∃R-completeness of Nash equilibria in Perfect Information Stochastic Games

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Basic definitions and utility functions

Nash Equilibria

Game Theory in Model Checking and Synthesis

 $\exists \mathbb{R}$ -complexity

The NP and $\operatorname{SqrtSum}$ Complexity Classes

The $\exists \mathbb{R}$ Complexity Class

Proof Sketch: ∃ℝ-Completeness of Nash equilibria

Gadgets

Reduction

Implications for Model Checking



An m-player perfect information stochastic game G is defined by

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■ Directed graph (V, A)



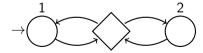
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- An initial node $v_o \in V$.



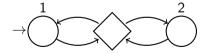
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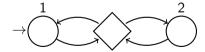


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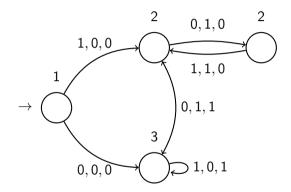
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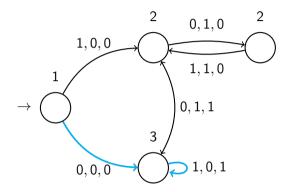
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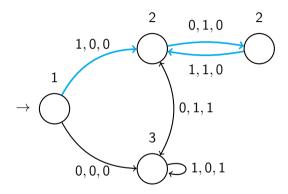
■ Utility functions u_i assigns a payoff $u_i(i)$ for Player i to a play $h \in \mathcal{H}_{\infty}$



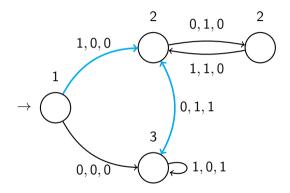
A simple mean-payoff game.



A simple mean-payoff game. Mean payoff for player 1: 1

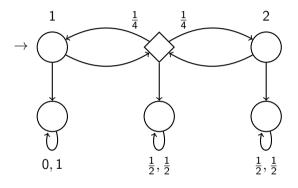


A simple mean-payoff game. Mean payoff for player 1: $\frac{1}{2}$



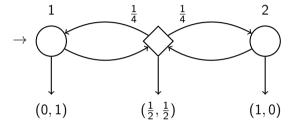
A simple mean-payoff game. Mean payoff for player 1: 0

Recursive games



A simple game with terminal rewards only. Ummels ${\rm ^{\prime}11}$

Recursive games



A simple game with terminal rewards only. Ummels $^\prime 11$

Strategies and Nash equilibria

A strategy τ_i assigns a probability distribution to the outgoing arcs of vertices $v \in V_i$ depending on the given history h.

■ A strategy is *stationary*, if the choice of the players at a vertice is independent of the prior history of play (i.e. the strategy is memoryless).

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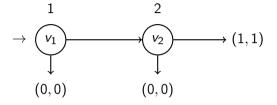
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We assume players are acting *rationally*. This is commonly captured by the following notion

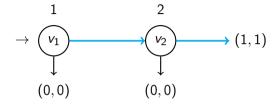
Definition (Nash equilibria)

A strategy profile $\tau = (\tau_1, \tau_2, \dots, \tau_m)$ is a *Nash equilibrium*, if no player i has a unilateral deviation available that strictly improves their payoff.

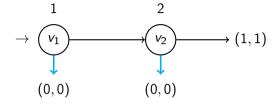
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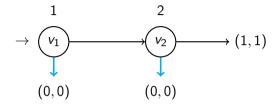
A two-player reachability game with an irrational Nash equilibrium. Ummels $^\prime 11$



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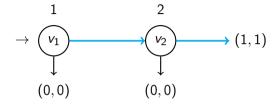
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A two-player reachability game with an irrational Nash equilibrium. Ummels '11

Definition (Subgame Perfect Nash equilibria)

A Subgame perfect Nash equilibrium is a NE of a game G, that is not only the best response from v_0 , but is a best response in G[h] given any history h of play.



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Game Theory in Model Checking

Games provide a well studied framework that can capture many model checking problems with *adversaries*.

