adiar does not use any hash tables nor recursion for its BDD manipulation. As a result, different BDD objects do not share common subtrees in Adiar. For the same reason, it neither uses pointers to traverse its BDDs. Instead, every BDD node v is uniquely identified by a pair (v.var, v.id) where v.id is v's index on level v.var. Lexicographically, this unique identifier (uid) imposes a total ordering of all BDD nodes into a levelised sequence of nodes. Here, the uid does not specify exactly where to find a BDD node in the input but when to expect it relative to the one currently read. For example, the BDD for $x_0 \vee x_1$ in Fig. 1c is represented in Adiar as the list of nodes in Fig. 3a: every node is a 3-tuple with its uid followed by the unique identifier of its low and its high children.

As depicted in Fig. 4, the previous BDD operations in Adiar, such as or, process a BDD with two sweeps. Both sweeps use time-forward processing [4,18] to achieve their I/O-efficiency: computation is deferred with one or more priority queues until all relevant data has been read. During the first sweep, the Apply, the entire recursion tree is unfolded top-down. Here, the priority queues also double as a computation cache [10,40] by merging separate paths to the same

$$[\{(0,0),(1,0),\top\}; \{(1,0),\bot,\top\}]$$
(a) Node-based Representation
$$[(0,0) \longrightarrow (1,0); (1,0) \longrightarrow \bot; (0,0) \dashrightarrow \top; (1,0) \dashrightarrow \top;]$$
(b) Arc-based Representation

Fig. 3: Representation of the $x_0 \vee \neg x_1$ BDD (Fig. 1c) in Adiar.

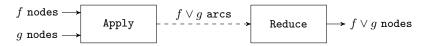


Fig. 4: The Apply–Reduce pipeline of or in Adiar.

recursion target. Hence, the resulting output is in fact not a tree but a DAG. Yet, it is only an OBDD and needs to be reduced. To do so, Adiar uses an I/O-efficient variant of the original bottom-up Reduce algorithm by Bryant [4, 13]. Here, a priority queue is used to forward the uid of reduced nodes t' in the final ROBDD to their to-be reduced parents s in the intermediate OBDD. Yet, to know the parents s, the Reduce needs the intermediate OBDD to be transposed. Luckily, the Apply sweep outputs its OBDD transposed and so no extra work is needed [5, 49]. For example, the or of Fig. 1a and Fig. 1b creates the arcbased representation in Fig. 3b. Here, the arcs end up sorted by their target. For all intents and purposes, this is a transposition of the DAG. This can then be reduced into the node-based representation in Fig. 3a.

The I/O and time complexity of this Apply–Reduce tandem is

$$\mathcal{O}(\operatorname{sort}(N+T))$$
,

where N is the size of the input(s) and T is the size of the unreduced output of the Apply sweep [49].

To catch up with conventional implementation's performance, major efforts have been dedicated to improve on this foundational design.

Levelised Cuts [51] The arcs placed in the above-mentioned priority queues correspond to cuts in the (R)OBDDs. These cuts have a particular shape that follows its levelisation. Hence, the maximum size of the priority queues is bounded by (heuristic over-approximations of) the maximum levelised cut in the input.

These sound upper bounds on the priority queues' size can in turn be used to determine a priori whether one can use a priority queue that is much faster but only works in internal memory.

In practice, this improves performance for smaller and moderate instances.

Levelised Random Access [50] Orthogonally, a product construction's Apply sweep, e.g. an or, can be simplified if one of its inputs is narrow, i.e. each level

fits into internal memory. In this case, one can load each level in its entirety into internal memory. Doing so, provides random access to all of its nodes on said level making one of the Apply's two priority queues in [49] obsolete.

In practice, this improves performance for larger instances.

3 I/O-efficient Multi-variable Quantification

The work in [49] only covers simple BDD operations that *recurse-once-and-return*, e.g. the or. Yet, this does not cover the exists in Fig. 2, where the nested call to or on line 8 depends on the recursions from lines 4 and 5.

To address this, we introduce the *nested sweeping* framework. As shown in Figs. 5 and 6, this wraps the algorithm(s) depicted in Fig. 4: after transposing the input in an initial Apply sweep, a single *outer* Reduce sweep accumulates the result of multiple *inner* Apply–Reduce sweeps. More precisely, nested sweeping consists of the following four phases.

Outer Apply: As shown in Fig. 7, inputs are combined (and possibly manipulated) in an Apply sweep into a single file, F_{outer} . This transposes and merges the inputs such that they are of the form needed by the Reduce of [49].

In the case of exists, this could be a simple transposition of f. In Section 3.3, we describe how this phase can do double duty to process some of the

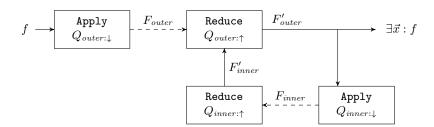


Fig. 5: The Apply-Reduce pipeline of exists with Nested Sweeping.

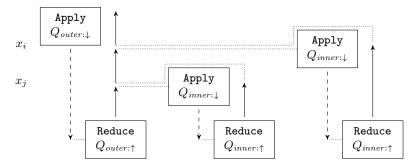


Fig. 6: Sweep direction (solid/dashed) and control-flow (dotted) of Nested Sweeping. The y-axis corresponds to the levels within the BDD.

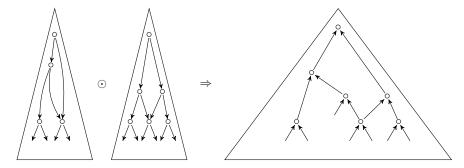


Fig. 7: Outer Apply one (or more) input BDD(s) are processed in a top-down sweep to create a single transposed BDD.

quantifications. In the case of the relational product, the conjunction of states and relation can be computed as part of this phase.

Outer Reduce: As in [49], each level of F_{outer} is reduced bottom-up by having a priority queue, $Q_{outer:\uparrow}$, forward the information about reduced nodes, t', to their unreduced parents, s. The reduced output is pushed into a new file, F'_{outer} .

