I/O-efficient Symbolic Model Checking

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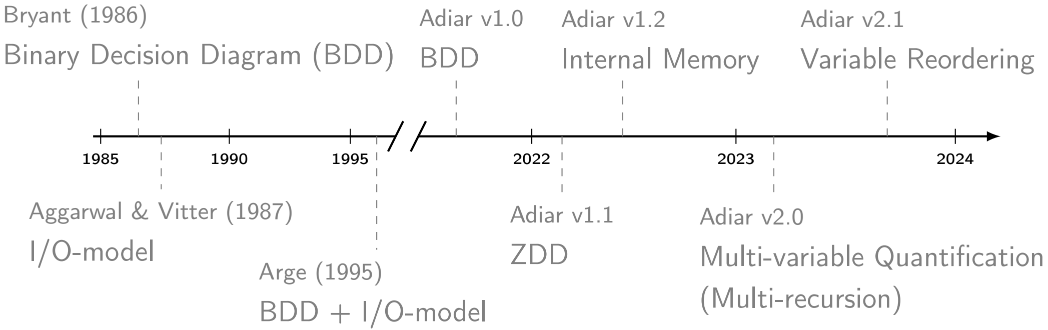
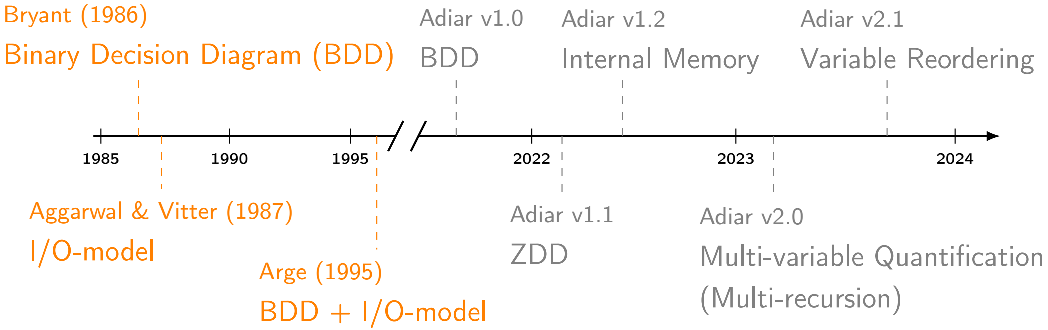
31st of August, 2022

AARHUS
UNIVERSITY

Adiar

github.com/ssoelvsten/adiar


Bryant (1986)
Binary Decision Diagram (BDD)
Adiar v1.0
BDD
Adiar v1.2
Internal Memory
Adiar v2.1
Variable Reordering
1985
1990
1995
2022
2023
2024
Aggarwal & Vitter (1987)
I/O-model
Arge (1995)
BDD + I/O-model
Adiar v1.1
ZDD
Adiar v2.0
Multi-variable Quantification
(Multi-recursion)



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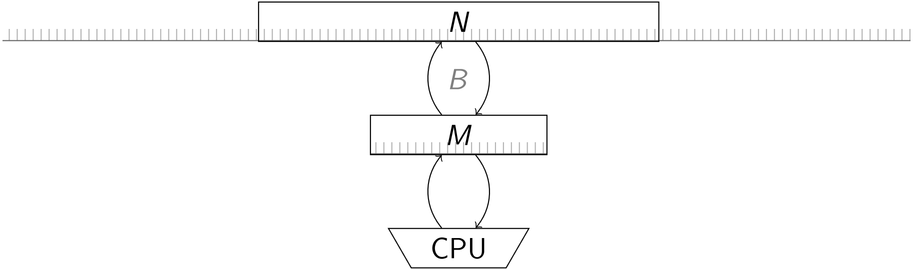
Aggarwal and Vitter '87 : I/O-model



The I/O-model by Aggarwal and Vitter '87

Aggarwal and Vitter '87 : I/O-model

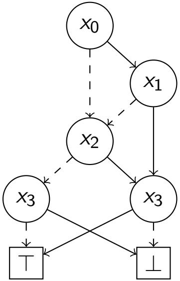
For any realistic values of N, M, and B we have that

$$N/B \quad < \quad \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \quad \ll \quad N \ ,$$

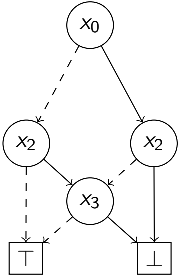
Theorem (Aggarwal and Vitter '87)
N elements can be sorted in $\Theta(sort(N))$ I/Os.

Theorem (Arge '95)
N elements can be inserted in and extracted from a Priority Queue in $\Theta(sort(N))$ I/Os.

Bryant '86 : Binary Decision Diagram



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

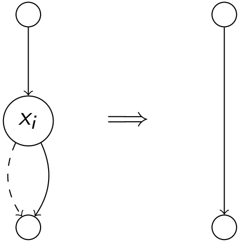


(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

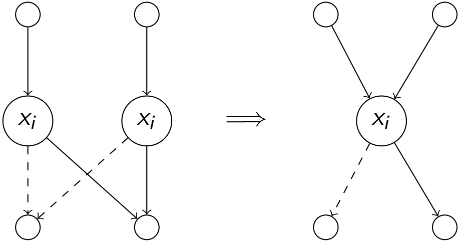
Examples of (Reduced Ordered) Binary Decision Diagrams.

Bryant '86 : Binary Decision Diagram

Theorem
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.

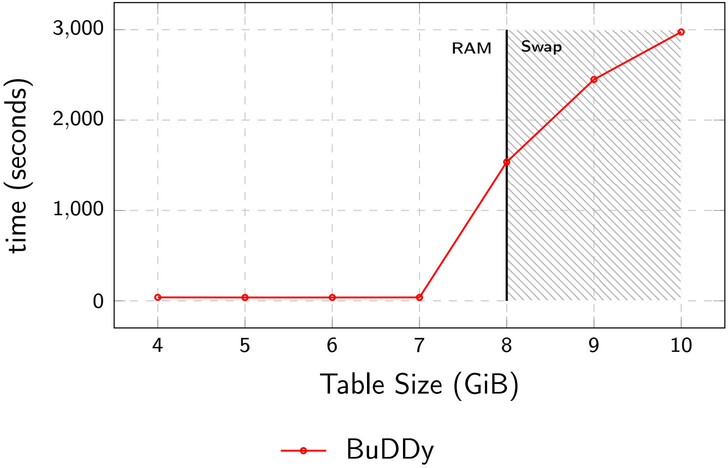


(1) Remove redundant nodes

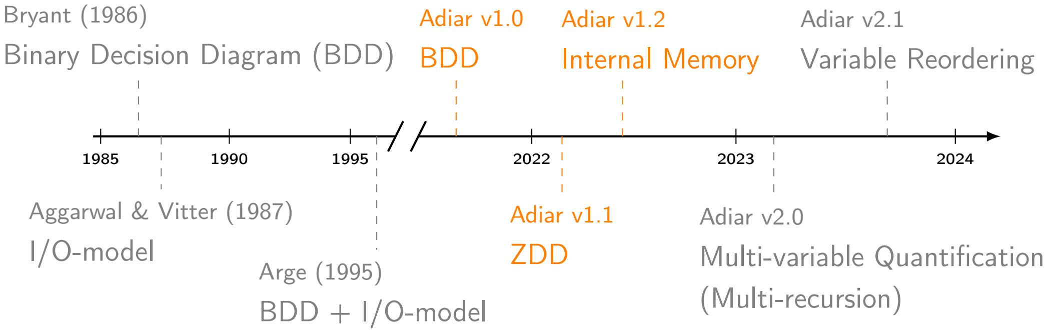


(2) Merge duplicate nodes

Arge '95 : BDD + I/O-model



Running time for solving a problem that does not need more than 3 GiB.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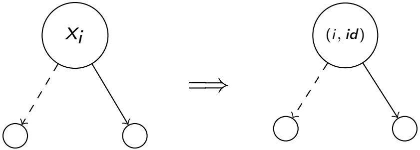
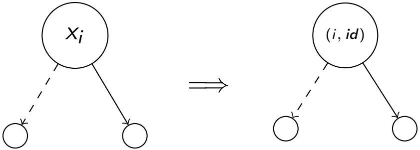
Adiar v1.0 : BDD

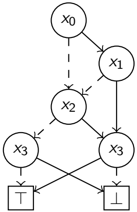


Adiar v1.0 : BDD



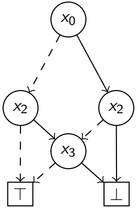
$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$

Adiar v1.0 : BDD



$$
\begin{array}{lcr}
[& ((0,0),(2,0),(1,0)) & , \\
 & ((1,0),(2,0),(3,1)) & , \\
 & ((2,0),(3,0),(3,1)) & , \\
 & ((3,0),\bot,\top) & , \\
 & ((3,1),\top,\bot) &]
\end{array}
$$

(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

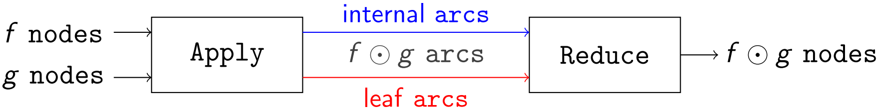


$$
\begin{array}{lcr}
[& ((0,0),(2,0),(2,1)) & , \\
 & ((2,0),\top,(3,0)) & , \\
 & ((2,1),(3,0),\bot) & , \\
 & ((3,0),\top,\bot) &]
\end{array}
$$

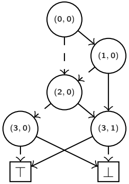
(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Node-based representation of prior shown BDDs

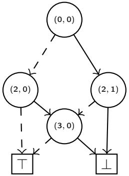
Adiar v1.0 : BDD



Adiar v1.0 : BDD



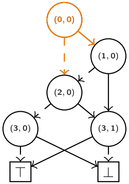
(a) $(x_{\mathbf{0}} \land x_{\mathbf{1}} \land x_{\mathbf{3}}) \lor (x_{\mathbf{2}} \oplus x_{\mathbf{3}})$



