Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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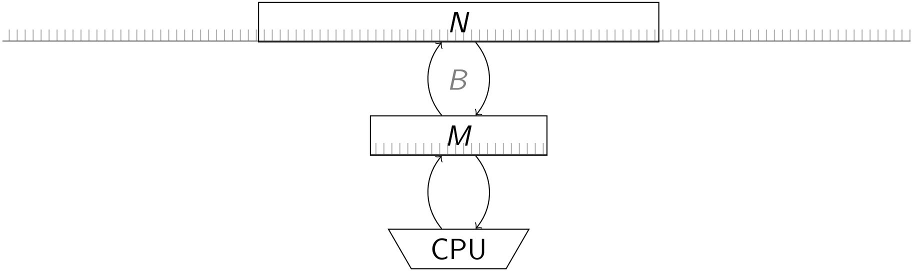


Figure 1: The I/O model by Aggarwal and Vitter '87

For any realistic values of N, M, and B we have that

$$N/B \quad < \quad \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \quad \ll \quad N \ ,$$

Theorem (Aggarwal and Vitter '87)
N elements can be sorted in $\Theta(\mathit{sort}(N))$ I/Os.

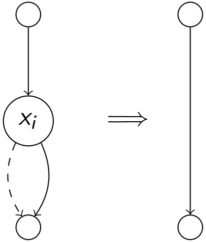
Theorem (Arge '95)
N elements can be inserted in and extracted from a Priority Queue in $\Theta(\mathit{sort}(N))$ I/Os.

(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

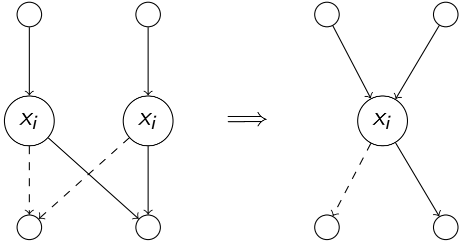
(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Figure 2: Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(a) Rule 1: Remove redundant nodes



(b) Rule 2: Merge duplicate nodes

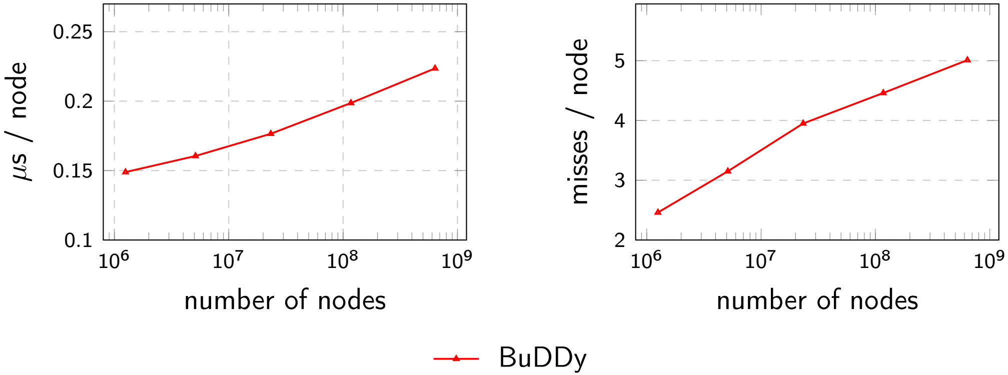


Figure 4: Cache behaviour for the *N*-Queens problem.

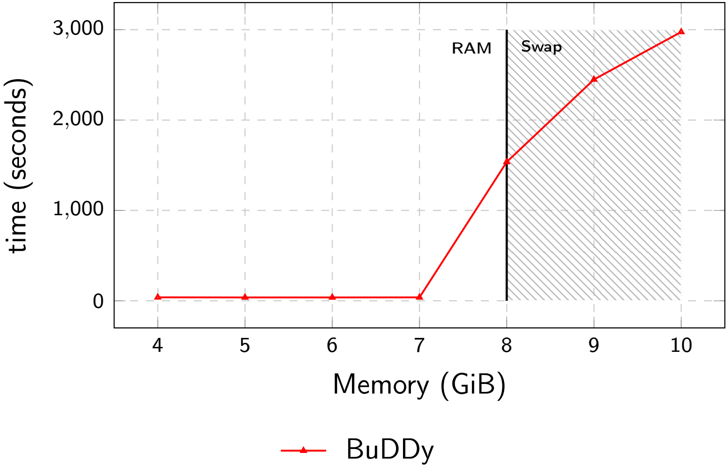
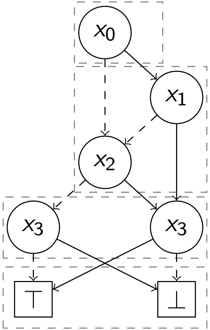


Figure 5: Running time for *Tic-Tac-Toe* with $N = 21$.

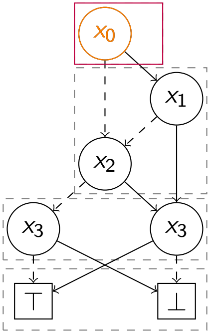


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 6: Blocks active in memory

$M = 4,\ B = 2$

node I/Os	cache lookups
0	0



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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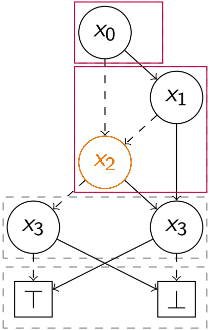
$$M = 4,\ B = 2$$

node I/Os	cache lookups
1	1



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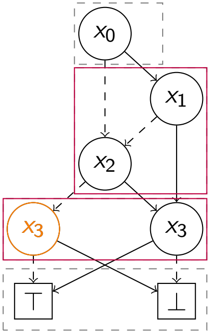
$$M = 4,\ B = 2$$

node I/Os	cache lookups
2	2



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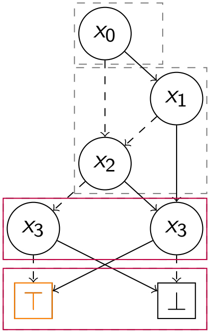
$$M = 4,\ B = 2$$

node I/Os	cache lookups
3	3



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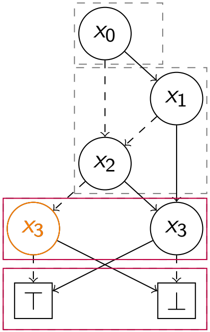
$M = 4,\ B = 2$

node I/Os	cache lookups
4	3



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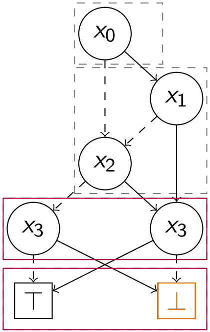
$$M = 4,\ B = 2$$

node I/Os	cache lookups
4	3



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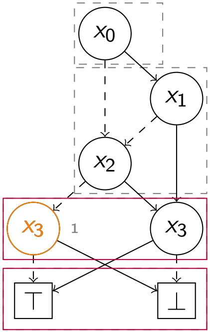
$$M = 4,\ B = 2$$

node I/Os	cache lookups
4	3



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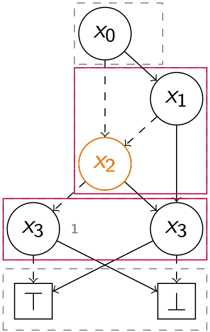
$$M = 4,\ B = 2$$

node I/Os	cache lookups
4	3



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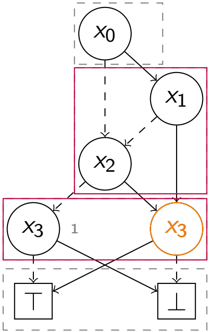
$$M = 4,\ B = 2$$

node I/Os	cache lookups
5	3



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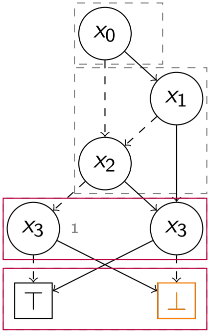
$$M = 4,\ B = 2$$

node I/Os	cache lookups
5	4



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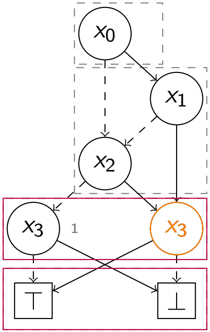
$$M = 4,\ B = 2$$

node I/Os	cache lookups
6	4



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

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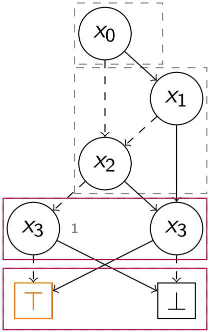
$$M = 4, \ B = 2$$

node I/Os	cache lookups
6	4



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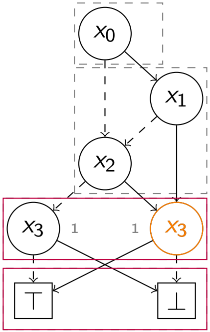
node I/Os	cache lookups
6	4



