

Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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The I/O model by Aggarwal and Vitter '87

For any realistic values of N , M , and B we have that

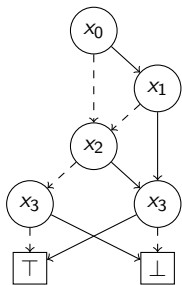
$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

Theorem (Aggarwal and Vitter '87)

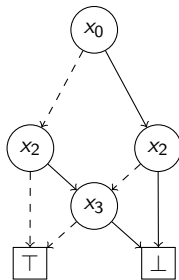
N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 \wedge x_2)$

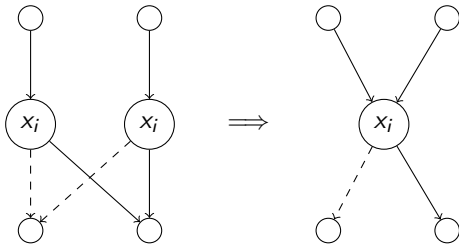
Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)

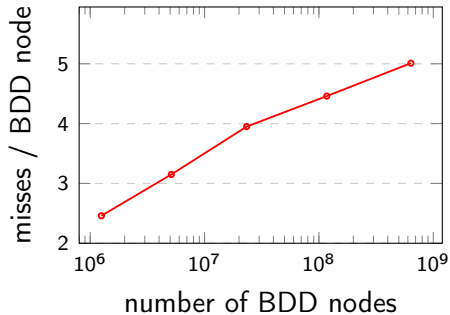
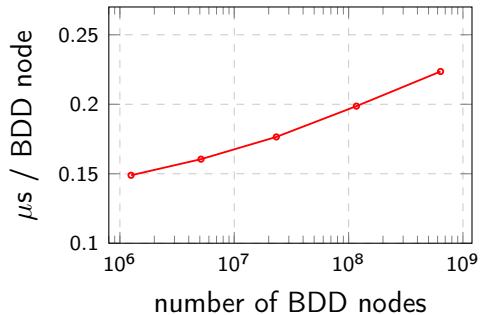
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



(2) Merge duplicate nodes

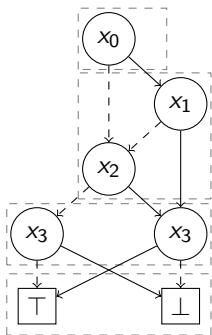


—●— BuDDy

Cache behaviour for the N -Queens problem.



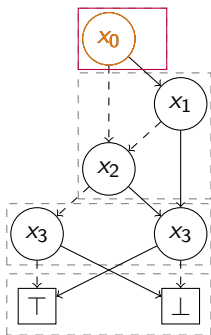
Running time for *Tic-Tac-Toe* with $N = 21$.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

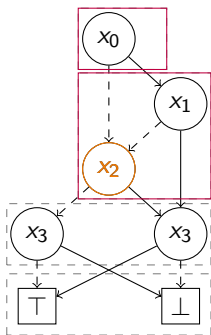
node I/Os	cache lookups
0	0



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

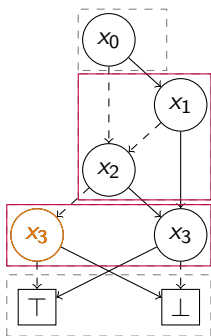
node I/Os	cache lookups
1	1



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

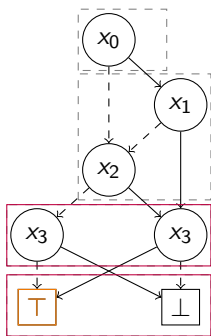
node I/Os	cache lookups
2	2



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

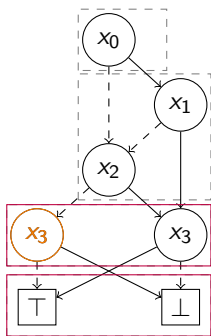
node I/Os	cache lookups
3	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

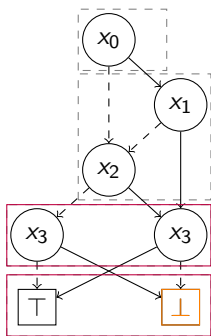
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

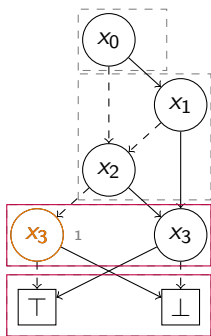
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

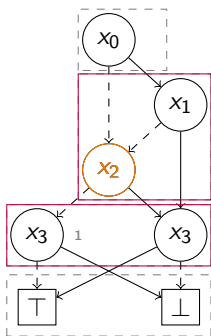
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

