

# Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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The I/O model by Aggarwal and Vitter '87

For any realistic values of  $N$ ,  $M$ , and  $B$  we have that

$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

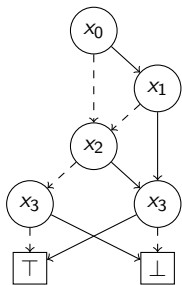
**Theorem (Aggarwal and Vitter '87)**

*$N$  elements can be sorted in  $\Theta(\text{sort}(N))$  I/Os.*

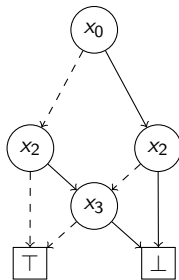
**Theorem (Arge '95)**

*$N$  elements can be inserted in and extracted from a Priority Queue in  $\Theta(\text{sort}(N))$  I/Os.*





**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 \oplus x_2 \vee x_3 \wedge x_2)$

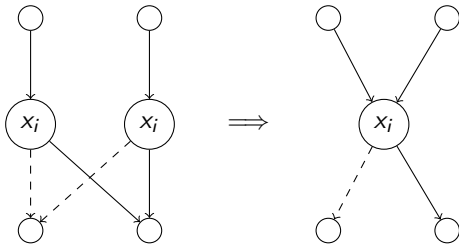
Examples of (Reduced Ordered) Binary Decision Diagrams.

### Theorem (Bryant '86)

*For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.*



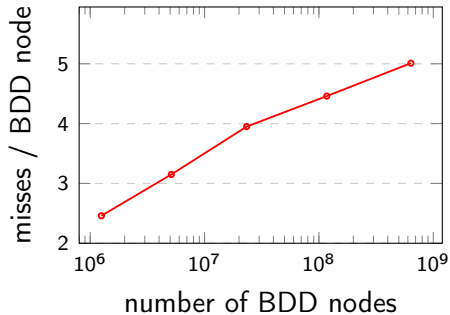
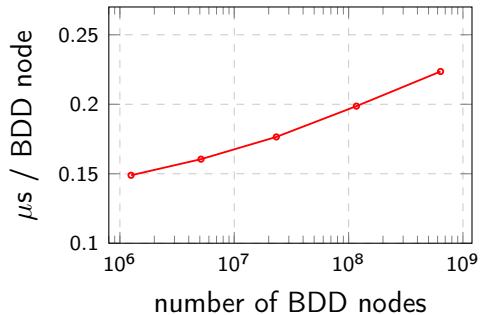
**(1)** Remove redundant nodes



**(2)** Merge duplicate nodes





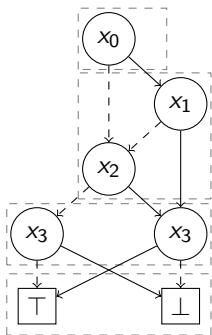


—●— BuDDy

Cache behaviour for the  $N$ -Queens problem.



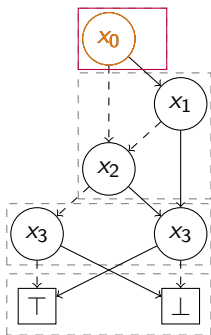
Running time for *Tic-Tac-Toe* with  $N = 21$ .



