Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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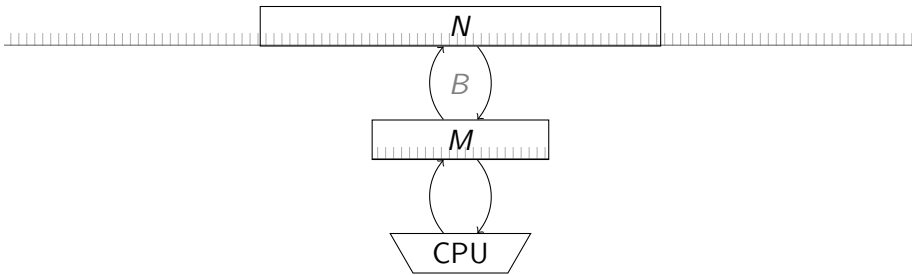
UNIVERSITY



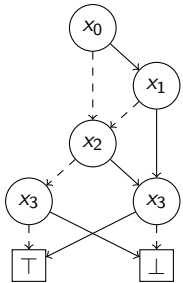
The I/O model by Aggarwal and Vitter '87

For any realistic values of N, M, and B we have that

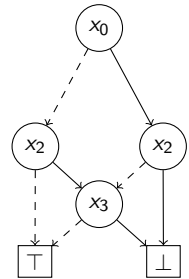
$$N/B \quad < \quad \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \quad \ll \quad N \ ,$$

Theorem (Aggarwal and Vitter '87)
N elements can be sorted in $\Theta(sort(N))$ I/Os.

Theorem (Arge '95)
N elements can be inserted in and extracted from a Priority Queue in $\Theta(sort(N))$ I/Os.



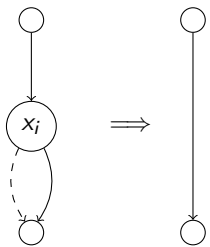
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



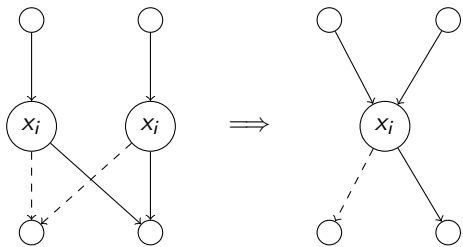
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Examples of (Reduced Ordered) Binary Decision Diagrams.

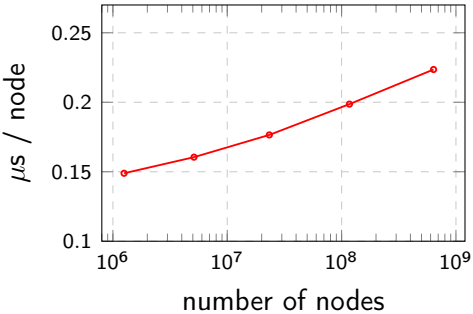
Theorem (Bryant '86)
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.

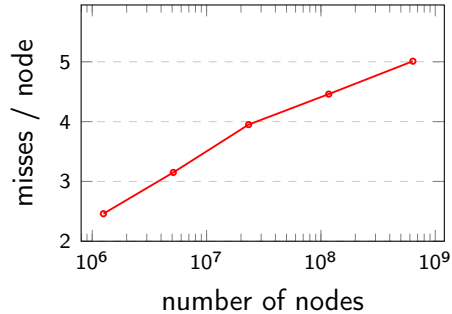


(1) Remove redundant nodes



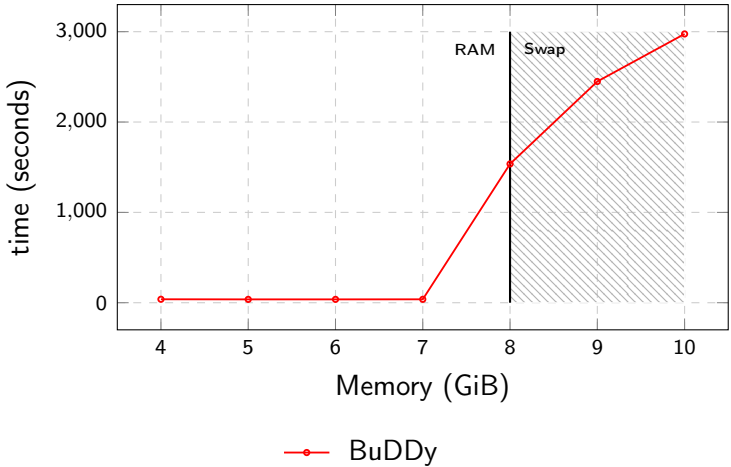
(2) Merge duplicate nodes



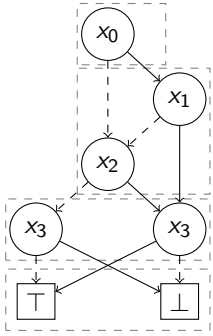


BuDDy

Cache behaviour for the *N*-Queens problem.



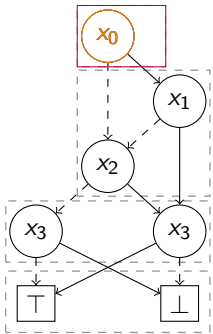
Running time for *Tic-Tac-Toe* with $N = 21$.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$M = 4,\ B = 2$

node I/Os	cache lookups
0	0



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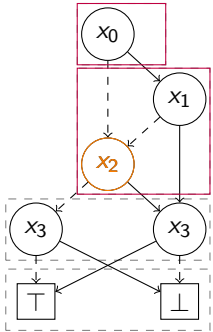
$M = 4,\ B = 2$

node I/Os	cache lookups
1	1



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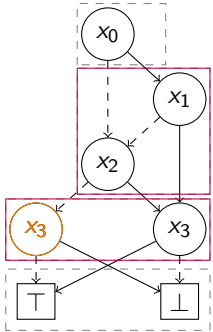
$M = 4,\ B = 2$

node I/Os	cache lookups
2	2



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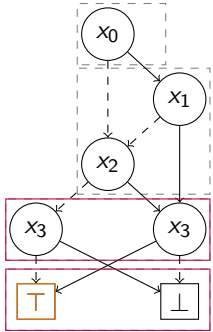
$M = 4,\ B = 2$

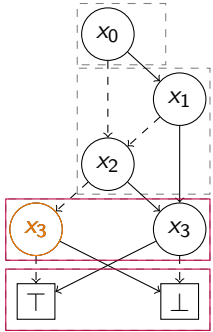
node I/Os	cache lookups
3	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$M = 4,\ B = 2$

node I/Os	cache lookups
4	3



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$M = 4,\ B = 2$

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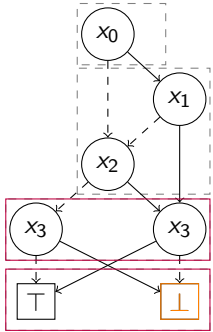
4	3



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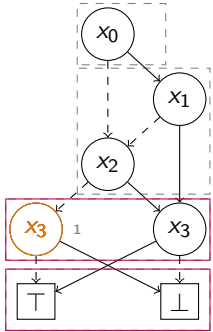

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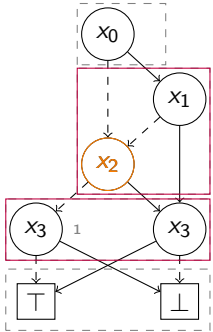
4	3