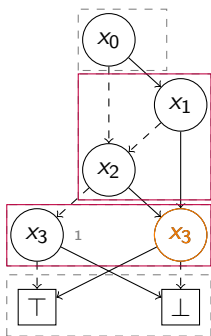
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

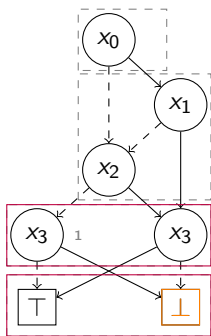
node I/Os	cache lookups
5	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

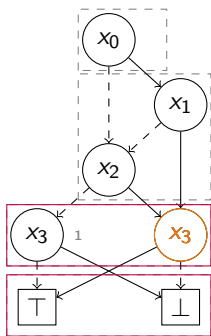
node I/Os	cache lookups
5	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

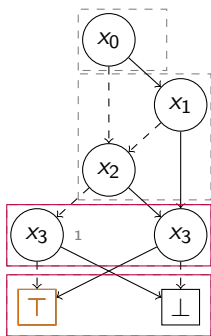
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

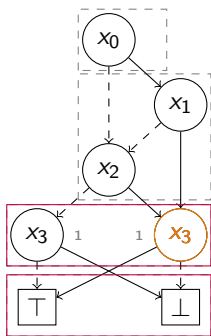
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

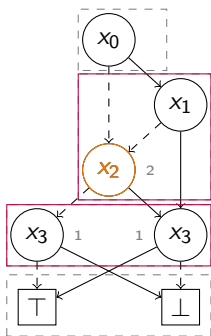
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

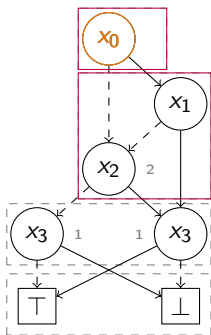
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

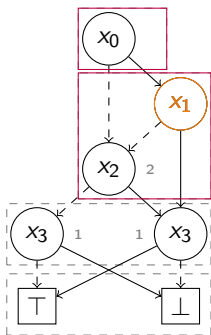
node I/Os	cache lookups
7	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

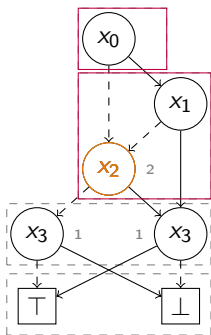
node I/Os	cache lookups
8	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

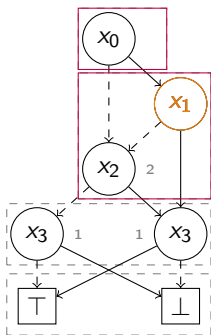
node I/Os	cache lookups
8	5



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

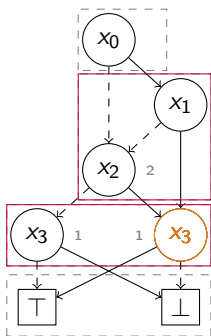
node I/Os	cache lookups
8	6



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

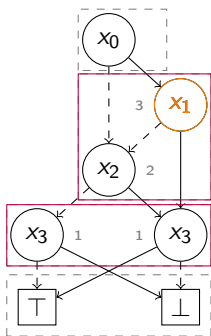
node I/Os	cache lookups
8	6



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

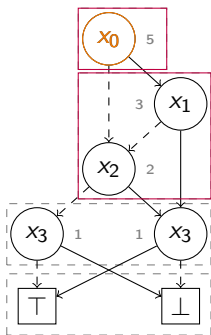
node I/Os	cache lookups
9	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

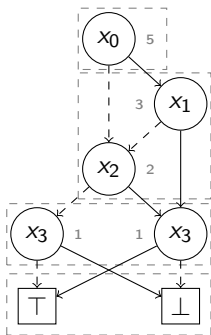
node I/Os	cache lookups
9	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

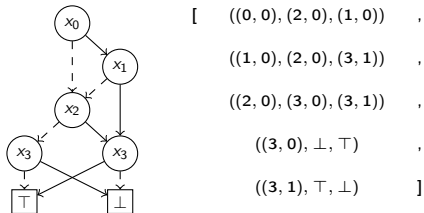
$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

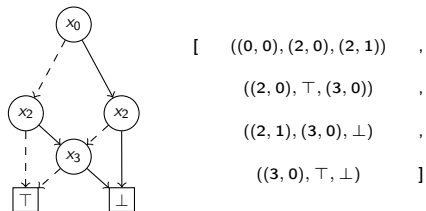




$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



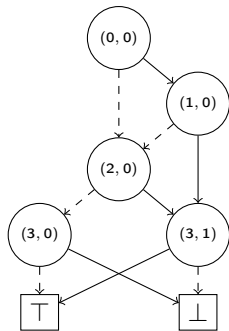
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