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

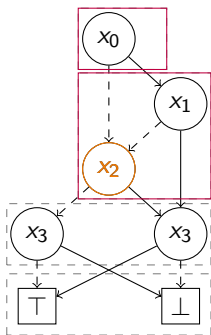
node I/Os	cache lookups
0	0



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

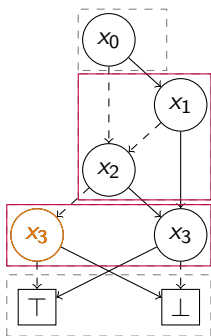
node I/Os	cache lookups
1	1



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

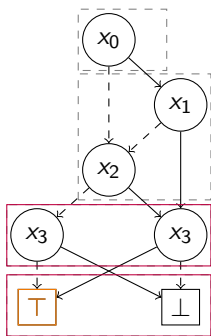
node I/Os	cache lookups
2	2



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

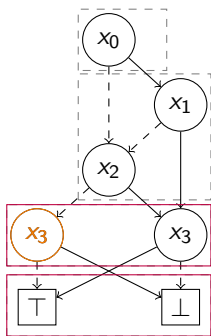
node I/Os	cache lookups
3	3



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

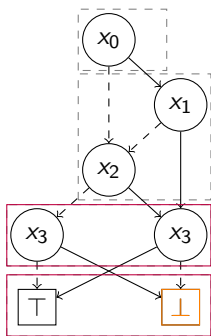


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

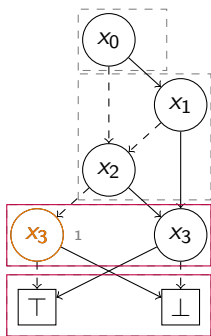




**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

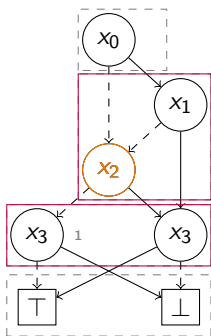
node I/Os	cache lookups
4	3



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$$M = 4, B = 2$$

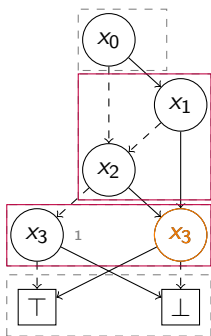
node I/Os	cache lookups
4	3



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

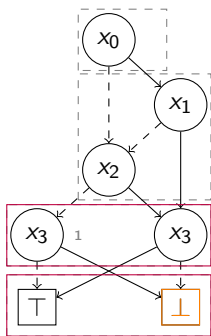
node I/Os	cache lookups
5	3



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

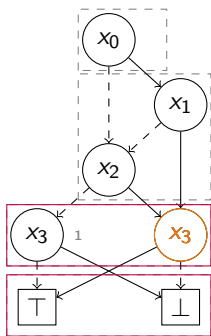
node I/Os	cache lookups
5	4



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

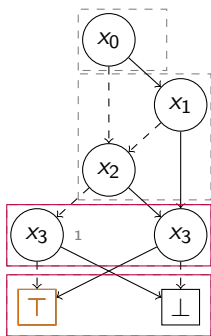
node I/Os	cache lookups
6	4



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

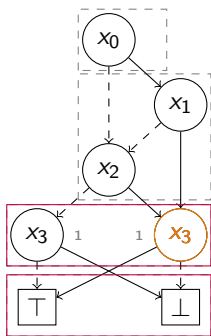
node I/Os	cache lookups
6	4



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

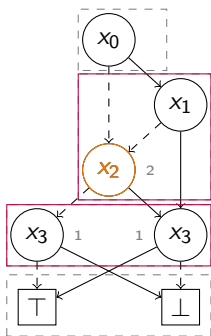


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

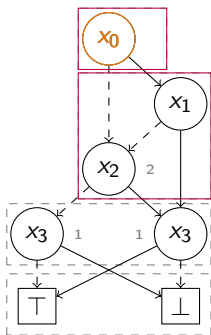




**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

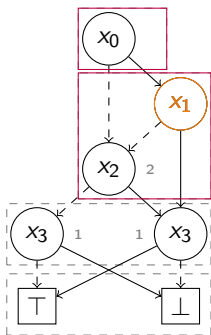
node I/Os	cache lookups
7	4



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

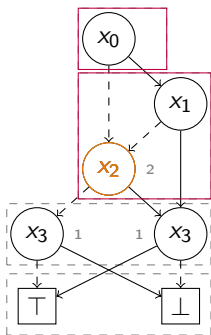
node I/Os	cache lookups
8	4



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

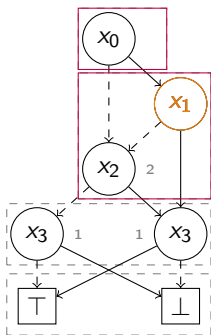
node I/Os	cache lookups
8	5



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

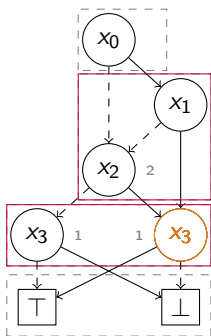
node I/Os	cache lookups
8	6



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

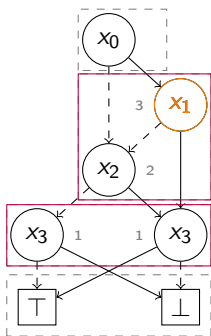
node I/Os	cache lookups
8	6



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

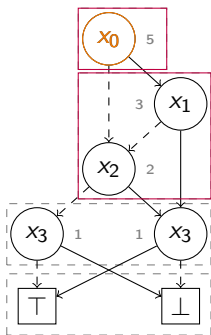
node I/Os	cache lookups
9	7



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
9	7

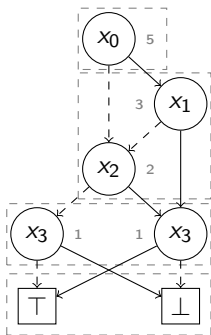


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7





**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

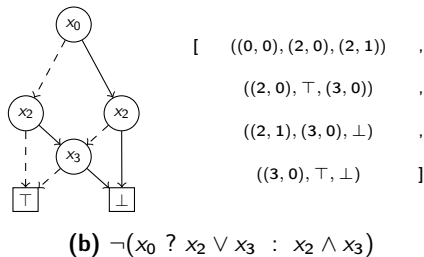
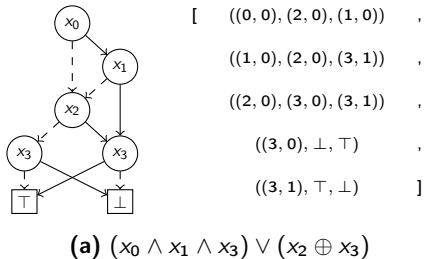
node I/Os	cache lookups
10	7





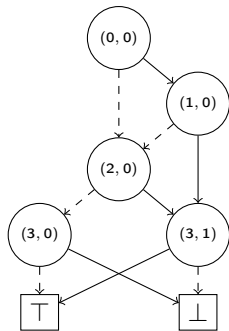


$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



Node-based representation of prior shown BDDs

## CountPaths Example



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

# CountPaths Example



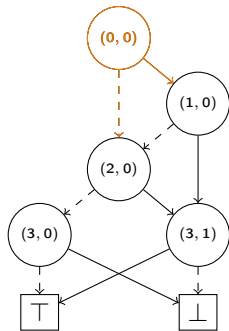
**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue:  $Q_{count}$ :

