

Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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The I/O model by Aggarwal and Vitter '87

For any realistic values of N , M , and B we have that

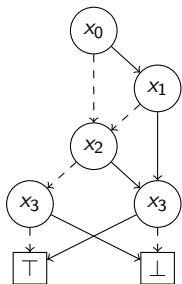
$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

Theorem (Aggarwal and Vitter '87)

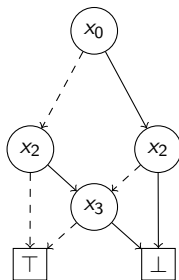
N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 \wedge x_2)$

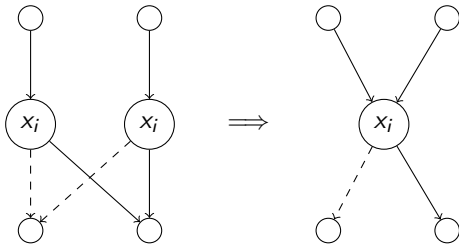
Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)

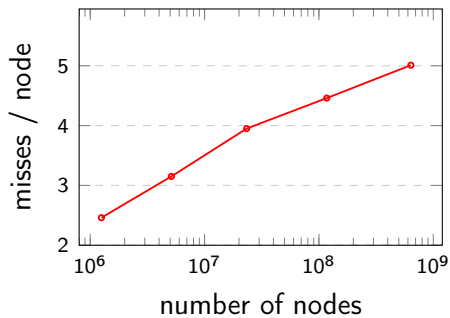
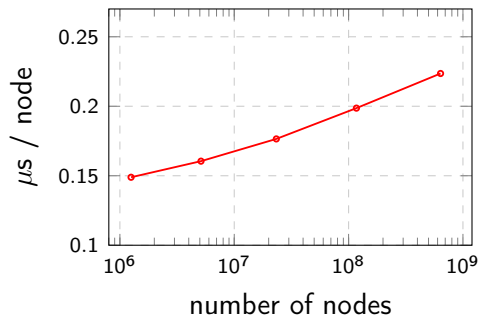
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



(2) Merge duplicate nodes

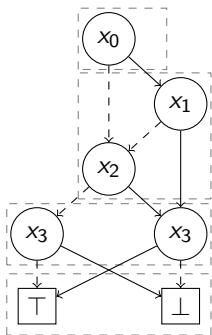


—●— BuDDy

Cache behaviour for the N -Queens problem.



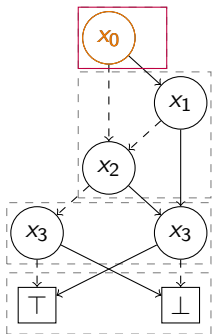
Running time for *Tic-Tac-Toe* with $N = 21$.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

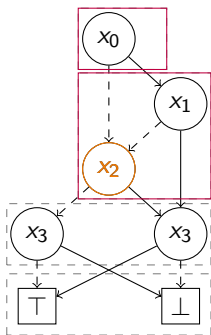
| node I/Os | cache lookups |
|-----------|---------------|
| 0 | 0 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

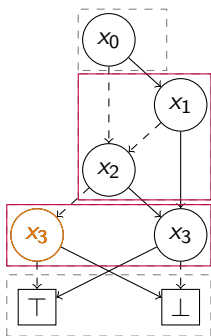
| node I/Os | cache lookups |
|-----------|---------------|
| 1 | 1 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

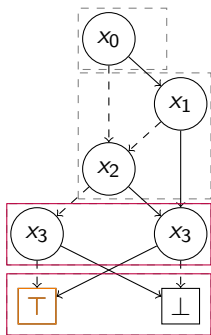
| node I/Os | cache lookups |
|-----------|---------------|
| 2 | 2 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

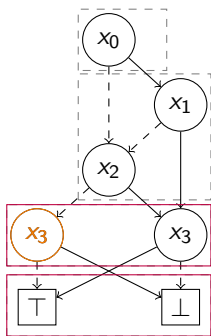
| node I/Os | cache lookups |
|-----------|---------------|
| 3 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

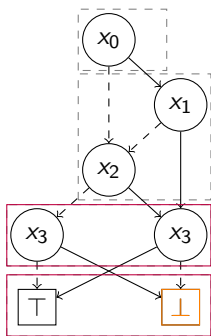
| node I/Os | cache lookups |
|-----------|---------------|
| 4 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

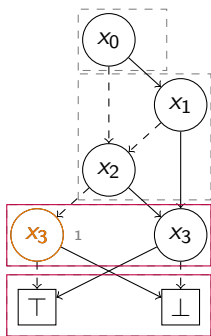
| node I/Os | cache lookups |
|-----------|---------------|
| 4 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

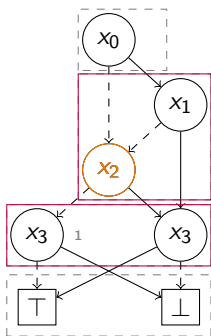
| node I/Os | cache lookups |
|-----------|---------------|
| 4 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

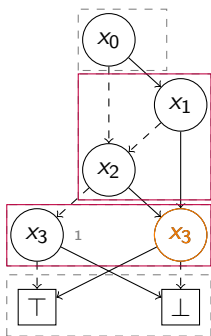
| node I/Os | cache lookups |
|-----------|---------------|
| 4 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

