I/O-efficient Manipulation of Binary Decision Diagrams

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Adiar: Binary Decision Diagrams in External Memory. 2022

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Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

- CountPaths
- Apply
- Equality Checking

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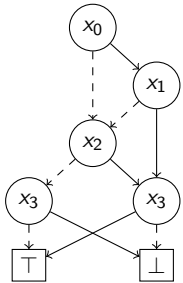
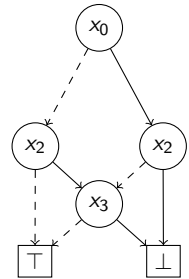


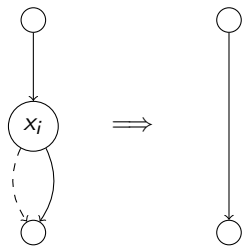
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



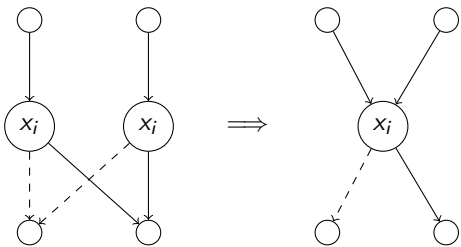
(b) $\neg(x_0 \; ? \; x_2 \wedge x_3 : x_2 \wedge x_3)$

Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



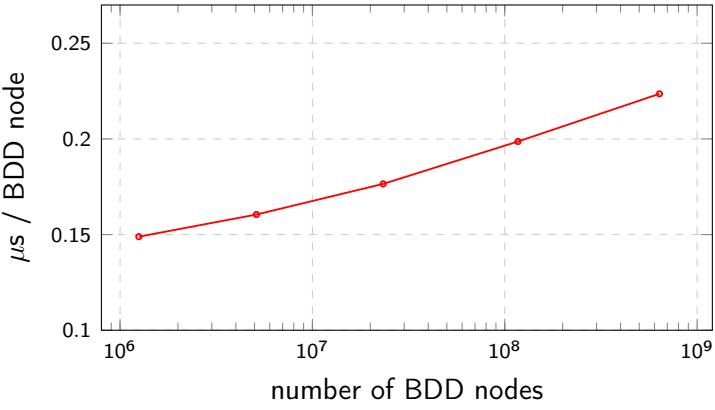
(2) Merge duplicate nodes

```
bdd_apply(f, g, ⊗):
  if f,g ∈ {⊥,⊤}
  then f ⊗ g
  else let i = top(f.var, g.var)
           t = bdd_apply(f[x_i := ⊤], g[x_i := ⊤], ⊗)
           e = bdd_apply(f[x_i := ⊥], g[x_i := ⊥], ⊗)
       in make_node(i,t,e)
```

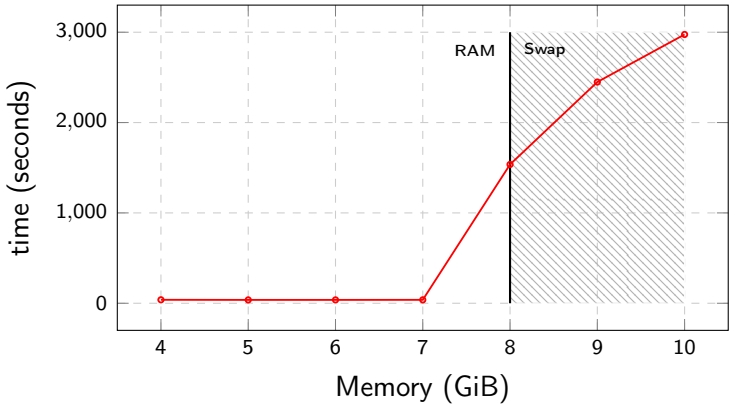
```
bdd_apply(f, g, ⊗):
  if f,g ∈ {⊥,⊤}
  then f ⊗ g
  else let i = top(f.var, g.var)
           t = bdd_apply(f[x_i := ⊤], g[x_i := ⊤], ⊗)
           e = bdd_apply(f[x_i := ⊥], g[x_i := ⊥], ⊗)
       in make_node(i,t,e)
```

Theorem
`bdd_apply` *runs in* $O(N_f \cdot N_g)$ *time.*

- Memoisation (*Computation Cache*) ensures each recursion is computed only once.
- Reduction Rules can be maintained within `make_node(i,t,e)` in $O(1)$ time.
  1. Redundancy is resolved with an if-statement.
  2. Duplication is avoided with a hash table (*Unique Node Table*).



Running time of *BuDDy* for the *N*-Queens problem.



Running time of *BuDDy* for *Tic-Tac-Toe* with $N = 21$.

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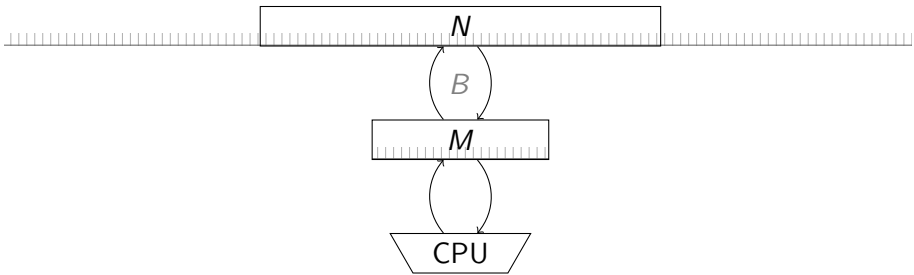


The I/O model by Aggarwal and Vitter '87

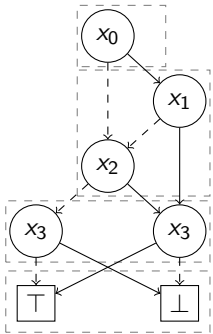
For any realistic values of N, M, and B we have that

$$N/B \quad < \quad \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \quad \ll \quad N \ ,$$

Theorem (Aggarwal and Vitter '87)
N elements can be sorted in $\Theta(sort(N))$ I/Os.

Theorem (Arge '95)
N elements can be inserted in and extracted from a Priority Queue in $\Theta(sort(N))$ I/Os.

CountPaths : *Example*

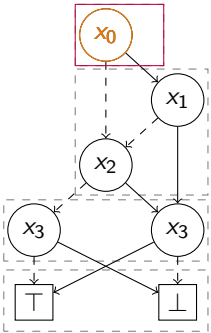


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$M = 4,\ B = 2$

node I/Os	cache lookups
0	0

CountPaths : *Example*

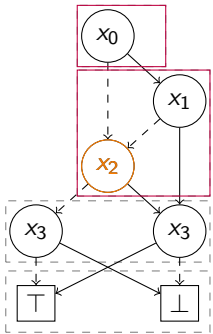


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$M = 4,\ B = 2$

node I/Os	cache lookups
1	1

CountPaths : *Example*



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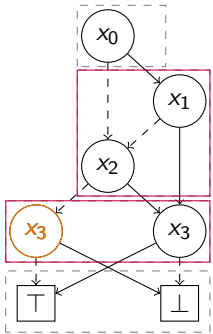
node I/Os	cache lookups
2	2

CountPaths : *Example*



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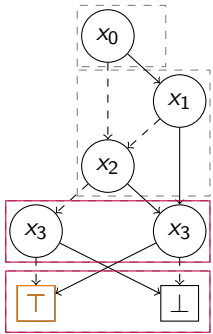
$M = 4,\ B = 2$

node I/Os	cache lookups
3	3

CountPaths : *Example*



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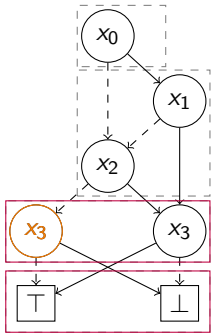
$M = 4,\ B = 2$

node I/Os	cache lookups
4	3

CountPaths : *Example*



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node I/Os	cache lookups
4	3

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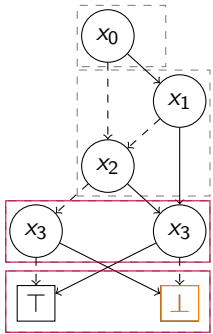

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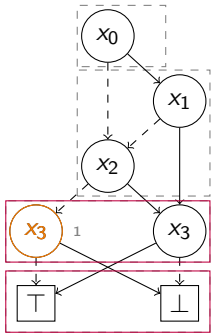
node I/Os	cache lookups
4	3

CountPaths : *Example*



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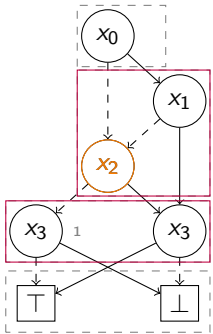