- \blacksquare A protocol between m entities can be described by a stochastic game of m players.
- lacktriangle A distributed system of m peers can be described by a *concurrent* game of m players.

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Classical model checking objectives can be encapsulated in the utility function.

- Reachability objectives can be captured by payoffs in $\{0,1\}$ in a recursive game.
- Safety objectives can be captured by payoffs in $\{-1,1\}$ in a recursive game, since an *infinite* game has payoff 0.
- Other Büchi objectives can also be described in general Mean-payoff games.

Game Theory in Synthesis

The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

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The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

=

Does there exist a strategy, such that Player 1 is surely winning?

The subject of this seminar

Consider the problem:

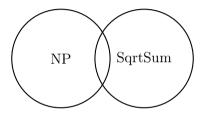
Given an m-player game G and payoff demands $L \in \mathbb{R}^m$, does there exist a stationary ¹ NE τ with $U(\tau) \geq L$?

We will show this is $\exists \mathbb{R}$ -complete.

 $^{^1}$ The problem of existence of a Nash equilibria satisfying some demands is undecidable for ≥ 10 players in recursive games, so we will only focus on *stationary* strategies.

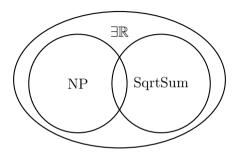
 $\exists \mathbb{R} ext{-complexity}$

∃R Complexity Class



The relation between NP, $\operatorname{SqrtSum},$ and $\exists \mathbb{R}$

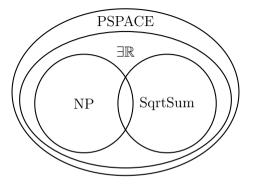
∃ℝ Complexity Class



The relation between NP, $\operatorname{SqrtSum}$, and $\exists \mathbb{R}$

The complexity class $\exists \mathbb{R}$ both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of $\operatorname{SqrtSum}$.

∃R Complexity Class



The relation between NP, SqrtSum, and $\exists \mathbb{R}$

The complexity class $\exists \mathbb{R}$ both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of $\operatorname{SqrtSum}$.

NP Complexity Class

Remember that the well-known class $\operatorname{NP}\mathsf{can}$ be captured by the ILP problem:

min
$$c^T x$$

s.t. $Ax \le b$
 $x \in \mathbb{N}^n$

where $A \in \mathbb{Z}^{n \times m}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

SqrtSum Complexity Class

Consider the following problem: Given $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_m \in \mathbb{R}$ is the following inequality satisfied?

$$\sum_{i=1}^n \sqrt{a_i} \le \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial...

SqrtSum Complexity Class

Consider the following problem: Given $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_m \in \mathbb{R}$ is the following inequality satisfied?

$$\sum_{i=1}^n \sqrt{a_i} \le \sum_{j=1}^m \sqrt{b_j}$$

Seems trivial... How many decimals do you have to compute, before you know the answer? 2

Definition (SqrtSum)

The complexity class $\hat{S}qrtSum$ consists of all problems that are polynomial time reducible to the problem above.

 $^{^1}$ This consistently comes up in Computational Geometry. Here, theoretical works solve this by assuming the \mathbb{R} -RAM computational model; leaving an adventure for implementors to experience later.

∃R Complexity Class

The Existential Theory of the Reals is the language of all true sentences of the form

$$\exists x_1, x_2, \ldots, x_n \in \mathbb{R} : \phi(x_1, x_2, \ldots, x_n)$$

where ϕ is a quantifier-free Boolean formula of inequalities and equalities.

∃R Complexity Class

The Existential Theory of the Reals is the language of all true sentences of the form

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Definition $(\exists \mathbb{R})$

The complexity class $\exists \mathbb{R}$ consists of all problems, that are polynomial time reducible to the existential theory of the reals.

∃R Complexity Class

We will consider the following $\exists \mathbb{R}$ -complete problem.

Definition (HomQuad)

Given a system S of I homogeneous quadratic polynomials 3 in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ for all $k \in \{1, 2, ..., I\}$ and x is a probability distribution?