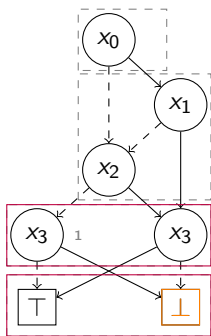
| node I/Os | cache lookups |
|-----------|---------------|
| 5 | 3 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

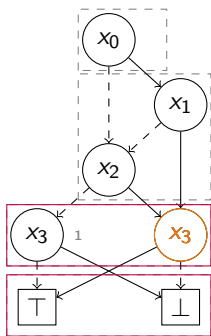
| node I/Os | cache lookups |
|-----------|---------------|
| 5 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

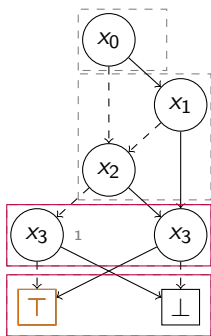
| node I/Os | cache lookups |
|-----------|---------------|
| 6 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

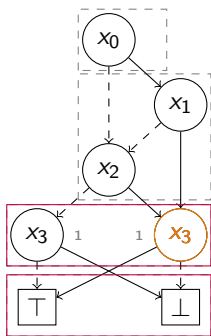
| node I/Os | cache lookups |
|-----------|---------------|
| 6 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

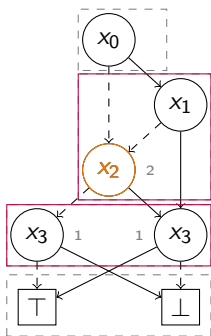
| node I/Os | cache lookups |
|-----------|---------------|
| 6 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

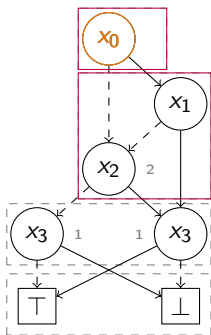
| node I/Os | cache lookups |
|-----------|---------------|
| 6 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

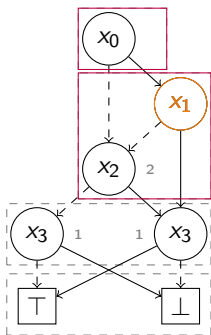
| node I/Os | cache lookups |
|-----------|---------------|
| 7 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

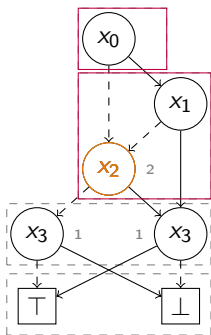
| node I/Os | cache lookups |
|-----------|---------------|
| 8 | 4 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

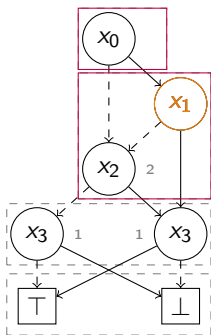
| node I/Os | cache lookups |
|-----------|---------------|
| 8 | 5 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

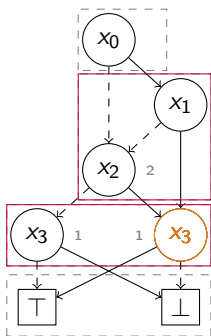
| node I/Os | cache lookups |
|-----------|---------------|
| 8 | 6 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

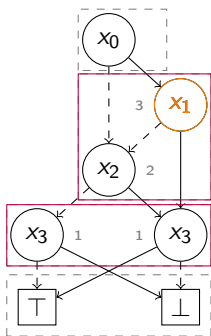
| node I/Os | cache lookups |
|-----------|---------------|
| 8 | 6 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

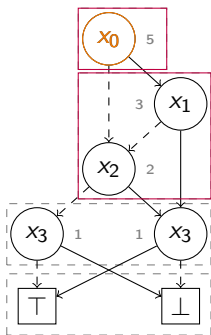
| node I/Os | cache lookups |
|-----------|---------------|
| 9 | 7 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

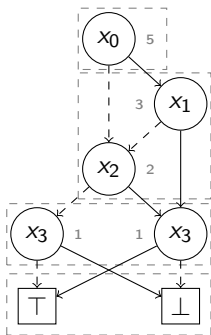
| node I/Os | cache lookups |
|-----------|---------------|
| 9 | 7 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

| node I/Os | cache lookups |
|-----------|---------------|
| 10 | 7 |



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

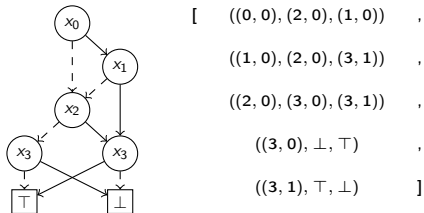
$$M = 4, B = 2$$

| node I/Os | cache lookups |
|-----------|---------------|
| 10 | 7 |

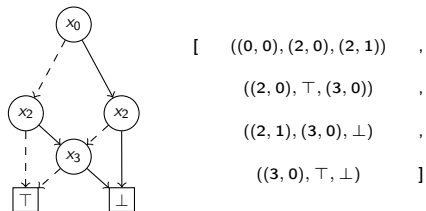




$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



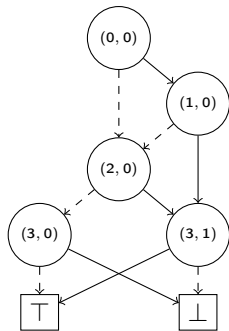
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

