

I/O-efficient Manipulation of Binary Decision Diagrams

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Adiar: Binary Decision Diagrams in External Memory. 2022



Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

- CountPaths

- Apply

- Equality Checking

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Apply

Equality Checking



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \text{ ? } x_2 \wedge x_3 : x_2 \wedge x_3)$

Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)

For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



(2) Merge duplicate nodes

```
bdd_apply( $f$ ,  $g$ ,  $\otimes$ ):  
  if  $f, g \in \{\perp, \top\}$   
  then  $f \otimes g$   
  else let  $i = \text{top}(f.\text{var}, g.\text{var})$   
         $t = \text{bdd\_apply}(f[x_i := \top], g[x_i := \top], \otimes)$   
         $e = \text{bdd\_apply}(f[x_i := \perp], g[x_i := \perp], \otimes)$   
  in make_node( $i, t, e$ )
```

```

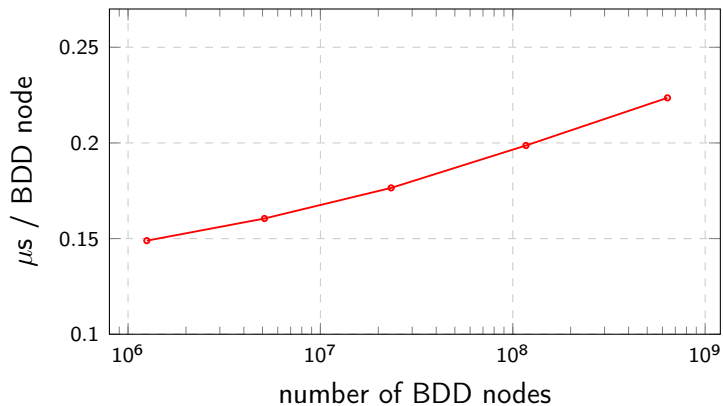
bdd_apply( $f, g, \otimes$ ):
  if  $f, g \in \{\perp, \top\}$ 
  then  $f \otimes g$ 
  else let  $i = \text{top}(f.\text{var}, g.\text{var})$ 
          $t = \text{bdd\_apply}(f[x_i := \top], g[x_i := \top], \otimes)$ 
          $e = \text{bdd\_apply}(f[x_i := \perp], g[x_i := \perp], \otimes)$ 
  in make_node( $i, t, e$ )

```

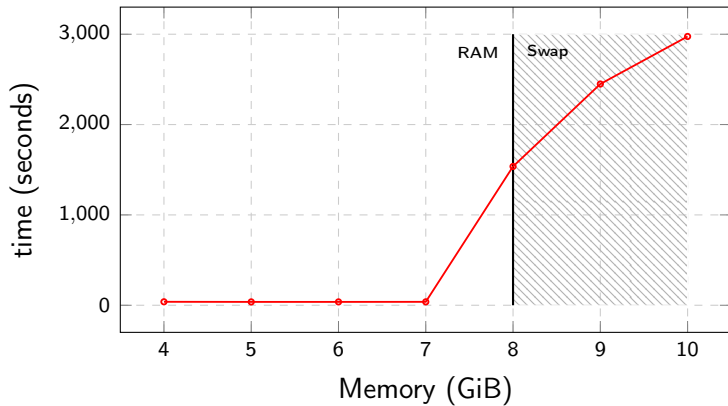
Theorem

`bdd_apply` runs in $O(N_f \cdot N_g)$ time.

- Memoisation (*Computation Cache*) ensures each recursion is computed only once.
- Reduction Rules can be maintained within `make_node(i, t, e)` in $O(1)$ time.
 - 1 Redundancy is resolved with an if-statement.
 - 2 Duplication is avoided with a hash table (*Unique Node Table*).



Running time of *BuDDy* for the *N*-Queens problem.



Running time of *BuDDy* for *Tic-Tac-Toe* with $N = 21$.

Contents

What are Binary Decision Diagrams?

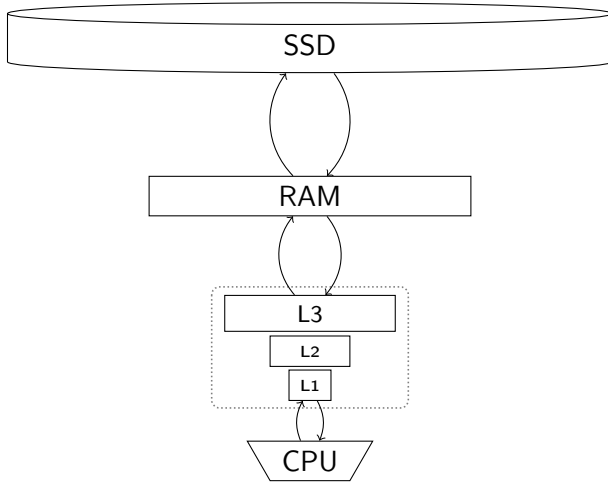
Why do they break?

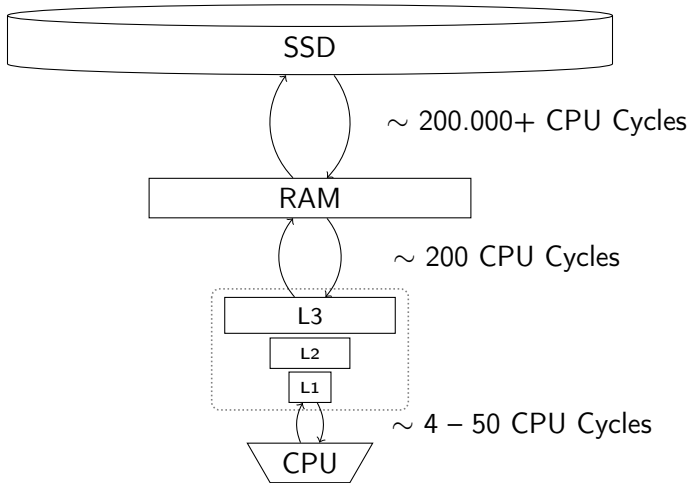
How can we fix it?

