

Anagram Trees

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Wordrow

wordrow.io

get_game();



`get_game();`



Reason: (1.) Simple backend (2.) Small file size

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`contains(x)`

`anagrams(x)`

`subanagrams(x)`

`insert(x)`

Multi-valued Anatree

Letter Ordering

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Anagrams

Consider an *alphabet* $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and words x, y, \dots from a *language* $L \subseteq \Sigma^*$.

Definition (Rohit Parikh, 1961)

The *Parikh vector* of a word $x \in \Sigma^*$ is $\Psi(x) \triangleq \langle |\sigma_1|, |\sigma_2|, \dots, |\sigma_k| \rangle$.

Example

For $\Sigma = \{a, b, c\}$, $\Psi(abb) = \langle 1, 2, 0 \rangle$ and $\Psi(abab) = \Psi(abba) = \langle 2, 2, 0 \rangle$.

Anagrams

Consider an *alphabet* $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and words x, y, \dots from a *language* $L \subseteq \Sigma^*$.

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Example

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Theorem (Rohit Parikh, 1961)

Given a Context-Free Language, $L \subseteq \Sigma^*$, one can efficiently construct the set of all Parikh vectors. One can use this to identify that $x \in \Sigma^*$ **cannot** be in the language.

More Details: cs.umu.se/kurser/TDBC92/VT06/final/3.pdf

Anagrams

Consider an *alphabet* $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_k\}$ and words x, y, \dots from a *language* $L \subseteq \Sigma^*$.

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Example

For $\Sigma = \{a, b, c\}$, $\Psi(abb) = \langle 1, 2, 0 \rangle$ and $\Psi(abab) = \Psi(abba) = \langle 2, 2, 0 \rangle$.

Definition (Anagram)

$x, y \in \Sigma^*$ are *anagrams* if $\Psi(x) = \Psi(y)$.

Definition (Subanagram)

$x \in \Sigma^*$ is a *subanagram* of $y \in \Sigma^*$ if $\Psi(x) \leq \Psi(y)$.

Anagrams

Lemma

Given $x, y \in \Sigma^n$, one can compute whether $\Psi(x) = \Psi(y)$ in $\mathcal{O}(n + |\Sigma|)$ time.

Lemma

Given $x, y \in \Sigma^$, one can compute whether $\Psi(x) \leq \Psi(y)$ in $\mathcal{O}(|x| + |y| + |\Sigma|)$ time.*

Proof.



Anagrams

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Lemma

Given $x, y \in \Sigma^*$, one can compute whether $\Psi(x) \leq \Psi(y)$ in $\mathcal{O}(|x| + |y| + |\Sigma|)$ time.

Proof.

Compute the Parikh vectors similar to the first half of *Counting Sort*.



Example

Counting the number of a 's, b 's, and c 's in aba and aab both yield $\langle 2, 1, 0 \rangle$.

Anagrams

Lemma

Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.



Anagrams

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Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.

Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x =$ b a a

$y =$ a b a

$x =$ c a b

$y =$ a b a

Anagrams

Lemma

Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

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Example

$x =$ a a b

$y =$ a a b

$x =$ c a b

$y =$ a b a

Anagrams

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Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.

Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x = \underline{a} \ a \ b$

$y = \underline{a} \ a \ b$

$x = \ c \ a \ b$

$y = \ a \ b \ a$

Anagrams

Lemma

Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.

Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x = \quad a \quad \underline{a} \quad b$
 $y = \quad a \quad \underline{a} \quad b$

$x = \quad c \quad a \quad b$
 $y = \quad a \quad b \quad a$

Anagrams

Lemma

Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.

Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x = \quad a \quad a \quad \underline{b}$

$y = \quad a \quad a \quad \underline{b}$

$x = \quad c \quad a \quad b$

$y = \quad a \quad b \quad a$

Anagrams

Lemma

Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

Proof.

Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

x =	a	a	b
y =	a	a	b

✓

x =	c	a	b
y =	a	b	a

Anagrams

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Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

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Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

x =	a	a	b
y =	a	a	b

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x =	a	b	c
y =	a	a	b

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Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x =$	a	a	b
$y =$	a	a	b

✓

$x =$	<u>a</u>	b	c
$y =$	<u>a</u>	a	b

Anagrams

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Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

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Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

$x =$	a	a	b
$y =$	a	a	b

✓

$x =$	a	<u>b</u>	c
$y =$	a	<u>a</u>	b

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Given $x, y \in \Sigma^n$, computing whether $\Psi(x) = \Psi(y)$ takes $\mathcal{O}(\text{sort}(n))$ time.

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Sort words x and y in $\mathcal{O}(\text{sort}(n))$ time. Then, check whether they now are the very same word in $\mathcal{O}(n)$ time. □

Example

x =	a	a	b
y =	a	a	b

✓

x =	a	b	c
y =	a	a	b

!

Anagrams

Lemma

Given $x, y \in \Sigma^$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.*

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Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$ b a

$y =$ a b a

$x =$ c a

$y =$ a b a

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$ a b

$y =$ a a b

$x =$ c a

$y =$ a b a

Anagrams

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Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x = \underline{a} \quad b$

$y = \underline{a} \quad a \quad b$

$x = \quad c \quad a$

$y = \quad a \quad b \quad a$

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Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

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$x =$ a b

$y =$ a a b

$x =$ c a

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Example

$x =$ a b

$y =$ a a b

$x =$ c a

$y =$ a b a

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Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	
$y =$	a	a	b

✓

$x =$	c	a	
$y =$	a	b	a

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

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Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	
$y =$	a	a	b

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$x =$	a	c	
$y =$	a	a	b

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Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

x =	a	b	
y =	a	a	b

✓

x =	<u>a</u>	c	
y =	<u>a</u>	a	b

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

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Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	
$y =$	a	a	b

✓

$x =$	a	<u>c</u>	
$y =$	a	<u>a</u>	b

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	✓
$y =$	a	a	

$x =$	a	<u>c</u>	
$y =$	a	a	<u>b</u>

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

Proof.

Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	
$y =$	a	a	b

✓

$x =$	a	<u>c</u>	
$y =$	a	a	b

Anagrams

Lemma

Given $x, y \in \Sigma^*$, checking $\Psi(x) \leq \Psi(y)$ takes $\mathcal{O}(\text{sort}(|x|) + \text{sort}(|y|))$ time.

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Again, sort words x and y . Now, match each symbol of x with ones in y ; skip symbols of y if x is “ahead”. □

Example

$x =$	a	b	
$y =$	a	a	b

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$x =$	a	c	
$y =$	a	a	b

!

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Multi-valued Anatree

Letter Ordering

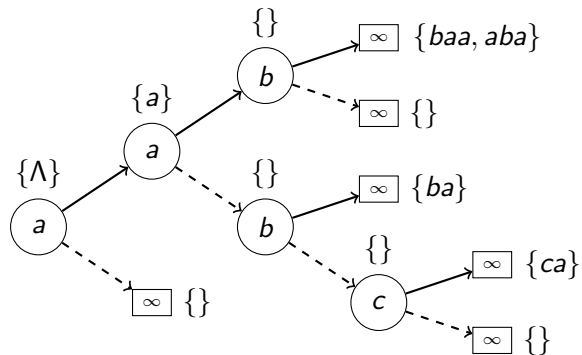
Anatree

Given an alphabet, Σ , and an ordering on its symbols, $< : \Sigma \times \Sigma \rightarrow \{\top, \perp\}$, the *Anatree* data structure manages a set of words $L \subseteq \Sigma^*$ on which one can do

Operation	
insert(x)	$\mathcal{O}(\text{sort}(x) + \Sigma)$
delete(x)	
contains(x)	$\mathcal{O}(\text{sort}(x) + \Sigma)$
anagrams(x)	$\mathcal{O}(\text{sort}(x) + \Sigma + T)$
subanagrams(x)	$\mathcal{O}(\text{sort}(x) + \min(N_{\text{Tree}}, 2^{ x } \cdot \Sigma) + T)$

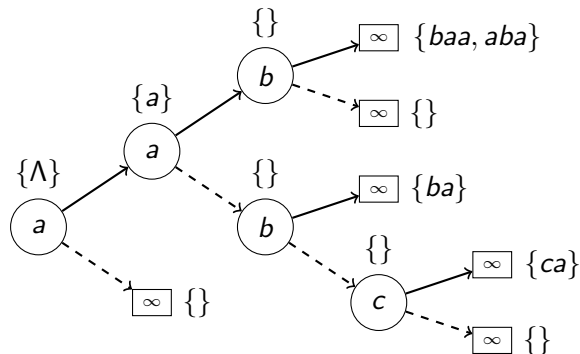
where N_{Tree} is the size of the Anagram tree and T is the output size.