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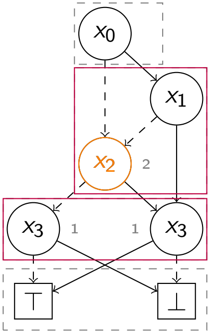
node I/Os	cache lookups
6	4



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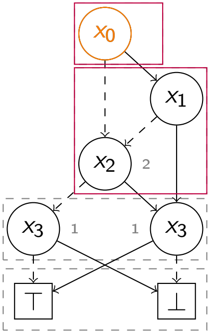
$M = 4,\ B = 2$

node I/Os	cache lookups
7	4



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$M = 4,\ B = 2$

node I/Os	cache lookups
8	4

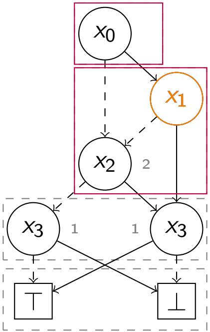

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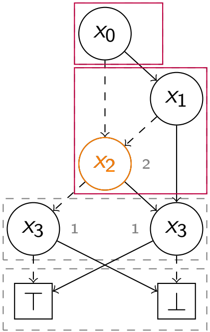
Figure 6: Blocks active in memory

$$M = 4,\ B = 2$$

node I/Os	cache lookups
8	5



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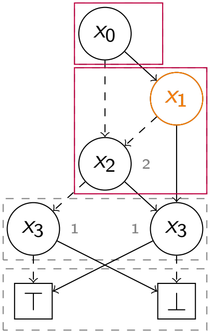
$$M = 4,\ B = 2$$

node I/Os	cache lookups
8	6



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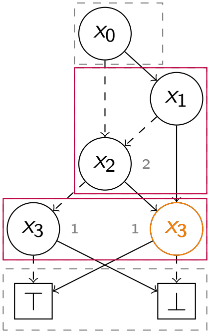
$$M = 4, \ B = 2$$

node I/Os	cache lookups
8	6



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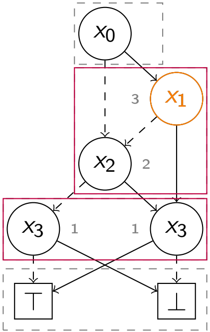
$$M = 4,\ B = 2$$

node I/Os	cache lookups
9	7



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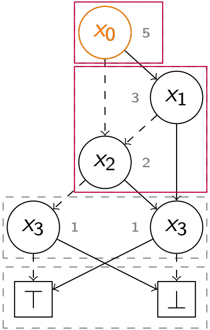
$$M = 4,\ B = 2$$

node I/Os	cache lookups
9	7



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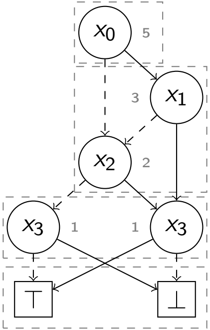
$$M = 4,\ B = 2$$

node I/Os	cache lookups
10	7



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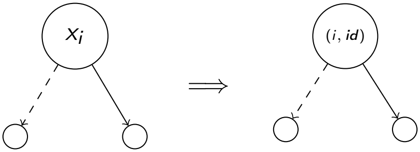
$M = 4,\ B = 2$

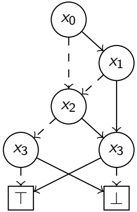
node I/Os	cache lookups
10	7

Let every node be uniquely identified by a tuple $(label, id) : \mathbb{N} \times \mathbb{N}$.



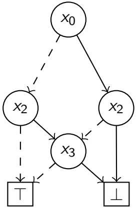
Nodes are ordered based on their *uid* as follows

$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



$$[\ ((0,0),(2,0),(1,0)),\ ((1,0),(2,0),(3,1)),\ ((2,0),(3,0),(3,1)),\ ((3,0),\bot,\top),\ ((3,1),\top,\bot)\ ]$$

(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

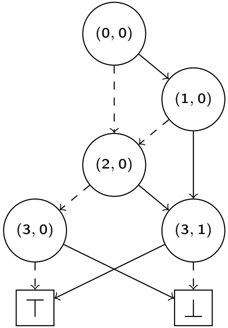


$$[\ ((0,0),(2,0),(2,1)),\ ((2,0),\top,(3,0)),\ ((2,1),(3,0),\bot),\ ((3,0),\top,\bot)\ ]$$

(b) $\neg(x_0 \ ?\ x_2 \vee x_3 \ :\ x_2 \wedge x_3)$

Figure 7: Node-based representation of prior shown BDDs

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

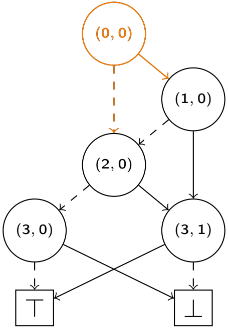
Figure 8: In-order traversal of BDD

Priority Queue: $Q_{\boldsymbol{count}}$:

[

]

CountPaths Example



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

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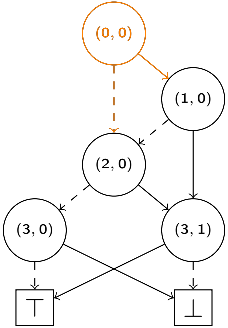
[

]

CountPaths Example



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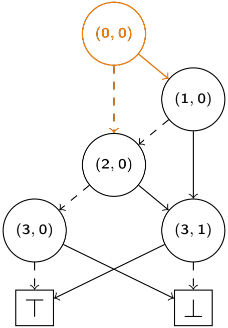
Priority Queue: $Q_{\textit{count}}$:

$$
\begin{array}{ll}
[& ((0,0) \xrightarrow{\top} (1,0),\quad 1) \quad , \\
  & ((0,0) \xrightarrow{\bot} (2,0),\quad 1) \quad , \\
  & \\
  & \qquad\qquad\qquad]
\end{array}
$$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

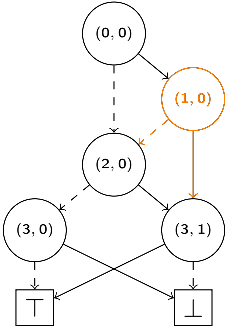
Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(1, 0)$	0	0

Priority Queue: $Q_{\boldsymbol{count}}$:

[$((0,0) \xrightarrow{\top} (1,0),\ 1)$,
$((0,0) \xrightarrow{\bot} (2,0),\ 1)$,
]

CountPaths Example



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[$((0,0) \xrightarrow{\top} (1,0),\ 1)$,
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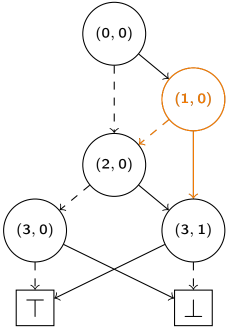
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

Seek	Sum	Result
(1, 0)	1	0

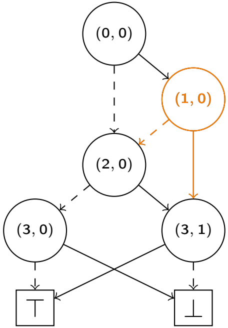
Priority Queue: Q_{count}:

[

$((0,0) \xrightarrow{\perp} (2,0),\ 1)$,

]

CountPaths Example



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(1,0)$	1	0

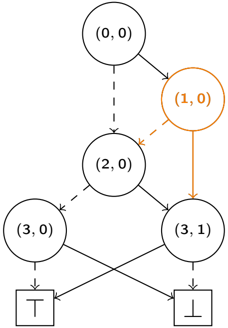
Priority Queue: Q_{count}:

[

$((0,0) \xrightarrow{\perp} (2,0),\quad 1)$,
$((1,0) \xrightarrow{\perp} (2,0),\quad 1)$,

$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
]

CountPaths Example



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

Seek	Sum	Result
(2, 0)	0	0

Priority Queue: $Q_{\boldsymbol{count}}$:

[

$((0,0) \xrightarrow{\perp} (2,0),\quad 1)$,
$((1,0) \xrightarrow{\perp} (2,0),\quad 1)$,

$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
]

CountPaths Example

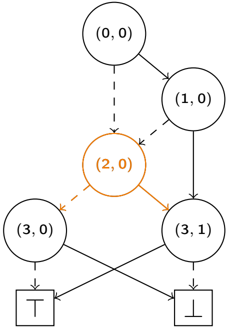


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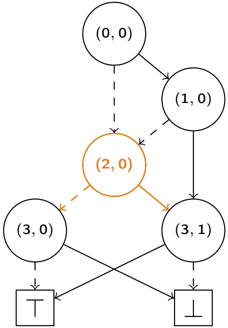
Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(2,0)$	0	0

Priority Queue: Q_{count}:

$$[\ ((0,0) \xrightarrow{\perp} (2,0),\ 1),\ ((1,0) \xrightarrow{\perp} (2,0),\ 1),\ ((1,0) \xrightarrow{\top} (3,1),\ 1),\ ]$$

CountPaths Example



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Priority Queue: $Q_{\boldsymbol{count}}$:

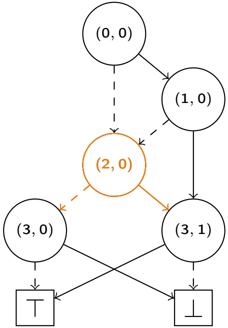
[

$((1,0) \xrightarrow{\perp} (2,0),\quad 1)$,

$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(2,0)$	2	0

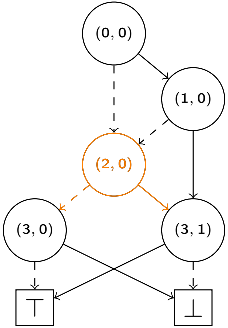
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

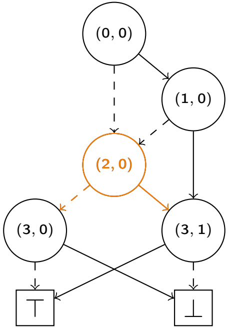
Seek	Sum	Result
(2, 0)	2	0

Priority Queue: $Q_{\textit{count}}$:

[

$((2,0) \xrightarrow{\perp} (3,0),\quad 2)$,
$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
$((2,0) \xrightarrow{\top} (3,1),\quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

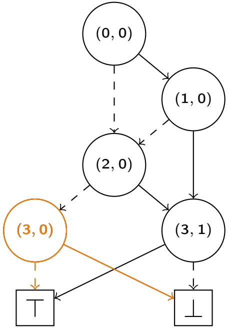
Seek	Sum	Result
(3, 0)	0	0

Priority Queue: $Q_{\textit{count}}$:

[

$((2,0) \xrightarrow{\perp} (3,0),\quad 2)$,
$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
$((2,0) \xrightarrow{\top} (3,1),\quad 2)$]

CountPaths Example



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

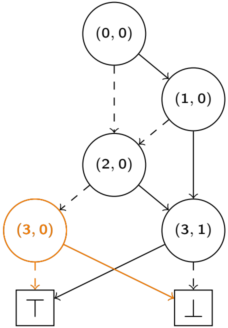
Seek	Sum	Result
$(3, 0)$	0	0

Priority Queue: $Q_{\boldsymbol{count}}$:

[

$((2,0) \xrightarrow{\bot} (3,0), \quad 2)$,
$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,
$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths Example



(a) $(x_0 \land x_1 \land x_3) \lor (x_2 \oplus x_3)$

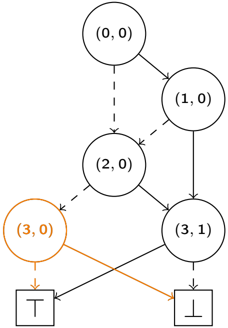
Figure 8: In-order traversal of BDD

Seek	Sum	Result
(3, 0)	2	0

Priority Queue: $Q_{\textit{count}}$:

[
$((1,0) \xrightarrow{\top} (3,1),\ 1)$,
$((2,0) \xrightarrow{\top} (3,1),\ 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

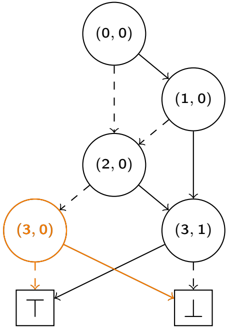
Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(3, 0)$	2	2

Priority Queue: $Q_{\textbf{count}}$:

$$[\ ((1,0) \xrightarrow{\top} (3,1),\quad 1)\ ,\quad ((2,0) \xrightarrow{\top} (3,1),\quad 2)\ ]$$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

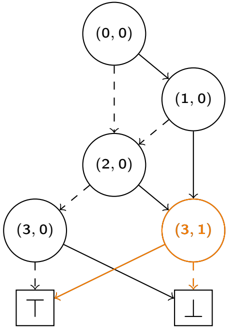
Figure 8: In-order traversal of BDD

Seek	Sum	Result
$(3, 1)$	0	2

Priority Queue: $Q_{\textit{count}}$:

[
$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
$((2,0) \xrightarrow{\top} (3,1),\quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

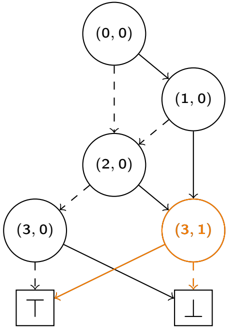
Seek	Sum	Result
(3, 1)	0	2

Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1),\quad 1)$,
$((2,0) \xrightarrow{\top} (3,1),\quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

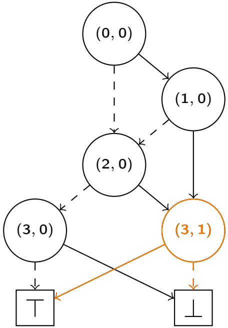
Seek	Sum	Result
$(3,1)$	1	2

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\top} (3,1),\quad 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

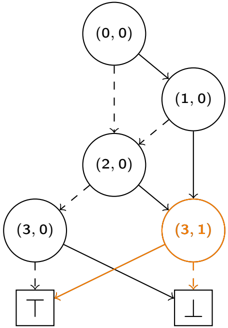
Seek	Sum	Result
$(3,1)$	3	2

Priority Queue: $Q_{\boldsymbol{count}}$:

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

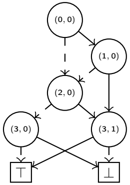
Seek	Sum	Result
$(3, 1)$	3	5

Priority Queue: $Q_{\boldsymbol{count}}$:

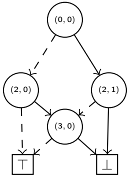
[

]

Apply Example ($\wedge$)

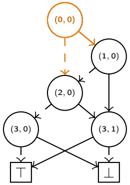


(a) $(x_{\mathbf{0}} \wedge x_{\mathbf{1}} \wedge x_{\mathbf{3}}) \vee (x_{\mathbf{2}} \oplus x_{\mathbf{3}})$

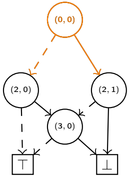


(b) $\neg(x_{\mathbf{0}} \ ? \ x_{\mathbf{2}} \vee x_{\mathbf{3}} \ : \ x_{\mathbf{2}} \wedge x_{\mathbf{3}})$

Apply Example (∧)

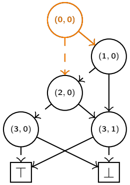


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

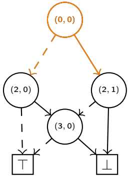


(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

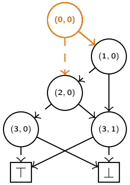


(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

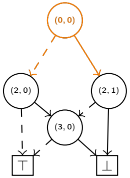
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

$$
\begin{array}{l}
[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad , \\
 \quad (0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad , \\
 \quad]
\end{array}
$$

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



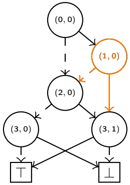
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
min((1, 0), (2, 1))

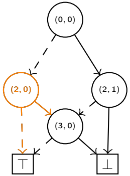
Priority Queue: $Q_{app:1}$:

$$[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad ,$$
$$(0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad ,$$
$$]$$

Apply Example ($\wedge$)



(a) $(x_{\mathbf{0}} \wedge x_{\mathbf{1}} \wedge x_{\mathbf{3}}) \vee (x_{\mathbf{2}} \oplus x_{\mathbf{3}})$