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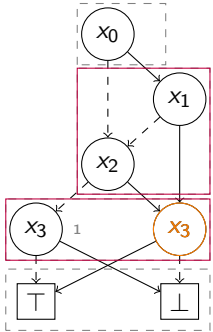
$M = 4,\ B = 2$

node I/Os	cache lookups
5	3



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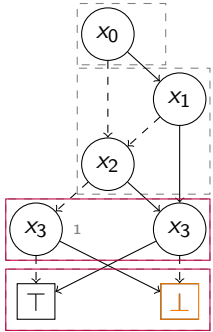
$M = 4,\ B = 2$

node I/Os	cache lookups
5	4



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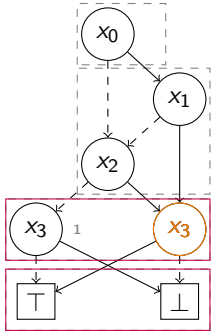
$M = 4,\ B = 2$

node I/Os	cache lookups
6	4



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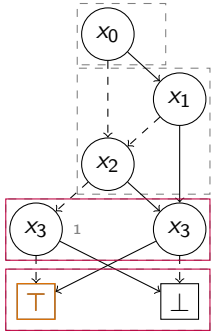
$M = 4,\ B = 2$

node I/Os	cache lookups
6	4



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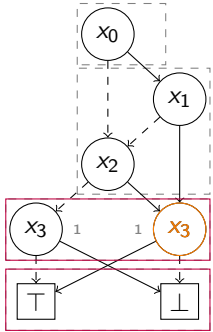
6	4



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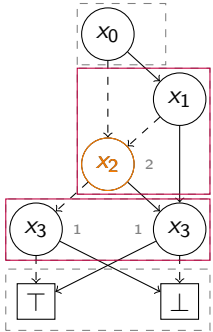

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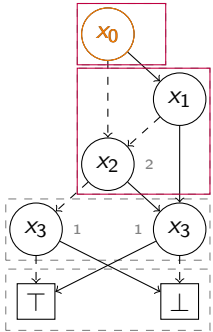
node I/Os	cache lookups
7	4



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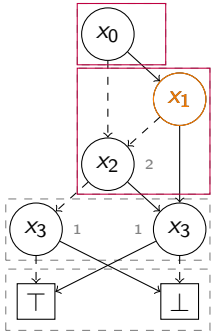
$M = 4,\ B = 2$

node I/Os	cache lookups
8	4



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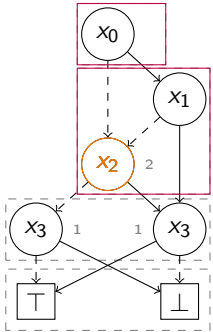
$M = 4,\ B = 2$

node I/Os	cache lookups
8	5



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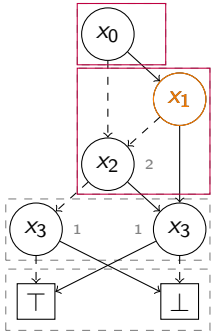
node I/Os	cache lookups
8	6



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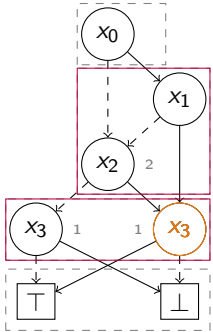

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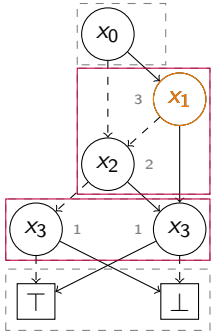
$M = 4,\ B = 2$

node I/Os	cache lookups
9	7



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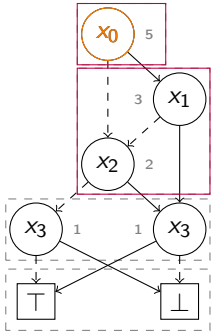

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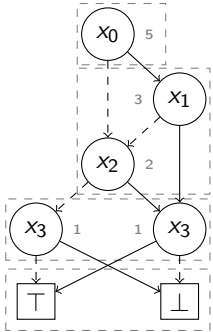
$M = 4,\ B = 2$

node I/Os	cache lookups
10	7



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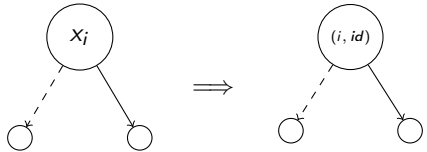
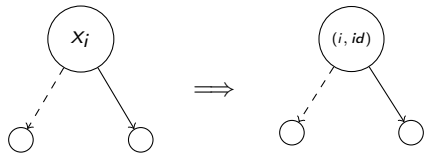
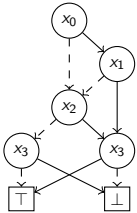
10	7


x_i
$\Longrightarrow$
(i, id)

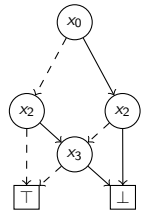


$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



$$[\ ((0,0),(2,0),(1,0))\ ,\ ((1,0),(2,0),(3,1))\ ,\ ((2,0),(3,0),(3,1))\ ,\ ((3,0),\bot,\top)\ ,\ ((3,1),\top,\bot)\ ]$$

(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

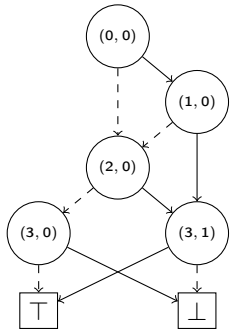


$$[\ ((0,0),(2,0),(2,1))\ ,\ ((2,0),\top,(3,0))\ ,\ ((2,1),(3,0),\bot)\ ,\ ((3,0),\top,\bot)\ ]$$

(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

CountPaths Example



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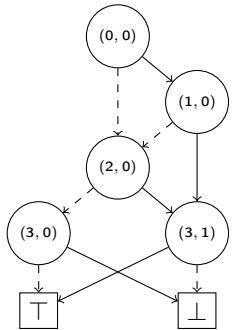
CountPaths Example



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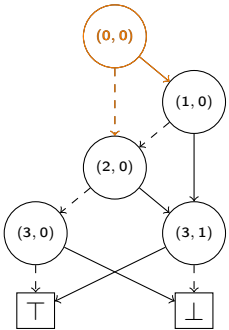
Priority Queue: Q_{count}:

[

]

CountPaths Example



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[

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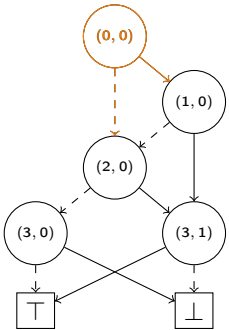
CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

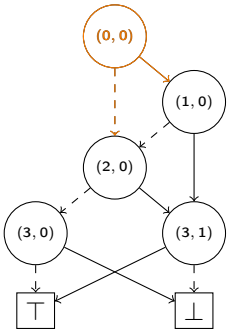
Priority Queue: Q_{count}:

[$((0,0) \xrightarrow{\top} (1,0), \quad 1)$,

$((0,0) \xrightarrow{\bot} (2,0), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(1, 0)$	0	0

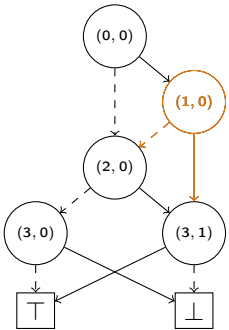
Priority Queue: Q_{count}:

$[\ ((0,0) \xrightarrow{\top} (1,0),\ 1)\ ,$

$((0,0) \xrightarrow{\bot} (2,0),\ 1)\ ,$

$]$

CountPaths Example



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Seek	Sum	Result
$(1, 0)$	0	0

