$\exists\mathbb{R}$-completeness of Nash equilibria in Perfect Information Stochastic Games

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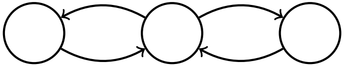

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- Directed graph (V, A)
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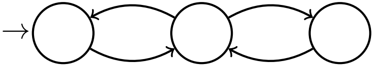

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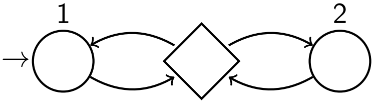
- An initial node $v_o \in V$.
- V is partitioned into disjoint sets $V_0, V_1, \ldots, V_m$, where V_i are controlled by Player i and V_0 are *chance* nodes.



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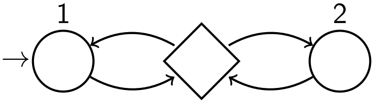

- A play $h \in \mathcal{H}_\infty$ is an infinite sequence $(h_t)_{t \geq 0}$ of vertices in V, where

$$h_0 = v_0, \quad (h_t, h_{t+1}) \in A$$

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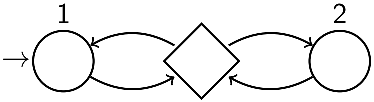


- A play $h \in \mathcal{H}_\infty$ is an infinite sequence $(h_t)_{t \geq 0}$ of vertices in V, where

$$h_0 = v_0, \quad (h_t, h_{t+1}) \in A$$

- Utility functions u_i assigns a payoff $u_i(i)$ for Player i to a play $h \in \mathcal{H}_\infty$

Mean-payoff games

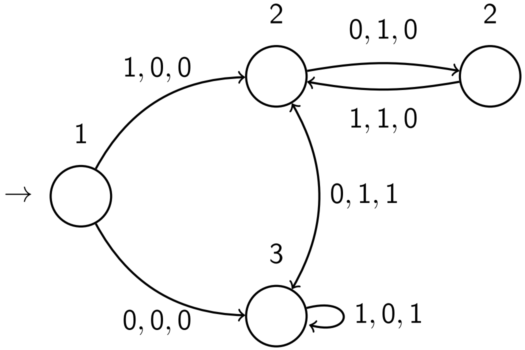


Figure 1: A simple mean-payoff game.

