Relational Product of BDDs in External Memory

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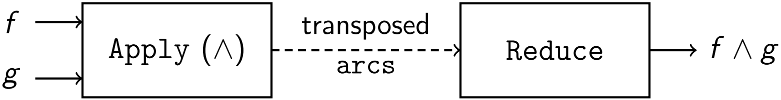

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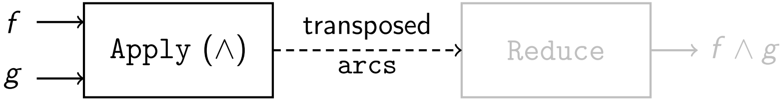

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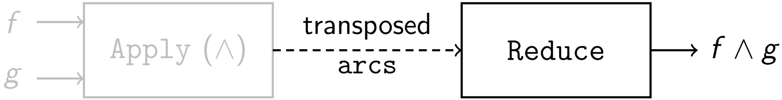

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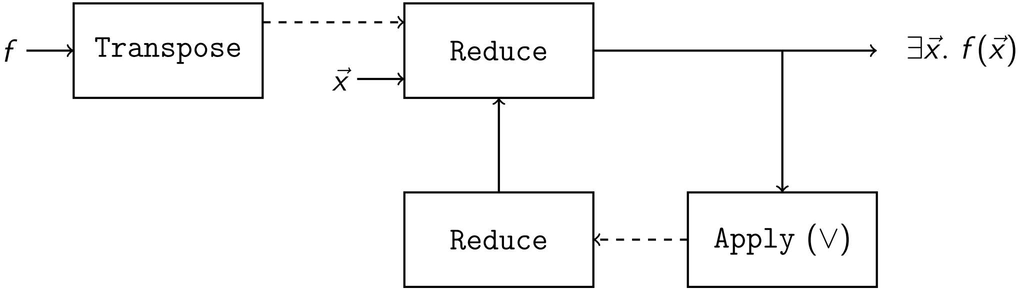

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[TACAS 25]

[SPIN 25]

Replace

Definition
A relabelling π is monotonic if $x_i < x_j \implies \pi(x_i) < \pi(x_j)$

Lemma
If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

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- **One can apply π in a single linear scan.**
  $\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.

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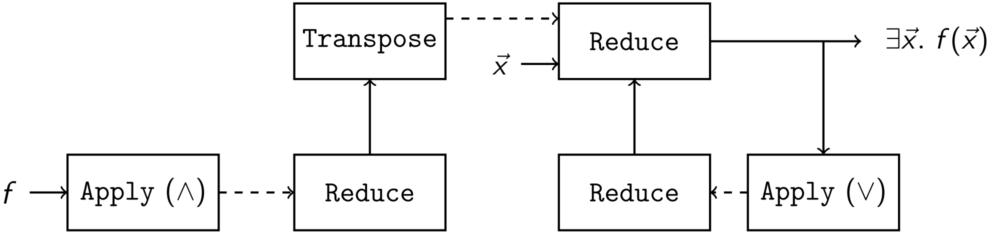
- **One can apply π in a single linear scan.**
  $\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.
- **One can incorporate π into a (succeeding) top-down `Apply` sweep.**
  $\mathcal{O}(N)$ time, 0 I/Os, and 0 external space.
- **One can incorporate π into a (preceeding) bottom-up `Reduce` sweep.**
  $\mathcal{O}(n)$ time, 0 I/Os, and 0 external space.

AndExists

Observation
The I/O-efficient `And` [TACAS 22] and `Exists` [TACAS 25] operations can be merged:



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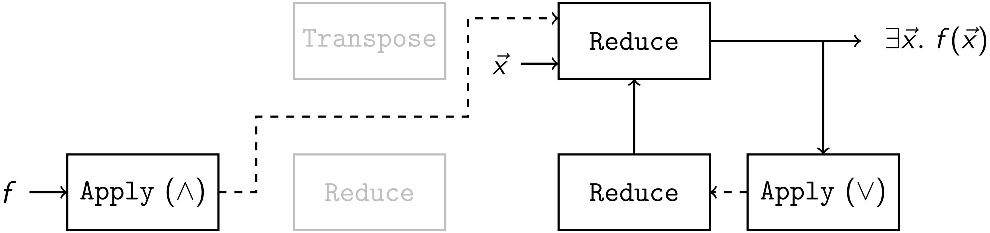

