

# Multi-variable Quantification of BDDs in External Memory using Nested Sweeping

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**Steffan Christ Sølvesten**, Jaco van de Pol

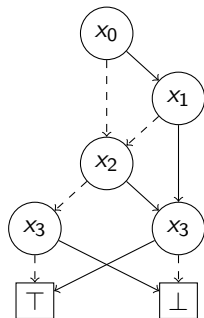
TACAS 2025





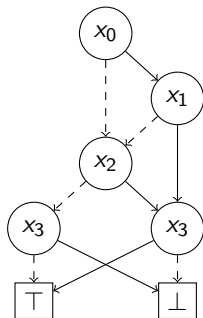
1986

# Binary Decision Diagrams



**(a)**  $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

# Binary Decision Diagrams

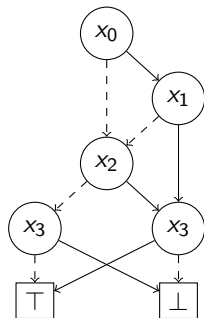


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## Theorem (Bryant '86)

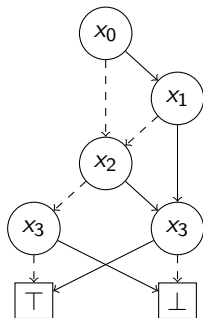
*Given a fixed variable order, a (Reduced Ordered) Binary Decision Diagram is a unique canonical representation of a Boolean function.*

# Binary Decision Diagrams



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## Theorem (Bryant '86)

Given BDDs  $\phi$  and  $\psi$ ,  $\phi \odot \psi$  is computable in  $\mathcal{O}(|\phi| \cdot |\psi|)$  time.

## Theorem (Bryant '86)

Given BDD  $\phi$  and Boolean  $b$ ,  $\phi[x_i \mapsto b]$  is computable in  $\mathcal{O}(|\phi|)$  time.

## Corollary

Given BDD  $\phi$ ,  $\exists x_i. \phi(x)$  requires  $\mathcal{O}(|\phi|^2)$  time.

## Proof.

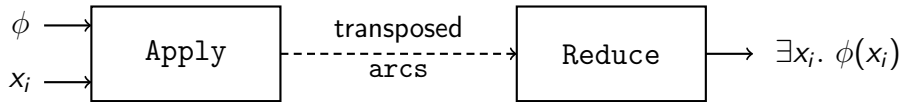
$$\exists x_i. \phi(x_i) \equiv \phi[x_i \mapsto \perp] \vee \phi[x_i \mapsto \top]$$