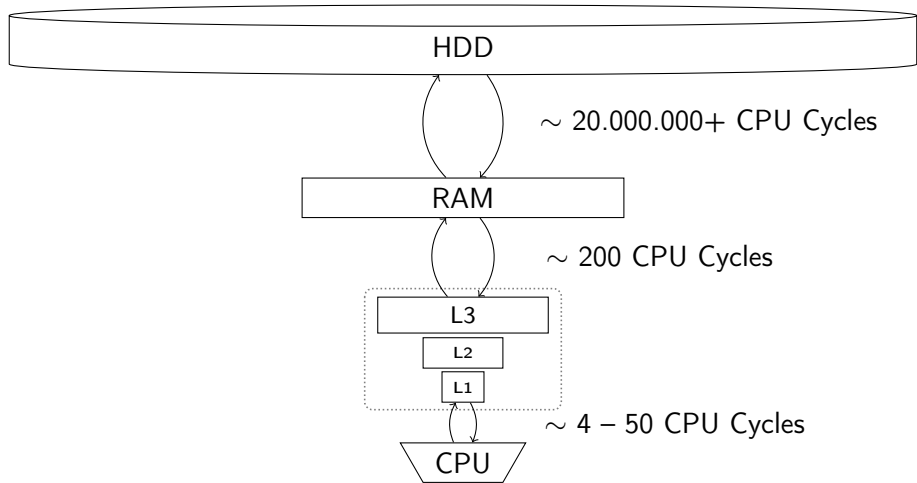
CountPaths

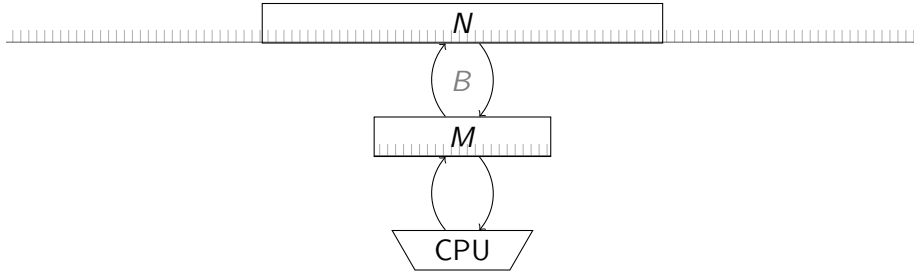
Apply

Equality Checking









The I/O model by Aggarwal and Vitter '87

For any realistic values of N , M , and B we have that

$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

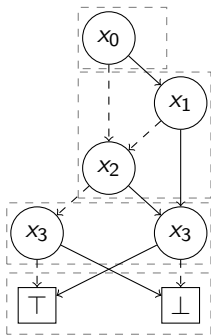
Theorem (Aggarwal and Vitter '87)

N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.

CountPaths : *Example*

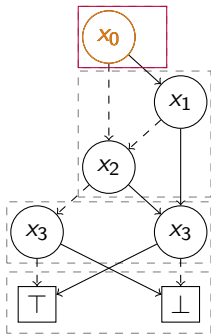


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
0	0

CountPaths : *Example*

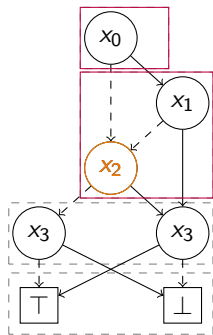


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
1	1

CountPaths : *Example*

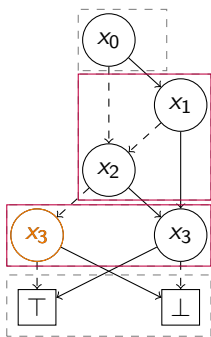


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
2	2

CountPaths : *Example*

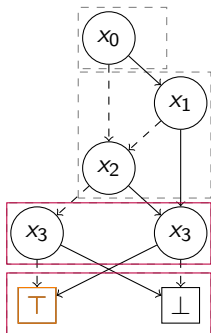


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
3	3

CountPaths : *Example*

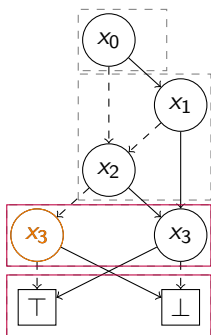


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

CountPaths : *Example*

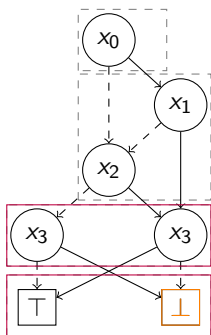


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node I/Os	cache lookups
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CountPaths : *Example*

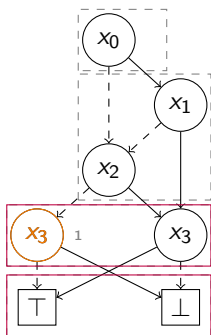


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node I/Os	cache lookups
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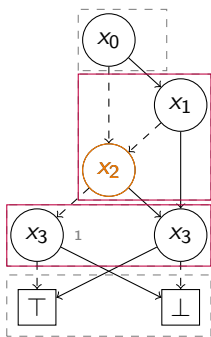


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

CountPaths : *Example*

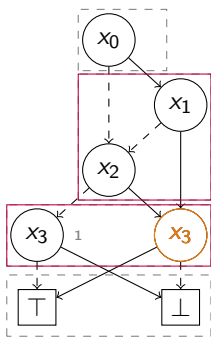


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
5	3

CountPaths : *Example*

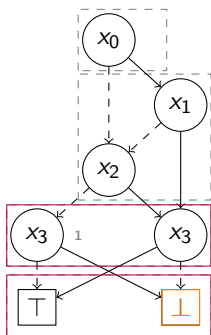


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
5	4

CountPaths : *Example*

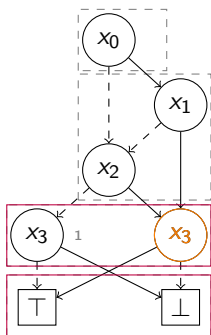


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

CountPaths : *Example*

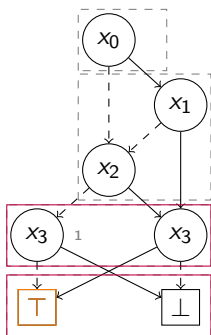


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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node I/Os	cache lookups
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CountPaths : *Example*

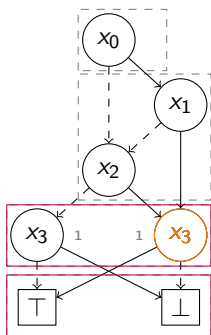


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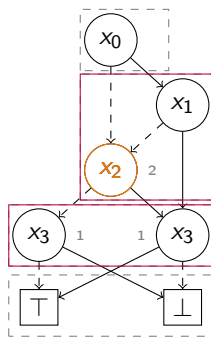