Mean-payoff games

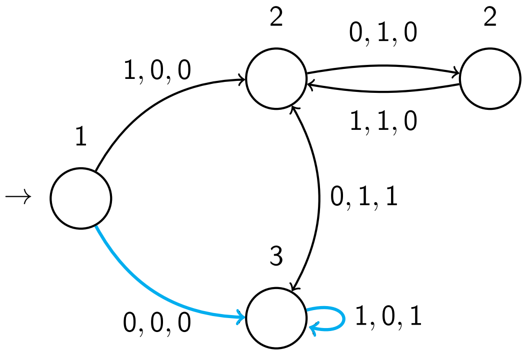


Figure 1: A simple mean-payoff game. Mean payoff for player 1: 1

Mean-payoff games

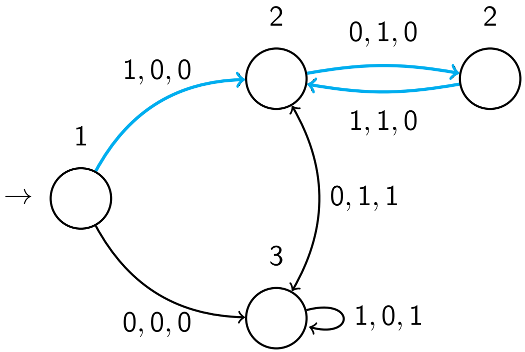


Figure 1: A simple mean-payoff game. Mean payoff for player 1: $\frac{1}{2}$

Mean-payoff games

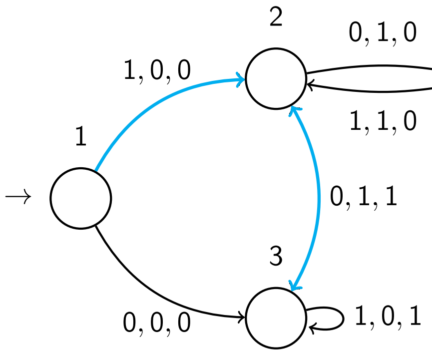


Figure 1: A simple mean-payoff game. Mean payoff for player 1: 0

Recursive games

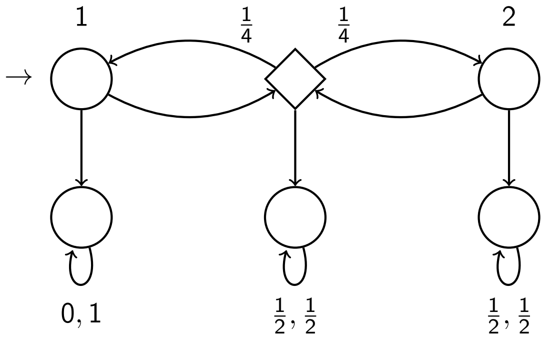
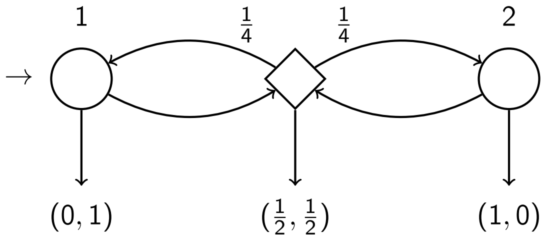


Figure 2: A simple game with terminal rewards only. Ummels '11

Recursive games



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Strategies and Nash equilibria

A strategy τ_i assigns a probability distribution to the outgoing arcs of vertices $v \in V_i$ depending on the given history h.

- A strategy is *stationary*, if the choice of the players at a vertice is independent of the prior history of play (i.e. the strategy is memoryless).

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We assume players are acting *rationally*. This is commonly captured by the following notion

Definition (Nash equilibria)
A strategy profile $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is a *Nash equilibrium*, if no player i has a unilateral deviation available that strictly improves their payoff.

Subgame Perfect Nash equilibria

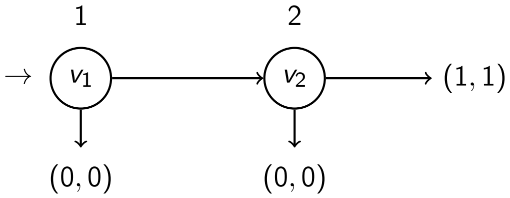
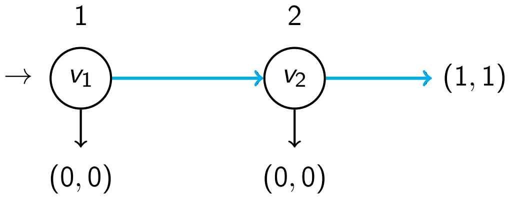


Figure 3: A two-player reachability game with an irrational Nash equilibrium. Ummels '11

Subgame Perfect Nash equilibria



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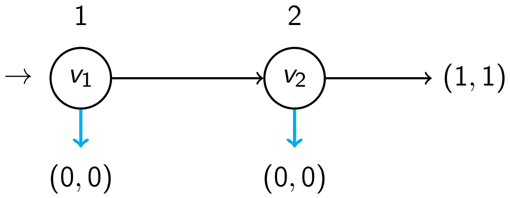

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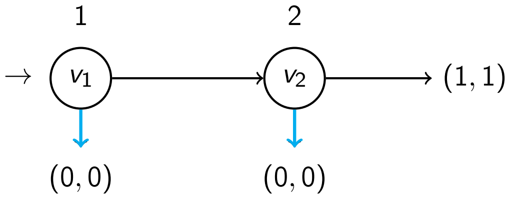

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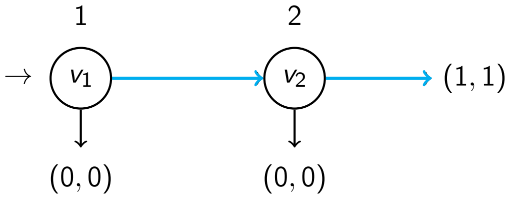
Definition (Subgame Perfect Nash equilibria)
A *Subgame perfect* Nash equilibrium is a NE of a game G, that is *not* only the best response from v_0, but is a best response in $G[h]$ given *any* history h of play.

