

Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

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Figure 1: The I/O model by Aggarwal and Vitter '87

For any realistic values of N , M , and B we have that

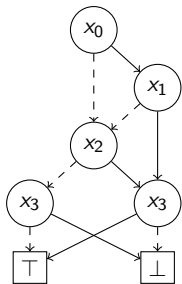
$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

Theorem (Aggarwal and Vitter '87)

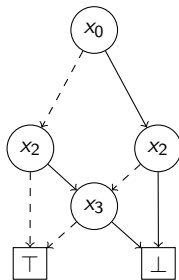
N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

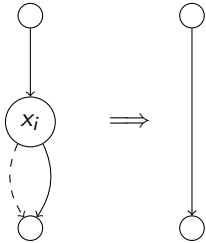


(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

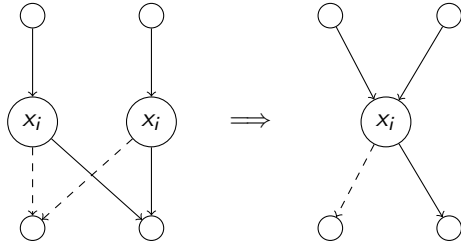
Figure 2: Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)

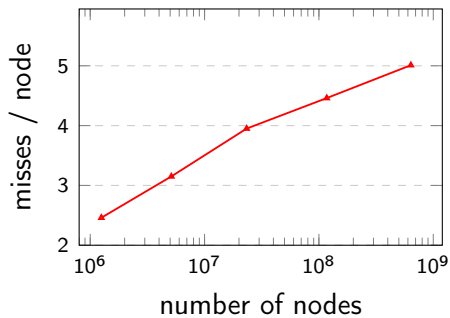
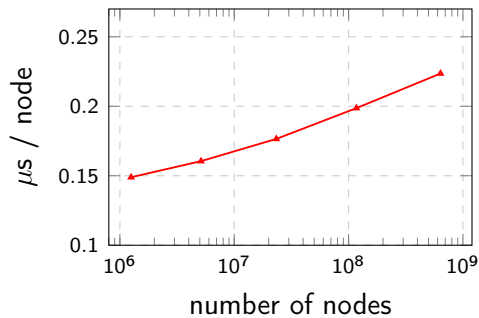
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(a) Rule 1: Remove redundant nodes



(b) Rule 2: Merge duplicate nodes



—▲— BuDDy

Figure 4: Cache behaviour for the N -Queens problem.

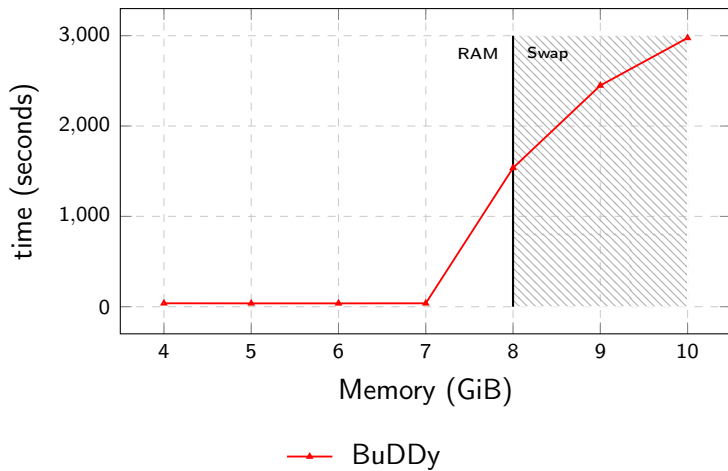
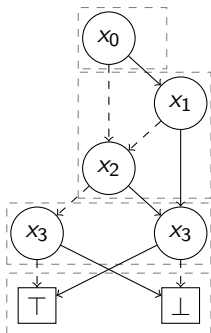


Figure 5: Running time for *Tic-Tac-Toe* with $N = 21$.

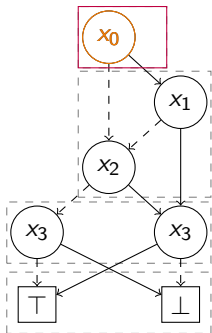


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
0	0

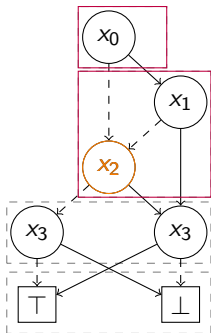


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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$$M = 4, B = 2$$

node I/Os	cache lookups
1	1

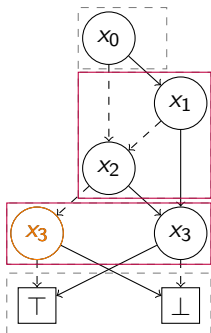


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$$M = 4, B = 2$$

node I/Os	cache lookups
2	2

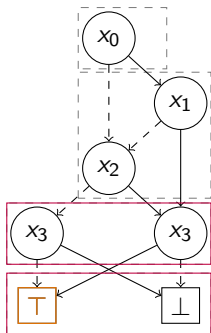


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$$M = 4, B = 2$$

node I/Os	cache lookups
3	3

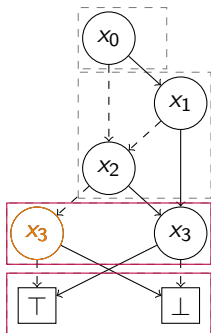


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$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

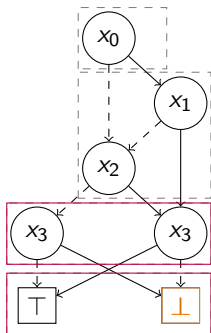


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4	3

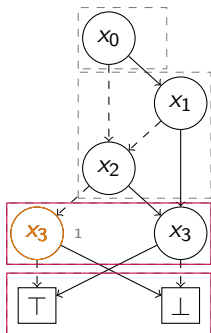


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$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

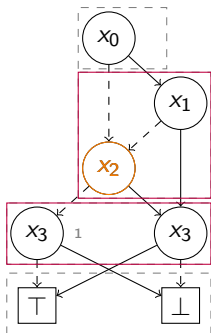


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$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