□



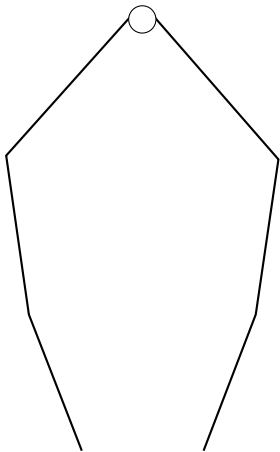
1996

$$\exists x_i. \phi(x_i)$$

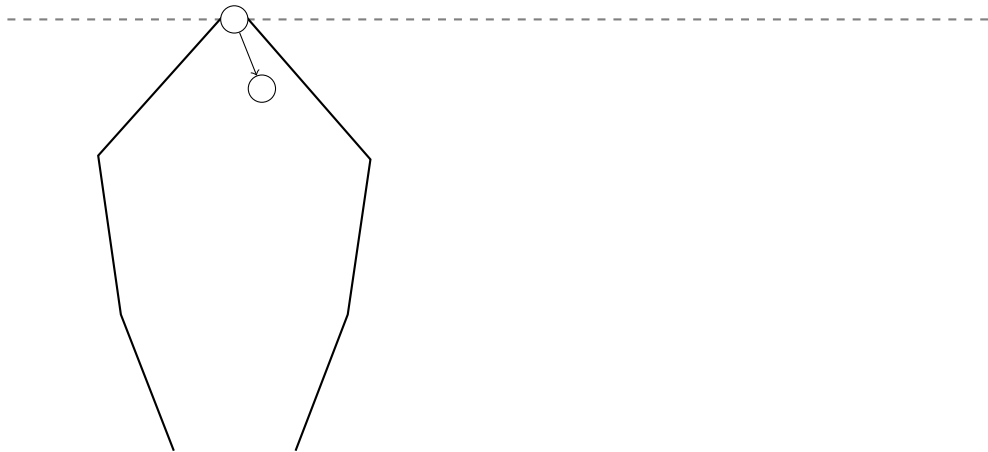




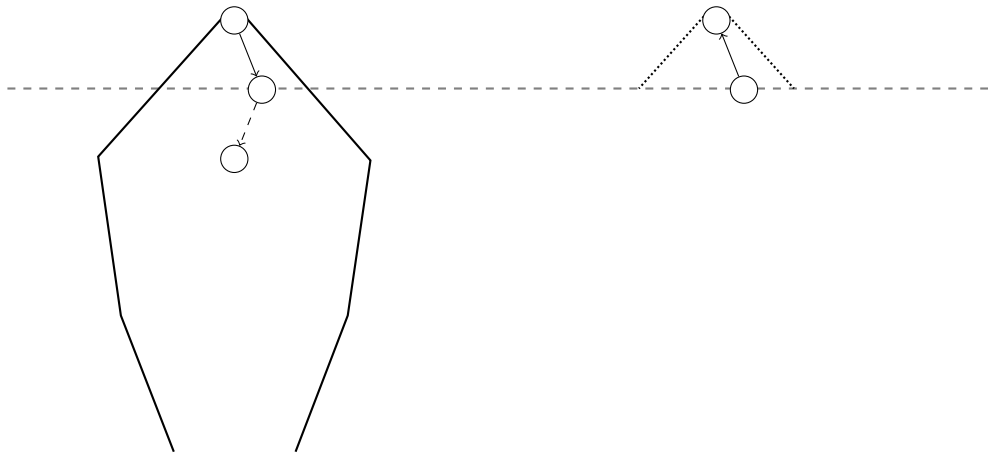
$\exists x_i. \phi(x_i) : \text{Apply}$



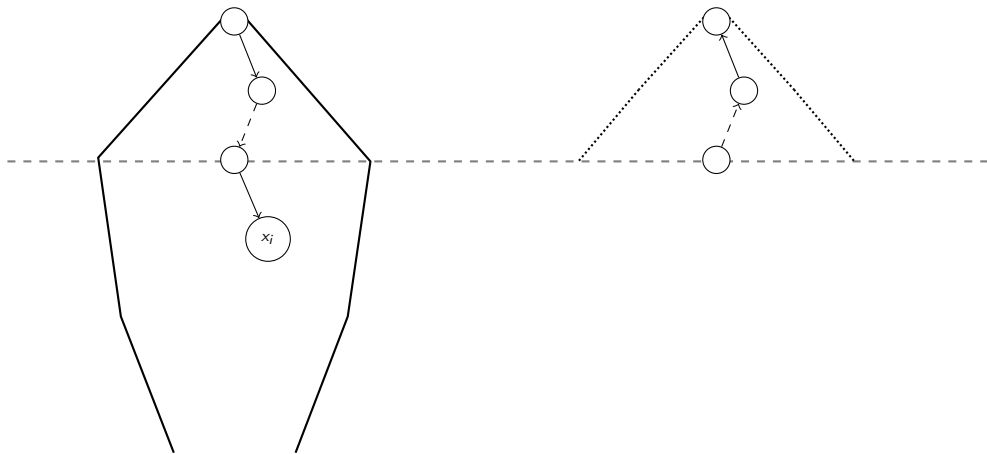
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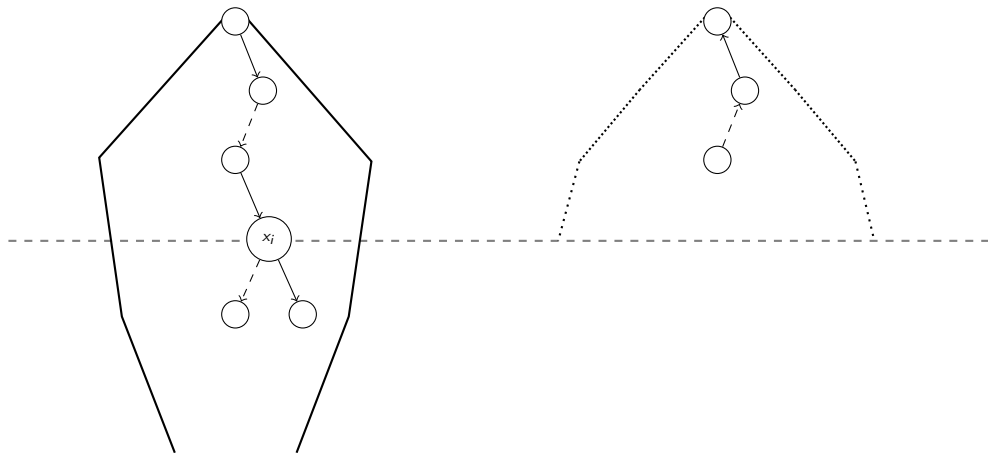
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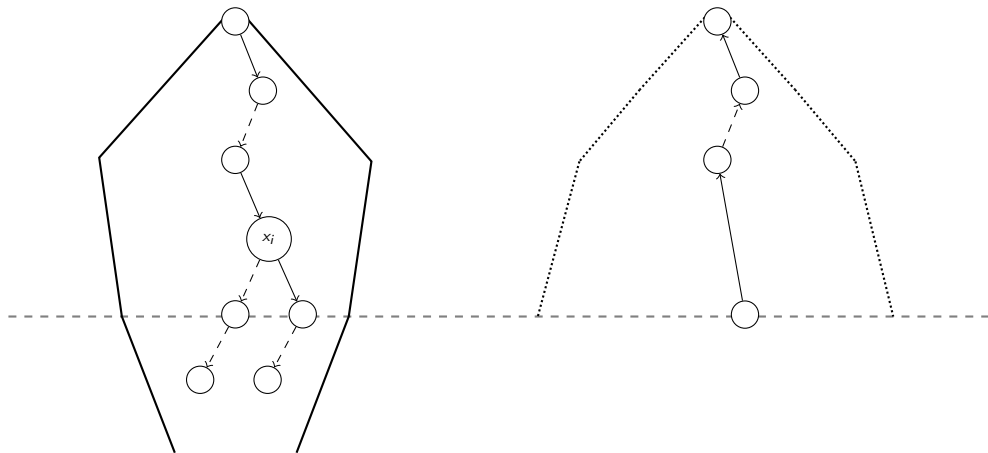
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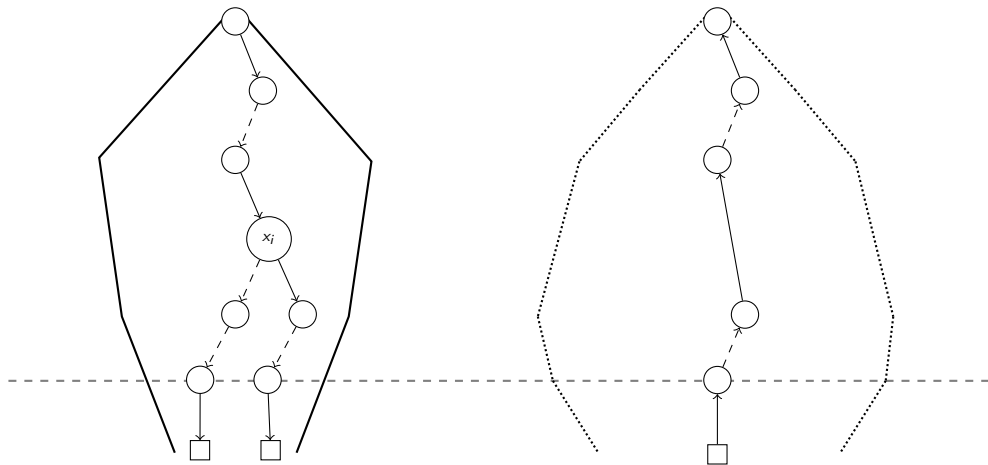
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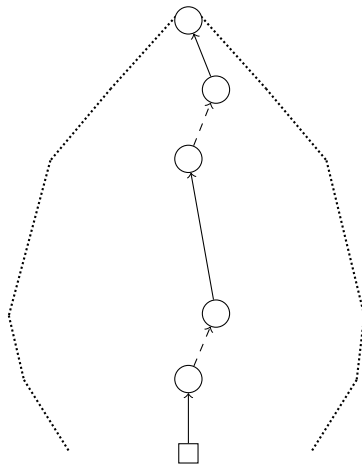
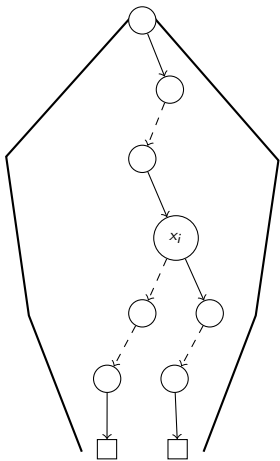
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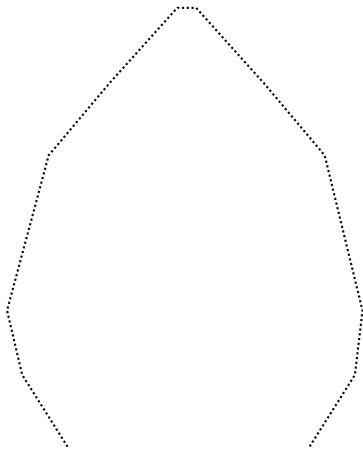