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The I/O-efficient `And` [TACAS 22] and `Exists` [TACAS 25] operations can be merged:

- **The outer accumulating `Reduce` sweep of the `Exists` can do double-duty as the `Reduce` sweep of the preceeding `And` operation.**
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The I/O-efficient `And` [TACAS 22] and `Exists` [TACAS 25] operations can be merged:

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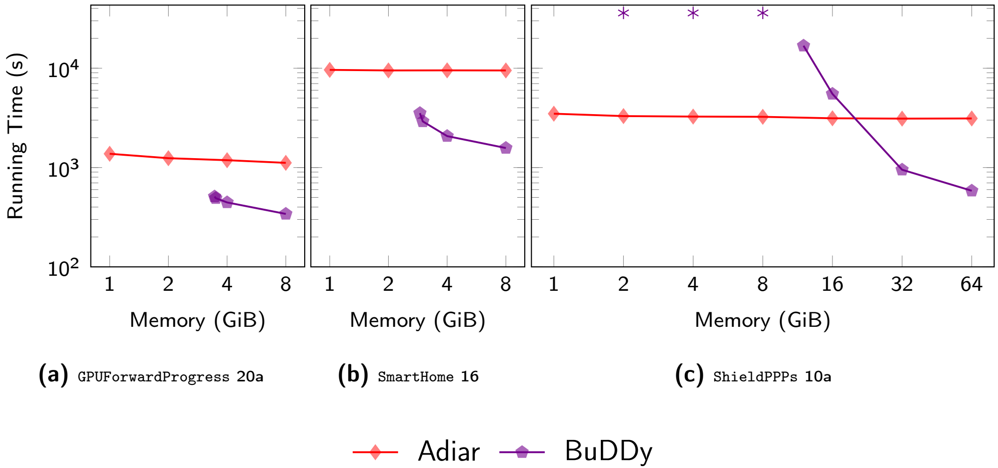
  This saves $\Theta(\text{sort}(N))$ time and I/Os.

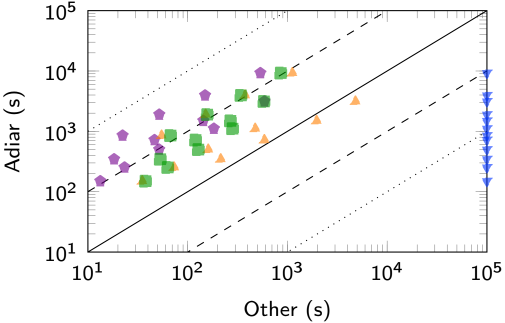
- **The `And` operation can prune subtrees that trivially will become redundant during the succeeding `Exists`.**
  This can save up to $\mathcal{O}(\text{sort}(N^{2^k}))$ time and I/Os.
  In practice, this only saves up to $\mathcal{O}(\text{sort}(N))$ time and I/Os.

Experiment: Next($S_{\vec{x}}$, $T_{\vec{x},\vec{x}'}$)

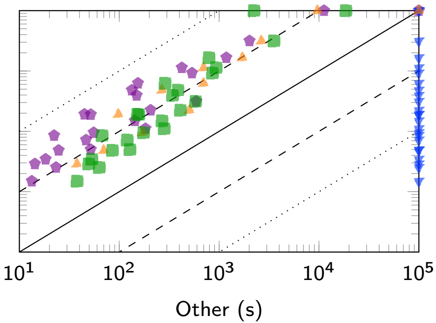


Relational Product for MCC models with a 2^{25} state space BDD. Timeouts are marked as stars.