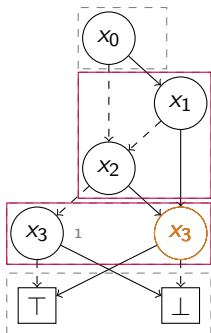


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
5	3

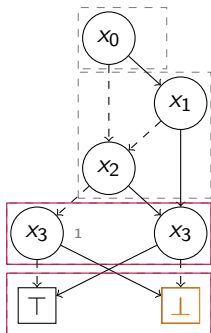


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
5	4

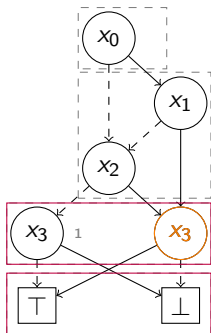


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

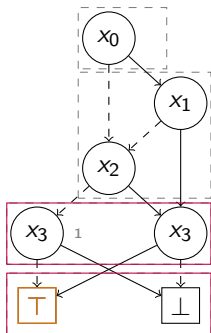


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

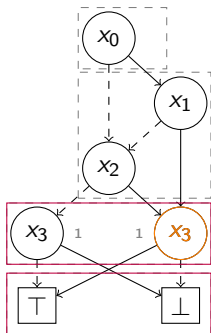


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node I/Os	cache lookups
6	4

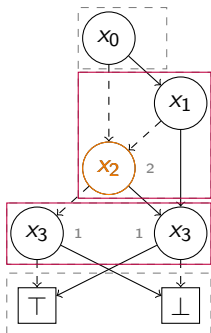


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$$M = 4, B = 2$$

node I/Os	cache lookups
6	4

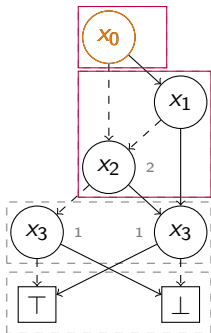


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$$M = 4, B = 2$$

node I/Os	cache lookups
7	4

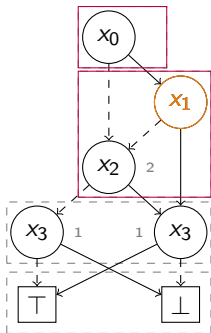


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
8	4

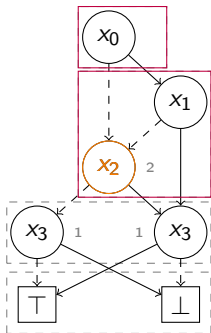


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
8	5

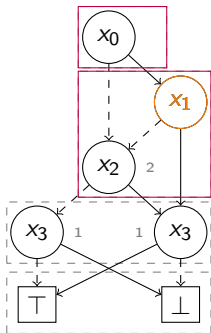


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
8	6

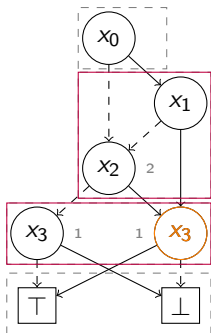


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$$M = 4, B = 2$$

node I/Os	cache lookups
8	6

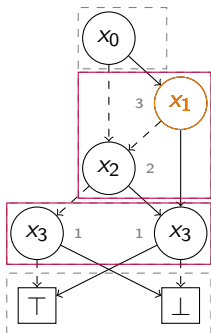


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$$M = 4, B = 2$$

node I/Os	cache lookups
9	7

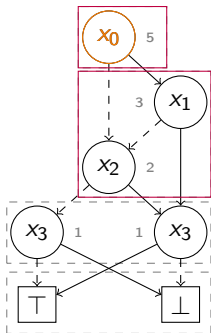


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
9	7

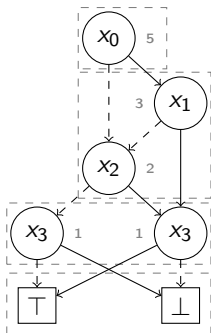


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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7



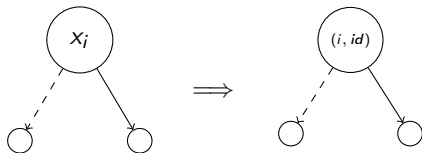
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Figure 6: Blocks active in memory

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

Let every node be uniquely identified by a tuple $(label, id) : \mathbb{N} \times \mathbb{N}$.



Nodes are ordered based on their *uid* as follows

$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$

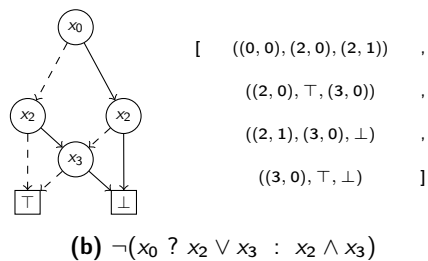
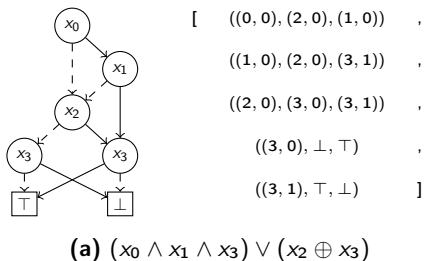
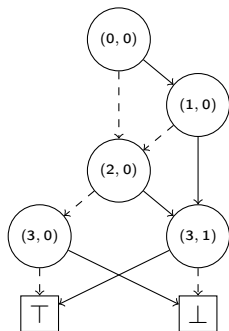


Figure 7: Node-based representation of prior shown BDDs

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

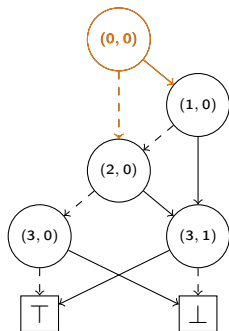
Priority Queue: Q_{count} :

[

]

Figure 8: In-order traversal of BDD

CountPaths Example



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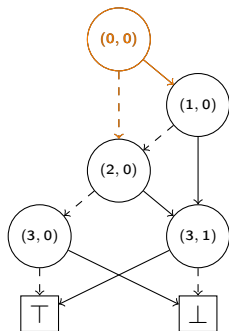
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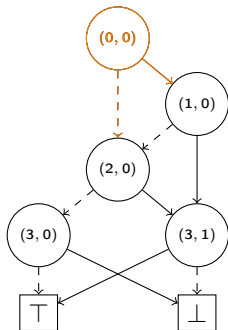
Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

