

Multi-variable Quantification of BDDs in External Memory using Nested Sweeping

Steffan Christ Sølvesten, Jaco van de Pol

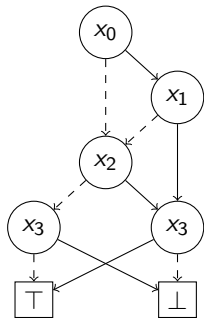
TACAS 2025





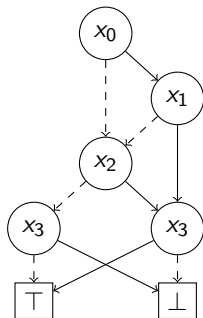
1986

Binary Decision Diagrams



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Binary Decision Diagrams

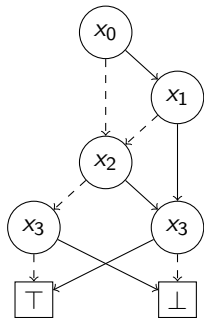


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Theorem (Bryant '86)

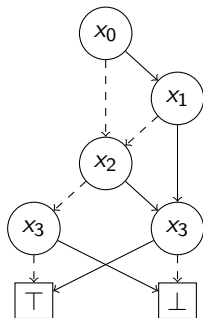
Given a fixed variable order, a (Reduced Ordered) Binary Decision Diagram is a unique canonical representation of a Boolean function.

Binary Decision Diagrams



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Given BDDs ϕ and ψ , $\phi \odot \psi$ is computable in $\mathcal{O}(|\phi| \cdot |\psi|)$ time.

Theorem (Bryant '86)

Given BDD ϕ and Boolean b , $\phi[x_i \mapsto b]$ is computable in $\mathcal{O}(|\phi|)$ time.

Corollary

Given BDD ϕ , $\exists x_i. \phi(x)$ requires $\mathcal{O}(|\phi|^2)$ time.

Proof.

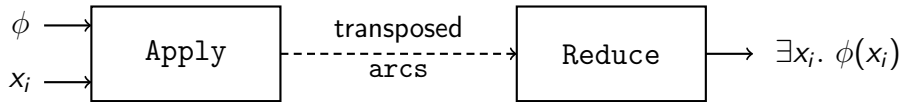
$$\exists x_i. \phi(x_i) \equiv \phi[x_i \mapsto \perp] \vee \phi[x_i \mapsto \top]$$