(b) $\neg(x_{\mathbf{0}}\ ?\ x_{\mathbf{2}} \vee x_{\mathbf{3}}\ :\ x_{\mathbf{2}} \wedge x_{\mathbf{3}})$

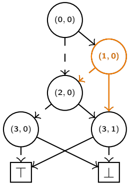
Seek:

$\min((1, 0), (2, 1))$

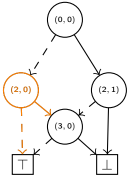
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

$$[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)),$$
$$(0,0) \xrightarrow{\perp} ((2,0),(2,0)),$$
$$]$$

Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



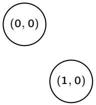
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

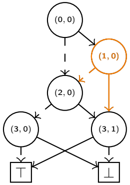
$\min((1,0),(2,1))$

Priority Queue: $Q_{app:1}$:

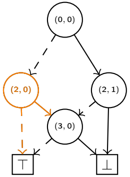
$$
\begin{array}{ll}
[& (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad , \\
 & (0,0) \xrightarrow{\bot} ((2,0),(2,0)) \quad , \\
 & (1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad , \\
 & (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad , \\
 & \quad]
\end{array}
$$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

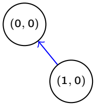
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

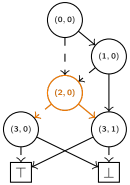
]

Output:

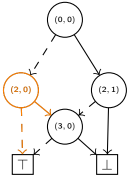
$(0, 0) \xrightarrow{\top} (1, 0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{\mathit{app}:1}$:

[

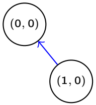
$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

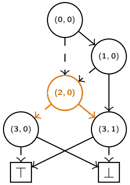
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

]

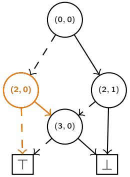
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((2,0),(2,0))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

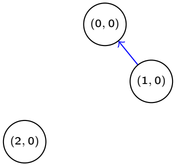
[

$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,
$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

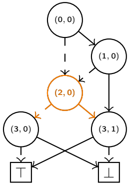
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

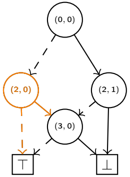
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

$(1,0) \xrightarrow{\bot} ((2,0),(2,1))$,

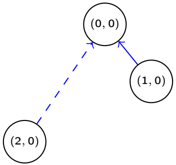
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

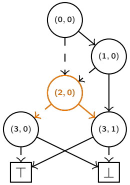
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Output:

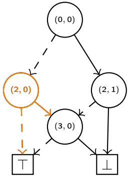
$(0,0) \xrightarrow{\bot} (2,0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



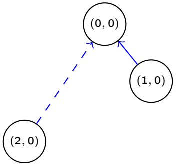
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((2,0),(2,1))$

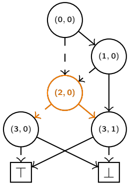
Priority Queue: $Q_{app:1}$:

$$
\begin{aligned}
[\quad & \\
& (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \\
& (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad , \\
& (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \\
& (2,0) \xrightarrow{\perp} ((3,0),\top) \quad]
\end{aligned}
$$

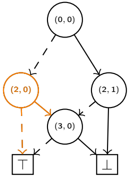
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,1))$

Priority Queue: $Q_{\mathbf{app:1}}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

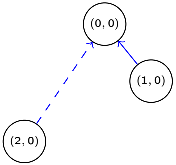
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{\mathbf{app:2}}$:

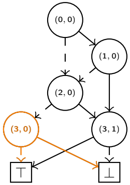
[$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$ $((3,0),(3,1))$,

]

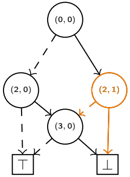
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

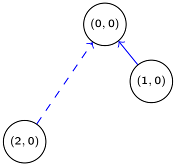
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

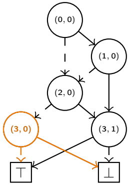
[$(1,0) \xrightarrow{\bot} ((2,0),(2,1))$ $((3,0),(3,1))$,

]

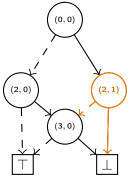
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:
$\max((2,0),(2,1))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

$[$

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

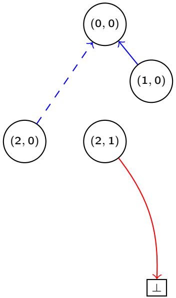
$(2,0) \xrightarrow{\bot} ((3,0),\top)$ $]$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

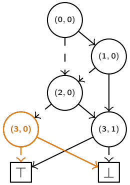
$[$ $(1,0) \xrightarrow{\bot} ((2,0),(2,1))$ $((3,0),(3,1))$,

$]$

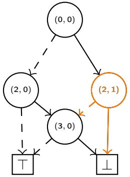
Output:
$(2,1) \xrightarrow{\top} \bot$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\max((2,0),(2,1))$

Priority Queue: $Q_{\mathit{app}:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

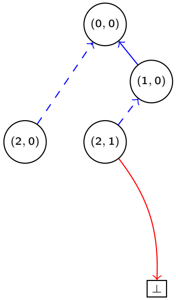
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{\mathit{app}:2}$:

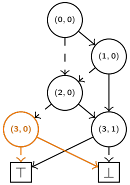
[

]

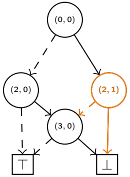
Output:
$(1,0) \xrightarrow{\bot} (2,1)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \vee x_3 \;:\; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

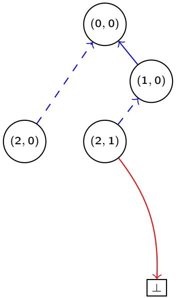
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

$$[\; (1,0) \xrightarrow{\top} ((3,1),(2,1)),\quad (2,1) \xrightarrow{\bot} ((3,0),(3,0)),\quad (2,0) \xrightarrow{\top} ((3,1),(3,0)),\quad (2,0) \xrightarrow{\bot} ((3,0),\top) \;]$$

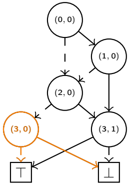
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

[ ]

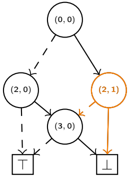
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0\ ?\ x_2 \vee x_3\ :\ x_2 \wedge x_3)$

Seek:
$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

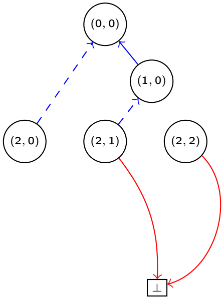
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

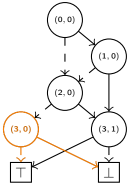
[

]

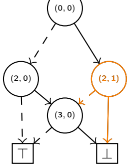
Output:
$(2,2) \xrightarrow{\top} \bot$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

[

$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

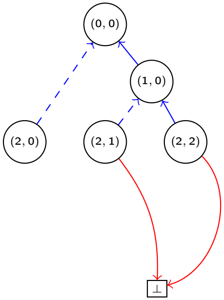
Priority Queue: $Q_{\mathbf{app}:\mathbf{2}}$:

[

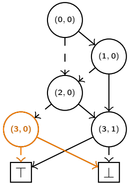
]

Output:

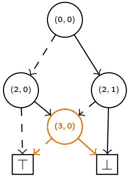
$(1,0) \xrightarrow{\top} (2,2)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

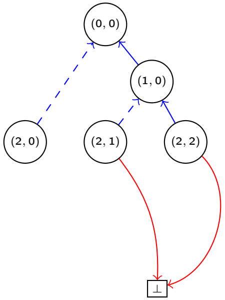
$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

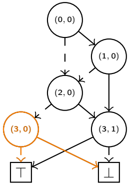
[

]

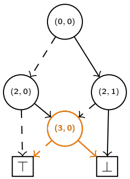
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\min((3,0),(3,0))$

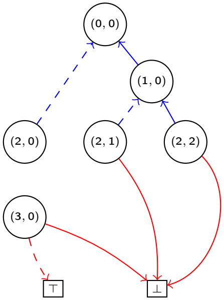
Priority Queue: $Q_{\textit{app}:1}$:

$$
\begin{aligned}
[\; & (2,1) \xrightarrow{\perp} ((3,0),(3,0)) \quad , \\
& (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \\
& (2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad , \\
& (2,0) \xrightarrow{\perp} ((3,0),\top) \quad]
\end{aligned}
$$