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CountPaths : *Example*

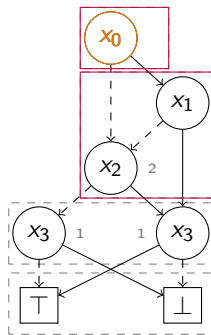


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
7	4

CountPaths : *Example*

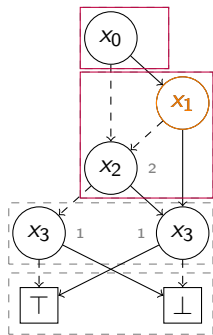


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
8	4

CountPaths : *Example*

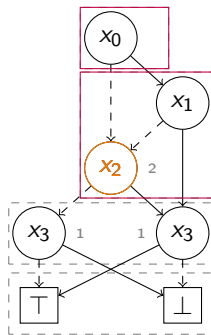


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
8	5

CountPaths : *Example*

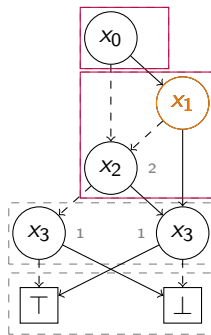


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
8	6

CountPaths : *Example*

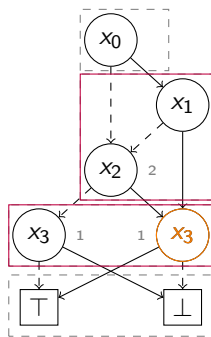


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node I/Os	cache lookups
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CountPaths : *Example*

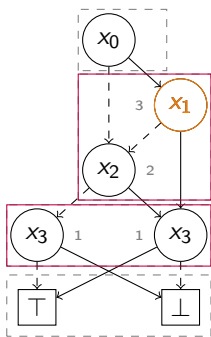


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
9	7

CountPaths : *Example*

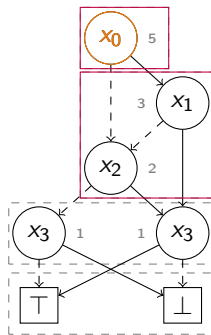


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node I/Os	cache lookups
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CountPaths : *Example*

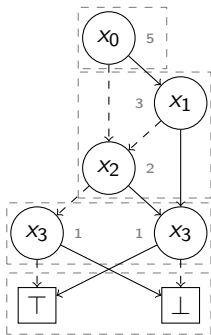


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

Algorithm	Time Complexity
bdd_pathcount	$O(N_f)$
bdd_not	$O(N_f)$
bdd_restrict	$O(N_f)$
bdd_apply	$O(N_f \cdot N_g)$
bdd_equal	$O(1)$

Algorithm	I/O-Complexity
bdd_pathcount	$O(N_f)$
bdd_not	$O(N_f)$
bdd_restrict	$O(N_f)$
bdd_apply	$O(N_f \cdot N_g)$
bdd_equal	$O(1)$

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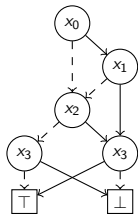
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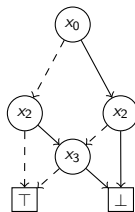


$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



[((0, 0), (2, 0), (1, 0)) ,
 ((1, 0), (2, 0), (3, 1)) ,
 ((2, 0), (3, 0), (3, 1)) ,
 ((3, 0), ⊤, ⊥) ,
 ((3, 1), ⊥, ⊤)]

(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

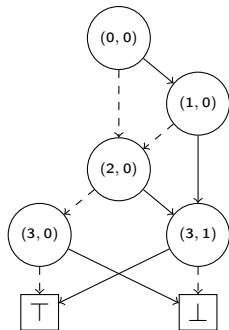


[((0, 0), (2, 0), (2, 1)) ,
 ((2, 0), ⊥, (3, 0)) ,
 ((2, 1), (3, 0), ⊤) ,
 ((3, 0), ⊤, ⊥)]

(b) $\neg(x_0 ? x_2 \wedge x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

CountPaths : *Example*



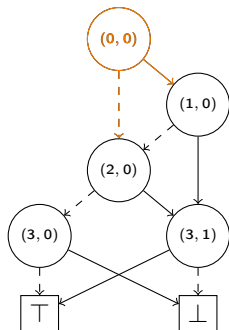
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths : *Example*



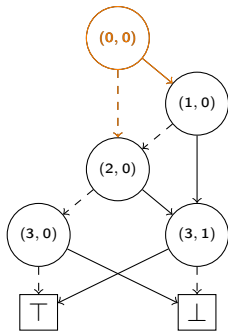
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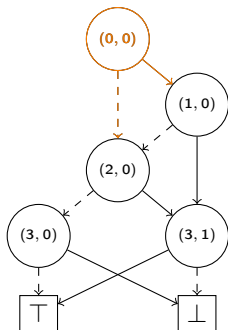
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[$((0,0) \xrightarrow{\top} (1,0), 1)$,
 $((0,0) \xrightarrow{\perp} (2,0), 1)$,

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	0	0

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
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]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

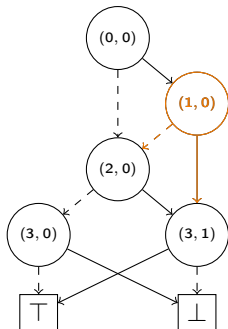
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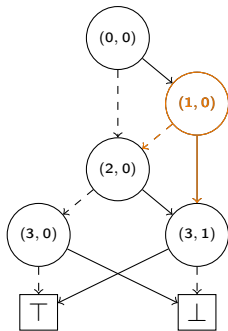
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CountPaths : *Example*



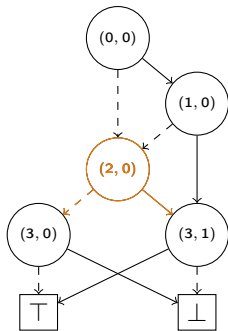
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	0	0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
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CountPaths : *Example*



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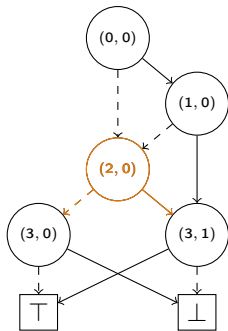
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
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Priority Queue: Q_{count} :

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 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
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]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

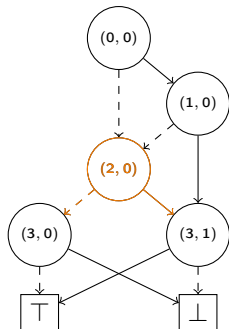
Seek	Sum	Result
(2, 0)	2	0

Priority Queue: Q_{count} :

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CountPaths : *Example*



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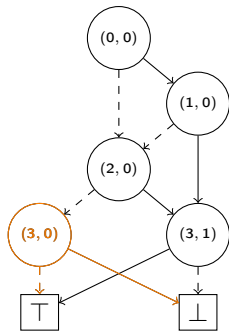
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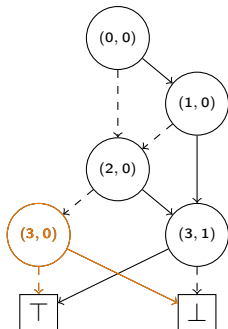
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[

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 $((1, 0) \xrightarrow{\top} (3, 1), \quad 1) \quad ,$
 $((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad]$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