□

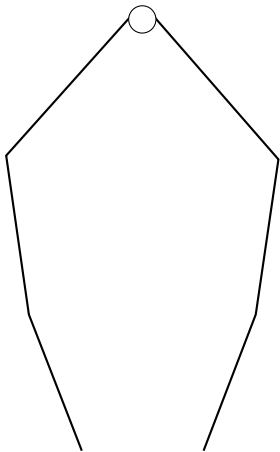


1996

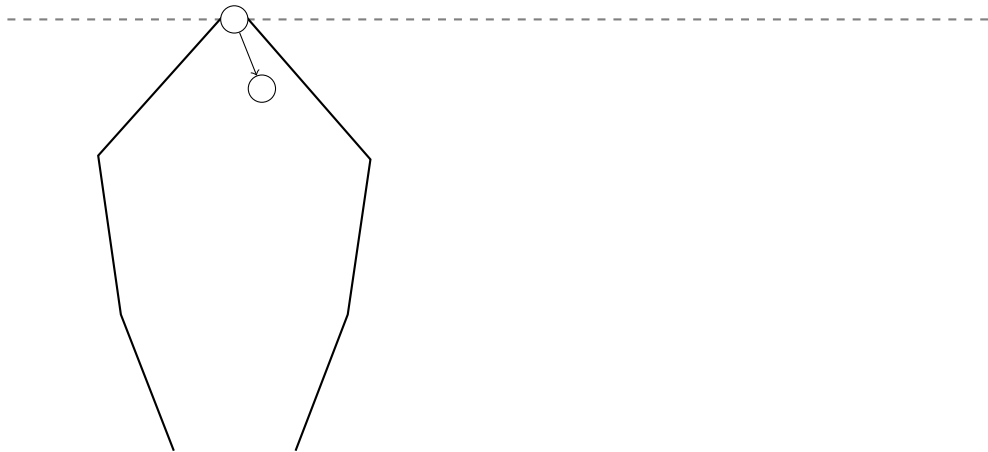
$$\exists x_i. \phi(x_i)$$



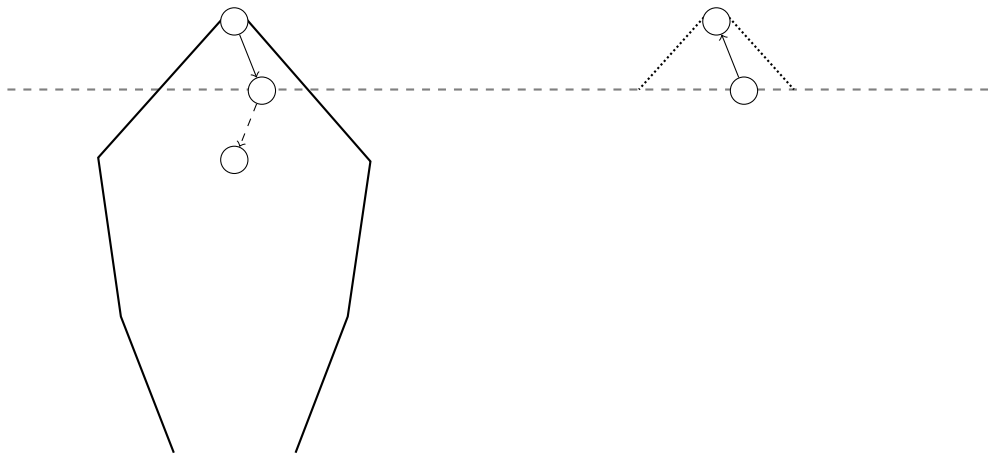
$\exists x_i. \phi(x_i) : \text{Apply}$



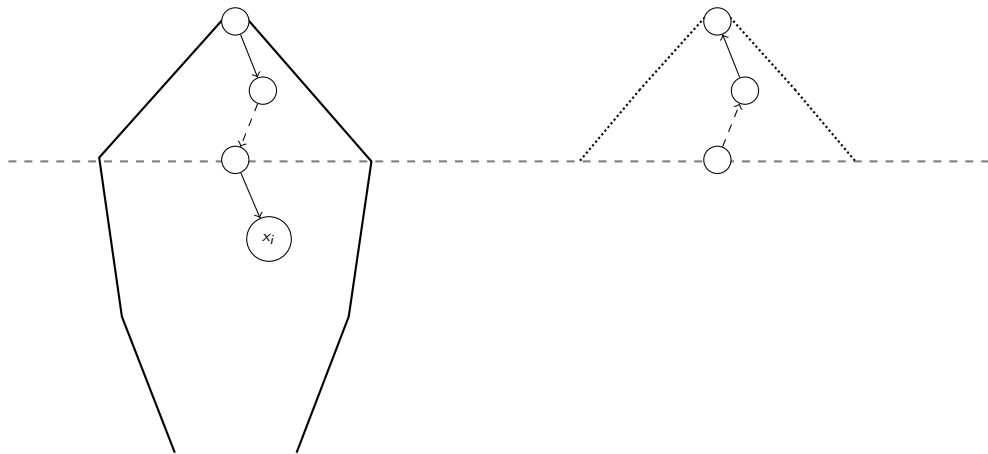
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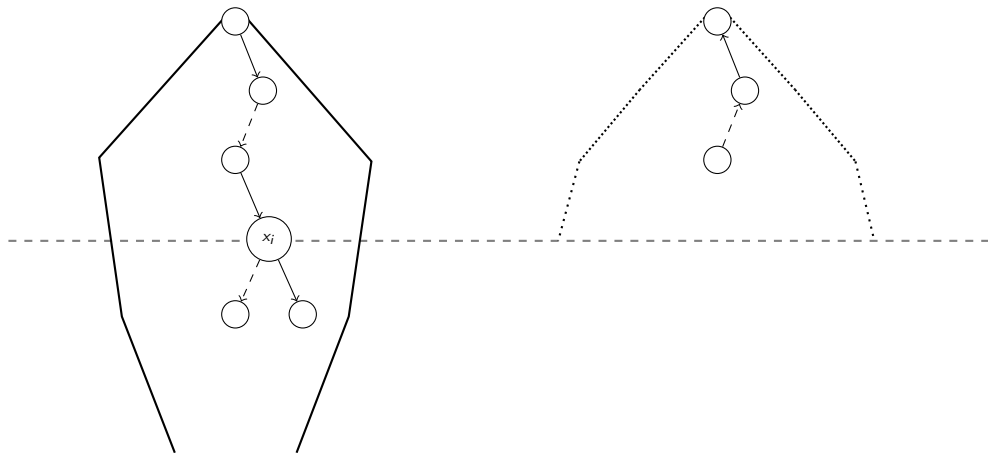
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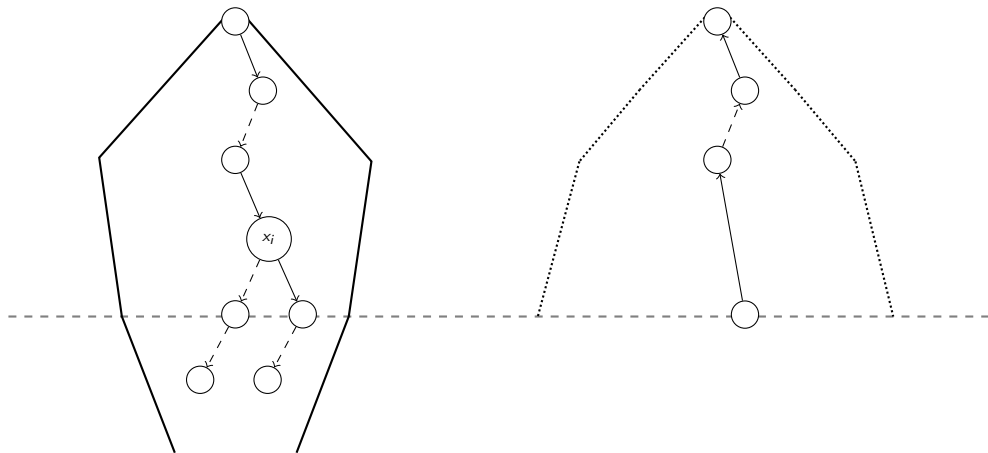
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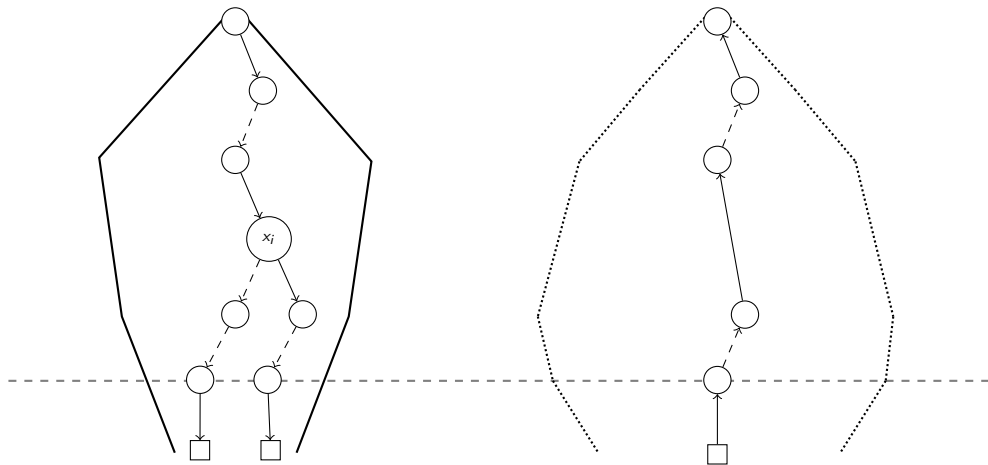