CountPaths Example



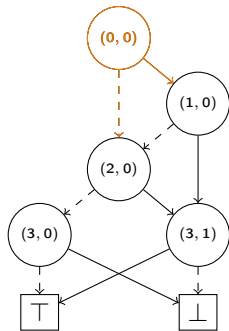
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example



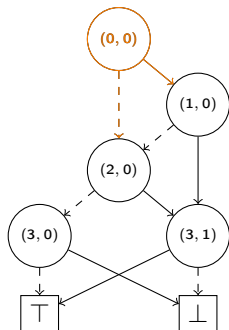
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example

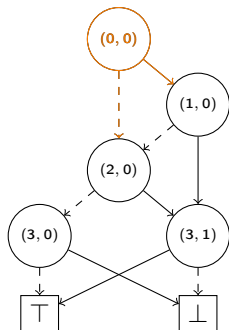


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(1,0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[$((0,0) \xrightarrow{\top} (1,0), 1)$,
 $((0,0) \xrightarrow{\perp} (2,0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(1, 0)

Sum
0

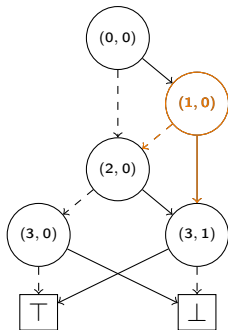
Result
0

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(1, 0)

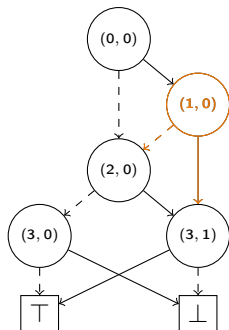
Sum
1

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

Priority Queue: Q_{count} :

[

$((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	0	0

Priority Queue: Q_{count} :

[

$((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
1

Result
0

Priority Queue: Q_{count} :

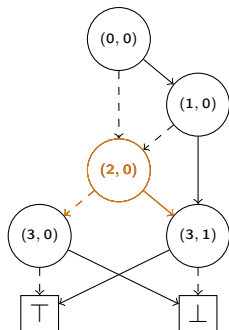
[

$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,

$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
2

Result
0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
2

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

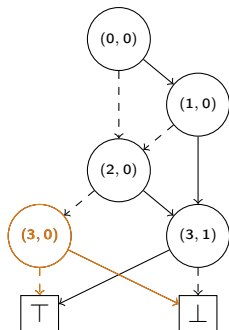
Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
2

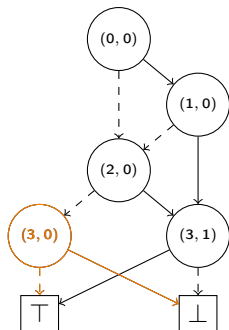
Result
0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	2	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

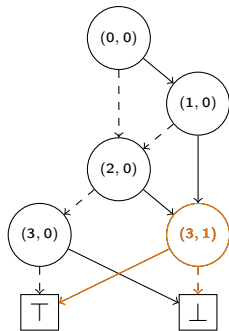
Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
1

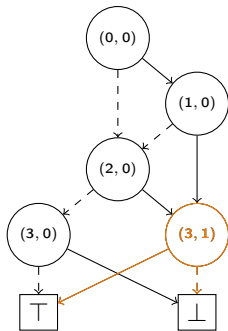
Result
2

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

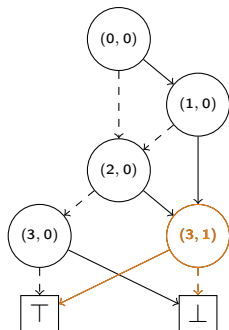
Result
2

Priority Queue: Q_{count} :

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

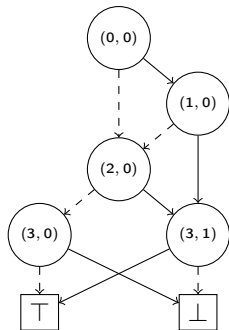
Result
5

Priority Queue: Q_{count} :

[

]

CountPaths Example



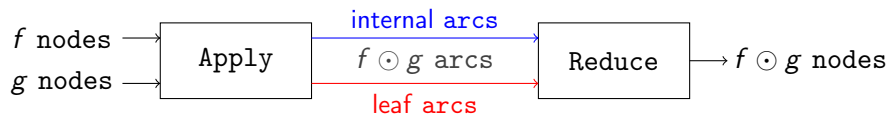
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result
5

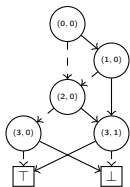
Priority Queue: Q_{count} :

