# Efficient Equality Checking for Non-Shared Binary Decision Diagrams

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$$\underbrace{O(\mathsf{sort}(N^2))}_{\texttt{Apply}} + \underbrace{O(\mathsf{sort}(N^2))}_{\texttt{Reduce}} + \underbrace{O(1))}_{\mathsf{check\ is}\ \top} = O(\mathsf{sort}(N^2))$$

**Theorem (Bryant '86)**
*Let $\pi$ be a variable order and $f : \mathbb{B}^n \to \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing $f$ with ordering $\pi$.*

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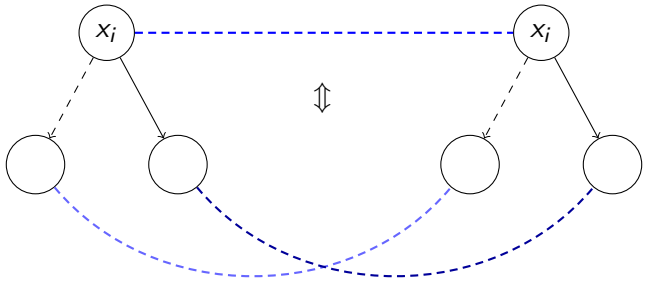
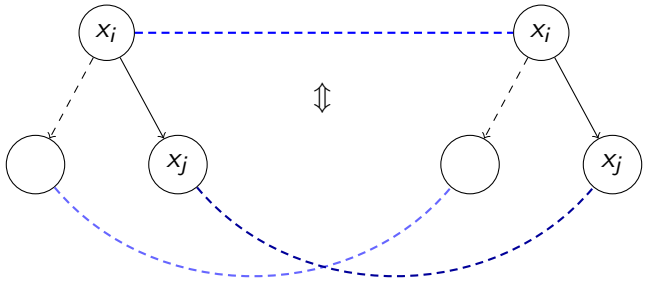
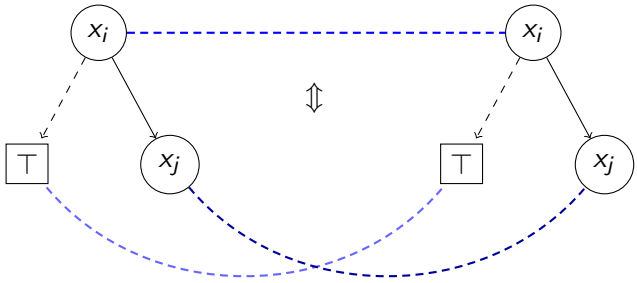
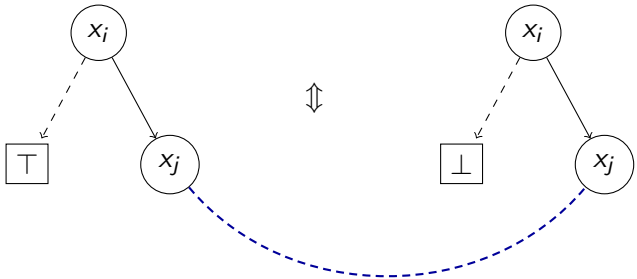
**Trivial cases: $f \not\equiv g$ if there is a mismatch in**

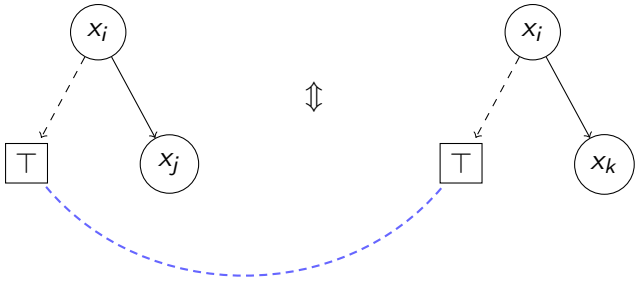
| | | |
|---|---|---|
| ■ $N_f \neq N_g$ | Number of nodes | $O(1)$ I/Os |
| ■ $L_f \neq L_g$ | Number of levels | $O(1)$ I/Os |
| ■ $N_{f,i} \neq N_{g,i}$ | Number of nodes on a level | $O(L/B)$ I/Os |
| ■ $L_{f,i} \neq L_{g,i}$ | Label of an $i$th level | $O(L/B)$ I/Os |

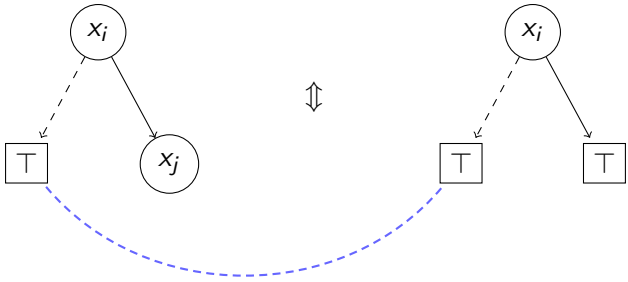

$x_i$
$x_i$


$x_i$
$x_i$
$x_j$
$x_j$


$x_i$
$x_i$
$\Updownarrow$
$\top$
$x_j$
$\top$
$x_j$


$x_i$
$\top$
$x_j$
⇕
$x_i$
$\bot$
$x_j$


$x_i$
$\top$
$x_j$
$\Updownarrow$
$x_i$
$\top$
$x_k$


$x_i$
$x_j$
⊤
⊤
$x_i$
⊤

`IsIsomorphic(`$f$`, `$g$`)`

- Check whether root $v_f$ of $f$ and root $v_g$ of $g$ have a local violation.
- Check $\mathit{low}(v_f) \sim \mathit{low}(v_g)$ and $\mathit{high}(v_f) \sim \mathit{high}(v_g)$ "recursively".