Anatree



$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

Anatree.contains(ba)



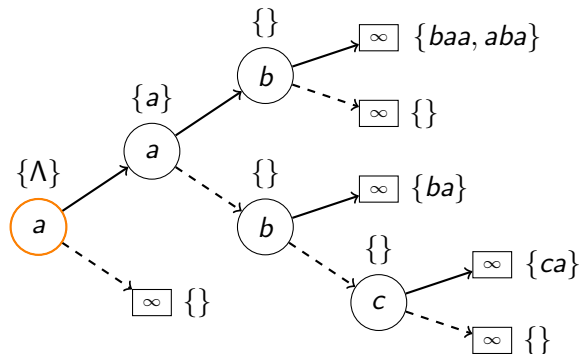
$L = \{\Lambda, a, ba, ca, aba, baa\}$

contains(x):

```
n := find(root, sort(x), 0)
return n ≠ NIL & n.contains(x)
```

find(n, x', i):

Anatree.contains(ba)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

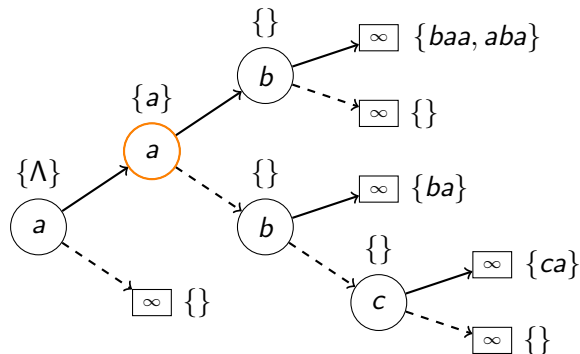
contains(x):

```
n := find(root, sort(x), 0)
return n ≠ NIL & n.contains(x)
```

find(n, x', i):

```
if x'[i] = n.char
    return find(n.true , x', i+1)
```

Anatree.contains(ba)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

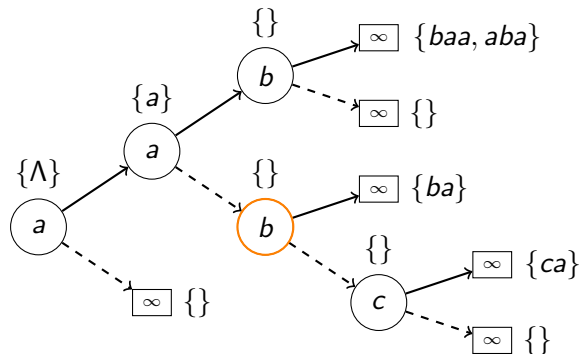
contains(x):

```
n := find(root, sort(x), 0)
return n  $\neq$  NIL & n.contains(x)
```

find(n, x', i):

```
if x'[i] > n.char
    return find(n.false, x', i)
if x'[i] = n.char
    return find(n.true, x', i+1)
```

Anatree.contains(ba)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

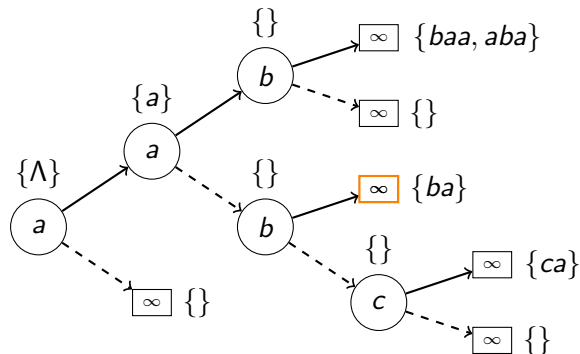
contains(x):

```
n := find(root, sort(x), 0)
return n ≠ NIL & n.contains(x)
```

find(n, x', i):

```
if x'[i] > n.char
  return find(n.false, x', i)
if x'[i] = n.char
  return find(n.true, x', i+1)
```

Anatree.contains(ba)



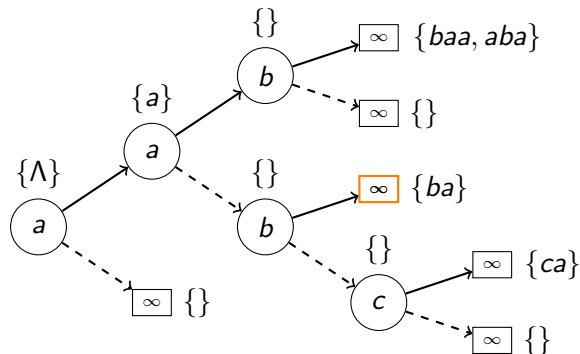
$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):
    n := find(root, sort(x), 0)
    return n ≠ NIL & n.contains(x)

find(n, x', i):
    if i = x'.length
        return n

    if x'[i] > n.char
        return find(n.false, x', i)
    if x'[i] = n.char
        return find(n.true, x', i+1)
```

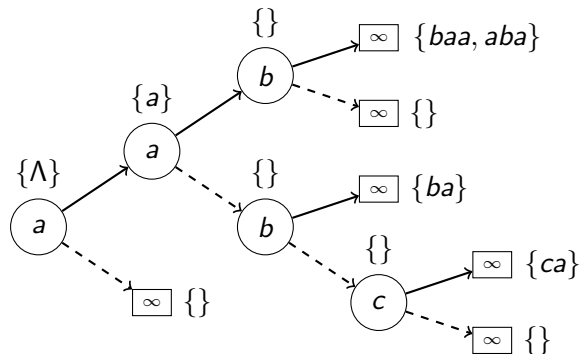
Anatree.contains(ba) = Yes



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):  
    n := find(root, sort(x), 0)  
    return n  $\neq$  NIL & n.contains(x)  
  
find(n, x', i):  
    if i = x'.length  
        return n  
  
    if x'[i] > n.char  
        return find(n.false, x', i)  
    if x'[i] = n.char  
        return find(n.true, x', i+1)
```

Anatree.contains(aca)



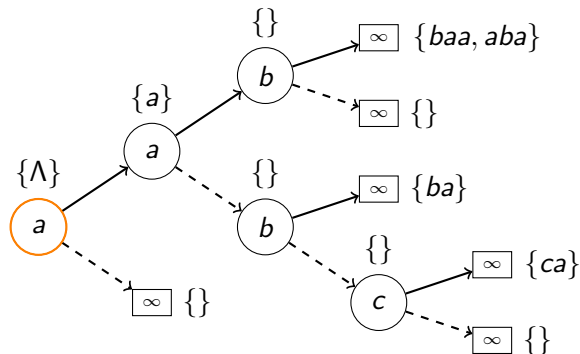
$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):
    n := find(root, sort(x), 0)
    return n ≠ NIL & n.contains(x)

find(n, x', i):
    if i = x'.length
        return n

    if x'[i] > n.char
        return find(n.false, x', i)
    if x'[i] = n.char
        return find(n.true, x', i+1)
```

Anatree.contains(aca)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

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contains(x):
    n := find(root, sort(x), 0)
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find(n, x', i):
    if i = x'.length
        return n

    if x'[i] > n.char
        return find(n.false, x', i)
    if x'[i] = n.char
        return find(n.true, x', i+1)
```

Anatree.contains(aca)