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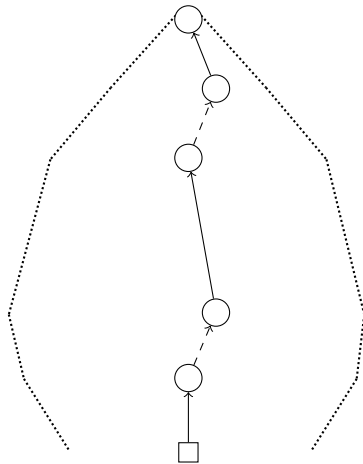
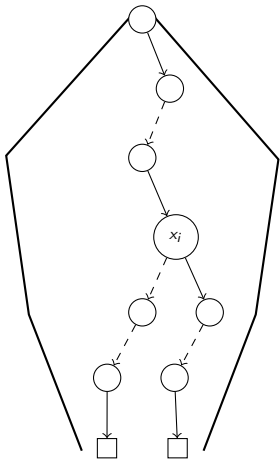
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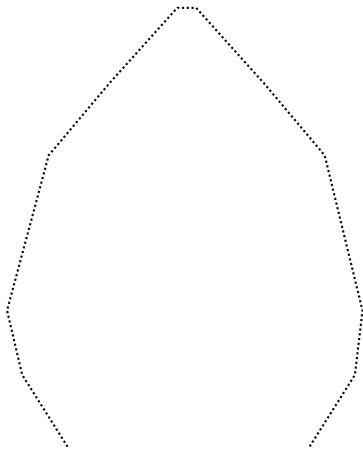
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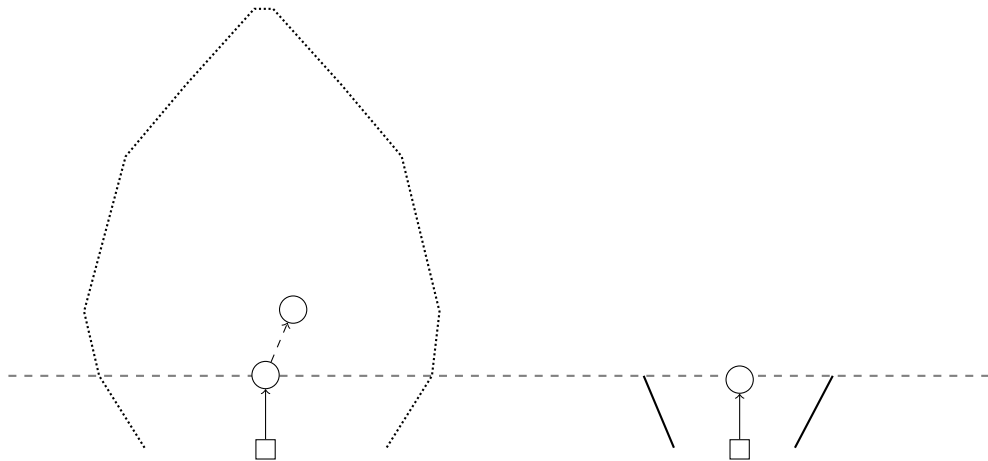
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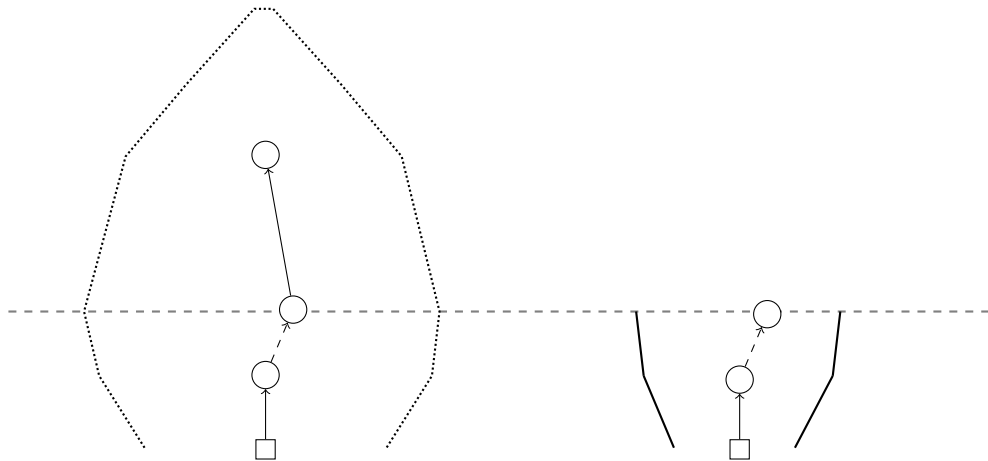
$\exists x_i. \phi(x_i) : \text{Reduce}$