[

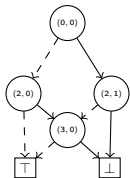
]



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Apply Example (\wedge)

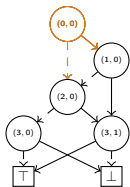


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

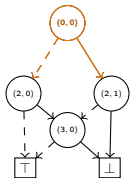
Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: $Q_{app:1}$:

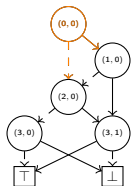
[$(0,0) \xrightarrow{\top} ((1,0), (2,1))$,
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$,



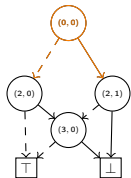
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

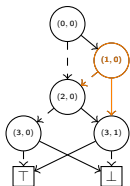
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

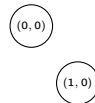
Seek:

$\min((1, 0), (2, 1))$

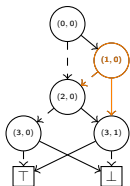
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{T} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{T} ((3, 1), (2, 1))$,

]



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

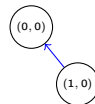
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

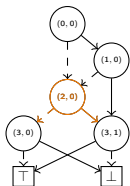
]

Output:

$(0, 0) \xrightarrow{\top} (1, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

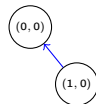
Priority Queue: $Q_{app:1}$:

[

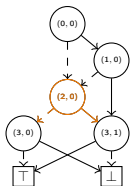
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

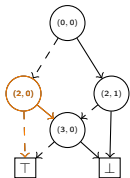
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



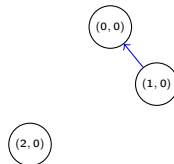
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

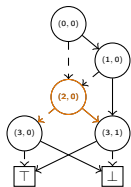
Priority Queue: $Q_{app:1}$:

[
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

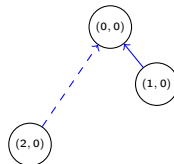
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

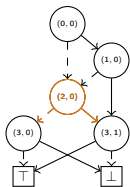
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:

$(0, 0) \xrightarrow{\perp} (2, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app}:1$:

[

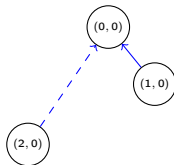
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

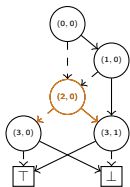
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

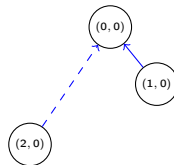
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

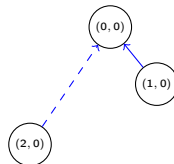
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

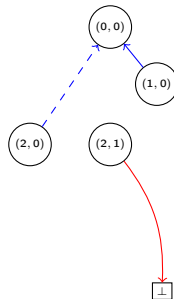
Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:

$(2, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

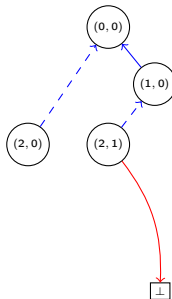
Priority Queue: $Q_{app:2}$:

[

]

Output:

$(1, 0) \xrightarrow{\perp} (2, 1)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

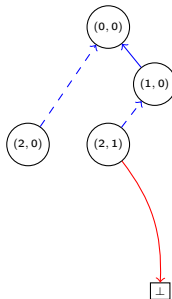
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

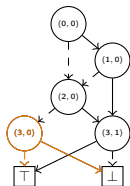
[

]

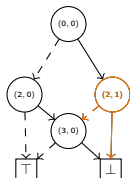
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

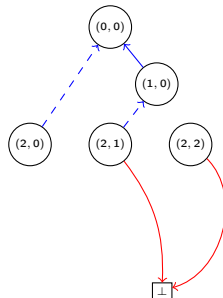
Priority Queue: $Q_{app:2}$:

[

]

Output:

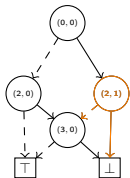
$(2, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

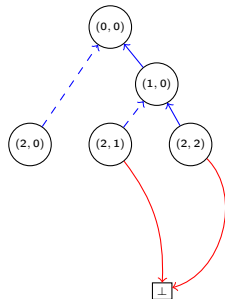
Priority Queue: $Q_{app:2}$:

[

]

Output:

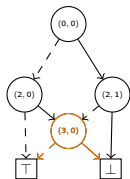
$(1, 0) \xrightarrow{\top} (2, 2)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

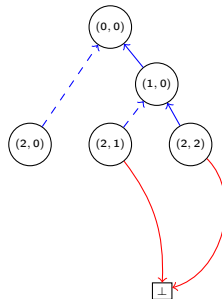
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