Figure 8: In-order traversal of BDD

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	0	0

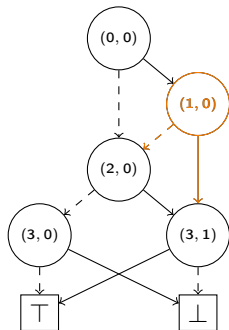
Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
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]

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CountPaths Example



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Seek	Sum	Result
(1, 0)	0	0

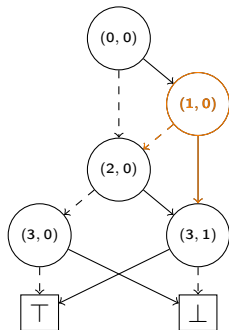
Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
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CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

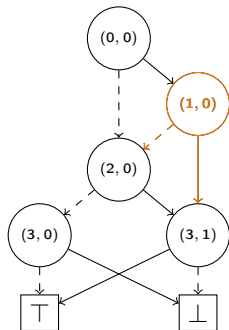
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CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

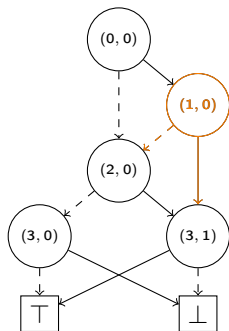
Seek	Sum	Result
(1, 0)	1	0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
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 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
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CountPaths Example



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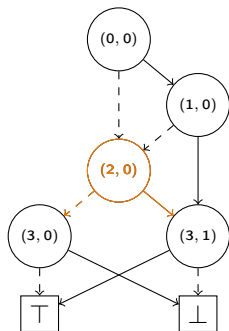
Seek	Sum	Result
(2, 0)	0	0

Priority Queue: Q_{count} :

[
	$((0, 0) \xrightarrow{\perp} (2, 0), 1)$,	
	$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,	
	$((1, 0) \xrightarrow{\top} (3, 1), 1)$,	
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CountPaths Example



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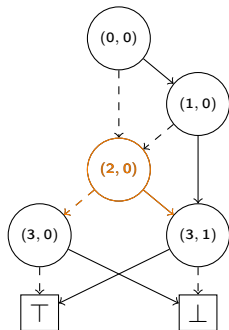
Figure 8: In-order traversal of BDD

Seek	Sum	Result
(2, 0)	0	0

Priority Queue: Q_{count} :

[
	$((0, 0) \xrightarrow{\perp} (2, 0), 1)$,	
	$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,	
	$((1, 0) \xrightarrow{\top} (3, 1), 1)$,	
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CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

Seek	Sum	Result
(2, 0)	1	0

Priority Queue: Q_{count} :

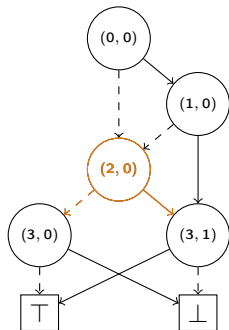
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CountPaths Example



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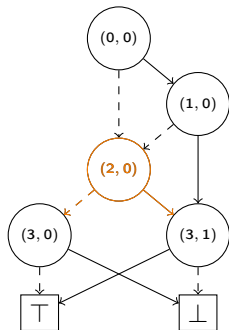
Seek	Sum	Result
(2, 0)	2	0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

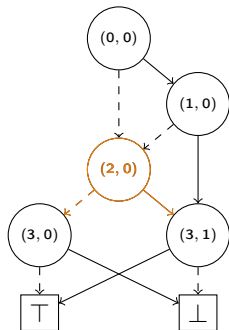
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 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

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CountPaths Example



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(3, 0)	0	0

Priority Queue: Q_{count} :

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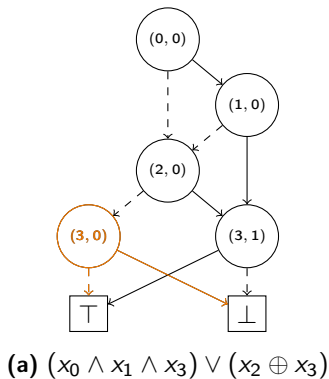


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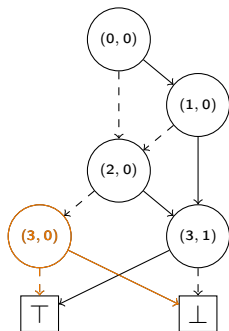
Seek	Sum	Result
(3, 0)	0	0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
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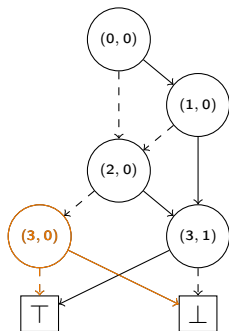
Seek	Sum	Result
(3, 0)	2	0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
$((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



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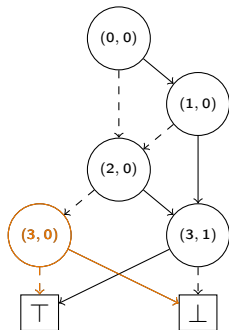
Seek	Sum	Result
(3, 0)	2	2

Priority Queue: Q_{count} :

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$((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

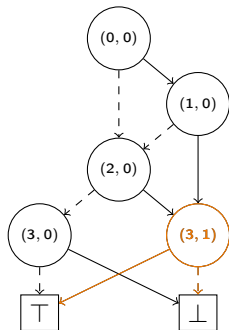
Seek	Sum	Result
(3, 1)	0	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	0	2

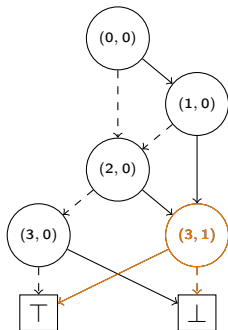
Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

Figure 8: In-order traversal of BDD

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

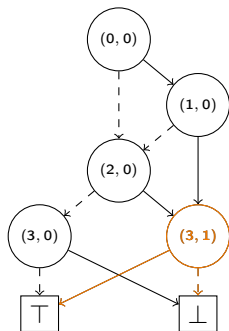
Seek	Sum	Result
(3, 1)	1	2

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	3	2

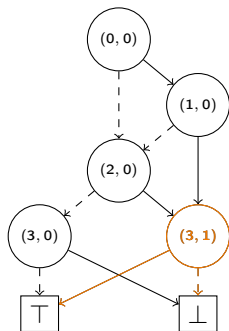
Priority Queue: Q_{count} :

