

# I/O-efficient Manipulation of Binary Decision Diagrams

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*Adiar: Binary Decision Diagrams in External Memory.* 2022





# Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

- CountPaths

- Apply

- Equality Checking

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What are Binary Decision Diagrams?

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**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 \oplus x_2 \wedge x_3 : x_2 \wedge x_3)$

Examples of (Reduced Ordered) Binary Decision Diagrams.

### Theorem (Bryant '86)

*For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.*



**(1)** Remove redundant nodes



**(2)** Merge duplicate nodes

```

bdd_apply( $f$ ,  $g$ ,  $\otimes$ ):
  if  $f, g \in \{\perp, \top\}$ 
  then  $f \otimes g$ 
  else let  $i = \text{top}(f.\text{var}), g.\text{var}$ 
          $t = \text{bdd\_apply}(f[x_i := \top], g[x_i := \top], \otimes)$ 
          $e = \text{bdd\_apply}(f[x_i := \perp], g[x_i := \perp], \otimes)$ 
  in make_node( $i, t, e$ )

```

```

bdd_apply( $f$ ,  $g$ ,  $\otimes$ ):
  if  $f, g \in \{\perp, \top\}$ 
  then  $f \otimes g$ 
  else let  $i = \text{top}(f.\text{var}, g.\text{var})$ 
          $t = \text{bdd\_apply}(f[x_i := \top], g[x_i := \top], \otimes)$ 
          $e = \text{bdd\_apply}(f[x_i := \perp], g[x_i := \perp], \otimes)$ 
  in make_node( $i, t, e$ )

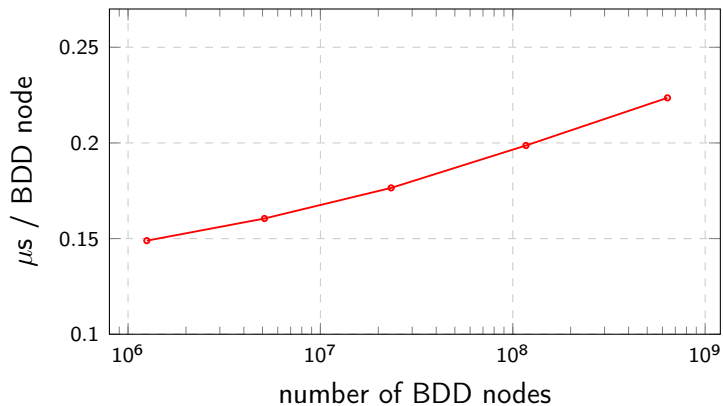
```

## Theorem

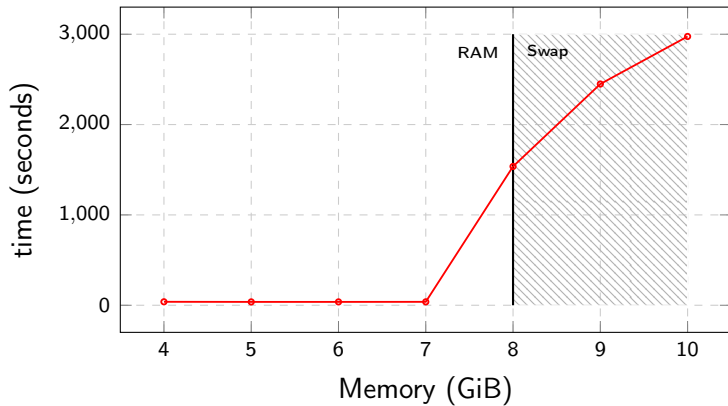
`bdd_apply` runs in  $O(N_f \cdot N_g)$  time.

- Memoisation (*Computation Cache*) ensures each recursion is computed only once.
- Reduction Rules can be maintained within `make_node( $i, t, e$ )` in  $O(1)$  time.
  - 1 Redundancy is resolved with an if-statement.
  - 2 Duplication is avoided with a hash table (*Unique Node Table*).





Running time of *BuDDy* for the *N*-Queens problem.



Running time of *BuDDy* for *Tic-Tac-Toe* with  $N = 21$ .

# Contents

What are Binary Decision Diagrams?

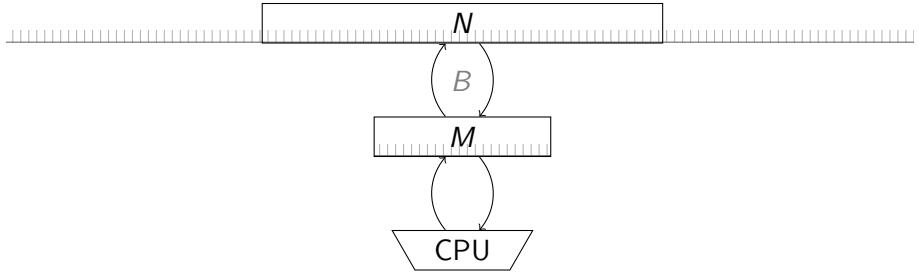
Why do they break?

How can we fix it?

CountPaths

Apply

Equality Checking



The I/O model by Aggarwal and Vitter '87

For any realistic values of  $N$ ,  $M$ , and  $B$  we have that

$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

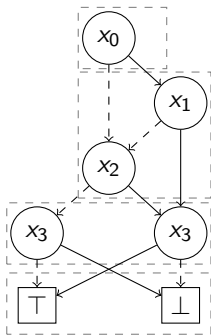
**Theorem (Aggarwal and Vitter '87)**

*$N$  elements can be sorted in  $\Theta(\text{sort}(N))$  I/Os.*

**Theorem (Arge '95)**

*$N$  elements can be inserted in and extracted from a Priority Queue in  $\Theta(\text{sort}(N))$  I/Os.*

## CountPaths : *Example*

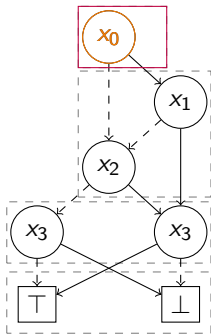


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
0	0

## CountPaths : *Example*

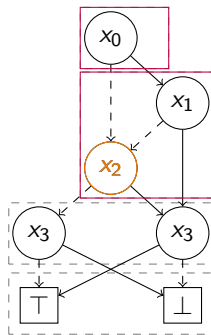


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
1	1

## CountPaths : *Example*



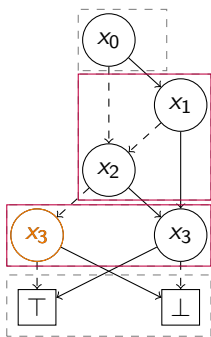
(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
2	2