Let x_j be the next level that needs a nested sweep. For exists, x_j is the largest still to-be quantified variable in X. As visualised in Fig. 8, the logic of [49] is extended as follows:

- 1. If the current level is x_j , each arc $s \to t'$ to a reduced node t' at this level is turned into a request and placed in a second priority queue, $Q_{inner:\downarrow}$. For exists, the requests are of the form $s \to (t'.low, t'.high)$.
- 2. If the current level is deeper than x_j , nodes are reduced as in [49] with a caveat: whether the arc $s \rightarrow t'$ to the reduced node t' is placed in $Q_{outer:\uparrow}$ or in $Q_{inner:\downarrow}$ depends on the level of the unreduced parent s as follows:
 - (a) If $x_j \leq s.\text{var}$, i.e. s is as deep or deeper than level x_j , then $s \rightarrow t'$ is placed in $Q_{outer:\uparrow}$ as normal.
 - (b) Otherwise, i.e. if $s.var < x_j, s \rightarrow t'$ is placed in $Q_{inner:\downarrow}$ instead.

For exists, Case 1 matches the invocation of or on line 8 of Fig. 2 whereas 2 is the return with an unquantified variable on line 7.

When level x_j has finished processing, $Q_{inner:\downarrow}$ is populated with all requests that span across level x_j . Now, the inner Apply sweep is invoked.

Inner Apply: As depicted in Fig. 9a, starting with the requests in $Q_{inner:\downarrow}$, the reduced nodes, t', placed in F'_{outer} by the outer Reduce sweep, are processed with an Apply sweep from [49]. The intermediate unreduced result is placed in a new file, F_{inner} .

For exists, this sweep is the execution of the or on line 8 of Fig. 2. Here, one can use the previous top-down algorithms from [49].

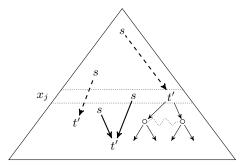
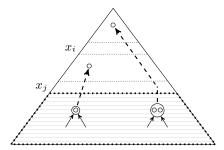
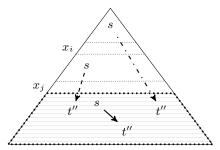


Fig. 8: Outer Reduce: solid arcs stay in $Q_{outer:\uparrow}$ (Case 2a) while dashed arcs are turned into requests for $Q_{inner:\downarrow}$ (Cases 2b on the left left and 1 on the right).



(a) Inner Apply: starting with root requests (dashed), F_{inner} is constructed with the to-be preserved subtrees (left) together with new nodes that are products of previous ones (right).



(b) Inner Reduce: arcs below x_j stay within the inner sweep (Case 1, solid). Arcs that cross x_j are given back to the outer (Case 2, dashed) or to the next inner sweep (Case 3, dash dotted).

Fig. 9: Visualization of the Inner Apply and the Inner Reduce.

Inner Reduce: After the inner Apply sweep, F_{inner} is reduced in another bottomup Reduce sweep of [49]. This creates the reduced nodes t'' placed in F'_{inner} . Let x_i be the next level above x_j that also needs a nested sweep. For exists, x_i is the largest variable smaller than x_j that also needs to be quantified. The arc $s \rightarrow t''$ is placed in a priority queue, $Q_{inner:\uparrow}$, as follows.

- 1. If $x_j < s.var$, i.e. the parent s is below level x_j , then $s \rightarrow t''$ is forwarded within this inner Reduce sweep's priority queue, $Q_{inner:\uparrow}$.
- 2. If $s.var \in [x_i, x_j]$, i.e. the parent s is between level x_i and x_j then $s \to t''$ is given back to the outer sweep, $Q_{outer:\uparrow}$. This matches Case 2a in the Outer Reduce.
- 3. If $s.var < x_i$, i.e. the parent s is above x_i , then $s \to t''$ is placed into $Q_{inner:\downarrow}$ to prepare the next invocation of an inner Apply sweep. This matches Case 2b in the Outer Reduce.

The three cases above are depicted in Fig. 9b. For exists, Cases 2 and 3 are equivalent to the return from or back to exists whereas Case 1 is equivalent to a return statement within the or's own recursion. Case 3 is needed to match 2b with x_i replaced with x_i .

Finally, F'_{inner} replaces F'_{outer} and control returns to the outer Reduce sweep to proceed with the levels above x_i .

 F_{inner} from the inner Apply sweep can be thought of as overlayed on top of F_{outer} from the outer Apply sweep; together they produce a valid (but unreduced) OBDD. The Outer and inner Reduce sweeps work together to reduce this into a single file, F'_{outer} . When no more levels, x_j , need to be processed and the outer Reduce sweep has finished processing, then F'_{outer} contains the final reduced BDD of all nested operations.

Whereas the Apply–Reduce algorithms in [49] only operate on a singly-rooted DAG, the inner Apply and Reduce sweeps have to operate on a multi-rooted one. Yet, these previous algorithms need not be changed since $Q_{inner:\downarrow}$ is prepopulated with all relevant roots in Cases 1 and 2b in the outer and Case 3 in the inner Reduce sweeps.

Since the result of the inner Apply and Reduce sweeps replaces the entire set of nodes in F'_{outer} , the priority queue $Q_{inner;\downarrow}$ not only needs to be populated with requests for the nodes that need to be changed but also with requests for the nodes one wishes to keep (see also Fig. 9a). This makes the inner Apply sweep not only compute the desired result but also act as a mark-and-sweep garbage collection. On the first glance, these additional non-modifying requests may seem too costly – especially if most requests do not modify subtrees. In practice, 33.3% of all requests created throughout our benchmarks (see Section 5 for a detailed presentation thereof) are subtree modifying. For each benchmark instance, 23.0% of all requests modify subtrees on average (median 35.6%). That is, a reasonable number of all requests (and hence BDD nodes processed) change the subgraph in F'_{outer} .

3.1 Complexity of Nested Sweeping

As mentioned in the description of the outer Apply sweep, nested sweeping works for multiple inputs. In this work, it suffices to assume it only has to deal with a single BDD f of N nodes as also depicted in Fig. 5.