[

]

Figure 8: In-order traversal of BDD

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Figure 8: In-order traversal of BDD

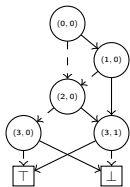
Seek	Sum	Result
(3, 1)	3	5

Priority Queue: Q_{count} :

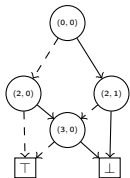
[

]

Apply Example (\wedge)

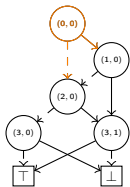


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

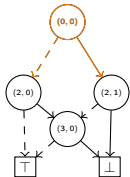


(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Apply Example (\wedge)

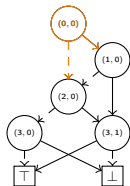


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

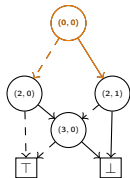


(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

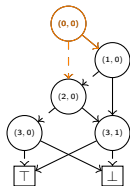
Priority Queue: $Q_{app:1}$:

[$(0,0) \xrightarrow{\top} ((1,0), (2,1))$,
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$,

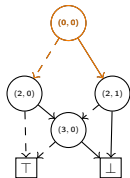


]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

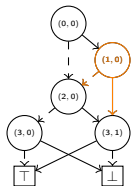
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

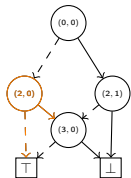


]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

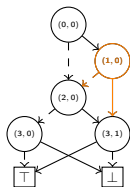
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

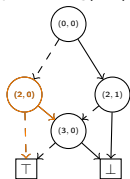


]

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

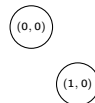
Seek:

$\min((1, 0), (2, 1))$

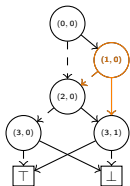
Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

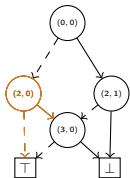
]



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

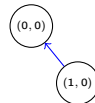
Priority Queue: $Q_{app:1}$:

[

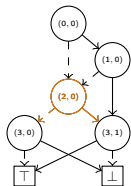
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

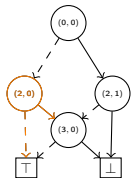
Output:
 $(0, 0) \xrightarrow{\top} (1, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

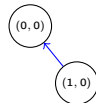
Priority Queue: $Q_{app:1}$:

[

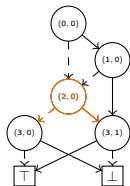
$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

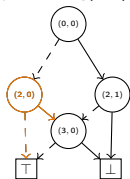
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,

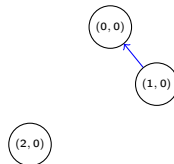
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

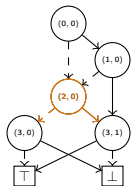
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

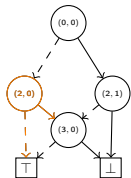
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

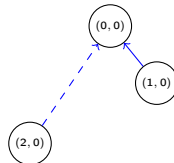
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

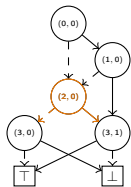
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:

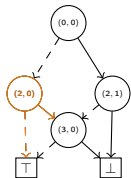
$(0, 0) \xrightarrow{\perp} (2, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

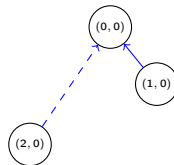
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

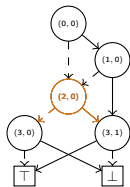
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

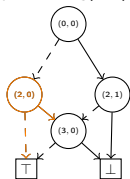
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

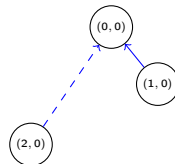
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

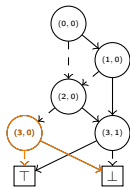
[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

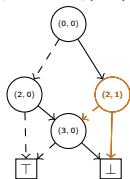
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

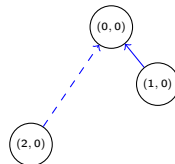
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

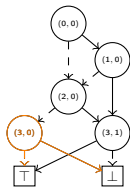
[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

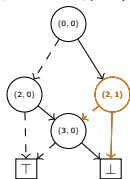
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

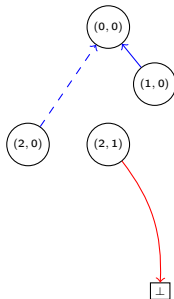
Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

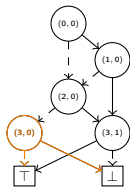
]

Output:

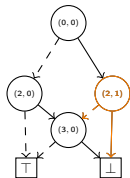
$(2, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

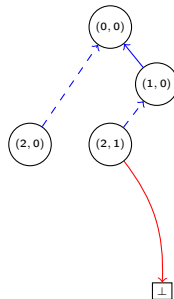
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

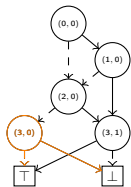
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

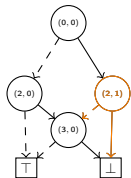
Output:
 $(1, 0) \xrightarrow{\perp} (2, 1)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

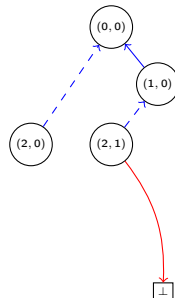
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

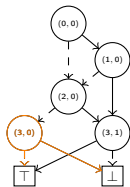
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

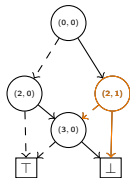
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

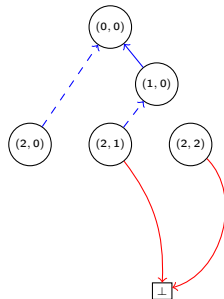
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

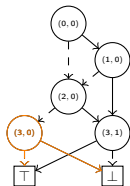
[

]

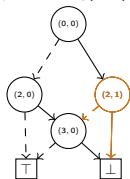
Output:
 $(2, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