## CountPaths : *Example*

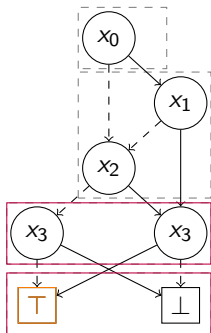


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
3	3

## CountPaths : *Example*

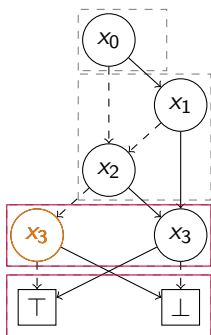


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
4	3

## CountPaths : *Example*

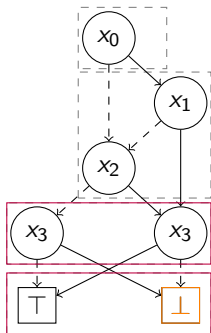


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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node I/Os	cache lookups
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## CountPaths : *Example*

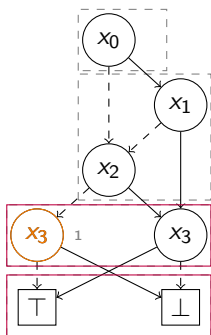


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node I/Os	cache lookups
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## CountPaths : *Example*

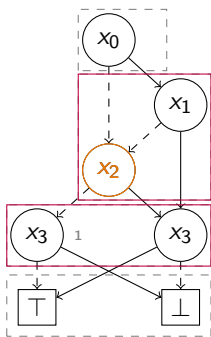


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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node I/Os	cache lookups
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## CountPaths : *Example*

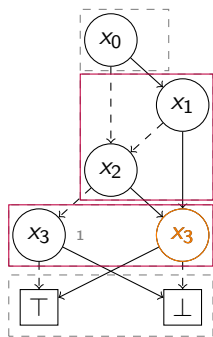


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
5	3

## CountPaths : *Example*

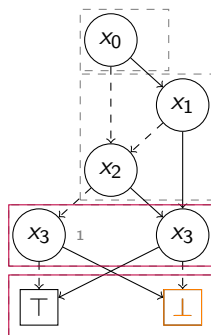


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
5	4

## CountPaths : *Example*



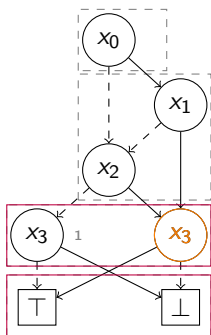
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$$M = 4, B = 2$$

node I/Os	cache lookups
6	4



## CountPaths : *Example*

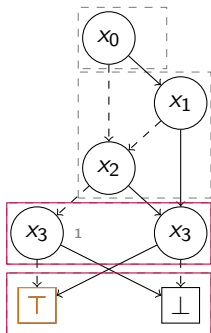


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

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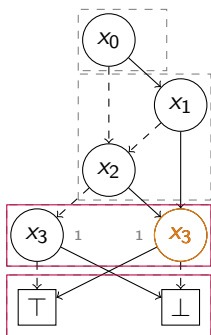


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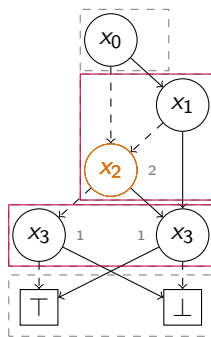


(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

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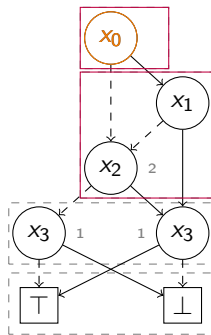


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
7	4

## CountPaths : *Example*



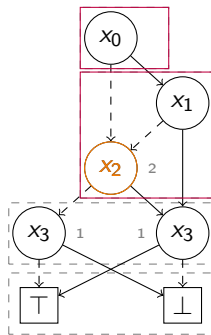
**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
8	4



## CountPaths : *Example*

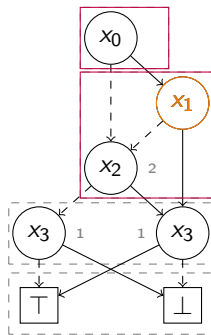


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$$M = 4, B = 2$$

node I/Os	cache lookups
8	6

## CountPaths : *Example*



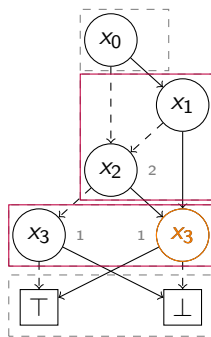
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node I/Os	cache lookups
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## CountPaths : *Example*

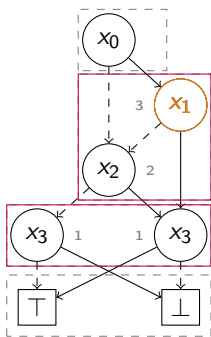


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node I/Os	cache lookups
9	7

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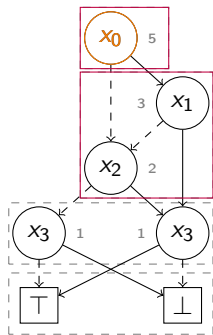


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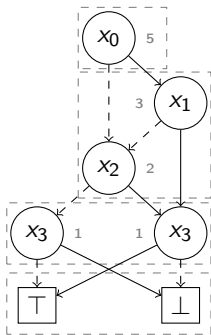


**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

Algorithm	Time Complexity
bdd_pathcount	$O(N_f)$
bdd_not	$O(N_f)$
bdd_restrict	$O(N_f)$
bdd_apply	$O(N_f \cdot N_g)$
bdd_equal	$O(1)$

Algorithm	I/O-Complexity
bdd_pathcount	$O(N_f)$
bdd_not	$O(N_f)$
bdd_restrict	$O(N_f)$
bdd_apply	$O(N_f \cdot N_g)$
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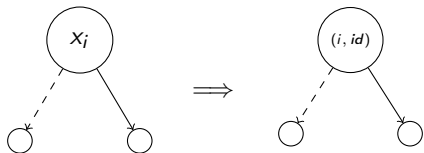
How can we fix it?