Lemma 1. A single BDD f with N nodes can be transposed in $\Theta(sort(N))$ I/Os and time and $\Theta(N)$ space.

Proof. In $\Theta(\operatorname{scan}(N))$ I/Os and time iterate over and split all nodes v in-order into the two arcs $v.\operatorname{uid} \to v.\operatorname{low}$ and $v.\operatorname{uid} \to v.\operatorname{high}$. Sort these 2N arcs on their target using $\Theta(\operatorname{sort}(N))$ I/Os and time and linear space transposes them.

In Section 3.3, we propose to embed valuable computations inside of the outer Apply sweep. This comes at the cost of potentially changing the BDD size. To encapsulate such cases too, let N' be the output size of the outer Apply sweep which may exceed $\mathcal{O}(N)$. Yet, this step is not the bottle-neck of the algorithm.

Lemma 2. Ignoring the work done within the inner sweeps, the outer Reduce sweep costs $\Theta(\operatorname{sort}(N'))$ I/Os and time and requires $\mathcal{O}(N')$ space.

Proof. This follows from the complexity of the Reduce algorithm in [49] (based on [6]) and the constant extra time spent for each of the N' nodes to resolve the additional logic in Cases 1 and 2 of the outer Reduce sweep.

By combining Lemmas 1 and 2 together with the fact that the last invocation of the inner Apply and Reduce sweeps constructs, together with the outer Reduce sweep, the output of size T, we obtain the following lower bound on the complexity of nested sweeping.

Corollary 1. Nested Sweeping uses $\Omega(N+T)$ space and $\Omega(sort(N+T))$ time and I/Os where N and T are the size of the input and output, respectively.

In particular for the exists BDD operation, let T_j be the size of F'_{outer} when the inner Apply sweep is invoked at level x_j .

Lemma 3. A single invocation of the inner Apply and Reduce sweeps at x_j costs $\Theta(sort(N'+T_i^2))$ I/Os and time and uses $O(N'+T_i^2)$ space.

Proof. As in [49], the algorithm's complexity depends on the number of elements placed in the priority queues [4]. In particular, a single nested or sweep deals with up to 2N' arcs from F_{outer} . On top of these, it also processes up to $2T_j^2$ arcs created during the product construction of F'_{outer} .

Since nested sweeping closely simulates the (parallelised) recursive BDD algorithm in Fig. 2, one should expect it achieves, similar to the algorithms in [49], major improvements in the number of I/Os at the cost of a log-factor in the running time when compared to the conventional recursive algorithms. This is indeed the case.

Proposition 1. Quantification of a set of variables, X, is computable with nested sweeping in $O(\operatorname{sort}(N^{2^{|X|}}))$ I/Os and time and $O(N^{2^{|X|}})$ space.

Proof. Due to Lemma 1, F_{outer} from the outer Apply sweep has up to 2N arcs. This is also the size of F'_{outer} without any inner sweeps. Each inner Apply and Reduce sweep may increase the size of F'_{outer} quadratically. The result follows from Lemmas 1 to 3.

Asymptotically, this is not an improvement over just quantifying each variable one-by-one using the algorithm already proposed in the full version of [49]. Yet, doing so would involve 2|X| sweeps over *all* levels of the input whereas, as highlighted in Fig. 6, nested sweeping only processes levels below each quantified variable. This difference is also evident in practice: throughout our benchmarks (see Section 5 for details), when quantifying with nested sweeping rather than each variable independently, the total number of requests processed with the or operation decreases by 13.9% while the share of 2-ary product constructions increases from 57.3% to 66.6%.

3.2 Optimisations for Nested Sweeping

While nested sweeping as described above is an improvement over previous work in [49], there are multiple avenues to further improve its performance in practice.

Terminal Arcs: No inner Apply sweep changes the value of terminals. Hence, requests of the form $s \to \top$ and $s \to \bot$ can be forwarded to s regardless of any nesting levels, x_j , in-between. Furthermore, the request based on t' in the outer Reduce sweep may trivially resolve into a terminal. In this case, the resulting terminal can be forwarded to its parents (regardless of their level). For exists, this would be if both t'.low and t'.nigh are terminals or either of them is the \top terminal.

This decreases the size of $Q_{inner:\downarrow}$. Furthermore, it makes the requests placed in $Q_{inner:\downarrow}$ compatible with an implicit invariant of the Apply sweep's in [49].

In practice, the number of requests skipped this way depends on the use-case and the scale. 3.7% of all requests processed as part of our benchmarks (see Section 5 for a description) are for terminals. On average, 6.9% of the requests (with a median of 7.0%) are for terminals in each benchmark. For the Garden of Eden (GoE) benchmark specifically, 15.3% of all requests are terminals on average (median 17.7%). On the other hand for the Quantified Boolean Formula (QBF) benchmark, only 5.0% (median 5.9%) of them were.

Bail-out of Inner Sweep: There is no need for the outer Reduce sweep to invoke the inner sweeps if $Q_{inner:\downarrow}$ only contains requests that preserve subtrees, i.e. if Case 1 in the Outer Reduce did not create any requests that manipulate the accumulated OBDD in F'_{outer} . On level x_j , such requests can stem from a redundant node t' being suppressed. For exists, this may also occur due to either t'.low or t'.high being the \bot terminal, which is neutral for the or operation, or being \top , which is short-circuiting it.

In this case, the entire content of $Q_{inner;\downarrow}$ can be redistributed between $Q_{outer;\uparrow}$ and $Q_{inner;\downarrow}$ as in for the next deepest to-be quantified level, x_i . After doing so, the outer Reduce sweep can immediately proceed processing the next level.

For exists, any short-circuiting by the or operation in Case 1 of the outer Reduce sweep can kill off some subtrees in F'_{outer} . In this case, one cannot skip the last invocation of the inner sweeps. Otherwise, the final result F'_{outer} otherwise could include dead nodes. Yet, even so, one can instead of the expensive top-down algorithm, e.g. or for exists, invoke the inner Apply sweep with a much simpler (and so faster) mark-and-sweep algorithm.