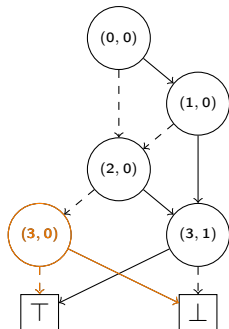
Seek	Sum	Result
(3, 0)	2	0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	2	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	0	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

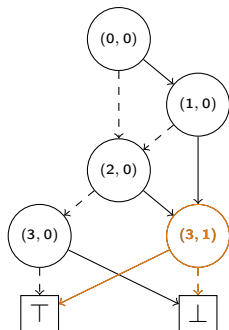
Seek	Sum	Result
(3, 1)	0	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

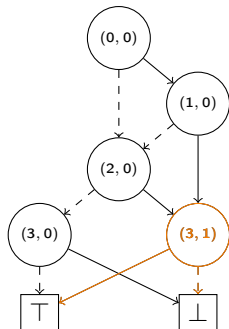
Seek	Sum	Result
(3, 1)	1	2

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\tau} (3, 1), \quad 2) \quad]$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

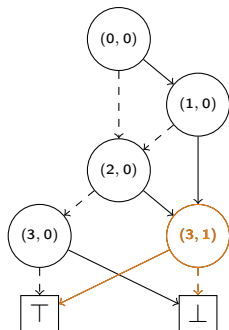
Seek	Sum	Result
(3, 1)	3	2

Priority Queue: Q_{count} :

[

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	3	5

Priority Queue: Q_{count} :

[

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result

5

Priority Queue: Q_{count} :

[

]

Contents

What are Binary Decision Diagrams?

Why do they break?

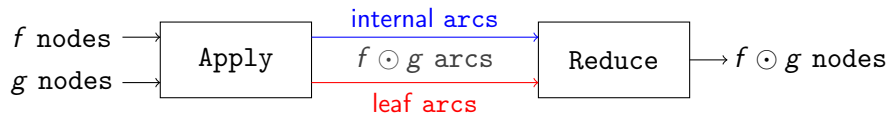
How can we fix it?

CountPaths

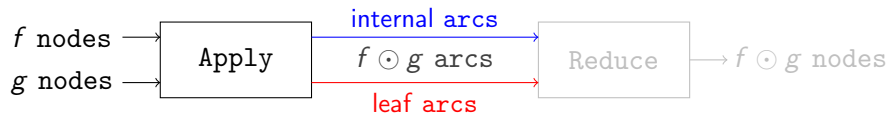
Apply

Equality Checking

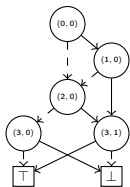
Apply



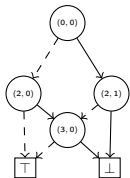
Apply



Apply : *Example*



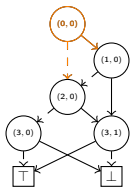
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



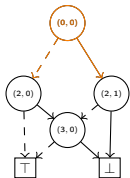
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Apply : *Example*



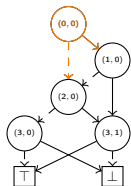
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



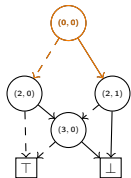
(b) $\neg(x_0 \oplus x_2 \vee x_3) \wedge (x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

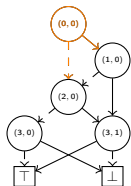
[$(0,0) \xrightarrow{\top} ((1,0), (2,1))$,
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$,



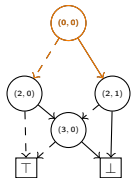
]

(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

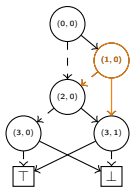
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



]

(c) $(a) \wedge (b)$

Apply : Example

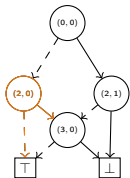


Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:
 [$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

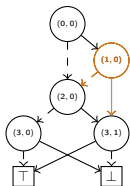


(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

]

(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

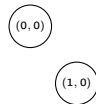
Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

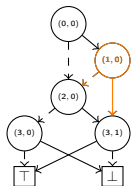
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

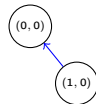
Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:
 $(0, 0) \xrightarrow{\top} (1, 0)$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

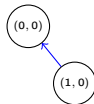
Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

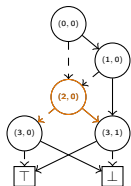
]

Output:

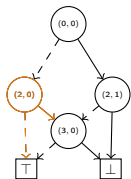


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app}:1$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

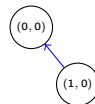
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

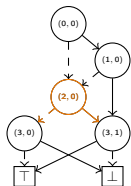
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

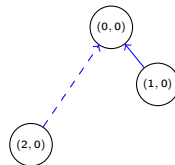
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

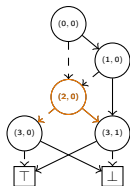
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:
 $(0, 0) \xrightarrow{\perp} (2, 0)$

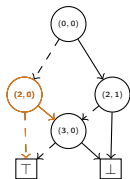


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

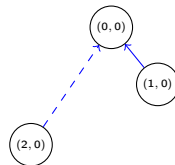
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

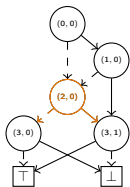
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

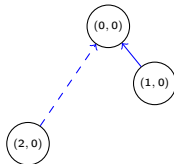
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

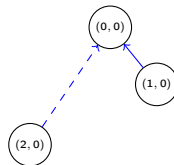
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:

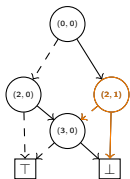


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

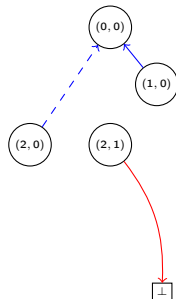
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:
 $(2, 1) \xrightarrow{\top} \perp$

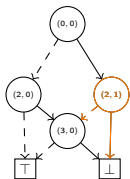


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

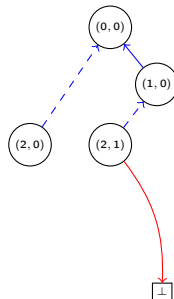
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(1, 0) \xrightarrow{\perp} (2, 1)$

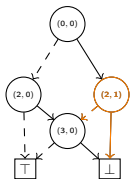


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

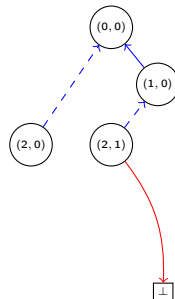
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

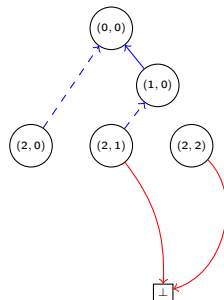
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(2, 2) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

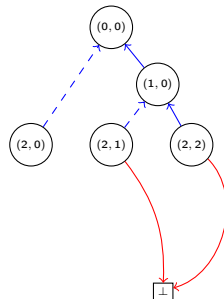
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