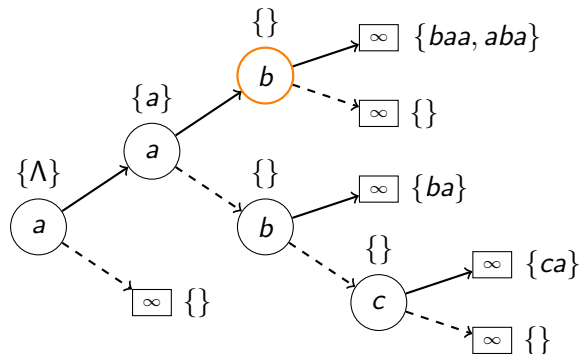
$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):
    n := find(root, sort(x), 0)
    return n ≠ NIL & n.contains(x)

find(n, x', i):
    if i = x'.length
        return n

    if x'[i] > n.char
        return find(n.false, x', i)
    if x'[i] = n.char
        return find(n.true, x', i+1)
```

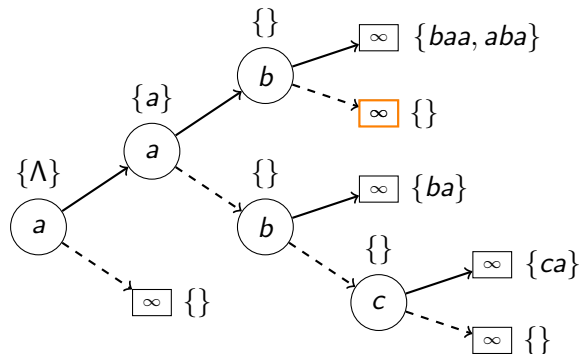

Anatree.contains(aca)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):  
    n := find(root, sort(x), 0)  
    return n  $\neq$  NIL & n.contains(x)  
  
find(n, x', i):  
    if i = x'.length  
        return n  
  
    if x'[i] > n.char  
        return find(n.false, x', i)  
    if x'[i] = n.char  
        return find(n.true, x', i+1)
```

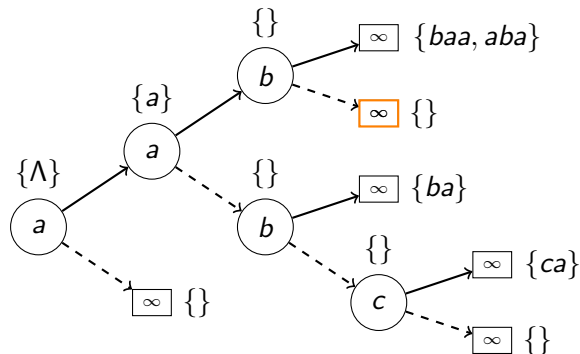
Anatree.contains(aca)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):  
    n := find(root, sort(x), 0)  
    return n  $\neq$  NIL & n.contains(x)  
  
find(n, x', i):  
    if i = x'.length  
        return n  
    if x'[i] < n.char  
        return NIL  
    if x'[i] > n.char  
        return find(n.false, x', i)  
    if x'[i] = n.char  
        return find(n.true, x', i+1)
```

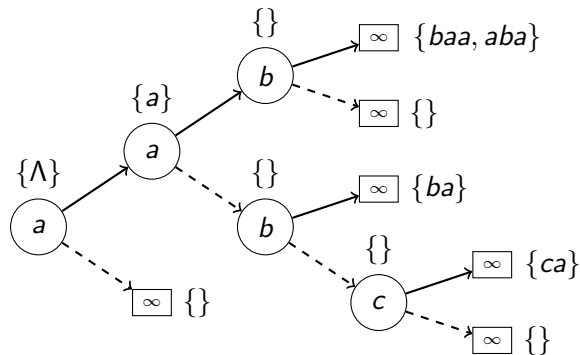
Anatree.contains(aca) = No



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
contains(x):  
    n := find(root, sort(x), 0)  
    return n  $\neq$  NIL & n.contains(x)  
  
find(n, x', i):  
    if i = x'.length  
        return n  
    if x'[i] < n.char  
        return NIL  
    if x'[i] > n.char  
        return find(n.false, x', i)  
    if x'[i] = n.char  
        return find(n.true, x', i+1)
```

Anatree.contains(...)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

Lemma

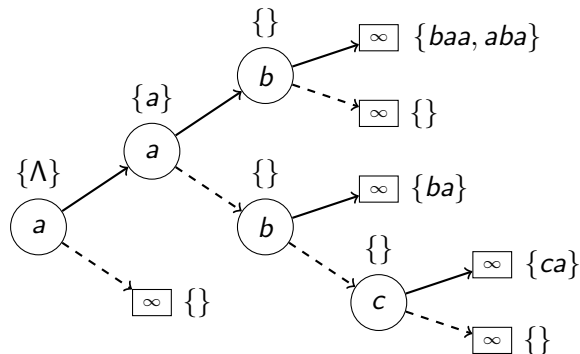
find(n , $sort(x)$, i) runs in $\mathcal{O}(sort(|x|) + |\Sigma|)$ time.

Proof.

$\mathcal{O}(1)$ time is spent per node. At most $|x|$ high edges and $|\Sigma|$ low edges are traversed, meaning at most $|x| + |\Sigma|$ nodes are visited.

On top of this, add the $\mathcal{O}(sort(|x|))$ time to sort x into x' . □

Anatree.contains(...)



$L = \{\Lambda, a, ba, ca, aba, baa\}$

Lemma

find(n , sort(x), i) runs in $\mathcal{O}(\text{sort}(|x|) + |\Sigma|)$ time.

Proof.

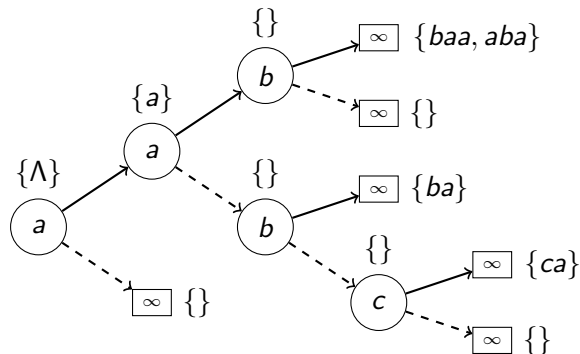
$\mathcal{O}(1)$ time is spent per node. . .

□

Corollary

contains(x) runs in $\mathcal{O}(\text{sort}(|x|) + |\Sigma|)$ time.

Anatree.anagrams(...)



$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

anagrams(x):

$n := \text{find}(\text{root}, \text{sort}(x), 0)$

if $n \neq \text{NIL}$

output words in n

Corollary

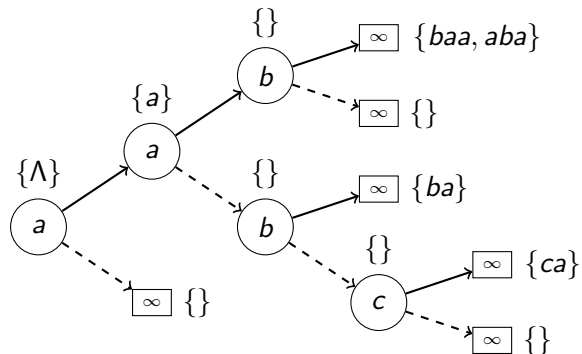
anagrams(x) runs in

$\mathcal{O}(\text{sort}(|x|) + |\Sigma| + T)$ time.

Proof.