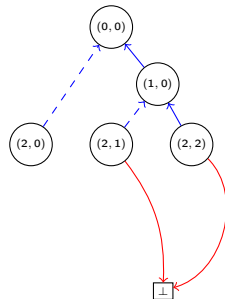
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$
 Priority Queue: $Q_{app:1}$:
 [

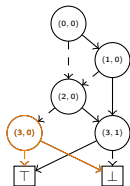
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

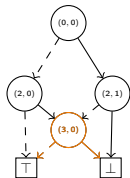
Output:
 $(1, 0) \xrightarrow{\top} (2, 2)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

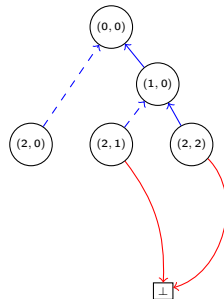
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

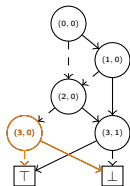
[

]

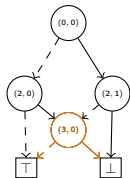
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

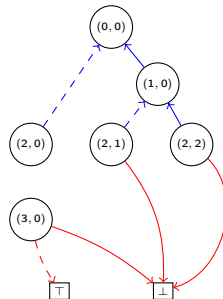
Priority Queue: $Q_{app:2}$:

[

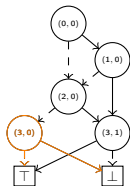
]

Output:

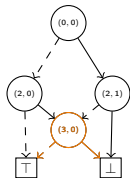
$(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



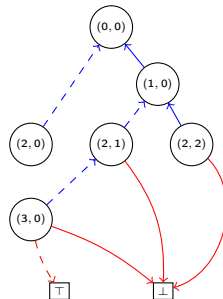
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$
 Priority Queue: $Q_{app:1}$:
 [

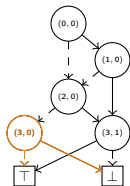
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

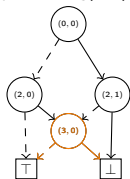
Output:
 $(2, 1) \xrightarrow{\perp} (3, 0)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



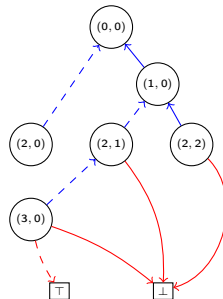
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$
 Priority Queue: $Q_{app:1}$:
 [

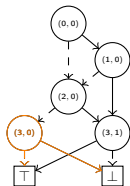
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

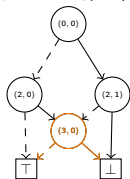
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

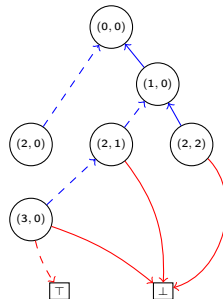
Seek:
 $\min((3, 1), (3, 0))$
 Priority Queue: $Q_{app:1}$
 [

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

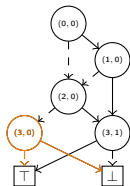
Priority Queue: $Q_{app:2}$

[
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

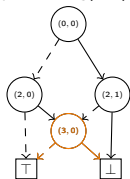
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

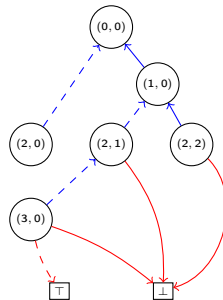
Priority Queue: $Q_{app:2}$:

[

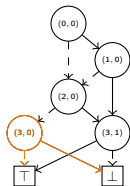
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

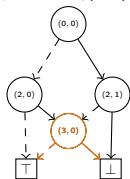
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

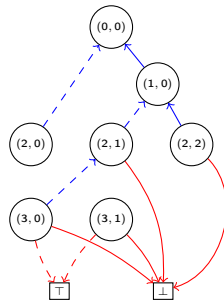
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

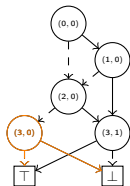
]

Output:

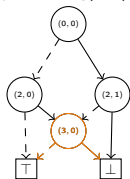
$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

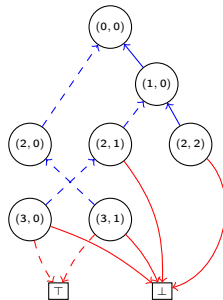
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

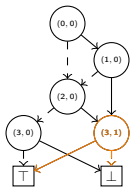
,

]

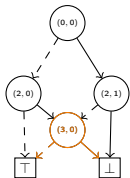
Output:
 $(2, 0) \xrightarrow{\perp} (3, 1)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

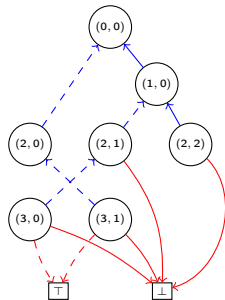
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

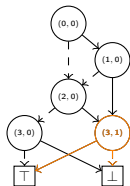
,

]

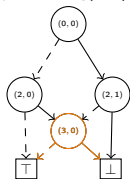
Output:



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

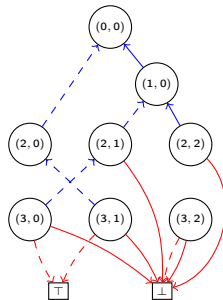
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

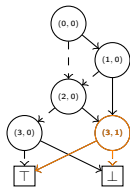
]

Output:

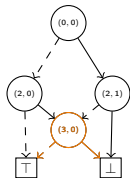
$(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$
 Priority Queue: $Q_{app:1}$:

[

Priority Queue: $Q_{app:2}$:

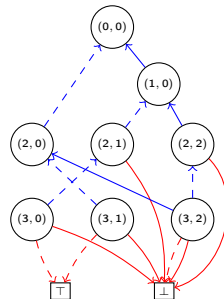
[

]

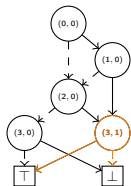
]

Output:

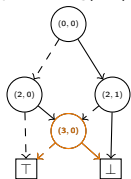
$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

[Priority Queue: $Q_{app:1}$:

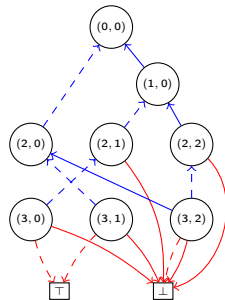
]]

Priority Queue: $Q_{app:2}$:

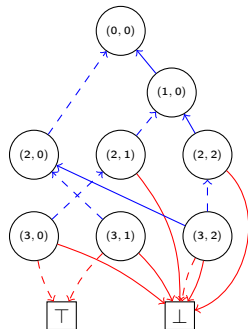
[

]

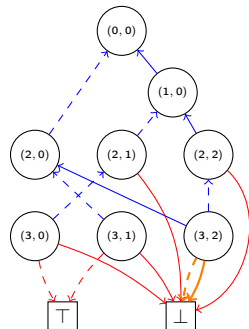
Output:



Reduce Example

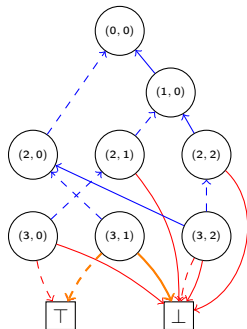


Reduce Example



[Level: 3
 [(3, 2) \mapsto \perp]]

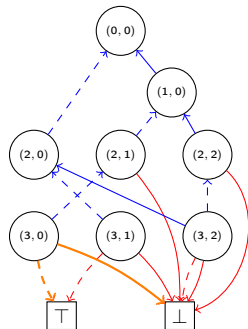
Reduce Example



Level: 3

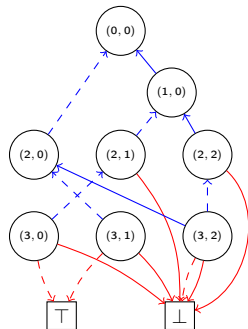
[$[(3, 2) \mapsto \perp]$]
[$((3, 1), \top, \perp)$,
]

Reduce Example



Level: 3
[$[(3, 2) \mapsto \perp]$]
[$((3, 1), \top, \perp)$,
 $((3, 0), \top, \perp)$]

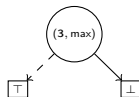
Reduce Example



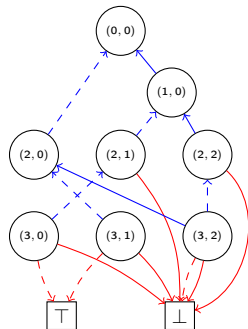
Level: 3

[$[(3, 2) \mapsto \perp]$]
[$[(3, 1) \mapsto (3, \max)]$,
	$((3, 0), \top, \perp)$]

Output:
 $((3, \max), \top, \perp)$



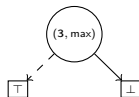
Reduce Example



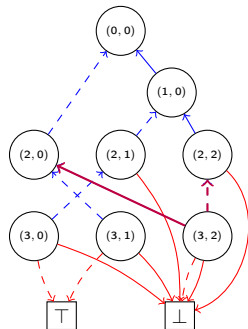
Level: 3

[$[(3, 2) \mapsto \perp]$]
[$[(3, 1) \mapsto (3, \max)]$,
	$[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

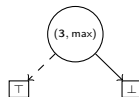
]

Level: 3

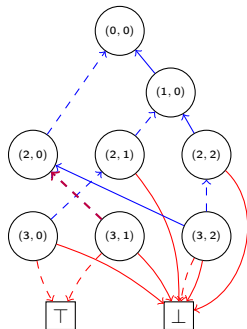
[]

[$[(3, 1) \mapsto (3, \max)]$,
 $[(3, 0) \mapsto (3, \max)]$]

Output:



Reduce Example



Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 3

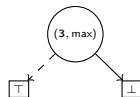
[]

[,

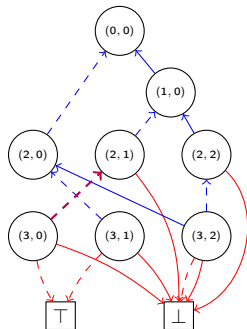
$[(3, 0) \mapsto (3, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

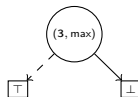
[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

]

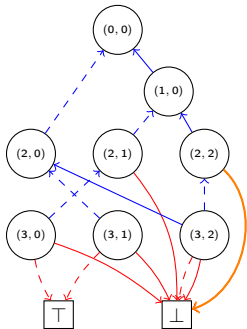
Level: 3

[
 [
 ,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$$(2, 1) \xrightarrow{\perp} (3, \max) \quad ,$$
$$(2, 0) \xrightarrow{T} \perp$$
$$(2, 0) \xrightarrow{\perp} (3, \max) \quad ,$$

1

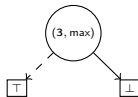
Level: 2

[

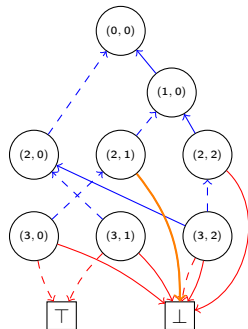
 $[(2, 2) \mapsto \perp]$

1

Output:



Reduce Example



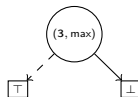
Priority Queue: Q_{red} :

[
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

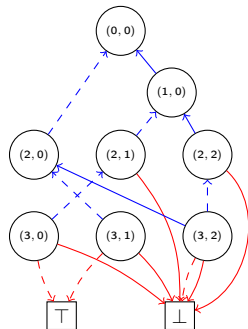
Level: 2

[$[(2, 2) \mapsto \perp]$]
 [$((2, 1), (3, \max), \perp)$,
]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

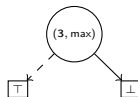
$((2, 1), (3, \max), \perp)$

,

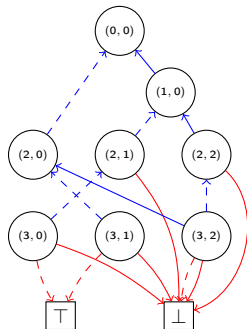
$((2, 0), (3, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

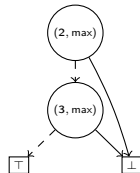
,

$((2, 0), (3, \max), \perp)$

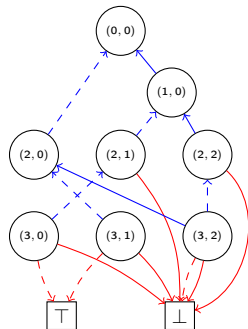
]