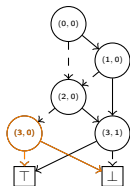
]

Output:
 $(1, 0) \xrightarrow{\top} (2, 2)$

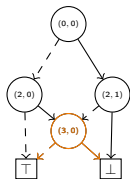


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

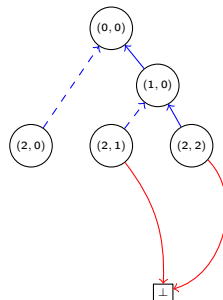
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

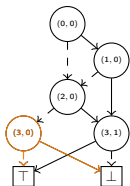
]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

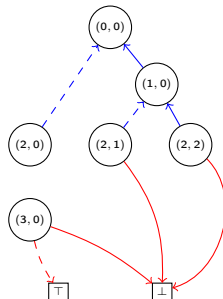
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

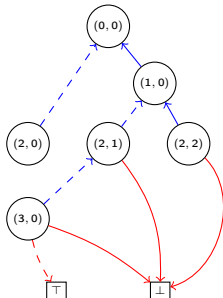
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(2, 1) \xrightarrow{\perp} (3, 0)$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

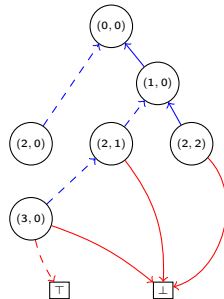
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:

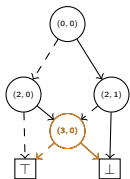


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

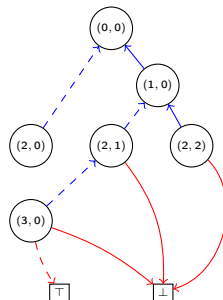
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

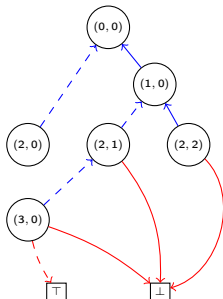
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

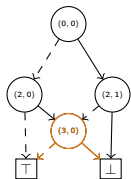


(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

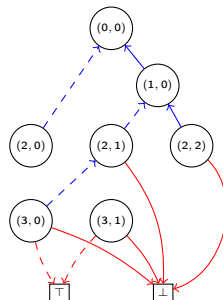
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

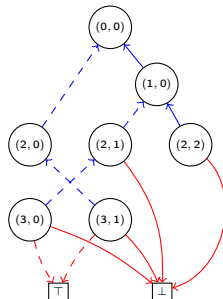
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(2, 0) \xrightarrow{\perp} (3, 1)$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

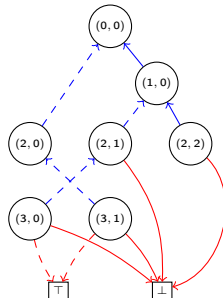
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

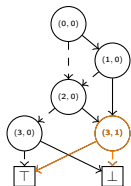
]

Output:



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

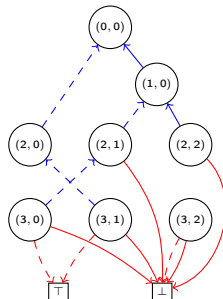
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

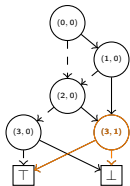
]

Output:
 $(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

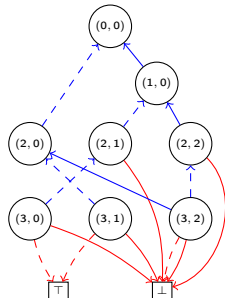
Priority Queue: $Q_{app:2}$:

[

]

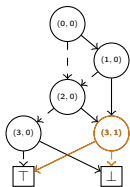
Output:

$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$



(c) $(a) \wedge (b)$

Apply : Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

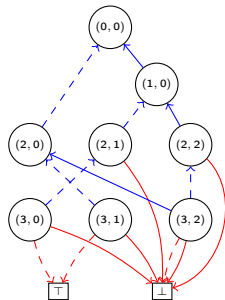
Priority Queue: $Q_{app:2}$:

[

]

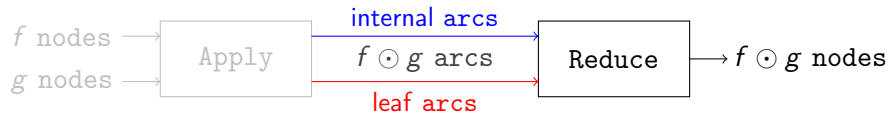
]

Output:



(c) $(a) \wedge (b)$

Apply



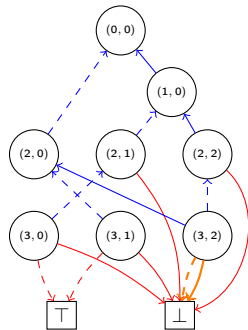
Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)

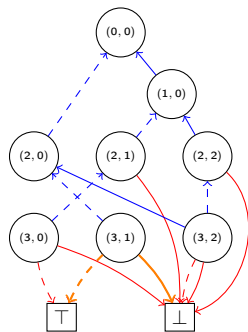


(c) $(a) \wedge (b)$

[Level: 3
 $[(3, 2) \mapsto \perp]$]

(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)

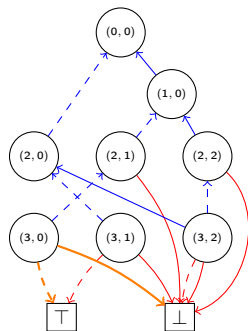


(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad]$

(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad$
 $\quad ((3, 0), \top, \perp) \quad]$

(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)

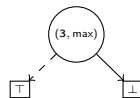


(c) $(a) \wedge (b)$

Level: 3

[$[(3, 2) \mapsto \perp]$]
[$[(3, 1) \mapsto (3, \max)]$,
	$((3, 0), \top, \perp)$]

Output:
 $((3, \max), \top, \perp)$



(d) $(a) \wedge (b)$ reduced

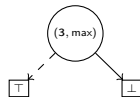
Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

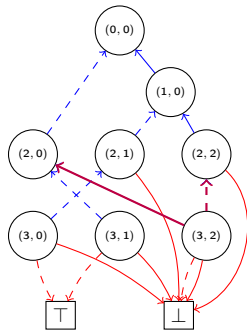
Level: 3
 $\left[\begin{array}{c} [(3, 2) \mapsto \perp] \end{array} \right]$
 $\left[\begin{array}{c} [(3, 1) \mapsto (3, \max)] \\ [(3, 0) \mapsto (3, \max)] \end{array} \right],$

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

]