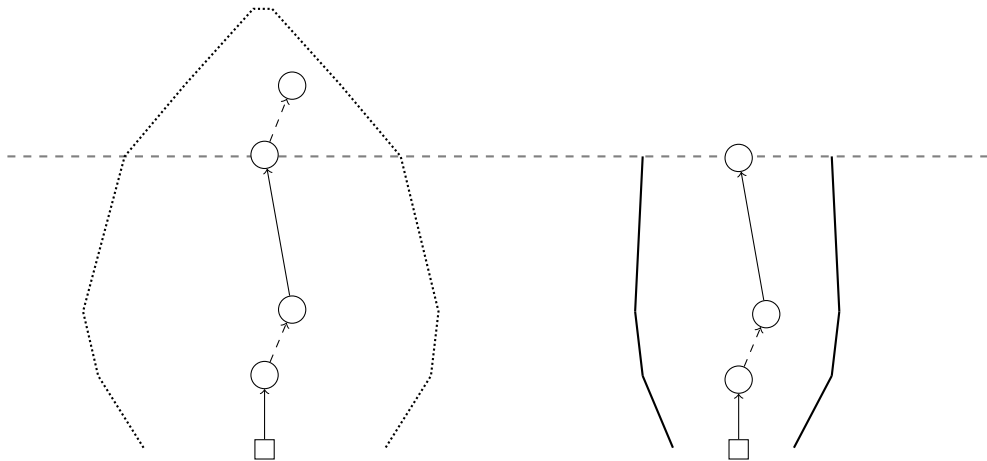
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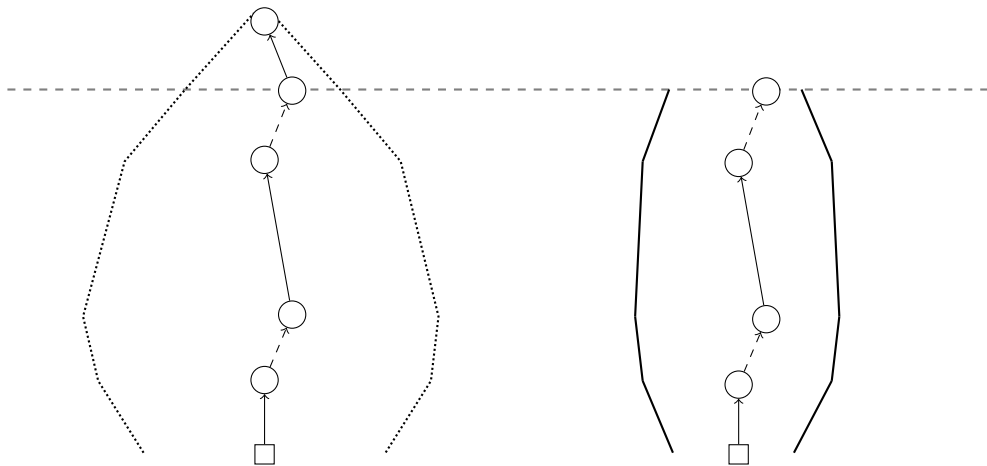
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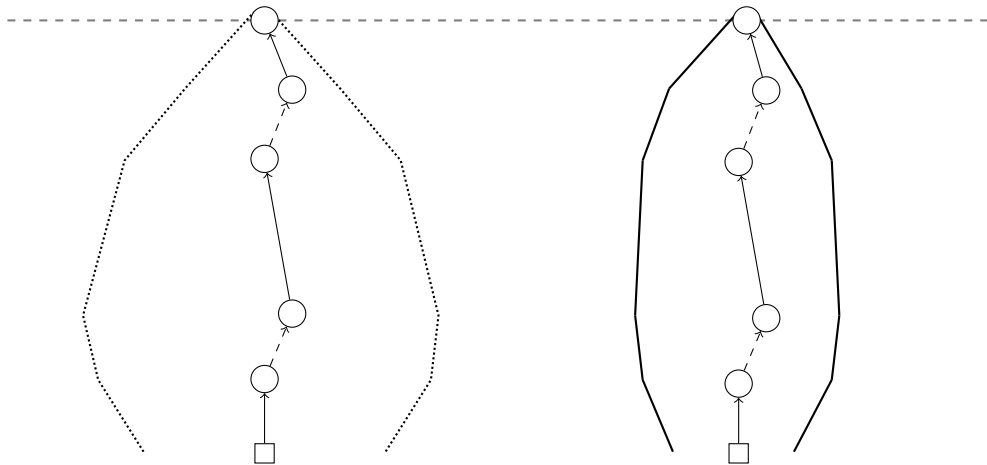
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$$\exists x_i. \phi(x_i)$$