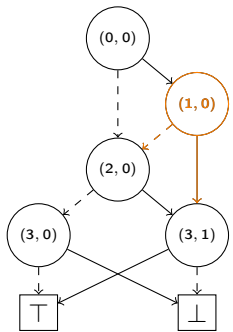
Priority Queue: Q_{count}:

[$((0,0) \xrightarrow{\top} (1,0), \quad 1)$,

$((0,0) \xrightarrow{\bot} (2,0), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(1, 0)$	1	0

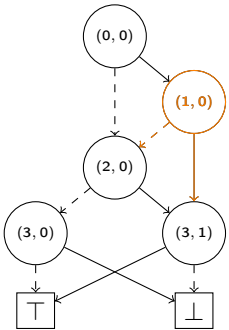
Priority Queue: Q_{count}:

[

$((0, 0) \xrightarrow{\perp} (2, 0), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(1, 0)$	1	0

Priority Queue: Q_{count}:

[

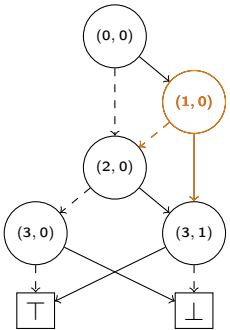
$((0,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	0	0

Priority Queue: Q_{count}:

[

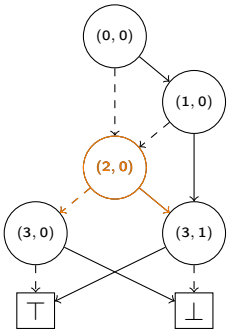
$((0,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	0	0

Priority Queue: Q_{count}:

[

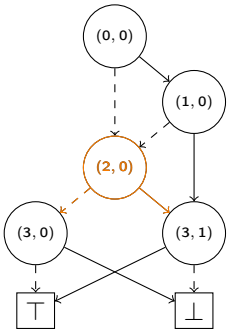
$((0,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	1	0

Priority Queue: Q_{count}:

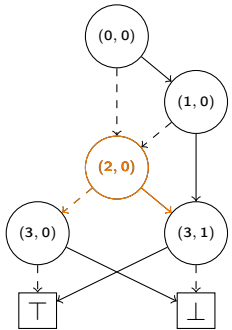
[

$((1, 0) \xrightarrow{\perp} (2, 0), \quad 1)$,

$((1, 0) \xrightarrow{\top} (3, 1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	2	0

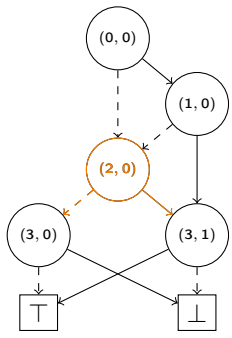
Priority Queue: Q_{count}:

[

$((1, 0) \xrightarrow{\top} (3, 1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

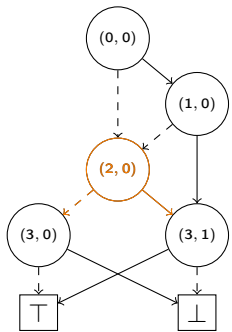
Seek	Sum	Result
$(2, 0)$	2	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2) \quad ,$
$((1,0) \xrightarrow{\top} (3,1), \quad 1) \quad ,$
$((2,0) \xrightarrow{\top} (3,1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

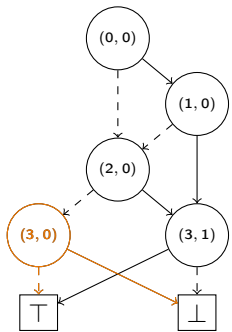
Seek	Sum	Result
$(3, 0)$	0	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2) \quad ,$
$((1,0) \xrightarrow{\top} (3,1), \quad 1) \quad ,$
$((2,0) \xrightarrow{\top} (3,1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

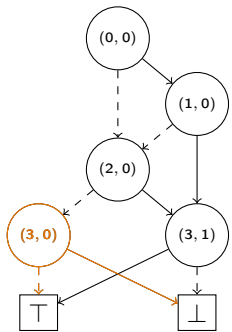
Seek	Sum	Result
$(3, 0)$	0	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2) \quad ,$
$((1,0) \xrightarrow{\top} (3,1), \quad 1) \quad ,$
$((2,0) \xrightarrow{\top} (3,1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 0)$	2	0

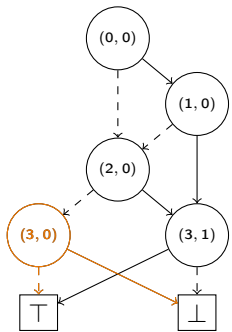
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 0)$	2	2

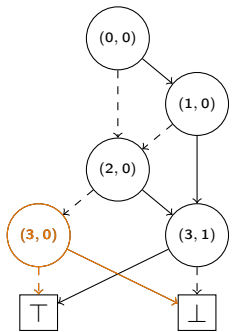
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 1)$	0	2

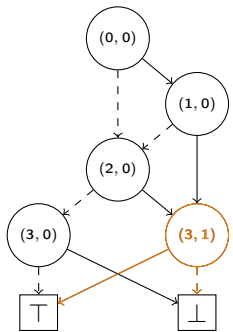
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 1)$	0	2

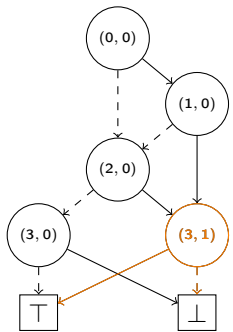
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

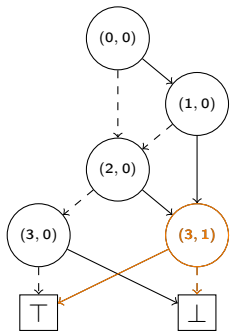
Seek	Sum	Result
$(3, 1)$	1	2

Priority Queue: Q_{count}:

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

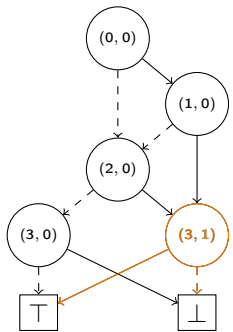
Seek	Sum	Result
(3, 1)	3	2

Priority Queue: Q_{count}:

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

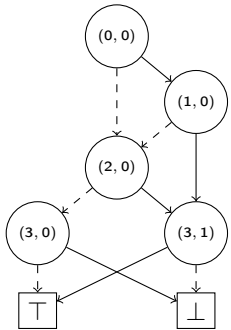
Seek	Sum	Result
(3, 1)	3	5

Priority Queue: Q_{count}:

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result

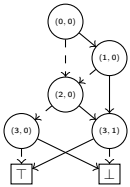
5

Priority Queue: Q_{count}:

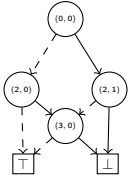
[

]

Apply Example ($\wedge$)

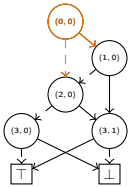


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

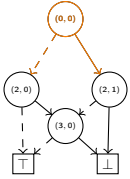


(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Apply Example ($\wedge$)

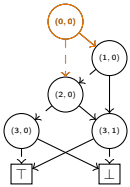


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

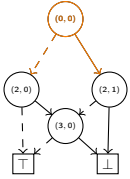


(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

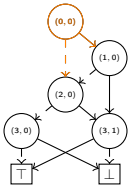
Priority Queue: $Q_{\textbf{app}:1}$:

[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,

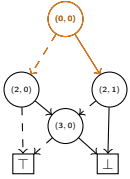
$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

]

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

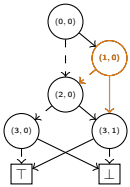
Seek:
$\min((1,0),(2,1))$

Priority Queue: $Q_{\textbf{app}:1}$:

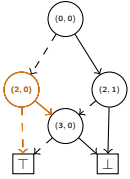
[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,
$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

]

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

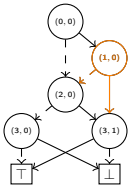
Seek:
$\min((1,0),(2,1))$

Priority Queue: $Q_{\textbf{app}:1}$:

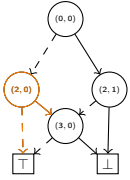
[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,
$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

]

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



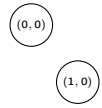
(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((1,0),(2,1))$

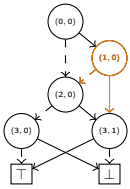
Priority Queue: $Q_{\textbf{app}:\mathbf{1}}$:

$$[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad ,$$
$$(0,0) \xrightarrow{\bot} ((2,0),(2,0)) \quad ,$$
$$(1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad ,$$
$$(1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad ,$$

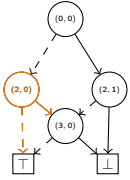
$]$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((1,0),(2,1))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

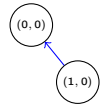
$$[$$

$$(0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad ,$$
$$(1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad ,$$
$$(1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad ,$$

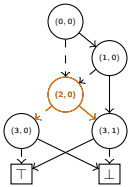
$$]$$

Output:

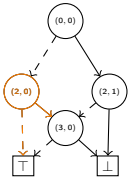
$(0,0) \xrightarrow{\top} (1,0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{\textbf{app}:1}$:

[

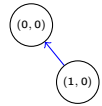
$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

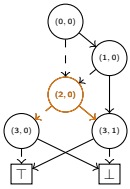
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

]

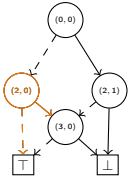
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{\textbf{app}:1}$:

[

$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

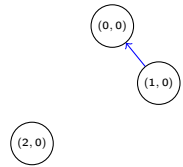
$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

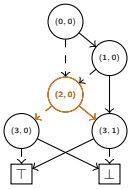
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

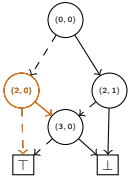
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((2,0),(2,0))$

Priority Queue: $Q_{\textbf{app}:1}$:

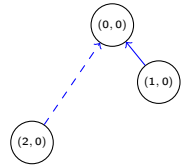
[

$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

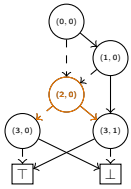
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

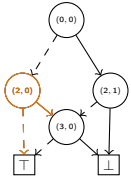
Output:
$(0,0) \xrightarrow{\perp} (2,0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

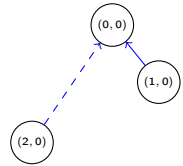
[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

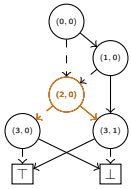
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

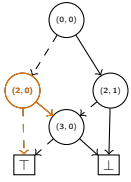
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

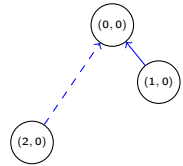
$(2, 0) \xrightarrow{\bot} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

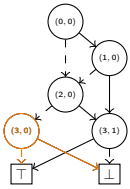
[$(1, 0) \xrightarrow{\bot} ((2, 0), (2, 1)) \quad ((3, 0), (3, 1))$,

]

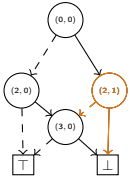
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

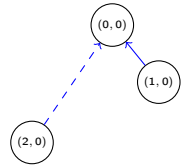
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

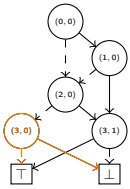
[$(1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad ((3,0),(3,1))$,

]

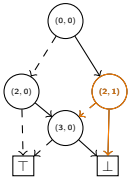
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

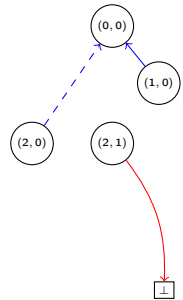
Priority Queue: $Q_{app:2}$:

[$(1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad ((3,0),(3,1))$,

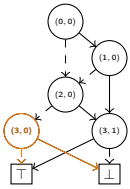
]

Output:

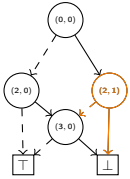
$(2,1) \xrightarrow{\top} \bot$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

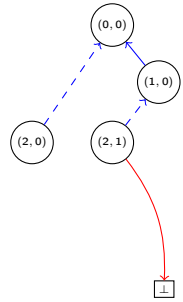
Priority Queue: $Q_{app:2}$:

[

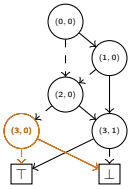
]

Output:

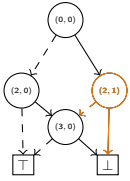
$(1,0) \xrightarrow{\perp} (2,1)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

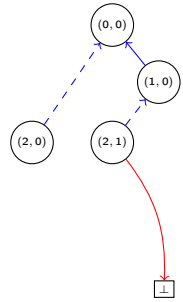
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

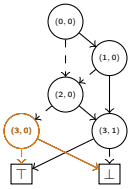
[

]

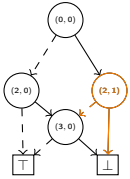
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

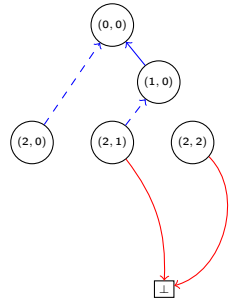
Priority Queue: $Q_{app:2}$:

[

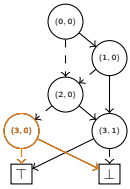
]

Output:

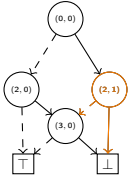
$(2,2) \xrightarrow{\top} \bot$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

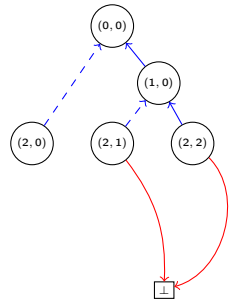
Priority Queue: $Q_{app:2}$:

[

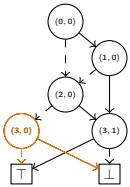
]

Output:

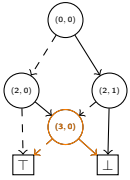
$(1,0) \xrightarrow{\top} (2,2)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

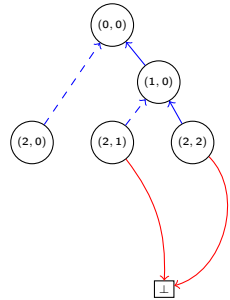
$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

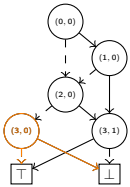
[

]

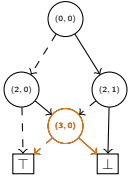
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

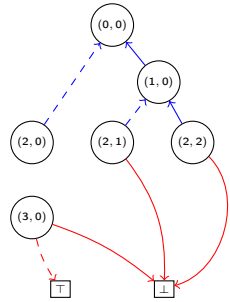
Priority Queue: $Q_{app:2}$:

[

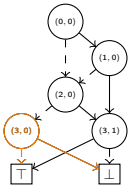
]

