

Relational Product of BDDs in External Memory

Steffan Christ Sølvesten, Jaco van de Pol

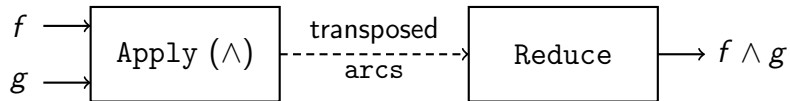
SPIN 2025

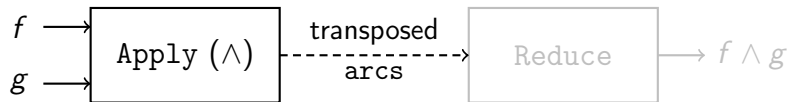


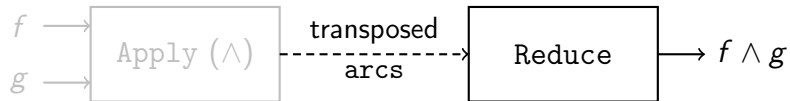
$$Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'})[\vec{x}' / \vec{x}]$$

[TACAS 22]

$$Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'}) [\vec{x}' / \vec{x}]$$



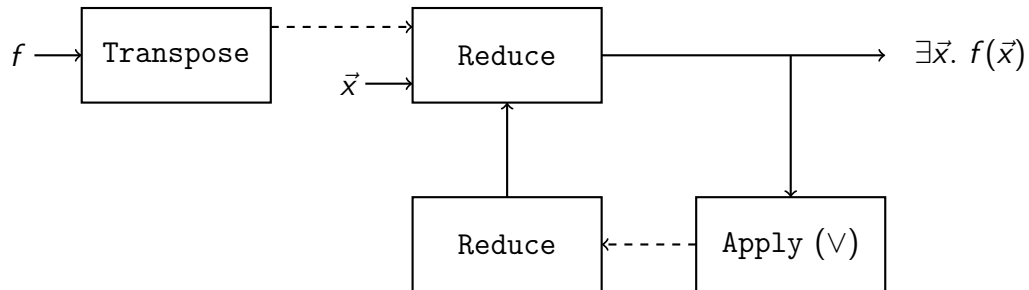




[TACAS 22]

$$\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'}) [\vec{x}' / \vec{x}]$$

[TACAS 25]



[TACAS 22]

$$Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'}) [\vec{x}' / \vec{x}]$$

[TACAS 25]

$$\begin{array}{c}
 \text{[TACAS 22]} \\
 | \\
 \text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\underbrace{\exists \vec{x}. S_{\vec{x}} \wedge}_{\text{[TACAS 25]}} T_{\vec{x}, \vec{x}'}) \underbrace{[\vec{x}' / \vec{x}]}_{\text{[SPIN 25]}}
 \end{array}$$

Replace

Definition

A relabelling π is monotonic if $x_i < x_j \implies \pi(x_i) < \pi(x_j)$

Lemma

If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

Replace

Definition

A relabelling π is monotonic if $x_i < x_j \implies \pi(x_i) < \pi(x_j)$

Lemma

If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

- One can apply π in a single linear scan.

$\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.

Replace

Definition

A relabelling π is monotonic if $x_i < x_j \implies \pi(x_i) < \pi(x_j)$

Lemma

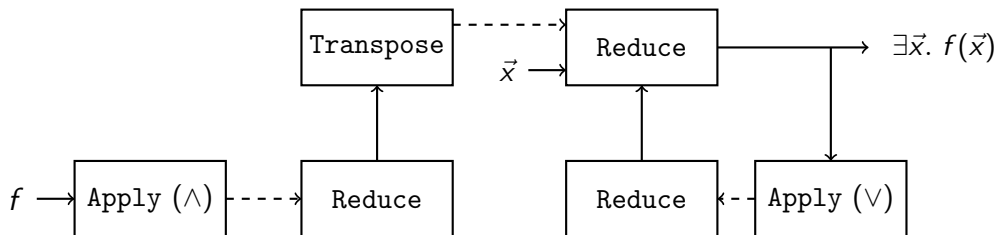
If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

- **One can apply π in a single linear scan.**
 $\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.
- **One can incorporate π into a (succeeding) top-down Apply sweep.**
 $\mathcal{O}(N)$ time, 0 I/Os, and 0 external space.
- **One can incorporate π into a (preceeding) bottom-up Reduce sweep.**
 $\mathcal{O}(n)$ time, 0 I/Os, and 0 external space.