CountPaths

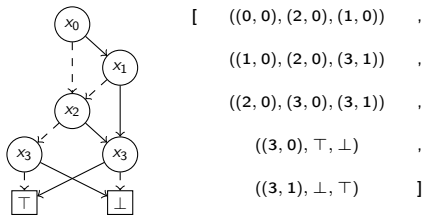
Apply

Equality Checking

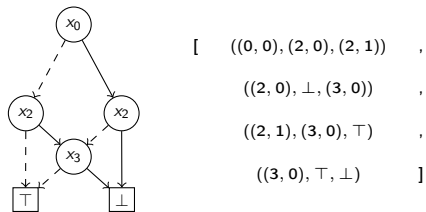




$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 \oplus x_2 \wedge x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

## CountPaths : *Example*



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

## CountPaths : *Example*



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Priority Queue:  $Q_{count}$ :

[

]

## CountPaths : *Example*



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## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue:  $Q_{count}$ :

[  $((0,0) \xrightarrow{\top} (1,0), 1)$  ,  
 $((0,0) \xrightarrow{\perp} (2,0), 1)$  ,

]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	0	0

Priority Queue:  $Q_{count}$ :

[  $((0, 0) \xrightarrow{\top} (1, 0), 1)$  ,  
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## CountPaths : *Example*



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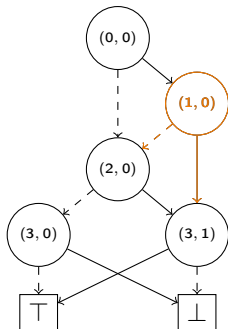
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Seek	Sum	Result
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Priority Queue:  $Q_{count}$ :

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 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

Priority Queue:  $Q_{count}$ :

[  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 ]

## CountPaths : *Example*



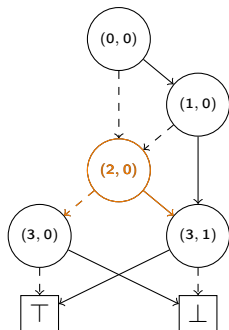
(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	0	0

Priority Queue:  $Q_{count}$ :

[  
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
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 ]

## CountPaths : *Example*



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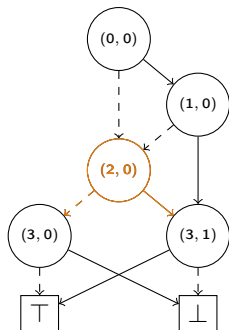
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 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$  ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	0	0

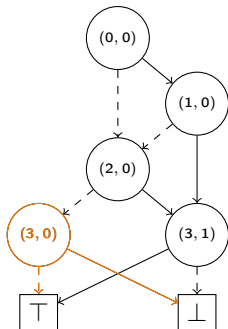
Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
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 $((2, 0) \xrightarrow{\top} (3, 1), 2)$  ]



## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	0	0

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$  ,  
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## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	2	0

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

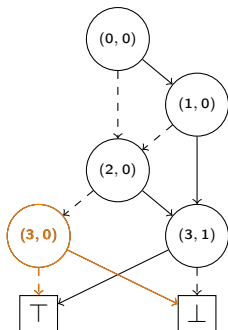
Seek	Sum	Result
(3, 0)	2	2

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	0	2

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	0	2

Priority Queue:  $Q_{count}$ :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$  ,  
 $((2, 0) \xrightarrow{T} (3, 1), 2)$  ]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	1	2

Priority Queue:  $Q_{count}$ :

[

$((2, 0) \xrightarrow{T} (3, 1), \quad 2) \quad ]$

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	3	2

Priority Queue:  $Q_{count}$ :

[

]

## CountPaths : *Example*



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 1)	3	5

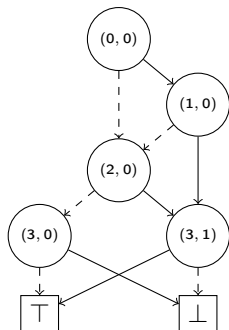
Priority Queue:  $Q_{count}$ :

[

]



## CountPaths : *Example*



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result

5

Priority Queue:  $Q_{count}$ :

[

]

# Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

CountPaths

Apply

Equality Checking

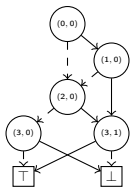
# Apply



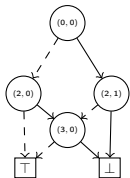
# Apply



# Apply : *Example*



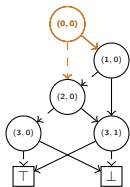
**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

**(c)**  $(a) \wedge (b)$

# Apply : *Example*



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



**(b)**  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

**(c)**  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

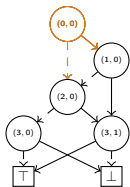
Priority Queue:  $Q_{app:1}$ :

[  $(0,0) \xrightarrow{\top} ((1,0), (2,1))$  ,  
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$  ,

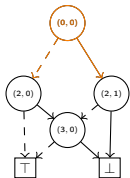


(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[  $(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$  ,  
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,

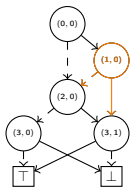


]

(c)  $(a) \wedge (b)$



# Apply : Example

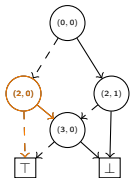


Seek:  
 $\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :  
 $[ (0, 0) \xrightarrow{\top} ((1, 0), (2, 1)) ,$   
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0)) ,$



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

]

(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

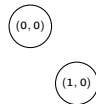
Seek:

$\min((1, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

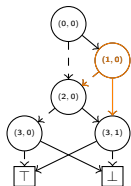
[  $(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$  ,  
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,  
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,  
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

]



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((1, 0), (2, 1))$

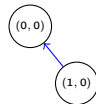
Priority Queue:  $Q_{app:1}$ :

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,  
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,  
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

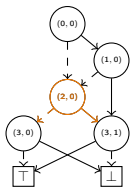
]

Output:  
 $(0, 0) \xrightarrow{\top} (1, 0)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



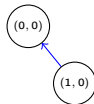
(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((2, 0), (2, 0))$

Priority Queue:  $Q_{app:1}$ :

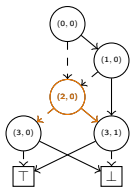
[  
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,  
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,  
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,  
 ]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((2, 0), (2, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$  ,

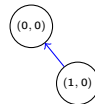
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((2, 0), (2, 0))$

Priority Queue:  $Q_{app:1}$ :

[

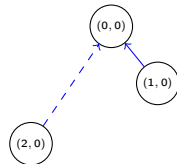
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:  
 $(0, 0) \xrightarrow{\perp} (2, 0)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

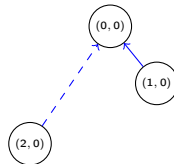
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$  ,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