Output:

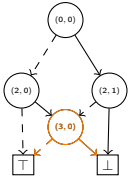
$(3,0) \xrightarrow{\perp} \top, (3,0) \xrightarrow{\top} \perp$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

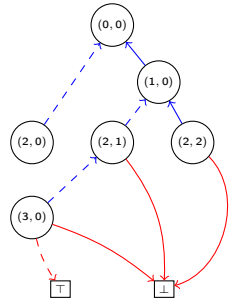
Priority Queue: $Q_{app:2}$:

[

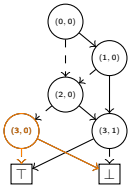
]

Output:

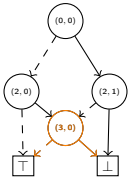
$(2,1) \xrightarrow{\bot} (3,0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,

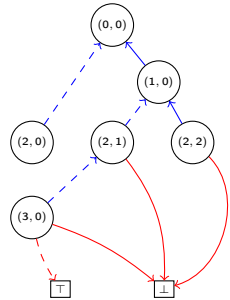
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

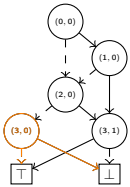
[

]

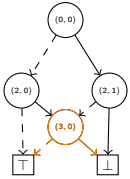
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

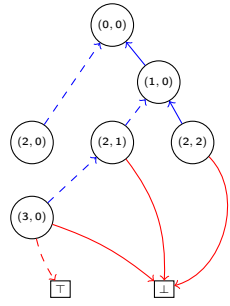
Priority Queue: $Q_{app:2}$:

[

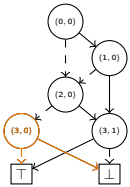
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

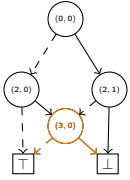
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

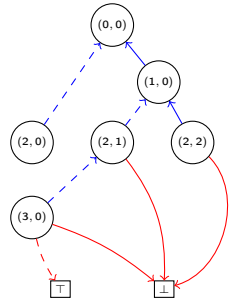
Priority Queue: $Q_{app:2}$:

[

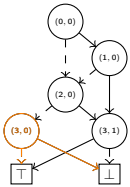
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

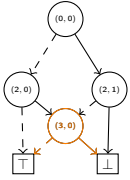
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

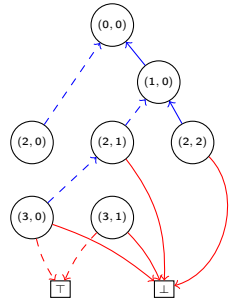
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

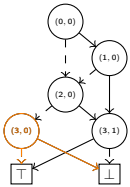
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:

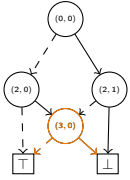
$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

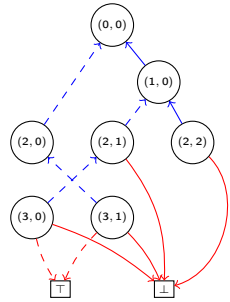
Priority Queue: $Q_{app:2}$:

[

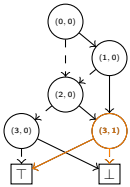
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \bot)$,

$(2, 2) \xrightarrow{\bot} ((3, 1), (3, 0)) \quad (\top, \bot)$]

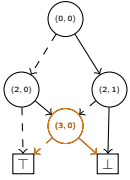
Output:
$(2, 0) \xrightarrow{\bot} (3, 1)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

]

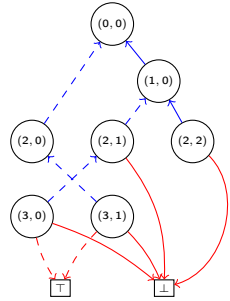
Priority Queue: $Q_{app:2}$:

[

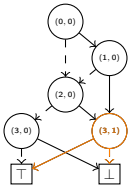
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

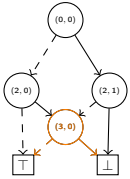
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

]

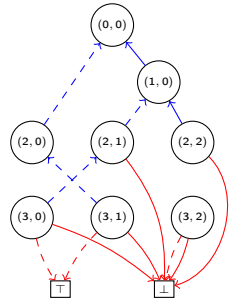
Priority Queue: $Q_{app:2}$:

[

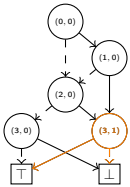
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

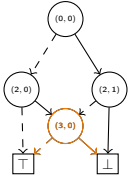
Output:
$(3,2) \xrightarrow{\perp} \perp, (3,2) \xrightarrow{\top} \perp$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

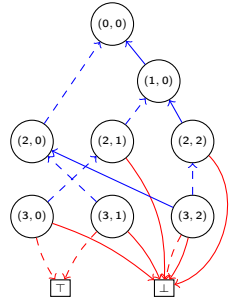
]

Priority Queue: $Q_{app:2}$:

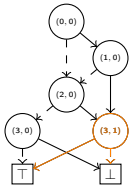
[

]

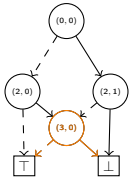
Output:
$(2,0) \xrightarrow{\top} (3,2), (2,2) \xrightarrow{\bot} (3,2)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

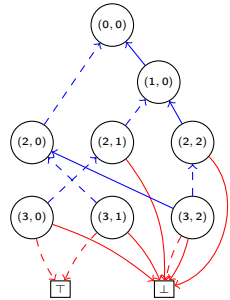
]

Priority Queue: $Q_{app:2}$:

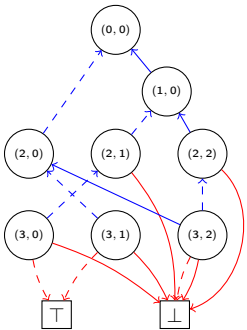
[

]

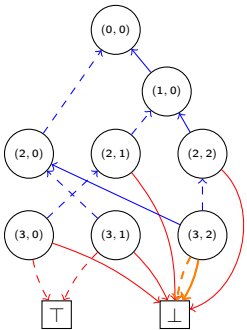
Output:



Reduce Example



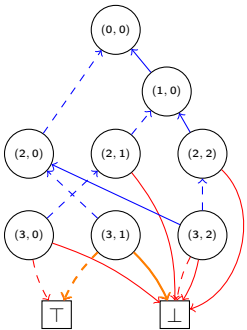
Reduce Example



Level: 3

$[\quad [(3,2) \mapsto \bot] \quad]$

Reduce Example

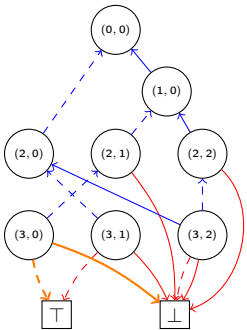


Level: 3

[$[(3,2) \mapsto \bot]$]

[$((3,1), \top, \bot)$,
]

Reduce Example

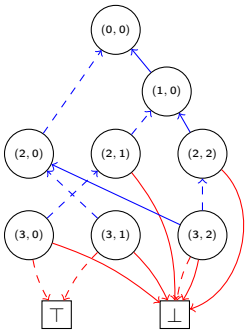


Level: 3

[$[(3,2) \mapsto \bot]$]

[$((3,1), \top, \bot)$,
$((3,0), \top, \bot)$]

Reduce Example

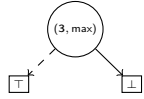


Level: 3

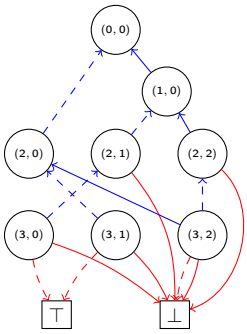
[$[(3,2) \mapsto \bot]$]