It takes $\mathcal{O}(\text{sort}(|x|) + |\Sigma|)$ time to find n and then another $\mathcal{O}(T)$ time to output its content. \square

Anatree.subanagrams(a) =

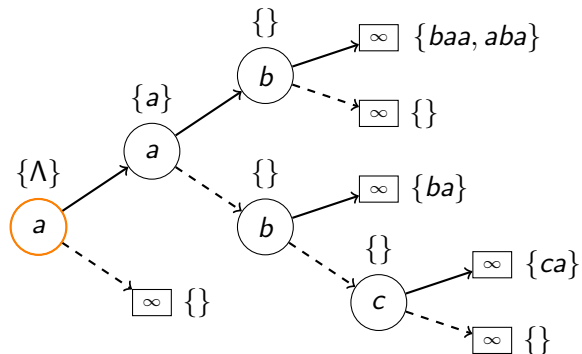


$L = \{\Lambda, a, ba, ca, aba, baa\}$

subanagrams(x):

subanagrams'(root, sort(x), 0)

`Anatree.subanagrams(a) = Λ`



$L = \{\Lambda, a, ba, ca, aba, baa\}$

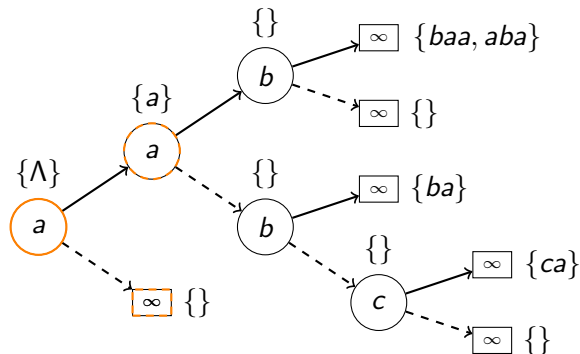
`subanagrams(x):`

`subanagrams'(root, sort(x), 0)`

`subanagrams'(n, x', i):`

output words in n

`Anatree.subanagrams(a) = Λ`



$L = \{\Lambda, a, ba, ca, aba, baa\}$

`subanagrams(x):`

`subanagrams'(root, sort(x), 0)`

`subanagrams'(n, x', i):`

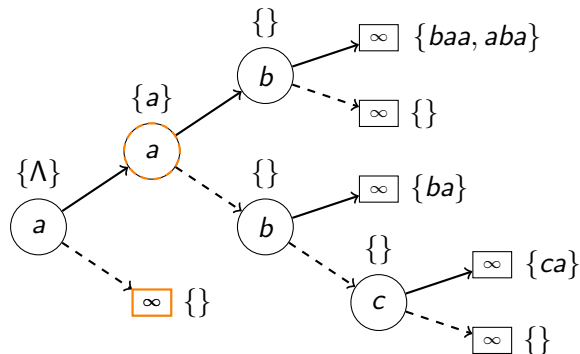
`output words in n`

`if x'[i] = n.char:`

`subanagrams'(n.false, x', i+1)`

`subanagrams'(n.true, x', i+1)`

`Anatree.subanagrams(a) = Λ`



`subanagrams(x):`

`subanagrams'(root, sort(x), 0)`

`subanagrams'(n, x', i):`

`output words in n`

`if n.char = ∞ :`

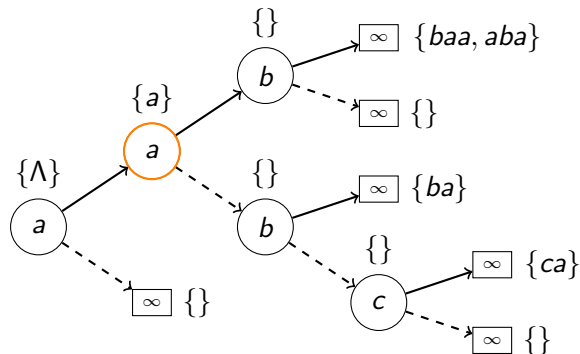
`return`

`if x'[i] = n.char:`

`subanagrams'(n.false, x', i+1)`

`subanagrams'(n.true, x', i+1)`

`Anatree.subanagrams(a) = Λ , a`



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
subanagrams(x):
```

```
    subanagrams'(root, sort(x), 0)
```

```
subanagrams'(n, x', i):
```

```
    output words in n
```

```
    if n.char = ∞:
```

```
        return
```

```
    if i = x'.length:
```

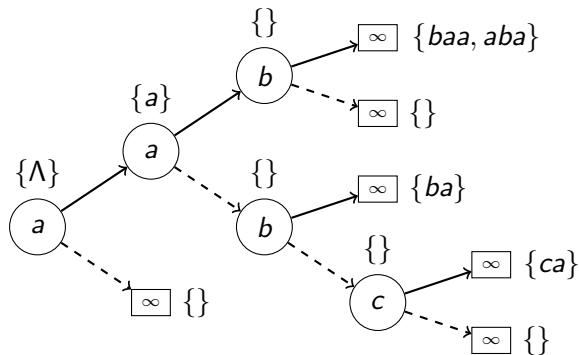
```
        return
```

```
    if x'[i] = n.char:
```

```
        subanagrams'(n.false, x', i+1)
```

```
        subanagrams'(n.true, x', i+1)
```

Anatree.subanagrams(a) = Λ , a



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
subanagrams(x):
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    subanagrams'(root, sort(x), 0)
```

```
subanagrams'(n, x', i):
```

```
    output words in n
```

```
    if n.char = ∞:
```

```
        return
```

```
    if i = x'.length:
```

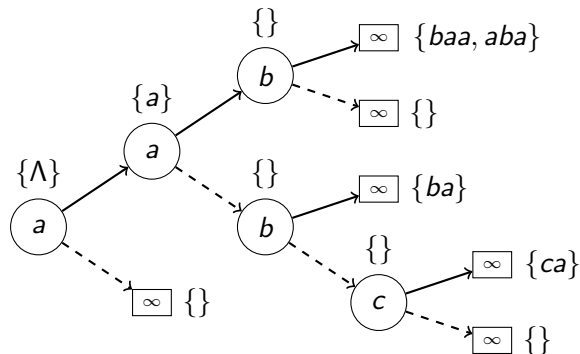
```
        return
```

```
    if x'[i] = n.char:
```

```
        subanagrams'(n.false, x', i+1)
```

```
        subanagrams'(n.true, x', i+1)
```

Anatree.subanagrams(abb) =



$L = \{\Lambda, a, ba, ca, aba, baa\}$

```

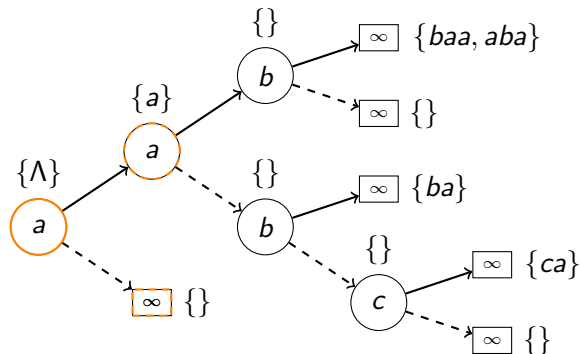
subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return

    if i = x'.length:
        return

    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
    
```

`Anatree.subanagrams(abb) = Λ`



$L = \{ \Lambda, a, ba, ca, aba, baa \}$

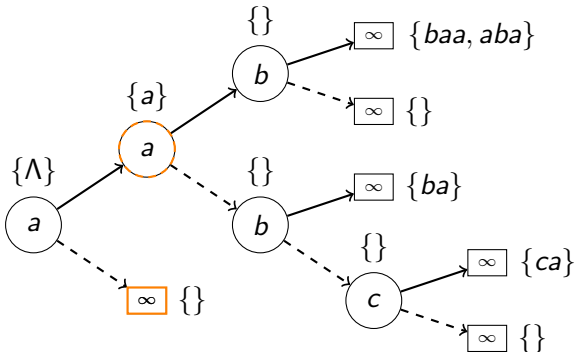
```

subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return

    if i = x'.length:
        return

    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
    
```

$$\text{Anatree.subanagrams}(\text{abb}) = \Lambda$$

$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

subanagrams(x):

```
subanagrams'(root, sort(x), 0)
```

```
subanagrams'(n, x', i):
```

output words in n

```
if n.char == ∞:
```

return

```
if i == x'.length:
```

return

```
if x'[i] == n.char:
```

```
subanagrams'(n.false, x', i+1)
```

```
subanagrams'(n.true,  x', i+1)
```