Subgame Perfect Nash equilibria



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Game Theory in Model Checking

Games provide a well studied framework that can capture many model checking problems with *adversaries*.

- A protocol between m entities can be described by a stochastic game of m players.
- A distributed system of m peers can be described by a *concurrent* game of m players.

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Classical model checking objectives can be encapsulated in the utility function.

- *Reachability* objectives can be captured by payoffs in $\{0, 1\}$ in a recursive game.
- *Safety* objectives can be captured by payoffs in $\{-1, 1\}$ in a recursive game, since an *infinite* game has payoff 0.
- Other Büchi objectives can also be described in general Mean-payoff games.

Game Theory in Synthesis

The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

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The problem of *synthesis* is to not only check a program satisfies a given specification, but to also generate parts of the program according to the specification.

Does there exist a controller, such that the system satisfies the specification?

$$\equiv$$

Does there exist a strategy, such that Player 1 is surely winning?

The subject of this seminar

Consider the problem:

Given an m-player game G and payoff demands $L \in \mathbb{R}^m$,
does there exist a stationary [1] NE τ with $U(\tau) \geq L$?

We will show this is $\exists\mathbb{R}$-complete.

---

[1] The problem of existence of a Nash equilibria satisfying some demands is undecidable for ≥ 10 players in recursive games, so we will only focus on *stationary* strategies.

$\exists\mathbb{R}$-complexity

$\exists\mathbb{R}$ Complexity Class

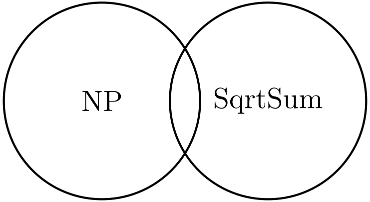

Figure 4: The relation between NP, $\mathrm{SqrtSum}$, and $\exists\mathbb{R}$

$\exists\mathbb{R}$ Complexity Class

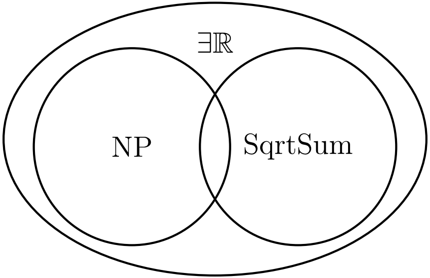

Figure 4: The relation between NP, $\mathrm{SqrtSum}$, and $\exists\mathbb{R}$

The complexity class $\exists\mathbb{R}$ both encapsulates the hardness of NP decision problems and the hardness of computing with real numbers of $\mathrm{SqrtSum}$.

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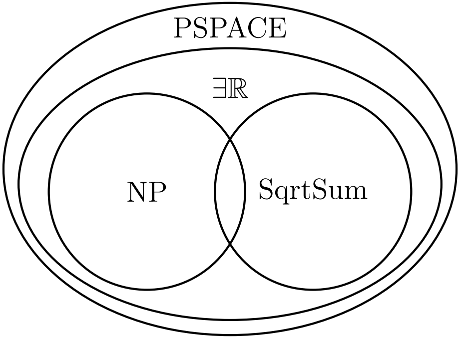

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NP **Complexity Class**

Remember that the well-known class NPcan be captured by the ILP problem:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{N}^n \end{aligned}$$

where $A \in \mathbb{Z}^{n \times m}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

SqrtSum **Complexity Class**

Consider the following problem: Given $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_m \in \mathbb{R}$ is the following inequality satisfied?

$$\sum_{i=1}^{n} \sqrt{a_i} \leq \sum_{j=1}^{m} \sqrt{b_j}$$

Seems trivial...

SqrtSum Complexity Class

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$$\sum_{i=1}^{n} \sqrt{a_i} \leq \sum_{j=1}^{m} \sqrt{b_j}$$

Seems trivial... How many decimals do you have to compute, before you know the answer? [2]

Definition (SqrtSum)
The complexity class SqrtSum consists of all problems that are polynomial time reducible to the problem above.

[1] This consistently comes up in Computational Geometry. Here, theoretical works solve this by assuming the $\mathbb{R}$-RAM computational model; leaving an adventure for implementors to experience later.