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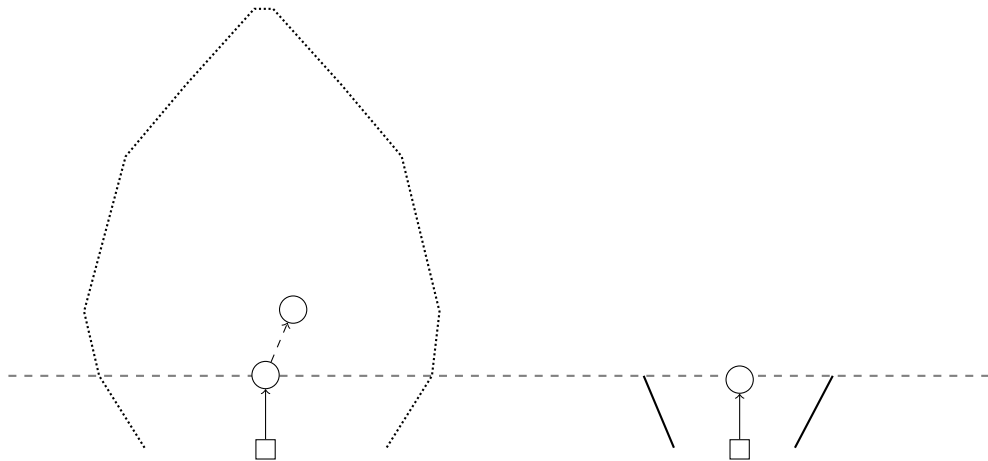