[

]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

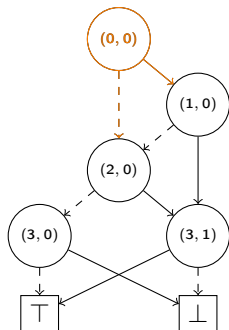
Priority Queue:  $Q_{count}$ :

[

]



# CountPaths Example

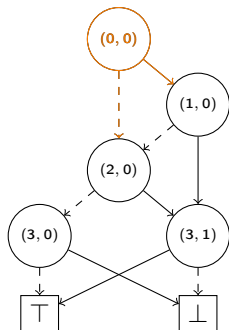


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue:  $Q_{count}$ :

[  $((0, 0) \xrightarrow{\top} (1, 0), 1)$  ,  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	0	0

Priority Queue:  $Q_{count}$ :

[  $((0, 0) \xrightarrow{\top} (1, 0), 1)$  ,  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,

]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(1, 0)

Sum  
0

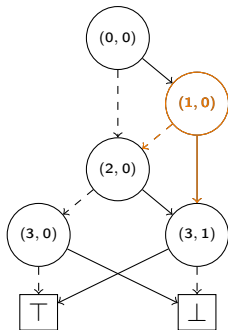
Result  
0

Priority Queue:  $Q_{count}$ :

[  $((0, 0) \xrightarrow{T} (1, 0), 1)$  ,  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,

]

# CountPaths Example



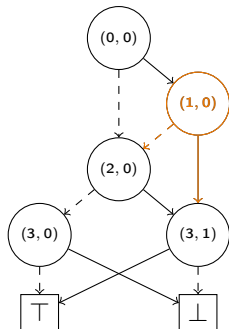
(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

Priority Queue:  $Q_{count}$ :

[  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

Priority Queue:  $Q_{count}$ :

[  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	0	0

Priority Queue:  $Q_{count}$ :

[

$((0, 0) \xrightarrow{\perp} (2, 0), 1)$	,
$((1, 0) \xrightarrow{\perp} (2, 0), 1)$	,
$((1, 0) \xrightarrow{\top} (3, 1), 1)$	,

]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(2, 0)

Sum  
0

Result  
0

Priority Queue:  $Q_{count}$ :

[  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	1	0

Priority Queue:  $Q_{count}$ :

[

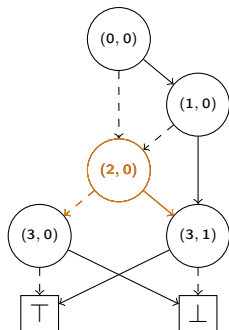
$((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,

$((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,

]



# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(2, 0)

Sum  
2

Result  
0

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3, 0)

Sum  
0

Result  
0

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3, 0)

Sum  
0

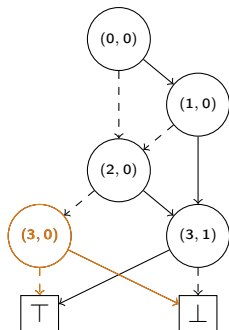
Result  
0

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3, 0)

Sum  
2

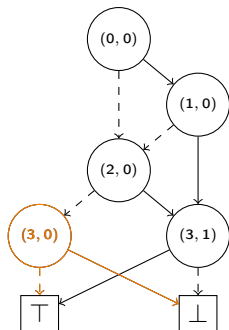
Result  
0

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	2	2

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3, 1)

Sum  
0

Result  
2

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
**(3, 1)**

Sum  
0

Result  
2

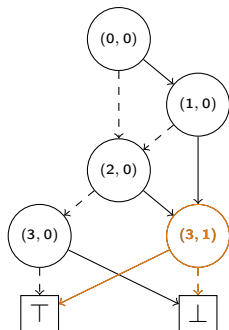
Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]



# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
**(3, 1)**

Sum  
1

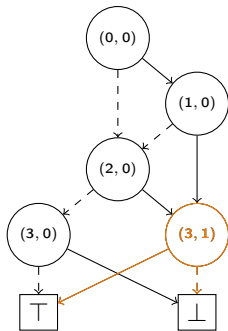
Result  
2

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad ]$

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3,1)

Sum  
3

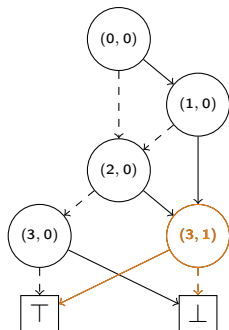
Result  
2

Priority Queue:  $Q_{count}$ :

[

]

# CountPaths Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek  
(3, 1)

Sum  
3

Result  
5

Priority Queue:  $Q_{count}$ :

[

]

# CountPaths Example



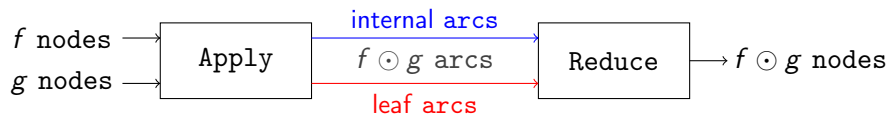
**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result  
5

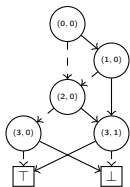
Priority Queue:  $Q_{count}$ :

[

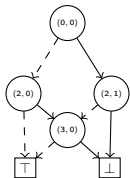
]