`Anatree.subanagrams(abb) = Λ , a`



$L = \{\Lambda, a, ba, ca, aba, baa\}$

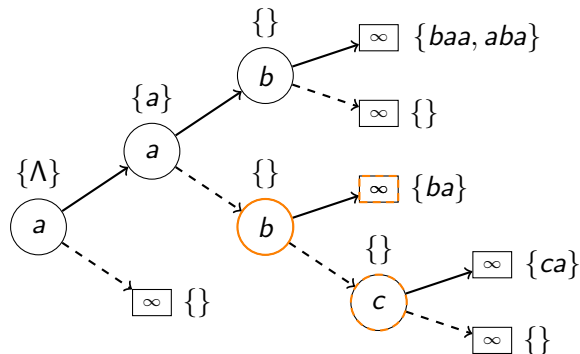
```

subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return

    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
    
```


`Anatree.subanagrams(abb) = Λ , a`



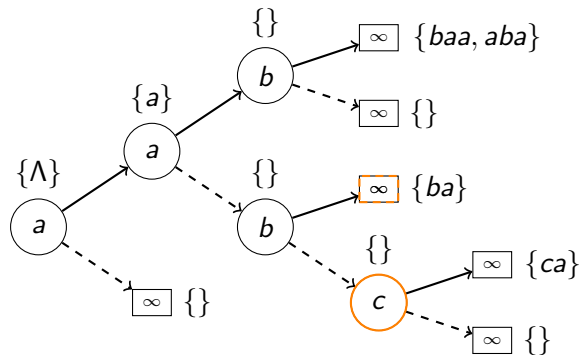
$L = \{\Lambda, a, ba, ca, aba, baa\}$

```
subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char =  $\infty$ :
        return

    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
```

`Anatree.subanagrams(abb) = Λ , a`

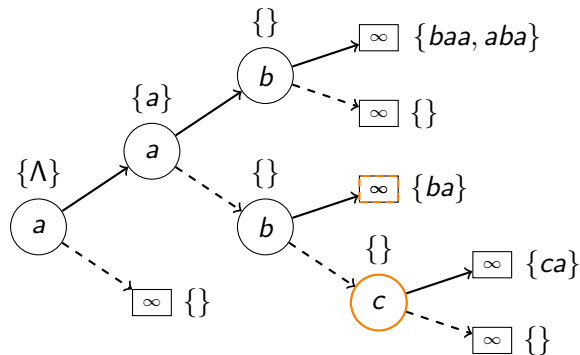


```

subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return
    while x'[i] < n.char:
        i++
    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
    
```

`Anatree.subanagrams(abb) = Λ , a`



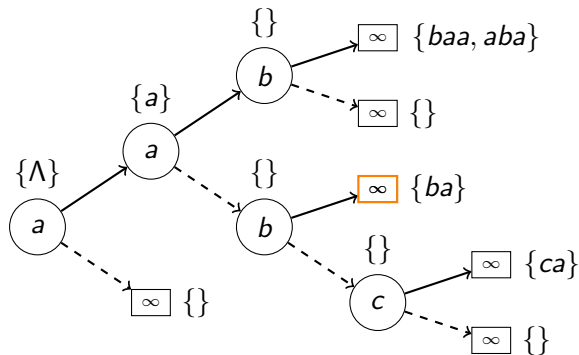
$L = \{\Lambda, a, ba, ca, aba, baa\}$

```

subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return
    while x'[i] < n.char:
        i++
    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
  
```

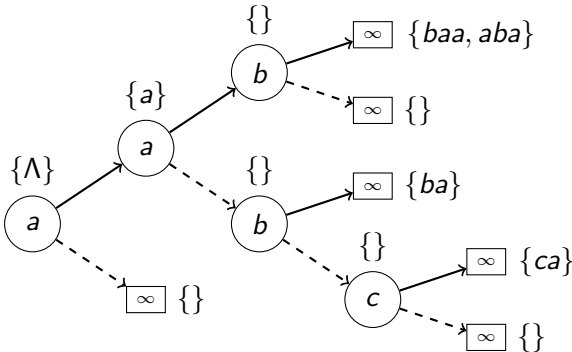
Anatree.subanagrams(abb) = Λ , a, ab


$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

```
subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char = ∞:
        return
    while x'[i] < n.char:
        i++
    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)
```

Anatree.subanagrams(abb) = Λ , a, ab


$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

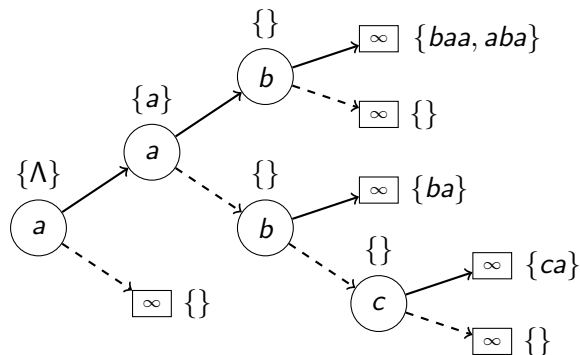
```

subanagrams(x):
    subanagrams'(root, sort(x), 0)

subanagrams'(n, x', i):
    output words in n
    if n.char =  $\infty$ :
        return
    while x'[i] < n.char:
        i++
    if i = x'.length:
        return
    if x'[i] > n.char:
        subanagrams'(n.false, x', i)
    if x'[i] = n.char:
        subanagrams'(n.false, x', i+1)
        subanagrams'(n.true, x', i+1)

```

Anatree.subanagrams(...)

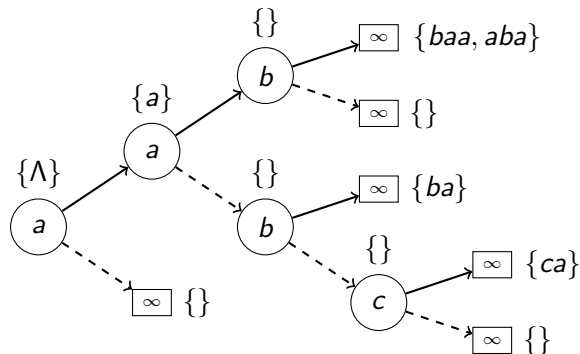


$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

Lemma

For $N = \sum_{i=1}^k |x_i|$, the anatree has size, N_{tree} , at most N .

Anatree.subanagrams(...)



$$L = \{\Lambda, a, ba, ca, aba, baa\}$$

Lemma

For $N = \sum_{i=1}^k |x_i|$, the anatree has size, N_{tree} , at most N .

Theorem

subanagrams(x) runs in $\mathcal{O}(\text{sort}(|x|) + \min(N_{tree}, 2^{|x|} \cdot |\Sigma|) + T)$ time.

Proof.

It takes $\mathcal{O}(\text{sort}(|x|))$ time to sort x and another $\mathcal{O}(T)$ to write the output.

For every match, the recursion splits in two. Each of these $2^{|x|}$ matches have $|\Sigma|$ or fewer mismatches. □

Anatree.keys(...)

Definition

The subset L' of $L \subseteq \Sigma^*$ is a set of keys w.r.t. Ψ if for all $x, y \in L'$ then $\Psi(x) \neq \Psi(y)$.

Anatree.keys(...)

Definition

The subset L' of $L \subseteq \Sigma^*$ is a set of keys w.r.t. Ψ if for all $x, y \in L'$ then $\Psi(x) \neq \Psi(y)$.

Theorem

keys(length) runs in $\mathcal{O}(\min(N_{tree}, 2^{length} \cdot |\Sigma|) + T)$ time.

Proof.

Left as an exercise to the reader...