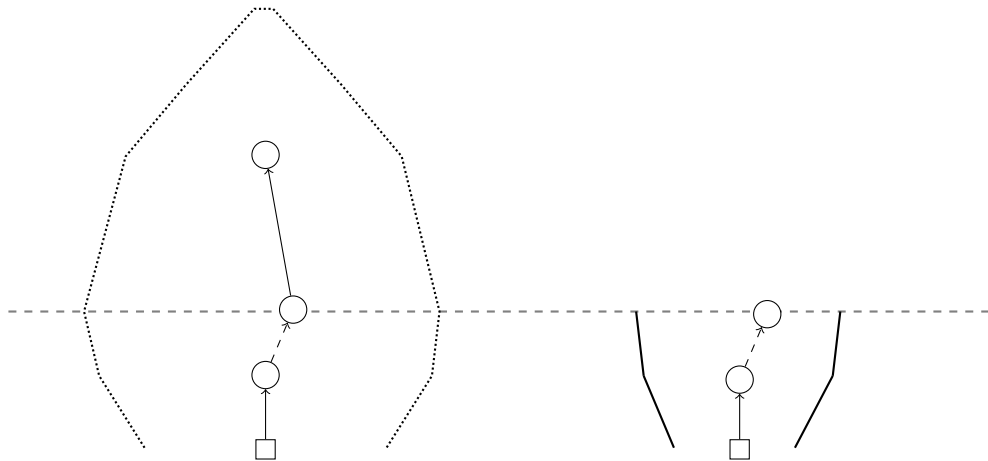
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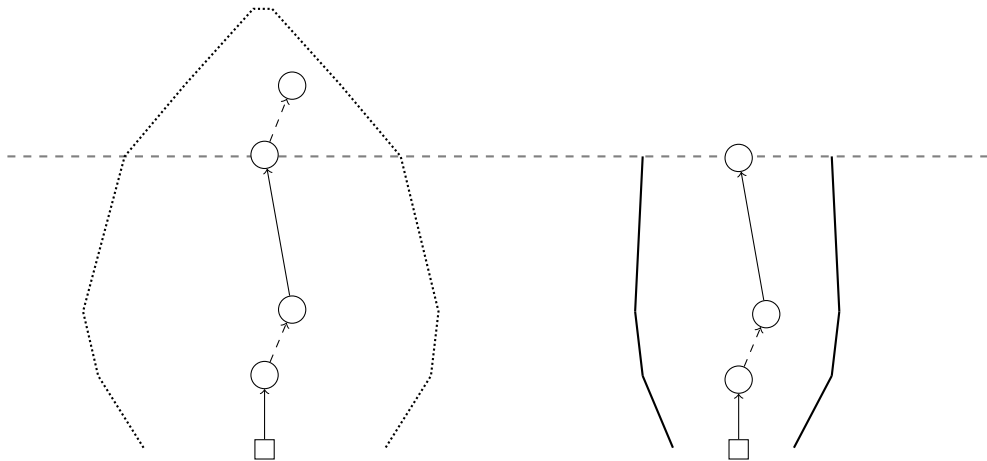
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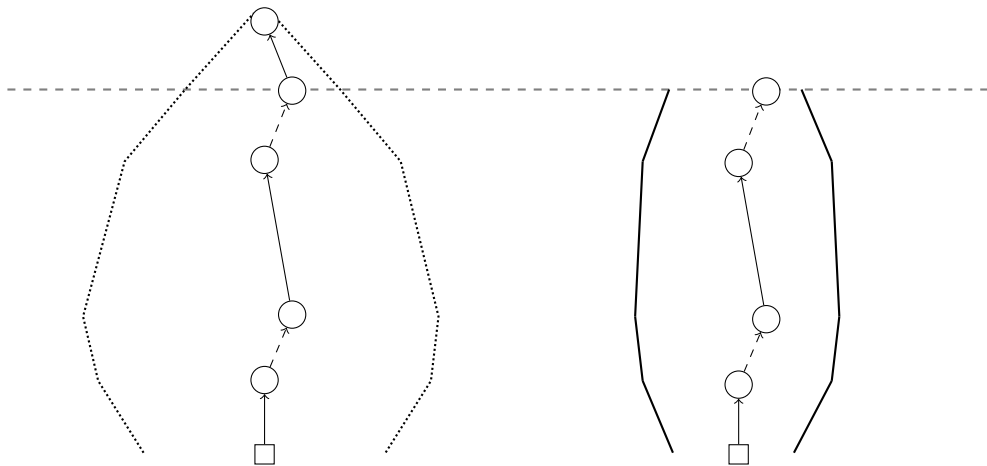
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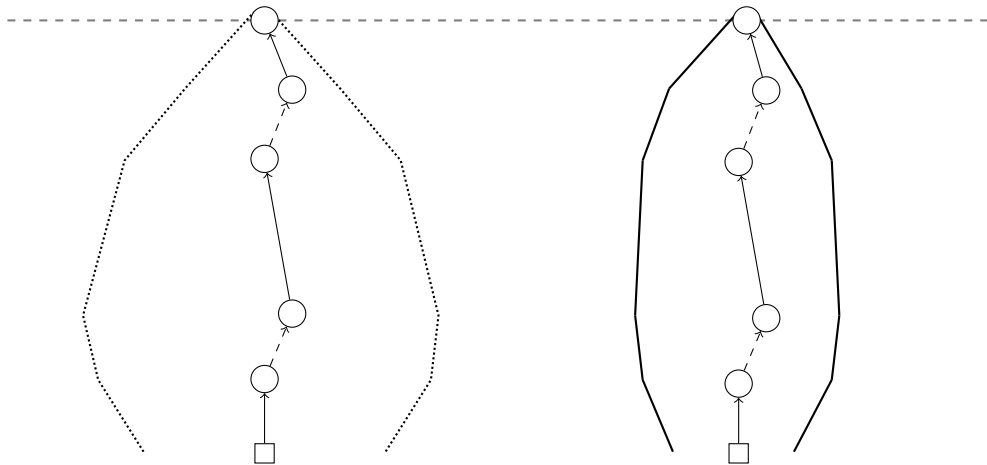
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**Theorem (Lars Arge '96)**

*Given BDDs  $\phi$  and  $\psi$ ,  $\phi \odot \psi$  is computable in  $\mathcal{O}(\text{sort}(|\phi| \cdot |\psi|))$  time and I/Os.*

**Theorem (Sølvsten et al. '22)**

*Given BDD  $\phi$  and Boolean  $b$ ,  $\phi[x_i \mapsto b]$  is computable in  $\mathcal{O}(\text{sort}(|\phi|))$  time and I/Os.*

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**Corollary (Sølvsten et al. '22)**

*Given BDD  $\phi$ , the time and I/O complexity of quantification is*

- $\mathcal{O}(\text{sort}(|\phi|^2))$  for a single variable.
- $\mathcal{O}(\text{sort}(|\phi|^{2^k}))$  for  $k$  variables.