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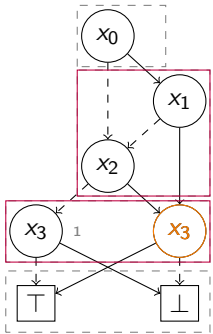
$M = 4,\ B = 2$

node I/Os	cache lookups
5	3

CountPaths : *Example*



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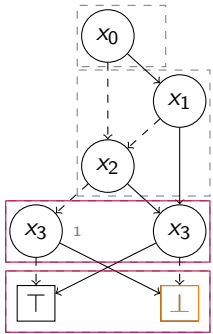
$M = 4,\ B = 2$

node I/Os	cache lookups
5	4

CountPaths : *Example*



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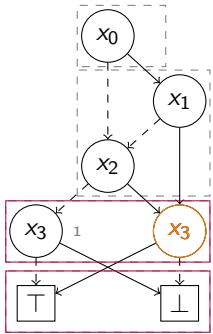
$M = 4,\ B = 2$

node I/Os	cache lookups
6	4

CountPaths : *Example*



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node I/Os	cache lookups
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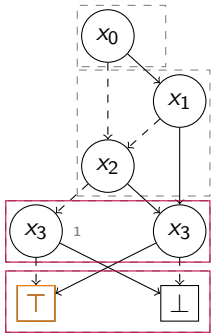

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node I/Os	cache lookups
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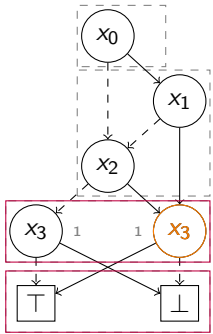

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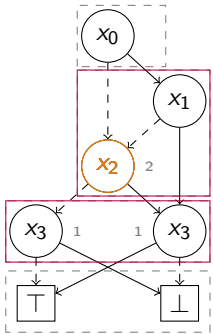

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node I/Os	cache lookups
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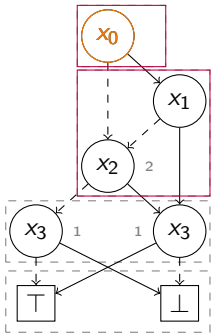
CountPaths : *Example*



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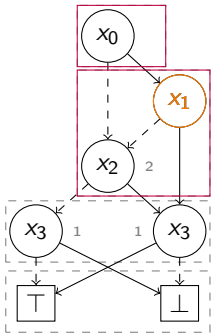
$M = 4,\ B = 2$

node I/Os	cache lookups
8	4

CountPaths : *Example*



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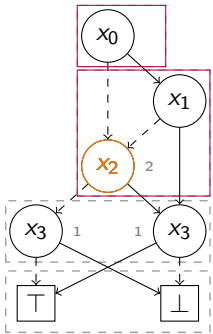
$M = 4,\ B = 2$

node I/Os	cache lookups
8	5

CountPaths : *Example*



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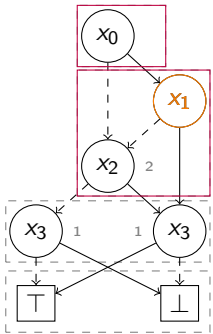

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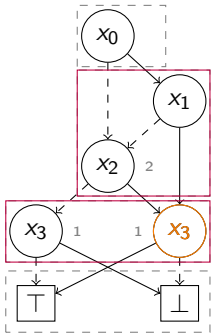

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node I/Os	cache lookups
9	7

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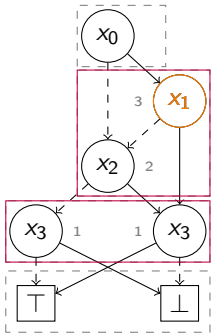

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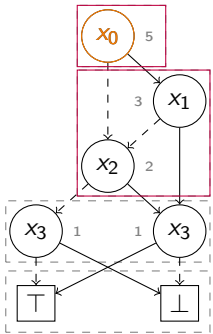

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node I/Os	cache lookups
10	7

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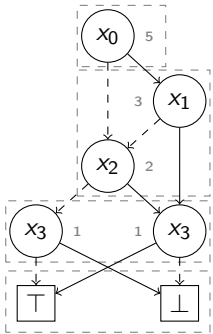

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$M = 4,\ B = 2$

node I/Os	cache lookups
10	7

Algorithm	Time Complexity
`bdd_pathcount`	$O(N_f)$
`bdd_not`	$O(N_f)$
`bdd_restrict`	$O(N_f)$
`bdd_apply`	$O(N_f \cdot N_g)$
`bdd_equal`	$O(1)$

Algorithm	I/O-Complexity
`bdd_pathcount`	$O(N_f)$
`bdd_not`	$O(N_f)$
`bdd_restrict`	$O(N_f)$
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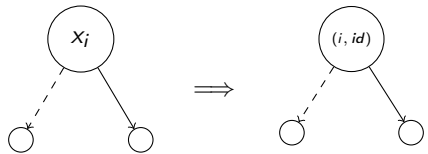
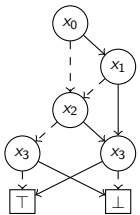
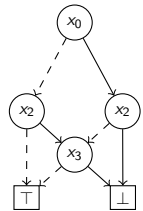


$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



$$[\ ((0,0),(2,0),(1,0))\ ,\ ((1,0),(2,0),(3,1))\ ,\ ((2,0),(3,0),(3,1))\ ,\ ((3,0),\top,\bot)\ ,\ ((3,1),\bot,\top)\ ]$$

(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

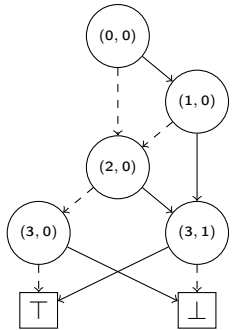


$$[\ ((0,0),(2,0),(2,1))\ ,\ ((2,0),\bot,(3,0))\ ,\ ((2,1),(3,0),\top)\ ,\ ((3,0),\top,\bot)\ ]$$

(b) $\neg(x_0\ ?\ x_2 \wedge x_3 : x_2 \wedge x_3)$

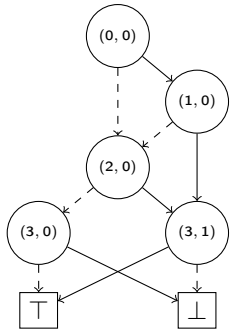
Node-based representation of prior shown BDDs

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

CountPaths : *Example*



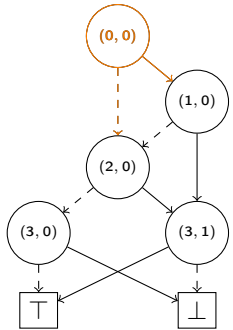
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count}:

[

]

CountPaths : *Example*



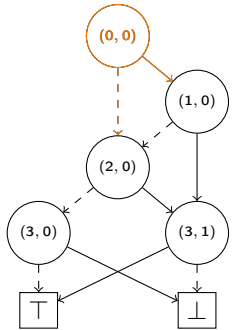
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count}:

[

]

CountPaths : *Example*



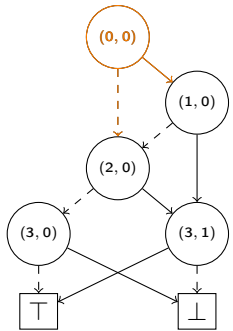
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count}:

$$\Big[\quad ((0,0) \xrightarrow{\top} (1,0), \quad 1) \quad ,$$
$$\quad\;\; ((0,0) \xrightarrow{\bot} (2,0), \quad 1) \quad ,$$

$$\Big]$$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

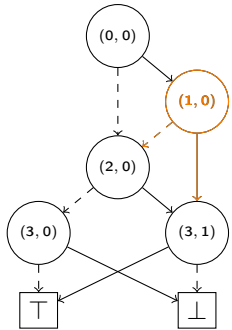
Seek	Sum	Result
$(1, 0)$	0	0

Priority Queue: Q_{count}:

$$[\quad ((0,0) \xrightarrow{\top} (1,0),\quad 1)\quad ,$$
$$\quad\ ((0,0) \xrightarrow{\bot} (2,0),\quad 1)\quad ,$$

$$]$$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

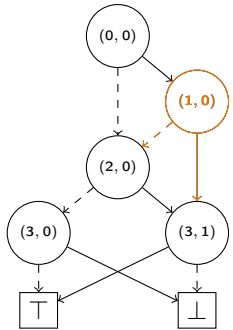
Seek	Sum	Result
$(1, 0)$	0	0

Priority Queue: Q_{count}:

$$[\quad ((0,0) \xrightarrow{\top} (1,0),\quad 1)\quad ,$$
$$\quad\ ((0,0) \xrightarrow{\bot} (2,0),\quad 1)\quad ,$$

$$]$$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(1, 0)$	1	0

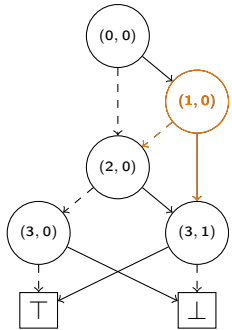
Priority Queue: Q_{count}:

[

$((0,0) \xrightarrow{\perp} (2,0), \quad 1)$,

]

CountPaths : *Example*



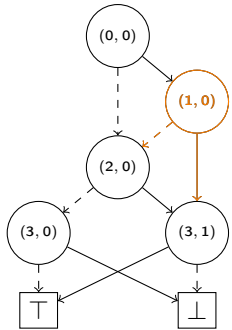
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(1, 0)$	1	0

Priority Queue: Q_{count}:

$$
\begin{bmatrix}
((0,0) \xrightarrow{\perp} (2,0), & 1) & , \\
((1,0) \xrightarrow{\perp} (2,0), & 1) & , \\
 & & \\
((1,0) \xrightarrow{\top} (3,1), & 1) & , \\
 & &]
\end{bmatrix}
$$

CountPaths : *Example*



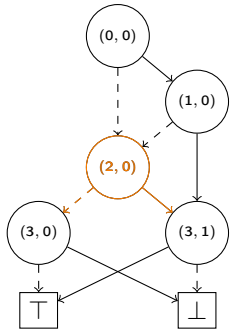
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	0	0

Priority Queue: Q_{count}:

$$
\begin{bmatrix}
((0,0) \xrightarrow{\perp} (2,0), & 1) & , \\
((1,0) \xrightarrow{\perp} (2,0), & 1) & , \\
 & & \\
((1,0) \xrightarrow{\top} (3,1), & 1) & , \\
 & &]
\end{bmatrix}
$$

CountPaths : *Example*



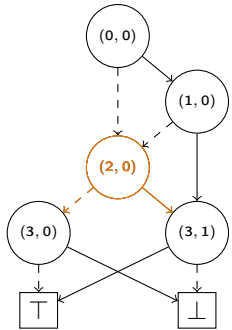
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	0	0

Priority Queue: Q_{count}:

$$[$$
$$((0,0) \xrightarrow{\perp} (2,0), \quad 1) \quad ,$$
$$((1,0) \xrightarrow{\perp} (2,0), \quad 1) \quad ,$$
$$((1,0) \xrightarrow{\top} (3,1), \quad 1) \quad ,$$
$$]$$

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	1	0

Priority Queue: Q_{count}:

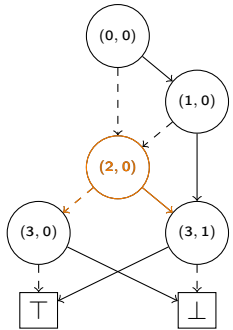
[

$((1,0) \xrightarrow{\perp} (2,0), \quad 1)$,

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(2, 0)$	2	0

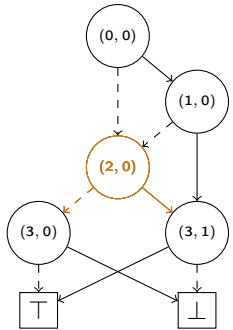
Priority Queue: Q_{count}:

[

$((1, 0) \xrightarrow{\top} (3, 1), \quad 1)$,

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

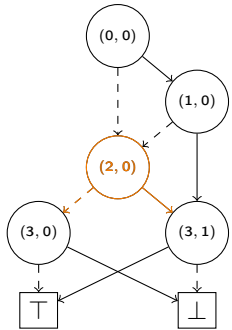
Seek	Sum	Result
$(2, 0)$	2	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2)$,
$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,
$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

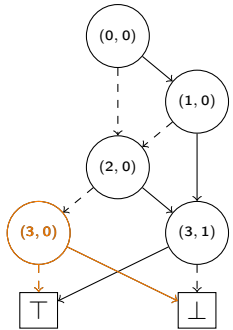
Seek	Sum	Result
$(3, 0)$	0	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2)$,
$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,
$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

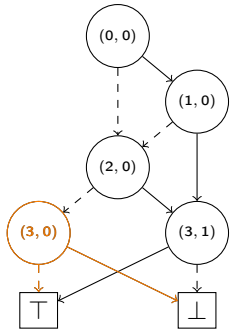
Seek	Sum	Result
$(3, 0)$	0	0

Priority Queue: Q_{count}:

[

$((2,0) \xrightarrow{\perp} (3,0), \quad 2)$,
$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,
$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 0)$	2	0

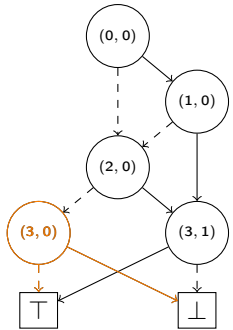
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 0)$	2	2

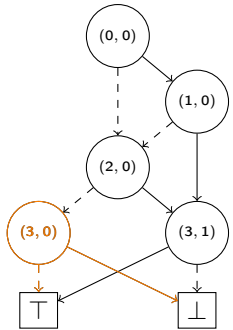
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3, 1)$	0	2

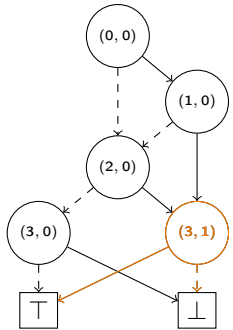
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
$(3,1)$	0	2

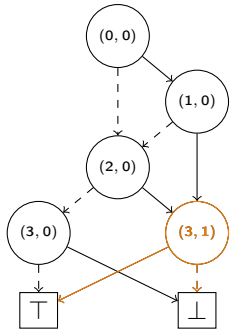
Priority Queue: Q_{count}:

[

$((1,0) \xrightarrow{\top} (3,1), \quad 1)$,

$((2,0) \xrightarrow{\top} (3,1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

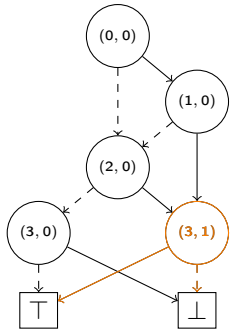
Seek	Sum	Result
$(3, 1)$	1	2

Priority Queue: Q_{count}:

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2)$]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

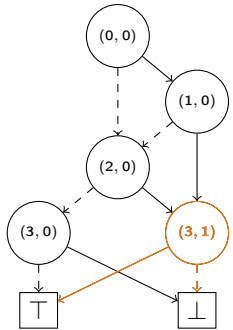
Seek	Sum	Result
$(3, 1)$	3	2

Priority Queue: Q_{count}:

[

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

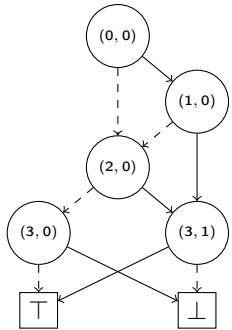
Seek	Sum	Result
$(3, 1)$	3	5

Priority Queue: Q_{count}:

[

]

CountPaths : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result

5

Priority Queue: Q_{count}:

[

]

Contents

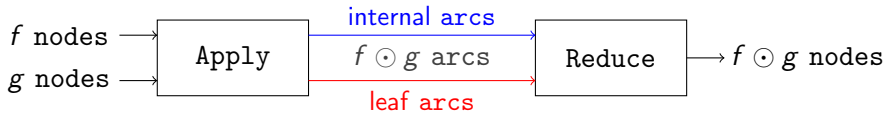
What are Binary Decision Diagrams?

Why do they break?

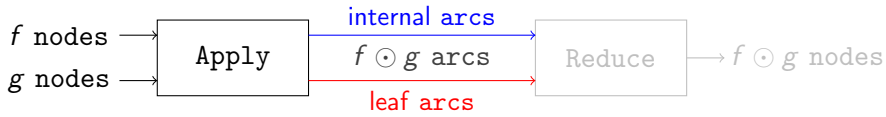
How can we fix it?