Theorem (Lars Arge '96)

Given BDDs ϕ and ψ , $\phi \odot \psi$ is computable in $\mathcal{O}(\text{sort}(|\phi| \cdot |\psi|))$ time and I/Os.

Theorem (Sølvsten et al. '22)

Given BDD ϕ and Boolean b , $\phi[x_i \mapsto b]$ is computable in $\mathcal{O}(\text{sort}(|\phi|))$ time and I/Os.

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Given BDD ϕ and Boolean b , $\phi[x_i \mapsto b]$ is computable in $\mathcal{O}(\text{sort}(|\phi|))$ time and I/Os.

Corollary (Sølvsten et al. '22)

Given BDD ϕ , the time and I/O complexity of quantification is

- $\mathcal{O}(\text{sort}(|\phi|^2))$ for a single variable.
- $\mathcal{O}(\text{sort}(|\phi|^{2^k}))$ for k variables.



2022

Adiar

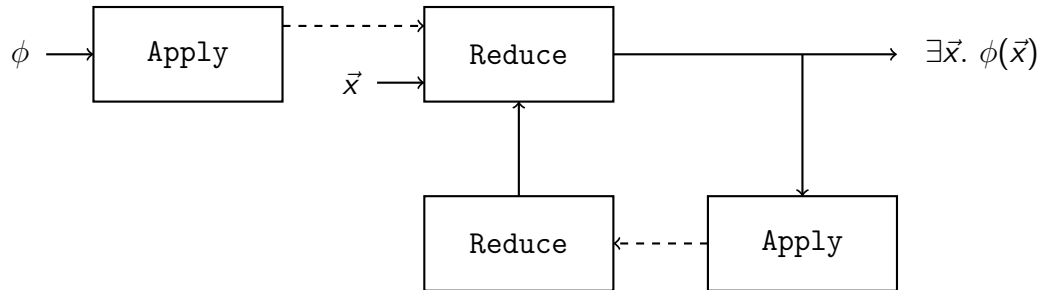
I/O-efficient Decision Diagrams

github.com/ssoelvsten/adiar

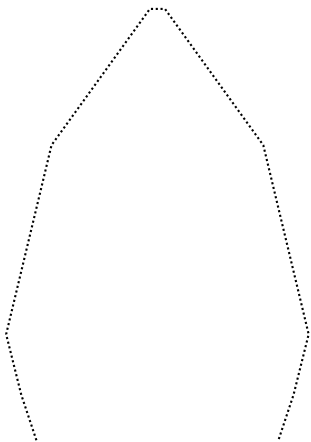


2025