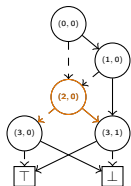
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Output:

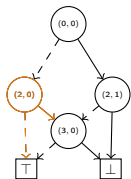


(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$   $((3, 0), (3, 1))$  ,

]

Output:



(c)  $(a) \wedge (b)$



# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

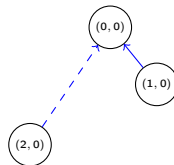
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad ((3, 0), (3, 1))$  ,

]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

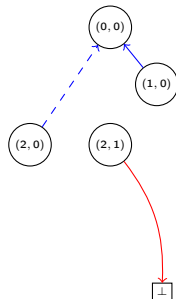
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[  $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$   $((3, 0), (3, 1))$  ,

]

Output:  
 $(2, 1) \xrightarrow{\top} \perp$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((2, 0), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

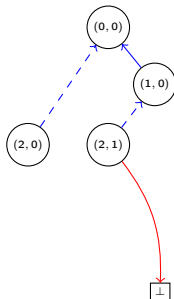
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:  
 $(1, 0) \xrightarrow{\perp} (2, 1)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

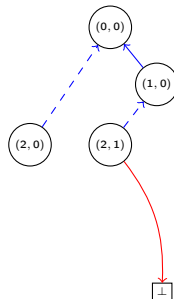
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,  
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

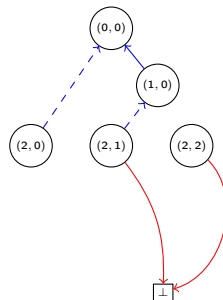
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$  ,  
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:  
 $(2, 2) \xrightarrow{\top} \perp$

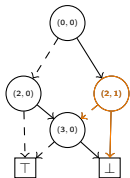


(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (2, 1))$

Priority Queue:  $Q_{app:1}$ :

[

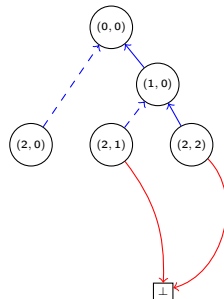
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:  
 $(1, 0) \xrightarrow{\top} (2, 2)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

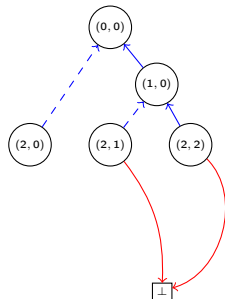
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

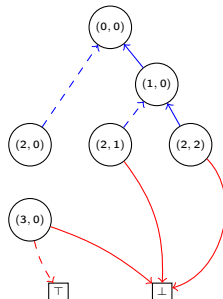
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:  
 $(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$



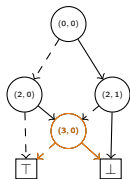
(c)  $(a) \wedge (b)$



# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

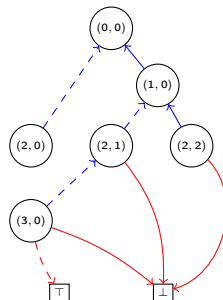
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:  
 $(2, 1) \xrightarrow{\perp} (3, 0)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

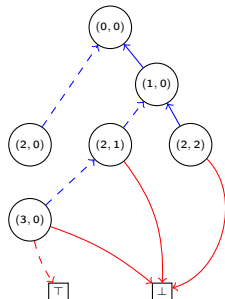
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$  ,  
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$  ,  
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

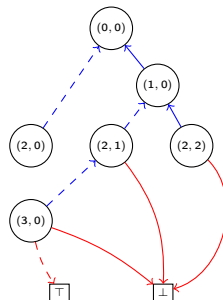
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

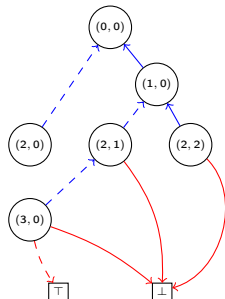
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$  ]

Priority Queue:  $Q_{app:2}$ :

[

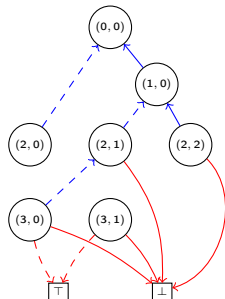
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:  
 $(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\min((3, 0), \top)$

Priority Queue:  $Q_{app:1}$ :

[

]

Priority Queue:  $Q_{app:2}$ :

[

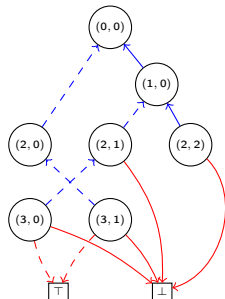
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:  
 $(2, 0) \xrightarrow{\perp} (3, 1)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

]

Priority Queue:  $Q_{app:2}$ :

[

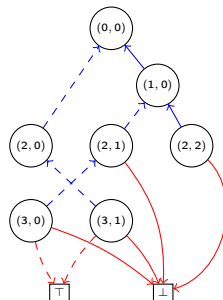
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

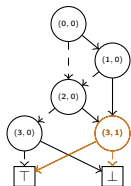
]

Output:

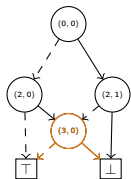


(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

]

Priority Queue:  $Q_{app:2}$ :

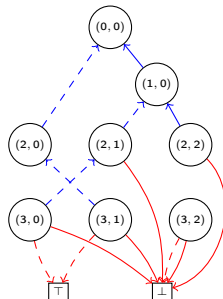
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$   
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

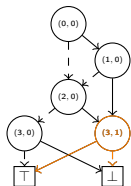
Output:  
 $(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



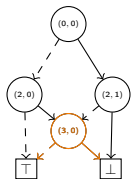
(c)  $(a) \wedge (b)$



# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:  
 $\max((3, 1), (3, 0))$

Priority Queue:  $Q_{app:1}$ :

[

]

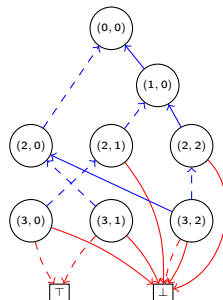
Priority Queue:  $Q_{app:2}$ :

[

]

Output:

$(2, 0) \xrightarrow{\top} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$



(c)  $(a) \wedge (b)$

# Apply : Example



(a)  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b)  $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue:  $Q_{app:1}$ :

[

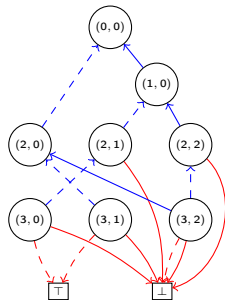
Priority Queue:  $Q_{app:2}$ :

