

# $\exists\mathbb{R}$ -completeness of Nash equilibria in Perfect Information Stochastic Games

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- A play  $h \in \mathcal{H}_\infty$  is an infinite sequence  $(h_t)_{t \geq 0}$  of vertices in  $V$ , where

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- Utility functions  $u_i$  assigns a payoff  $u_i(i)$  for Player  $i$  to a play  $h \in \mathcal{H}_\infty$

## Mean-payoff games



**Figure 1:** A simple mean-payoff game.

## Mean-payoff games



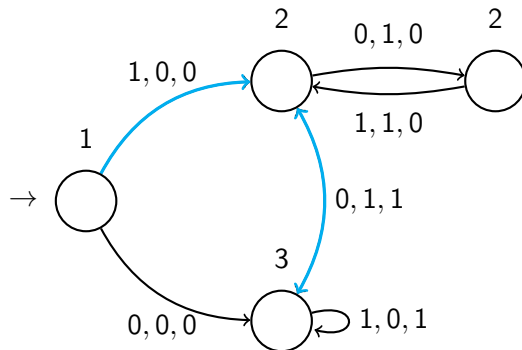
**Figure 1:** A simple mean-payoff game. Mean payoff for player 1: 1

## Mean-payoff games



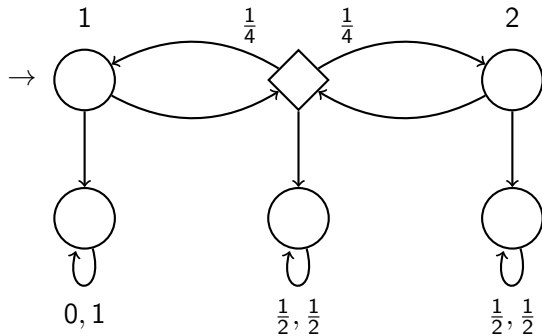
**Figure 1:** A simple mean-payoff game. Mean payoff for player 1:  $\frac{1}{2}$

## Mean-payoff games



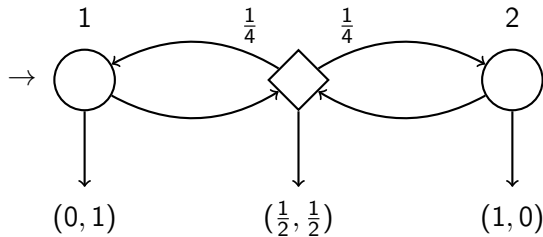
**Figure 1:** A simple mean-payoff game. Mean payoff for player 1: 0

## Recursive games



**Figure 2:** A simple game with terminal rewards only. Ummels '11

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**Figure 2:** A simple game with terminal rewards only. Ummels '11

## Strategies and Nash equilibria

A strategy  $\tau_i$  assigns a probability distribution to the outgoing arcs of vertices  $v \in V_i$  depending on the given history  $h$ .

- A strategy is *stationary*, if the choice of the players at a vertice is independent of the prior history of play (i.e. the strategy is memoryless).



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We assume players are acting *rationally*. This is commonly captured by the following notion

### **Definition (Nash equilibria)**

A strategy profile  $\tau = (\tau_1, \tau_2, \dots, \tau_m)$  is a *Nash equilibrium*, if no player  $i$  has a unilateral deviation available that strictly improves their payoff.

## Subgame Perfect Nash equilibria



**Figure 3:** A two-player reachability game with an irrational Nash equilibrium. Ummels '11

## Subgame Perfect Nash equilibria



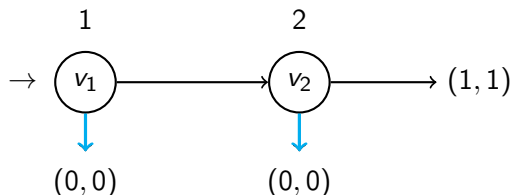
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### Definition (Subgame Perfect Nash equilibria)

A *Subgame perfect* Nash equilibrium is a NE of a game  $G$ , that is *not* only the best response from  $v_0$ , but is a best response in  $G[h]$  given *any* history  $h$  of play.