Output:

$((2, \max), (3, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

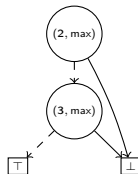
$[(2, 1) \mapsto (2, \max)]$

,

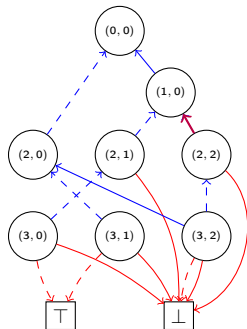
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

]

Level: 2

[

]

[

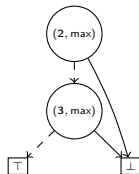
$[(2, 1) \mapsto (2, \max)]$

,

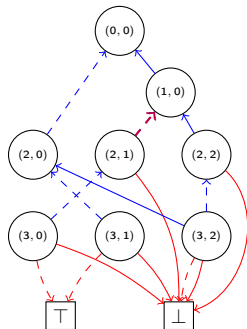
$[(2, 0) \mapsto (2, \max)]$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

[

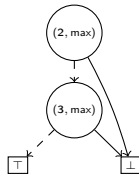
$[(2, 0) \mapsto (2, \max)]$

]

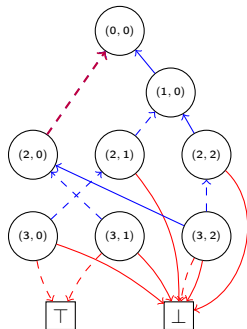
,

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

[

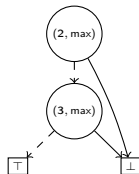
[

]

,

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

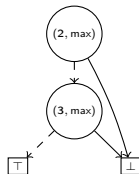
]

[

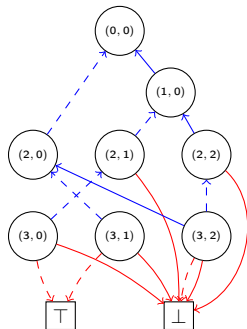
$((1, 0), (2, \max), \perp)$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

]

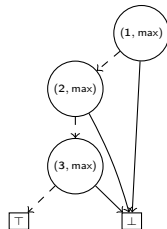
[

$[(1, 0) \mapsto (1, \max)]$

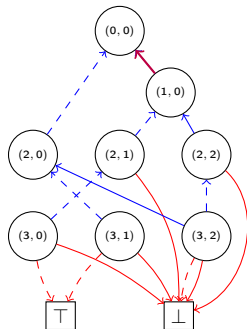
]

Output:

$((1, \max), (2, \max), \perp)$



Reduce Example



Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{T} (1, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

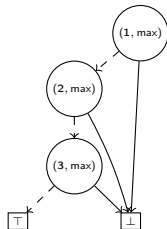
[

]

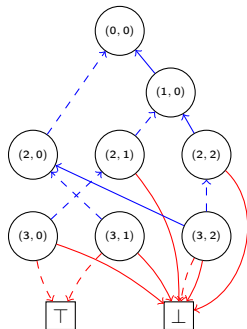
[

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

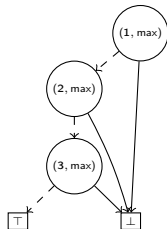
]

[

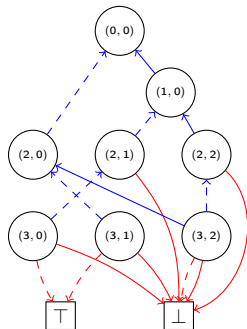
$((0, 0), (2, \max), (1, \max))$

]

Output:



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

[

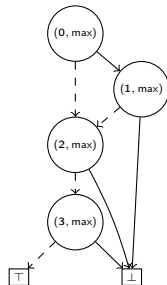
]

$[(0, 0) \mapsto (0, \max)]$

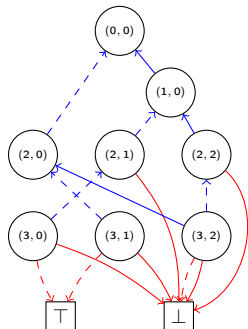
]

Output:

$((0, \max), (2, \max), (1, \max))$



Reduce Example



Priority Queue: Q_{red} :

[

]

Level: 0

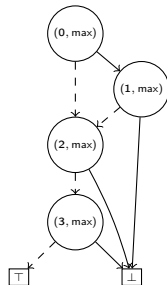
[

]

[

]

Output:



Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$[\quad]$$

Level: 2

[, ,]

Level: 3

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$\left[\quad (0, 0) \xrightarrow{\mathbb{T}} ((1, 0), (2, 1)) \quad \right]$$

Level: 2

$$\left[\begin{array}{c} (0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \end{array} \right] \quad ,$$

Level: 3

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$\left[(0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) \right]$$

Level: 2

$$\left[(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) \quad , \quad (1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad , \quad (1, 0) \xrightarrow{\top} ((3, 1), (2, 1)) \right]$$

Level: 3

$$\left[\quad , \quad , \quad , \quad \right]$$

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[, , ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$, $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, (1, 0) $\xrightarrow{\top}$ ((3, 1), (2, 1))]

Level: 3

[(2, 0) $\xrightarrow{\perp}$ ((3, 0), \top) , (2, 0) $\xrightarrow{\top}$ ((3, 1), (3, 0)) , (2, 1) $\xrightarrow{\perp}$ ((3, 0), (3, 0)) ,]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, (1, 0) $\xrightarrow{\top}$ ((3, 1), (2, 1))]

Level: 3

[(2, 0) $\xrightarrow{\perp}$ ((3, 0), \top) , (2, 0) $\xrightarrow{\top}$ ((3, 1), (3, 0)) , (2, 1) $\xrightarrow{\perp}$ ((3, 0), (3, 0)) , (2, 2) $\xrightarrow{\perp}$ ((3, 1), (3, 0))]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$, $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, ,]

Level: 3

[, $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$, $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$, $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[,]

Level: 3

[, (2, 2) $\xrightarrow{\perp}$ ((3, 1), (3, 0)) , (2, 0) $\xrightarrow{\perp}$ ((3, 0), \top)]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

[]

Level: 2

[, ,]

Level: 3

[, , , $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Levelized Priority Queue: $Q_{app:1}$:

Level: 1

$$[\quad]$$

Level: 2

[, ,]

Level: 3

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Memory layout and efficient sorting

The unique identifier of nodes and leafs can be represented in a single 64-bit integer.