Anatree.insert(Λ)

```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):
```

{ }

∞

Anatree.insert(Λ)

```
insert(x):  
    root = insert'(root, sort(x), 0, x)
```

```
insert'(n, x', i, x):  
    if i = x'.length:  
        n.insert(x)
```

```
    return n
```

$\{\Lambda\}$

∞

Anatree.insert(*ba*)

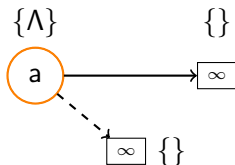
```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i = x'.length:  
        n.insert(x)
```

$\{\Lambda\}$

∞

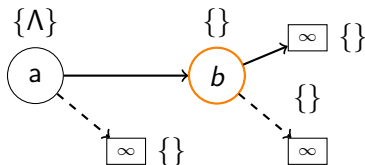
```
return n
```

Anatree.insert(*ba*)



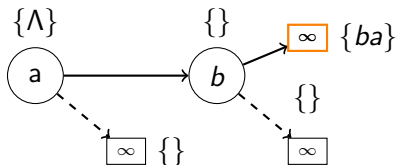
```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i = x'.length:  
        n.insert(x)  
    else if n.char =  $\infty$ :  
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }  
        n.true = insert'(n.true, x', i+1, x)  
  
    return n
```

Anatree.insert(*ba*)



```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i = x'.length:  
        n.insert(x)  
    else if n.char = ∞:  
        n = node{ char: x'[i], false: ∞, true: ∞ }  
        n.true = insert'(n.true, x', i+1, x)  
  
return n
```

Anatree.insert(*ba*)



```
insert(x):
```

```
    root = insert'(root, sort(x), 0, x)
```

```
insert'(n, x', i, x):
```

```
    if i = x'.length:
```

```
        n.insert(x)
```

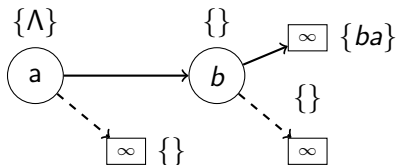
```
    else if n.char =  $\infty$ :
```

```
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }
```

```
        n.true = insert'(n.true, x', i+1, x)
```

```
    return n
```

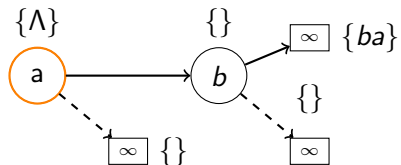
Anatree.insert(*a*)



```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i = x'.length:  
        n.insert(x)  
    else if n.char =  $\infty$ :  
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }  
        n.true = insert'(n.true, x', i+1, x)
```

```
return n
```

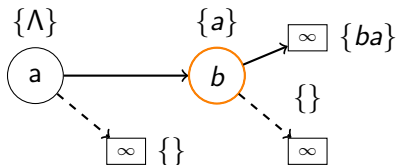

Anatree.insert(a)



```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i == x'.length:  
        n.insert(x)  
    else if n.char ==  $\infty$ :  
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }  
        n.true = insert'(n.true, x', i+1, x)
```

```
else if x'[i] == m.char:  
    n.true = insert'(n.true, x', i+1, x)  
return n
```

Anatree.insert(a)



```
insert(x):
```

```
    root = insert'(root, sort(x), 0, x)
```

```
insert'(n, x', i, x):
```

```
    if i = x'.length:
```

```
        n.insert(x)
```

```
    else if n.char =  $\infty$ :
```

```
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }
```

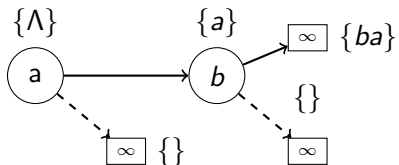
```
        n.true = insert'(n.true, x', i+1, x)
```

```
    else if x'[i] == m.char:
```

```
        n.true = insert'(n.true, x', i+1, x)
```

```
    return n
```

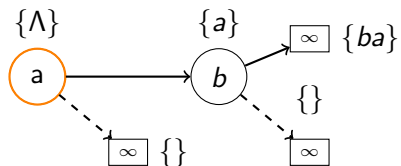
Anatree.insert(*baa*)



```
insert(x):  
    root = insert'(root, sort(x), 0, x)  
  
insert'(n, x', i, x):  
    if i == x'.length:  
        n.insert(x)  
    else if n.char ==  $\infty$ :  
        n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }  
        n.true = insert'(n.true, x', i+1, x)
```

```
    else if x'[i] == m.char:  
        n.true = insert'(n.true, x', i+1, x)  
    return n
```

Anatree.insert(*baa*)

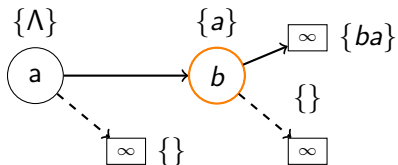


```
insert(x):
    root = insert'(root, sort(x), 0, x)

insert'(n, x', i, x):
    if i = x'.length:
        n.insert(x)
    else if n.char = ∞:
        n = node{ char: x'[i], false: ∞, true: ∞ }
        n.true = insert'(n.true, x', i+1, x)
```

```
else if x'[i] == m.char:
    n.true = insert'(n.true, x', i+1, x)
return n
```

Anatree.insert(*baa*)

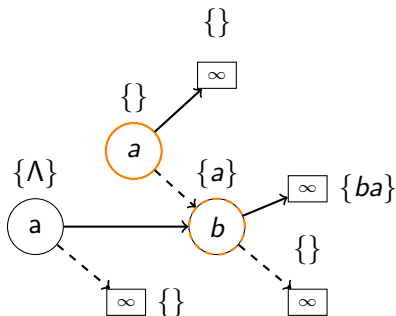


```
insert(x):
    root = insert'(root, sort(x), 0, x)

insert'(n, x', i, x):
    if i == x'.length:
        n.insert(x)
    else if n.char == ∞:
        n = node{ char: x'[i], false: ∞, true: ∞ }
        n.true = insert'(n.true, x', i+1, x)
    else if x'[i] < m.char:
        n' = node{ char: x'[i], false: n, true: ∞ }
        move n.words into n'.words
        n'.true = insert'(n'.true, x', i+1, x)
        return n'

    else if x'[i] == m.char:
        n.true = insert'(n.true, x', i+1, x)
    return n
```

Anatree.insert(*baa*)



```
insert(x):
```

```
root = insert'(root, sort(x), 0, x)
```

```
insert'(n, x', i, x):
```

```
if i == x'.length:
```

```
n.insert(x)
```

```
else if n.char =  $\infty$ :
```

```
n = node{ char: x'[i], false:  $\infty$ , true:  $\infty$  }
```

```
n.true = insert'(n.true, x', i+1, x)
```

```
else if x'[i] < m.char:
```

```
n' = node{ char: x'[i], false: n, true:  $\infty$  }
```

```
move n.words into n'.words
```

```
n'.true = insert'(n'.true, x', i+1, x)
```

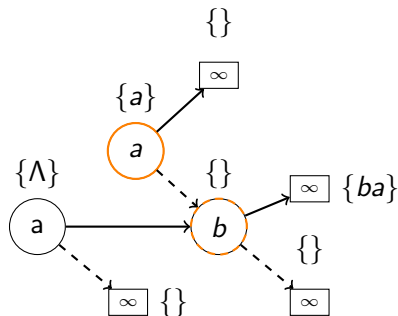
```
return n'
```

```
else if x'[i] == m.char:
```

```
n.true = insert'(n.true, x', i+1, x)
```

```
return n
```