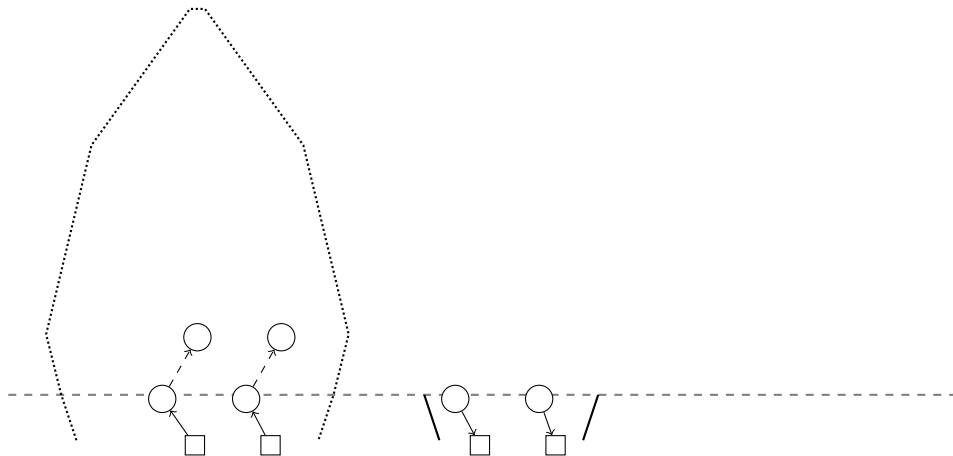
$\exists \vec{x}. \phi(\vec{x})$



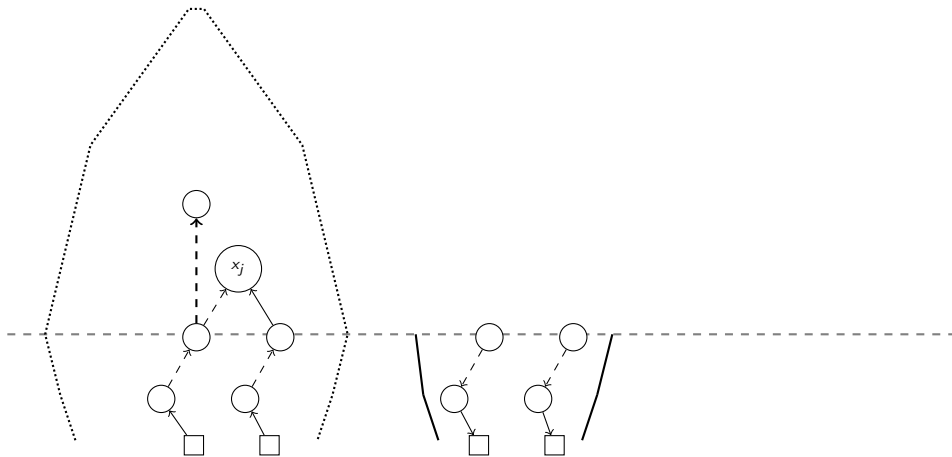
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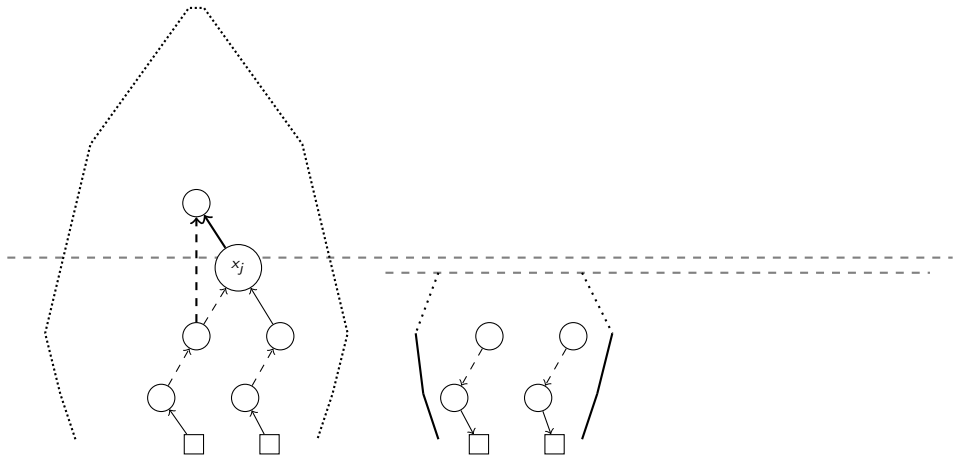
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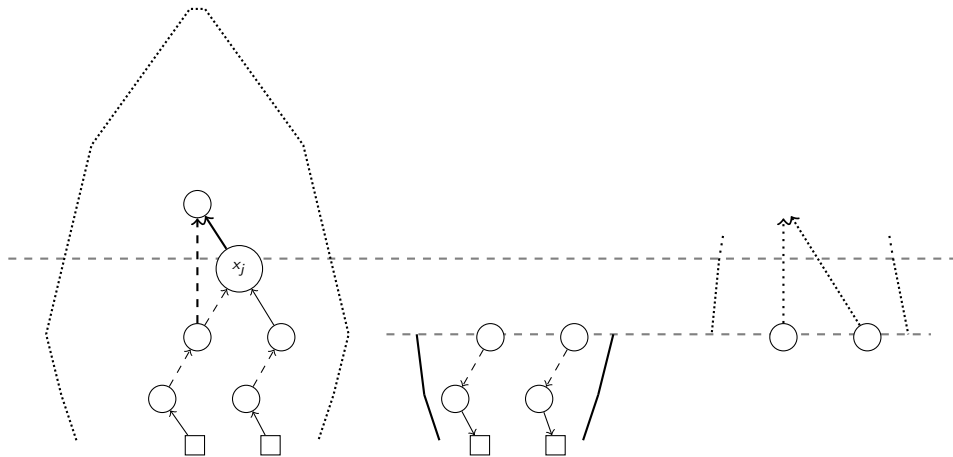
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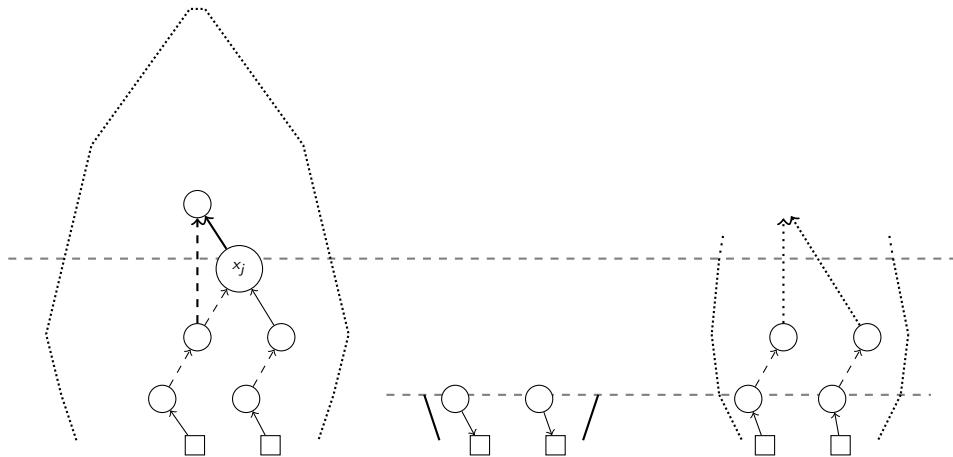
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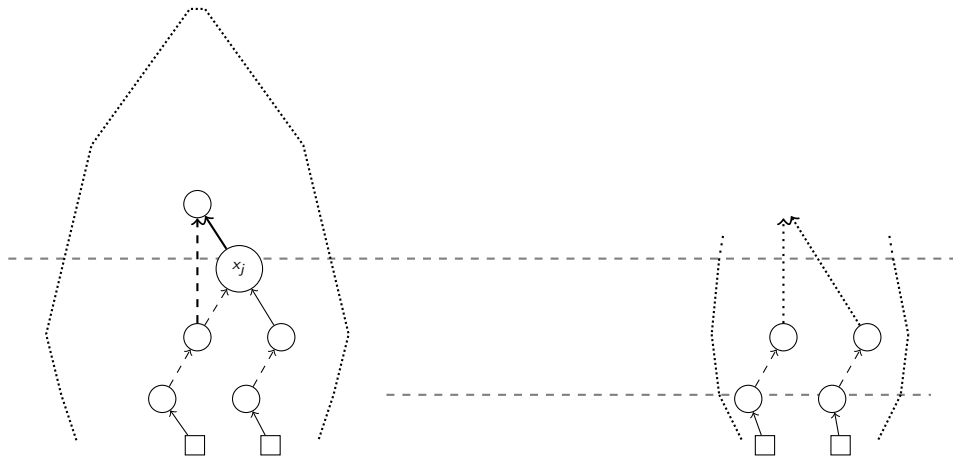
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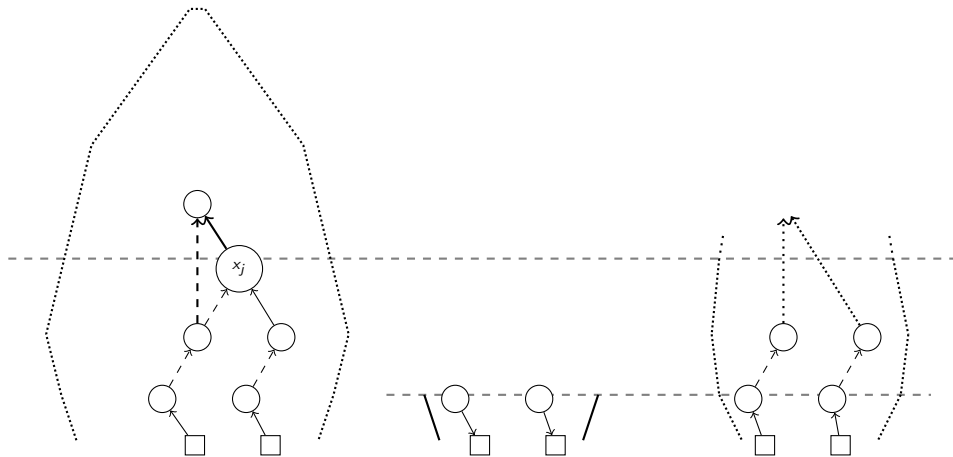
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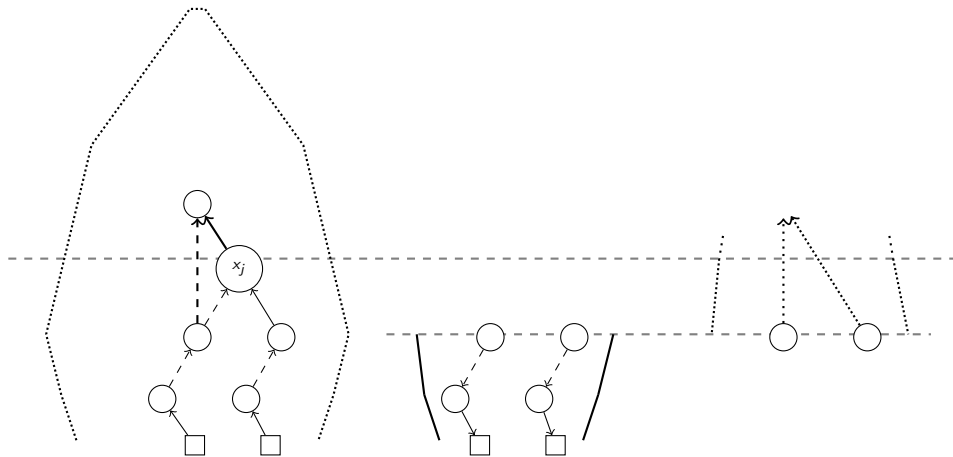
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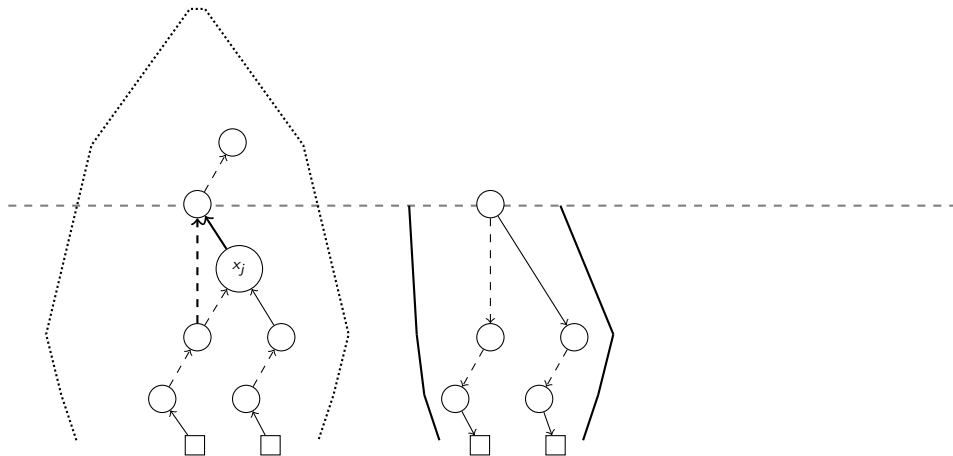
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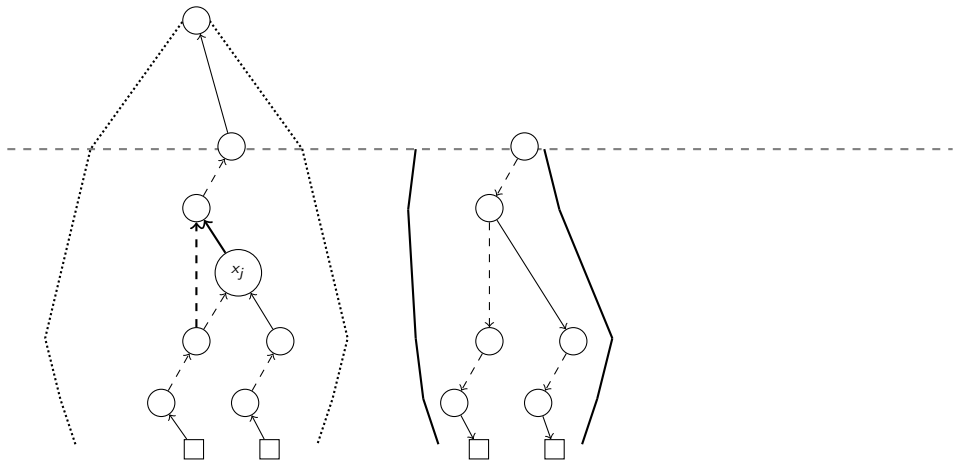