CountPaths Example



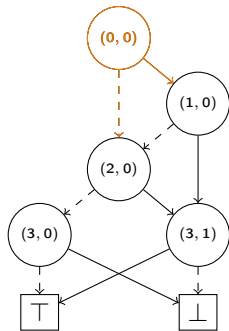
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example



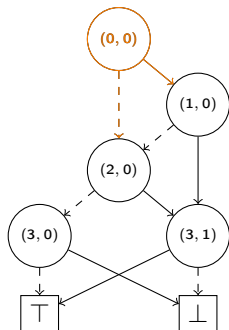
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example

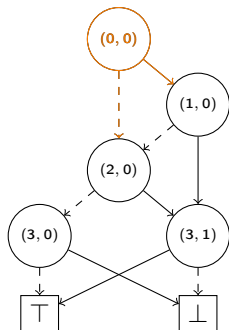


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

| Seek | Sum | Result |
|--------|-----|--------|
| (1, 0) | 0 | 0 |

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

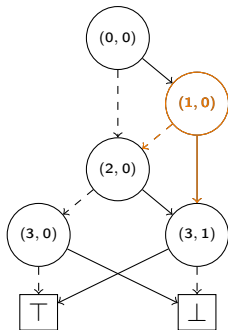
| Seek | Sum | Result |
|--------|-----|--------|
| (1, 0) | 0 | 0 |

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(1, 0)

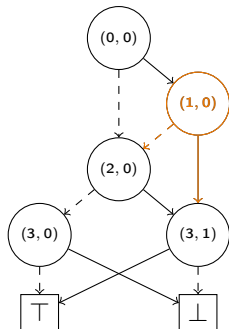
Sum
1

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

| Seek | Sum | Result |
|--------|-----|--------|
| (1, 0) | 1 | 0 |

Priority Queue: Q_{count} :

[

| | |
|--|---|
| $((0, 0) \xrightarrow{\perp} (2, 0), 1)$ | , |
| $((1, 0) \xrightarrow{\perp} (2, 0), 1)$ | , |
| $((1, 0) \xrightarrow{\top} (3, 1), 1)$ | , |

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

| Seek | Sum | Result |
|--------|-----|--------|
| (2, 0) | 1 | 0 |

Priority Queue: Q_{count} :

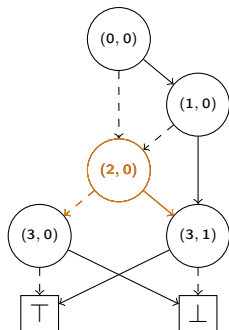
[

$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,

$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
2

Result
0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
2

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

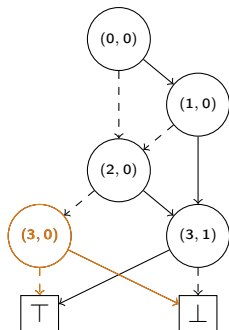
Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
2

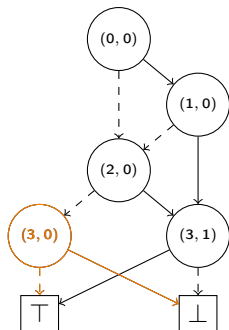
Result
0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
2

Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

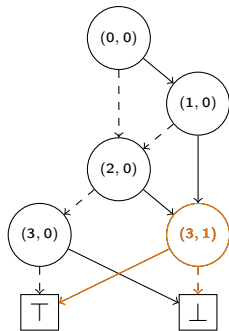
Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
1

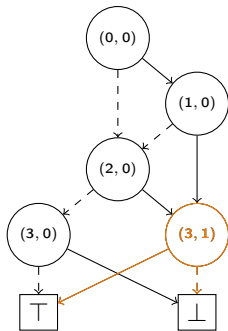
Result
2

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

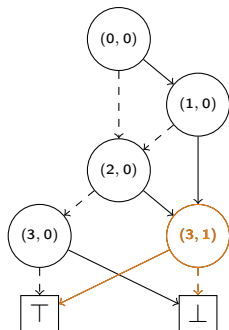
Result
2

Priority Queue: Q_{count} :

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

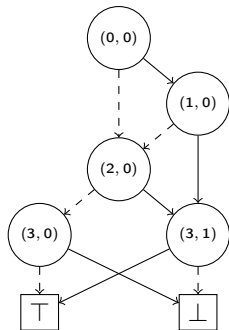
Result
5

Priority Queue: Q_{count} :

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

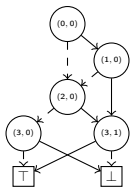
Result
5

Priority Queue: Q_{count} :

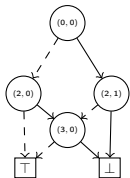
[

]

Apply Example (\wedge)

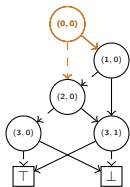


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

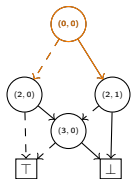
Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: $Q_{app:1}$:

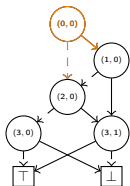
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



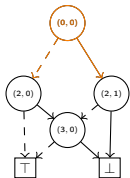
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



]

Apply Example (\wedge)

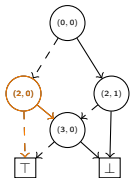


Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:
 $[\quad (0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) \quad ,$
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \quad ,$



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

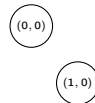
Seek:

$\min((1, 0), (2, 1))$

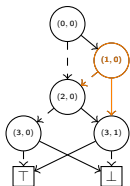
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{T} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{T} ((3, 1), (2, 1))$,

]



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

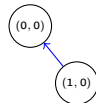
Priority Queue: $Q_{app:1}$:

[

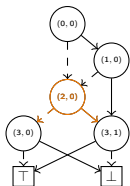
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:
 $(0, 0) \xrightarrow{\top} (1, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

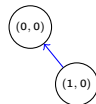
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

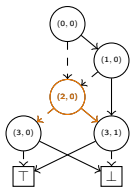
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

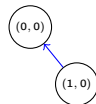
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

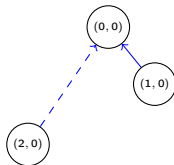
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

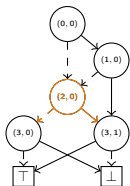
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:

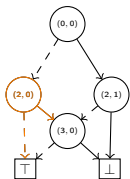
$(0, 0) \xrightarrow{\perp} (2, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app}:1$:

[

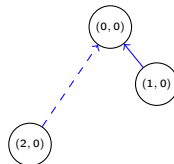
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

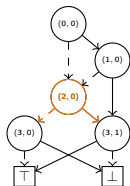
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

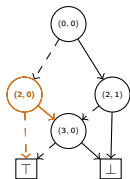
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

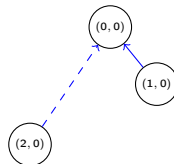
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

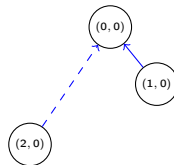
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

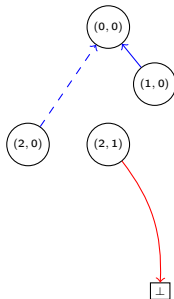
Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:

$(2, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

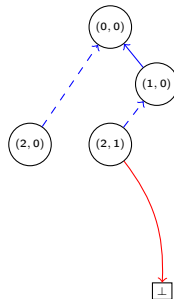
Priority Queue: $Q_{app:2}$:

[

]

Output:

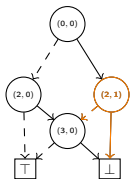
$(1, 0) \xrightarrow{\perp} (2, 1)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

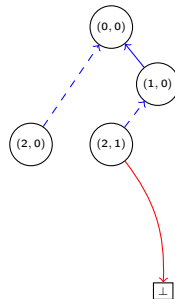
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

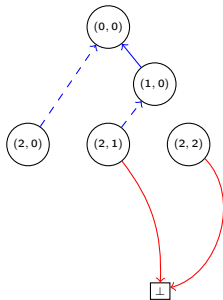
Priority Queue: $Q_{app:2}$:

[

]

Output:

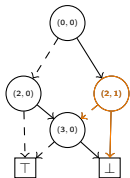
$(2, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

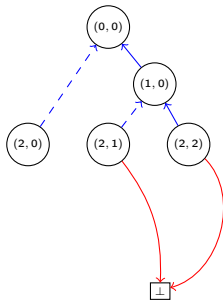
Priority Queue: $Q_{app:2}$:

[

]

Output:

$(1, 0) \xrightarrow{\top} (2, 2)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

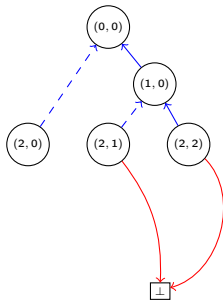
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

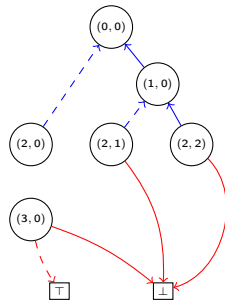
Priority Queue: $Q_{app:2}$:

[

]

Output:

$(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

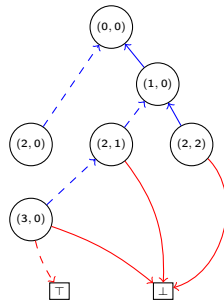
Priority Queue: $Q_{app:2}$:

[

]

Output:

$(2, 1) \xrightarrow{\perp} (3, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

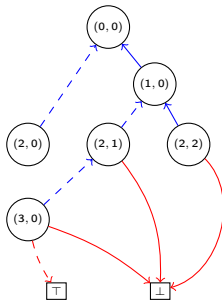
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

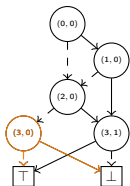
[

]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

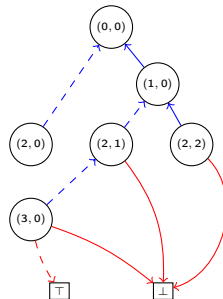
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