# Apply Example ( $\wedge$ )



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

# Apply Example ( $\wedge$ )

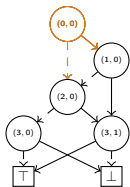


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

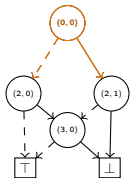
# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue:  $Q_{app:1}$ :

[  $(0,0) \xrightarrow{\top} ((1,0), (2,1))$  ,  
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$  ,

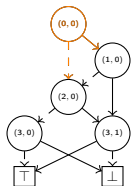


(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

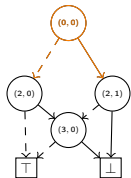
]



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[  $(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$  ,  
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,



]

# Apply Example ( $\wedge$ )

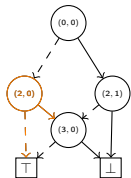


Seek:  
 $\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :  
 $[ \quad (0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) \quad ,$   
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \quad ,$

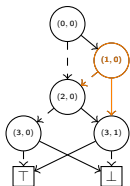


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

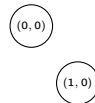
Seek:

$\min((1, 0), (2, 1))$

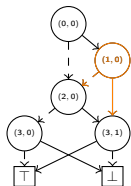
Priority Queue:  $Q_{app:1}$ :

[  $(0, 0) \xrightarrow{T} ((1, 0), (2, 1))$  ,  
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,  
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,  
 $(1, 0) \xrightarrow{T} ((3, 1), (2, 1))$  ,

]



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,

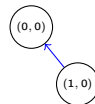
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

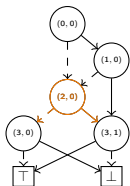
]

Output:

$(0, 0) \xrightarrow{\top} (1, 0)$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((2, 0), (2, 0))$

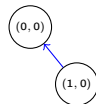
Priority Queue:  $Q_{app:1}$ :

[

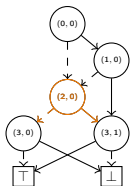
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,  
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,  
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

]

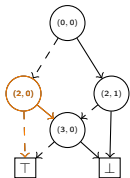
Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,

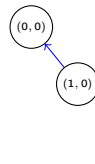
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

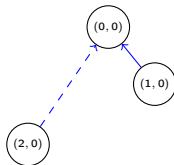
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

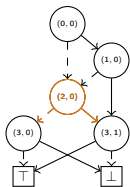
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:

$(0, 0) \xrightarrow{\perp} (2, 0)$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue:  $Q_{app}:1$ :

[

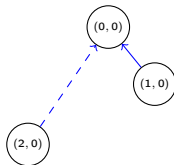
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

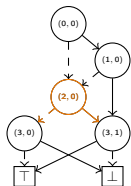
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:





# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

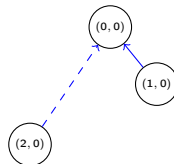
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$   $((3, 0), (3, 1))$  ,

]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

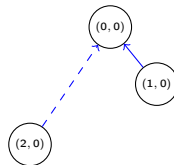
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$   $((3, 0), (3, 1))$  ,

]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

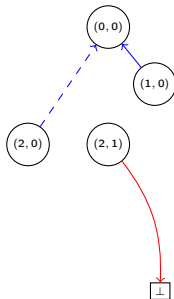
Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$   $((3, 0), (3, 1))$  ,

]

Output:

$(2, 1) \xrightarrow{\top} \perp$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

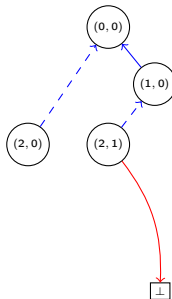
Priority Queue:  $Q_{app:2}$ :

[

]

Output:

$(1, 0) \xrightarrow{\perp} (2, 1)$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

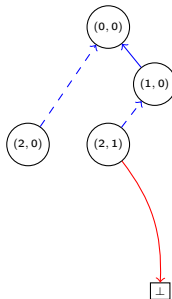
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

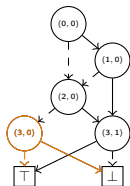
[

]

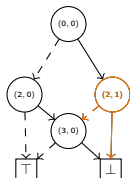
Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

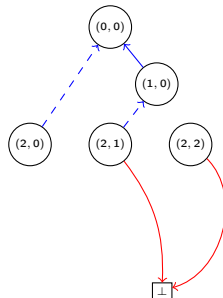
Priority Queue:  $Q_{app:2}$ :

[

]

Output:

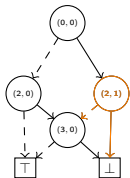
$(2, 2) \xrightarrow{\top} \perp$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

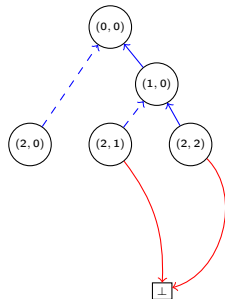
Priority Queue:  $Q_{app:2}$ :

[

]

Output:

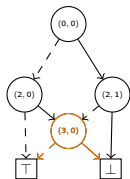
$(1, 0) \xrightarrow{\top} (2, 2)$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,

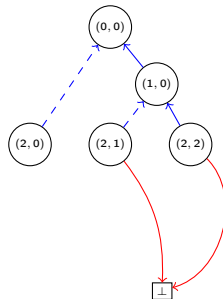
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:





# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

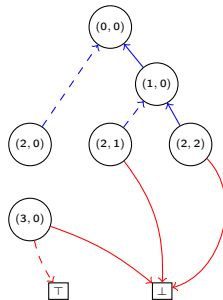
Priority Queue:  $Q_{app:2}$ :