[$[(3,1) \mapsto (3, \max)]$, $((3,0), \top, \bot)$]

Output:
$((3, \max), \top, \bot)$



Reduce Example

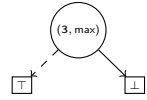


Level: 3

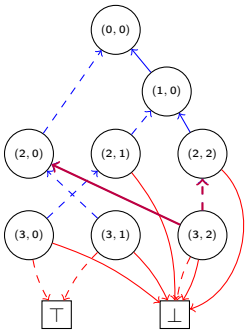
[$[(3,2) \mapsto \bot]$]

[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,

$(2,0) \xrightarrow{\top} \perp$,

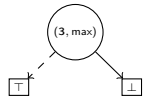
]

Level: 3

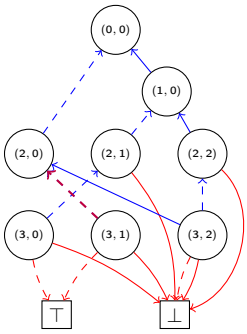
[ ]

[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,

$(2,0) \xrightarrow{\top} \perp$,

$(2,0) \xrightarrow{\perp} (3, \max)$,

]

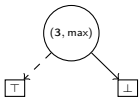
Level: 3

[ ]

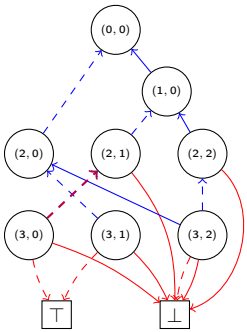
[,

$[(3,0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,
$(2,1) \xrightarrow{\perp} (3, \max)$,
$(2,0) \xrightarrow{\top} \perp$,
$(2,0) \xrightarrow{\perp} (3, \max)$,

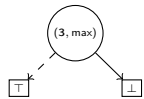
]

Level: 3

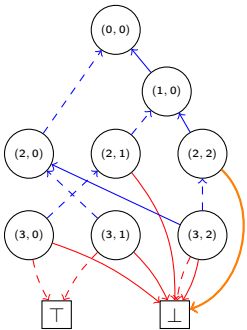
[ ]

[,
]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(2,1) \xrightarrow{\bot} (3, \max)$,

$(2,0) \xrightarrow{\top} \bot$,

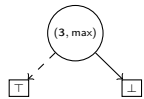
$(2,0) \xrightarrow{\bot} (3, \max)$,

]

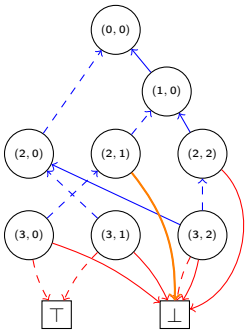
Level: 2

[$[(2,2) \mapsto \bot]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(2,0) \xrightarrow{\top} \bot$,

$(2,0) \xrightarrow{\bot} (3, \max)$,

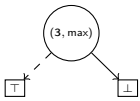
]

Level: 2

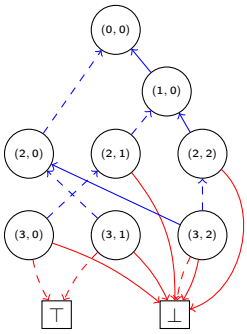
[$[(2,2) \mapsto \bot]$]

[$((2,1),(3,\max),\bot)$,
]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

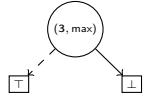
]

Level: 2

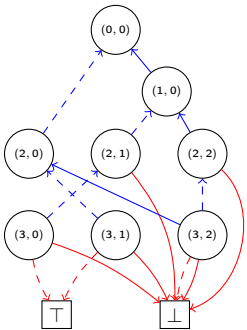
[$[(2,2) \mapsto \bot]$]

[$((2,1),(3,\text{max}),\bot)$,
$((2,0),(3,\text{max}),\bot)$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

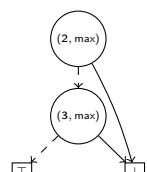
]

Level: 2

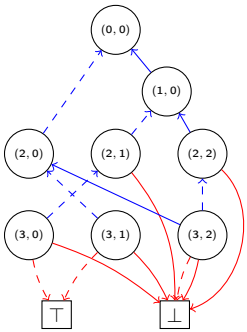
[$[(2,2) \mapsto \bot]$]

[$[(2,1) \mapsto (2, \max)]$,
$((2,0), (3, \max), \bot)$]

Output:
$((2, \max), (3, \max), \bot)$



Reduce Example



Priority Queue: Q_{red}:

[

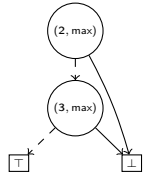
]

Level: 2

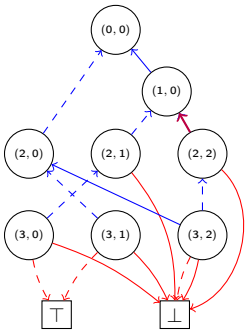
[$[(2,2) \mapsto \bot]$]

[$[(2,1) \mapsto (2, \max)]$,
$[(2,0) \mapsto (2, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

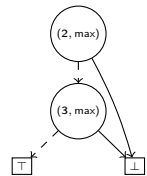
]

Level: 2

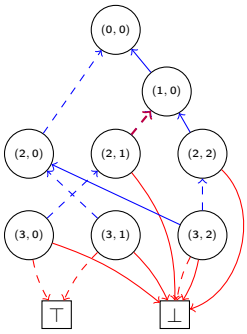
[ ]

[$[(2,1) \mapsto (2,\max)]$,
$[(2,0) \mapsto (2,\max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

]

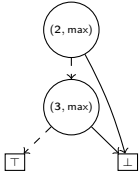
Level: 2

[ ]

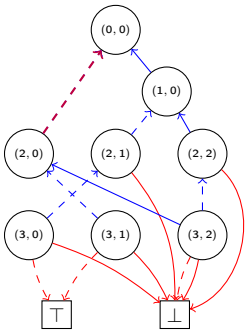
[,

$[(2,0) \mapsto (2, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

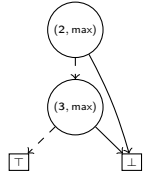
$(0,0) \xrightarrow{\bot} (2, \max)$]

Level: 2

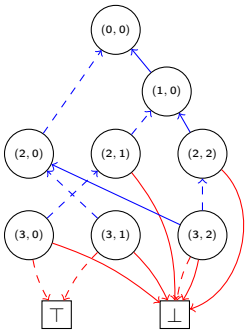
[ ]

[,
]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

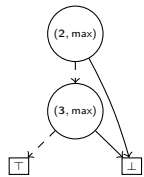
$(0,0) \xrightarrow{\perp} (2, \max)$]

Level: 1

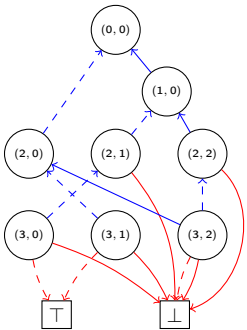
[ ]

[$((1,0),(2,\max),\perp)$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

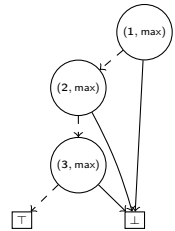
$(0,0) \xrightarrow{\perp} (2, \max)$]