- CountPaths
- Apply
- Equality Checking

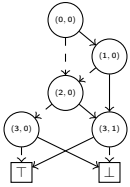
Apply


f nodes
g nodes
Apply
internal arcs
$f \odot g$ arcs
leaf arcs
Reduce
$f \odot g$ nodes

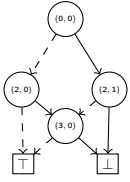
Apply


f nodes
g nodes
Apply
internal arcs
$f \odot g$ arcs
leaf arcs
Reduce
$f \odot g$ nodes

Apply : *Example*



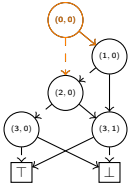
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



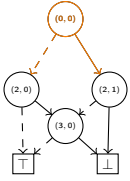
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Apply : *Example*



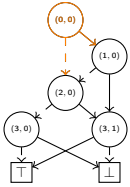
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



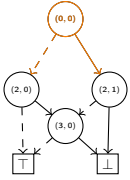
(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \ ? \ x_2 \vee x_3 \ : \ x_2 \wedge x_3)$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

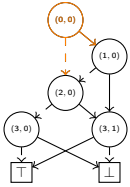
[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,

$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

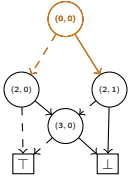
]

(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((1,0),(2,1))$

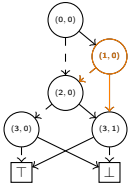
Priority Queue: $Q_{app:1}$:

[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,
$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

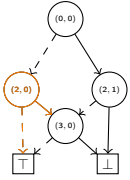
]

(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((1,0),(2,1))$

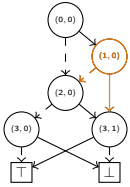
Priority Queue: $Q_{\textbf{app}:1}$:

[$(0,0) \xrightarrow{\top} ((1,0),(2,1))$,
$(0,0) \xrightarrow{\bot} ((2,0),(2,0))$,

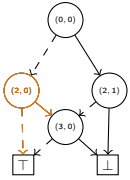
]

(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



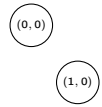
(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((1,0),(2,1))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

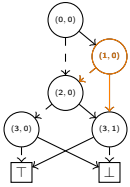
$$[\quad (0,0) \xrightarrow{\top} ((1,0),(2,1)) \quad ,$$
$$(0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad ,$$
$$(1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad ,$$
$$(1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad ,$$

$]$

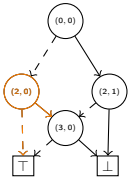


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((1,0),(2,1))$

Priority Queue: $Q_{\mathbf{app}:\mathbf{1}}$:

[

$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

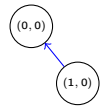
$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

]

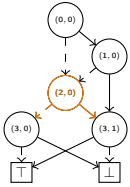
Output:

$(0,0) \xrightarrow{\top} (1,0)$

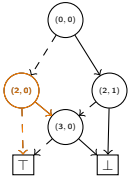


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{app:1}$:

[

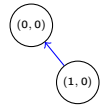
$(0,0) \xrightarrow{\perp} ((2,0),(2,0))$,

$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

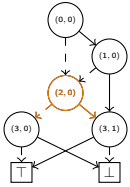
]

Output:

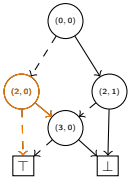


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,0))$

Priority Queue: $Q_{\mathbf{app}:1}$:

$$[$$

$$(0,0) \xrightarrow{\perp} ((2,0),(2,0)) \quad ,$$

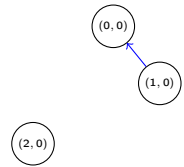
$$(1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad ,$$

$$(1,0) \xrightarrow{\top} ((3,1),(2,1)) \quad ,$$

$$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad ,$$

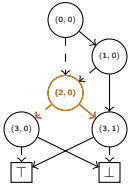
$$(2,0) \xrightarrow{\perp} ((3,0),\top) \quad]$$

Output:

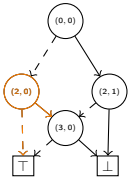


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((2,0),(2,0))$

Priority Queue: $Q_{\textbf{app}:\mathbf{1}}$:

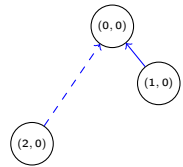
[

$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,
$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

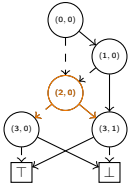
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Output:
$(0,0) \xrightarrow{\perp} (2,0)$

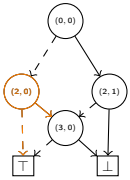


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

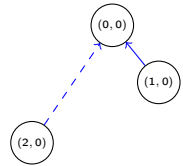
$(1,0) \xrightarrow{\perp} ((2,0),(2,1))$,

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

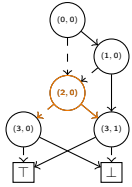
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Output:

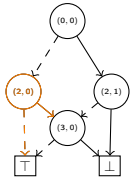


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

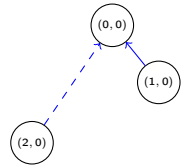
$(2, 0) \xrightarrow{\bot} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\bot} ((2, 0), (2, 1)) \quad ((3, 0), (3, 1))$,

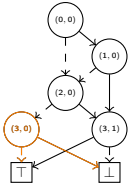
]

Output:

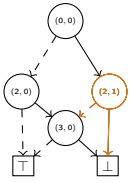


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

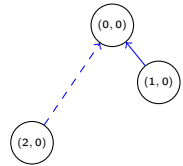
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[$(1,0) \xrightarrow{\bot} ((2,0),(2,1)) \quad ((3,0),(3,1))$,

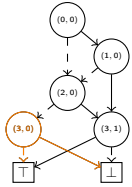
]

Output:

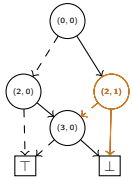


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

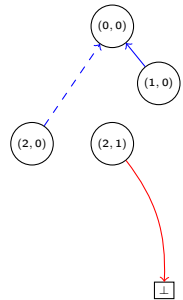
Priority Queue: $Q_{app:2}$:

[$(1,0) \xrightarrow{\perp} ((2,0),(2,1)) \quad ((3,0),(3,1))$,

]

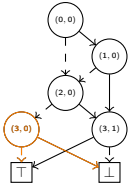
Output:

$(2,1) \xrightarrow{\top} \perp$

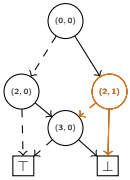


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\max((2,0),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

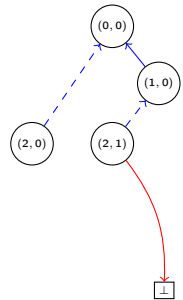
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[

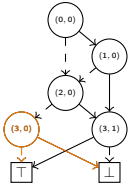
]

Output:
$(1,0) \xrightarrow{\bot} (2,1)$

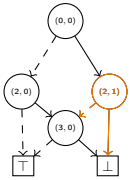


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\bot} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

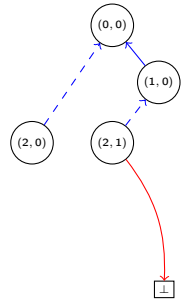
$(2, 0) \xrightarrow{\bot} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

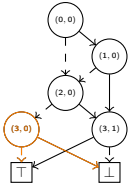
]

Output:

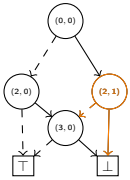


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(1,0) \xrightarrow{\top} ((3,1),(2,1))$,
$(2,1) \xrightarrow{\bot} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

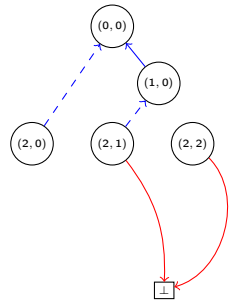
Priority Queue: $Q_{app:2}$:

[

]

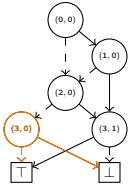
Output:

$(2,2) \xrightarrow{\top} \bot$

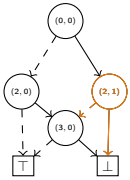


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(2,1))$

Priority Queue: $Q_{app:1}$:

[

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

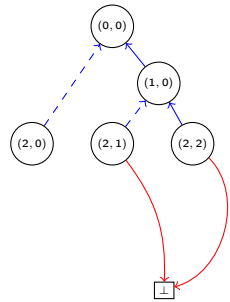
Priority Queue: $Q_{app:2}$:

[

]

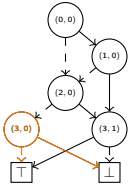
Output:

$(1,0) \xrightarrow{\top} (2,2)$

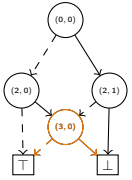


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

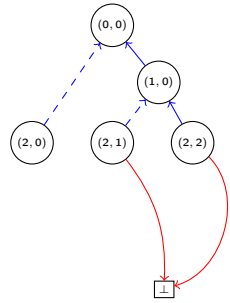
$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[

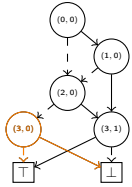
]