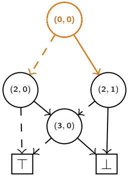
(b) $\neg(x_{\mathbf{0}}\ ?\ x_{\mathbf{2}} \lor x_{\mathbf{3}}\ :\ x_{\mathbf{2}} \land x_{\mathbf{3}})$

(c) $(a) \land (b)$

Adiar v1.0 : BDD



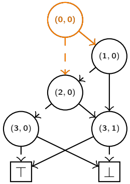
(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



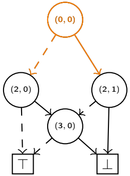
(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

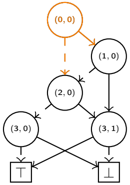
[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,

$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

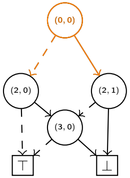
]

(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \lor x_3\ :\ x_2 \land x_3)$

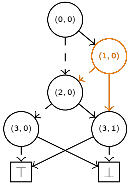
Seek:
min((1, 0), (2, 1))

Priority Queue: $Q_{app:1}$:

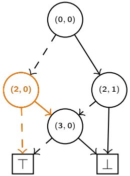
$$[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad ,$$
$$(0,0) \xrightarrow{\bot} ((2,0),(2,0)) \quad , \quad]$$

(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:

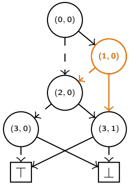
$\min((1,0),(2,1))$

Priority Queue: $Q_{\textit{app}:1}$:

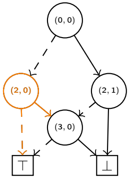
$$\left[\begin{array}{ll} (0,0) \xrightarrow{\top} ((1,0),(2,1)) & , \\ (0,0) \xrightarrow{\perp} ((2,0),(2,0)) & , \end{array}\right]$$

(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



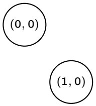
(b) $\neg(x_0 \; ? \; x_2 \lor x_3 \; : \; x_2 \land x_3)$

Seek:

$\min((1, 0), (2, 1))$

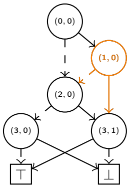
Priority Queue: $Q_{app:1}$:

$$
\begin{array}{l}
[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad , \\
\quad\;\; (0,0) \xrightarrow{\bot} ((2,0),(2,0)) \quad , \\
\quad\;\; (1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad , \\
\quad\;\; (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad , \\
\hspace{12em}]
\end{array}
$$

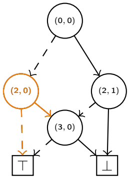


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



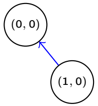
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((1,0),(2,1))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

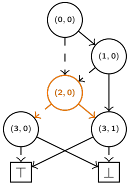
$$[\ (0,0) \xrightarrow{\bot} ((2,0),(2,0)),\ (1,0) \xrightarrow{\bot} ((2,0),(2,1)),\ (1,0) \xrightarrow{\top} ((3,1),(2,1)),\ ]$$

Output:
$(0,0) \xrightarrow{\top} (1,0)$

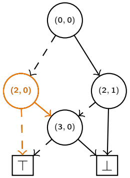


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_{\mathbf{0}} \wedge x_{\mathbf{1}} \wedge x_{\mathbf{3}}) \vee (x_{\mathbf{2}} \oplus x_{\mathbf{3}})$



(b) $\neg(x_{\mathbf{0}} \ ? \ x_{\mathbf{2}} \vee x_{\mathbf{3}} \ : \ x_{\mathbf{2}} \wedge x_{\mathbf{3}})$

Seek:
$\min((2,0),(2,0))$

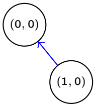
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,
$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

]

Output:



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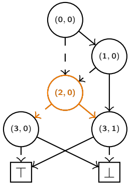
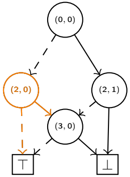
Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



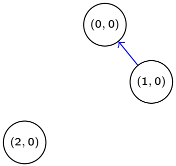
(b) $\neg(x_0 \; ? \; x_2 \lor x_3 \; : \; x_2 \land x_3)$

Seek:
$\min((2,0),(2,0))$

Priority Queue: $Q_{app:1}$:

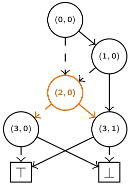
$$[\; (0,0) \xrightarrow{\bot} ((2,0),(2,0)) \;,\; (1,0) \xrightarrow{\bot} ((2,0),(2,1)) \;,\; (1,0) \xrightarrow{\top} ((3,1),(2,1)) \;,\; (2,0) \xrightarrow{\top} ((3,1),(3,0)) \;,\; (2,0) \xrightarrow{\bot} ((3,0),\top) \;]$$

Output:



(c) $(a) \land (b)$

Adiar v1.0 : BDD



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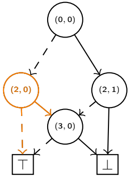


(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:
$\min((2,0),(2,0))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

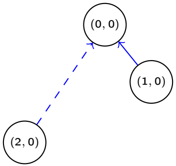
[

$(1,0) \xrightarrow{\bot} ((2,0),(2,1))$,
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

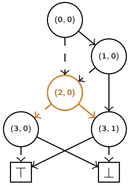
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Output:
$(0,0) \xrightarrow{\bot} (2,0)$

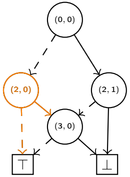


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



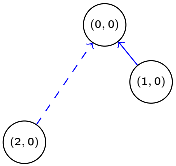
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((2, 0), (2, 1))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

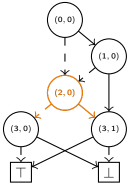
$$
\begin{aligned}
[\quad & \\
& (1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad , \\
& (1, 0) \xrightarrow{\top} ((3, 1), (2, 1)) \quad , \\
& (2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad , \\
& (2, 0) \xrightarrow{\perp} ((3, 0), \top) \quad]
\end{aligned}
$$

Output:

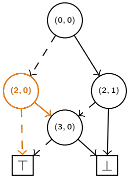


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:
$\min((2,0),(2,1))$

Priority Queue: $Q_{\mathit{app}:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

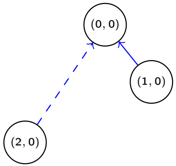
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{\mathit{app}:2}$:

[$(1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad ((3,0),(3,1))$,

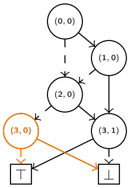
]

Output:

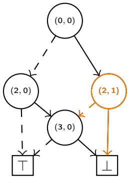


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:
$\max((2,0),(2,1))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

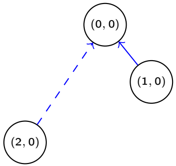
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

[$(1,0) \xrightarrow{\bot} ((2,0),(2,1))$ $((3,0),(3,1))$,

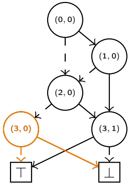
]

Output:

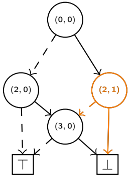


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \lor x_3\ :\ x_2 \land x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

$[$

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\bot} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\bot} ((3, 0), \top)$ $]$

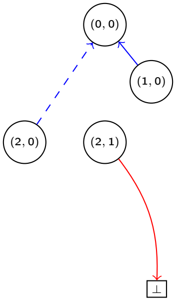
Priority Queue: $Q_{app:2}$:

$[$ $(1, 0) \xrightarrow{\bot} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

$]$

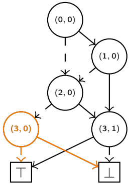
Output:

$(2, 1) \xrightarrow{\top} \bot$

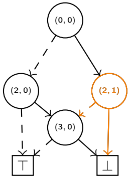


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

$$\begin{aligned}
[\quad & \\
& (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad , \\
& (2,1) \xrightarrow{\bot} ((3,0),(3,0)) \quad , \\
& (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \\
& (2,0) \xrightarrow{\bot} ((3,0),\top) \quad]
\end{aligned}$$

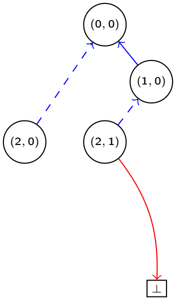
Priority Queue: $Q_{\mathbf{app}:\mathbf{2}}$:

[

]

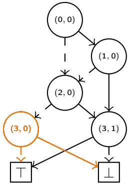
Output:

$(1,0) \xrightarrow{\bot} (2,1)$

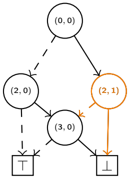


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:
$\min((3,1),(2,1))$

Priority Queue: $Q_{\mathit{app}:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

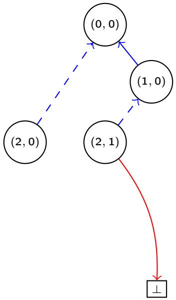
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{\mathit{app}:2}$:

[

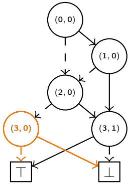
]

Output:

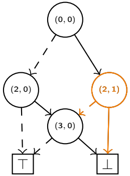


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((3,1),(2,1))$

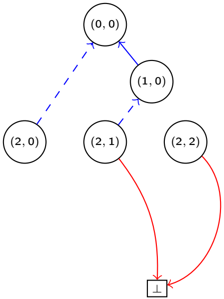
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

$$
\begin{array}{ll}
[& \\
(1,0) \xrightarrow{\top} ((3,1),(2,1)) & , \\
(2,1) \xrightarrow{\bot} ((3,0),(3,0)) & , \\
(2,0) \xrightarrow{\top} ((3,1),(3,0)) & , \\
(2,2) \xrightarrow{\bot} ((3,1),(3,0)) & , \\
(2,0) \xrightarrow{\bot} ((3,0),\top) &]
\end{array}
$$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

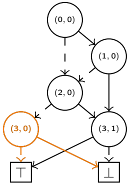
[ ]

Output:
$(2,2) \xrightarrow{\top} \bot$

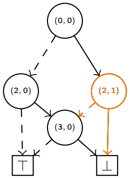


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

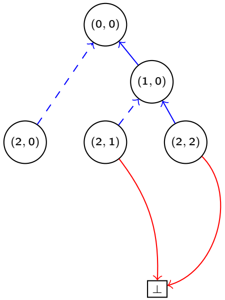
$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[