$\exists\mathbb{R}$ Complexity Class

The *Existential Theory of the Reals* is the language of all true sentences of the form

$$\exists x_1, x_2, \ldots, x_n \in \mathbb{R} : \phi(x_1, x_2, \ldots, x_n)$$

where ϕ is a quantifier-free Boolean formula of inequalities and equalities.

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where ϕ is a quantifier-free Boolean formula of inequalities and equalities.

Definition ($\exists\mathbb{R}$)
The complexity class $\exists\mathbb{R}$ consists of all problems, that are polynomial time reducible to the existential theory of the reals.

$\exists\mathbb{R}$ Complexity Class

We will consider the following $\exists\mathbb{R}$-complete problem.

Definition (HomQuad)
Given a system $\mathcal{S}$ of l *homogeneous quadratic* polynomials [3] in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ for all $k \in \{1, 2, \ldots, l\}$ and x is a probability distribution?

[3] A homogenous quadratic polynomial is of the form $\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$ where $A \in [-1, 1]^{n \times n}$.

Proof Sketch: $\exists\mathbb{R}$-Completeness of Nash equilibria

$\exists\mathbb{R}$-Completness of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands $L \in \mathbb{R}^m$,
does there exist a stationary NE τ with $U(\tau) \geq L$?

$\exists\mathbb{R}$-Completness of Nash equilibria

Consider the problem:

Given an m-player game G and payoff demands $L \in \mathbb{R}^m$,
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It has already been shown to be NP-hard for ≥ 2 players and $\mathrm{SqrtSum}$-hard for ≥ 4 players. Furthermore, it is contained within $\exists\mathbb{R}$.

It is $\exists\mathbb{R}$-complete! We will show this by reduction to:

Definition ($\mathrm{HomQuad}$)
Given a system $\mathcal{S}$ of l *homogeneous quadratic* polynomials in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ and x is a probability distribution?

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Definition ($\mathrm{HomQuad}$)
Given a system $\mathcal{S}$ of l *homogeneous quadratic* polynomials in n variables, does there exist an $x \in \mathbb{R}^n$ such that $q_k(x) = 0$ and x is a probability distribution?

That is, given a system $\mathcal{S}$ of l polynomials of the form

$$q_k(x) = a_{1,1}x_1x_1 + a_{1,2}x_1x_2 + \cdots + a_{ij}x_ix_j + \ldots a_{nn}x_nx_n$$

we will construct a game $\mathcal{G}(\mathcal{S})$ such that all $q_k(x) = 0$ if and only if $\mathcal{G}(\mathcal{S})$ has a stationary Nash equilibria that satisfies some payoff demand.

Proof Sketch: $\mathcal{G}_{\text{var}}$

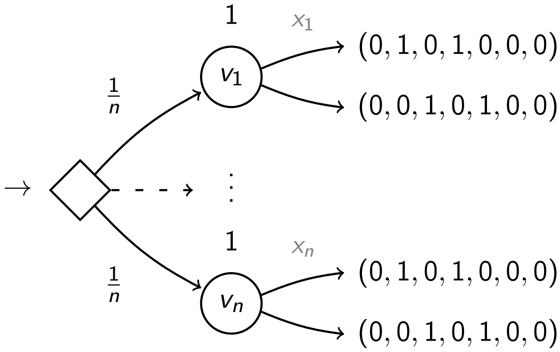
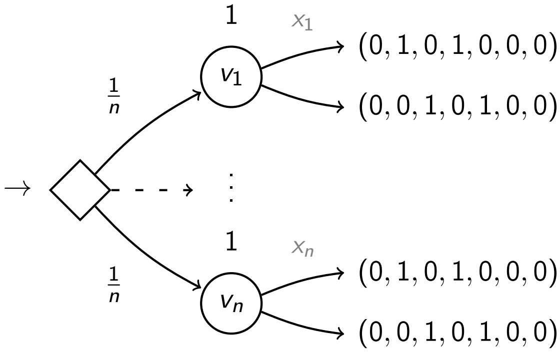


Figure 5: The gadget game $\mathcal{G}_{\text{var}}$

At each v_i, Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Proof Sketch: $\mathcal{G}_{\text{var}}$



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At each v_i, Player 1 can choose to either give payoff 1 to players 2 and 4 or 3 and 5.