³A homogenous quadratic polynomial is of the form $\sum_{i=1}^n \sum_{j=1}^n A_{ij}x_ix_j$ where $A \in [-1,1]^{n \times n}$.

Proof Sketch: ∃ℝ-Completeness of Nash equilibria

∃R-Completness of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands $L \in \mathbb{R}^m$, does there exist a stationary NE τ with $U(\tau) \geq L$?

 $\exists \mathbb{R}\text{-Completness}$ of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands $L \in \mathbb{R}^m$, does there exist a stationary NE τ with $U(\tau) \geq L$?

It has already been shown to be NP-hard for ≥ 2 players and $\operatorname{SqrtSum}$ -hard for ≥ 4 players. Furthermore, it is contained within $\exists \mathbb{R}$.

It is $\exists \mathbb{R}$ -complete! We will show this by reduction to:

Definition (HomQuad)

Given a systèm S of I homogeneous quadratic polynomials in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ and x is a probability distribution?

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Definition (HomQuad)

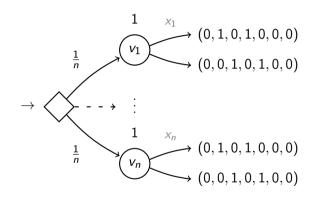
Given a system S of I homogeneous quadratic polynomials in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ and x is a probability distribution?

That is, given a system S of I polynomials of the form

$$q_k(x) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{ij}x_ix_j + \ldots + a_{nn}x_nx_n$$

we will construct a game $\mathcal{G}(\mathcal{S})$ such that all $q_k(x) = 0$ if and only if $\mathcal{G}(\mathcal{S})$ has a stationary Nash equilibria that satisfies some payoff demand.

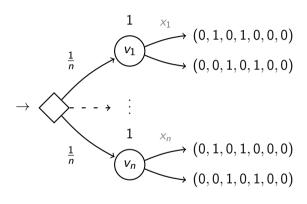
Proof Sketch: \mathcal{G}_{var}



The gadget game $\mathcal{G}_{\mathrm{var}}$

At each v_i , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Proof Sketch: \mathcal{G}_{var}



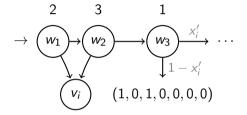
The gadget game $\mathcal{G}_{\mathrm{var}}$

At each v_i , Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Player 1 strategy corresponds to a probability distribution if it satisfies the payoff demand

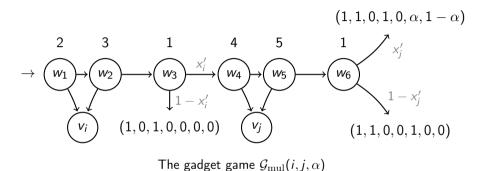
$$\left(0,\frac{1}{n},\frac{n-1}{n},\ldots\right)$$

Proof Sketch: $\mathcal{G}_{\mathrm{mul}}(i,j,\alpha)$



The gadget game $\mathcal{G}_{\mathrm{mul}}(i,j,lpha)$

Proof Sketch: $\mathcal{G}_{\text{mul}}(i, j, \alpha)$



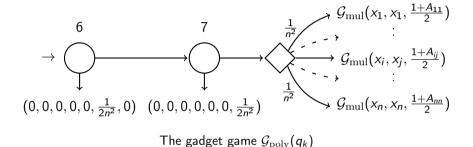
If Player 1 receives payoff 1, then Player-6 gets $\alpha x_i x_j$ and Player-7 gets $(1 - \alpha)x_i x_j$.

$$\max_{ au_1} \min_{ au_2} \Pr\left[u_1(v_0(au_1, au_2)) = 1\right]$$

$$\forall \tau_2 : \Pr[u_1(v_0(\tau_1, \tau_2)) = 1] \ge \text{value}$$

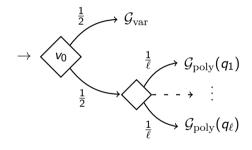
Proof Sketch: $\mathcal{G}_{\text{poly}}(q)$

For a homogenous quadratic polynomial $q_k(x) = \sum_{i,j=1}^n A_{ij} x_i x_j$.