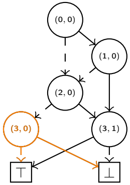
]

Output:
$(1,0) \xrightarrow{\top} (2,2)$

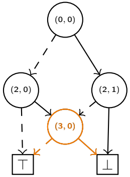


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

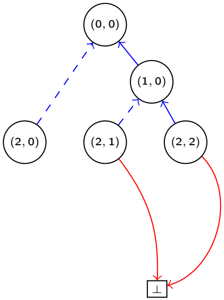
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[

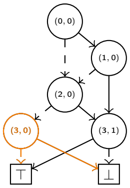
]

Output:

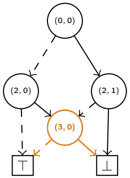


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \lor x_3 \;:\; x_2 \land x_3)$

Seek:
$\min((3,0),(3,0))$

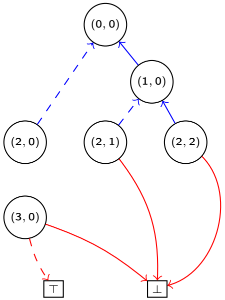
Priority Queue: $Q_{app:1}$:

$$\begin{array}{ll}
[& \\
(2,1) \xrightarrow{\bot} ((3,0),(3,0)) & , \\
(2,0) \xrightarrow{\top} ((3,1),(3,0)) & , \\
(2,2) \xrightarrow{\bot} ((3,1),(3,0)) & , \\
(2,0) \xrightarrow{\bot} ((3,0),\top) &]
\end{array}$$

Priority Queue: $Q_{app:2}$:

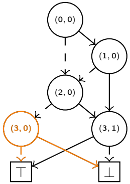
[ ]

Output:
$(3,0) \xrightarrow{\bot} \top, (3,0) \xrightarrow{\top} \bot$

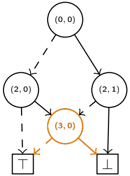


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((3,0),(3,0))$

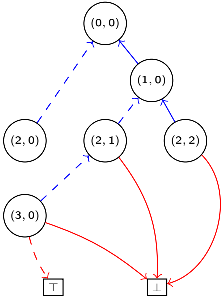
Priority Queue: $Q_{\mathit{app}:1}$:

$$[\; (2,0) \xrightarrow{\top} ((3,1),(3,0)) \;,\; (2,2) \xrightarrow{\perp} ((3,1),(3,0)) \;,\; (2,0) \xrightarrow{\perp} ((3,0),\top) \;]$$

Priority Queue: $Q_{\mathit{app}:2}$:

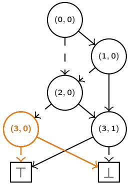
$$[\;]$$

Output:
$(2,1) \xrightarrow{\perp} (3,0)$

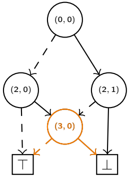


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
min((3, 1), (3, 0))

Priority Queue: $Q_{\mathit{app}:1}$:

[

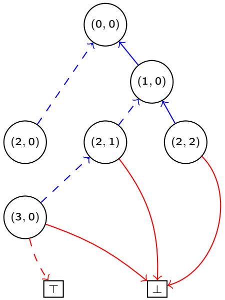
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{\mathit{app}:2}$:

[

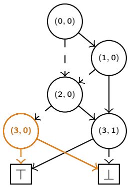
]

Output:

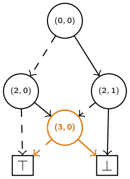


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:

$\min((3,1),(3,0))$

Priority Queue: $Q_{\mathit{app}:1}$:

[

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

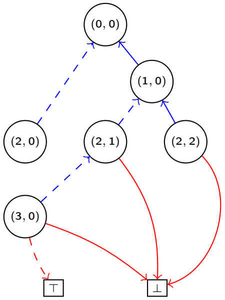
Priority Queue: $Q_{\mathit{app}:2}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\perp)$,

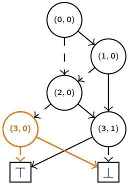
$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top,\perp)$]

Output:

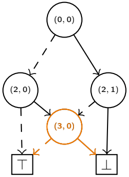


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:
$\min((3, 0), \top)$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

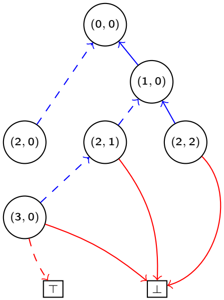
Priority Queue: $Q_{\mathbf{app}:\mathbf{2}}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

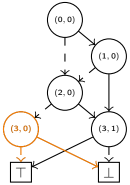
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:

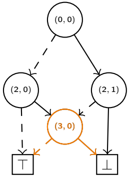


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((3,0), \top)$

Priority Queue: $Q_{\mathbf{app:1}}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

Priority Queue: $Q_{\mathbf{app:2}}$:

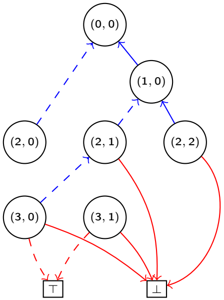
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top,\perp)$]

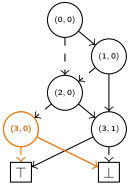
Output:

$(3,1) \xrightarrow{\perp} \top, (3,1) \xrightarrow{\top} \perp$

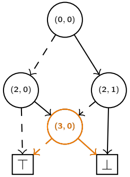


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \lor x_3 \;:\; x_2 \land x_3)$

Seek:

$\min((3,0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

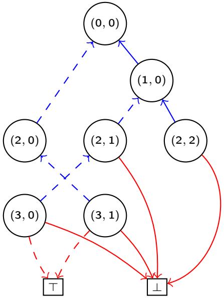
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \bot)$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top, \bot)$]

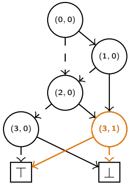
Output:

$(2,0) \xrightarrow{\bot} (3,1)$

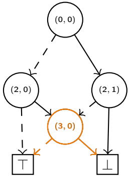


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
max((3, 1), (3, 0))

Priority Queue: $Q_{\mathbf{app}:1}$:

[

]

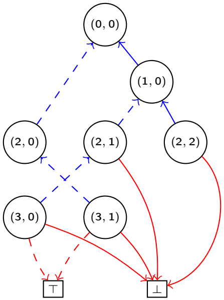
Priority Queue: $Q_{\mathbf{app}:2}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\bot)$,

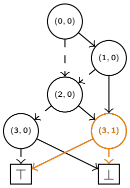
$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top,\bot)$]

Output:

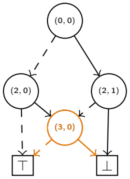


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:
max((3, 1), (3, 0))

Priority Queue: $Q_{app:1}$:

[

]

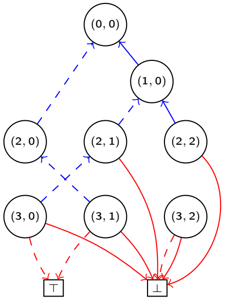
Priority Queue: $Q_{app:2}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\bot)$,

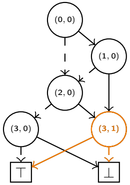
$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top,\bot)$]

Output:
$(3,2) \xrightarrow{\bot} \bot, (3,2) \xrightarrow{\top} \bot$

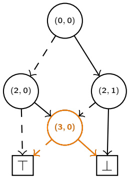


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Seek:

max((3, 1), (3, 0))

Priority Queue: $Q_{app:1}$:

[

]

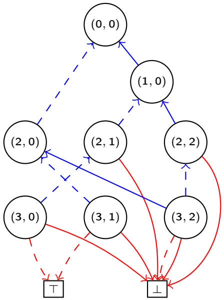
Priority Queue: $Q_{app:2}$:

[

]

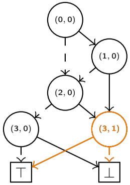
Output:

$(2,0) \xrightarrow{\top} (3,2), (2,2) \xrightarrow{\bot} (3,2)$

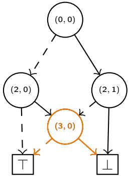


(c) $(a) \land (b)$

Adiar v1.0 : BDD



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \lor x_3 \ : \ x_2 \land x_3)$

Priority Queue: $Q_{app:1}$:

[

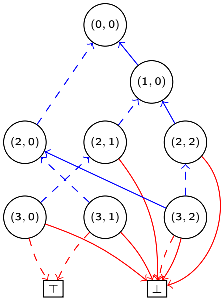
]

Priority Queue: $Q_{app:2}$:

[

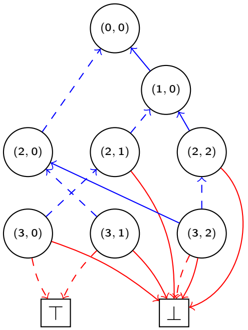
]

Output:



(c) $(a) \land (b)$

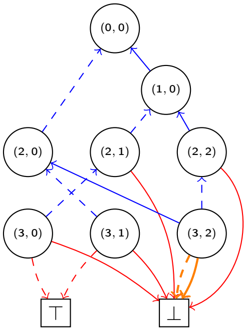
Adiar v1.0 : BDD



(c) $(a) \land (b)$

(d) $(a) \land (b)$ **reduced**

Adiar v1.0 : BDD



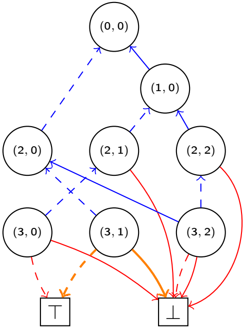
(c) $(a) \wedge (b)$

Level: 3

[$[(3,2) \mapsto \bot]$]

(d) $(a) \wedge (b)$ **reduced**

Adiar v1.0 : BDD



(c) $(a) \land (b)$

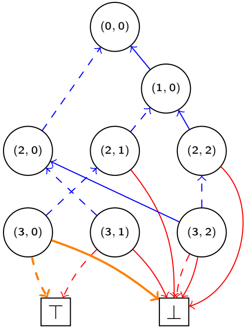
Level: 3

[$[(3,2) \mapsto \bot]$]