In practice, 75.6% of all nested sweeps in our benchmarks (see Section 5 for their presentation) are skippable. For each benchmark, between 6.8% and 93.5% of all nested computations were skipped with an average of 59.2% (median of 81.0%). The number of nested levels depends on the problem domain and its instance. For the Garden of Eden (GoE) benchmark, only 26.8% of all nested computations were skipped on average (median 29.0%), whereas 82.7% (median 84.3%) of all levels of the Quantified Boolean Formulas (QBFs) could be skipped.

Root Requests Sorter: Instead of making the outer Reduce sweep push requests directly into $Q_{inner:\downarrow}$, it can push it into an intermediate list of requests, $L_{outer:\downarrow}$. The content of $L_{outer:\downarrow}$ is sorted using the same ordering as $Q_{inner:\downarrow}$ as the inner Apply sweep is invoked, to then merge it on the fly with theinner Apply sweep's priority queue. This allows one to postpone initialising this priority queu until the inner Apply sweep is invoked. This has multiple benefits:

- $Q_{inner:\downarrow}$, resp. $Q_{inner:\uparrow}$, only exists and uses internal memory during the inner Apply sweep, resp. the inner Reduce sweep. Hence, the memory otherwise dedicated to $Q_{inner:\downarrow}$ can be used in the outer Reduce sweep for the Reduce's per-level data structures in [49]. Furthermore, this also increases the amount of space available to the inner Reduce sweep. Hence, this ought to improve the running time of both the inner and the outer Reduce sweeps.
- In practice, sorting a list of elements once is significantly faster than maintaining an order in a priority queue [41]. Merging $L_{outer:\downarrow}$ on the fly with $Q_{inner:\downarrow}$ is faster than passing requests to the inner sweep's priority queue.
- If $Q_{inner:\downarrow}$, resp. $Q_{inner:\uparrow}$, is initialised for each inner Apply sweep, resp. inner Reduce sweep, then the monotonic and faster *levelised priority queue* in the full version of [49] can be used instead of a regular non-monotonic priority queue.
- Levelised cuts [51] bound the size of each individual inner Apply and Reduce sweep. Hence, for each nested sweep, one can, if it is safe to do so, replace $Q_{inner:\downarrow}$ and/or $Q_{inner:\uparrow}$ with a faster internal memory variant.
- Levelised random access [50] may need to change the sorting predicate in $Q_{inner:\downarrow}$. Hence, $L_{outer:\downarrow}$ allows this optimisation to be applied for each invocation of the inner Apply sweep depending on the width of F'_{outer} .

Furthermore, levelised cuts not only bound the size of $Q_{outer:\uparrow}$ in the outer Reduce sweep but also the size of $L_{outer:\downarrow}$. Hence, while deciding whether $Q_{outer:\uparrow}$ fits into memory, one can also decide whether $L_{outer:\downarrow}$ does.

All in all, this allows the optimisations in [50,51] to be applied on a sweep-by-sweep basis. In practice, if one neither uses faster internal memory variants of $Q_{inner:\downarrow}$ and $Q_{inner:\uparrow}$ nor levelised random access, then Adiar needs a total of 32.1 h to solve 145 out of the 147 benchmarks in Section 5. Using these two optimisations shaves 13.0 h off the total computation time (speedup of 1.68). For each individual instance, this improves Adiar's performance between a factor of 1.07 and 5.05 (1.77 on average)¹. Furthermore, without $L_{outer:\downarrow}$, the exponential blow-up in Proposition 1 implies $Q_{inner:\downarrow}$ would almost always have to use external memory. As the optimisations in [49-51] would then not be applicable, one would expect a slowdown of several orders of magnitude similar to [50,51].

¹ Compared to [51], the *external* memory sorters in this comparison still use the levelised cuts to circumvent wasting time with initialising too much internal memory. This is why, there is not a speedup of several orders of magnitude. If this use of levelised cuts is also reverted to obtain its state back in [49], then preliminary experiments on a machine with 8 GiB of memory exhibits a speedup of 2.71 on average. As memory increases, one should expect a difference similar to the one reported in [51]

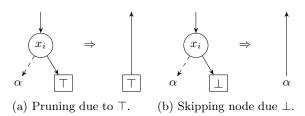


Fig. 10: Example of pruning quantification of a to-be quantified level x_i .

3.3 Optimisations for Quantification

As presented above, the outer Apply sweep merely transposes the input BDD f. Yet, doing so may not make the most out of having to touch the entire BDD graph; as long as the result is a transposed BDD for, one can incorporate additional computations inside of this sweep. Hence, we now explore possible top-down sweeps that can be used instead of the algorithm in Lemma 1.

Pruning \top **Siblings:** Corollary 1 and Proposition 1 show a possibly wide gap in the potential performance of the nested exists algorithm. Lemma 3 shows this stems from the possibility of some partially quantified result explodes exponentially in size. Yet, T_j can only be larger than T if it contains subtrees that will be pruned or merged later when another variable is quantified. This can only happen due to the \top terminal shortcutting an or. Hence, to be closer to the lower bound in Corollary 1, we need to identify redundant computation by pushing information about the \top terminal down through the BDD of f.

As shown in Fig. 10a, one can collapse to-be quantified nodes at a level x_i if one of their children is the \top terminal. Similarly, as shown in Fig. 10b, one can skip over nodes with a \bot terminal as its child. This can be done as part of a simple top-down sweep similar to the Restrict in the full version of [49].

In the worst-case, this does not apply to any node in f and so the output is similar to the algorithm in Lemma 1. Our preliminary experiments indicate this approach introduces an overhead of up to 2%. Yet, if nodes are prunable, making N' < N, then total performance can improve with up to 21%.

Deepest Variable Quantification: The single-variable quantification in the full version of [49] can also be used to transpose f. This removes one of the to-be quantified variables $x_i \in X$ in $\mathcal{O}(\operatorname{sort}(N^2))$ I/Os and time and $\mathcal{O}(N^2)$ space. The resulting transposed graph, F_{outer} , has size $N' \leq N^2$. To not change the overall memory usage, one can choose x_i to be the largest to-be quantified variable. Doing so makes the levels at x_i and below equivalent to F_{inner} after the first inner Apply and Reduce sweep. That is, $N' \leq N + T_i$ and one saves an entire nested sweep at no cost to memory usage.