Level: 1

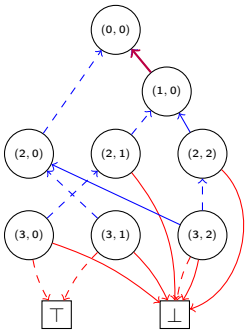
[ ]

[$[(1,0) \mapsto (1, \max)]$]

Output:
$((1, \max), (2, \max), \perp)$



Reduce Example



Priority Queue: Q_{red}:

[

$(0,0) \xrightarrow{\top} (1, \max)$,

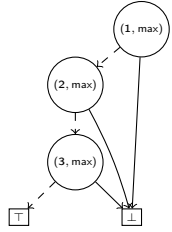
$(0,0) \xrightarrow{\bot} (2, \max)$]

Level: 1

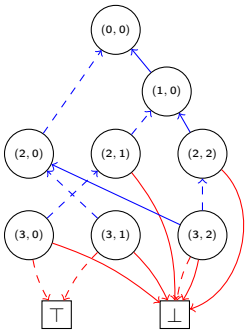
[ ]

[ ]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

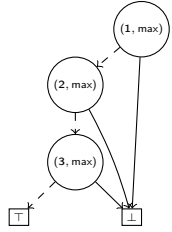
]

Level: 0

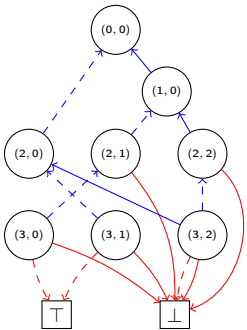
[ ]

[((0, 0), (2, max), (1, max))]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

]

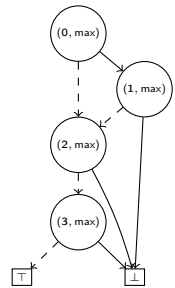
Level: 0

[ ]

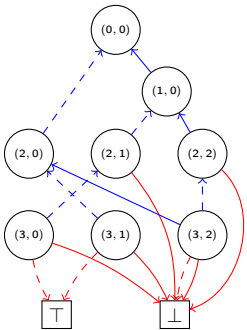
[$[(0,0) \mapsto (0, \max)]$]

Output:

((0, max), (2, max), (1, max))



Reduce Example



Priority Queue: Q_{red}:

[

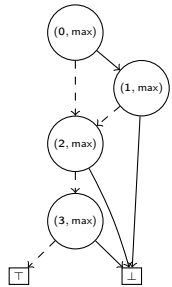
]

Level: 0

[ ]

[ ]

Output:



Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[, , ,]

Levelized Priority Queue: $Q_{\mathit{app}:\mathbf{1}}$:

Level: 1

$$\left[\ (0,0) \xrightarrow{\top} ((1,0),(2,1))\ \right]$$

Level: 2

$$\left[\ (0,0) \xrightarrow{\bot} ((2,0),(2,0))\ ,\ ,\ \right]$$

Level: 3

$$\left[\ ,\ ,\ ,\ \right]$$

Levelized Priority Queue: $Q_{\textbf{\textit{app}}:\mathbf{1}}$:

Level: 1

$$\left[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad\right]$$

Level: 2

$$\left[\quad (0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad\right]$$

Level: 3

$$\left[\qquad\qquad , \qquad\qquad , \qquad\qquad , \qquad\qquad\right]$$

Levelized Priority Queue: $Q_{\textbf{\textit{app}}:\mathbf{1}}$:

Level: 1

$[\quad\quad\quad

Levelized Priority Queue: $Q_{\textbf{\textit{app}}:\mathbf{1}}$:

Level: 1

$$[\quad\quad\quad

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad]$$

Level: 2

$$[\quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad]$$

Level: 3

$$[\quad (2,0) \xrightarrow{\perp} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad , \quad]$$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad]$$

Level: 2

$$[\quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad]$$

Level: 3

$$[\quad (2,0) \xrightarrow{\perp} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad (2,1) \xrightarrow{\perp} ((3,0),(3,0)) \quad , \quad]$$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad\quad\quad

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad\quad\quad

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad\quad\quad

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad\quad\quad

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[, $(2,0) \xrightarrow{\top} ((3,1),(3,0))$, $(2,2) \xrightarrow{\bot} ((3,1),(3,0))$, $(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[, , $(2,2) \xrightarrow{\perp} ((3,1),(3,0))$, $(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Levelized Priority Queue: $Q_{\textbf{\textit{app}}:\mathtt{1}}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[, , , $(2,0) \xrightarrow{\perp} ((3,0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[ ]

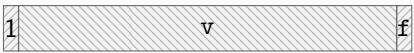
Level: 2

[, ,]

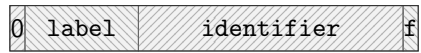
Level: 3

[, , ,]

The unique identifier of nodes and leaves can be represented in a single 64-bit integer.



(a) Unique identifier of a leaf `v`

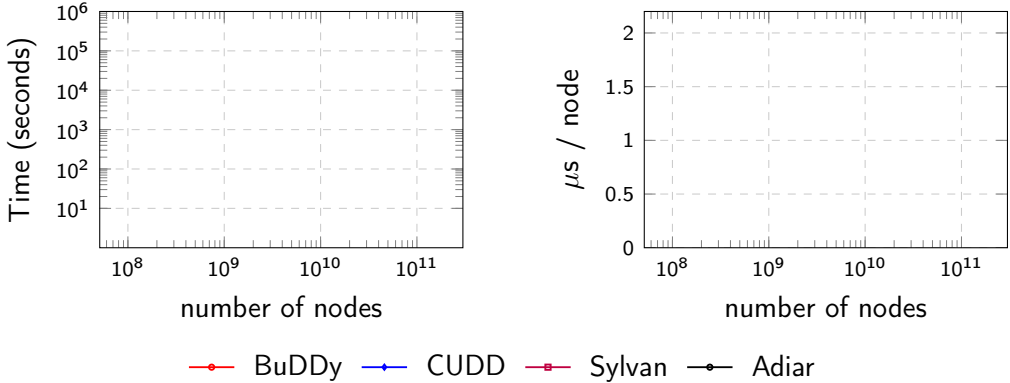


(b) Unique identifier of an internal node

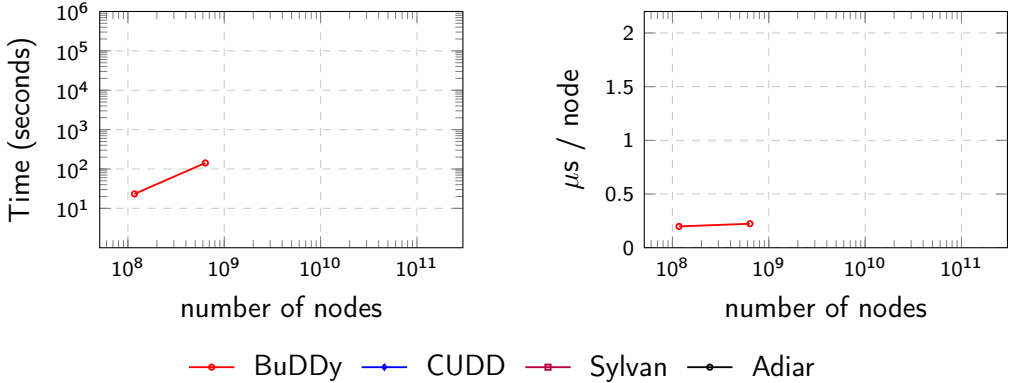
The `f` bit-flag is used to store the *is_high* boolean inside of the `source` of an arc.