[$((3,1), \top, \bot)$,
]

(d) $(a) \land (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

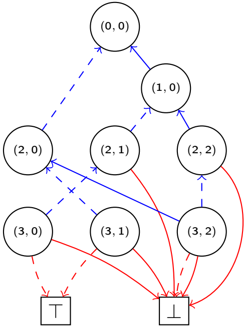
Level: 3

[$[(3,2) \mapsto \bot]$]

[$((3,1), \top, \bot)$,
$((3,0), \top, \bot)$]

(d) $(a) \wedge (b)$ **reduced**

Adiar v1.0 : BDD



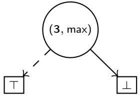
(c) $(a) \wedge (b)$

Level: 3

[$[(3, 2) \mapsto \bot]$]

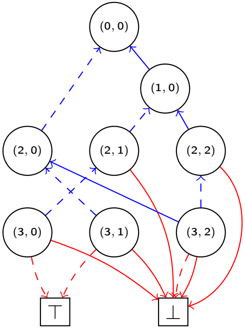
[$[(3, 1) \mapsto (3, \text{max})]$,
$((3, 0), \top, \bot)$]

Output:
$((3, \text{max}), \top, \bot)$



(d) $(a) \wedge (b)$ **reduced**

Adiar v1.0 : BDD



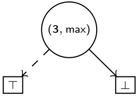
(c) $(a) \land (b)$

Level: 3

[$[(3, 2) \mapsto \bot]$]

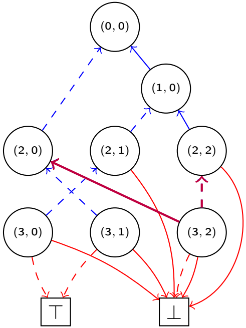
[$[(3, 1) \mapsto (3, \text{max})]$,
$[(3, 0) \mapsto (3, \text{max})]$]

Output:



(d) $(a) \land (b)$ **reduced**

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\boldsymbol{red}}$:

[$(2,2) \xrightarrow{\bot} \bot$,

$(2,0) \xrightarrow{\top} \bot$,

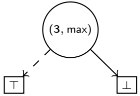
]

Level: 3

[ ]

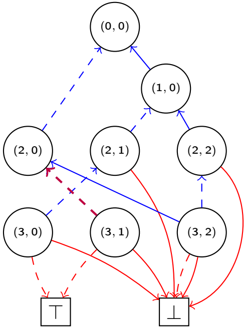
[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

$$[\quad (2,2) \xrightarrow{\perp} \perp \quad ,$$

$$(2,0) \xrightarrow{\top} \perp \quad ,$$

$$(2,0) \xrightarrow{\perp} (3, \mathrm{max}) \quad ,$$

$$]$$

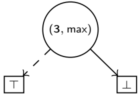
Level: 3

[ ]

[,

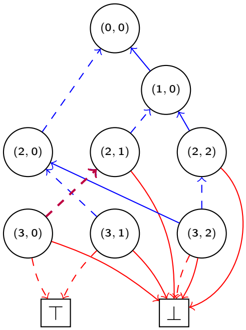
$[(3,0) \mapsto (3, \mathrm{max})]$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\boldsymbol{red}}$:

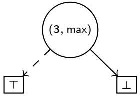
[$(2,2) \xrightarrow{\perp} \perp$,
$(2,1) \xrightarrow{\perp} (3, \max)$,
$(2,0) \xrightarrow{\top} \perp$,
$(2,0) \xrightarrow{\perp} (3, \max)$,
]

Level: 3

[ ]

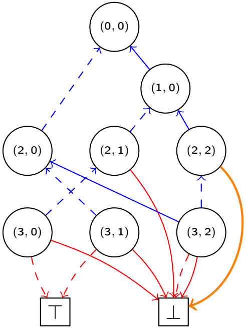
[,
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(2, 1) \xrightarrow{\perp} (3, \mathrm{max})$,

$(2, 0) \xrightarrow{\top} \perp$,

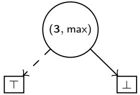
$(2, 0) \xrightarrow{\perp} (3, \mathrm{max})$,

]

Level: 2

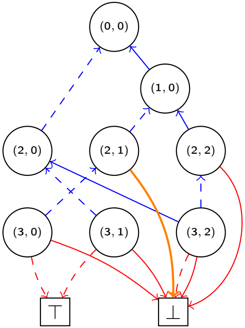
[$[(2, 2) \mapsto \perp]$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\boldsymbol{red}}$:

[

$(2,0) \xrightarrow{\top} \bot$,

$(2,0) \xrightarrow{\bot} (3,\mathrm{max})$,

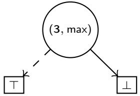
]

Level: 2

[$[(2,2) \mapsto \bot]$]

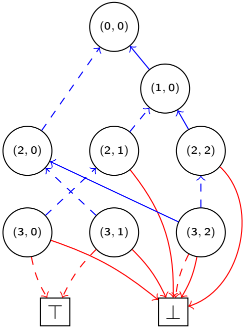
[$((2,1),(3,\mathrm{max}),\bot)$,
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\mathbf{red}}$:

[

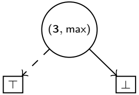
]

Level: 2

[$[(2,2) \mapsto \bot]$]

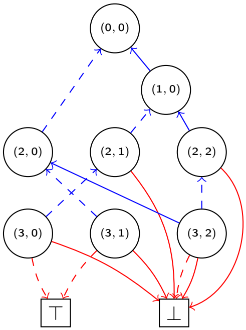
[$((2,1), (3, \text{max}), \bot)$,
$((2,0), (3, \text{max}), \bot)$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\boldsymbol{red}}$:

[

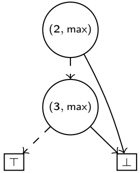
]

Level: 2

[$[(2,2) \mapsto \bot]$]

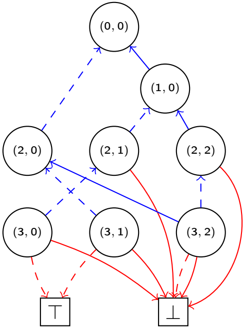
[$[(2,1) \mapsto (2,\text{max})]$,
$((2,0),(3,\text{max}),\bot)$]

Output:
$((2,\text{max}),(3,\text{max}),\bot)$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \land (b)$

Priority Queue: Q_{red}:

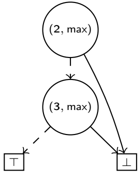
[ ]

Level: 2

[$[(2,2) \mapsto \bot]$]

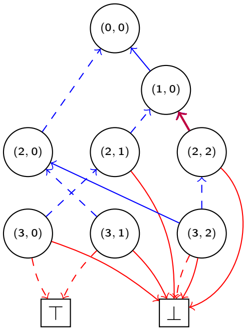
[$[(2,1) \mapsto (2,\text{max})]$,
$[(2,0) \mapsto (2,\text{max})]$]

Output:



(d) $(a) \land (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\textit{red}}$:

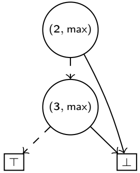
[$(1,0) \xrightarrow{\top} \bot$,]

Level: 2

[ ]

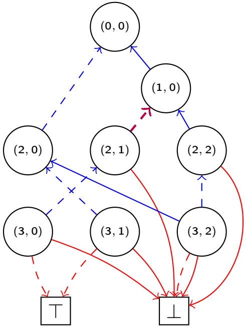
[$[(2,1) \mapsto (2,\mathrm{max})]$,
$[(2,0) \mapsto (2,\mathrm{max})]$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

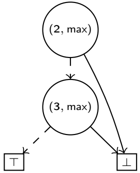
]

Level: 2

[ ]

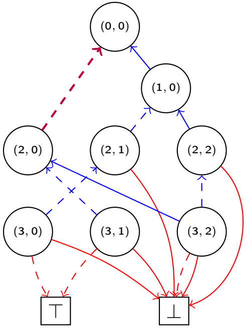
[$[(2,0) \mapsto (2,\max)]$,]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

$(0,0) \xrightarrow{\bot} (2, \max)$]

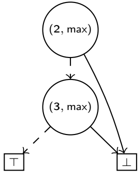
Level: 2

[ ]

[,

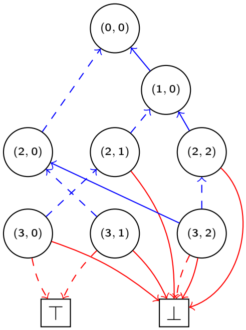
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\boldsymbol{red}}$:

[

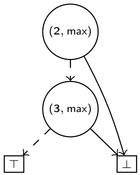
$(0,0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[ ]

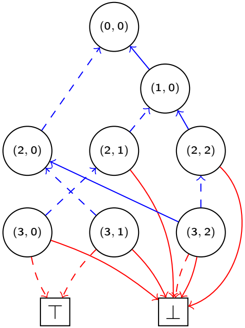
[$((1,0), (2, \max), \perp)$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \land (b)$

Priority Queue: $Q_{\textbf{\textit{red}}}$:

[

$(0,0) \xrightarrow{\bot} (2, \text{max})$]

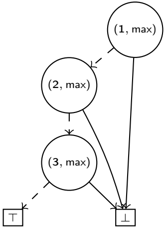
Level: 1

[ ]

[$[(1,0) \mapsto (1, \text{max})]$]

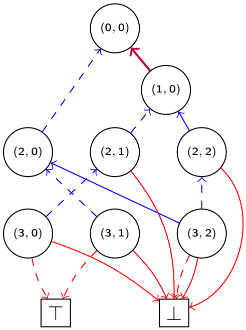
Output:

$((1, \text{max}), (2, \text{max}), \bot)$



(d) $(a) \land (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\textit{red}}$:

[

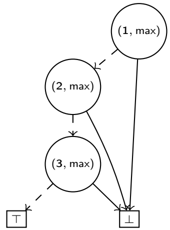
$(0,0) \xrightarrow{\top} (1, \text{max})$,
$(0,0) \xrightarrow{\bot} (2, \text{max})$]