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# Game Theory in Model Checking

Games provide a well studied framework that can capture many model checking problems with *adversaries*.

- A protocol between  $m$  entities can be described by a stochastic game of  $m$  players.
- A distributed system of  $m$  peers can be described by a *concurrent* game of  $m$  players.

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Classical model checking objectives can be encapsulated in the utility function.

- *Reachability* objectives can be captured by payoffs in  $\{0, 1\}$  in a recursive game.
- *Safety* objectives can be captured by payoffs in  $\{-1, 1\}$  in a recursive game, since an *infinite* game has payoff 0.
- Other Büchi objectives can also be described in general Mean-payoff games.



# Game Theory in Synthesis

The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

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Does there exist a controller, such that the system satisfies the specification?

≡

Does there exist a strategy, such that Player 1 is surely winning?

# The subject of this seminar

Consider the problem:

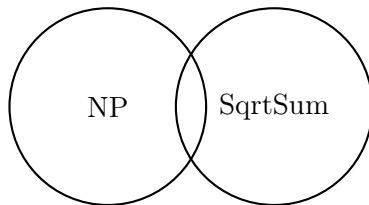
Given an  $m$ -player game  $G$  and payoff demands  $L \in \mathbb{R}^m$ ,  
does there exist a stationary <sup>1</sup> NE  $\tau$  with  $U(\tau) \geq L$ ?

We will show this is  $\exists\mathbb{R}$ -complete.

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<sup>1</sup>The problem of existence of a Nash equilibria satisfying some demands is undecidable for  $\geq 10$  players in recursive games, so we will only focus on *stationary* strategies.

$\exists\mathbb{R}$ -complexity



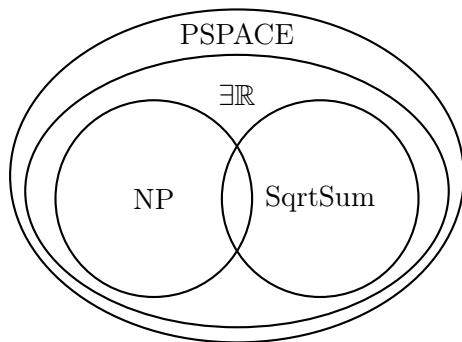
**Figure 4:** The relation between NP, SqrtSum, and  $\exists\mathbb{R}$



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The complexity class  $\exists\mathbb{R}$  both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of SqrtSum.

## $\exists\mathbb{R}$ Complexity Class



**Figure 4:** The relation between  $\text{NP}$ ,  $\text{SqrtSum}$ , and  $\exists\mathbb{R}$

The complexity class  $\exists\mathbb{R}$  both encapsulates the hardness of  $\text{NP}$  decision problems and the hardness of computing with real numbers of  $\text{SqrtSum}$ .

## NP Complexity Class

Remember that the well-known class NP can be captured by the ILP problem:

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \in \mathbb{N}^n\end{array}$$

where  $A \in \mathbb{Z}^{n \times m}$ ,  $b \in \mathbb{Z}^m$ ,  $c \in \mathbb{Z}^n$



## SqrtSum Complexity Class

Consider the following problem: Given  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in \mathbb{R}$  is the following inequality satisfied?

$$\sum_{i=1}^n \sqrt{a_i} \leq \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial...

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## SqrtSum Complexity Class

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$$\sum_{i=1}^n \sqrt{a_i} \leq \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial... How many decimals do you have to compute, before you know the answer? <sup>2</sup>

### **Definition** (SqrtSum)

The complexity class SqrtSum consists of all problems that are polynomial time reducible to the problem above.

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<sup>1</sup>This consistently comes up in Computational Geometry. Here, theoretical works solve this by assuming the  $\mathbb{R}$ -RAM computational model; leaving an adventure for implementors to experience later.

The *Existential Theory of the Reals* is the language of all true sentences of the form

$$\exists x_1, x_2, \dots, x_n \in \mathbb{R} : \phi(x_1, x_2, \dots, x_n)$$

where  $\phi$  is a quantifier-free Boolean formula of inequalities and equalities.