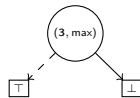
Level: 3

[

$[(3, 1) \mapsto (3, \max)]$,

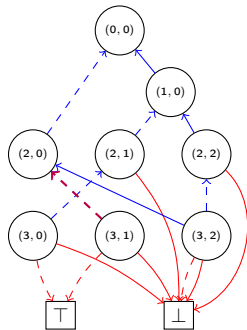
$[(3, 0) \mapsto (3, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{T} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 3

[

]

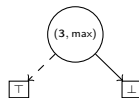
[

$[(3, 0) \mapsto (3, \max)]$

,

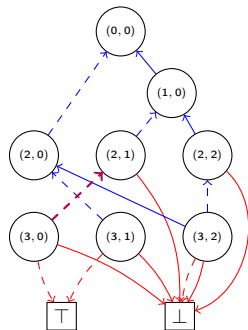
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

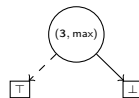
Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

Level: 3

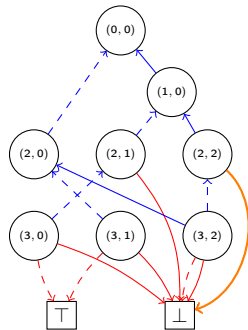
[
 [
 ,
]
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

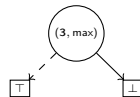
Priority Queue: Q_{red} :

[
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

Level: 2

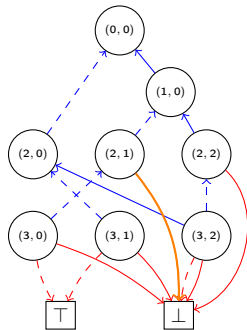
[$((2, 2) \mapsto \perp)$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

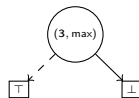
[

$((2, 1), (3, \max), \perp)$

,

]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

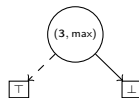
$((2, 1), (3, \max), \perp)$

,

$((2, 0), (3, \max), \perp)$

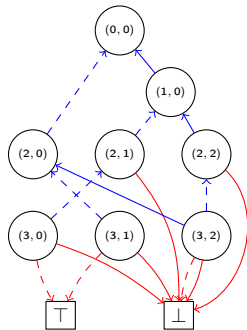
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

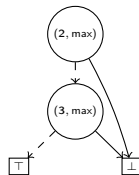
$[(2, 1) \mapsto (2, \max)]$

,

$((2, 0), (3, \max), \perp)$

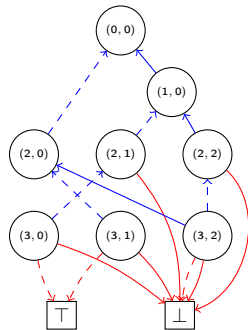
]

Output:
 $((2, \max), (3, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

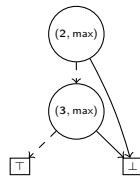
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

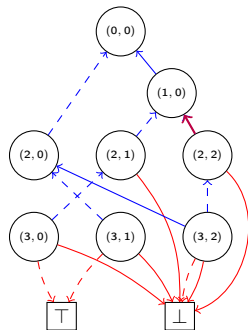
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

]

Level: 2

[

]

[

$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

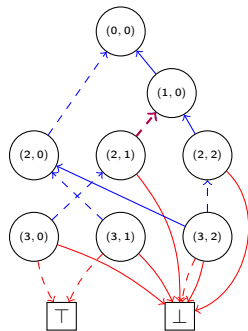
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

]

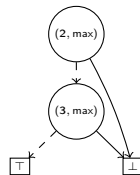
[

$[(2, 0) \mapsto (2, \max)]$

,

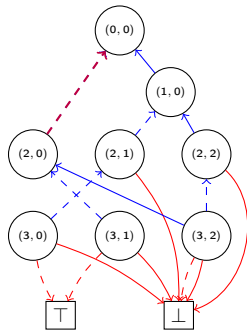
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

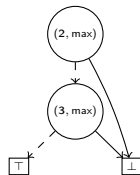
[

[

]

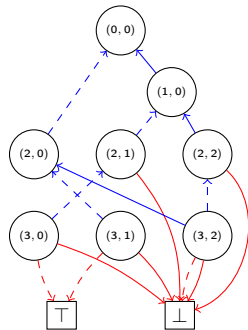
,
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0,0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

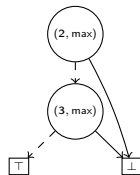
]

[

$((1,0), (2, \max), \perp)$

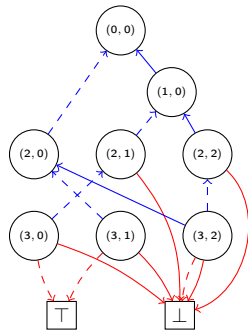
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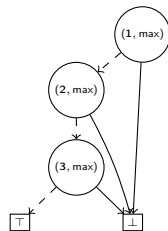
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[

$[(1, 0) \mapsto (1, \max)]$

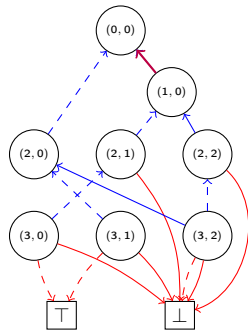
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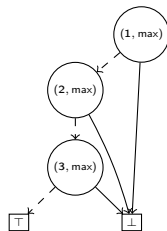
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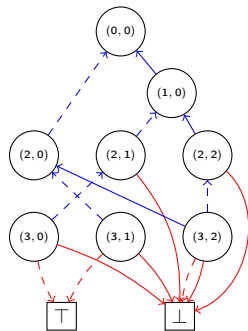
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Output:



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Apply : *Example* (Continued)



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

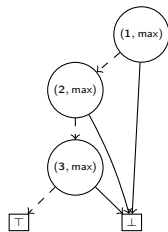
[

]

[$((0, 0), (2, \max), (1, \max))$

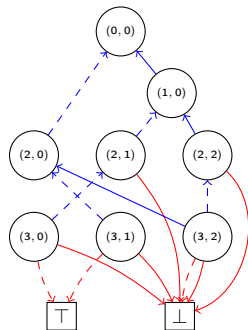
]

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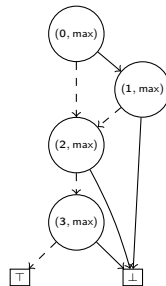
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]

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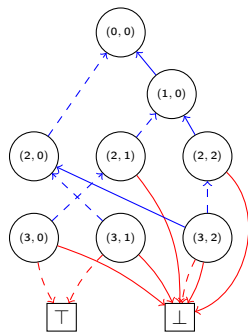
]

Output:
 $((0, \max), (2, \max), (1, \max))$



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Apply : *Example* (Continued)