Player 1 strategy corresponds to a probability distribution if it satisfies the payoff demand

$$\left(0, \frac{1}{n}, \frac{n-1}{n}, \dots\right)$$

Proof Sketch: $\mathcal{G}_{\mathrm{mul}}(i, j, \alpha)$

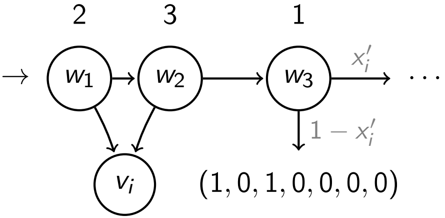


Figure 6: The gadget game $\mathcal{G}_{\mathrm{mul}}(i, j, \alpha)$

Proof Sketch: $\mathcal{G}_{\mathrm{mul}}(i, j, \alpha)$

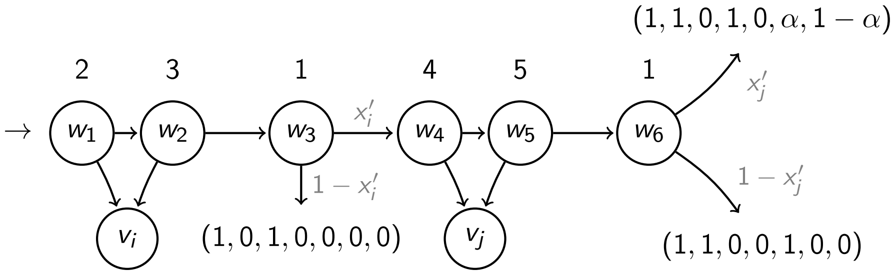


Figure 6: The gadget game $\mathcal{G}_{\mathrm{mul}}(i, j, \alpha)$

If Player 1 receives payoff 1, then Player-6 gets $\alpha x_i x_j$ and Player-7 gets $(1 - \alpha) x_i x_j$.

$$\max_{\tau_1} \min_{\tau_2} \Pr\left[u_1(v_0(\tau_1, \tau_2)) = 1\right]$$

$$\forall \tau_2 : \Pr\left[u_1(v_0(\tau_1, \tau_2)) = 1\right] \geq \text{value}$$

Proof Sketch: $\mathcal{G}_{\text{poly}}(q)$

For a homogenous quadratic polynomial $q_k(x) = \sum_{i,j=1}^{n} A_{ij} x_i x_j$.

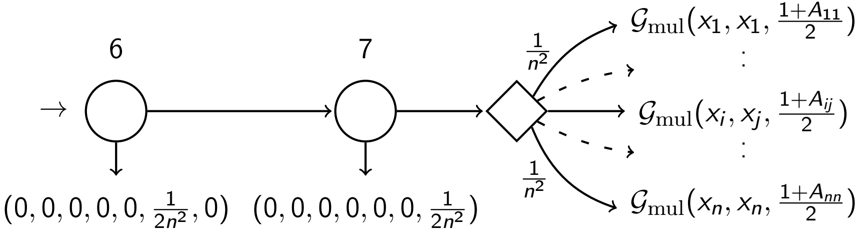


Figure 7: The gadget game $\mathcal{G}_{\text{poly}}(q_k)$

If Player 1 receives payoff 1, then Player 6 gains payoff $\frac{1}{2n^2}(\|x\|_1^2 + q_k(x))$.

If also $\|x\|_1$ is 1, then $q_k(x) = 0$.

Proof Sketch: Final reduction

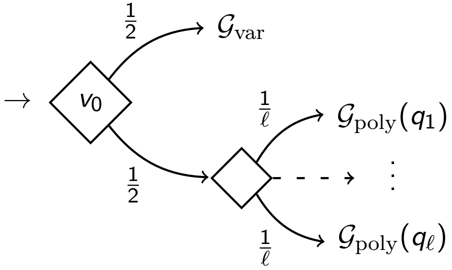


Figure 8: The game $\mathcal{G}(\mathcal{S})$ of the reduction

$\mathcal{S}$ is a "yes"-instance of $\mathrm{HomQuad}$ if and only if the game $\mathcal{G}(\mathcal{S})$ has a Nash Equilibria that satisfies the demands

$$\left(\frac{1}{2}, \frac{1}{n}, \frac{n-1}{n}, 0, 0, \ldots, 0\right)$$

Theorem
It is $\exists\mathbb{R}$-complete to decide whether for a given m-player recursive game G and payoff demands $L \in \mathbb{R}^m$ there exists a stationary Nash equilibria τ with $U(\tau) \geq L$.