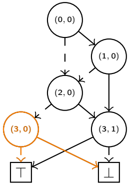
Priority Queue: $Q_{\textit{app}:2}$:

[ ]

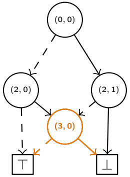
Output:
$(3,0) \xrightarrow{\perp} \top, (3,0) \xrightarrow{\top} \perp$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{\mathit{app}:1}$:

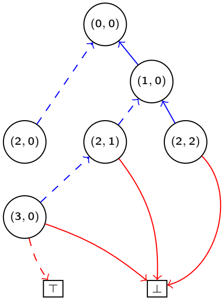
$$
\begin{aligned}
[\quad & \\
& (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \\
& (2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad , \\
& (2,0) \xrightarrow{\bot} ((3,0),\top) \quad]
\end{aligned}
$$

Priority Queue: $Q_{\mathit{app}:2}$:

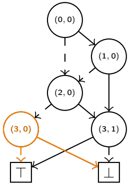
[ ]

Output:

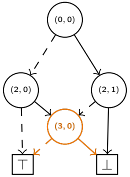
$(2,1) \xrightarrow{\bot} (3,0)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \vee x_3 \;:\; x_2 \wedge x_3)$

Seek:
$\min((3,1),(3,0))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

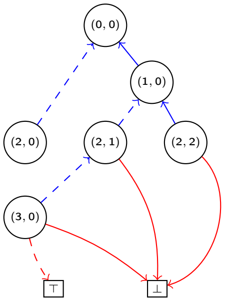
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

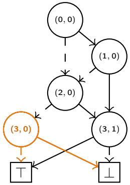
[

]

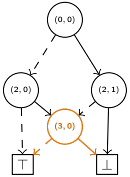
Output:



Apply Example (∧)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

min((3, 1), (3, 0))

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\bot} ((3, 0), \top)$]

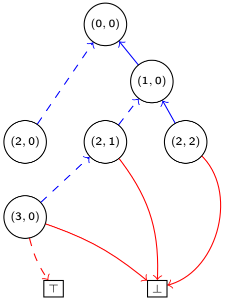
Priority Queue: $Q_{app:2}$:

[

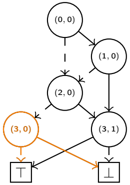
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \bot)$,

$(2, 2) \xrightarrow{\bot} ((3, 1), (3, 0)) \quad (\top, \bot)$]

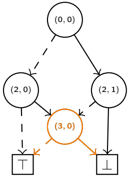
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \vee x_3 \;:\; x_2 \wedge x_3)$

Seek:
$\min((3,0), \top)$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

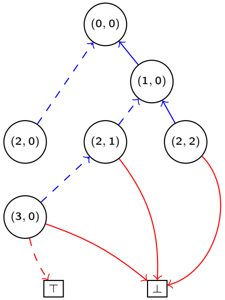
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

[

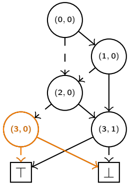
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

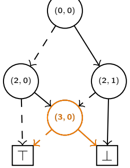
Output:



Apply Example (∧)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0), \top)$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

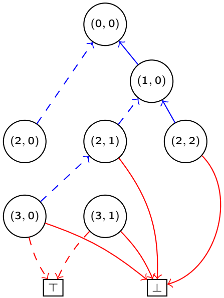
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

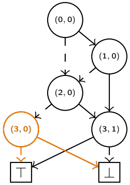
$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

Output:

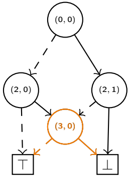
$(3,1) \xrightarrow{\perp} \top, (3,1) \xrightarrow{\top} \perp$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((3,0), \top)$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

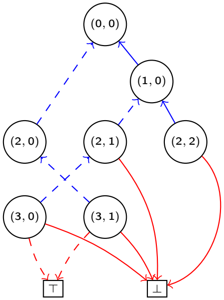
]

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

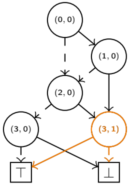
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\bot)$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top,\bot)$]

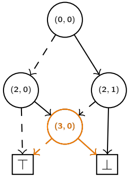
Output:
$(2,0) \xrightarrow{\bot} (3,1)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \;?\; x_2 \vee x_3 \;:\; x_2 \wedge x_3)$

Seek:
$\max((3,1),(3,0))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

]

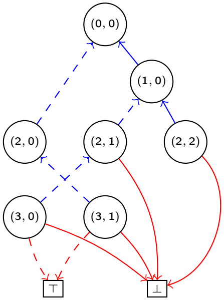
Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

[

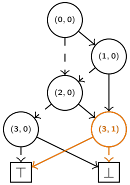
$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\bot)$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top,\bot)$]

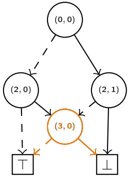
Output:



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:

$\max((3,1),(3,0))$

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

[

]

Priority Queue: $Q_{\boldsymbol{app}:\mathbf{2}}$:

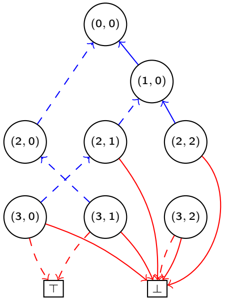
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top,\bot)$,

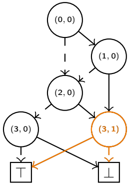
$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top,\bot)$]

Output:

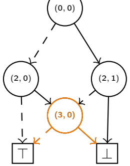
$(3,2) \xrightarrow{\bot} \bot, (3,2) \xrightarrow{\top} \bot$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Seek:
$\max((3, 1), (3, 0))$

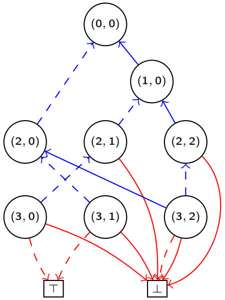
Priority Queue: $Q_{app:1}$:

[ ]

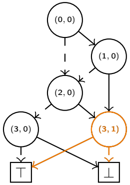
Priority Queue: $Q_{app:2}$:

[ ]

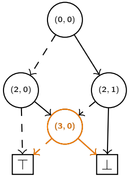
Output:
$(2, 0) \xrightarrow{\top} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$



Apply Example ($\wedge$)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

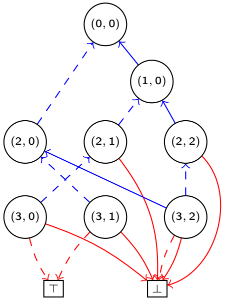
]

Priority Queue: $Q_{app:2}$:

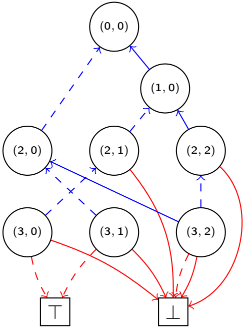
[

]

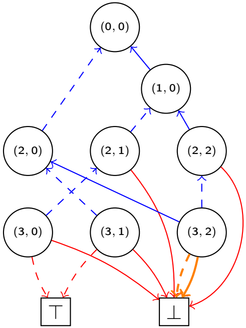
Output:



Reduce Example



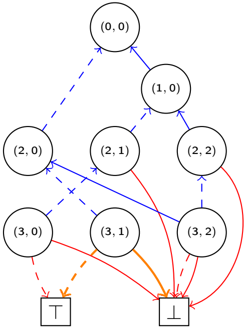
Reduce Example



Level: 3

[$[(3,2) \mapsto \perp]$]

Reduce Example



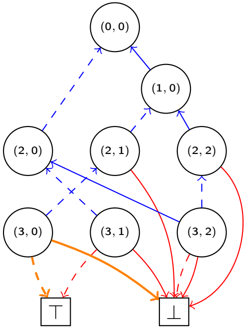
Level: 3

$[\qquad [(3,2) \mapsto \perp] \qquad]$

$[\qquad ((3,1), \top, \perp) \qquad ,$

$]$

Reduce Example

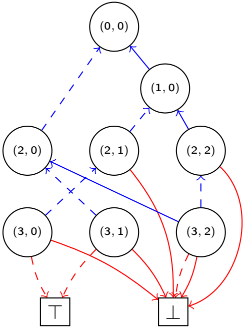


Level: 3

[$[(3,2) \mapsto \bot]$]

[$((3,1), \top, \bot)$,
$((3,0), \top, \bot)$]