AndExists

Observation

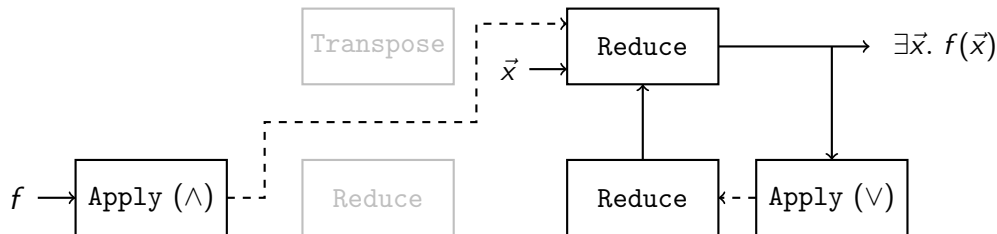
The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:



AndExists

Observation

The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:



AndExists

Observation

The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:

- **The outer accumulating Reduce sweep of the Exists can do double-duty as the Reduce sweep of the preceeding And operation.**

This saves $\Theta(\text{sort}(N))$ time and I/Os.

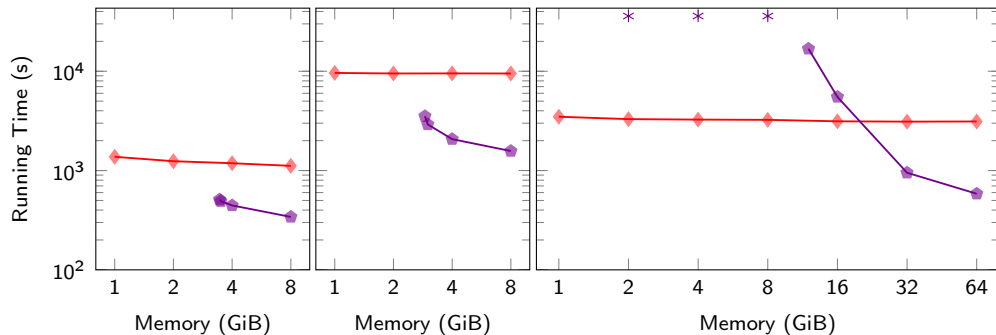
AndExists

Observation

The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:

- **The outer accumulating Reduce sweep of the Exists can do double-duty as the Reduce sweep of the preceeding And operation.**
This saves $\Theta(\text{sort}(N))$ time and I/Os.
- **The And operation can prune subtrees that trivially will become redundant during the succeeding Exists.**
This can save up to $\mathcal{O}(\text{sort}(N^{2^k}))$ time and I/Os.
In practice, this only saves up to $\mathcal{O}(\text{sort}(N))$ time and I/Os.

Experiment: $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$



(a) GPUForwardProgress 20a

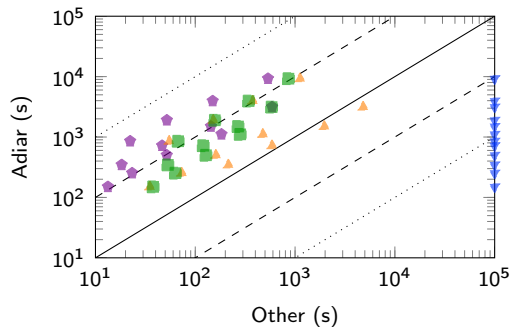
(b) SmartHome 16

(c) ShieldPPPs 10a

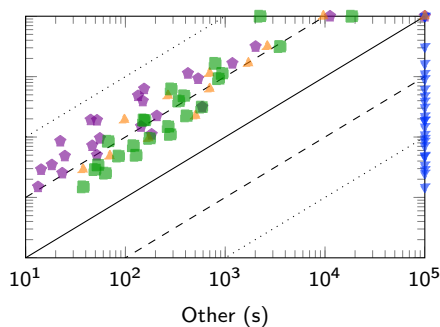
—◆— Adiar —■— BuDDy

Relational Product for MCC models with a 2^{25} state space BDD. Timeouts are marked as stars.