[

]

Output:

$(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

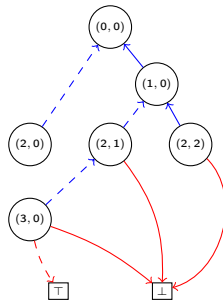
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

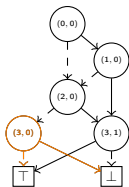
[

Output:  
 $(2, 1) \xrightarrow{\perp} (3, 0)$

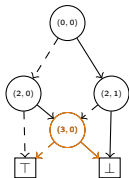


]

# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

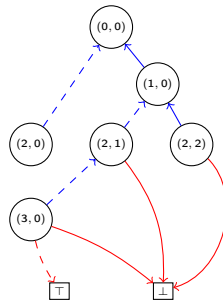
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

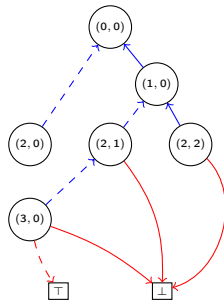
Priority Queue:  $Q_{app:2}$ :

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$  ,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$  ]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

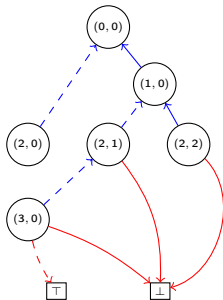
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

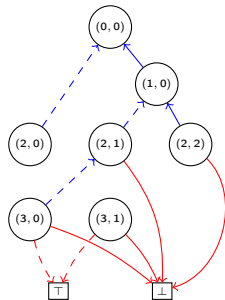
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

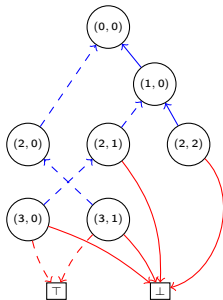
Seek:  
 $\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

Output:

$(2, 0) \xrightarrow{\perp} (3, 1)$



Priority Queue:  $Q_{app:2}$ :

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



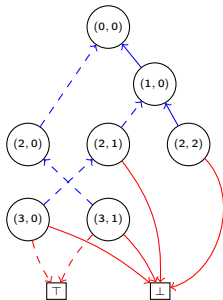
(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$   
 Priority Queue:  $Q_{app:1}$   
 [

Priority Queue:  $Q_{app:2}$

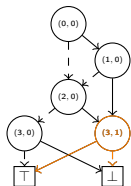
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$   
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

Output:





# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

Priority Queue:  $Q_{app:2}$ :

[

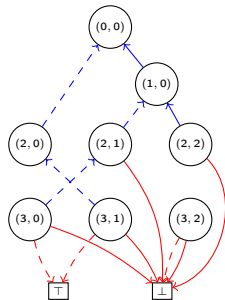
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$   
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

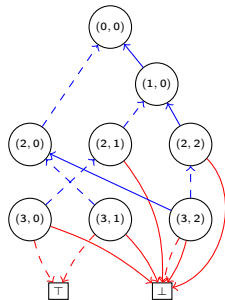
Output:

$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$

]

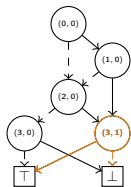
Priority Queue:  $Q_{app:2}$ :

[

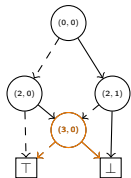


]

# Apply Example ( $\wedge$ )



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue:  $Q_{app:1}$ :

[

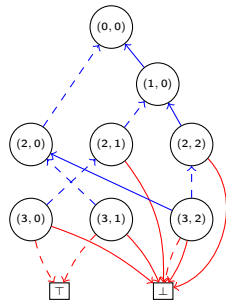
Priority Queue:  $Q_{app:2}$ :

[

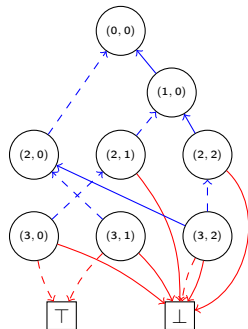
]

]

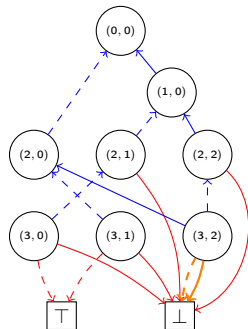
Output:



## Reduce Example

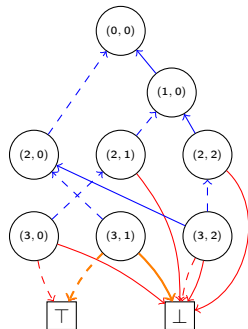


# Reduce Example



[                      Level: 3  
                     $[(3, 2) \mapsto \perp]$                       ]

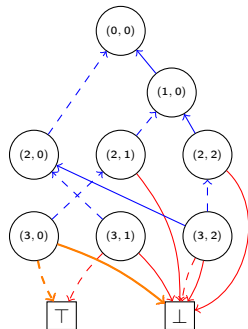
# Reduce Example



Level: 3

[	$[(3, 2) \mapsto \perp]$	]
[	$((3, 1), \top, \perp)$	,
		]

# Reduce Example



Level: 3

[  $[(3, 2) \mapsto \perp]$  ]