Our preliminary experiments indicate this only slows down computation time on average by 4.7%. We hypothesise this is due to the deepest variable x_i is often

close to the bottom of the BDD and so, this sweep is primarily transposing the graph with more complex logic.

One can also incorporate the above pruning of \top siblings inside this quantification sweep. This improves performance for applicable cases. But, it does not offset the additional overhead in the remaining cases.

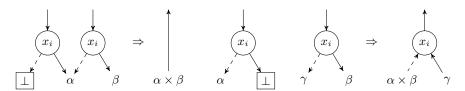
Partial Quantification: The single-variable quantification in the full version of [49] can be generalised to partially resolve all $x_i \in X$ in a single sweep. The requests in [49] are for pairs of nodes $(t_1, t_2) \in f \times f$.

Without loss of generality, assume $t_1.var = t_2.var = x_i \in X$. In this case, the 2-ary product construction should turn into $(t_1.low, t_1.high, t_2.low, t_2.high)$. If any of these four uids are the \top terminal then the entire request can immediately be resolved to \top . Furthermore, any \bot terminal is neutral to the or operation and can be pruned from the 4-tuple. Similarly, any duplicate uids can be merged. Since or is commutative, one can sort the 4-tuple to quickly identify these cases. If the resulting tuple has 2 or fewer entries remaining, the product construction can proceed as in the full version of [49] (Fig. 11a). Otherwise, a new node with x_i is created with the result of each half of the 4-tuple as its children (Fig. 11b). Inductively, this is correct (after later quantification of the new x_i node and its subtrees) as the or operation is associative.

The resulting DAG is a 2-ary product construction of f, and so F_{outer} has size $N' \leq N^2$. As per [49], this single sweep is computable in $\mathcal{O}(\operatorname{sort}(N^2))$ I/Os and time and $\mathcal{O}(N^2)$ space.

Similar to the two \top terminal pruning above, partial quantification prunes shortcutted subtrees across all levels of f. Furthermore, similar to deepest quantification, it leaves at least one fewer levels of to-be quantified variables to be processed later.

Our preliminary experiments indicate, partial quantification can in practice improve performance up to 61%. Yet, many other intances slow down just as much (up to 115%, i.e. also a bit more than a factor of two). We hypothesise this is due to partial quantification pairing nodes with a conflicting assignment. For example, in Fig. 11b α is paired with β rather than γ . Oddly enough, in our preliminary experiments, the instances that were improved by \top pruning



(a) Fully quantified pair of nodes

(b) Partially quantified pair of nodes

Fig. 11: Example of partial quantification of a level x_i .

are disjoint from the ones improved by partial quantification. Further research is needed to investigate why and when partial quantification is useful.

Repeated Partial Quantification: The top-down sweep aboves produces a transposed and unreduced OBDD. Yet, it is in practice possible that N' is smaller than $(1+\epsilon)\cdot N$ for some $\epsilon\in\mathbb{Q}$, i.e. its size has not grown considerable. In this case, the resulting product construction has very few new BDD nodes that are potentially reducible. Hence, it may be more beneficial to untranspose the OBDD and then immediately rerun another transposing top-down sweep. Doing so with pruning or partial quantification can propagate the \top terminal further and so prune more subtrees. Yet, it is unlikely that pruning \top terminals in Fig. 10a makes said terminal available for another to-be quantified variable. That is, it is unlikely in this case that a second sweep would further prune subtrees. Hence, this is most promising to do with partial quantification.

Since there are very few new BDD nodes, it is unlikely that the Reduce sweep of [49] will do much more than just untranspose the DAG. Hence, one would want to untranspose it with a simpler and faster algorithm. As can be seen in Fig. 3, one can instead merely sort all arcs on their source and then merge them on the fly into nodes. Asymptotically, this is still a $\Theta(\text{sort}(N'))$ operation. But, the constant involved is smaller than the Reduce of [49].

Hence, one can repeat the above partial quantification operation until N' exceeds $(1+\epsilon)\cdot N$, it has run δ times, or there are no more to-be quantified variables left in F_{outer} . In practice, we have not yet found any instance where more than a single quantification sweep further improves performance. Hence, as further research hopefully uncovers when it is beneficial to use partial quantification, we can extrapolate this into a value of δ .

4 Implementation of Nested Sweeping in Adiar

Most of the logic in Section 3 can be implemented by wrapping the priority queues $Q_{outer:\uparrow}$, $Q_{inner:\downarrow}$, and $Q_{inner:\uparrow}$ and $L_{outer:\downarrow}$ with additional logic on how to merge and whereto split requests.

- The logic of whether to push to $Q_{outer:\uparrow}$ or $L_{outer:\downarrow}$ in the outer Reduce sweep, resp. Cases 2 and 3 in the inner Reduce sweep, is a conditional on level x_j , resp. x_i .
- During the inner Apply sweep, requests from $L_{outer:\downarrow}$ are merged on the fly with the ones pushed to $Q_{inner:\downarrow}$.
- When placing requests in $L_{outer:\downarrow}$, they are marked as originating from the outer Reduce sweep. During the inner Reduce sweep, requests are forwarded to $Q_{outer:\uparrow}$ or $Q_{inner:\uparrow}$ depending on whether they are marked to be from the outer sweep or not.

This has been implemented in Adiar v2.0 with (compile-time known) decorators: a class with the same interface as the priority queues runs the above logic before

passing it onto the wrapped priority queues and sorters. This makes the logic of each sweep agnostic to and reusable in the context of nested sweeping.

The nested sweeping framework, i.e. the decorators, $L_{outer:\downarrow}$, and the algorithm and its optimisations, has been implemented with 1287 lines of templated C++ classes and functions. Similar to [48,51], the use of templates completely remove any indirection and abstraction introduced for the sake of code quality. The entire framework has been tested separately from the remaining codebase with 104 unit tests. The quantification algorithms themselves grew from 548 lines of code and 84 unit tests to 1152 lines of code and 152 unit tests (without any of the optimisations in Section 3.3).