Experiment: $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$ & $\text{Prev}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$



(a) $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$

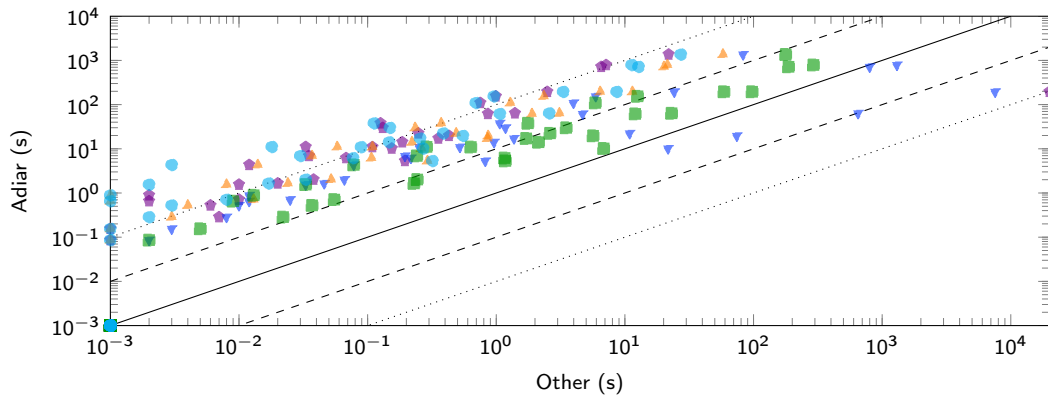


(b) $\text{Prev}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$

◆ BuDDy ▼ CAL ▲ CUDD ■ LibBDD

Relational Product for MCC models, 384 GiB of memory, and $2^{22}, \dots, 2^{25}$ state space BDDs.

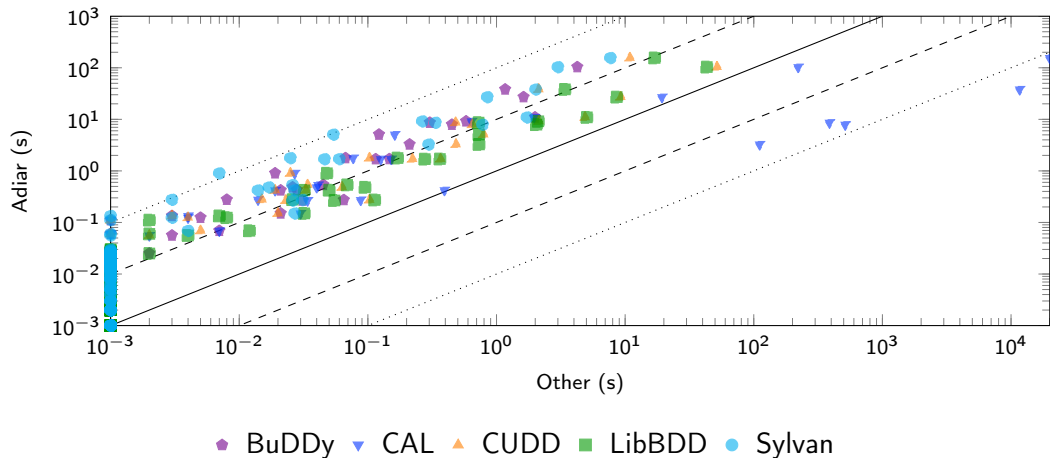
Experiment: Reachability



◆ BuDDy ▼ CAL ▲ CUDD ■ LibBDD ● Sylvan

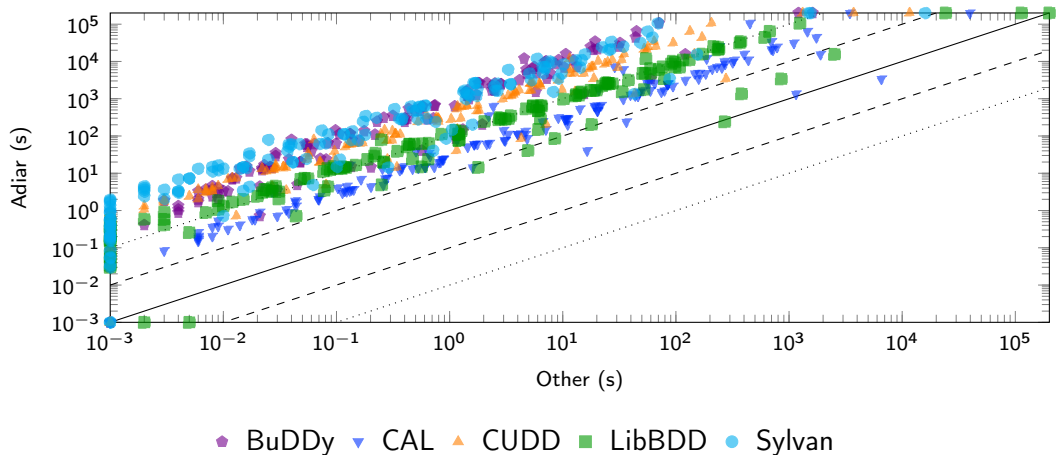
16 Petri Nets [MCC 21–23] with 384 GiB of memory.