Output:

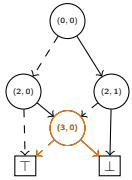


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,1) \xrightarrow{\perp} ((3,0),(3,0))$,
$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,
$(2,2) \xrightarrow{\perp} ((3,1),(3,0))$,
$(2,0) \xrightarrow{\perp} ((3,0),\top)$]

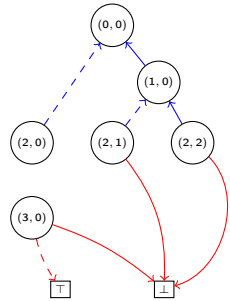
Priority Queue: $Q_{app:2}$:

[

]

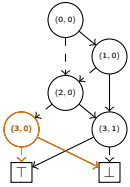
Output:

$(3,0) \xrightarrow{\perp} \top, (3,0) \xrightarrow{\top} \perp$

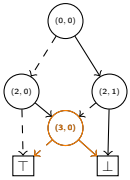


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,

$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

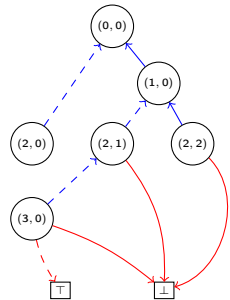
Priority Queue: $Q_{app:2}$:

[

]

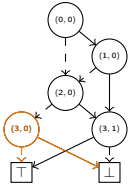
Output:

$(2,1) \xrightarrow{\bot} (3,0)$

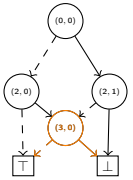


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0))$,

$(2,2) \xrightarrow{\bot} ((3,1),(3,0))$,

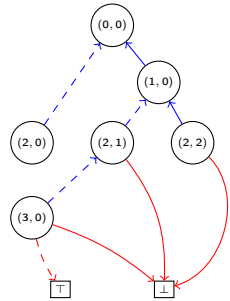
$(2,0) \xrightarrow{\bot} ((3,0),\top)$]

Priority Queue: $Q_{app:2}$:

[

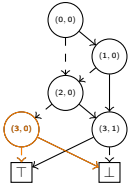
]

Output:

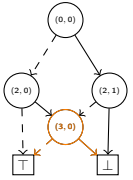


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

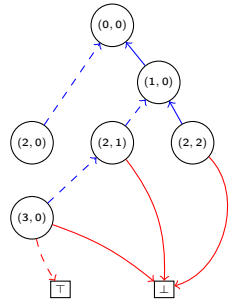
Priority Queue: $Q_{app:2}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

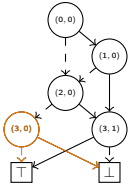
$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

Output:

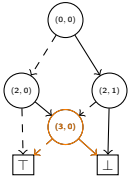


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

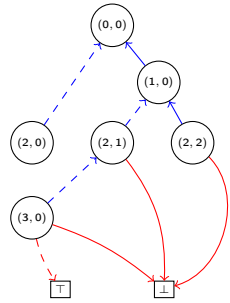
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$,

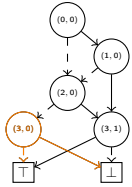
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$]

Output:

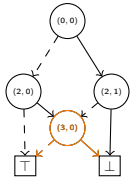


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\min((3,0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2,0) \xrightarrow{\perp} ((3,0), \top)$]

Priority Queue: $Q_{app:2}$:

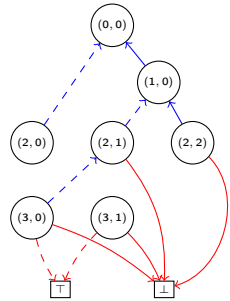
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

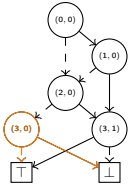
Output:

$(3,1) \xrightarrow{\perp} \top, (3,1) \xrightarrow{\top} \perp$

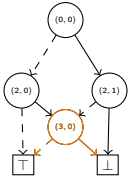


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:
$\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

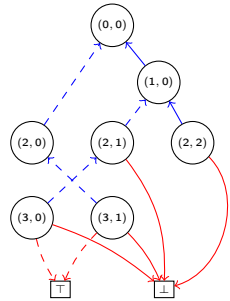
Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \bot)$,

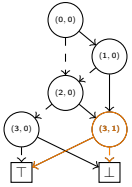
$(2, 2) \xrightarrow{\bot} ((3, 1), (3, 0)) \quad (\top, \bot)$]

Output:
$(2, 0) \xrightarrow{\bot} (3, 1)$

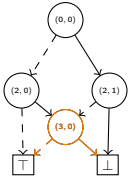


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

]

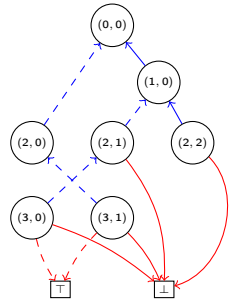
Priority Queue: $Q_{app:2}$:

[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \bot)$,

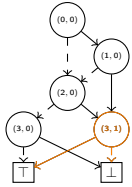
$(2,2) \xrightarrow{\bot} ((3,1),(3,0)) \quad (\top, \bot)$]

Output:

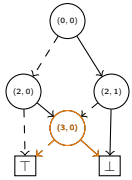


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

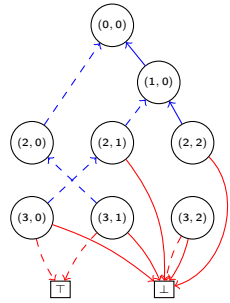
[

$(2,0) \xrightarrow{\top} ((3,1),(3,0)) \quad (\top, \perp)$,

$(2,2) \xrightarrow{\perp} ((3,1),(3,0)) \quad (\top, \perp)$]

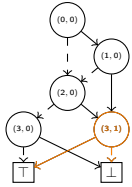
Output:

$(3,2) \xrightarrow{\perp} \perp, (3,2) \xrightarrow{\top} \perp$

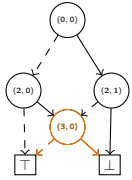


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Seek:

$\max((3,1),(3,0))$

Priority Queue: $Q_{app:1}$:

[

]

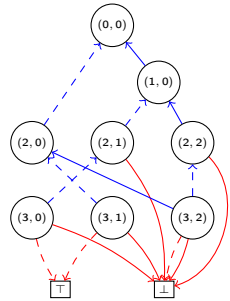
Priority Queue: $Q_{app:2}$:

[

]

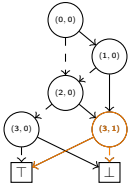
Output:

$(2,0) \xrightarrow{\top} (3,2), (2,2) \xrightarrow{\perp} (3,2)$

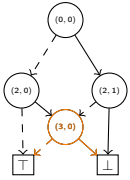


(c) $(a) \wedge (b)$

Apply : *Example*



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \; ? \; x_2 \vee x_3 \; : \; x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

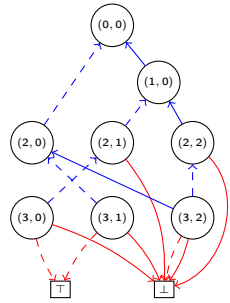
]

Priority Queue: $Q_{app:2}$:

[

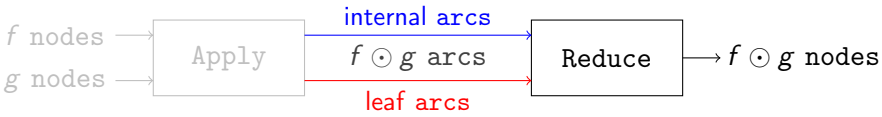
]

Output:

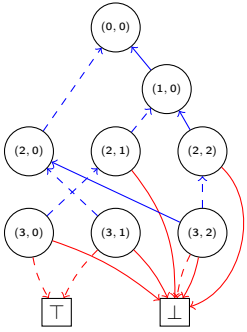


(c) $(a) \wedge (b)$

Apply


f nodes
g nodes
Apply
internal arcs
$f \odot g$ arcs
leaf arcs
Reduce
$f \odot g$ nodes

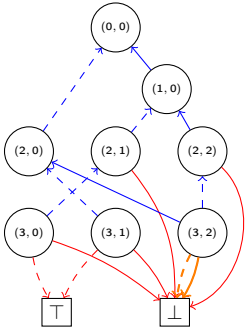
Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



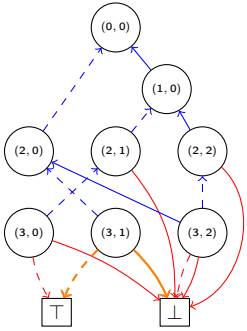
(c) $(a) \wedge (b)$

Level: 3

[$[(3, 2) \mapsto \bot]$]