Level: 1

[ ]

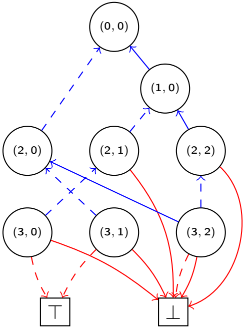
[ ]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: $Q_{\textbf{\textit{red}}}$:

[

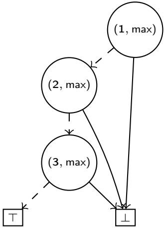
]

Level: 0

[ ]

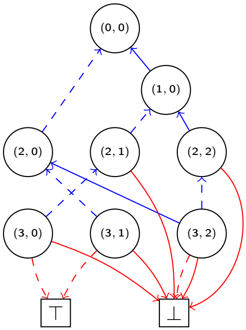
[((0, 0), (2, max), (1, max))]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \land (b)$

Priority Queue: Q_{red}:

[

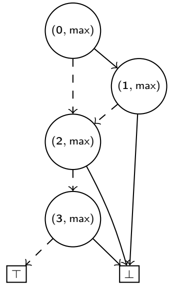
]

Level: 0

[ ]

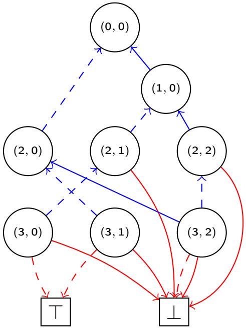
[[(0, 0) $\mapsto$ (0, max)]]

Output:
((0, max), (2, max), (1, max))



(d) $(a) \land (b)$ reduced

Adiar v1.0 : BDD



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[ ]

[ ]

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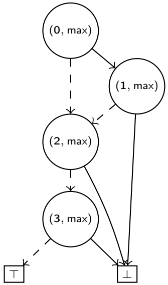

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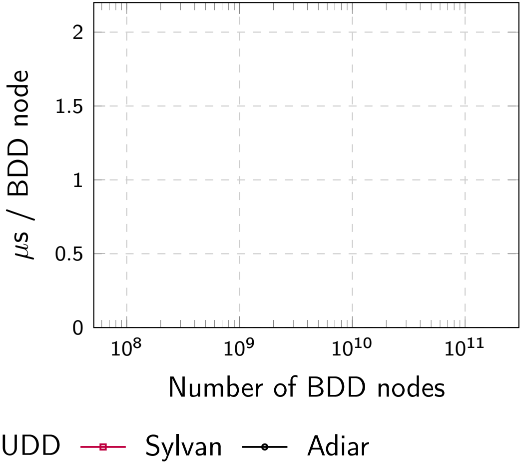
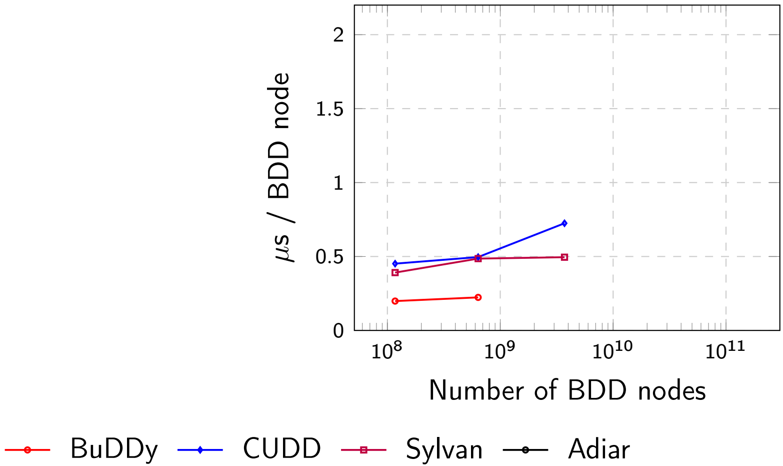


Minimal running time for the *Queens* problems.

Adiar v1.0 : BDD



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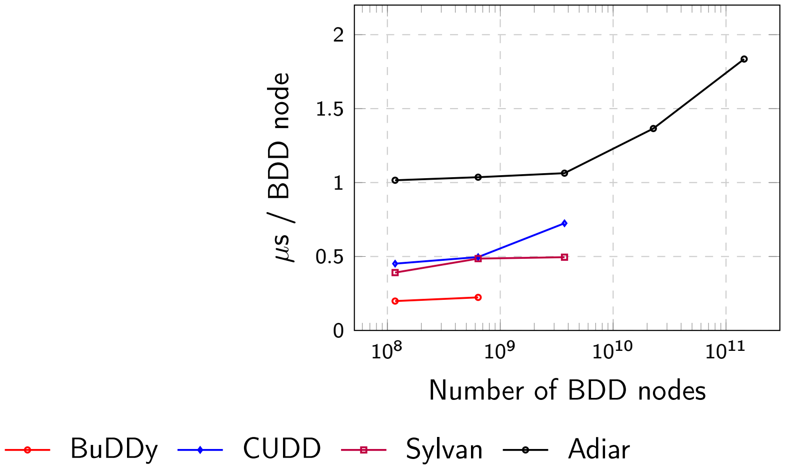

Minimal running time for the *Queens* problems.

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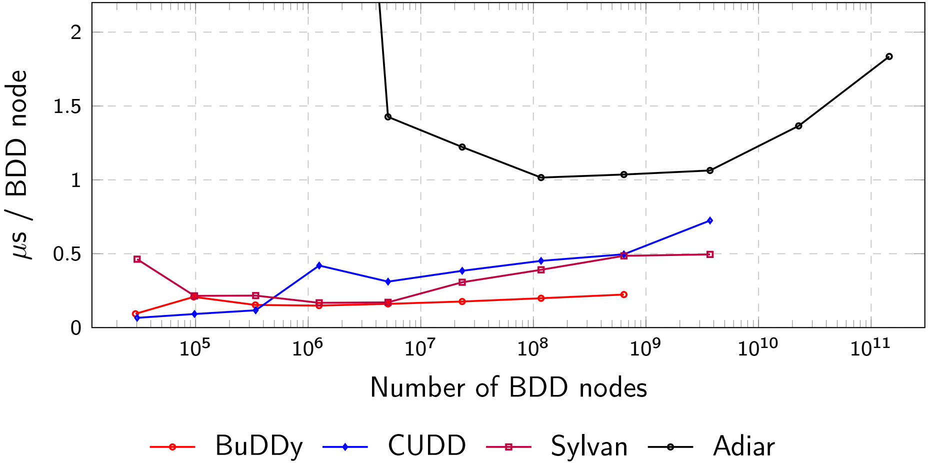
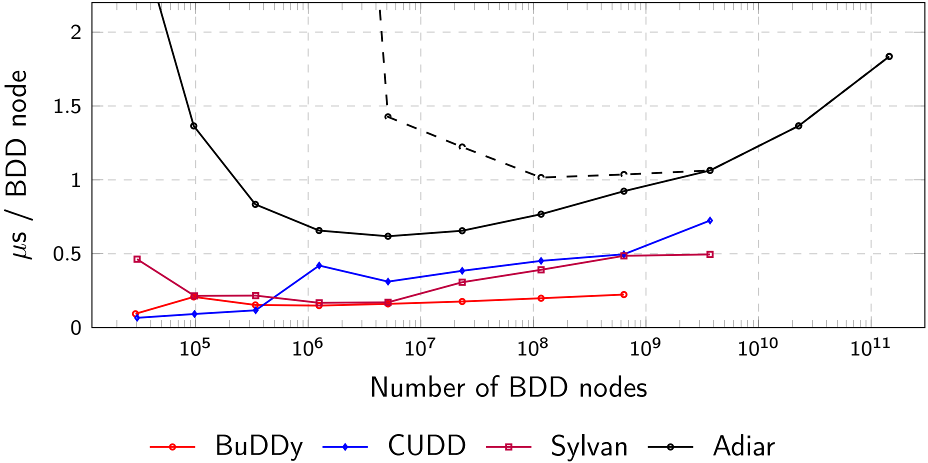


Minimal running time for the *Queens* problems.

Adiar v1.2 : Internal Memory



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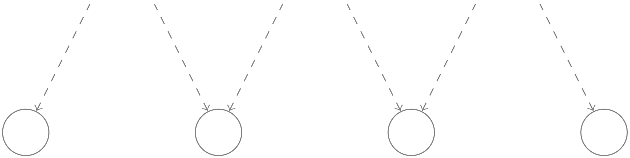

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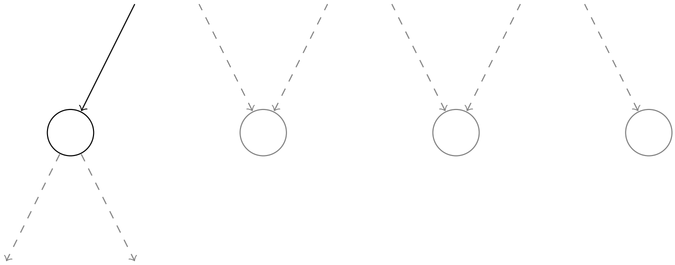

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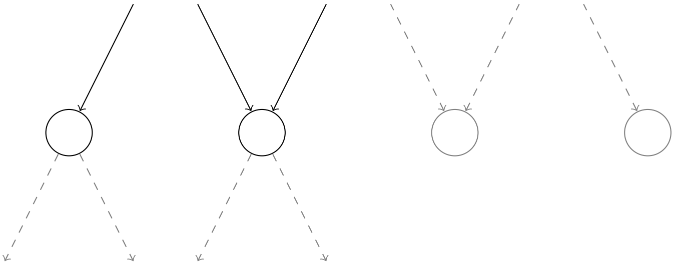

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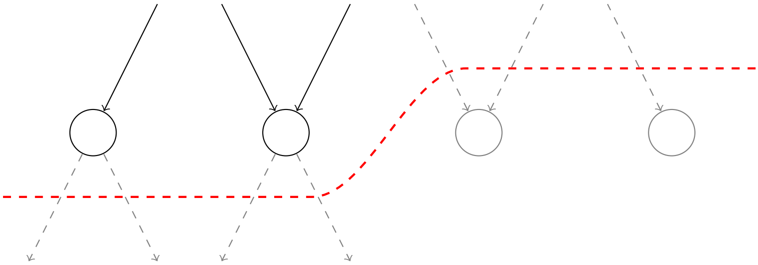