Experiment: Deadlock Detection



16 Petri Nets [MCC 21–23] and 59 Boolean Networks [AEON, PyBoolNet] with 384 GiB RAM.

Experiment: SCC Decomposition



16 Petri Nets [MCC 21–23] and 59 Boolean Networks [AEON, PyBoolNet] with 384 GiB RAM.

Conclusions and Future Work

- To improve the I/O-efficient $Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$, focus on *AndExists*.
 - Factor of $\sim 2\times$ by using a *AndExists* instead of *And* and *Exists* for conventional depth-first implementations [1]. This may explain the sudden performance gap.
 - For larger instances, less than $\frac{1}{10}$ th of the time is spent on the *And*.

-
- [1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
 - [2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
 - [3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
 - [4] Brand et al.: *A Decision Diagram Operation for Reachability*. (2023).
 - [5] Marmorstein & Siminiceanu: *The Saturation algorithm for Symbolic State-space Exploration*. (2006).

Conclusions and Future Work

- To improve the I/O-efficient $Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$, focus on *AndExists*.
 - Factor of $\sim 2\times$ by using a *AndExists* instead of *And* and *Exists* for conventional depth-first implementations [1]. This may explain the sudden performance gap.
 - For larger instances, less than $\frac{1}{10}$ th of the time is spent on the *And*.
- Deal with small BDDs using Depth-first Recursion.

-
- [1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
 - [2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
 - [3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
 - [4] Brand et al.: *A Decision Diagram Operation for Reachability*. (2023).
 - [5] Marmorstein & Siminiceanu: *The Saturation algorithm for Symbolic State-space Exploration*. (2006).

Conclusions and Future Work

- To improve the I/O-efficient $Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$, focus on *AndExists*.
 - Factor of $\sim 2\times$ by using a *AndExists* instead of *And* and *Exists* for conventional depth-first implementations [1]. This may explain the sudden performance gap.
 - For larger instances, less than $\frac{1}{10}$ th of the time is spent on the *And*.
- Deal with small BDDs using Depth-first Recursion.
- Apply ideas from recent and more advanced BDD algorithms.

For example the ones in [2], [3], [4], and [5] .

-
- [1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
 - [2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
 - [3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
 - [4] Brand et al.: *A Decision Diagram Operation for Reachability*. (2023).
 - [5] Marmorstein & Siminiceanu: *The Saturation algorithm for Symbolic State-space Exploration*. (2006).

Conclusions and Future Work

- To improve the I/O-efficient $Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$, focus on *AndExists*.
 - Factor of $\sim 2\times$ by using a *AndExists* instead of *And* and *Exists* for conventional depth-first implementations [1]. This may explain the sudden performance gap.
 - For larger instances, less than $\frac{1}{10}$ th of the time is spent on the *And*.
- Deal with small BDDs using Depth-first Recursion.
- Apply ideas from recent and more advanced BDD algorithms.
For example the ones in [2], [3], [4], and [5] .
- Design a $Replace(\pi)$ for Non-monotone Variable Substitutions.

-
- [1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
 - [2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
 - [3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
 - [4] Brand et al.: *A Decision Diagram Operation for Reachability*. (2023).
 - [5] Marmorstein & Siminiceanu: *The Saturation algorithm for Symbolic State-space Exploration*. (2006).

Steffan Christ Sølvesten

✉ soelvsten@cs.au.dk

🌐 ssoelvsten.github.io

Adiar

🔗 github.com/ssoelvsten/adiar

📖 ssoelvsten.github.io/adiar