[

]

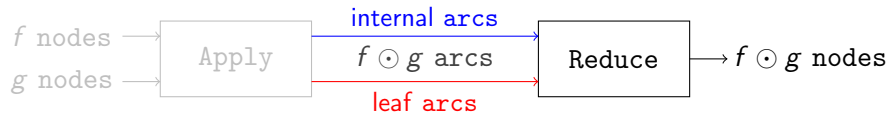
]

Output:



(c)  $(a) \wedge (b)$

# Apply



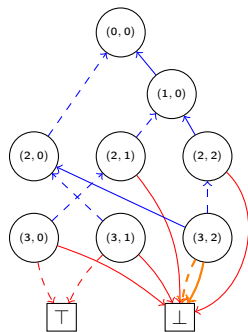
## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)

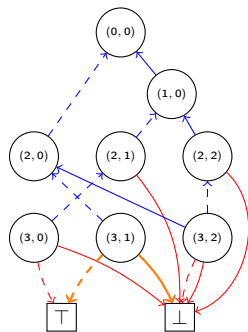


(c)  $(a) \wedge (b)$

[                      Level: 3  
                      $[(3, 2) \mapsto \perp]$                       ]

(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)

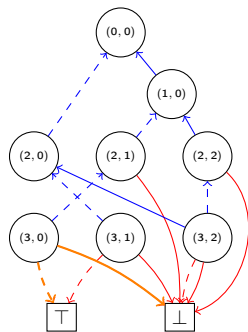


(c)  $(a) \wedge (b)$

Level: 3  
 $[ \quad [(3, 2) \mapsto \perp] \quad ]$   
 $[ \quad ((3, 1), \top, \perp) \quad , \quad ]$

(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Level: 3  
 $[ \quad [(3, 2) \mapsto \perp] \quad ]$   
 $[ \quad ((3, 1), \top, \perp) \quad , \quad$   
 $\quad ((3, 0), \top, \perp) \quad ]$

(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)

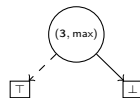


(c)  $(a) \wedge (b)$

Level: 3

[	$[(3, 2) \mapsto \perp]$	]
[	$[(3, 1) \mapsto (3, \max)]$	,
	$((3, 0), \top, \perp)$	]

Output:  
 $((3, \max), \top, \perp)$



(d)  $(a) \wedge (b)$  reduced



## Apply : *Example* (Continued)

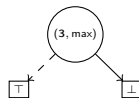


(c)  $(a) \wedge (b)$

Level: 3

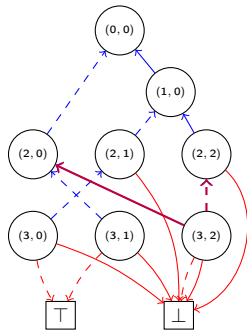
[	$[(3, 2) \mapsto \perp]$	]
[	$[(3, 1) \mapsto (3, \max)]$	,
	$[(3, 0) \mapsto (3, \max)]$	]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[  $(2, 2) \xrightarrow{\perp} \perp$  ,

$(2, 0) \xrightarrow{T} \perp$  ,

]

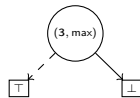
Level: 3

[

$[(3, 1) \mapsto (3, \max)]$  ,

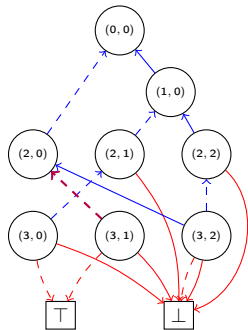
$[(3, 0) \mapsto (3, \max)]$  ]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[  $(2, 2) \xrightarrow{\perp} \perp$  ,

$(2, 0) \xrightarrow{T} \perp$  ,

$(2, 0) \xrightarrow{\perp} (3, \max)$  ,

]

Level: 3

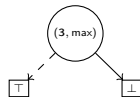
[

[  $(3, 0) \mapsto (3, \max)$  ]

,

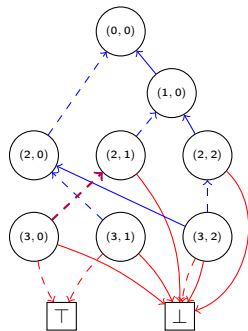
]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

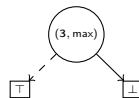
Priority Queue:  $Q_{red}$ :

[  $(2, 2) \xrightarrow{\perp} \perp$  ,  
 $(2, 1) \xrightarrow{\perp} (3, \max)$  ,  
 $(2, 0) \xrightarrow{\top} \perp$  ,  
 $(2, 0) \xrightarrow{\perp} (3, \max)$  ,

Level: 3

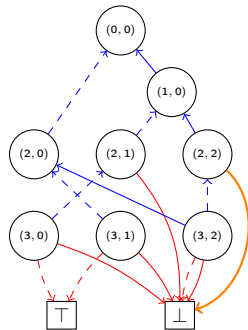
[  
 [  
 ,  
 ]  
 ]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



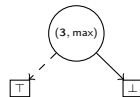
(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[  
 $(2, 1) \xrightarrow{\perp} (3, \max)$  ,  
 $(2, 0) \xrightarrow{\top} \perp$  ,  
 $(2, 0) \xrightarrow{\perp} (3, \max)$  ,  
 ]

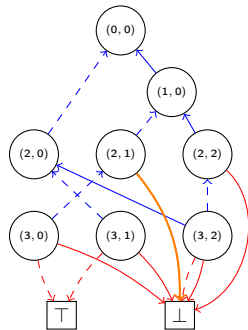
[  
 Level: 2  
 $[(2, 2) \mapsto \perp]$   
 ]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(2, 0) \xrightarrow{\top} \perp$  ,

$(2, 0) \xrightarrow{\perp} (3, \max)$  ,

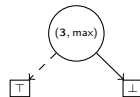
]

Level: 2

[  $[(2, 2) \mapsto \perp]$  ]

[  $((2, 1), (3, \max), \perp)$  , ]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

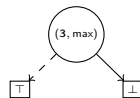
$((2, 1), (3, \max), \perp)$

,

$((2, 0), (3, \max), \perp)$

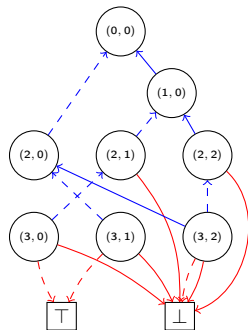
]