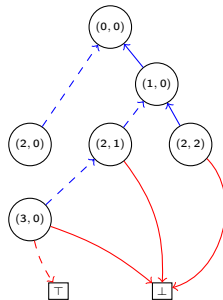
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

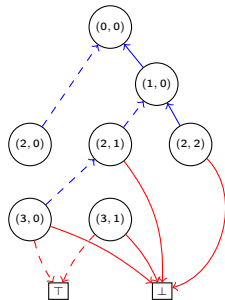
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

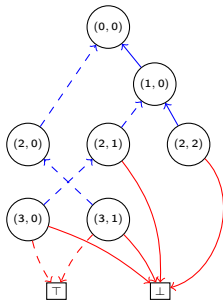
Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

Output:

$(2, 0) \xrightarrow{\perp} (3, 1)$



Priority Queue: $Q_{app:2}$:

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

Priority Queue: $Q_{app:2}$:

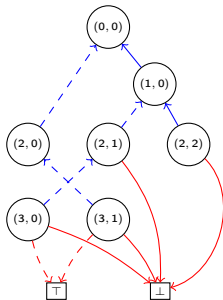
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

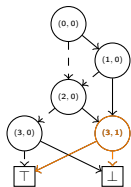
,

]

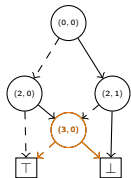
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

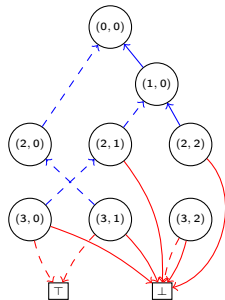
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

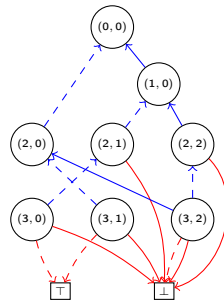
Output:

$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$

]

Priority Queue: $Q_{app:2}$:

[



]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

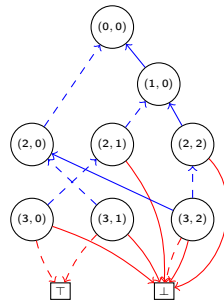
Priority Queue: $Q_{app:2}$:

[

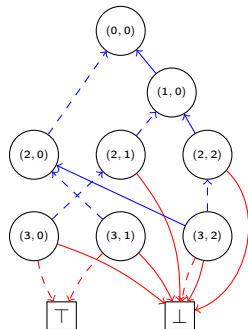
]

]

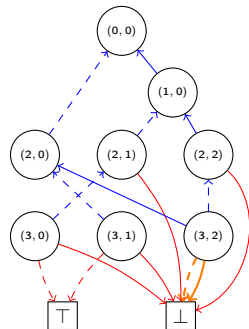
Output:



Reduce Example

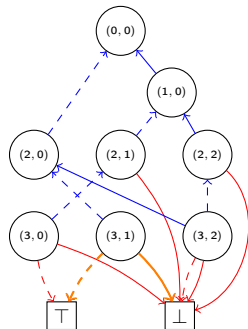


Reduce Example



[Level: 3
 $[(3, 2) \mapsto \perp]$]

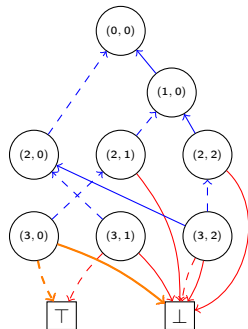
Reduce Example



Level: 3

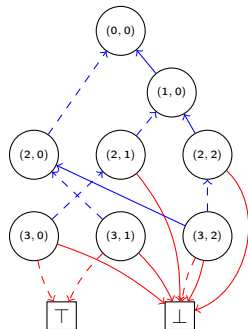
| | | |
|---|--------------------------|---|
| [| $[(3, 2) \mapsto \perp]$ |] |
| [| $((3, 1), \top, \perp)$ | , |
| | |] |

Reduce Example



Level: 3
[$[(3, 2) \mapsto \perp]$]
[$((3, 1), \top, \perp)$,
 $((3, 0), \top, \perp)$]

Reduce Example



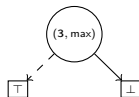
Level: 3

[$[(3, 2) \mapsto \perp]$]

[$[(3, 1) \mapsto (3, \max)]$,

$((3, 0), \top, \perp)$]

Output:
 $((3, \max), \top, \perp)$



Reduce Example



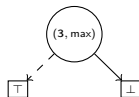
Level: 3

[$[(3, 2) \mapsto \perp]$]

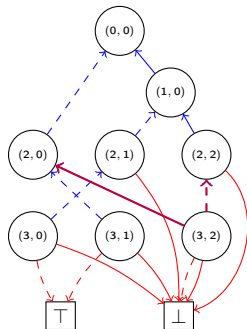
[$[(3, 1) \mapsto (3, \max)]$,

$[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{T} \perp$,

]

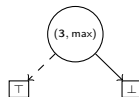
Level: 3

[]

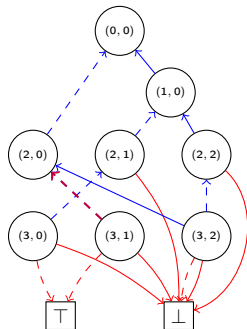
[$[(3, 1) \mapsto (3, \max)]$,

$[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 3

[

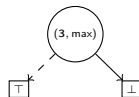
]