[  $((3, 1), \top, \perp)$  ,

$((3, 0), \top, \perp)$  ]

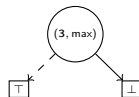
# Reduce Example



Level: 3

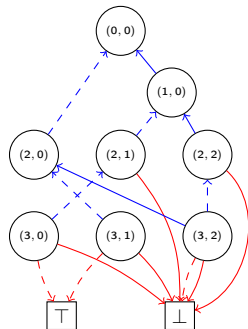
[	$[(3, 2) \mapsto \perp]$	]
[	$[(3, 1) \mapsto (3, \max)]$	,
	$((3, 0), \top, \perp)$	]

**Output:**  
 $((3, \max), \top, \perp)$





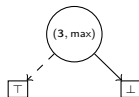
# Reduce Example



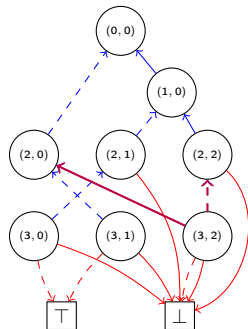
Level: 3

[	$[(3, 2) \mapsto \perp]$	]
[	$[(3, 1) \mapsto (3, \max)]$	,
	$[(3, 0) \mapsto (3, \max)]$	]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[  $(2, 2) \xrightarrow{\perp} \perp$  ,

$(2, 0) \xrightarrow{T} \perp$  ,

]

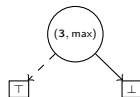
Level: 3

[ ]

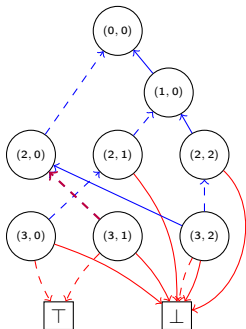
[  $[(3, 1) \mapsto (3, \max)]$  ,

$[(3, 0) \mapsto (3, \max)]$  ]

Output:



## Reduce Example



Priority Queue:  $Q_{red}$ :

$$[(2, 2) \xrightarrow{\perp} \perp]$$
$$(2, 0) \xrightarrow{T} \perp$$
$$(2, 0) \xrightarrow{\perp} (3, \max) \quad ,$$

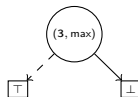
1

Level: 3

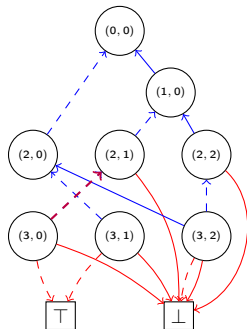
[ ]

$$[(3, 0) \mapsto (3, \max)]$$

**Output:**



# Reduce Example



Priority Queue:  $Q_{red}$ :

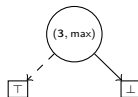
[  $(2, 2) \xrightarrow{\perp} \perp$  ,  
 $(2, 1) \xrightarrow{\perp} (3, \max)$  ,  
 $(2, 0) \xrightarrow{\top} \perp$  ,  
 $(2, 0) \xrightarrow{\perp} (3, \max)$  ,

]

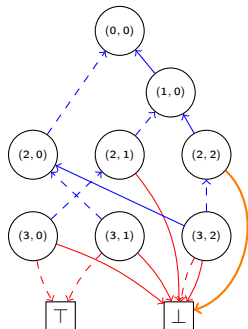
Level: 3

[  
 [  
 ,  
 ]

Output:



## Reduce Example



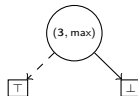
Priority Queue:  $Q_{red}$ :

$$\begin{array}{l} (2, 1) \xrightarrow{\perp} (3, \max) \quad , \\ (2, 0) \xrightarrow{\top} \perp \quad , \\ (2, 0) \xrightarrow{\perp} (3, \max) \quad , \end{array}$$

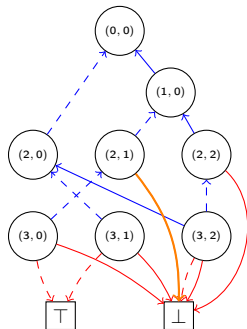
Level: 2

[  $((2, 2) \mapsto \perp)$  ]

**Output:**



# Reduce Example



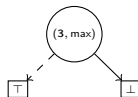
Priority Queue:  $Q_{red}$ :

[  
 $(2, 0) \xrightarrow{\top} \perp$  ,  
 $(2, 0) \xrightarrow{\perp} (3, \max)$  ,  
 ]

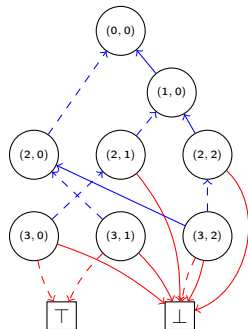
Level: 2

[  $[(2, 2) \mapsto \perp]$  ]  
 [  $((2, 1), (3, \max), \perp)$  ,  
 ]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

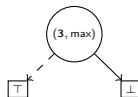
$((2, 1), (3, \max), \perp)$

,

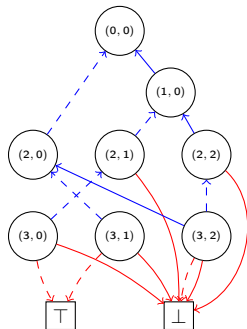
$((2, 0), (3, \max), \perp)$

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

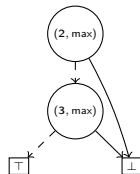
,

$((2, 0), (3, \max), \perp)$

]