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# Adiar

*I/O-efficient Decision Diagrams*

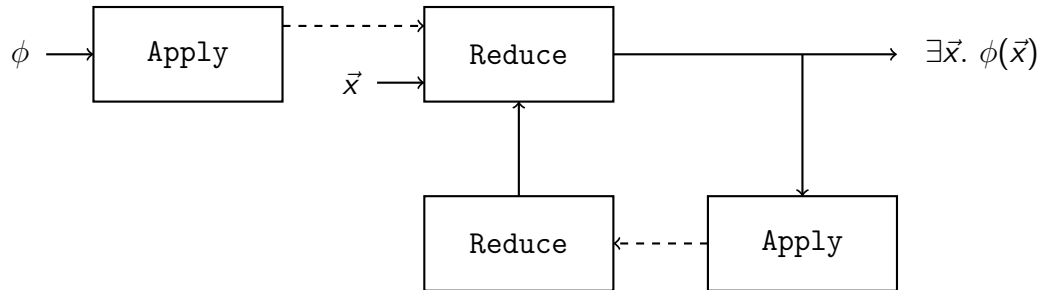
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[github.com/ssoelvsten/adiar](https://github.com/ssoelvsten/adiar)

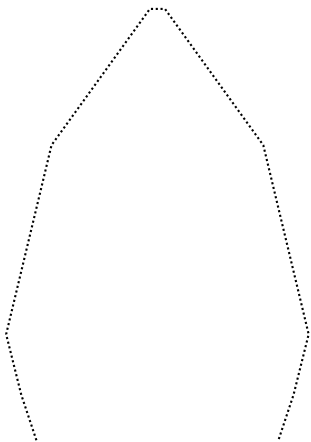


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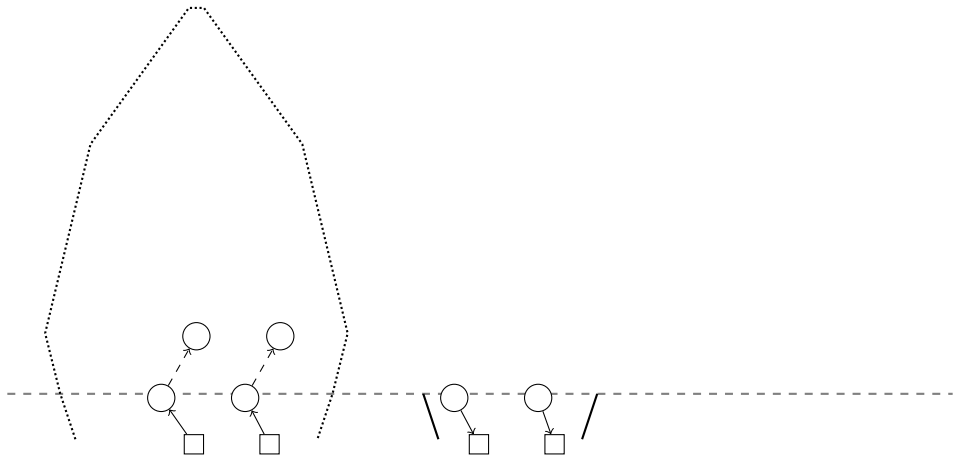
$\exists \vec{x}. \phi(\vec{x})$



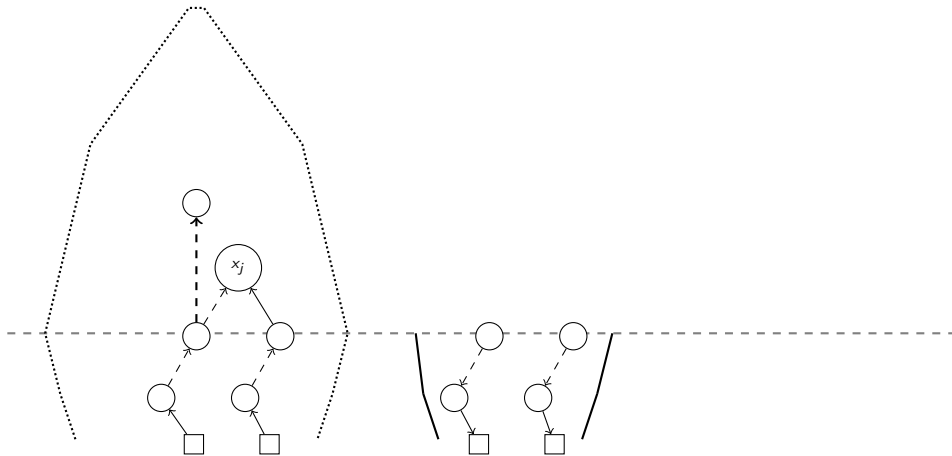
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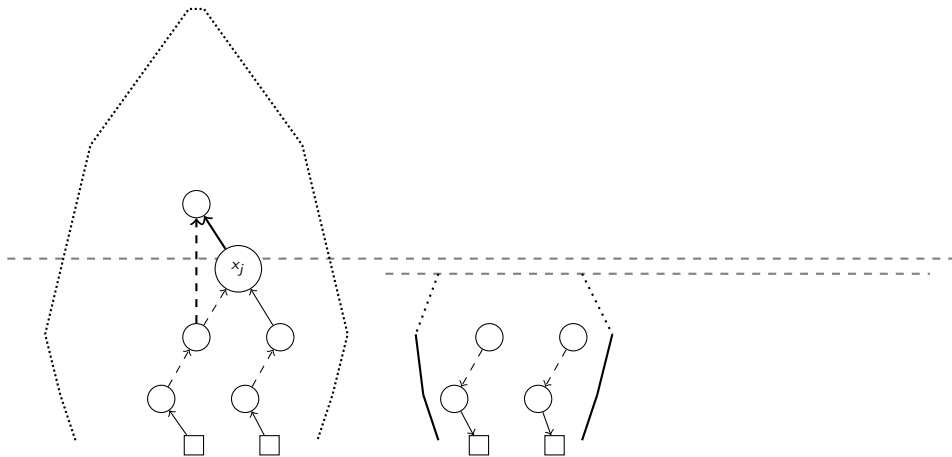
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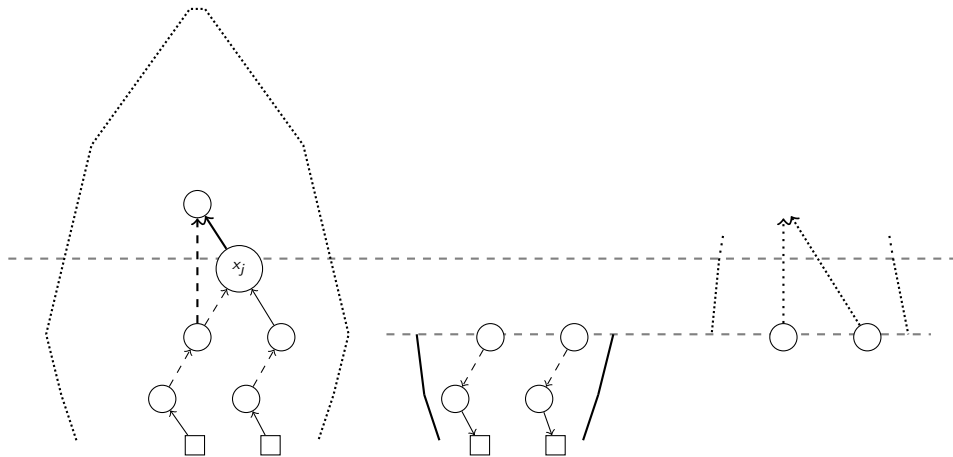