- *The problem is $\exists\mathbb{R}$-complete even for acyclic 7-player recursive games with non-negative rewards.*
- *It even holds for stationary Subgame Perfect Equilibria.*

Implications for Model Checking

$\exists\mathbb{R}$-Completeness of Stationary NE without Payoff Demands

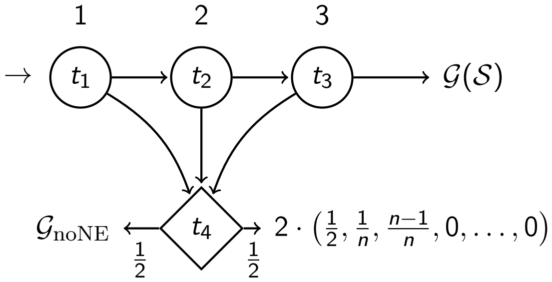
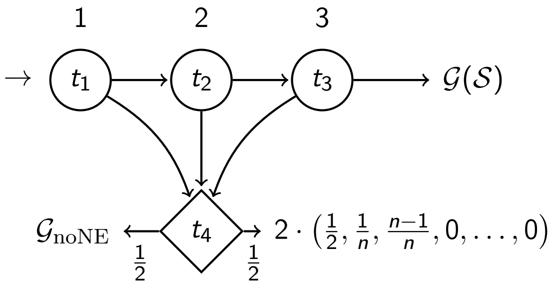


Figure 9: The game $\mathcal{G}_{\exists\mathrm{NE}}(\mathcal{S})$

$\mathcal{G}_{\mathrm{noNE}}$ is an independent sub-game,

- Has *no* stationary Nash equilibria
- Players 1, 2, 3 always get payoff 0.

$\exists\mathbb{R}$-Completeness of Stationary NE without Payoff Demands



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$\mathcal{S}$ is a "yes"-instance of $\textsc{HomQuad}$ if and only if the game $\mathcal{G}_{\exists\mathrm{NE}}(\mathcal{S})$ has a stationary Nash Equilibria.

$\exists\mathbb{R}$-Completeness of Reachability and Safety objectives

There exists different $\mathcal{G}_{\mathrm{noNE}}$ gadget games for the different restrictions of the utility function:

- Reachability objective
- Safety objective

Theorem
It is $\exists\mathbb{R}$-complete to decide whether a given m-player game with Reachability or Safety objectives has a stationary NE.

- *even for $m = 7$ players.*

$\exists\mathbb{R}$-Completeness of being almost surely winning

Consider the game $\mathcal{G}_{\exists\mathrm{NE}}(\mathcal{S})$ in which another player is added, who is always winning in $\mathcal{G}(\mathcal{S})$, but not in $\mathcal{G}_{\mathrm{noNE}}$.

Theorem
For any i, it is $\exists\mathbb{R}$-complete to decide whether a given m-player recursive game has a stationary NE in which Player i is almost surely winning.

- *even for $m = 8$ players.*

Final remarks

It is $\exists\mathbb{R}$-complete to decide in an m-player perfect information recursive game.

- exists Subgame Perfect Nash equilibria satisfying demand $L \in \mathbb{R}^m$
- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.

Final remarks

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- exists any Nash equilibria for *Reachability* and *Safety* objectives
- exists any Nash equilibria such that Player 1 is surely winning.

Notice here that

- This problem is already shown by Ummels '11 to be NP-hard for ≥ 2 players and $\mathrm{SqrtSum}$-hard for ≥ 4 players so this completeness result could not become much tighter.
- There have been recent results of $\exists\mathbb{R}$-completeness in *imperfect information* games. The complexity of these results stem from the structure of the game, not the lack of information.