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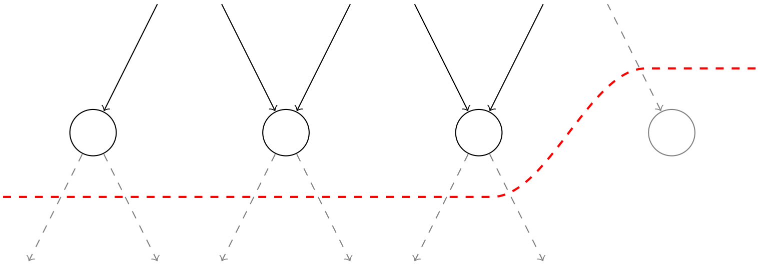

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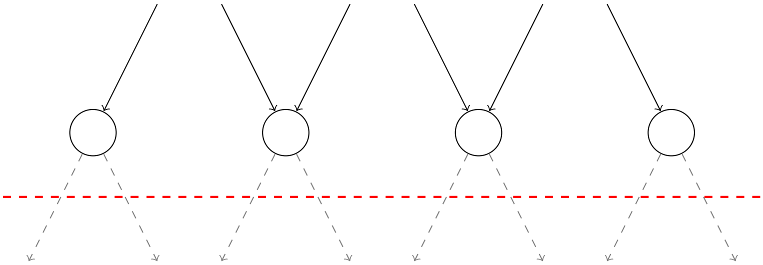
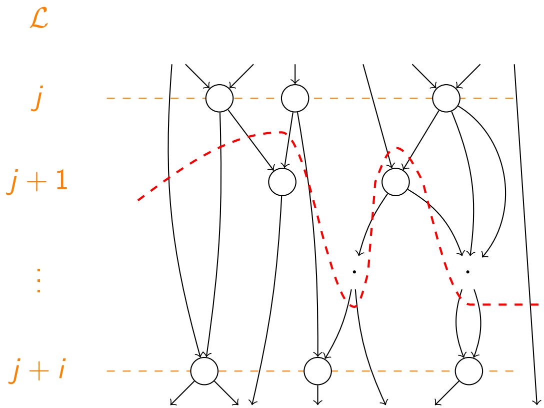
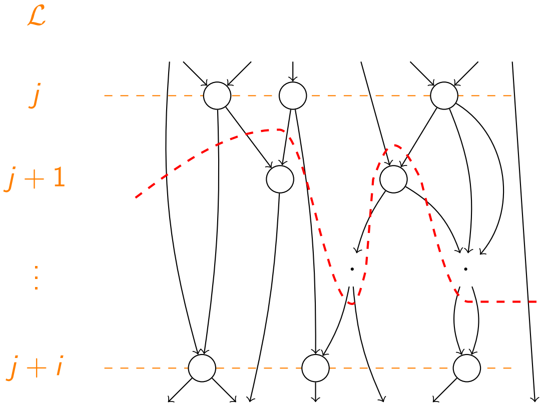


Definition (i-level cut)



Adiar v1.2 : Internal Memory

Definition (i-level cut)



Lemma
The maximum i-level cut problem is in P for $i \in \{1, 2\}$.

Adiar v1.2 : Internal Memory

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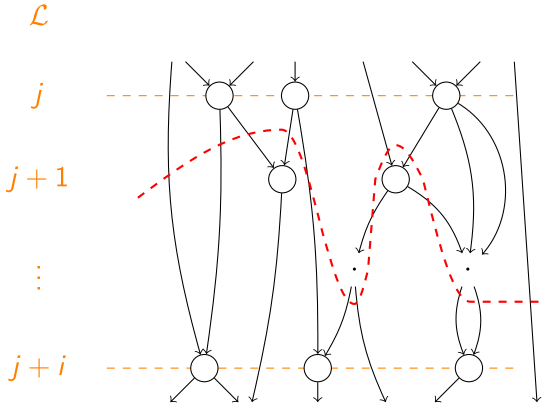

Lemma
The maximum i-level cut problem is in P for $i \in \{1, 2\}$.

Theorem (Lampis, Kaouri, Mitsou 2011)
The maximum i-level cut problem is NP-complete for $i \geq 4$.

Adiar v1.2 : Internal Memory

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Theorem
The maximum (i-level) cut of a BDD with N internal nodes is $N + 1$.

Adiar v1.2 : Internal Memory

Theorem
The maximum (i-level) cut of a BDD with N internal nodes is $N + 1$*.*

Theorem
For $i \in \{1, 2\}$*, the maximum i-level cut of the (unreduced) output of Apply is upper bounded by the product of the inputs' corresponding i-level cuts.*

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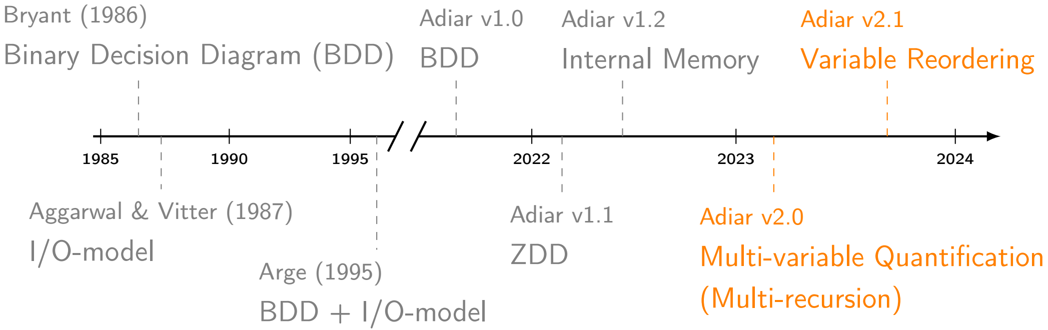
Theorem
For $i \in \{1, 2\}$, *the maximum i-level cut of the (unreduced) output of Apply is upper bounded by the product of the inputs' corresponding i-level cuts.*

Lemma
The maximum 2*-level cut of a BDD is upper bounded by* $\frac{3}{2}$ *its* 1*-level cut.*


Bryant (1986)
Binary Decision Diagram (BDD)
Adiar v1.0
BDD
Adiar v1.2
Internal Memory
Adiar v2.1
Variable Reordering
1985
1990
1995
2022
2023
2024
Aggarwal & Vitter (1987)
I/O-model
Arge (1995)
BDD + I/O-model
Adiar v1.1
ZDD
Adiar v2.0
Multi-variable Quantification
(Multi-recursion)

Adiar v2.0 : Multi-variable Quantification

$$RelProd(S, T) \equiv (\ \exists \vec{x}.\ S(\vec{x}) \land T(\vec{x}, \vec{x'})\ )[\vec{x'}/\vec{x}]$$

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Adiar v2.0 : Multi-variable Quantification

```
exists(f, V)
  if f = ⊥ ∨ f = ⊤
       then f
  else if V ∩ {i ∈ ℕ | i ≥ top(f)} = ∅
       then f
  else if top(f) ∉ V
       then Node { top(f), exists(f.low, V), exists(f.high, V) }
  else let low  = exists(f.low, V)
           high = exists(f.high, V)
       in or(low, high)
```

A recursive multi-variable **exists** operation.

Adiar v2.0 : Multi-variable Quantification

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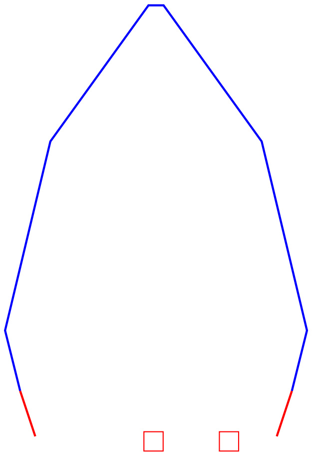

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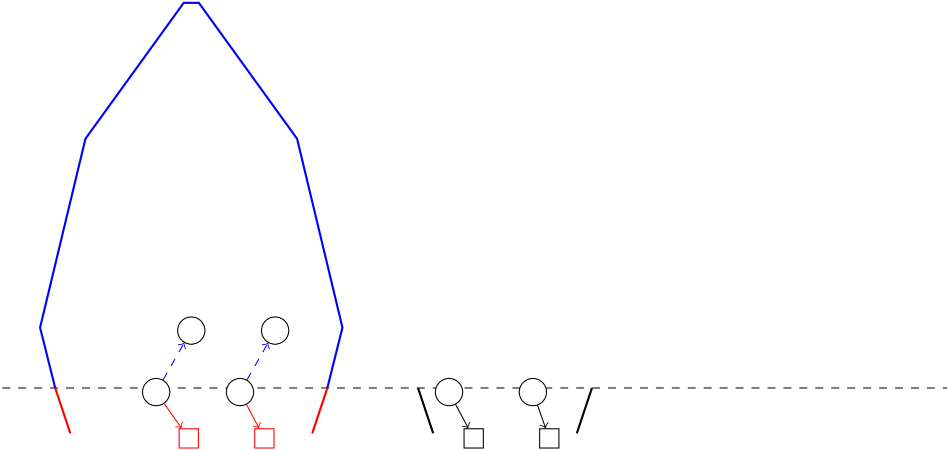