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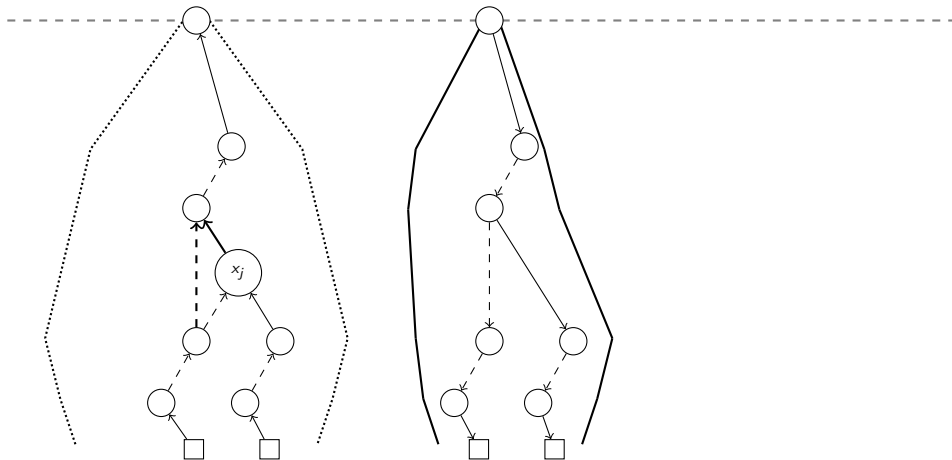


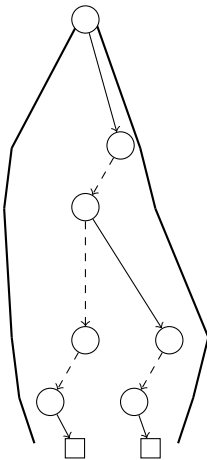
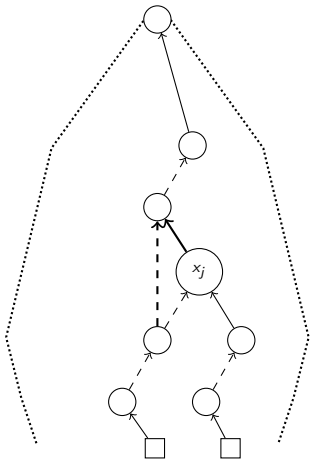
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Theorem (Sølvsten et al. '25)

Given BDD ϕ , the quantification of k variables, $\exists \vec{x}. \phi(\vec{x})$, is computable in $\mathcal{O}(\text{sort}(|\phi|^{2^k}))$ time and I/Os.

Benchmarks

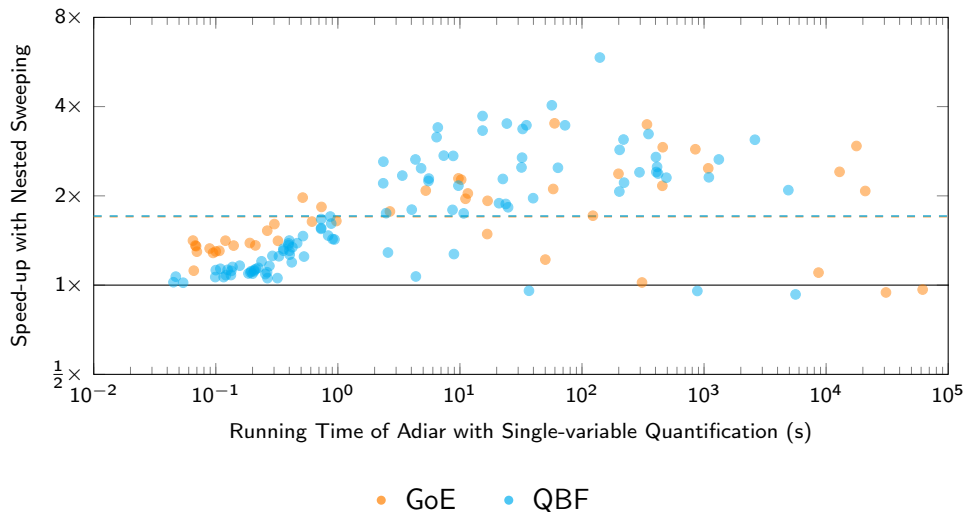
Garden-of-Eden

Given dimensions $N_1, N_2 \in \mathbb{N}$, determine whether there exists in Conway's *Game of Life* an initial state of size $N_1 \times N_2$ that is a *Garden of Eden*, i.e. is otherwise unreachable.