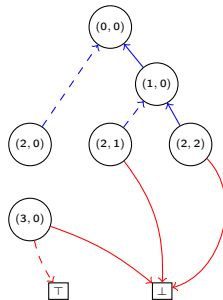
Priority Queue: $Q_{app:2}$:

[

]

Output:

$(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

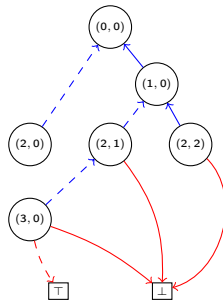
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

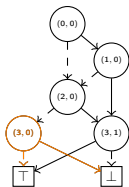
[

Output:
 $(2, 1) \xrightarrow{\perp} (3, 0)$

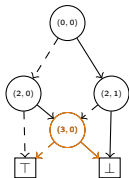


]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

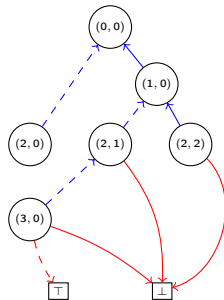
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

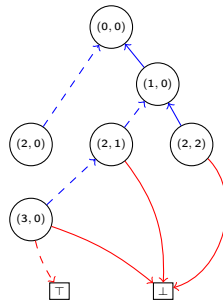
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

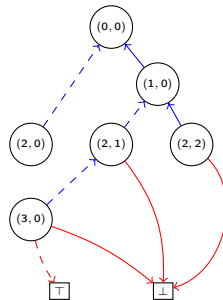
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

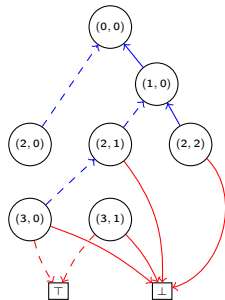
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

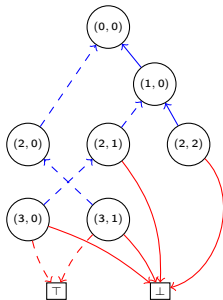
Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

Output:

$(2, 0) \xrightarrow{\perp} (3, 1)$



Priority Queue: $Q_{app:2}$:

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

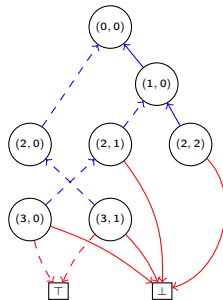
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

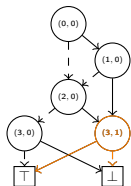
,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

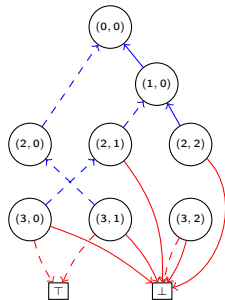
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

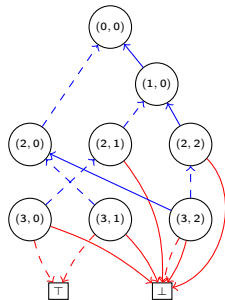
Output:

$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$

]

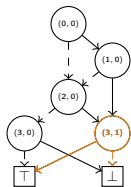
Priority Queue: $Q_{app:2}$:

[

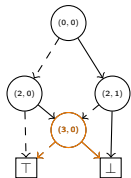


]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

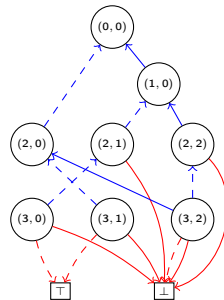
Priority Queue: $Q_{app:2}$:

[

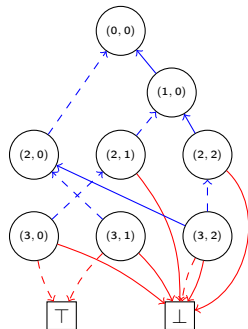
]

]

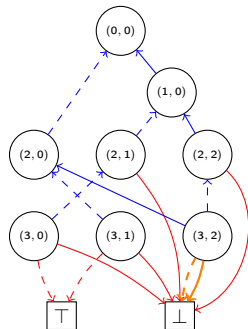
Output:



Reduce Example

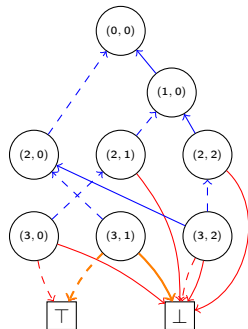


Reduce Example



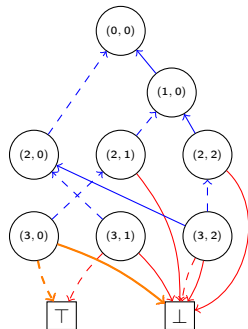
[Level: 3
 [(3, 2) \mapsto \perp]]