Reduce Example

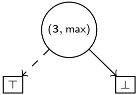


Output:
$((3, \text{max}), \top, \bot)$

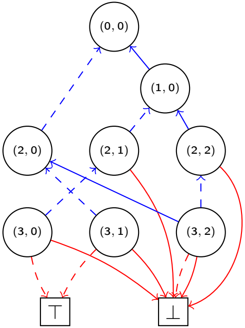
Level: 3

[$[(3, 2) \mapsto \bot]$ **]**

[$[(3, 1) \mapsto (3, \text{max})]$,
$((3, 0), \top, \bot)$ **]**



Reduce Example

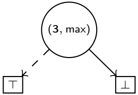


Output:

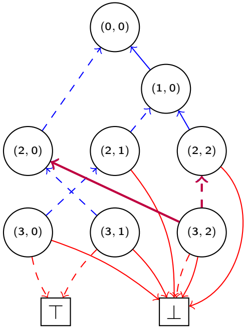
Level: 3

[$[(3,2) \mapsto \bot]$ **]**

[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$ **]**



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

$[\quad (2,2) \xrightarrow{\perp} \perp \quad ,$

$\quad (2,0) \xrightarrow{\top} \perp \quad ,$

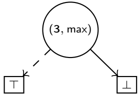
$]$

Level: 3

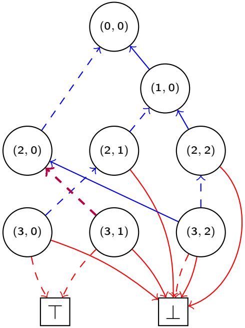
$[\quad]$

$[\quad [(3,1) \mapsto (3, \mathsf{max})] \quad ,$
$\quad [(3,0) \mapsto (3, \mathsf{max})] \quad]$

Output:



Reduce Example



Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,

$(2,0) \xrightarrow{\top} \perp$,

$(2,0) \xrightarrow{\perp} (3, \text{max})$,

]

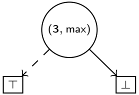
Level: 3

[ ]

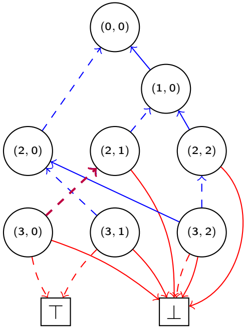
[,

$[(3,0) \mapsto (3, \text{max})]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

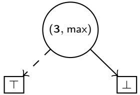
[$(2,2) \xrightarrow{\bot} \bot$,
$(2,1) \xrightarrow{\bot} (3, \text{max})$,
$(2,0) \xrightarrow{\top} \bot$,
$(2,0) \xrightarrow{\bot} (3, \text{max})$,
]

Level: 3

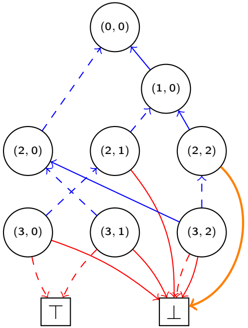
[ ]

[,
]

Output:



Reduce Example



Priority Queue: $Q_{\textit{red}}$:

[

$(2, 1) \xrightarrow{\perp} (3, \mathsf{max})$,

$(2, 0) \xrightarrow{\top} \perp$,

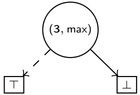
$(2, 0) \xrightarrow{\perp} (3, \mathsf{max})$,

]

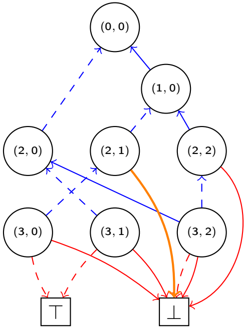
Level: 2

[$[(2, 2) \mapsto \perp]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(2,0) \xrightarrow{\top} \bot$,

$(2,0) \xrightarrow{\bot} (3, \mathrm{max})$,

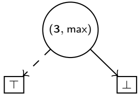
]

Level: 2

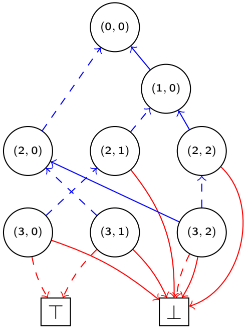
[$[(2,2) \mapsto \bot]$]

[$((2,1), (3, \mathrm{max}), \bot)$,
]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

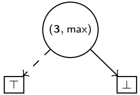
]

Level: 2

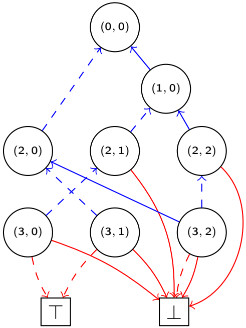
[$[(2,2) \mapsto \perp]$]

[$((2,1),(3,\mathrm{max}),\perp)$,
$((2,0),(3,\mathrm{max}),\perp)$]

Output:



Reduce Example



Priority Queue: $Q_{\textit{\textbf{red}}}$:

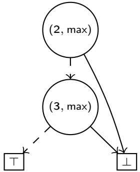
[ ]

Level: 2

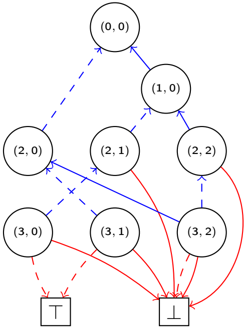
[$[(2,2) \mapsto \perp]$]

[$[(2,1) \mapsto (2, \text{max})]$,
$((2,0), (3, \text{max}), \perp)$]

Output:
$((2, \text{max}), (3, \text{max}), \perp)$



Reduce Example



Priority Queue: $Q_{\textbf{\textit{red}}}$:

[

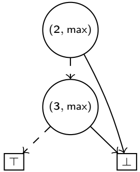
]

Level: 2

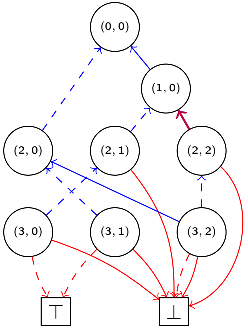
[$[(2,2) \mapsto \bot]$]

[$[(2,1) \mapsto (2,\text{max})]$,
$[(2,0) \mapsto (2,\text{max})]$]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

$(1,0) \xrightarrow{\top} \bot$,

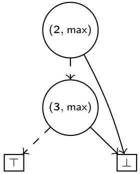
]

Level: 2

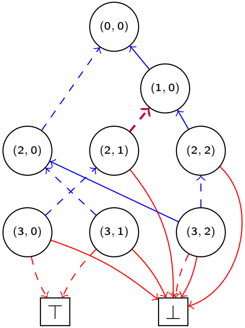
[ ]

[$[(2,1) \mapsto (2,\max)]$,
$[(2,0) \mapsto (2,\max)]$]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \mathrm{max})$,

]

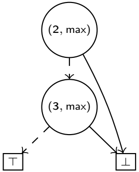
Level: 2

[ ]

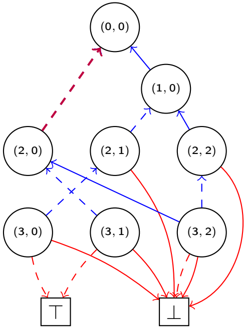
[,

$[(2,0) \mapsto (2, \mathrm{max})]$]

Output:



Reduce Example



Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,
$(1,0) \xrightarrow{\bot} (2, \text{max})$,

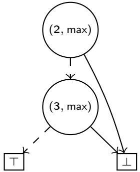
$(0,0) \xrightarrow{\bot} (2, \text{max})$]

Level: 2

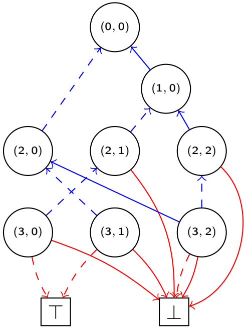
[ ]

[,
]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

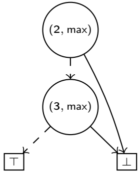
$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

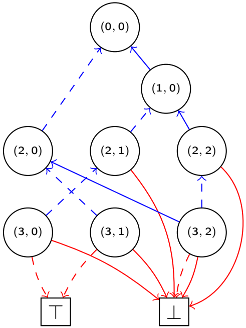
[ ]

[$((1, 0), (2, \max), \perp)$]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

$(0, 0) \xrightarrow{\perp} (2, \text{max})$]

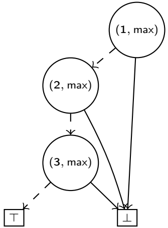
Level: 1

[ ]