5 Experimental Evaluation

To evaluate the impact of using nested sweeping, we have run experiments aiming at answering the following three research questions:

- 1. How does nested sweeping compare to the repeated use of the single-variable quantification from the full version of [49]?
- 2. How does Adiar with nested sweeping compare to the external memory BDD package, CAL [46]?
- 3. How does Adiar with nested sweeping compare to conventional BDD packages that use depth-first recursion and memoisation [9, 21, 27, 33, 52]?

5.1 Benchmarks

For this evaluation, we have implemented the following two benchmarks that rely on multi-variable quantification. Similar to [48, 49], all benchmarks have been implemented on top of C++-templated adapters for each BDD package. This makes each BDD package run the exact same set of operations without introducing any indirection. The source code for all benchmarks can be found at the following url:

github.com/ssoelvsten/bdd-benchmark

QBF Solving: Given a Quantified Boolean Formula (QBF) in the QCIR [54] format, each gate of the given circuit is recursively transformed into a BDD. For inputs, we use the 102 encodings from [47] of 2-player games on a grid. In our experience, the symbolic style of these inputs makes them well suited to be solved with BDDs. Hence, they provide a typical use-case of quantification in BDDs. Furthermore, though these inputs are not in CNF they are in prenex form. In practice, resolving these prenex quantifications at the end is computationally much more expensive than computing the to-be quantified circuit, i.e. the matrix.

Based on preliminary experiments, we use a variable order based on a depthfirst traversal of the given circuit. In the prenex, we merge adjacent blocks with the same quantifier to increase the number of concurrently quantified variables. Garden-of-Eden: In a cellular automaton, a Garden-of-Eden [42] (GoE) is any configuration without a predecessor. In Conway's Game of Life [24], recent results show there exists no GoE of size 8×8 or smaller [8]. Hence, the BDD for an $n_r\times n_c \leq 8\times 8$ sized transition relation will collapse to \top when all of its previous state variables are existentially quantified. Yet, a row-major encoding of the transition relation requires only a polynomially sized BDD. Hence, the complexity of this problem manifests as an explosion of the BDD's size during the existential quantification.

The transition relation on a grid of size $n_r \times n_c$ is encoded with $(n_r+2) \cdot (n_c+2)$ previous state variables, \vec{x} , and (up to) $n_r \cdot n_c$ next state variables, $\vec{x'}$. By reusing next state variables for multiple cells, one can restrict the search for symmetric GoEs. Post state variables, $x'_i \in \vec{x'}$, follow a row-major order. Previous state variables, $x_i \in \vec{x}$, are interleaved to directly precede their respective post state variable, x'_i .

5.2 Hardware and Settings

As in [48–51], we have run our experiments on the *Grendel* cluster at the Centre for Scientific Computing Aarhus. In particular, we ran both benchmarks on machines with 48-core 3.0 GHz Intel Xeon Gold 6248R processors, 384 GiB of RAM, 3.5 TiB of SSD disk, and which run Rocky Linux (Kernel 4.18.0-513). All code was compiled with GCC 10.1.0 or rustc 1.72.1. Each BDD package was given $\frac{9}{10}$ th of the available RAM, i.e. 345 GiB, leaving $\frac{1}{10}$ th to other data structures and the operating system. Next to that, the BDD packages use a single thread and their default/recommended settings.

Note that these machines have vasts amounts of memory. This is to ensure that depth-first implementations are not slowed down by external factors. If less memory is available, then depth-first implementations would have to run multiple garbage collections to stay within the memory limits (cf. the largest instances solved by BuDDy [33] in Fig. 14a). This, in turn, clears their memoisation tables and forces them to recompute previous results. Furthermore, this large amount of memory ensures they can solve larger problems without using the swap partition. If they had to use it then they would slow down by about two orders of magnitude (see the full paper of [49] for an example). Hence, machines of this scale allow us to measure the algorithms' running time without the noise otherwise introduced by their execution environment. Finally, this biases the running time in favour of the depth-first implementations, which in turn makes the numbers we report on Adiar's relative performance close to the worst-case.

5.3 Experimental Results

The computing cluster's scheduler does not let many long-running jobs run concurrently. To obtain all 1176 data points reported below within only a few months, we had to place each of the 147 instances in buckets of instances with a common timeout. In particular, an instance is placed in the bucket with the

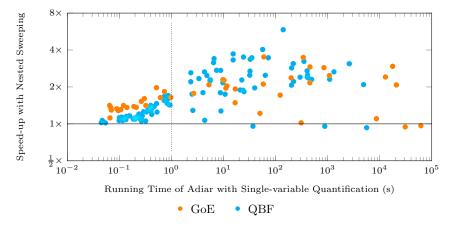


Fig. 12: Relative time $(T_{\text{old}}/T_{\text{new}})$ of quantification with nested sweeping (T_{new}) compared to the previous repeated single-variable quantification (T_{old}) .

smallest timeout that is four times larger than Adiar needed during preliminary experiments. That is, a BDD package timing out should only be understood as it (possibly) being considerably slower than Adiar.

Depending on an instance bucket placement, running time measurements were made 1 to 3 times. Due to node failures on the cluster, Adiar with nested sweeping was run once more, resulting in its measurements being repeated on many instances 4 times. On average, all data points had 3.0 measurements. Similar to [48,49,51], we report for each benchmark the minimum time recorded as it is the measurement with the least noise [17].

Adiar needs less than 1 s to solve 27 out of the 45 GoE instances, resp. 50 out of the 102 QBF instances. As will become evident later with Figs. 13 and 14, this makes them so small that they are not within the current scope of Adiar. For completeness, we still show and discuss these results.

RQ 1: Improvement by Nested Sweeping Figure 12 shows the speed-up of using Adiar with nested sweeping (without any optimisations in Section 3.3) relative to quantifying each variable individually. Across all instance sizes, nested sweeping is in general an improvement in performance. We have recorded a slowdown of up a factor of 1.05 for 5 instances. Yet, we also recorded speed-ups up to a factor of 5.88 for the 142 remaining instances. On average, performance improves by a factor of 1.7 for both QBF and GoE. The total computation time was decreased by 21% from 49.4 h to 39.1 h.