Experiment: Next($S_{\vec{x}}$, $T_{\vec{x},\vec{x}'}$) & Prev($S_{\vec{x}}$, $T_{\vec{x},\vec{x}'}$)



(a) Next($S_{\vec{x}}$, $T_{\vec{x},\vec{x}'}$)

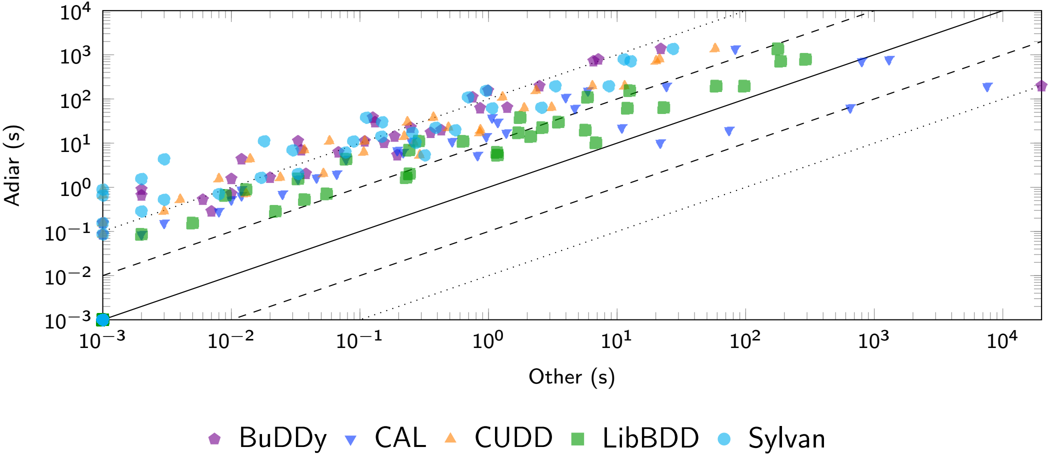


(b) Prev($S_{\vec{x}}$, $T_{\vec{x},\vec{x}'}$)

BuDDy CAL CUDD LibBDD

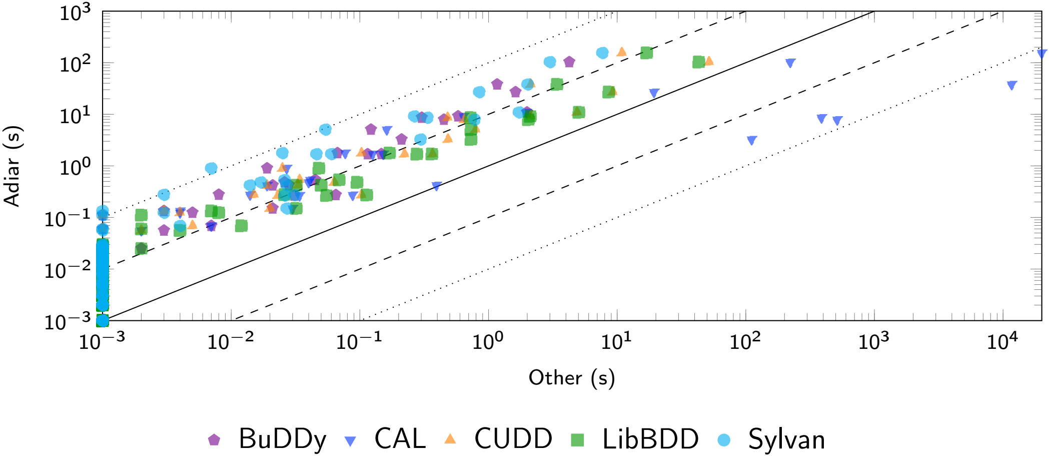
Relational Product for MCC models, 384 GiB of memory, and $2^{22}, \ldots, 2^{25}$ state space BDDs.