Anatree.insert(*baa*)



```
insert(x):
```

```
    root = insert'(root, sort(x), 0, x)
```

```
insert'(n, x', i, x):
```

```
    if i = x'.length:
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        n.insert(x)
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    else if n.char = ∞:
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        n = node{ char: x'[i], false: ∞, true: ∞ }
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        n.true = insert'(n.true, x', i+1, x)
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    else if x'[i] < m.char:
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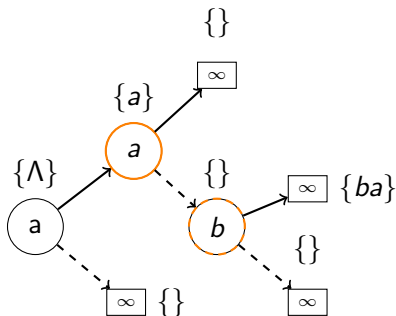
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Anatree.insert(*baa*)



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    root = insert'(root, sort(x), 0, x)
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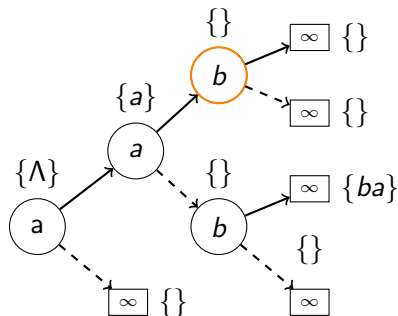
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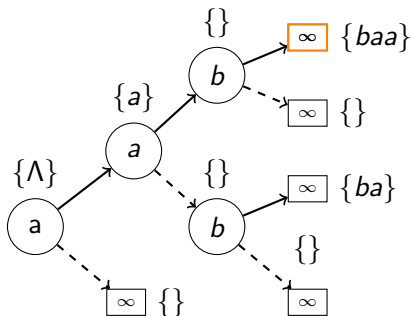
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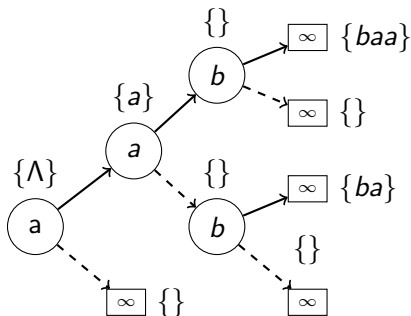
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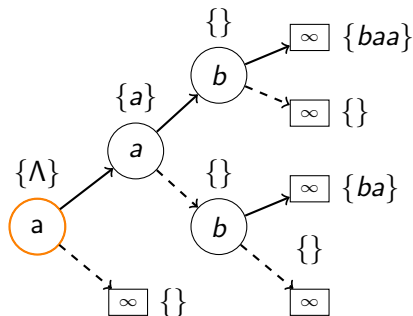
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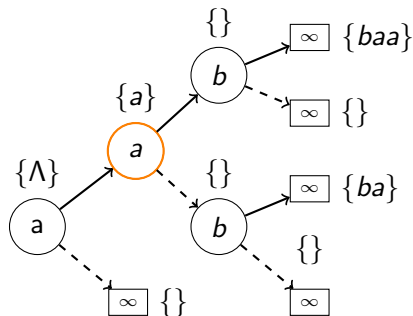
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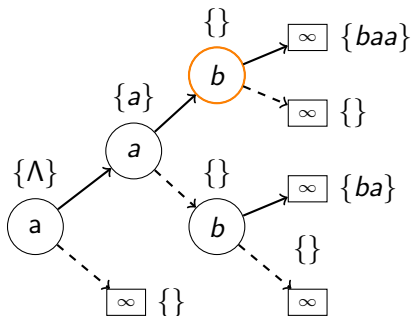
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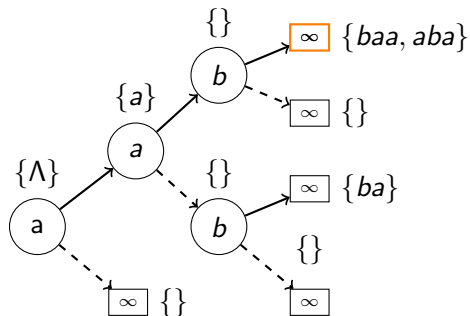
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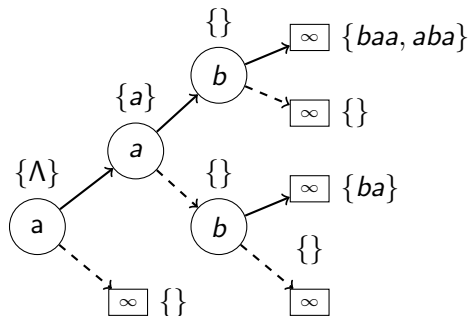
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Anatree.insert(ca)



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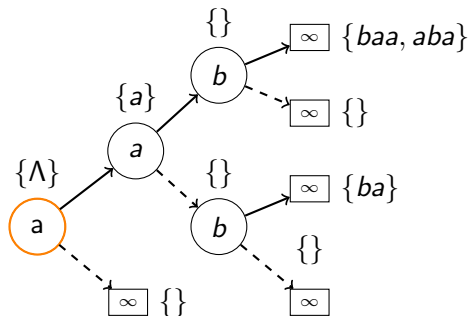
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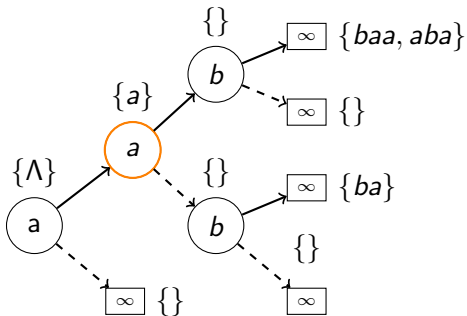
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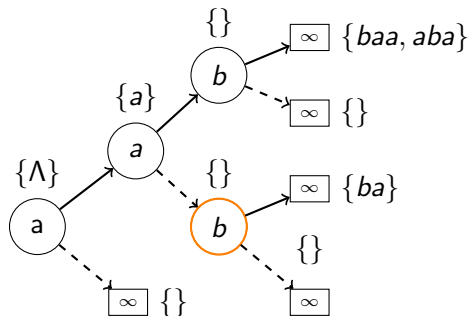
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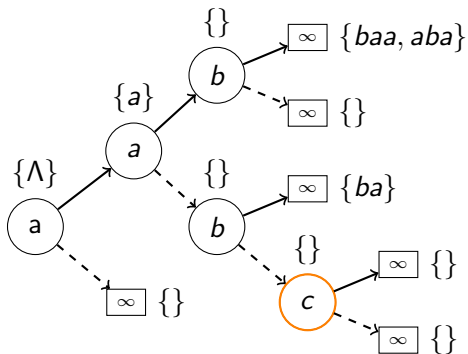
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Anatree.insert(ca)

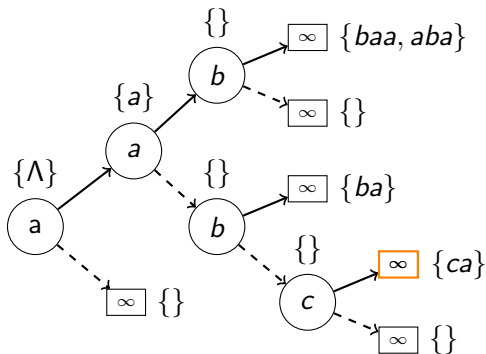


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Anatree.insert(ca)



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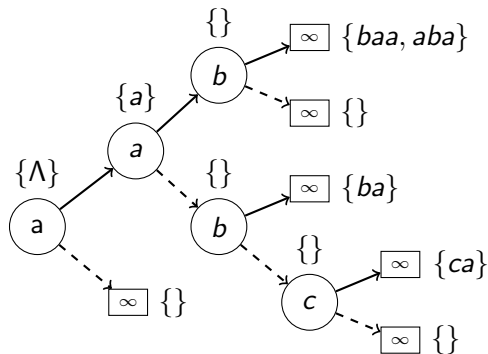
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Anatree.insert(...)



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Anatree.insert(...)

Theorem

insert(x) runs in $\mathcal{O}(\text{sort}(|x|) + \Sigma)$ time.

Proof.

Similar argument as for `find(n, x', i)`.



Anatree.insert(...)