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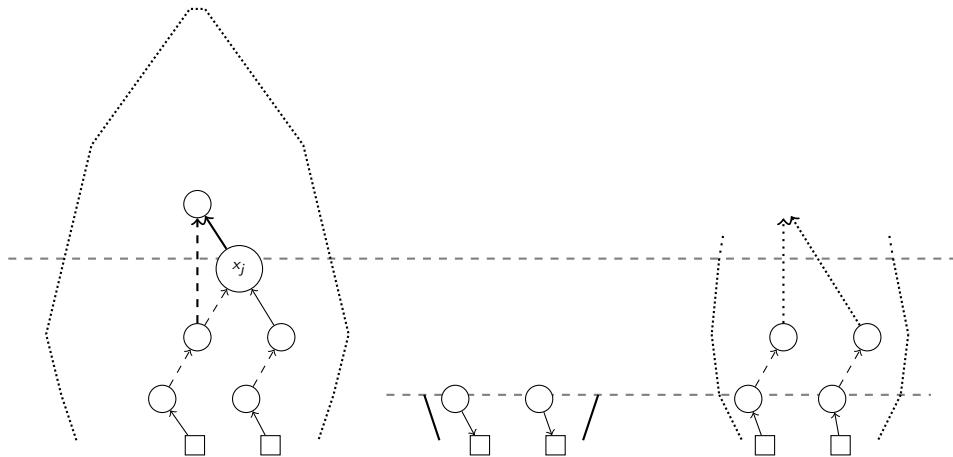




