# ∃R-completeness of Nash equilibria in Perfect Information Stochastic Games

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Basic definitions and utility functions

Nash Equilibria

Game Theory in Model Checking and Synthesis

 $\exists \mathbb{R}$ -complexity

The  $\operatorname{NP}$  and  $\operatorname{SqrtSum}$  Complexity Classes

The  $\exists \mathbb{R}$  Complexity Class

Proof Sketch: ∃ℝ-Completeness of Nash equilibria

Gadgets

Reduction

Implications for Model Checking



An m-player perfect information stochastic game G is defined by

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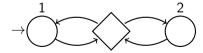
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- An initial node  $v_o \in V$ .



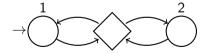
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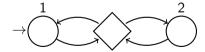


■ A play  $h \in \mathcal{H}_{\infty}$  is an infinite sequence  $(h_t)_{t\geq 0}$  of vertices in V, where

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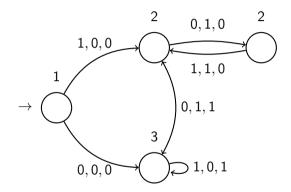
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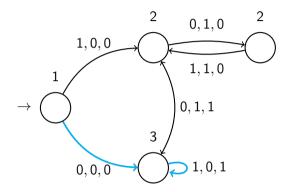
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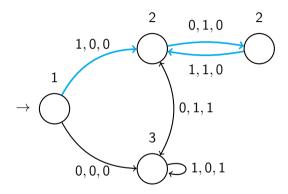
■ Utility functions  $u_i$  assigns a payoff  $u_i(i)$  for Player i to a play  $h \in \mathcal{H}_{\infty}$ 



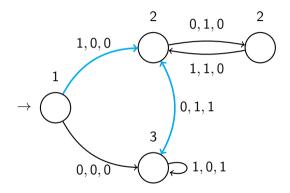
A simple mean-payoff game.



A simple mean-payoff game. Mean payoff for player 1: 1

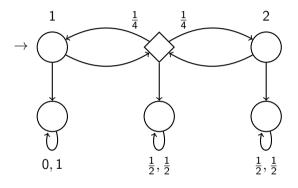


A simple mean-payoff game. Mean payoff for player 1:  $\frac{1}{2}$ 



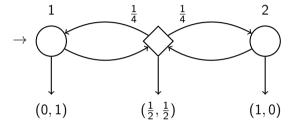
A simple mean-payoff game. Mean payoff for player 1: 0

# Recursive games



A simple game with terminal rewards only. Ummels  ${\rm ^{\prime}11}$ 

# Recursive games



A simple game with terminal rewards only. Ummels  $^\prime 11$ 

# Strategies and Nash equilibria

A strategy  $\tau_i$  assigns a probability distribution to the outgoing arcs of vertices  $v \in V_i$  depending on the given history h.

■ A strategy is *stationary*, if the choice of the players at a vertice is independent of the prior history of play (i.e. the strategy is memoryless).

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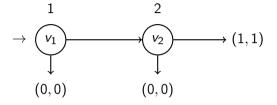
■ A strategy is *stationary*, if the choice of the players at a vertice is independent of the prior history of play (i.e. the strategy is memoryless).

We assume players are acting *rationally*. This is commonly captured by the following notion

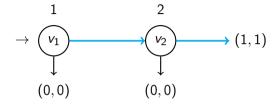
#### Definition (Nash equilibria)

A strategy profile  $\tau = (\tau_1, \tau_2, \dots, \tau_m)$  is a *Nash equilibrium*, if no player i has a unilateral deviation available that strictly improves their payoff.

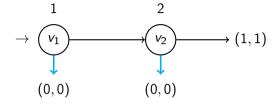
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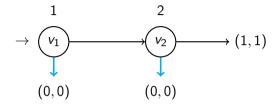
A two-player reachability game with an irrational Nash equilibrium. Ummels  $^\prime 11$ 



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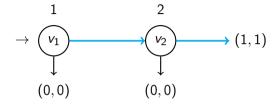
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A two-player reachability game with an irrational Nash equilibrium. Ummels '11

## Definition (Subgame Perfect Nash equilibria)

A Subgame perfect Nash equilibrium is a NE of a game G, that is not only the best response from  $v_0$ , but is a best response in G[h] given any history h of play.