Output:



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

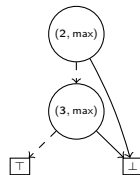
$[(2, 1) \mapsto (2, \max)]$

,

$((2, 0), (3, \max), \perp)$

]

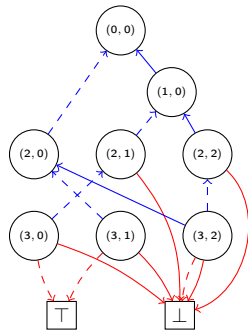
Output:  
 $((2, \max), (3, \max), \perp)$



(d)  $(a) \wedge (b)$  reduced



# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

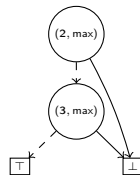
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

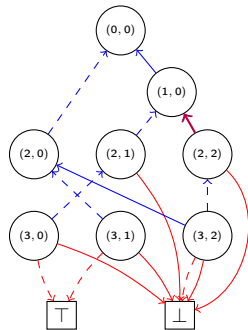
]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(1, 0) \xrightarrow{T} \perp$  ,

]

Level: 2

[

]

[

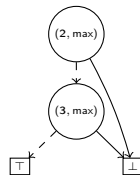
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

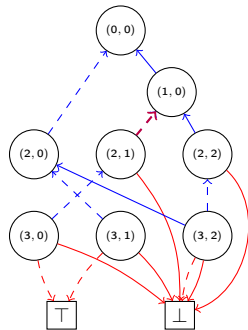
]

Output:



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(1,0) \xrightarrow{\top} \perp$  ,

$(1,0) \xrightarrow{\perp} (2, \max)$  ,

]

Level: 2

[

]

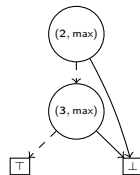
[

$[(2,0) \mapsto (2, \max)]$

,

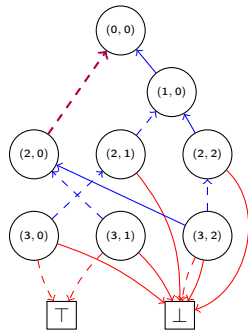
]

Output:



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(1, 0) \xrightarrow{T} \perp$  ,

$(1, 0) \xrightarrow{\perp} (2, \max)$  ,

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 2

[

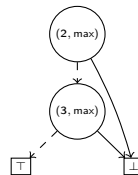
[

]

,

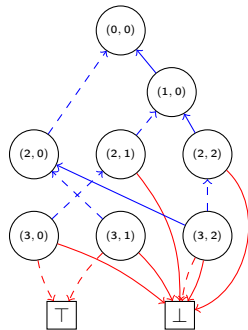
]

Output:



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(0,0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

[

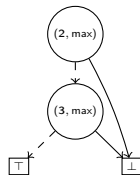
]

[

$((1,0), (2, \max), \perp)$

]

Output:



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(0,0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

[

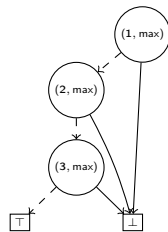
]

[

$[(1,0) \mapsto (1, \max)]$

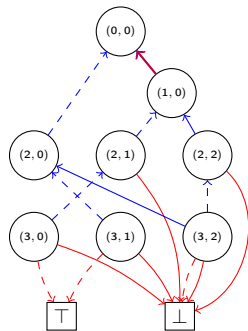
]

Output:  
 $((1, \max), (2, \max), \perp)$



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

$(0, 0) \xrightarrow{T} (1, \max)$  ,

$(0, 0) \xrightarrow{\perp} (2, \max)$  ]

Level: 1

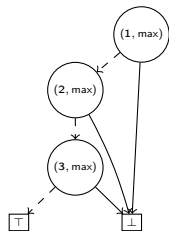
[

]

[

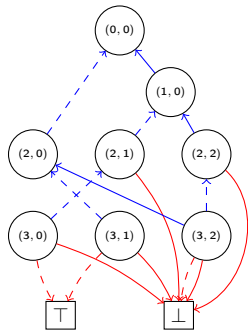
]

Output:



(d)  $(a) \wedge (b)$  reduced

## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

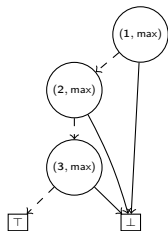
]

Level: 0

[

]

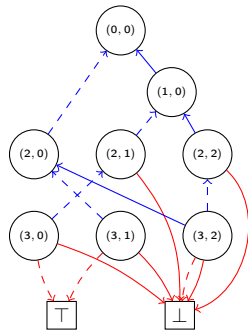
[  $((0,0), (2, \max), (1, \max))$  ]



(d)  $(a) \wedge (b)$  reduced



## Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

]

Level: 0

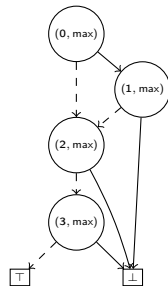
[

]

$[(0,0) \mapsto (0, \max)]$

]

Output:  
 $((0, \max), (2, \max), (1, \max))$



(d)  $(a) \wedge (b)$  reduced

# Apply : *Example* (Continued)



(c)  $(a) \wedge (b)$

Priority Queue:  $Q_{red}$ :

[

]

Level: 0

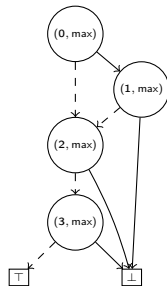
[

]

[

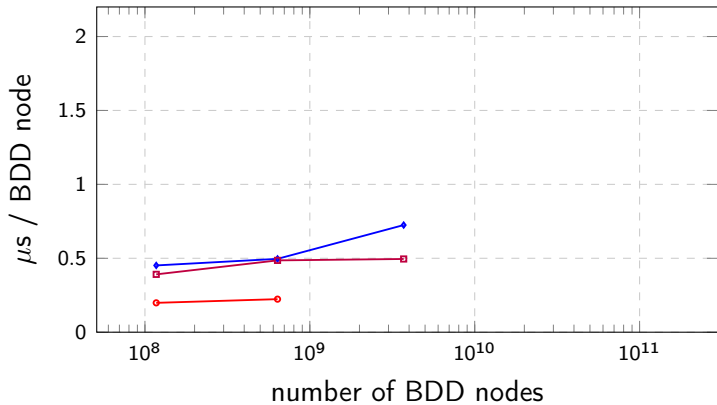
]

Output:



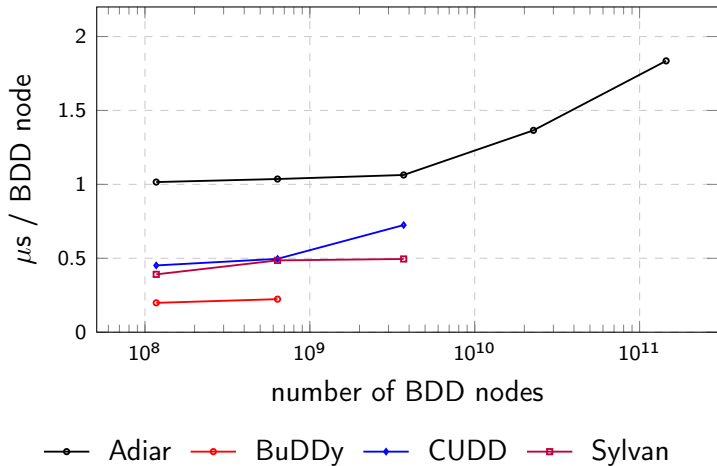
(d)  $(a) \wedge (b)$  reduced

Algorithm	I/O-Complexity
bdd_pathcount	$O(\text{sort}(N_f))$
bdd_not	$O(N_f/B)$
bdd_restrict	$O(\text{sort}(N_f))$
bdd_apply	$O(\text{sort}(N_f \cdot N_g))$



—○— Adiar    —○— BuDDy    —♦— CUDD    —□— Sylvan

Running time for the *N-Queens* problems.



Running time for the *N-Queens* problems.

# Contents

What are Binary Decision Diagrams?

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bdd_equal	?



## Equality Checking

$$f \leftrightarrow g \equiv \top$$

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$$\underbrace{O(\text{sort}(N^2))}_{\text{Apply}} + \underbrace{O(\text{sort}(N^2))}_{\text{Reduce}} + \underbrace{O(1)}_{\text{check is } \top} = O(\text{sort}(N^2))$$

## Equality Checking

### Theorem (Bryant '86)

*Let  $\pi$  be a variable order and  $f : \mathbb{B}^n \rightarrow \mathbb{B}$  then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing  $f$  with ordering  $\pi$ .*

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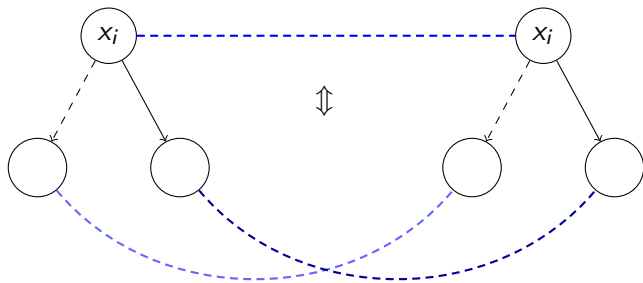
**Trivial cases:  $f \neq g$  if there is a mismatch in**

- $N_f \neq N_g$       Number of nodes       $O(1)$  I/Os
- $L_f \neq L_g$       Number of levels       $O(1)$  I/Os
- $N_{f,i} \neq N_{g,i}$       Number of nodes on a level       $O(L/B)$  I/Os
- $L_{f,i} \neq L_{g,i}$       Label of an  $i$ th level       $O(L/B)$  I/Os

# Equality Checking

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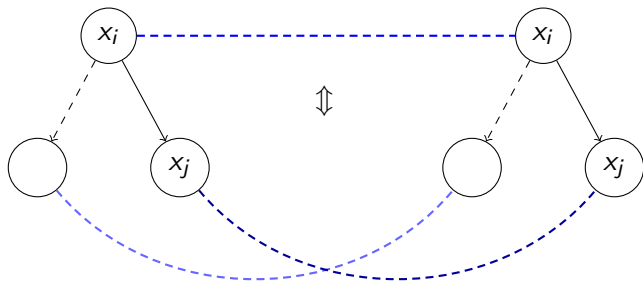
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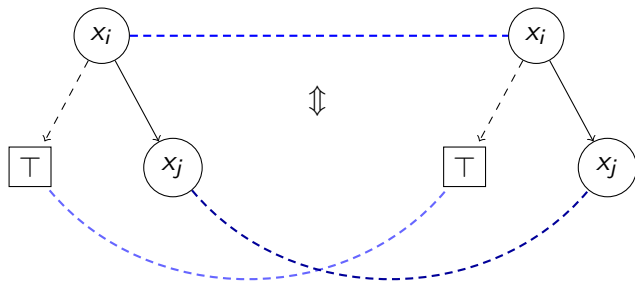
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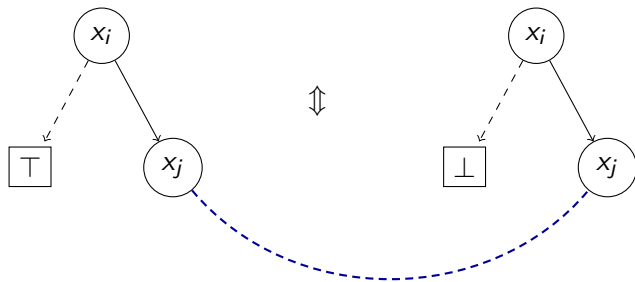
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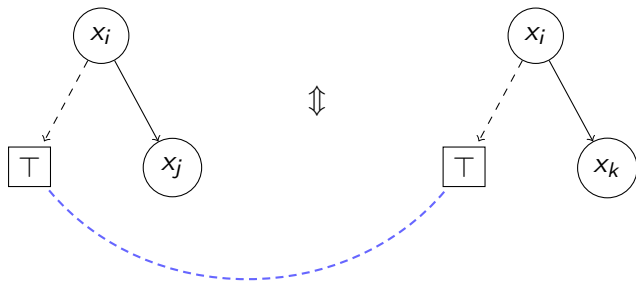




## Equality Checking

### Theorem (Bryant '86)

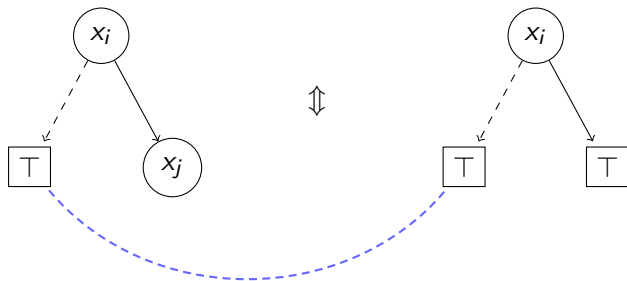
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`IsIsomorphic( $f$ ,  $g$ )`

- Check whether root  $v_f$  of  $f$  and root  $v_g$  of  $g$  have a local violation.
- Check  $low(v_f) \sim low(v_g)$  and  $high(v_f) \sim high(v_g)$  “recursively”.