(a) Unique identifier of a leaf v

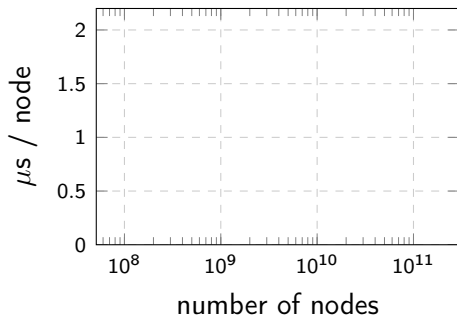
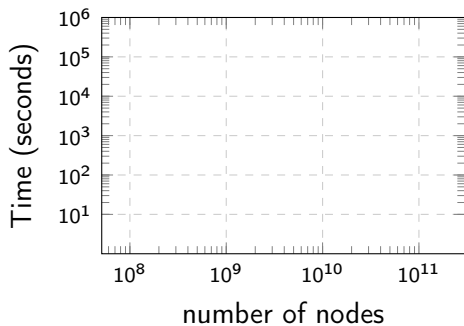


(b) Unique identifier of an internal node

The f bit-flag is used to store the *is_high* boolean inside of the source of an arc.

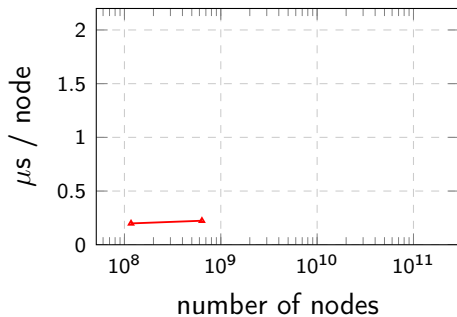
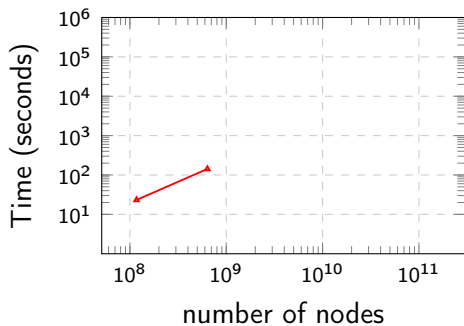
Adiar

github.com/ssoelvsten/adiar



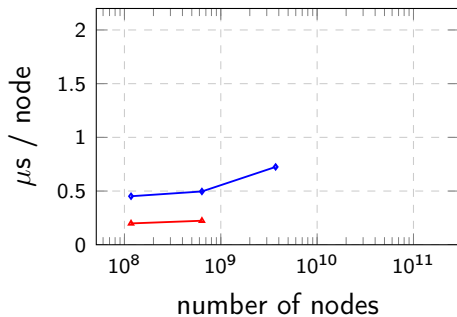
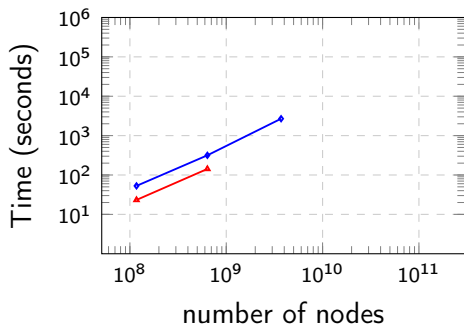
—▲— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Figure 11: Minimum running times for the N -Queens problem.



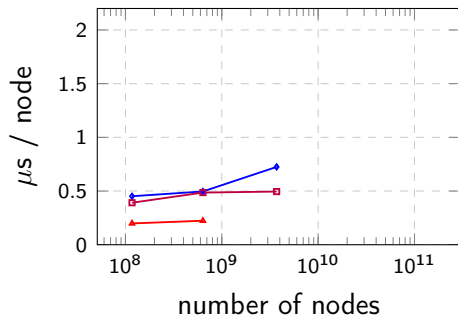
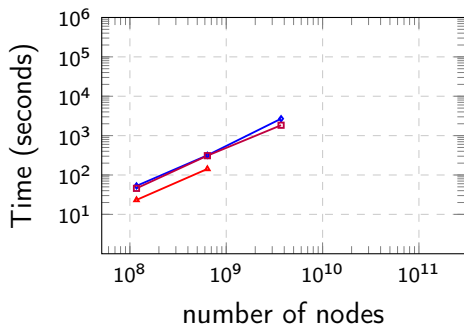
—▲— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Figure 11: Minimum running times for the N -Queens problem.



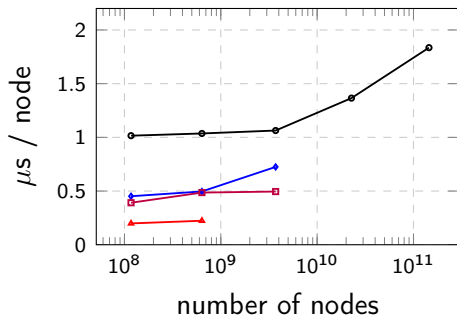
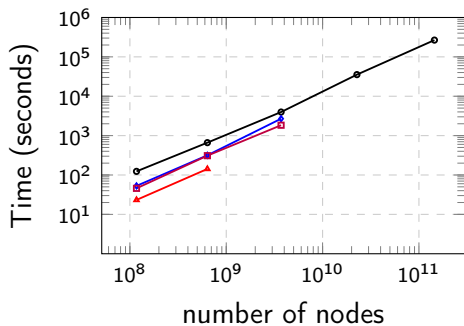
—▲— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Figure 11: Minimum running times for the N -Queens problem.



—▲— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Figure 11: Minimum running times for the N -Queens problem.



—▲— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Figure 11: Minimum running times for the N -Queens problem.

Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f ? g : h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f _{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f _{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	#paths in f to \top	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min / \max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$

Table 1: I/O-complexity of depth-first algorithms compared to our time-forwarded.