[

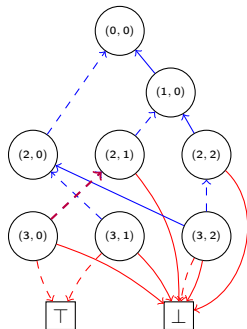
$[(3, 0) \mapsto (3, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

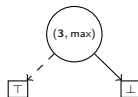
[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

]

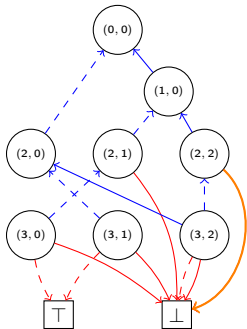
Level: 3

[
 [
 ,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$$(2, 1) \xrightarrow{\perp} (3, \max) \quad ,$$
$$(2, 0) \xrightarrow{T} \perp$$
$$(2, 0) \xrightarrow{\perp} (3, \max) \quad ,$$

1

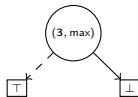
Level: 2

[

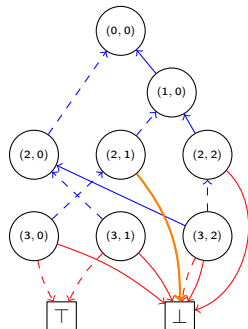
 $[(2, 2) \mapsto \perp]$

1

Output:



Reduce Example



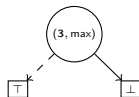
Priority Queue: Q_{red} :

[
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

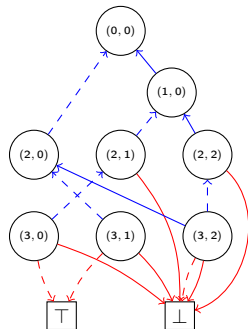
Level: 2

[$[(2, 2) \mapsto \perp]$]
 [$((2, 1), (3, \max), \perp)$,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

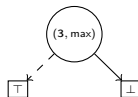
$((2, 1), (3, \max), \perp)$

,

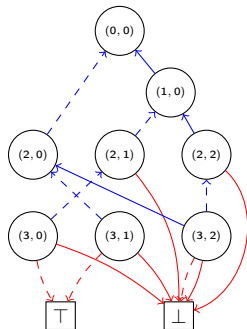
$((2, 0), (3, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

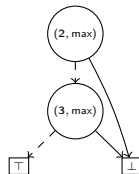
,

$((2, 0), (3, \max), \perp)$

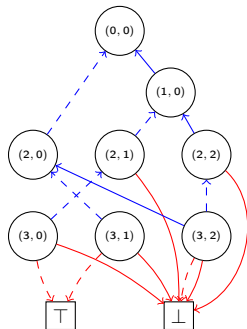
]

Output:

$((2, \max), (3, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

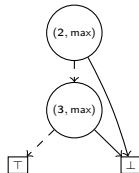
$[(2, 1) \mapsto (2, \max)]$

,

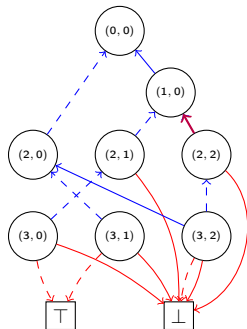
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

]

Level: 2

[

]

[

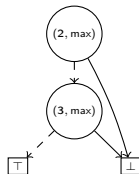
$[(2, 1) \mapsto (2, \max)]$

,

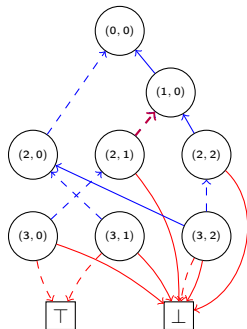
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

[

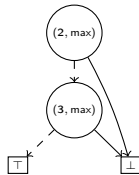
$[(2, 0) \mapsto (2, \max)]$

]

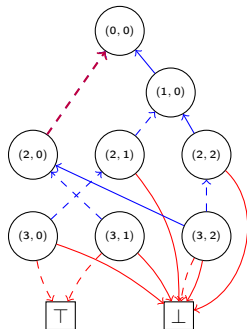
,

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

[

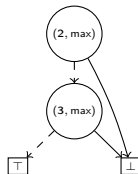
[

]

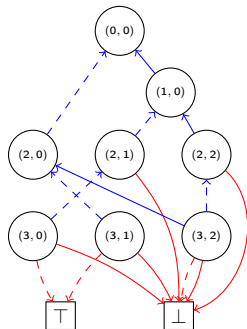
,

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

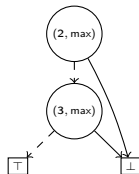
]

[

$((1, 0), (2, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

]

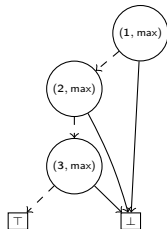
[

$[(1, 0) \mapsto (1, \max)]$

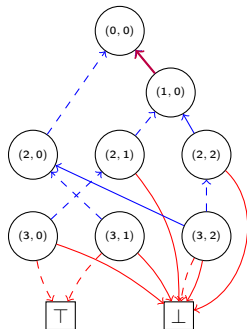
]