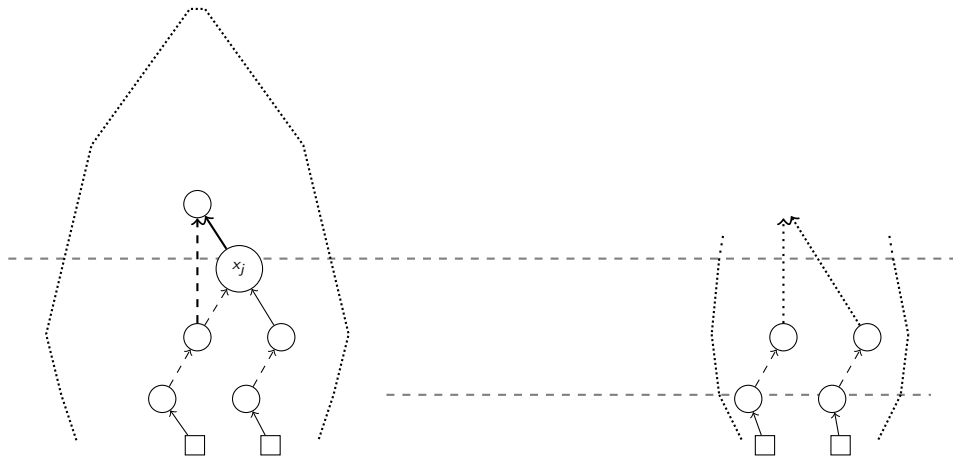
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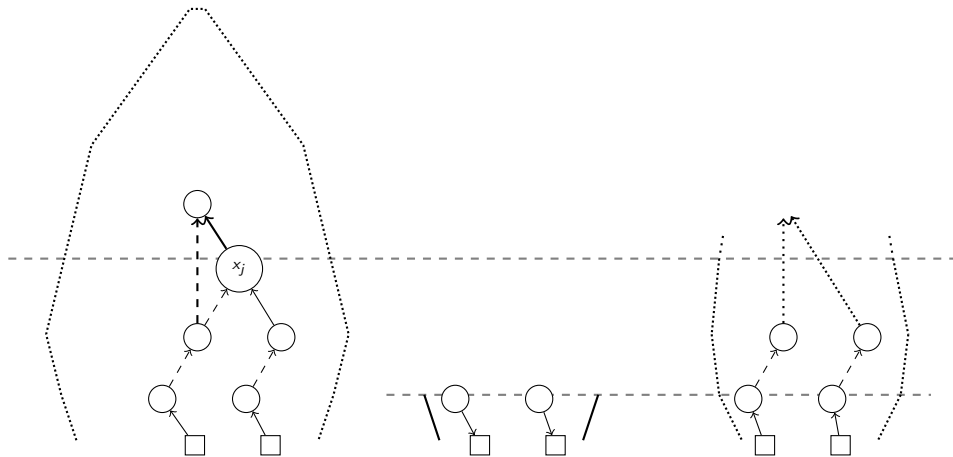
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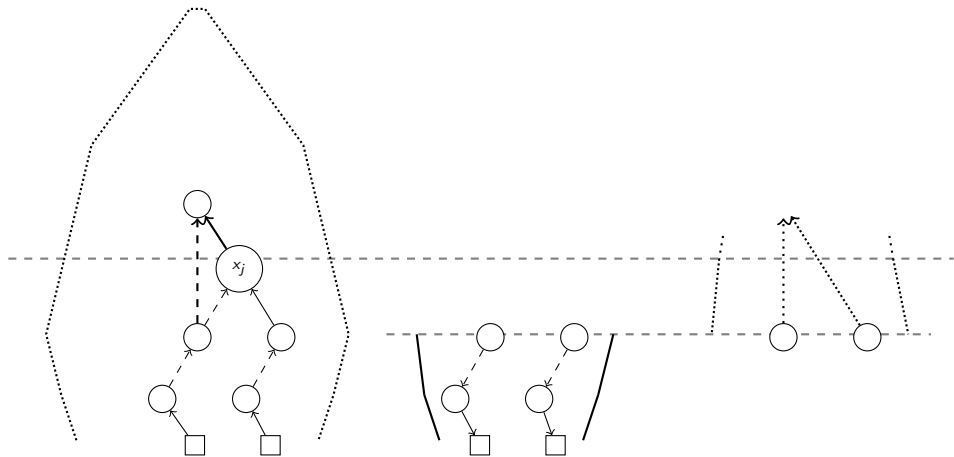
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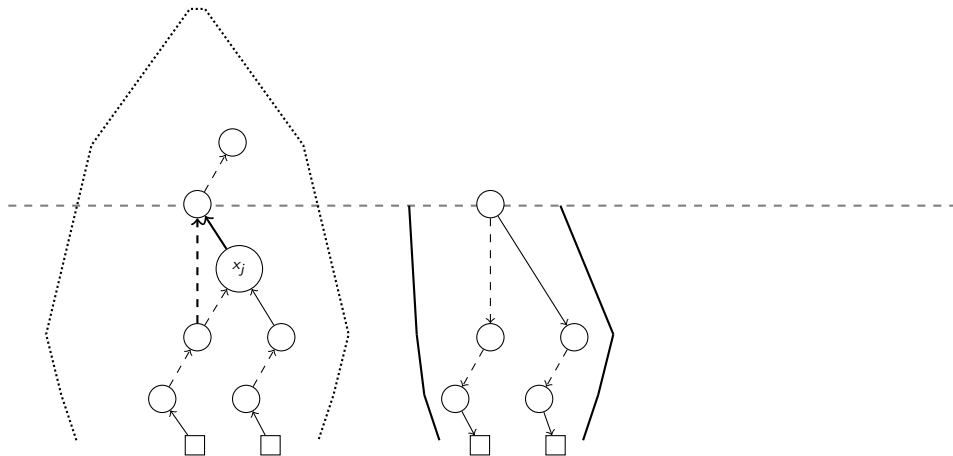
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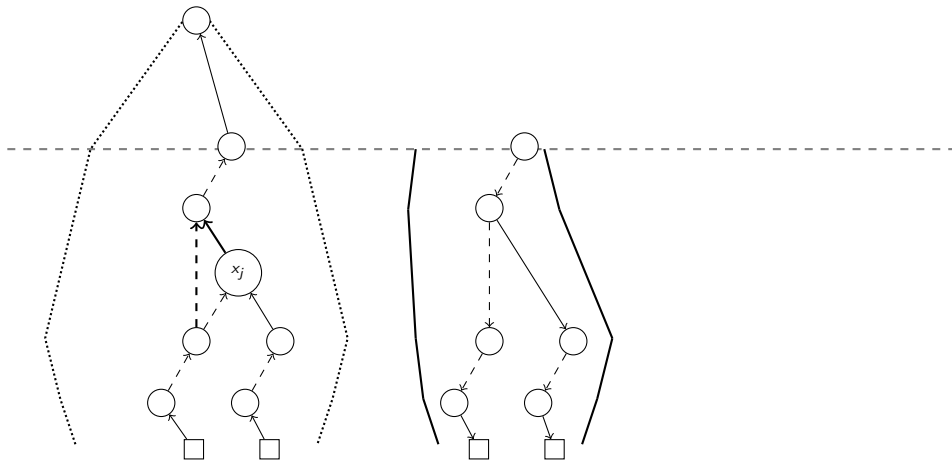
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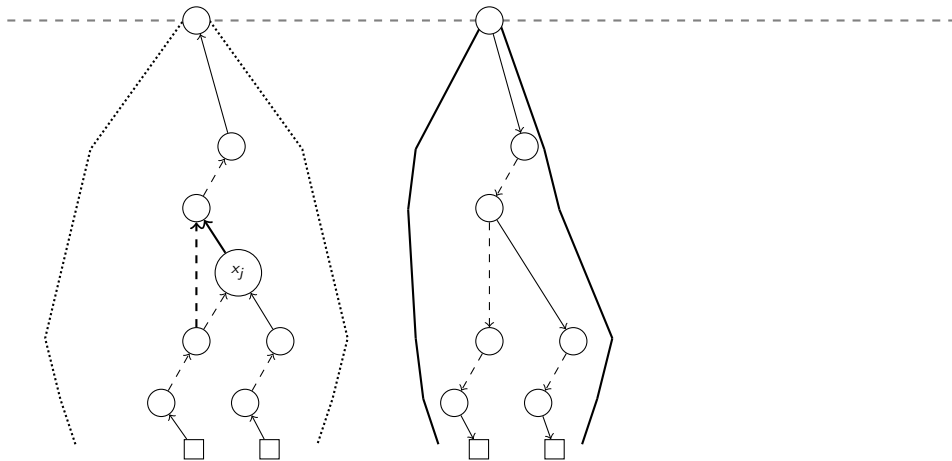
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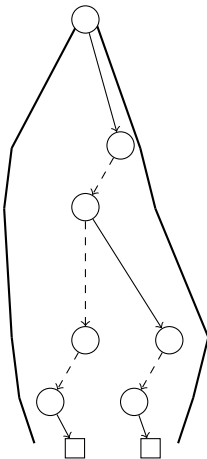
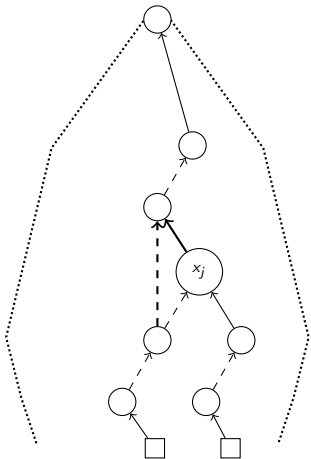
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## Theorem (Sølvsten et al. '25)