Experiment: Reachability



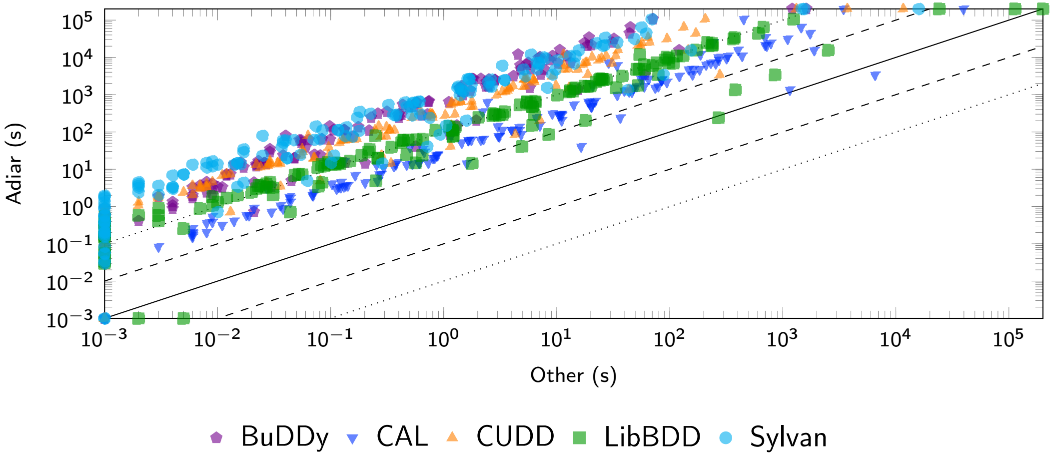
16 Petri Nets [MCC 21–23] with 384 GiB of memory.

Experiment: Deadlock Detection



16 Petri Nets [MCC 21–23] and 59 Boolean Networks [AEON, PyBoolNet] with 384 GiB RAM.

Experiment: SCC Decomposition



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Conclusions and Future Work

- **To improve the I/O-efficient *Next*($S_{\vec{x}}$, $T_{\vec{x},\vec{x'}}$), focus on *AndExists*.**
  - Factor of $\sim 2\times$ by using a `AndExists` instead of `And` and `Exists` for conventional depth-first implementations [1]. This may explain the sudden performance gap.
  - For larger instances, less than $\frac{1}{10}$th of the time is spent on the `And`.

[1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
[2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
[3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
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- **Deal with small BDDs using Depth-first Recursion.**
- **Apply ideas from recent and more advanced BDD algorithms.**

  For example the ones in [2], [3], [4], and [5] .

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- **Design a *Replace*(π) for Non-monotone Variable Substitutions.**

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Adiar

github.com/ssoelvsten/adiar

ssoelvsten.github.io/adiar

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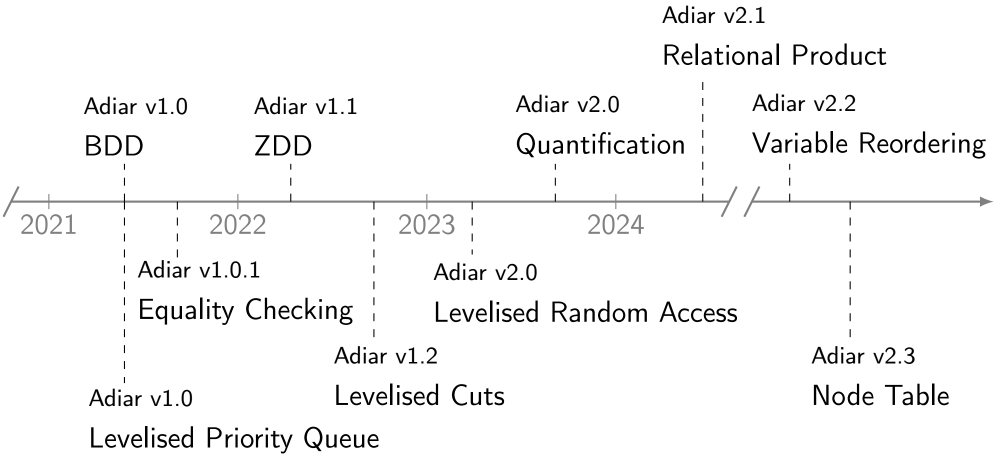
Adiar v2.1
Relational Product
Adiar v1.0
BDD
Adiar v1.1
ZDD
Adiar v2.0
Quantification
Adiar v2.2
Variable Reordering
2021
2022
2023
2024
Adiar v1.0.1
Equality Checking
Adiar v2.0
Levelised Random Access
Adiar v1.2
Levelised Cuts
Adiar v2.3
Node Table
Adiar v1.0
Levelised Priority Queue