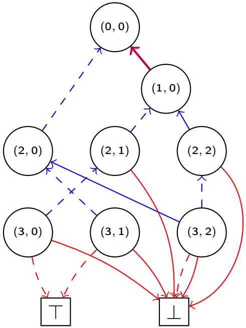
[$[(1, 0) \mapsto (1, \text{max})]$]

Output:

$((1, \text{max}), (2, \text{max}), \perp)$



Reduce Example



Priority Queue: $Q_{\mathbf{red}}$:

[

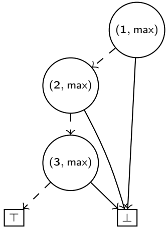
$(0, 0) \xrightarrow{\top} (1, \mathrm{max})$,
$(0, 0) \xrightarrow{\perp} (2, \mathrm{max})$]

Level: 1

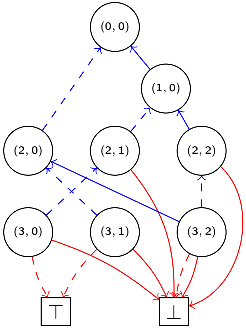
[ ]

[ ]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

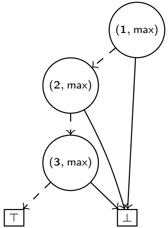
]

Level: 0

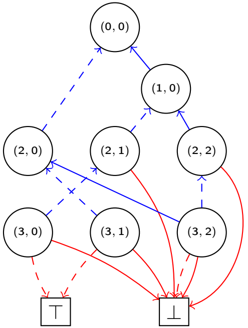
[ ]

[((0, 0), (2, max), (1, max))]

Output:



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[ ]

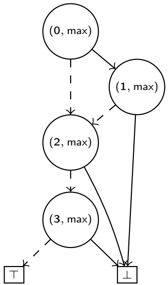
Level: 0

[ ]

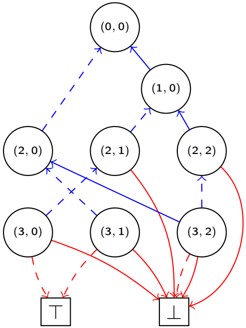
[$[(0,0) \mapsto (0, \text{max})]$]

Output:

$((0, \text{max}), (2, \text{max}), (1, \text{max}))$



Reduce Example



Priority Queue: $Q_{\boldsymbol{red}}$:

[

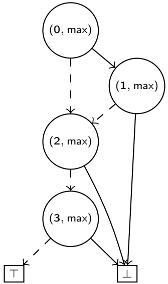
]

Level: 0

[ ]

[ ]

Output:



Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\qquad\qquad]$

Level: 2

$[\qquad , \qquad , \qquad]$

Level: 3

$[\qquad , \qquad , \qquad , \qquad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$\Big[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad \Big]$

Level: 2

$\Big[\quad (0,0) \xrightarrow{\bot} ((2,0),(2,0)) \quad , \quad , \quad \Big]$

Level: 3

$\Big[\quad , \quad , \quad , \quad \Big]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$\left[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad\right]$$

Level: 2

$$\left[\quad (0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad\right]$$

Level: 3

$$\left[\quad , \quad , \quad , \quad\right]$$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

[ ]

Level: 2

[$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$, $(1,0) \xrightarrow{\perp} ((2,0),(2,1))$, $(1,0) \xrightarrow{\top} ((3,1),(2,1))$]

Level: 3

[, , ,]

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

[ ]

Level: 2

[$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$, $(1,0) \xrightarrow{\perp} ((2,0),(2,1))$, $(1,0) \xrightarrow{\top} ((3,1),(2,1))$]

Level: 3

[$(2,0) \xrightarrow{\perp} ((3,0),\top)$, $(2,0) \xrightarrow{\top} ((3,1),(3,0))$, ,]

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad]$

Level: 2

$[\quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad]$

Level: 3

$[\quad (2,0) \xrightarrow{\perp} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad , \quad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad]$

Level: 2

$[\quad , \quad (1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad]$

Level: 3

$[\quad (2,0) \xrightarrow{\perp} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad (2,1) \xrightarrow{\perp} ((3,0),(3,0)) \quad , \quad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad]$

Level: 2

$[\quad , \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad]$

Level: 3

$[\quad (2,0) \xrightarrow{\bot} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad (2,1) \xrightarrow{\bot} ((3,0),(3,0)) \quad , \quad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad]$$

Level: 2

$$\left[\quad , \quad , \quad (1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad \right]$$

Level: 3

$$\left[(2,0) \xrightarrow{\bot} ((3,0),\top) \quad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad (2,1) \xrightarrow{\bot} ((3,0),(3,0)) \quad , \quad (2,2) \xrightarrow{\bot} ((3,1),(3,0)) \right]$$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[$(2,0) \xrightarrow{\perp} ((3,0),\top)$, $(2,0) \xrightarrow{\top} ((3,1),(3,0))$, $(2,1) \xrightarrow{\perp} ((3,0),(3,0))$, $(2,2) \xrightarrow{\perp} ((3,1),(3,0))$]

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$$[\quad]$$

Level: 2

$$[\quad , \quad , \quad]$$

Level: 3

$$\left[\ (2,1) \xrightarrow{\perp} ((3,0),(3,0)) \ ,\ (2,0) \xrightarrow{\top} ((3,1),(3,0)) \ ,\ (2,2) \xrightarrow{\perp} ((3,1),(3,0)) \ ,\ (2,0) \xrightarrow{\perp} ((3,0),\top)\ \right]$$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\qquad\qquad\qquad]$

Level: 2

$[\qquad\qquad , \qquad\qquad , \qquad\qquad]$

Level: 3

$[\qquad\qquad , \quad (2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad , \quad (2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad , \quad (2,0) \xrightarrow{\bot} ((3,0),\top) \quad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\quad]$

Level: 2

$[\quad , \quad , \quad]$

Level: 3

$[\quad , \quad , \quad (2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad , \quad (2,0) \xrightarrow{\perp} ((3,0),\top) \quad]$

Levelized Priority Queue: $Q_{\boldsymbol{app}:\mathbf{1}}$:

Level: 1

$[\qquad\qquad]$

Level: 2

$[\qquad , \qquad , \qquad]$

Level: 3

$[\qquad , \qquad , \qquad , \quad (2,0) \xrightarrow{\perp} ((3,0), \top) \quad]$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[ ]

Level: 2

[, ,]

Level: 3

[, , ,]

Memory layout and efficient sorting

The unique identifier of nodes and leafs can be represented in a single 64-bit integer.

(a) Unique identifier of a leaf `v`

(b) Unique identifier of an internal node

The `f` bit-flag is used to store the *is_high* boolean inside of the `source` of an arc.

Adiar

github.com/ssoelvsten/adiar

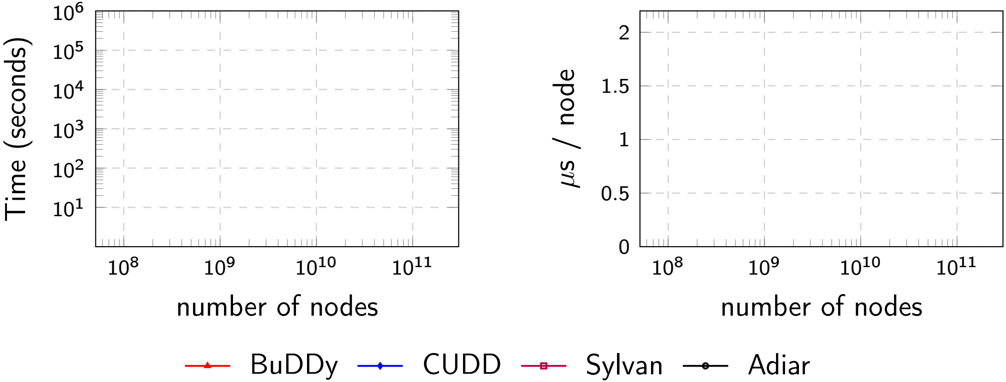


Figure 11: Minimum running times for the *N*-Queens problem.