Adiar

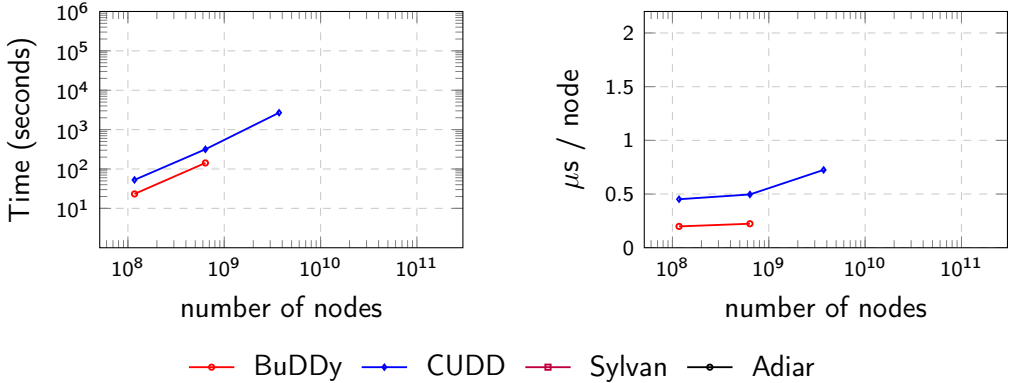
github.com/ssoelvsten/adiar



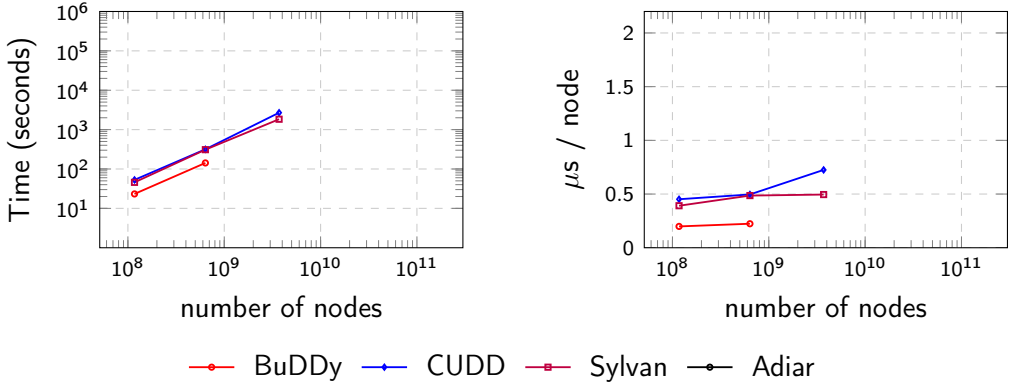
Minimum running times for the N-Queens problem.



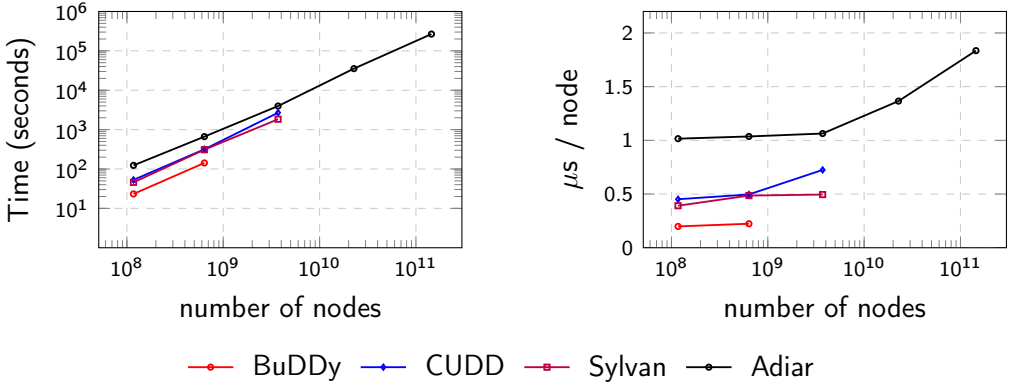
Minimum running times for the N-Queens problem.



Minimum running times for the N-Queens problem.

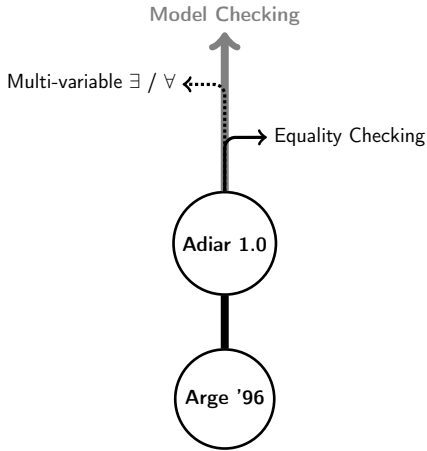


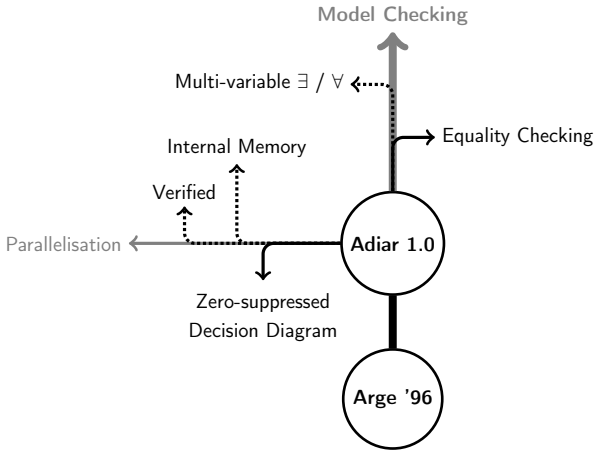
Minimum running times for the N-Queens problem.

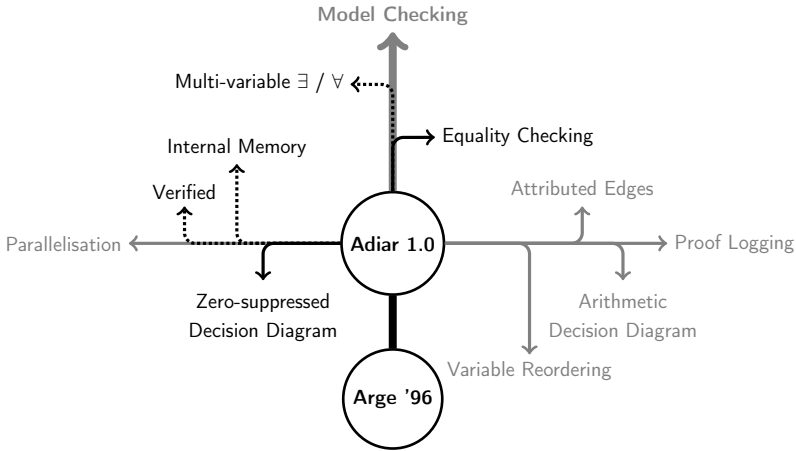


Minimum running times for the N-Queens problem.

Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f \ ? \ g \ : \ h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f\|_{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f\|_{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	#paths in f to $\top$	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min$ / $\max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Adiar 1.0
Arge '96


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Internal Memory
Verified
Parallelisation
Adiar 1.0
Zero-suppressed
Decision Diagram
Arge '96


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Internal Memory
Verified
Attributed Edges
Parallelisation
Adiar 1.0
Proof Logging
Zero-suppressed
Decision Diagram
Arithmetic
Decision Diagram
Variable Reordering
Arge '96