## $\exists\mathbb{R}$ Complexity Class

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### **Definition** ( $\exists\mathbb{R}$ )

The complexity class  $\exists\mathbb{R}$  consists of all problems, that are polynomial time reducible to the existential theory of the reals.

We will consider the following  $\exists\mathbb{R}$ -complete problem.

**Definition (HomQuad)**

Given a system  $S$  of  $l$  *homogeneous quadratic* polynomials<sup>3</sup> in  $n$  variables, does there exist an  $x \in \mathbb{R}^n$  such that  $q_k(x) = 0$  for all  $k \in \{1, 2, \dots, l\}$  and  $x$  is a probability distribution?

---

<sup>3</sup>A homogenous quadratic polynomial is of the form  $\sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j$  where  $A \in [-1, 1]^{n \times n}$ .

**Proof Sketch:  $\exists\mathbb{R}$ -Completeness of Nash equilibria**

## $\exists \mathbb{R}$ -Completeness of Nash equilibria

Consider the problem:

Given an  $m$ -player game  $G$  and payoff demands  $L \in \mathbb{R}^m$ ,  
does there exist a stationary NE  $\tau$  with  $U(\tau) \geq L$ ?

## $\exists\mathbb{R}$ -Completeness of Nash equilibria

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Given an  $m$ -player game  $G$  and payoff demands  $L \in \mathbb{R}^m$ ,  
does there exist a stationary NE  $\tau$  with  $U(\tau) \geq L$ ?

It has already been shown to be NP-hard for  $\geq 2$  players and SqrtSum-hard for  $\geq 4$  players. Furthermore, it is contained within  $\exists\mathbb{R}$ .



It is  $\exists\mathbb{R}$ -complete! We will show this by reduction to:

**Definition** (HomQuad)

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That is, given a system  $\mathcal{S}$  of  $l$  polynomials of the form

$$q_k(x) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{ij}x_ix_j + \cdots a_{nn}x_nx_n$$

we will construct a game  $\mathcal{G}(\mathcal{S})$  such that all  $q_k(x) = 0$  if and only if  $\mathcal{G}(\mathcal{S})$  has a stationary Nash equilibria that satisfies some payoff demand.

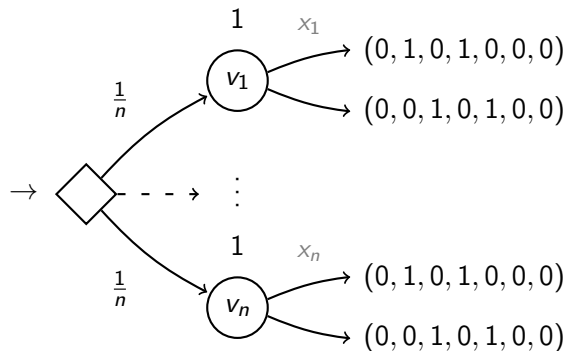
## Proof Sketch: $\mathcal{G}_{\text{var}}$



At each  $v_i$ , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

**Figure 5:** The gadget game  $\mathcal{G}_{\text{var}}$

## Proof Sketch: $\mathcal{G}_{\text{var}}$



**Figure 5:** The gadget game  $\mathcal{G}_{\text{var}}$

At each  $v_i$ , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Player 1 strategy corresponds to a probability distribution if it satisfies the payoff demand

$$\left(0, \frac{1}{n}, \frac{n-1}{n}, \dots\right)$$

**Proof Sketch:**  $\mathcal{G}_{\text{mul}}(i, j, \alpha)$



**Figure 6:** The gadget game  $\mathcal{G}_{\text{mul}}(i, j, \alpha)$

## Proof Sketch: $\mathcal{G}_{\text{mul}}(i, j, \alpha)$



**Figure 6:** The gadget game  $\mathcal{G}_{\text{mul}}(i, j, \alpha)$

If Player 1 receives payoff 1, then Player-6 gets  $\alpha x_i x_j$  and Player-7 gets  $(1 - \alpha) x_i x_j$ .

$$\max_{\tau_1} \min_{\tau_2} \Pr[u_1(v_0(\tau_1, \tau_2)) = 1]$$

$$\forall \tau_2 : \Pr[u_1(v_0(\tau_1, \tau_2)) = 1] \geq \text{value}$$

## Proof Sketch: $\mathcal{G}_{\text{poly}}(q)$

For a homogenous quadratic polynomial  $q_k(x) = \sum_{i,j=1}^n A_{ij}x_i x_j$ .