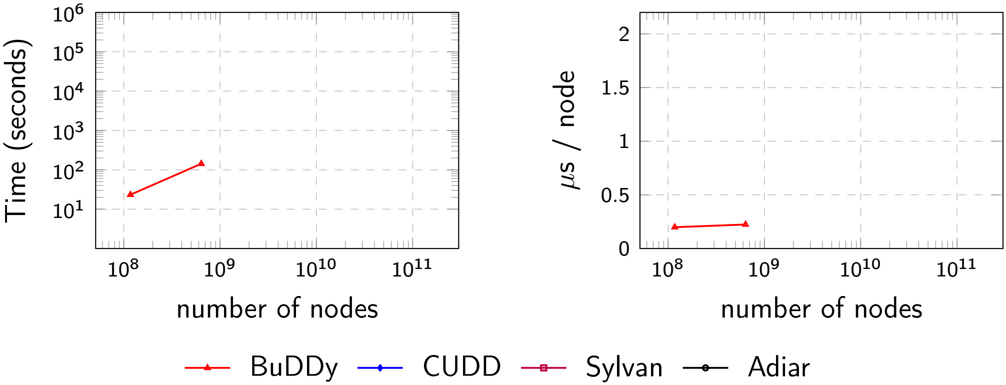


Figure 11: Minimum running times for the *N*-Queens problem.

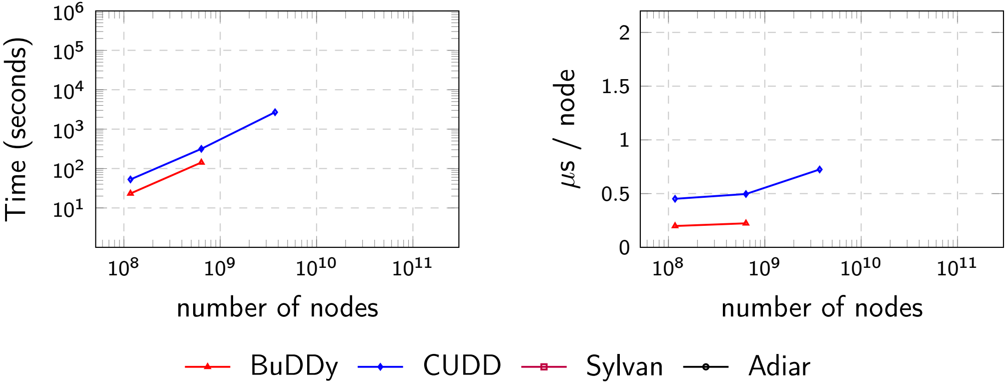


Figure 11: Minimum running times for the *N*-Queens problem.

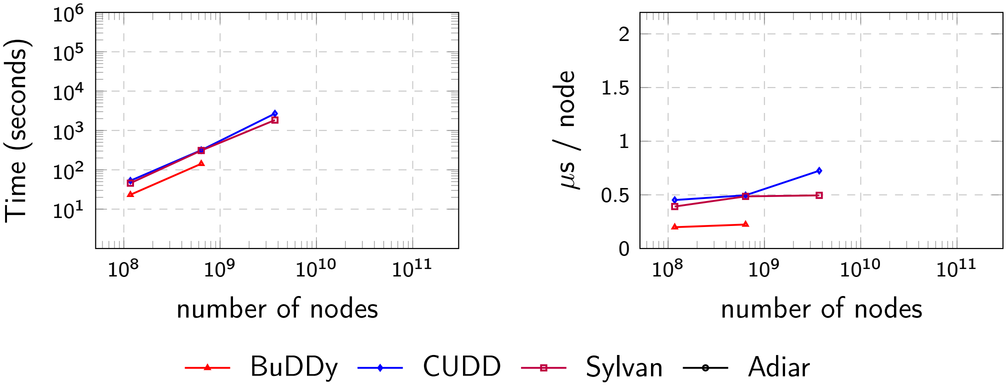


Figure 11: Minimum running times for the *N*-Queens problem.

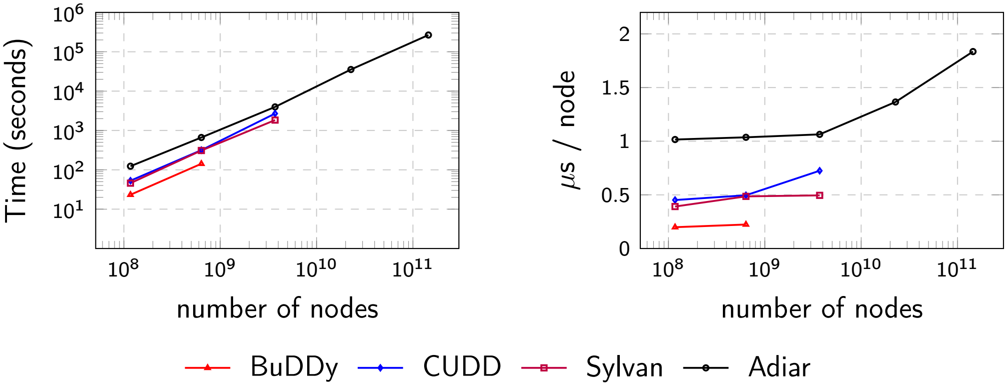
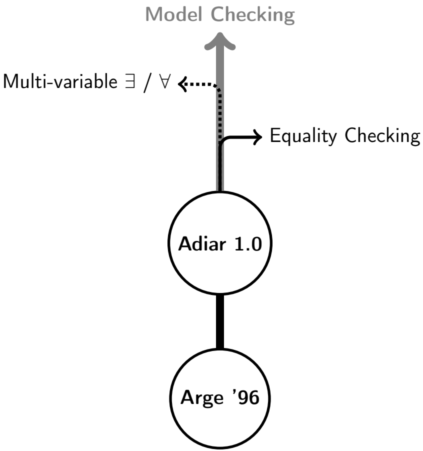
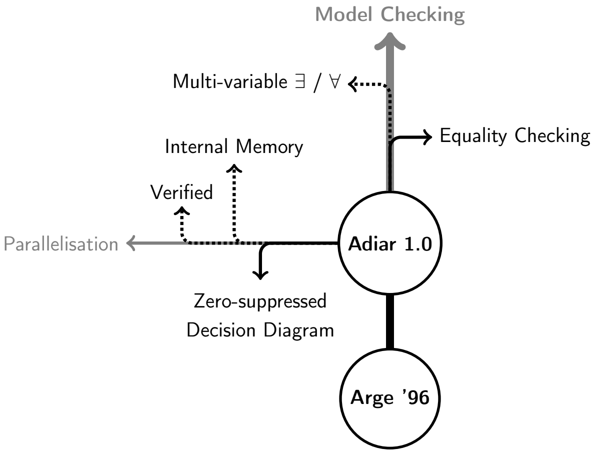


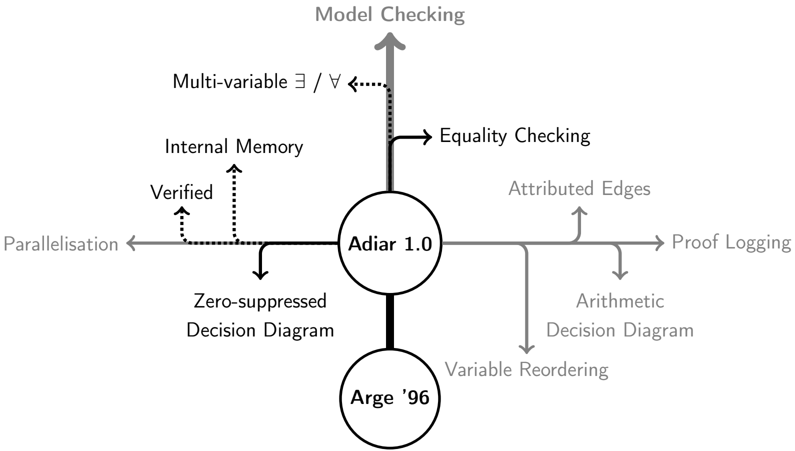
Figure 11: Minimum running times for the N-Queens problem.

Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f \ ? \ g \ : \ h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f\|_{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f\|_{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	$\#$paths in f to $\top$	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min / \max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$

Table 1: I/O-complexity of depth-first algorithms compared to our time-forwarded.


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Adiar 1.0
Arge '96


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Internal Memory
Verified
Parallelisation
Adiar 1.0
Zero-suppressed
Decision Diagram
Arge '96


Model Checking
Multi-variable ∃ / ∀
Equality Checking
Internal Memory
Verified
Attributed Edges
Parallelisation
Adiar 1.0
Proof Logging
Zero-suppressed
Decision Diagram
Arithmetic
Decision Diagram
Variable Reordering
Arge '96