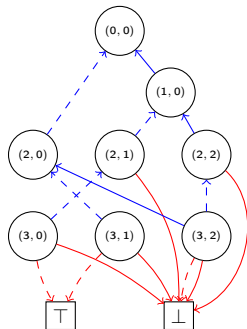
Output:

$((2, \max), (3, \max), \perp)$





# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

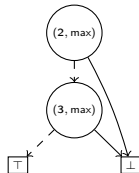
$[(2, 1) \mapsto (2, \max)]$

,

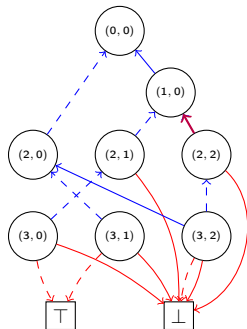
$[(2, 0) \mapsto (2, \max)]$

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(1, 0) \xrightarrow{T} \perp$  ,

]

Level: 2

[

]

[

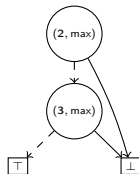
$[(2, 1) \mapsto (2, \max)]$

,

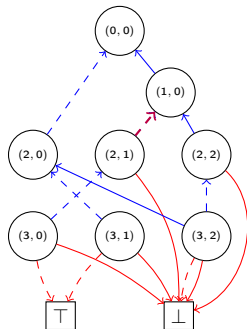
$[(2, 0) \mapsto (2, \max)]$

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(1, 0) \xrightarrow{T} \perp$  ,

$(1, 0) \xrightarrow{\perp} (2, \max)$  ,

]

Level: 2

[

[

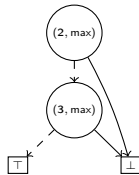
$[(2, 0) \mapsto (2, \max)]$

]

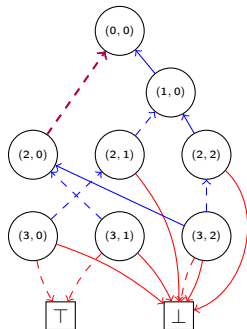
,

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(1, 0) \xrightarrow{T} \perp$  ,

$(1, 0) \xrightarrow{\perp} (2, \max)$  ,

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 2

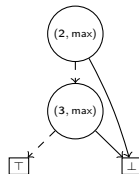
[

[

]

,  
]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

[

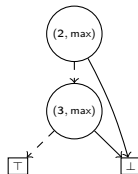
]

[

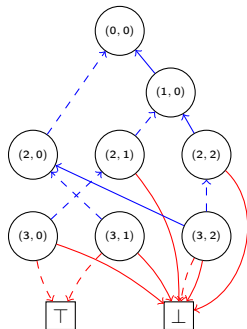
$((1, 0), (2, \max), \perp)$

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

[

]

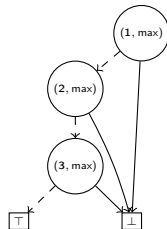
[

$[(1, 0) \mapsto (1, \max)]$

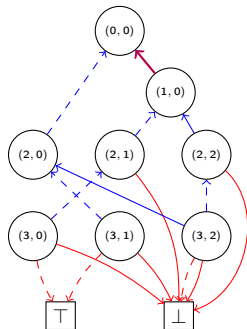
]

Output:

$((1, \max), (2, \max), \perp)$



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

$(0, 0) \xrightarrow{T} (1, \max)$  ,

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

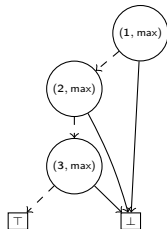
[

]

[

]

Output:



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 0

[

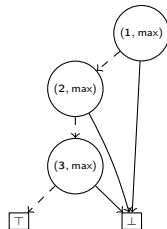
]

[

((0, 0), (2, max), (1, max))

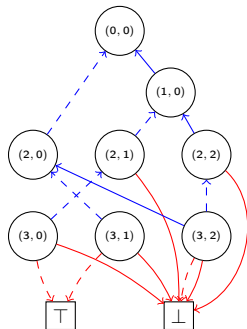
]

Output:





# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 0

[

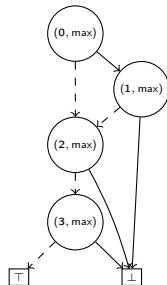
]

$[(0, 0) \mapsto (0, \max)]$

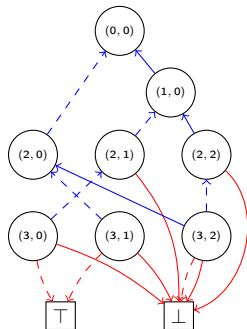
]

Output:

$((0, \max), (2, \max), (1, \max))$



# Reduce Example



Priority Queue:  $Q_{red}$ :

[

]

Level: 0

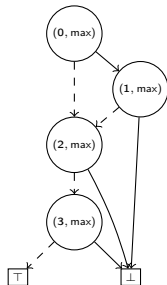
[

]

[

]

Output:



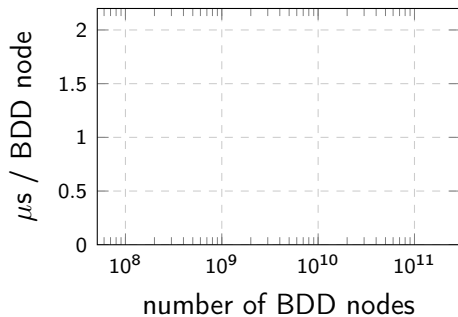
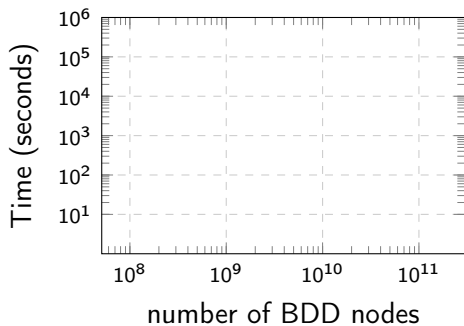


Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f ? g : h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f _{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f _{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	#paths in $f$ to $\top$	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min / \max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$



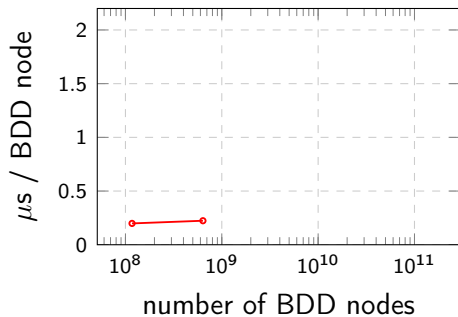
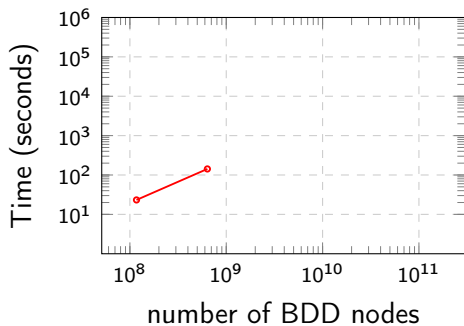
# Adiar

[github.com/ssoelvsten/adiar](https://github.com/ssoelvsten/adiar)



—○— BuDDy    —●— CUDD    —■— Sylvan    —●— Adiar

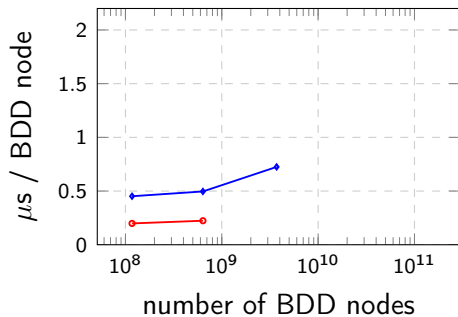
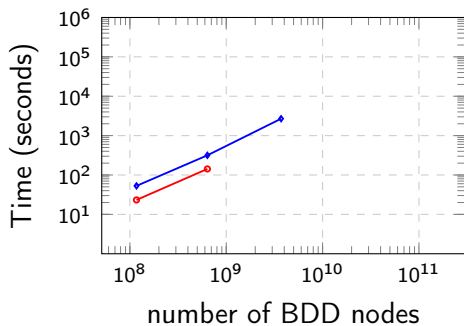
Minimum running times for the  $N$ -Queens problem.



—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

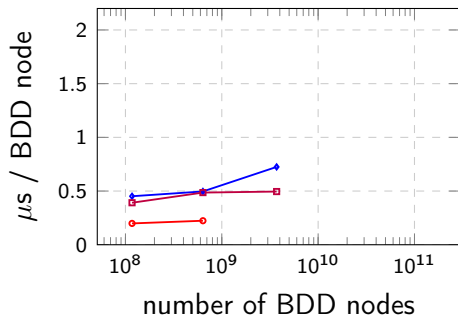
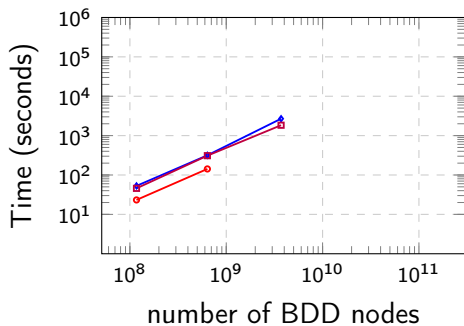
Minimum running times for the  $N$ -Queens problem.





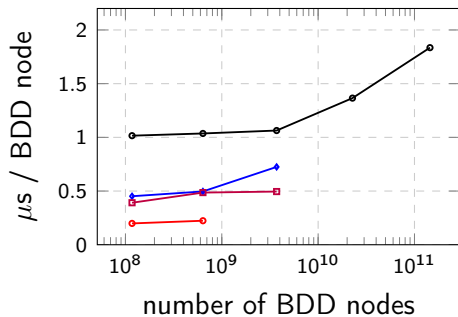
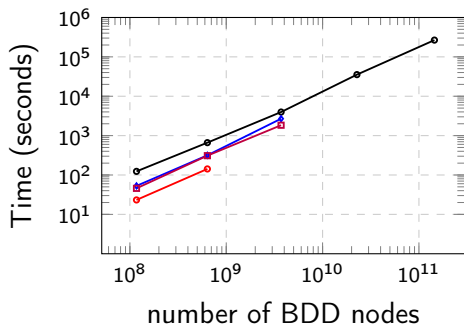
—○— BuDDy —◇— CUDD —■— Sylvan —●— Adiar

Minimum running times for the  $N$ -Queens problem.



—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

Minimum running times for the  $N$ -Queens problem.



—○— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimum running times for the  $N$ -Queens problem.