Reduce Example



Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad]$

Reduce Example



Level: 3
 [$[(3, 2) \mapsto \perp]$]
 [$((3, 1), \top, \perp)$,
 $((3, 0), \top, \perp)$]

Reduce Example



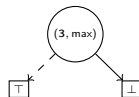
Level: 3

[[(3, 2) \mapsto \perp]]

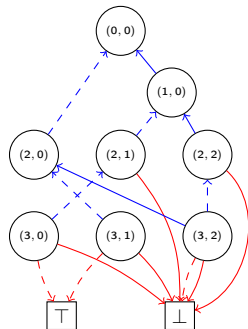
[[(3, 1) \mapsto (3, max)] ,

[((3, 0), \top , \perp)]

Output:
 ((3, max), \top , \perp)



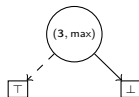
Reduce Example



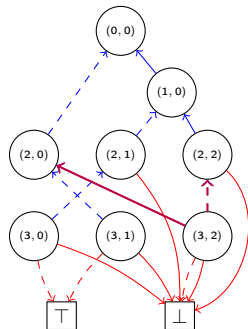
Level: 3

[$[(3, 2) \mapsto \perp]$]
[$[(3, 1) \mapsto (3, \max)]$,
	$[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{T} \perp$,

]

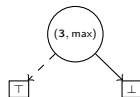
Level: 3

[]

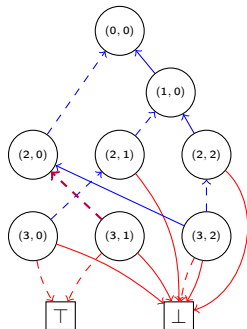
[$[(3, 1) \mapsto (3, \max)]$,

$[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 3

[

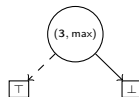
]

[

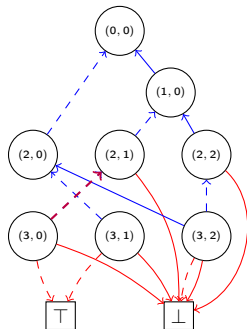
$[(3, 0) \mapsto (3, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

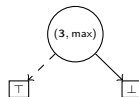
[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

]

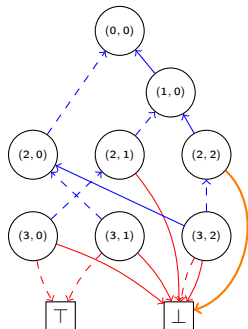
Level: 3

[
 [
 ,
]

Output:



Reduce Example



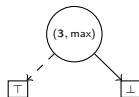
Priority Queue: Q_{red} :

$$\begin{array}{l} (2, 1) \xrightarrow{\perp} (3, \max) \quad , \\ (2, 0) \xrightarrow{\top} \perp \quad , \\ (2, 0) \xrightarrow{\perp} (3, \max) \quad , \end{array}$$

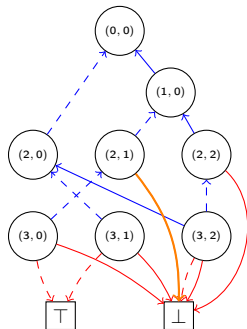
Level: 2

[$[(2, 2) \mapsto \perp]$]

Output:



Reduce Example



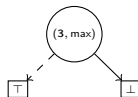
Priority Queue: Q_{red} :

[
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

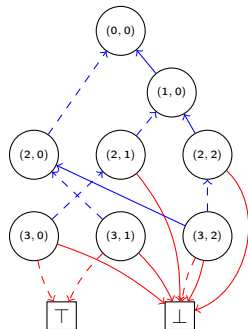
Level: 2

[$[(2, 2) \mapsto \perp]$]
 [$((2, 1), (3, \max), \perp)$,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

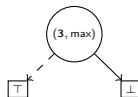
$((2, 1), (3, \max), \perp)$

,

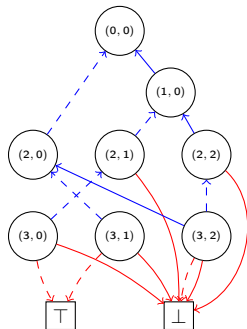
$((2, 0), (3, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

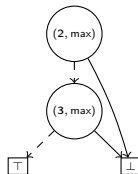
,

$((2, 0), (3, \max), \perp)$

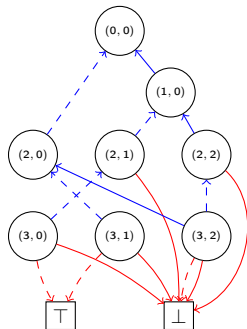
]