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# Game Theory in Model Checking

Games provide a well studied framework that can capture many model checking problems with *adversaries*.

- $\blacksquare$  A protocol between m entities can be described by a stochastic game of m players.
- lacktriangle A distributed system of m peers can be described by a *concurrent* game of m players.

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Classical model checking objectives can be encapsulated in the utility function.

- Reachability objectives can be captured by payoffs in  $\{0,1\}$  in a recursive game.
- Safety objectives can be captured by payoffs in  $\{-1,1\}$  in a recursive game, since an *infinite* game has payoff 0.
- Other Büchi objectives can also be described in general Mean-payoff games.

## Game Theory in Synthesis

The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

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Does there exist a controller, such that the system satisfies the specification?

=

Does there exist a strategy, such that Player 1 is surely winning?

# The subject of this seminar

Consider the problem:

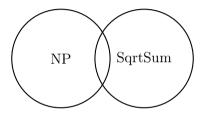
Given an m-player game G and payoff demands  $L \in \mathbb{R}^m$ , does there exist a stationary <sup>1</sup> NE  $\tau$  with  $U(\tau) \geq L$ ?

We will show this is  $\exists \mathbb{R}$ -complete.

 $<sup>^1</sup>$ The problem of existence of a Nash equilibria satisfying some demands is undecidable for  $\geq 10$  players in recursive games, so we will only focus on *stationary* strategies.

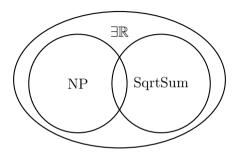
 $\exists \mathbb{R} ext{-complexity}$ 

# ∃R Complexity Class



The relation between NP,  $\operatorname{SqrtSum},$  and  $\exists \mathbb{R}$ 

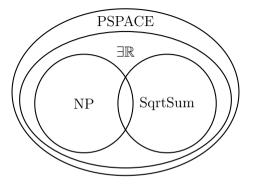
## **∃**ℝ Complexity Class



The relation between NP,  $\operatorname{SqrtSum}$ , and  $\exists \mathbb{R}$ 

The complexity class  $\exists \mathbb{R}$  both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of  $\operatorname{SqrtSum}$ .

## ∃R Complexity Class



The relation between NP, SqrtSum, and  $\exists \mathbb{R}$ 

The complexity class  $\exists \mathbb{R}$  both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of  $\operatorname{SqrtSum}$ .

# **NP Complexity Class**

Remember that the well-known class  $\operatorname{NP}\mathsf{can}$  be captured by the ILP problem:

min 
$$c^T x$$
  
s.t.  $Ax \le b$   
 $x \in \mathbb{N}^n$ 

where  $A \in \mathbb{Z}^{n \times m}$ ,  $b \in \mathbb{Z}^m$ ,  $c \in \mathbb{Z}^n$ 

# SqrtSum Complexity Class

Consider the following problem: Given  $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_m \in \mathbb{R}$  is the following inequality satisfied?

$$\sum_{i=1}^n \sqrt{a_i} \le \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial...

# **SqrtSum Complexity Class**

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$$\sum_{i=1}^n \sqrt{a_i} \le \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial... How many decimals do you have to compute, before you know the answer?  $^2$ 

#### Definition (SqrtSum)

The complexity class  $\hat{S}qrtSum$  consists of all problems that are polynomial time reducible to the problem above.

 $<sup>^1</sup>$ This consistently comes up in Computational Geometry. Here, theoretical works solve this by assuming the  $\mathbb{R}$ -RAM computational model; leaving an adventure for implementors to experience later.

# ∃R Complexity Class

The Existential Theory of the Reals is the language of all true sentences of the form

$$\exists x_1, x_2, \ldots, x_n \in \mathbb{R} : \phi(x_1, x_2, \ldots, x_n)$$

where  $\phi$  is a quantifier-free Boolean formula of inequalities and equalities.

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#### Definition $(\exists \mathbb{R})$

The complexity class  $\exists \mathbb{R}$  consists of all problems, that are polynomial time reducible to the existential theory of the reals.

# ∃R Complexity Class

We will consider the following  $\exists \mathbb{R}$ -complete problem.

## **Definition (HomQuad)**

Given a system S of I homogeneous quadratic polynomials  $^3$  in n variables, does there exist an  $x \in \mathbb{R}^n$  such that  $q_k(x) = 0$  for all  $k \in \{1, 2, ..., I\}$  and x is a probability distribution?