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Level: 0

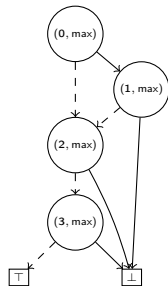
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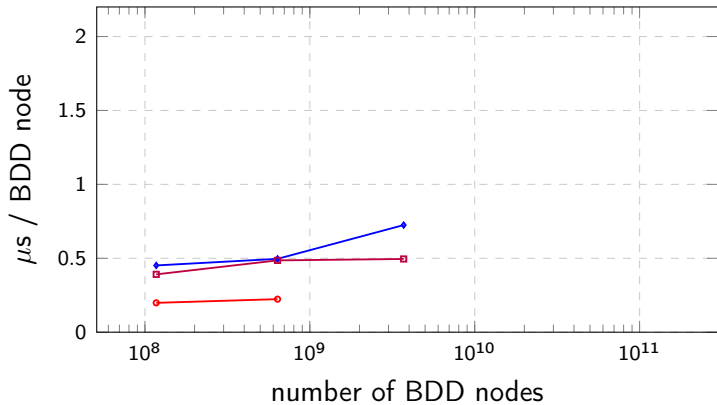
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Output:



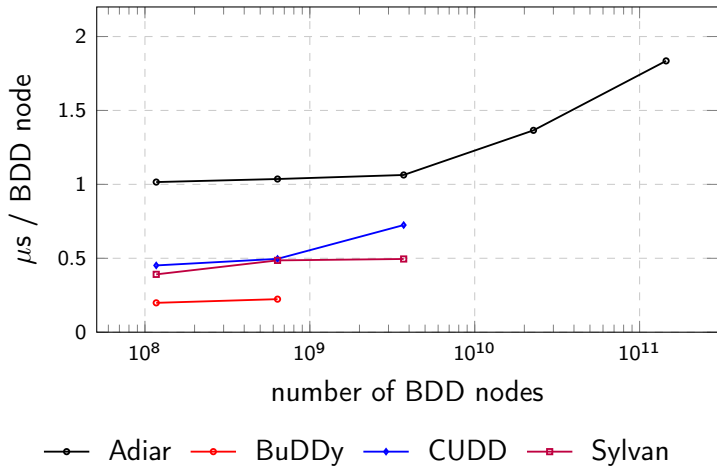
(d) $(a) \wedge (b)$ reduced

Algorithm	I/O-Complexity
bdd_pathcount	$O(\text{sort}(N_f))$
bdd_not	$O(N_f/B)$
bdd_restrict	$O(\text{sort}(N_f))$
bdd_apply	$O(\text{sort}(N_f \cdot N_g))$



—○— Adiar —○— BuDDy —♦— CUDD —□— Sylvan

Running time for the *N-Queens* problems.



Running time for the *N-Queens* problems.

Contents

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Equality Checking

$$f \leftrightarrow g \equiv \top$$

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$$\underbrace{O(\text{sort}(N^2))}_{\text{Apply}} + \underbrace{O(\text{sort}(N^2))}_{\text{Reduce}} + \underbrace{O(1)}_{\text{check is } \top} = O(\text{sort}(N^2))$$

Equality Checking

Theorem (Bryant '86)

Let π be a variable order and $f : \mathbb{B}^n \rightarrow \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π .

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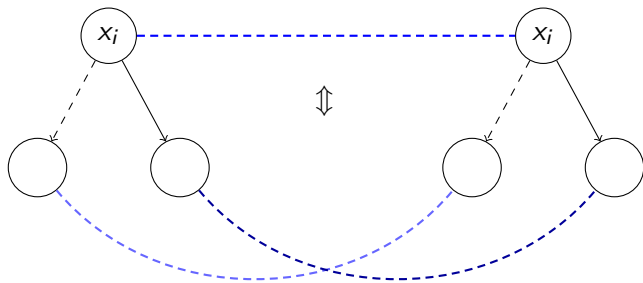
Trivial cases: $f \neq g$ if there is a mismatch in

- $N_f \neq N_g$ Number of nodes $O(1)$ I/Os
- $L_f \neq L_g$ Number of levels $O(1)$ I/Os
- $N_{f,i} \neq N_{g,i}$ Number of nodes on a level $O(L/B)$ I/Os
- $L_{f,i} \neq L_{g,i}$ Label of an i th level $O(L/B)$ I/Os

Equality Checking

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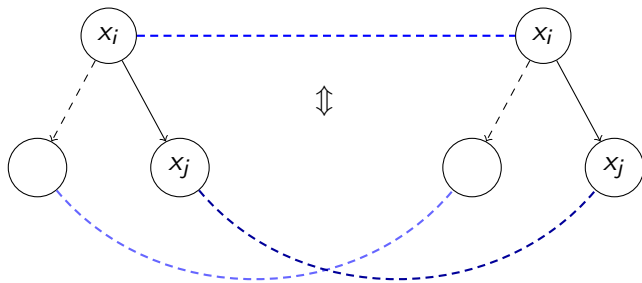
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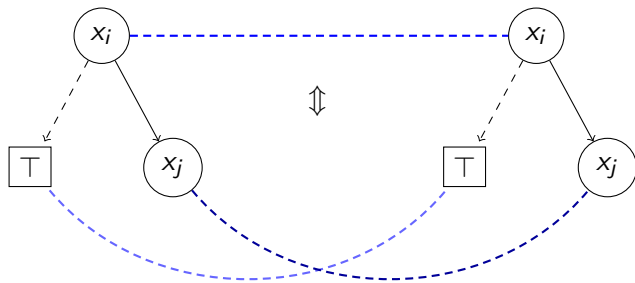
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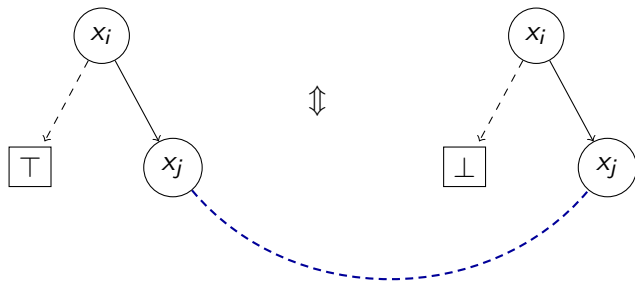
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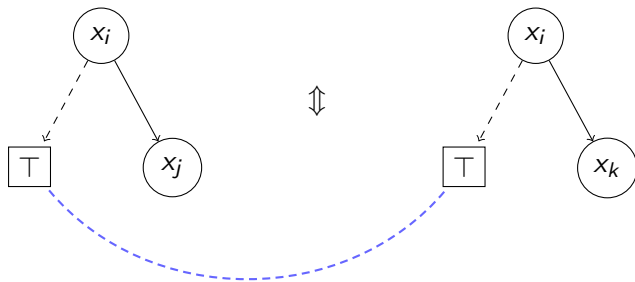
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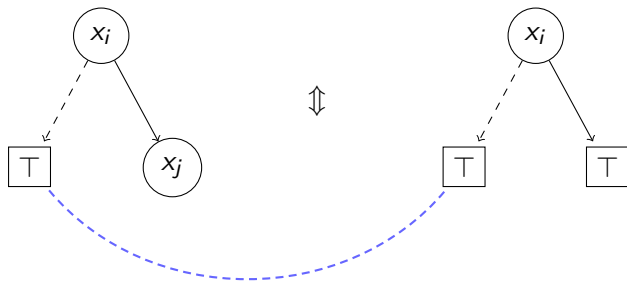
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Let π be a variable order and $f : \mathbb{B}^n \rightarrow \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π .