Output:

$((2, \max), (3, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

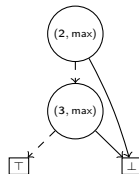
$[(2, 1) \mapsto (2, \max)]$

,

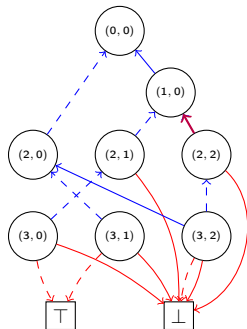
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

]

Level: 2

[

]

[

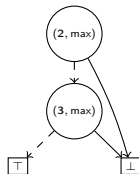
$[(2, 1) \mapsto (2, \max)]$

,

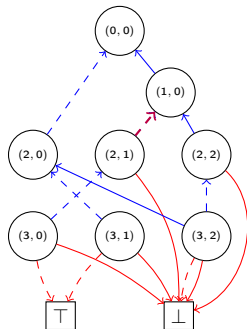
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

[

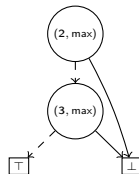
$[(2, 0) \mapsto (2, \max)]$

]

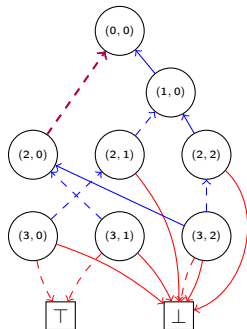
,

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

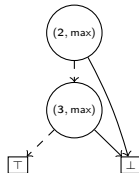
[

[

]

,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

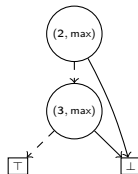
]

[

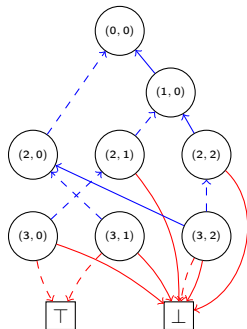
$((1, 0), (2, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

]

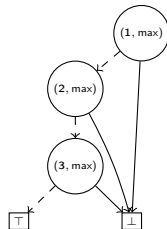
[

$[(1, 0) \mapsto (1, \max)]$

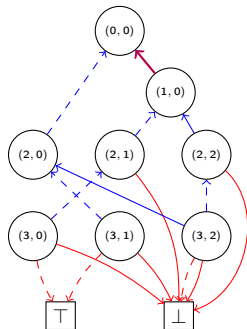
]

Output:

$((1, \max), (2, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{T} (1, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

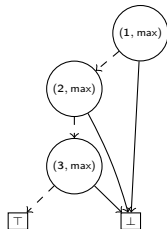
[

]

[

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

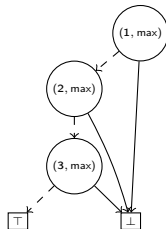
]

[

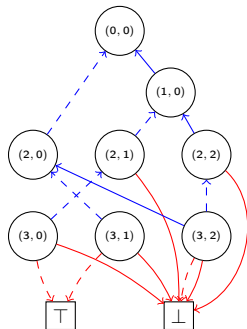
((0, 0), (2, max), (1, max))

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

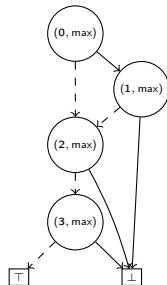
]

$[(0, 0) \mapsto (0, \max)]$

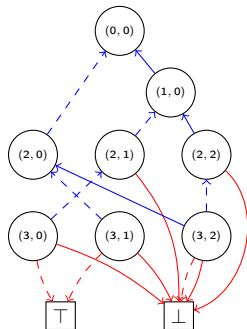
]

Output:

$((0, \max), (2, \max), (1, \max))$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

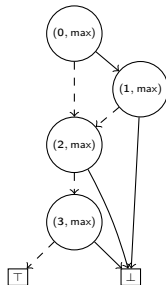
[

]

[

]

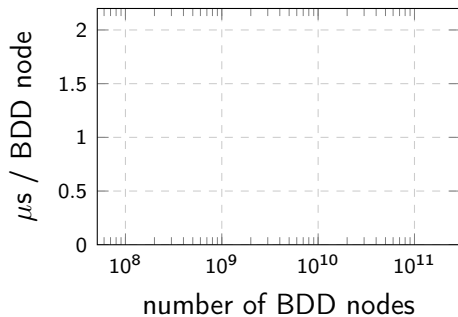
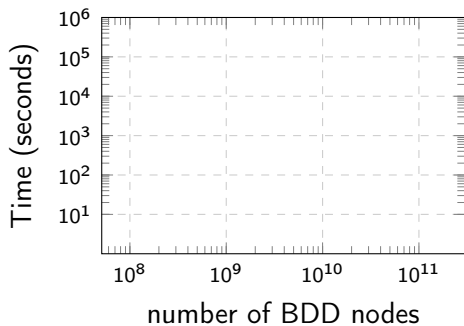
Output:



Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f ? g : h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f _{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f _{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	#paths in f to \top	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min / \max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$

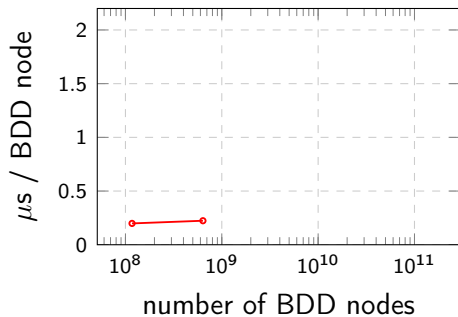
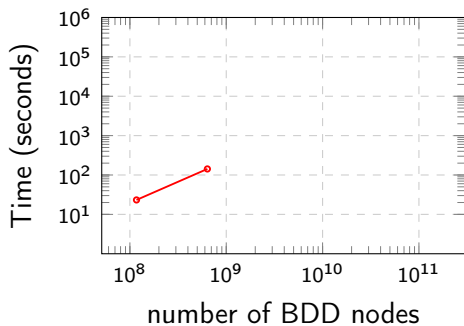
Adiar

github.com/ssoelvsten/adiar



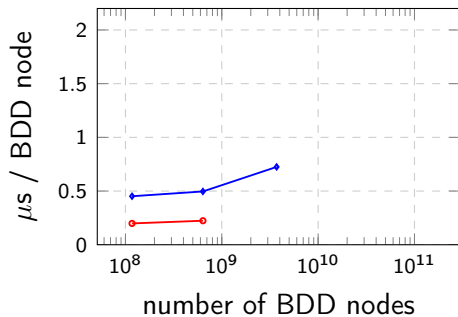
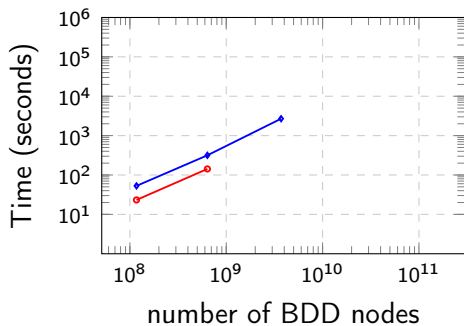
—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



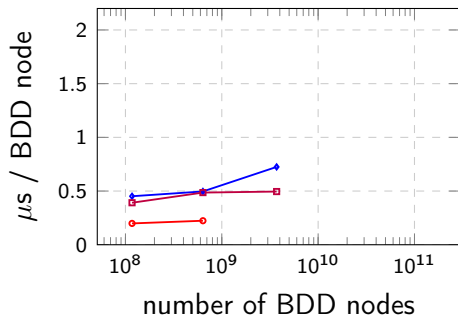
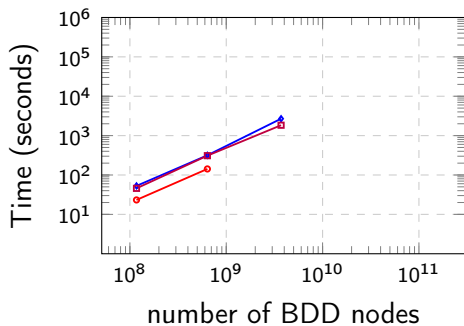
—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



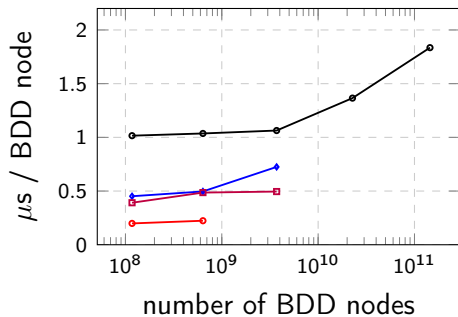
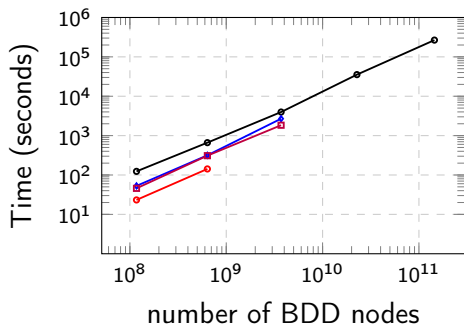
—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.