RQ 2: Comparison to CAL To the best of our knowledge, CAL [46] (based on [7,43]) is the only other BDD package also designed to manipulate BDDs larger than main memory. To do so, it uses breadth-first algorithms that should work well with BDDs stored on disk via the operating system's swap memory [46].

Table 1: Time taken and the average ratio between Adiar and CAL for the 124 commonly solved instances. The average pertains only to instances where Adiar needed at least 1 s to solve them. Ratios larger than 1.00 means Adiar is faster.

	Time		# Solved		Avg. Ratio (1+s)	
	• GoE	• QBF	• GoE	• QBF	• GoE	• QBF
Adiar	7431.9s	688.0s	45	102	_	-
CAL	184688.3s	295660.0s	38	86	5.0	25.2

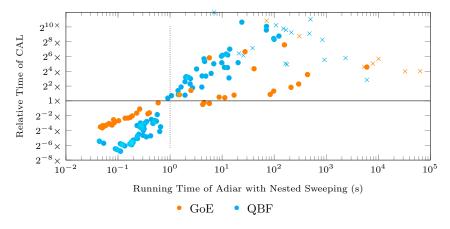


Fig. 13: Relative performance $(T_{\text{CAL}}/T_{\text{Adiar}})$ of CAL (T_{CAL}) compared to Adiar with Nested Sweeping (T_{Adiar}) . Time-/Memouts are marked as crosses.

CAL also includes algorithms to support multi-variable quantification [46]. For more details, see [46] and Section 6. The machines for our experiments provide a 48 GiB swap partition, i.e. a 12.5% increase in available space.

Preliminary experiments indicated CAL's breadth-first algorithms are much slower than Adiar's time-forward processing. Hence, we multiplied the timeout for CAL by a factor of 3. But as is evident in Fig. 13, this increase turned out to still overestimate CAL's performance on larger instances. Hence, the running times and averages in Fig. 13 and Table 1 pertain only to the 124 instances which CAL can solve within the given RAM, SWAP, and the time limits.

Even though this discards the instances where CAL struggles, i.e. the data points that remain are in CAL's favour, Fig. 13 shows Adiar heavily outperforms CAL for instances where Adiar takes 1 s or longer to solve. Where CAL uses 133.4 h to solve 124 instances, Adiar, by solving them in only 2.3 h, is 59.1 times faster. On these larger instances, CAL is on average 14.7 times slower than Adiar. As is evident in Fig. 13 and Table 1, Adiar especially outperforms CAL on the QBF benchmark. For example, the largest difference was measured for the hex/hein_15_5x5-13 QBF instance, where CAL is 1081 times slower than the 71.2 s Adiar needs to solve it.

On the other hand, CAL is considerably faster for the instances where Adiar takes less than 1 s to solve. At this smaller scale, CAL primarily uses conventional depth-first algorithms; doing the same for Adiar is still left as future work [50,51].

RQ 3: Comparison to Depth-First Implementations For this comparison, we have have compared performance with BuDDy 2.4 [33], CUDD 3.0.0 [52], LibBDD 0.5 [9], OxiDD 0.6 [27], and Sylvan 1.8.1 [21]. Their individual performance relative to Adiar is shown in Fig. 14. Out of the 147 instances, 140 are solved by all five depth-first BDD packages, i.e. the remaining 7 instances have at least one BDD package running out of memory (MO) or time (TO). Adiar solves all of them. Running out of time is most likely due to repeated need for garbage collection, which essentially is equivalent to an MO. Yet for fairness, Table 2 shows the total time for these 140 commonly solved instances. The average ratio, on the other hand, pertains to all instances solved by the respective BDD package.

As shown in Table 2, Adiar solves the 40 common GoE instances in 2.7 h. This makes it 1.13 times faster than CUDD and 2.20 times faster than OxiDD on this particular benchmark. Adiar further solves the 100 common QBF instances in 1.25 h. This makes it as fast as CUDD and 1.3 times faster than Sylvan at solving these QBF instances. On average, all five depth-first BDD packages are at most 4 times faster if Adiar needs more than 1 s to solve the same instance. This relative difference is similar to the results in [48–51]. That is, nested sweeping allows Adiar to compute quantifications at no additional cost to previous work. As was also evident with CAL, Adiar is yet not designed for the instances that need less than 1 s to solve and so the other BDD packages heavily outperform it in these cases.

LibBDD does not use a unique node table to save on memory. It neither uses complement edges [29]. Yet when comparing Fig. 14c to the others in Fig. 14, it is the only BDD package consistently faster than Adiar (ignoring its three

Table 2: Total time needed by Adiar and conventional depth-first implementations to solve the 140 commonly solved instances. The average ratio covers all instances that were commonly solved by all BDD packages and where Adiar needed at least 1 s to solve. Ratios larger than 1.00 means Adiar is faster.

	Time		# Solved		Avg. Ratio (1+s)	
	• GoE	• QBF	• GoE	• QBF	• GoE	• QBF
Adiar	9655.7s	4499.4s	45	102	-	-
BuDDy	4725.5s	3793.1s	40	100	0.30	0.25
CUDD	10892.8s	$4591.9\mathrm{s}$	40	101	0.61	0.75
Lib-BDD	$4365.7\mathrm{s}$	$2687.3\mathrm{s}$	43	101	0.54	0.45
OxiDD	$21223.9\mathrm{s}$	$2379.6\mathrm{s}$	41	101	0.48	0.39
Sylvan	2925.4s	5841.4s	44	102	0.46	0.70

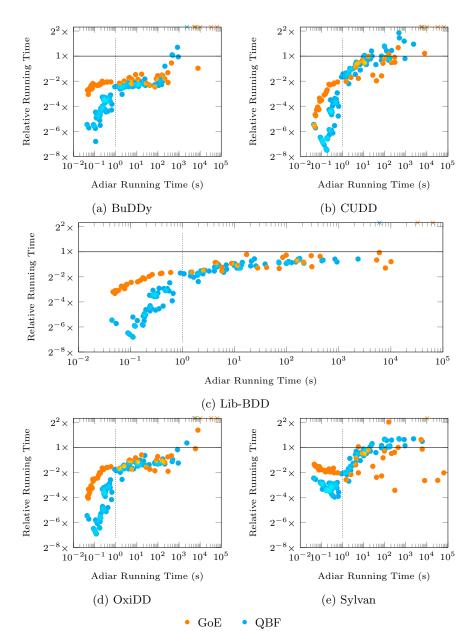


Fig. 14: Relative performance of depth-first implementations compared to Adiar with nested sweeping.