(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

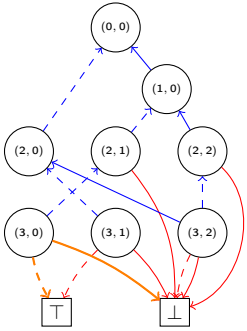
Level: 3

[$[(3, 2) \mapsto \bot]$]

[$((3, 1), \top, \bot)$,]

(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

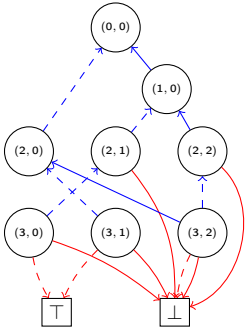
Level: 3

[$[(3, 2) \mapsto \bot]$]

[$((3, 1), \top, \bot)$,
$((3, 0), \top, \bot)$]

(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



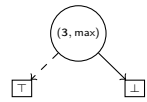
(c) $(a) \wedge (b)$

Level: 3

[$[(3,2) \mapsto \bot]$]

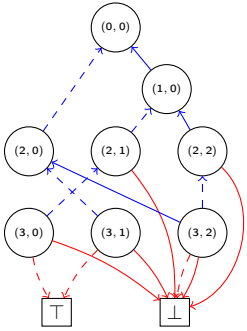
[$[(3,1) \mapsto (3, \max)]$, $((3,0), \top, \bot)$]

Output:
$((3, \max), \top, \bot)$



(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



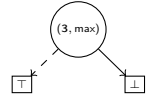
(c) $(a) \wedge (b)$

Level: 3

[$[(3,2) \mapsto \bot]$]

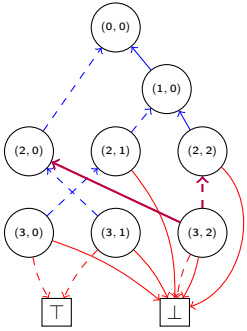
[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$]

Output:



(d) $(a) \wedge (b)$ **reduced**

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,

$(2,0) \xrightarrow{\top} \perp$,

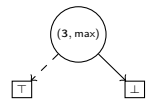
]

Level: 3

[ ]

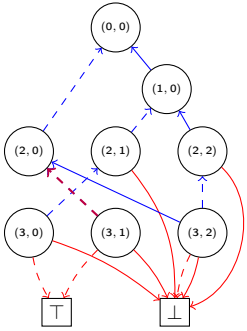
[$[(3,1) \mapsto (3, \max)]$,
$[(3,0) \mapsto (3, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,

$(2,0) \xrightarrow{\top} \perp$,

$(2,0) \xrightarrow{\perp} (3, \max)$,

]

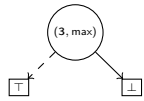
Level: 3

[ ]

[,

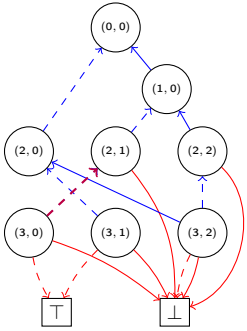
$[(3,0) \mapsto (3, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[$(2,2) \xrightarrow{\perp} \perp$,
$(2,1) \xrightarrow{\perp} (3, \max)$,
$(2,0) \xrightarrow{\top} \perp$,
$(2,0) \xrightarrow{\perp} (3, \max)$,

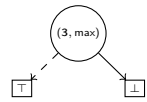
]

Level: 3

[ ]

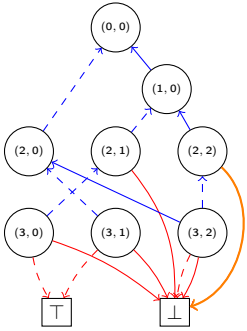
[,
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(2,1) \xrightarrow{\perp} (3, \max)$,

$(2,0) \xrightarrow{\top} \perp$,

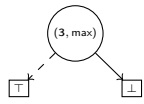
$(2,0) \xrightarrow{\perp} (3, \max)$,

]

Level: 2

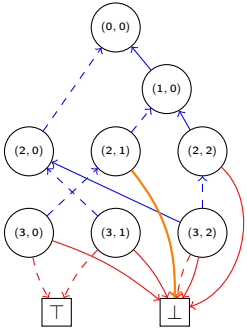
[$[(2,2) \mapsto \perp]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(2, 0) \xrightarrow{\top} \bot$,

$(2, 0) \xrightarrow{\bot} (3, \max)$,

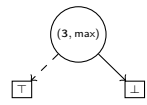
]

Level: 2

[$[(2, 2) \mapsto \bot]$]

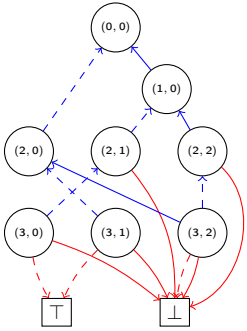
[$((2, 1), (3, \max), \bot)$,
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

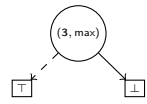
]

Level: 2

[$[(2, 2) \mapsto \bot]$]

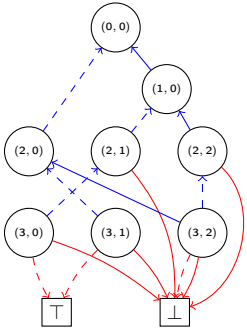
[$((2, 1), (3, \text{max}), \bot)$,
$((2, 0), (3, \text{max}), \bot)$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

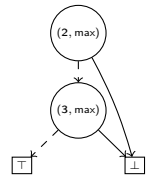
]

Level: 2

[$[(2, 2) \mapsto \bot]$]

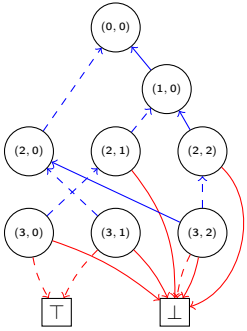
[$[(2, 1) \mapsto (2, \max)]$,
$((2, 0), (3, \max), \bot)$]

Output:
$((2, \max), (3, \max), \bot)$



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

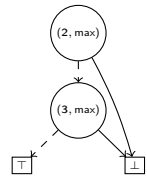
]

Level: 2

[$[(2, 2) \mapsto \bot]$]

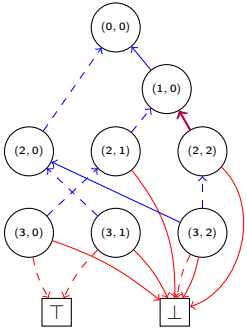
[$[(2, 1) \mapsto (2, \max)]$,
$[(2, 0) \mapsto (2, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(1, 0) \xrightarrow{\top} \bot$,

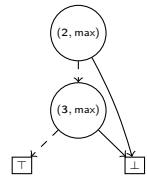
]

Level: 2

[ ]

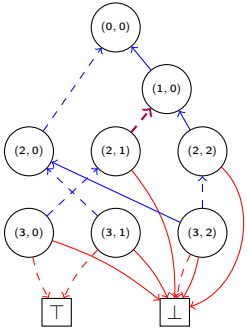
[$[(2, 1) \mapsto (2, \max)]$,
$[(2, 0) \mapsto (2, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

]

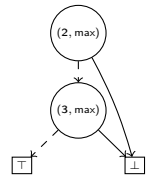
Level: 2

[ ]

[,

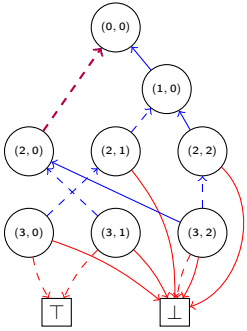
$[(2,0) \mapsto (2, \max)]$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(1,0) \xrightarrow{\top} \bot$,

$(1,0) \xrightarrow{\bot} (2, \max)$,

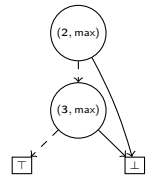
$(0,0) \xrightarrow{\bot} (2, \max)$]

Level: 2

[ ]

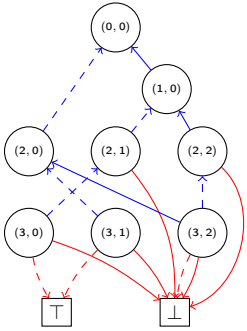
[,
]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

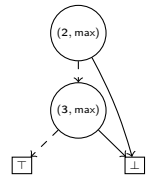
$(0, 0) \xrightarrow{\perp} (2, \text{max})$]

Level: 1

[ ]