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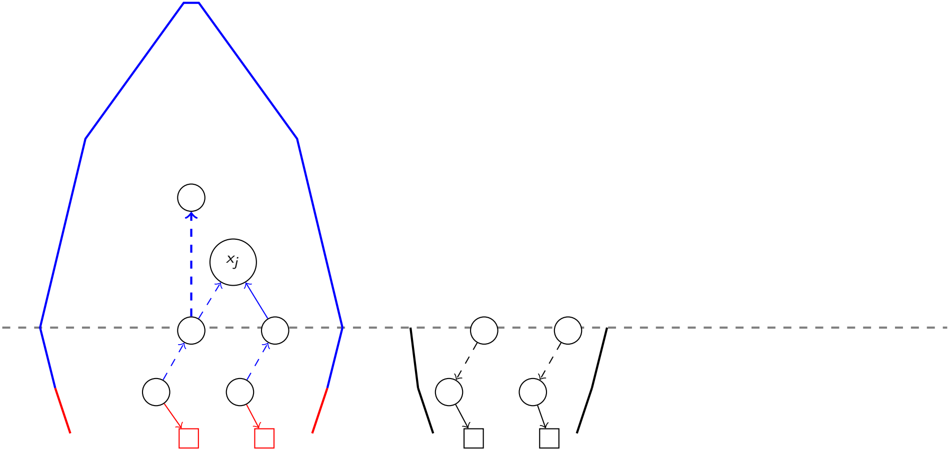

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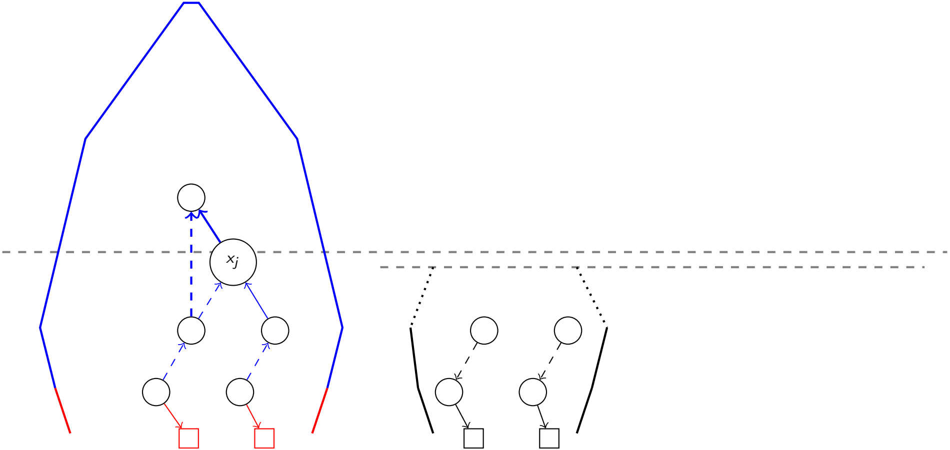

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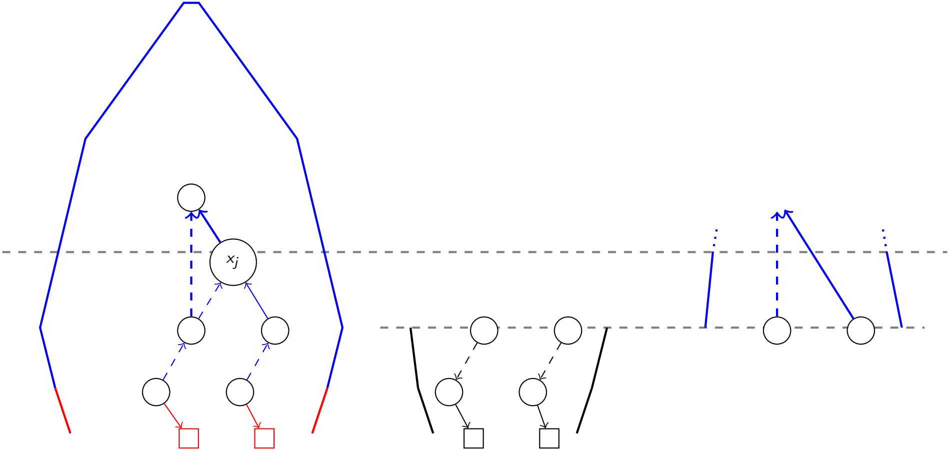

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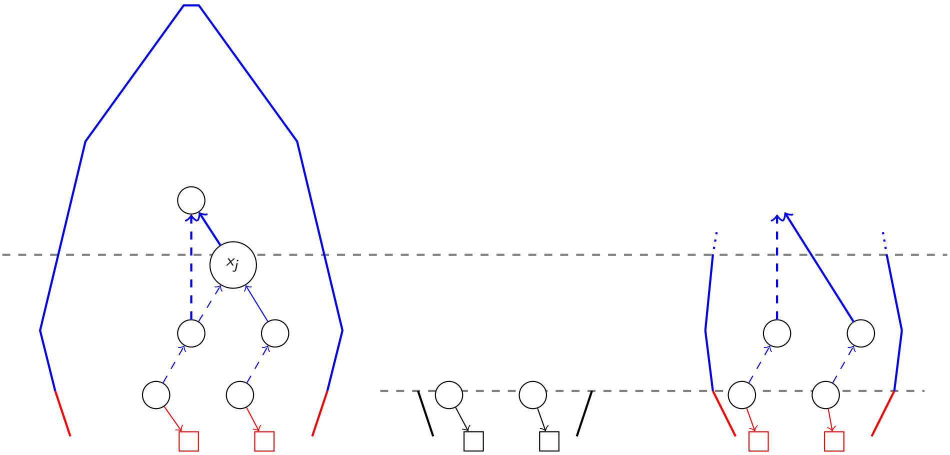

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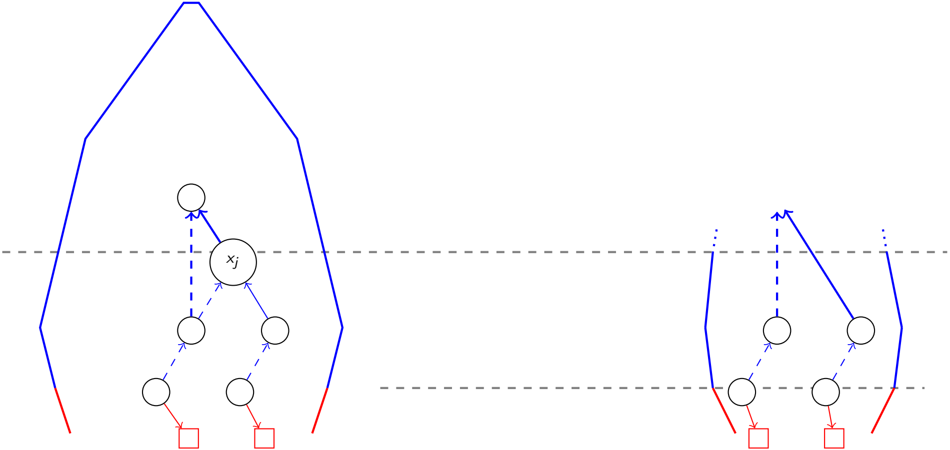

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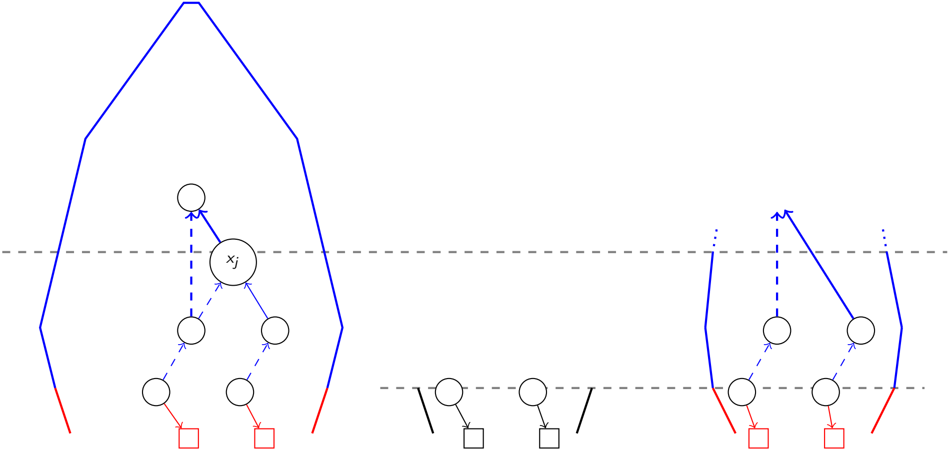

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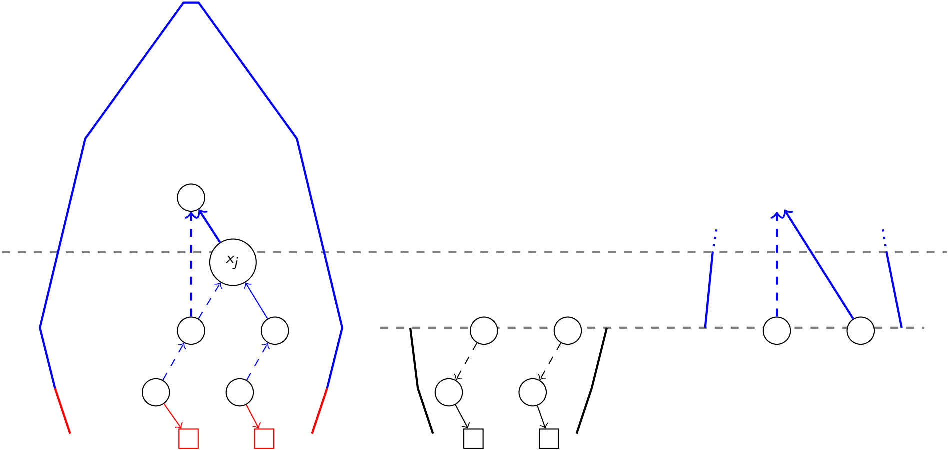