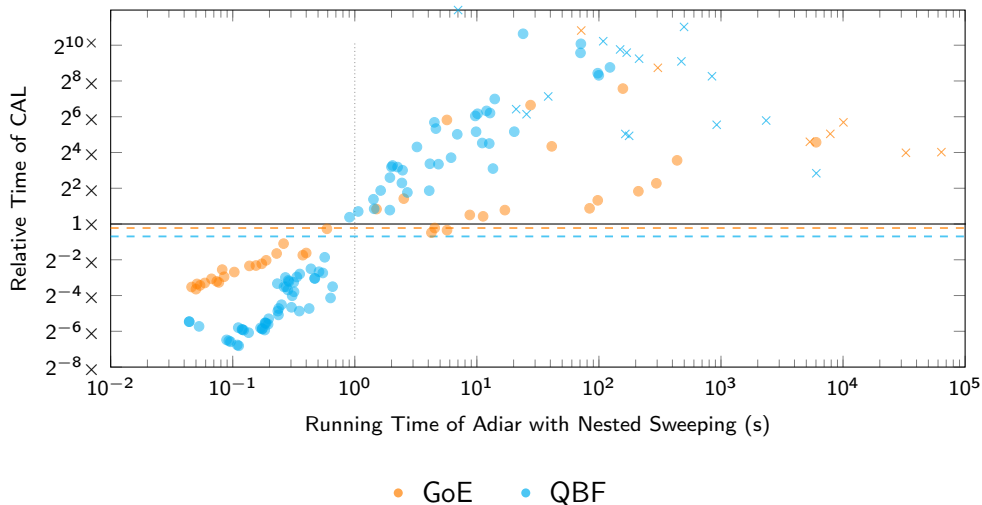
Quantified Boolean Formula

Determine whether a Boolean formula $\exists \vec{x}_1 \forall \vec{x}_2 \dots \exists \vec{x}_k. \phi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k)$ (or any order of quantifiers) evaluates to \top or \perp . For inputs, we use the two-player games from: Irfansha Shaik and Jaco van de Pol: “*Concise QBF encodings for games on a grid (extended version)*”. arXiv (2023).

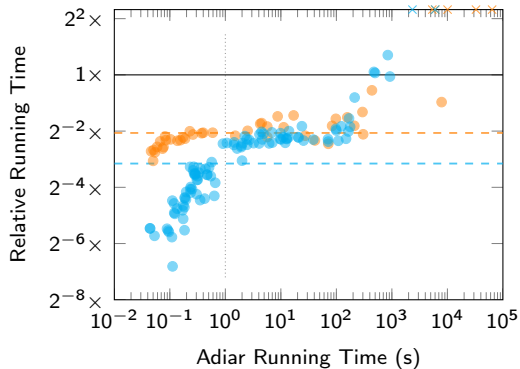
Benchmarks : Single vs. Nested Quantification



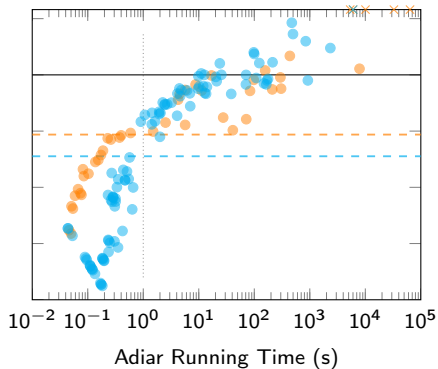
Benchmarks : Comparison to CAL



Benchmarks : Comparison to BuDDy and CUDD



(a) BuDDy



(b) CUDD

● GoE ● QBF

Steffan Christ Sølvesten

✉ soelvsten@cs.au.dk

🌐 ssoelvsten.github.io

Adiar

📄 github.com/ssoelvsten/adiar

📖 ssoelvsten.github.io/adiar



Nested Sweeping Framework

New BDD algorithms:

- ✓ Multi-variable Quantification
- ✓ Relational Product
- Functional Composition
- Variable Reordering

Other Decision Diagrams:

- Quantum Multi-valued Decision Diagrams
- Polynomial Boolean Rings

Better Transposition: Deepest Variable Quantification

$$bdd_exists(f, \max(\vec{x}))$$

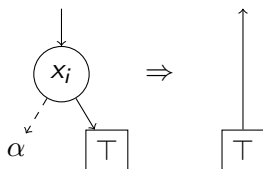
Motivation

Includes the first nested sweep inside of the transposition step.

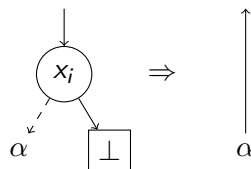
In Practice

Slows down computation time on average by 4.7%.

Better Transposition: Pruning \top Siblings



(a) Pruning due to \top .

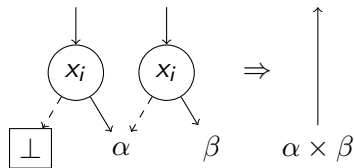


(b) Pruning due to \perp .

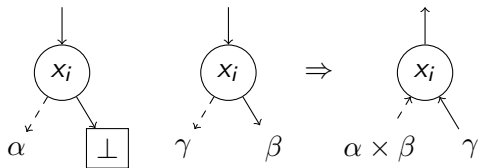
In Practice

If it prunes subtrees, running time can improve up to 21%. Otherwise, it adds an overhead of up to 2%.

Better Transposition: Partial Quantification



(a) Fully quantified pair of nodes.



(b) Partially quantified pair of nodes.

In Practice

In some instances, it improves performance by a factor of $\sim 2\times$.

In others, it slows down by $\sim 2\times$.

Observation

Instances improved by \top Pruning were disjoint from *Partial Quantification*.

Optimisations for Nested Sweeping

- **Leave terminal arcs out of nested sweeps.**

Required to satisfy invariants in nested algorithms from [TACAS 22].

- **Bail-out of Inner Sweep if all edges are subtree-preserving.**

In practice, 75.6% of all levels are skippable (median of 81.9% for each benchmark).

- **Use a sorted list as a bridge from the outer to the inner sweep.**

Postpones initialising data structures for the nested sweep until it is invoked.

- More memory available for the outer Reduce sweep.
- Sorting once and then merging on-the-fly with a priority queue can be faster than maintaining a larger priority queue.
- This enables the *levelised priority queues* [TACAS* 22], *levelised cuts* [ATVA 23], and *levelised random access* [SPIN 24] optimisations.

| | Single Quantification | | Nested Sweeping | |
|-------------------|-----------------------|---------|-----------------|---------|
| | LOC | # Tests | LOC | # Tests |
| nested_sweeping.h | – | – | 1287 | 104 |
| quantify | 548 | 84 | 1152 | 152 |
| core/... | 326 | – | 904 | – |
| bdd/... | 122 | 64 | 157 | 108 |
| zdd/... | 100 | 20 | 20 | 44 |