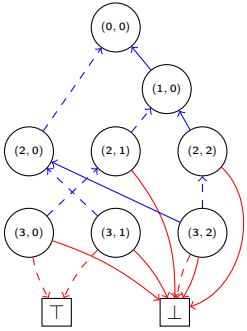
[$((1, 0), (2, \text{max}), \perp)$]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

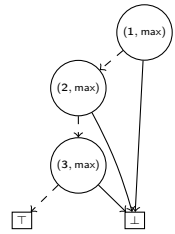
$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[ ]

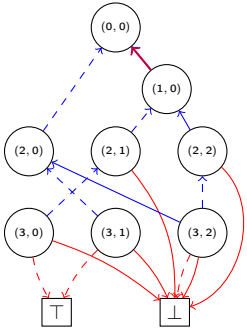
[$[(1, 0) \mapsto (1, \max)]$]

Output:
$((1, \max), (2, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

$(0,0) \xrightarrow{\top} (1, \text{max})$,

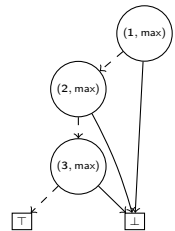
$(0,0) \xrightarrow{\bot} (2, \text{max})$]

Level: 1

[ ]

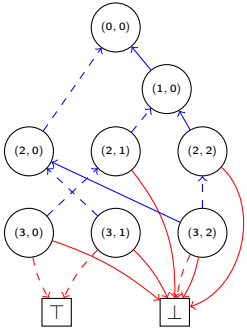
[ ]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

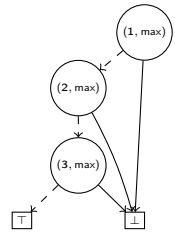
]

Level: 0

[ ]

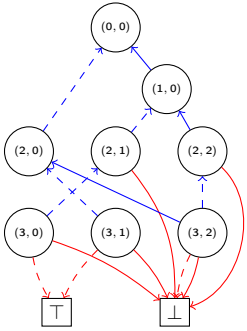
[((0, 0), (2, max), (1, max))]

Output:



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



(c) $(a) \wedge (b)$

Priority Queue: Q_{red}:

[

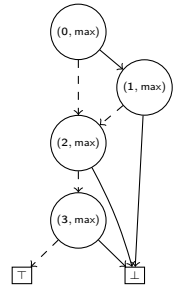
]

Level: 0

[ ]

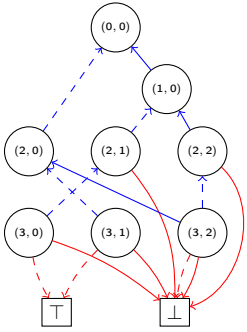
[$[(0, 0) \mapsto (0, \max)]$]

Output:
((0, max), (2, max), (1, max))



(d) $(a) \wedge (b)$ reduced

Apply : *Example* *(Continued)*



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Priority Queue: Q_{red}:

[

]

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[ ]

[ ]

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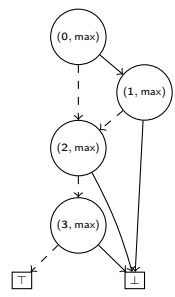
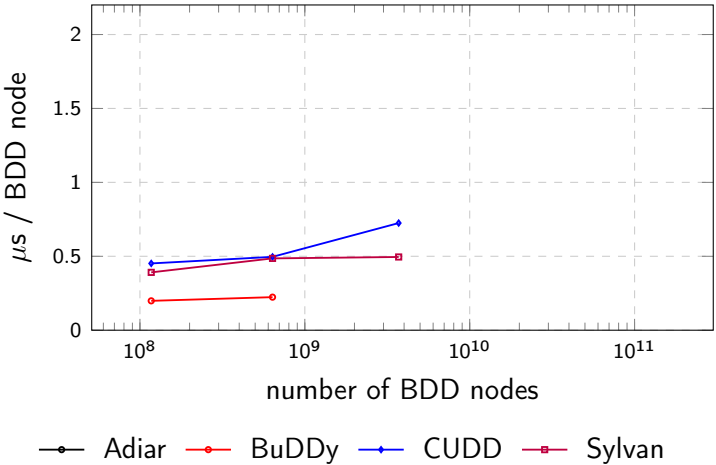
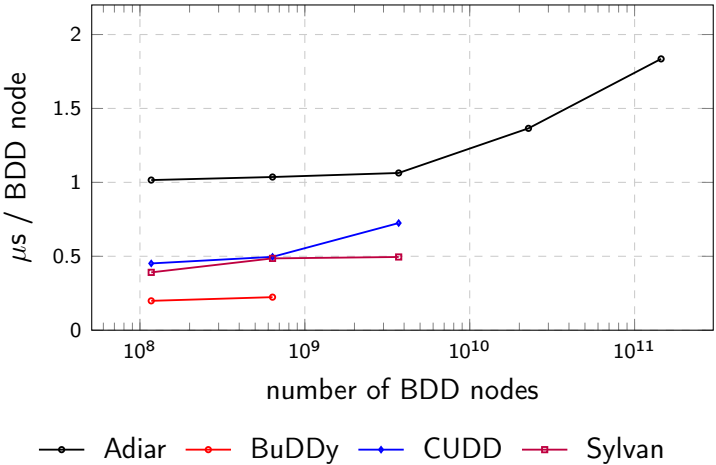


(d) $(a) \wedge (b)$ reduced

Algorithm	I/O-Complexity
`bdd_pathcount`	$O(\mathrm{sort}(N_f))$
`bdd_not`	$O(N_f/B)$
`bdd_restrict`	$O(\mathrm{sort}(N_f))$
`bdd_apply`	$O(\mathrm{sort}(N_f \cdot N_g))$



Running time for the *N-Queens* problems.



Running time for the *N-Queens* problems.

Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

- CountPaths
- Apply
- Equality Checking

Algorithm	I/O-Complexity
`bdd_pathcount`	$O(\mathrm{sort}(N_f))$
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`bdd_equal`	?

Equality Checking

$$f \leftrightarrow g \equiv \top$$

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$$f \leftrightarrow g \equiv \top$$

$$\underbrace{O(\text{sort}(N^2))}_{\texttt{Apply}} + \underbrace{O(\text{sort}(N^2))}_{\texttt{Reduce}} + \underbrace{O(1))}_{\text{check is } \top} = O(\text{sort}(N^2))$$

Equality Checking

Theorem (Bryant '86)
Let π be a variable order and $f : \mathbb{B}^n \to \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π.

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Let π be a variable order and $f : \mathbb{B}^n \to \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π.

Trivial cases: $f \not\equiv g$ if there is a mismatch in

■ $N_f \neq N_g$	Number of nodes	$O(1)$ I/Os
■ $L_f \neq L_g$	Number of levels	$O(1)$ I/Os
■ $N_{f,i} \neq N_{g,i}$	Number of nodes on a level	$O(L/B)$ I/Os
■ $L_{f,i} \neq L_{g,i}$	Label of an ith level	$O(L/B)$ I/Os

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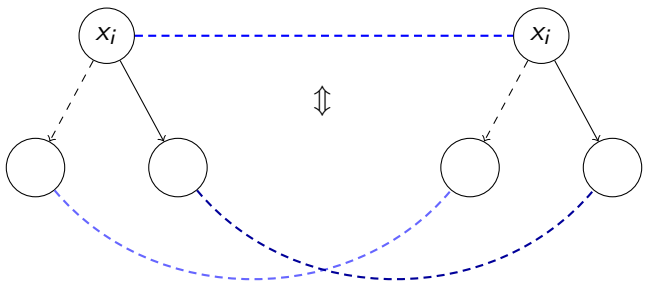

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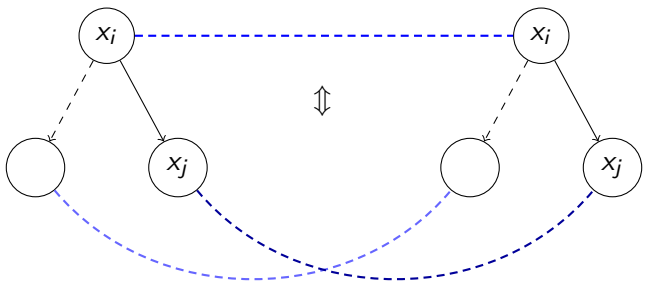

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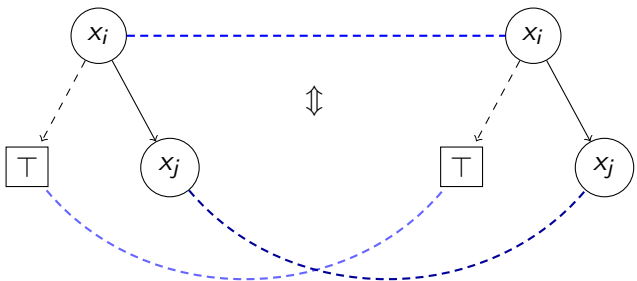