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insert(x) runs in $\mathcal{O}(\text{sort}(|x|) + \Sigma)$ time.

Proof.

Similar argument as for `find(n, x', i)`. □

Corollary

For $N = \sum_{i=1}^k |x_i|$, `insert(x_1, x_2, \dots, x_k)` requires $\mathcal{O}(\text{sort}(N) + k \cdot |\Sigma|)$ time.

Proof.

Follows from complexity of `insert(x_i)` and `sort` distributes over $+$ in \mathcal{O} -notation:

$$\mathcal{O}(\text{sort}(N_1) + \text{sort}(N_2)) = \mathcal{O}(\text{sort}(N_1 + N_2))$$
□

Anatree.delete(...)

Theorem





delete(x) runs in $\mathcal{O}(\text{sort}(|x|) + |\Sigma|)$ time.

Proof.

Left as an exercise to the reader...



Anatree

		Dictionary		Anatree			
		# Words	# Symbols	Size	#Keys	insert (s)	subanagrams (s)
	DK	32863	177308	62687	8513	12.62	1.05
	DE	23587	127562	55047	8201	9.46	0.88
	EN	40804	218342	75697	11741	10.62	1.43
	ES	39650	219776	56103	7502	8.45	0.89

Contents

Motivation

Wordrow

Anagrams

Binary Anatree

`contains(x)`

`anagrams(x)`

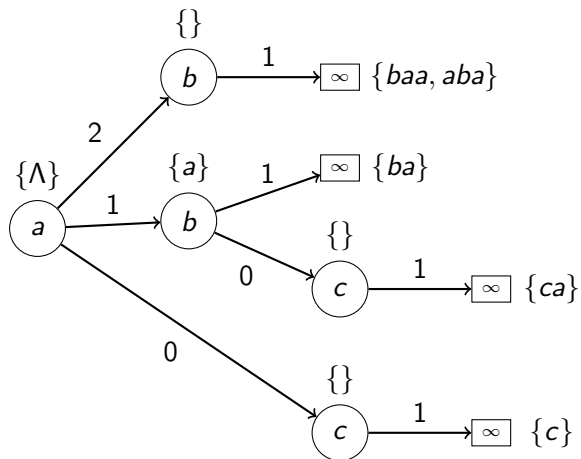
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Multi-valued Anatree

Letter Ordering

Multi-valued Anatrie



$$L = \{\Lambda, a, c, ba, ca, aba, baa\}$$

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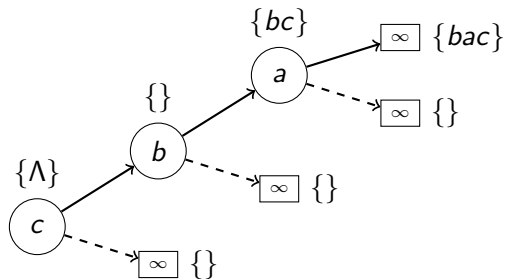
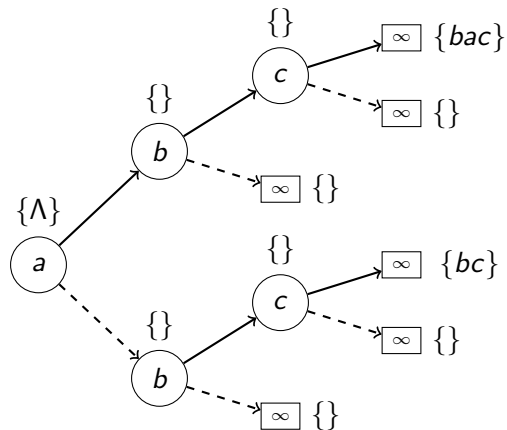
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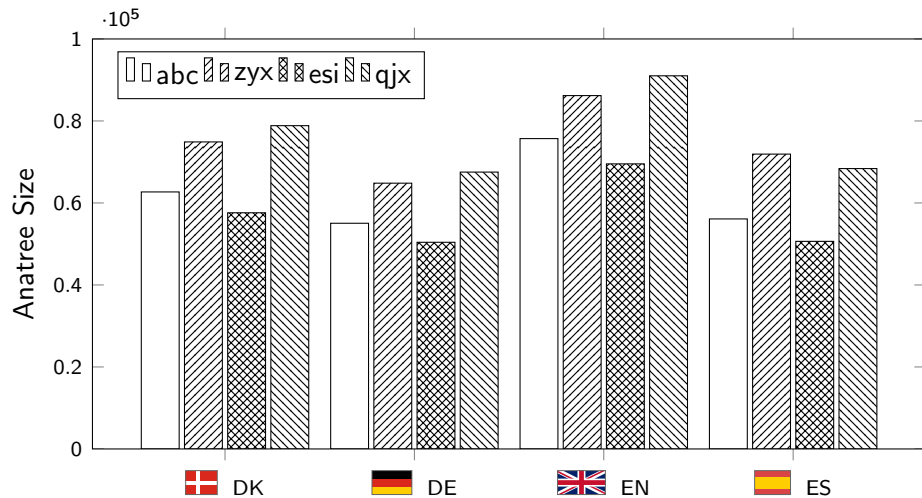
Letter Ordering

Letter Ordering

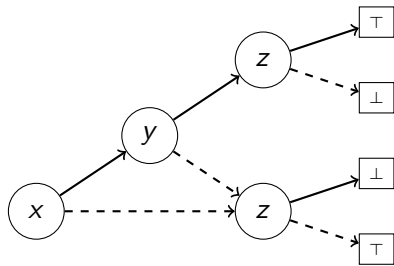


$$L = \{\Lambda, bc, bac\}$$

Letter Ordering

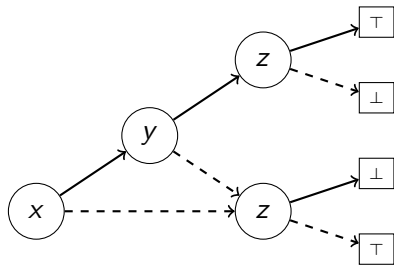


Binary Decision Diagrams



$$f(x, y, z) \equiv \neg((x \wedge y) \oplus z)$$

Binary Decision Diagrams

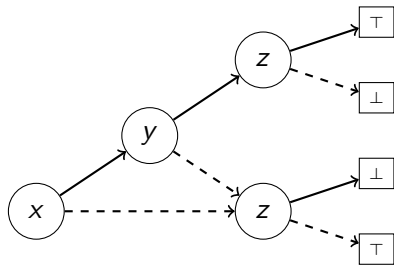


$$f(x, y, z) \equiv \neg((x \wedge y) \oplus z)$$

Used in the context of:

- Model Checking
- Compilers
- Game Solving

Binary Decision Diagrams



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Used in the context of:

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Features of BDDs:

- (Often) Smaller than Formula/Set
- Operation Complexity depends on BDD Size
- Size depends on Variable Ordering

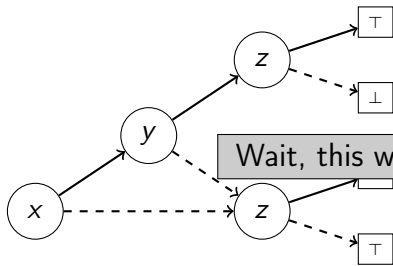
Binary Decision Diagrams

Always has been.

Used in the context of:

- Model Checking
- Compilers
- Game Solving

Wait, this was about BDDs?



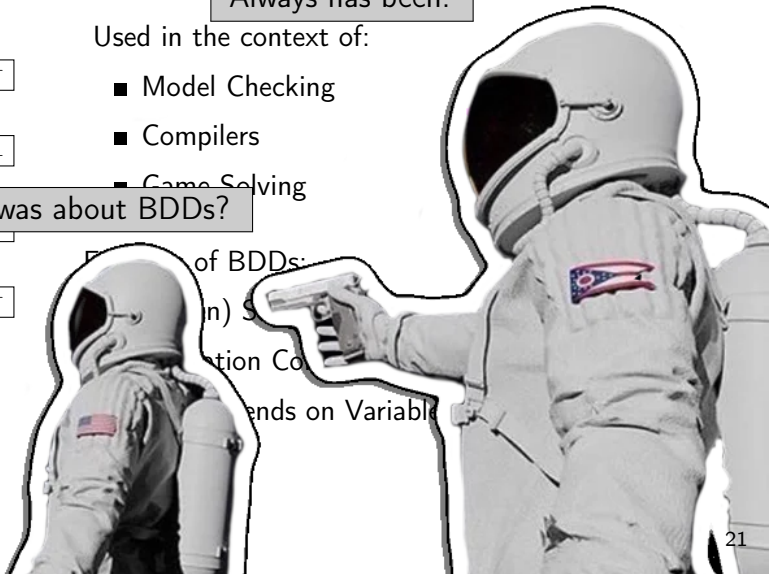
$$f(x, y, z) \equiv \neg((x \wedge y) \oplus z)$$

Properties of BDDs:

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ation Co

ends on Variable



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Wordrow

🎲 wordrow.io

🔗 github.com/ssoelvsten/wordrow

Anatree

🔗 github.com/ssoelvsten/anatree