Output:

$((1, \max), (2, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{T} (1, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

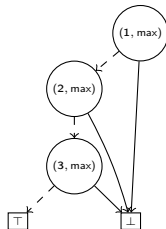
[

]

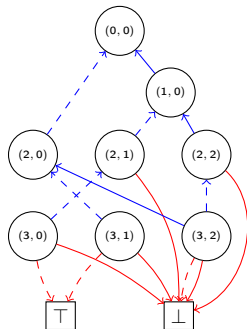
[

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

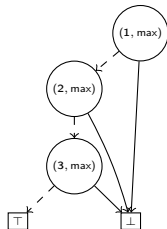
]

[

((0, 0), (2, max), (1, max))

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

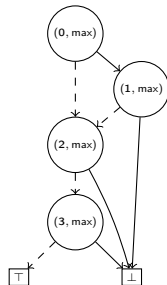
]

$[(0, 0) \mapsto (0, \max)]$

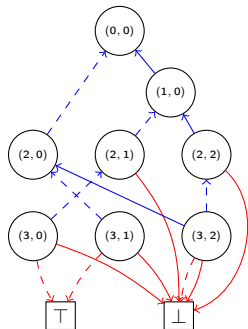
]

Output:

$((0, \max), (2, \max), (1, \max))$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

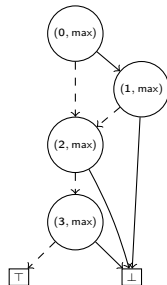
[

]

[

]

Output:



Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$[\quad]$$

Level: 2

[, ,]

Level: 3

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$\left[(0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) \right]$$

Level: 2

$$\left[(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \quad , \quad \quad \quad \right]$$

Level: 3

$$\left[\quad , \quad , \quad , \quad \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$\left[(0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) \right]$$

Level: 2

$$\left[(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \quad , \quad (1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad , \quad (1, 0) \xrightarrow{\top} ((3, 1), (2, 1)) \right]$$

Level: 3

$$\left[\quad , \quad , \quad , \quad \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[, , ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, (1, 0) $\xrightarrow{\top}$ ((3, 1), (2, 1))]

Level: 3

[(2, 0) $\xrightarrow{\perp}$ ((3, 0), \top) , (2, 0) $\xrightarrow{\top}$ ((3, 1), (3, 0)) , (2, 1) $\xrightarrow{\perp}$ ((3, 0), (3, 0)) ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, (1, 0) $\xrightarrow{\top}$ ((3, 1), (2, 1))]

Level: 3

[(2, 0) $\xrightarrow{\perp}$ ((3, 0), \top) , (2, 0) $\xrightarrow{\top}$ ((3, 1), (3, 0)) , (2, 1) $\xrightarrow{\perp}$ ((3, 0), (3, 0)) , (2, 2) $\xrightarrow{\perp}$ ((3, 1), (3, 0))]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$, $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, ,]

Level: 3

[, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$, $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[, (2, 2) $\xrightarrow{\perp}$ ((3, 1), (3, 0)) , (2, 0) $\xrightarrow{\perp}$ ((3, 0), \top)]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, ,]

Level: 3

[, , , $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$[\quad]$$

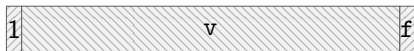
Level: 2

[, ,]

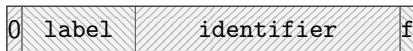
Level: 3

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The unique identifier of nodes and leaves can be represented in a single 64-bit integer.



(a) Unique identifier of a leaf v

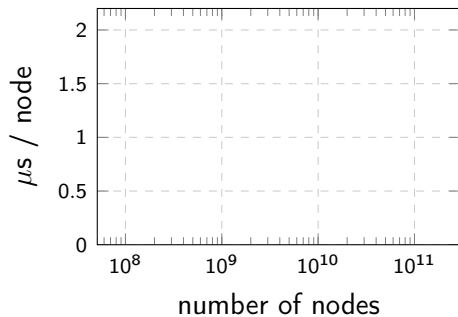
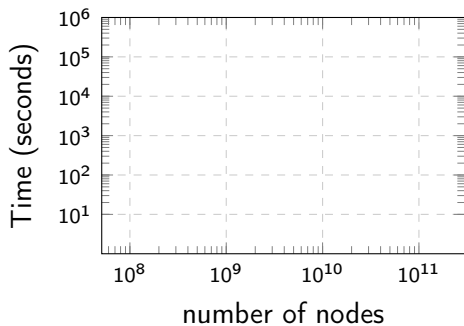


(b) Unique identifier of an internal node

The f bit-flag is used to store the *is_high* boolean inside of the source of an arc.