MOs). Furthermore, as is evident in Table 2, next to Sylvan it was second best at solving the most instances within the given time limit.

As is evident from Fig. 14e, Sylvan is comparatively good at some of the larger GoE instances, thereby beating all other BDD packages in the total time to solve the GoE benchmarks. As the BDD collapses to \top , one may expect this is due to Sylvan skipping the second recursive calls to exists if the first recursion resulted in \top . Yet, CUDD also includes this optimisation without exhibiting the same behaviour. Further investigation is needed to identify how Sylvan excels on these instances. Sylvan is also the only other BDD package able to solve all QBF instances within the given time limit, in parts thanks to its small memory footprint per BDD node [21]. Yet, Sylvan requires a total of 5.0 h to solve all 102 QBF instances whereas Adiar only needed 3.6 h, making Sylvan 1.4 slower than Adiar.

6 Related Work

Many other implementations of BDDs also support quantification of multiple variables. All these are based on a nested (inner) operation being accumulated in an (outer) traversal of the input; the nested sweeping framework achieves the same within the time-forward processing paradigm [4, 18] of Adiar's algorithms.

CAL: The CAL [46] BDD package (based on [7, 43]) is to the best of our knowledge the only implementation of BDDs also designed to compute on BDDs whose size exceed main memory. To do so, it uses breadth-first algorithms that are resolved level by level. For each level it still follows the conventional approach: a unique node table is used to manage BDD nodes while a polynomial running time is guaranteed by use of a memoisation table. These per-level hash tables, both in theory and in practice, put an upper bound on the maximum BDD width that CAL can support with a certain amount of internal memory [6].

Its quantification operation also required additional ideas particular to the design of CAL. Since it uses a single breadth-first queue for each level, each queue contains requests for both the outer and the nested inner traversals. Hence, both can be – and are – processed simultaneously [46]. Furthermore, it switches between breadth- and depth-first evaluation of subtrees to improve performance: the outer traversal is depth-first for the BDD nodes with to-be quantified variables and breadth-first otherwise. If the first subtree's quantification makes computing the other ones redundant, then all computation of the second is skipped. These depth-first steps are also placed in the very same queues as the breadth-first steps; doing so ensures no additional random access is introduced.

By the nature of nested sweeping, our proposed algorithm is, unlike CAL, not easily able to skip redundant computations. In Section 3.3 we investigate multiple promising avenues to achieve similar pruning of redundant computation. Furthermore, the lack of a unique node table in Adiar requires our algorithms to retraverse and copy the subtrees that are unchanged. Even so, as evident in Section 5, Adiar with nested sweeping outperforms CAL by up to several orders

of magnitude. Moreover, the I/O-efficient approach in [6,49], and by extension the ones in this work, are, unlike CAL, I/O efficient despite a BDD's level is wider than main memory.

Distribution Sweeping: In the context of computational geometry, distribution sweeping [25] is an I/O-efficient translations of internal memory sweepline algorithms. Here, the recursion is turned on its head: the recursive but I/O inefficient data structure is replaced with an I/O-efficient list and the iterative algorithm is instead turned into a recursive one. Specifically, all the points in the plane are sorted on the x-axis and distributed into M/B vertical strips (see Section 2.1 on the I/O-model). After these strips have been solved recursively, an M/B-way merge procedure both merges and prunes all strips into one while simultaneously recreating a vertical sweepline moving across all strips [12,25].

In our case of translating the exists algorithm (see Fig. 2), we also intend to move the recursion out of a data structure, namely out of the BDD. Unlike for distribution sweeping, we do not intend to divide-and-conquer the input but instead recurse through the dependencies of the algorithm's recursion, e.g. between the independent calls to exists and the nested or operation that depends on their result. Independent recursions are resolved simultaneously with regular time-forward processing sweeps as in [49]. Dependencies are handled by moving requests from the priority queue of one time-forward processing sweep to the priority queue of another. When all dependencies have been moved, the current sweep is paused to then start a nested sweep – the results of which are in turn parsed to its dependencies.

7 Conclusions and Future Work

Each sweep in [49] is independent of the others. Using only this approach, one can only quantify a single variable a time but not multiple at once. In this work, we enable multi-variable quantification with the *nested sweeping* framework. Here, multiple sweeps work together: each sweep forwards information within priority queues to itself, its parent, or its child in a recursion stack.

In practice, nested sweeping has improved the total time that Adiar needs to solve our quantification benchmarks by 21%. On average, it improves each instance's running time by a factor of 1.7. This allows us to extend the results in [48–51] to Adiar's quantification operations: Adiar can now solve larger problems that include quantification than conventional depth-first implementations can; in some cases it can even solve these problems faster. For example, whereas CUDD [52] could solve 141 out of our 147 benchmark instances in 5.6 h, Adiar only needed 4.6 h to do the same; Adiar could also solve the remaining 6 instances. Finally, Adiar is also faster, often by one or more orders of magnitude, than the only other existing external memory BDD package, CAL [46].

Adiar's nested sweeping framework has already been generalised to pave the way for the implementation of other multi-recursive BDD operations. We intend to use it for the relational product and functional composition which are both

used in model checking such as [28]. Furthermore, we hope to also use it as a fundamental building block in a novel variable reordering procedure. Finally, nested sweeping opens up the possiblity to create an I/O-efficient implementation of other types of decision diagrams. For example, both Quantum Multiple-valued Decision Diagrams [38] and Polymial Boolean Rings [11] require nested sweeps to implement their multiplication operations.

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