Return false on first violation. If there are no violations then return true.

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$\text{IsIsomorphic}(f, g)$

- Check whether root  $v_f$  of  $f$  and root  $v_g$  of  $g$  have a local violation.
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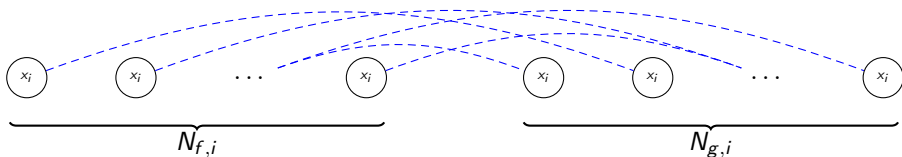
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$$\underbrace{O(\text{sort}(N^2))}_{\text{Apply'}} + \underbrace{\cancel{O(\text{sort}(N^2))}}_{\text{Reduce}} + \underbrace{\cancel{O(1)}}_{\text{check is } \top} = O(\text{sort}(N^2))$$

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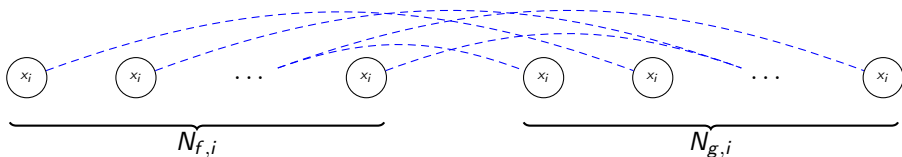
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## Theorem (Bryant '86)

Let  $\pi$  be a variable order and  $f : \mathbb{B}^n \rightarrow \mathbb{B}$  then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing  $f$  with ordering  $\pi$ .



Return false if more than  $N_{f,i} = N_{g,i}$  pairs of nodes are checked on level  $i$ .

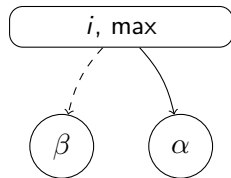
$$\underbrace{O(\text{sort}(\sum_i N_{f,i}))}_{\text{Apply}''} = O(\text{sort}(N))$$

# Equality Checking

## Observation

Each level output by the Reduce algorithm has the following properties:

## Equality Checking

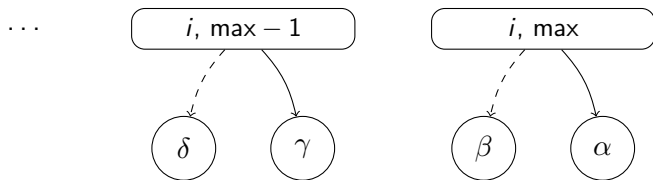


### Observation

Each level output by the Reduce algorithm has the following properties:



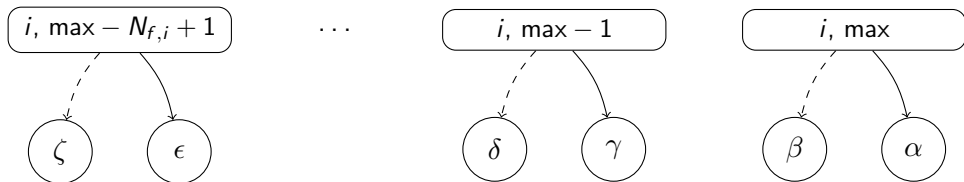
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Each level output by the Reduce algorithm has the following properties:

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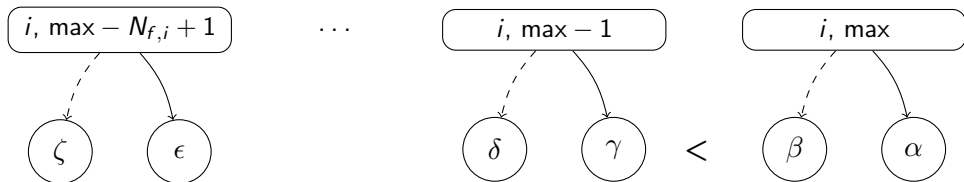


## Observation

Each level output by the Reduce algorithm has the following properties:

- Nodes on level  $i$  have their identifiers *consecutively* numbered.

# Equality Checking

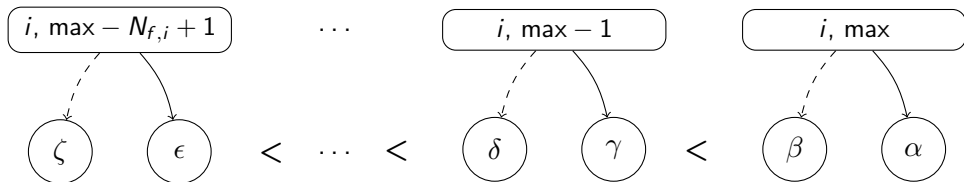


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## Observation

Each level output by the Reduce algorithm has the following properties:

- Nodes on level  $i$  have their identifiers *consecutively* numbered.
- Nodes on level  $i$  are output sorted by their children.

# Equality Checking

## Theorem

*If  $G_f$  and  $G_g$  are outputs of Reduce.*

$G_f \sim G_g \iff \text{For all } i \in [0; N_f) \text{ the node } G_f[i] \text{ matches } G_g[i] \text{ numerically.}$

## Proof.

$\Leftarrow$  : Must describe the exact same graph.

$\Rightarrow$  : Strong induction on BDD levels bottom-up.



# Equality Checking

## Theorem

*If  $G_f$  and  $G_g$  are outputs of Reduce.*

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$\Leftarrow$  : Must describe the exact same graph.

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## Corollary

*If  $G_f$  and  $G_g$  are outputs of Reduce then  $f \equiv g$  is computable using  $2 \cdot N/B$  I/Os.*

## Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38

Checking the (EPFL Benchmark) *voter* circuit's single output gate ( $|N_f| = |N_g| = 5.76$  MiB).

## Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38
$O(\text{sort}(N))$	0.058

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$O(\text{sort}(N))$	0.058
$2N/B$	0.006

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*Depth-First*

*Time-Forwarded*

$O(N_f)$

$O(\text{sort}(N_f))$

$O(N_f \cdot N_g)$

$O(\text{sort}(N_f))$

$O(1)$

$2N/B$