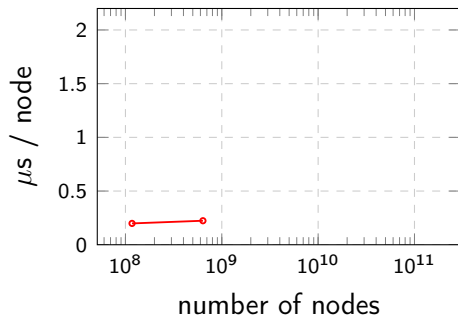
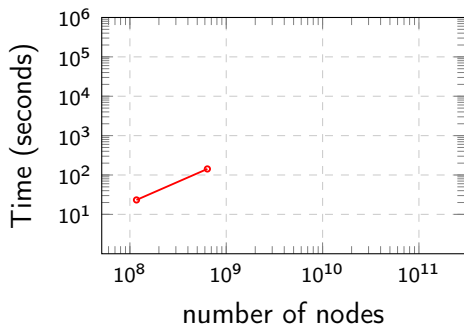
Adiar

github.com/ssoelvsten/adiar



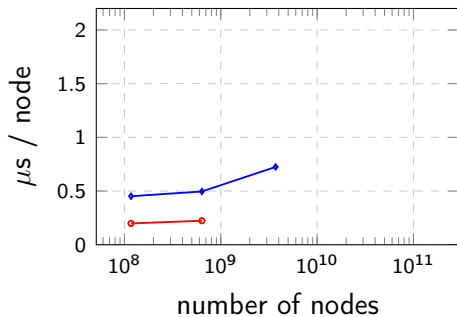
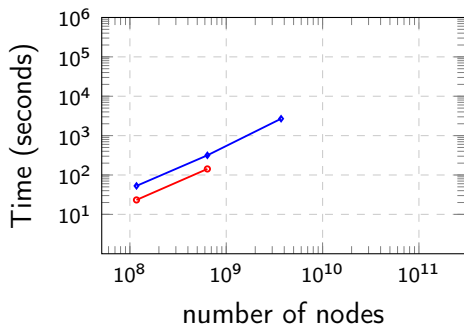
—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



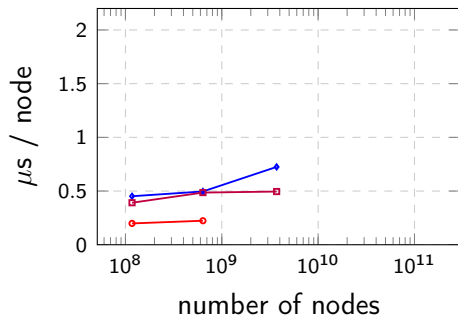
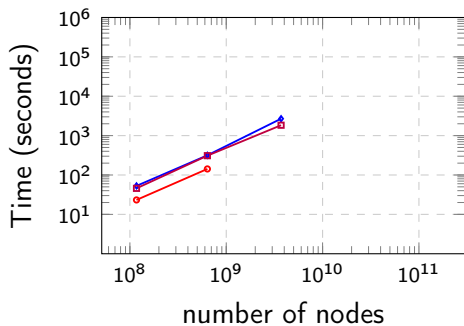
—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



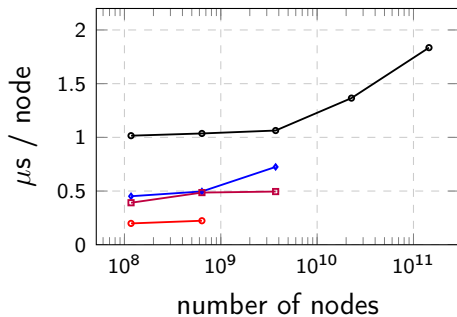
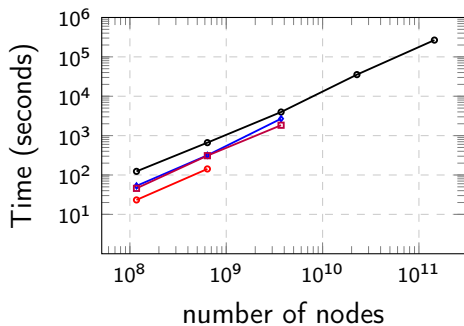
—○— BuDDy —◇— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



—○— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.

| Algorithm | | Depth-first | Time-forwarded |
|------------------|----------------------------------|------------------------------|---|
| Reduce | | $O(N)$ | $O(\text{sort}(N))$ |
| BDD Manipulation | | | |
| Apply | $f \odot g$ | $O(N_f \cdot N_g)$ | $O(\text{sort}(N_f \cdot N_g))$ |
| If-Then-Else | $f ? g : h$ | $O(N_f \cdot N_g \cdot N_h)$ | $O(\text{sort}(N_f \cdot N_g \cdot N_h))$ |
| Restrict | $f _{x_i=v}$ | $O(N)$ | $O(\text{sort}(N))$ |
| Negation | $\neg f$ | $O(1)$ | $O(1)$ |
| Quantification | $\exists/\forall v : f _{x_i=v}$ | $O(N^2)$ | $O(\text{sort}(N^2))$ |
| Counting | | | |
| Count Paths | #paths in f to \top | $O(N)$ | $O(\text{sort}(N))$ |
| Count SAT | $\#x : f(x)$ | $O(N)$ | $O(\text{sort}(N))$ |
| Other | | | |
| Equality | $f \equiv g$ | $O(1)$ | $O(\text{sort}(N))$ |
| Evaluate | $f(x)$ | $O(L)$ | $O(N/B)$ |
| Min/Max SAT | $\min / \max\{x \mid f(x)\}$ | $O(L)$ | $O(N/B)$ |