<sup>&</sup>lt;sup>3</sup>A homogenous quadratic polynomial is of the form  $\sum_{i=1}^n \sum_{j=1}^n A_{ij}x_ix_j$  where  $A \in [-1,1]^{n \times n}$ .

# Proof Sketch: ∃ℝ-Completeness of Nash equilibria

∃R-Completness of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands  $L \in \mathbb{R}^m$ , does there exist a stationary NE  $\tau$  with  $U(\tau) \geq L$ ?

 $\exists \mathbb{R}\text{-Completness}$  of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands  $L \in \mathbb{R}^m$ , does there exist a stationary NE  $\tau$  with  $U(\tau) \geq L$ ?

It has already been shown to be NP-hard for  $\geq 2$  players and  $\operatorname{SqrtSum}$ -hard for  $\geq 4$  players. Furthermore, it is contained within  $\exists \mathbb{R}$ .

It is  $\exists \mathbb{R}$ -complete! We will show this by reduction to:

### **Definition** (HomQuad)

Given a systèm S of I homogeneous quadratic polynomials in n variables, does there exist an  $x \in \mathbb{R}^n$  such that  $q_k(x) = 0$  and x is a probability distribution?

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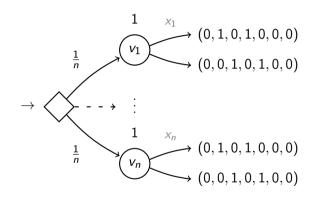
Given a system S of I homogeneous quadratic polynomials in n variables, does there exist an  $x \in \mathbb{R}^n$  such that  $q_k(x) = 0$  and x is a probability distribution?

That is, given a system S of I polynomials of the form

$$q_k(x) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{ij}x_ix_j + \ldots + a_{nn}x_nx_n$$

we will construct a game  $\mathcal{G}(\mathcal{S})$  such that all  $q_k(x) = 0$  if and only if  $\mathcal{G}(\mathcal{S})$  has a stationary Nash equilibria that satisfies some payoff demand.

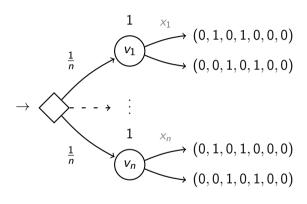
# Proof Sketch: $\mathcal{G}_{var}$



The gadget game  $\mathcal{G}_{\mathrm{var}}$ 

At each  $v_i$ , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Proof Sketch:  $\mathcal{G}_{var}$ 



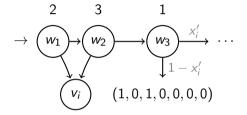
The gadget game  $\mathcal{G}_{\mathrm{var}}$ 

At each  $v_i$ , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Player 1 strategy corresponds to a probability distribution if it satisfies the payoff demand

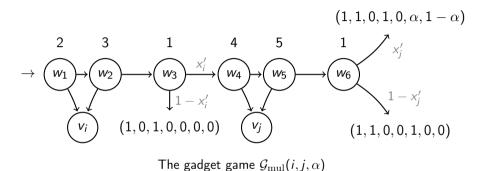
$$\left(0,\frac{1}{n},\frac{n-1}{n},\ldots\right)$$

Proof Sketch:  $\mathcal{G}_{\mathrm{mul}}(i,j,\alpha)$ 



The gadget game  $\mathcal{G}_{\mathrm{mul}}(i,j,lpha)$ 

# Proof Sketch: $\mathcal{G}_{\text{mul}}(i, j, \alpha)$



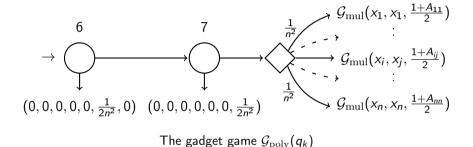
If Player 1 receives payoff 1, then Player-6 gets  $\alpha x_i x_j$  and Player-7 gets  $(1 - \alpha)x_i x_j$ .

$$\max_{ au_1} \min_{ au_2} \Pr\left[u_1(v_0( au_1, au_2)) = 1\right]$$

$$\forall \tau_2 : \Pr[u_1(v_0(\tau_1, \tau_2)) = 1] \ge \text{value}$$

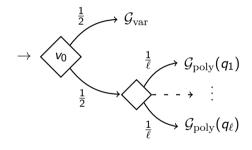
# Proof Sketch: $\mathcal{G}_{\text{poly}}(q)$

For a homogenous quadratic polynomial  $q_k(x) = \sum_{i,j=1}^n A_{ij} x_i x_j$ .