**Figure 7:** The gadget game  $\mathcal{G}_{\text{poly}}(q_k)$

If Player 1 receives payoff 1, then Player 6 gains payoff  $\frac{1}{2n^2}(\|x\|_1^2 + q_k(x))$ .

If also  $\|x\|_1$  is 1, then  $q_k(x) = 0$ .



## Proof Sketch: Final reduction



**Figure 8:** The game  $\mathcal{G}(\mathcal{S})$  of the reduction

$\mathcal{S}$  is a “yes”-instance of HomQuad if and only if the game  $\mathcal{G}(\mathcal{S})$  has a Nash Equilibria that satisfies the demands

$$\left(\frac{1}{2}, \frac{1}{n}, \frac{n-1}{n}, 0, 0, \dots, 0\right)$$

## Theorem

*It is  $\exists\mathbb{R}$ -complete to decide whether for a given  $m$ -player recursive game  $G$  and payoff demands  $L \in \mathbb{R}^m$  there exists a stationary Nash equilibria  $\tau$  with  $U(\tau) \geq L$ .*

- *The problem is  $\exists\mathbb{R}$ -complete even for acyclic 7-player recursive games with non-negative rewards.*
- *It even holds for stationary Subgame Perfect Equilibria.*

## Implications for Model Checking

## $\exists \mathbb{R}$ -Completeness of Stationary NE without Payoff Demands



**Figure 9:** The game  $\mathcal{G}_{\exists \text{NE}}(S)$

$\mathcal{G}_{\text{noNE}}$  is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

## $\exists \mathbb{R}$ -Completeness of Stationary NE without Payoff Demands



**Figure 9:** The game  $\mathcal{G}_{\exists \text{NE}}(S)$

$\mathcal{G}_{\text{noNE}}$  is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

$S$  is a “yes”-instance of HomQuad if and only if the game  $\mathcal{G}_{\exists \text{NE}}(S)$  has a stationary Nash Equilibria.

## $\exists\mathbb{R}$ -Completeness of Reachability and Safety objectives

There exists different  $\mathcal{G}_{\text{noNE}}$  gadget games for the different restrictions of the utility function:

- Reachability objective
- Safety objective

### Theorem

*It is  $\exists\mathbb{R}$ -complete to decide whether a given  $m$ -player game with Reachability or Safety objectives has a stationary NE.*

- *even for  $m = 7$  players.*

## $\exists\mathbb{R}$ -Completeness of being almost surely winning

Consider the game  $\mathcal{G}_{\exists\text{NE}}(\mathcal{S})$  in which another player is added, who is always winning in  $\mathcal{G}(\mathcal{S})$ , but not in  $\mathcal{G}_{\text{noNE}}$ .

### Theorem

*For any  $i$ , it is  $\exists\mathbb{R}$ -complete to decide whether a given  $m$ -player recursive game has a stationary NE in which Player  $i$  is almost surely winning.*

- *even for  $m = 8$  players.*

## Final remarks

It is  $\exists\mathbb{R}$ -complete to decide in an  $m$ -player perfect information recursive game.

- exists Subgame Perfect Nash equilibria satisfying demand  $L \in \mathbb{R}^m$
- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.



## Final remarks

It is  $\exists\mathbb{R}$ -complete to decide in an  $m$ -player perfect information recursive game.

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- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.

Notice here that

- This problem is already shown by Ummels '11 to be NP-hard for  $\geq 2$  players and SqrtSum-hard for  $\geq 4$  players so this completeness result could not become much tighter.
- There have been recent results of  $\exists\mathbb{R}$ -completeness in *imperfect information* games. The complexity of these results stem from the structure of the game, not the lack of information.