If Player 1 receives payoff 1, then Player 6 gains payoff $\frac{1}{2n^2}(\|x\|_1^2 + q_k(x))$. If also $\|x\|_1$ is 1, then $q_k(x) = 0$.

Proof Sketch: Final reduction



The game $\mathcal{G}(\mathcal{S})$ of the reduction

 ${\cal S}$ is a "yes"-instance of ${\rm HomQuad}$ if and only if the game ${\cal G}({\cal S})$ has a Nash Equilibria that satisfies the demands

$$\left(\frac{1}{2},\frac{1}{n},\frac{n-1}{n},0,0,\ldots,0\right)$$

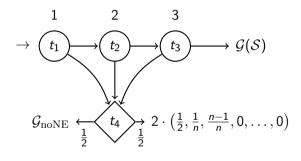
Theorem

It is $\exists \mathbb{R}$ -complete to decide whether for a given m-player recursive game G and payoff demands $L \in \mathbb{R}^m$ there exists a stationary Nash equilibria τ with $U(\tau) \geq L$.

- The problem is $\exists \mathbb{R}$ -complete even for acyclic 7-player recursive games with non-negative rewards.
- It even holds for stationary Subgame Perfect Equilibria.

Implications for Model Checking

$\exists \mathbb{R}\text{-}\mathsf{Completeness}$ of Stationary NE without Payoff Demands

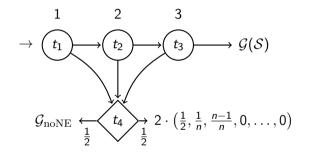


The game $\mathcal{G}_{\exists \mathrm{NE}}(\mathcal{S})$

 $\mathcal{G}_{\mathrm{noNE}}$ is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

$\exists \mathbb{R}\text{-}\mathsf{Completeness}$ of Stationary NE without Payoff Demands



The game $\mathcal{G}_{\exists \mathrm{NE}}(\mathcal{S})$

 $\mathcal{G}_{\mathrm{noNE}}$ is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

 ${\mathcal S}$ is a "yes"-instance of $\operatorname{Hom} \operatorname{Quad}$ if and only if the game ${\mathcal G}_{\exists \operatorname{NE}}({\mathcal S})$ has a stationary Nash Equilibria.

∃R-Completeness of Reachability and Safety objectives

There exists different $\mathcal{G}_{\mathrm{noNE}}$ gadget games for the different restrictions of the utility function:

- Reachability objective
- Safety objective

Theorem

It is $\exists \mathbb{R}$ -complete to decide whether a given m-player game with Reachability or Safety objectives has a stationary NE.

• even for m = 7 players.

∃R-Completeness of being almost surely winning

Consider the game $\mathcal{G}_{\exists NE}(\mathcal{S})$ in which another player is added, who is always winning in $\mathcal{G}(\mathcal{S})$, but not in \mathcal{G}_{noNE} .

Theorem

For any i, it is $\exists \mathbb{R}$ -complete to decide whether a given m-player recursive game has a stationary NE in which Player i is almost surely winning.

• even for m = 8 players.

Final remarks

It is $\exists \mathbb{R}$ -complete to decide in an *m*-player perfect information recursive game.

- lacktriangle exists Subgame Perfect Nash equilibria satisfying demand $L \in \mathbb{R}^m$
- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.

Final remarks

It is $\exists \mathbb{R}$ -complete to decide in an *m*-player perfect information recursive game.

- lacktriangle exists Subgame Perfect Nash equilibria satisfying demand $L \in \mathbb{R}^m$
- exists any Nash equilibria for Reachability and Safety objectives
- exists any Nash equilibria such that Player 1 is surely winning.

Notice here that

- This problem is already shown by Ummels '11 to be NP-hard for \geq 2 players and SqrtSum-hard for \geq 4 players so this completeness result could not become much tighter.
- There have been recent results of $\exists \mathbb{R}$ -completeness in *imperfect information* games. The complexity of these results stem from the structure of the game, not the lack of information.