`IsIsomorphic(f , g)`

- Check whether root v_f of f and root v_g of g have a local violation.
- Check $low(v_f) \sim low(v_g)$ and $high(v_f) \sim high(v_g)$ “recursively”.

Return false on first violation. If there are no violations then return true.

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$\text{IsIsomorphic}(f, g)$

- Check whether root v_f of f and root v_g of g have a local violation.
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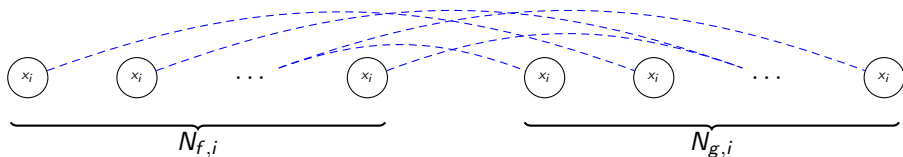
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$$\underbrace{O(\text{sort}(N^2))}_{\text{Apply'}} + \underbrace{\cancel{O(\text{sort}(N^2))}}_{\text{Reduce}} + \underbrace{\cancel{O(1)}}_{\text{check is } \top} = O(\text{sort}(N^2))$$

Equality Checking

Theorem (Bryant '86)

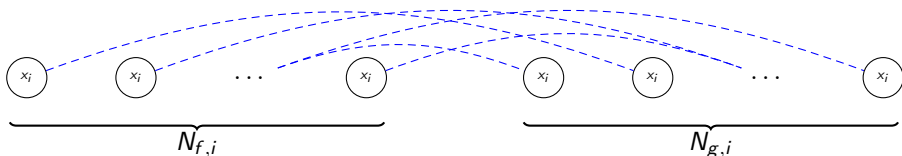
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Theorem (Bryant '86)

Let π be a variable order and $f : \mathbb{B}^n \rightarrow \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π .



Return false if more than $N_{f,i} = N_{g,i}$ pairs of nodes are checked on level i .

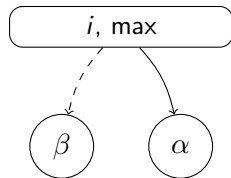
$$\underbrace{O(\text{sort}(\sum_i N_{f,i}))}_{\text{Apply}''} = O(\text{sort}(N))$$

Equality Checking

Observation

Each level output by the Reduce algorithm has the following properties:

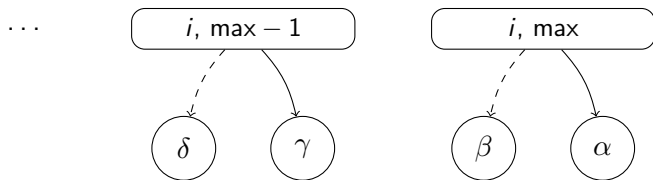
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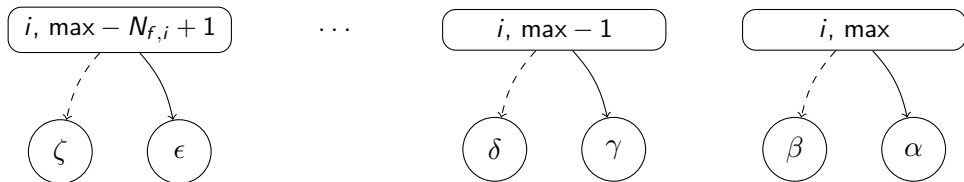
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Equality Checking

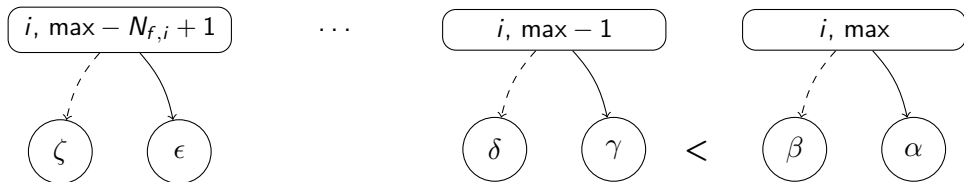


Observation

Each level output by the Reduce algorithm has the following properties:

- Nodes on level i have their identifiers *consecutively* numbered.

Equality Checking

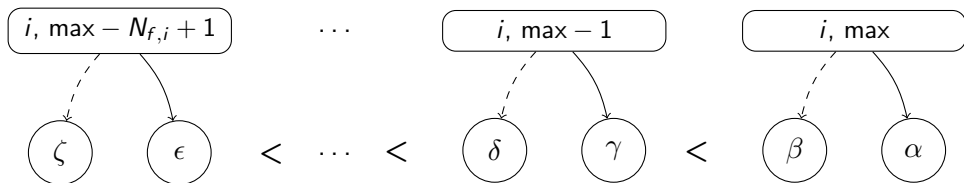


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Equality Checking



Observation

Each level output by the Reduce algorithm has the following properties:

- Nodes on level i have their identifiers *consecutively* numbered.
- Nodes on level i are output sorted by their children.

Equality Checking

Theorem

If G_f and G_g are outputs of Reduce.

$G_f \sim G_g \iff$ For all $i \in [0; N_f)$ the node $G_f[i]$ matches $G_g[i]$ numerically.

Proof.

\Leftarrow : Must describe the exact same graph.

\Rightarrow : Strong induction on BDD levels bottom-up.



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Proof.

\Leftarrow : Must describe the exact same graph.

\Rightarrow : Strong induction on BDD levels bottom-up. □

Corollary

If G_f and G_g are outputs of Reduce then $f \equiv g$ is computable using $2 \cdot N/B$ I/Os.

Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38

Checking the (EPFL Benchmark) *voter* circuit's single output gate ($|N_f| = |N_g| = 5.76$ MiB).

Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38
$O(\text{sort}(N))$	0.058

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Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38
$O(\text{sort}(N))$	0.058
$2N/B$	0.006

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Depth-First

Time-Forwarded

$O(N_f)$

$O(\text{sort}(N_f))$

$O(N_f \cdot N_g)$

$O(\text{sort}(N_f \cdot N_g))$

$O(1)$

$2N/B$