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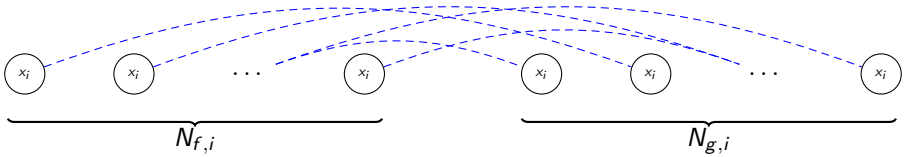
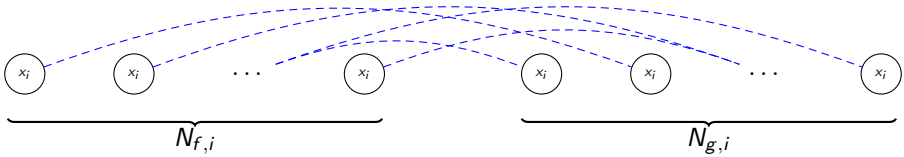
$$\underbrace{O(\mathsf{sort}(N^2))}_{\texttt{Apply}'} + \underbrace{\cancel{O(\mathsf{sort}(N^2))}}_{\texttt{Reduce}} + \underbrace{\cancel{O(1)}}_{\text{check is } \top} = O(\mathsf{sort}(N^2))$$


$x_i$
$x_i$
...
$x_i$
$x_i$
$x_i$
...
$x_i$
$N_{f,i}$
$N_{g,i}$




Return `false` if more than $N_{f,i} = N_{g,i}$ pairs of nodes $(v_f, v_g)$ are checked on level $i$.

$$O(\text{sort}(N))$$

**Observation**
The output of `Reduce` has the following properties

- Nodes on level $i$ have their identifiers *consecutively* numbered

$$MAX - N_{f,i} + 1, \ldots, MAX - 1, MAX \ .$$

- Nodes on level $i$ are output sorted by their children

$$((i_1, id_1), low_1, high_1) <_{lex(i,low,high)} ((i_2, id_2), low_2, high_2) \ ,$$

where

$$\forall (i, id) \ : \ (i, id) < \bot < \top \ ^{1}.$$

[1] Assuming the BDD is not negated. If that is the case then $(i, id) < \top < \bot$.

**Theorem**
*If $G_f$ and $G_g$ are outputs of* `Reduce`*.*

$$G_f \sim G_g \iff \textit{For all } i \in [0; N) \textit{ the node } G_f[i] \textit{ matches } G_g[i] \textit{ numerically.}$$

**Proof.**
$\Leftarrow$ : Must describe the exact same graph.

$\Rightarrow$ : Strong induction on BDD levels bottom-up ... $\square$

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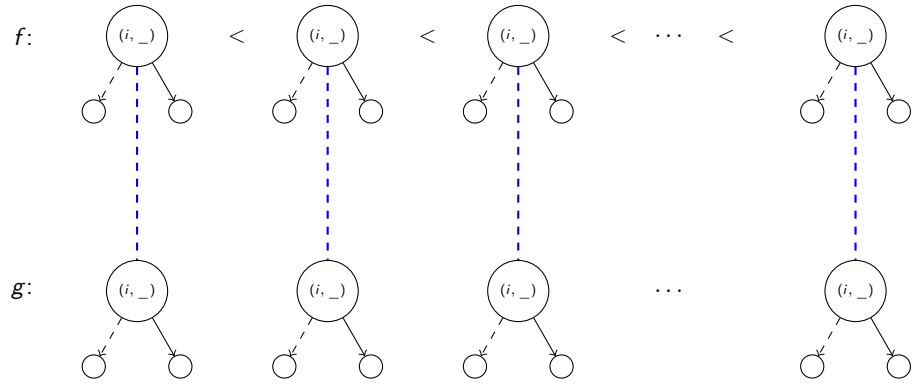
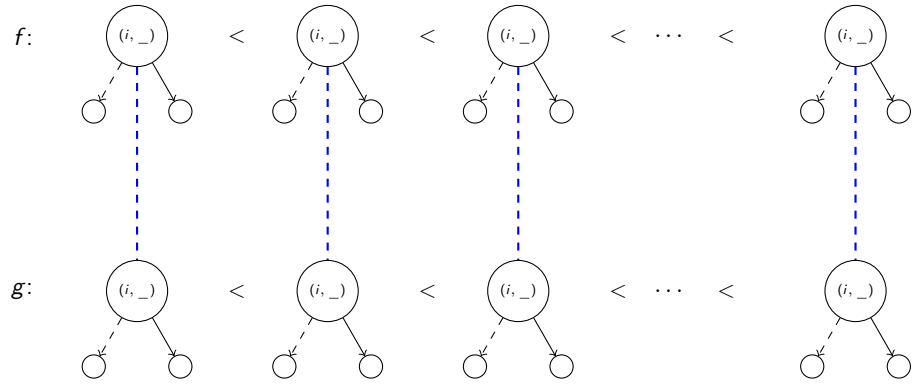
**Corollary**
*If $G_f$ and $G_g$ are outputs of* `Reduce` *then $f \equiv g$ is computable using $2 \cdot N/B$ I/Os.*[2]

[2] Assuming they are both unnegated (or both negated).

Base case: $n$




Induction case: $i < n$

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| Algorithm | Time (s) |
|---|---|
| $f \leftrightarrow g \equiv \top$ | 0.38 |
| $O(\mathsf{sort}(N))$ | 0.058 |
| $2 \cdot N/B$ | 0.006 |

Checking the (EPFL Benchmark) *voter* circuit's single output gate ($|N_f| = |N_g| = 5.76$ MiB).