If Player 1 receives payoff 1, then Player 6 gains payoff  $\frac{1}{2n^2}(\|x\|_1^2 + q_k(x))$ . If also  $\|x\|_1$  is 1, then  $q_k(x) = 0$ .

### **Proof Sketch: Final reduction**



The game  $\mathcal{G}(\mathcal{S})$  of the reduction

 ${\cal S}$  is a "yes"-instance of  ${\rm HomQuad}$  if and only if the game  ${\cal G}({\cal S})$  has a Nash Equilibria that satisfies the demands

$$\left(\frac{1}{2},\frac{1}{n},\frac{n-1}{n},0,0,\ldots,0\right)$$

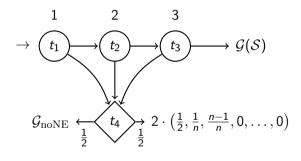
#### Theorem

It is  $\exists \mathbb{R}$ -complete to decide whether for a given m-player recursive game G and payoff demands  $L \in \mathbb{R}^m$  there exists a stationary Nash equilibria  $\tau$  with  $U(\tau) \geq L$ .

- The problem is  $\exists \mathbb{R}$ -complete even for acyclic 7-player recursive games with non-negative rewards.
- It even holds for stationary Subgame Perfect Equilibria.

# Implications for Model Checking

# $\exists \mathbb{R}\text{-}\mathsf{Completeness}$ of Stationary NE without Payoff Demands

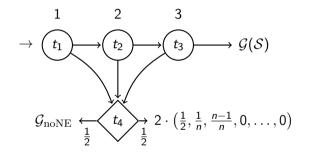


The game  $\mathcal{G}_{\exists \mathrm{NE}}(\mathcal{S})$ 

 $\mathcal{G}_{\mathrm{noNE}}$  is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

# $\exists \mathbb{R}\text{-}\mathsf{Completeness}$ of Stationary NE without Payoff Demands



The game  $\mathcal{G}_{\exists \mathrm{NE}}(\mathcal{S})$ 

 $\mathcal{G}_{\mathrm{noNE}}$  is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

 ${\mathcal S}$  is a "yes"-instance of  $\operatorname{Hom} \operatorname{Quad}$  if and only if the game  ${\mathcal G}_{\exists \operatorname{NE}}({\mathcal S})$  has a stationary Nash Equilibria.

# ∃R-Completeness of Reachability and Safety objectives

There exists different  $\mathcal{G}_{\mathrm{noNE}}$  gadget games for the different restrictions of the utility function:

- Reachability objective
- Safety objective

### **Theorem**

It is  $\exists \mathbb{R}$ -complete to decide whether a given m-player game with Reachability or Safety objectives has a stationary NE.

• even for m = 7 players.

# ∃R-Completeness of being almost surely winning

Consider the game  $\mathcal{G}_{\exists NE}(\mathcal{S})$  in which another player is added, who is always winning in  $\mathcal{G}(\mathcal{S})$ , but not in  $\mathcal{G}_{noNE}$ .

### **Theorem**

For any i, it is  $\exists \mathbb{R}$ -complete to decide whether a given m-player recursive game has a stationary NE in which Player i is almost surely winning.

• even for m = 8 players.

### Final remarks

It is  $\exists \mathbb{R}$ -complete to decide in an *m*-player perfect information recursive game.

- lacktriangle exists Subgame Perfect Nash equilibria satisfying demand  $L \in \mathbb{R}^m$
- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.

### Final remarks

It is  $\exists \mathbb{R}$ -complete to decide in an *m*-player perfect information recursive game.

- lacktriangle exists Subgame Perfect Nash equilibria satisfying demand  $L \in \mathbb{R}^m$
- exists any Nash equilibria for Reachability and Safety objectives
- exists any Nash equilibria such that Player 1 is surely winning.

### Notice here that

- This problem is already shown by Ummels '11 to be NP-hard for  $\geq$  2 players and SqrtSum-hard for  $\geq$  4 players so this completeness result could not become much tighter.
- There have been recent results of  $\exists \mathbb{R}$ -completeness in *imperfect information* games. The complexity of these results stem from the structure of the game, not the lack of information.