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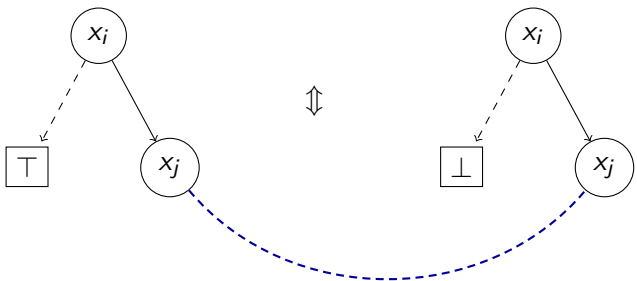

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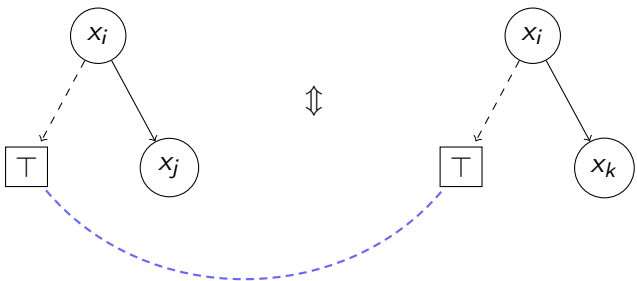

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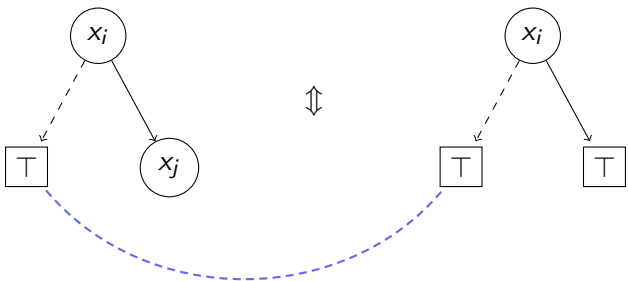

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Theorem (Bryant '86)
Let π be a variable order and $f : \mathbb{B}^n \to \mathbb{B}$ then there exists a unique (up to isomorphism) Reduced Ordered Binary Decision Diagram representing f with ordering π.

`IsIsomorphic(`f`,` g`)`

- Check whether root v_f of f and root v_g of g have a local violation.
- Check $low(v_f) \sim low(v_g)$ and $high(v_f) \sim high(v_g)$ "recursively".

Return `false` on first violation. If there are no violations then return `true`.

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$$\underbrace{O(\text{sort}(N^2))}_{\texttt{Apply}'} + \underbrace{\cancel{O(\text{sort}(N^2))}}_{\texttt{Reduce}} + \underbrace{\cancel{O(1)}}_{\text{check is } \top} = O(\text{sort}(N^2))$$

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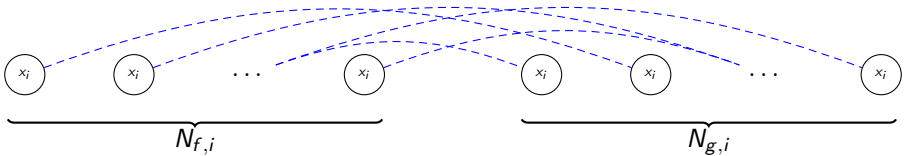

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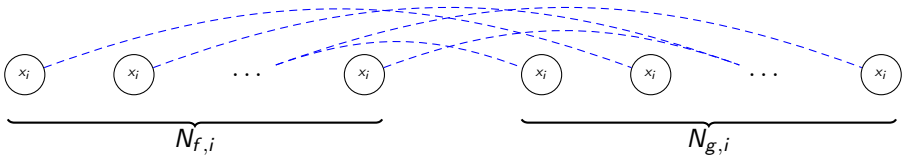


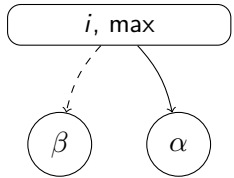
Return `false` if more than $N_{f,i} = N_{g,i}$ pairs of nodes are checked on level i.

$$\underbrace{O(\text{sort}(\Sigma_i\ N_{f,i}))}_{\texttt{Apply}''} = O(\text{sort}(N))$$

Equality Checking

Observation
Each level output by the `Reduce` algorithm has the following properties:

Equality Checking



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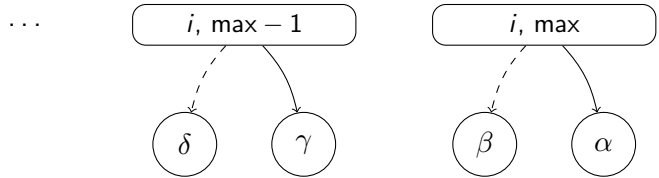

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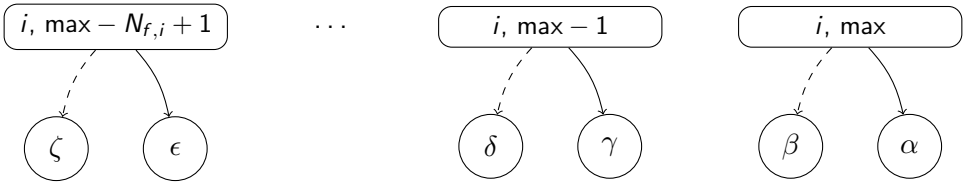
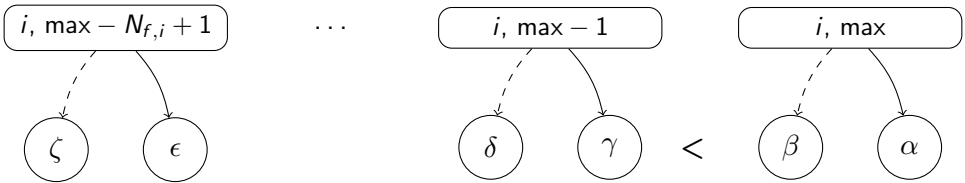


Observation
Each level output by the `Reduce` algorithm has the following properties:

- Nodes on level i have their identifiers *consecutively* numbered.

Equality Checking



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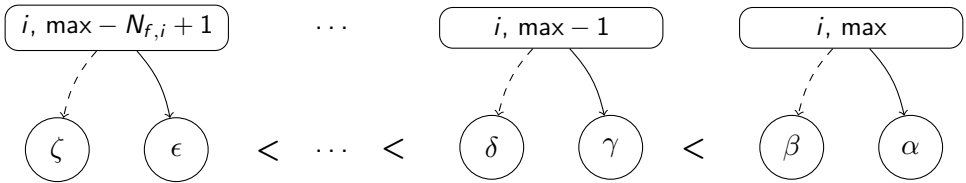
Equality Checking



Observation
Each level output by the `Reduce` algorithm has the following properties:

- Nodes on level i have their identifiers *consecutively* numbered.
- Nodes on level i are output sorted by their children.

Equality Checking

Theorem
If G_f and G_g are outputs of `Reduce`*.*

$$G_f \sim G_g \iff \textit{For all } i \in [0; N_f) \textit{ the node } G_f[i] \textit{ matches } G_g[i] \textit{ numerically.}$$

Proof.
$\Leftarrow$: Must describe the exact same graph.

$\Rightarrow$: Strong induction on BDD levels bottom-up. $\square$

Equality Checking

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If G_f and G_g are outputs of `Reduce`*.*

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Proof.
$\Leftarrow$: Must describe the exact same graph.

$\Rightarrow$: Strong induction on BDD levels bottom-up. □

Corollary
If G_f and G_g are outputs of `Reduce` *then $f \equiv g$ is computable using $2 \cdot N/B$ I/Os.*

Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38

Checking the (EPFL Benchmark) *voter* circuit's single output gate ($|N_f| = |N_g| = 5.76$ MiB).

Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38
$O(\text{sort}(N))$	0.058

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Equality Checking

Algorithm	Time (s)
$f \leftrightarrow g \equiv \top$	0.38
$O(\text{sort}(N))$	0.058
$2N/B$	0.006

Checking the (EPFL Benchmark) *voter* circuit's single output gate ($|N_f| = |N_g| = 5.76$ MiB).

Contents

What are Binary Decision Diagrams?

Why do they break?

How can we fix it?

- CountPaths
- Apply
- Equality Checking

	Depth-First	*Time-Forwarded*
CountPaths	$O(N_f)$	$O(\text{sort}(N_f))$
Apply	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
Equality Checking	$O(1)$	$2N/B$