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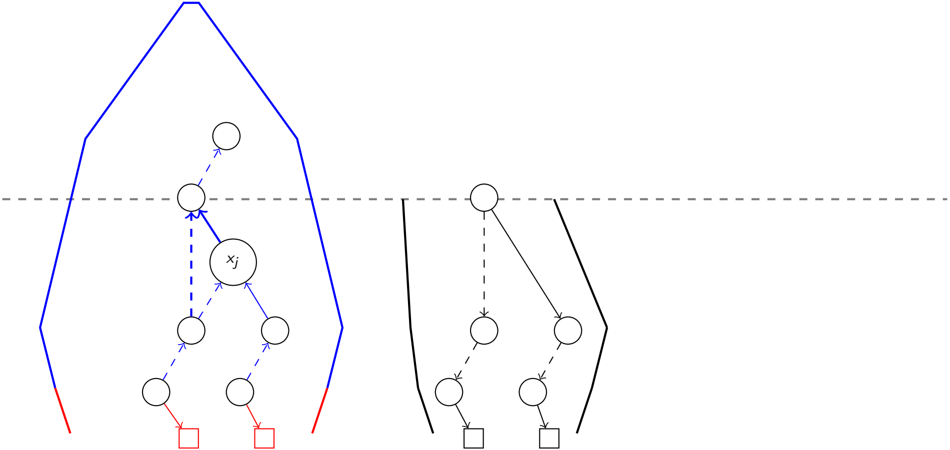

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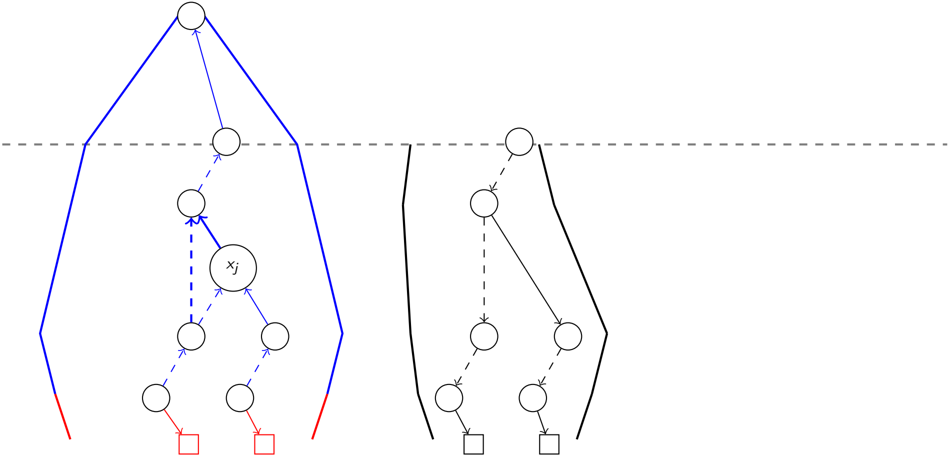

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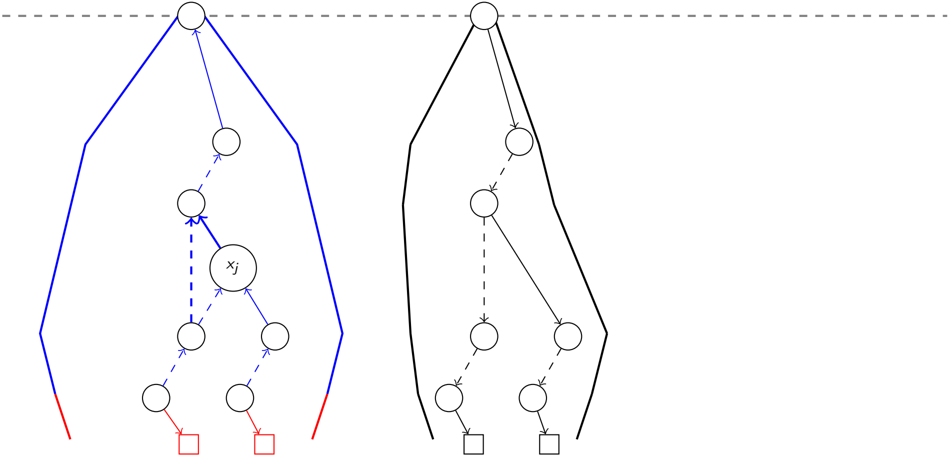

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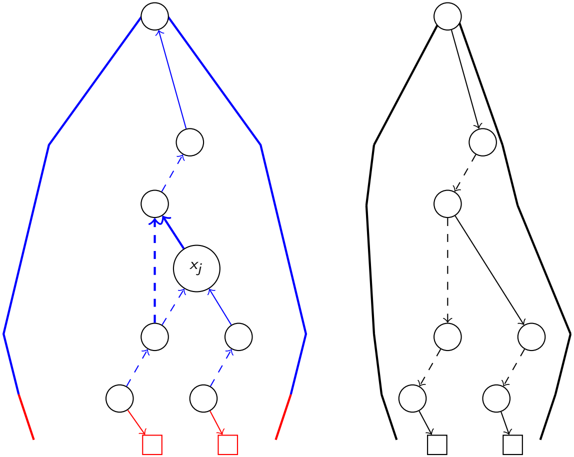


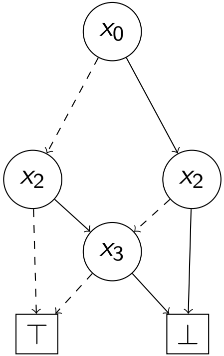
$$\mathit{RelProd}(S, T) \equiv (\ \exists \vec{x}.\ S(\vec{x}) \land T(\vec{x}, \vec{x'})\ )[\vec{x'}/\vec{x}]$$

Adiar v2.1 : Variable Reordering

```
substitute(f , i_map)
  let low  = substitute(f.low)
      high = substitute(f.high)
      i'   = i_map[top(f)]
  in bubble(i', low, high)
```

A recursive **substitute** operation.

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$



$\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

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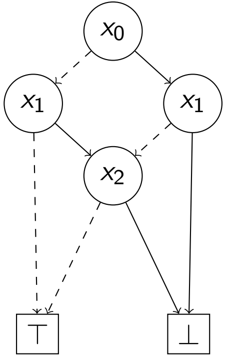

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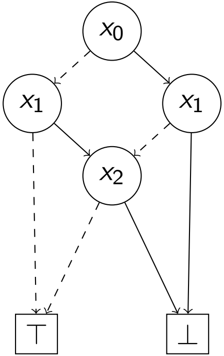

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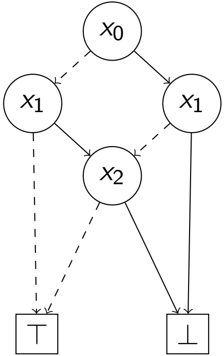

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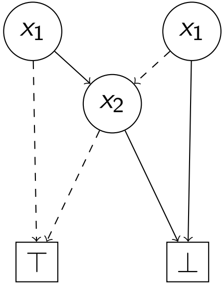

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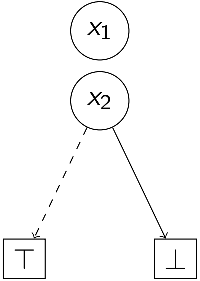

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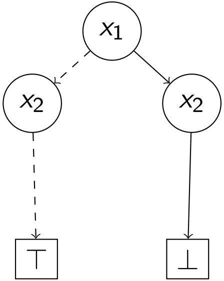

. . .

A recursive **substitute** operation.

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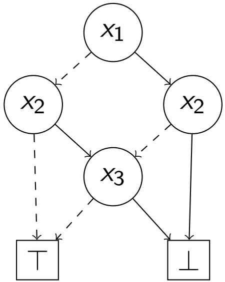

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A recursive **substitute** operation.

Time	Space	I/O
$O(NT)$	$O(NT)$	$O(NT)$

Complexity of depth-first **substitute**

Adiar v2.1 : Variable Reordering

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Time	Space	I/O
$O(NT)$	$O(NT)$	$O(NT)$

Complexity of depth-first **substitute**

Time	Space	I/O
$O(NT \log T)$	$O(N + T)$	$O(N \cdot \text{sort}(T))$

Complexity of level-by-level **substitute**

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)
Given BDD f_π with variable ordering π and remapping of variables $m : \mathbb{N} \to \mathbb{N}$, construct $f'_\pi \equiv f_\pi[x/m(x)]$.

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)
Given BDD f_π *with variable ordering* π *and remapping of variables* $m : \mathbb{N} \to \mathbb{N}$*,*
construct $f'_\pi \equiv f_\pi[x/m(x)]$*.*

Problem (Static Variable Reordering)
Given BDD f_π *with variable ordering* π *and another variable ordering* π'*,*
construct $f_{\pi'} \equiv f_\pi$*.*

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)
Given BDD f_π *with variable ordering* π *and remapping of variables* $m : \mathbb{N} \to \mathbb{N}$*,*
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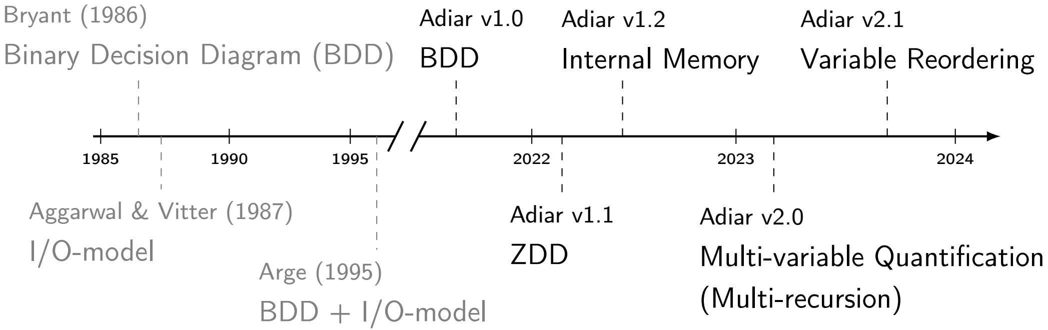
Problem (Static Variable Reordering)
Given BDD f_π *with variable ordering* π *and another variable ordering* π'*,*
construct $f_{\pi'} \equiv f_\pi$*.*

Problem (Dynamic Variable Reordering)
Given BDD f_π *with variable ordering* π*,*
find π' *and construct* $f_{\pi'} \equiv f_\pi$ *such that* $|f_{\pi'}|$ *is minimal.*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Adiar v1.2
Internal Memory
Adiar v2.1
Variable Reordering
1985
1990
1995
2022
2023
2024
Aggarwal & Vitter (1987)
I/O-model
Arge (1995)
BDD + I/O-model
Adiar v1.1
ZDD
Adiar v2.0
Multi-variable Quantification
(Multi-recursion)