Given BDD  $\phi$ , the quantification of  $k$  variables,  $\exists \vec{x}. \phi(\vec{x})$ , is computable in  $\mathcal{O}(\text{sort}(|\phi|^{2^k}))$  time and I/Os.

# Benchmarks

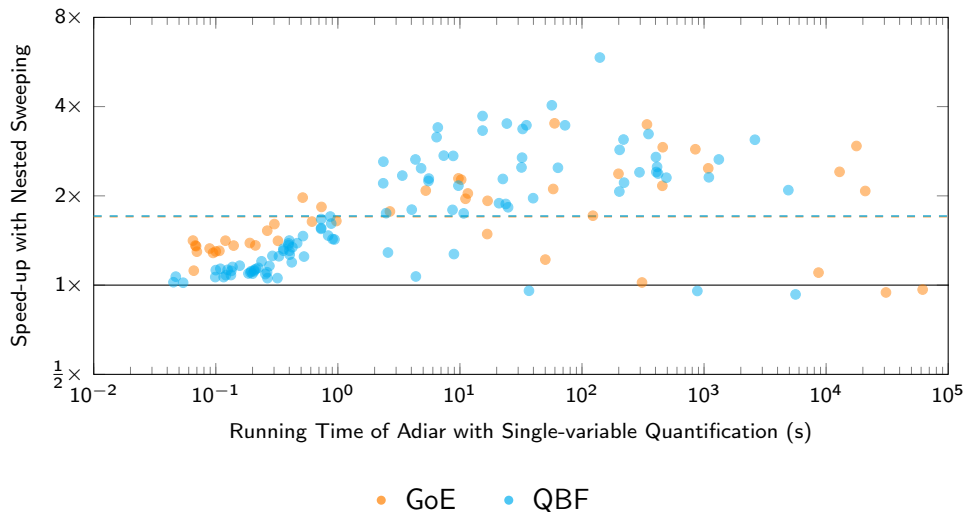
## Garden-of-Eden

Given dimensions  $N_1, N_2 \in \mathbb{N}$ , determine whether there exists in Conway's *Game of Life* an initial state of size  $N_1 \times N_2$  that is a *Garden of Eden*, i.e. is otherwise unreachable.

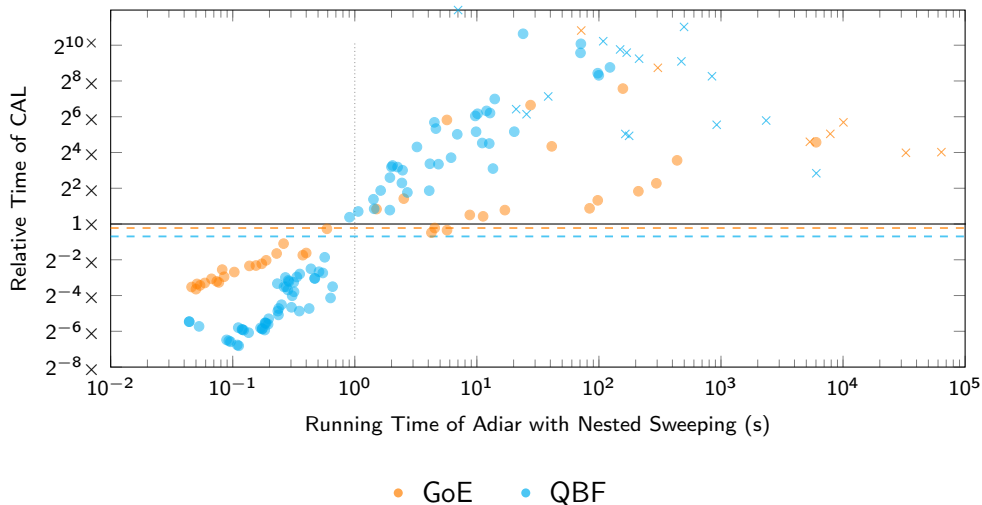
## Quantified Boolean Formula

Determine whether a Boolean formula  $\exists \vec{x}_1 \forall \vec{x}_2 \dots \exists \vec{x}_k. \phi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k)$  (or any order of quantifiers) evaluates to  $\top$  or  $\perp$ . For inputs, we use the two-player games from: Irfansha Shaik and Jaco van de Pol: “*Concise QBF encodings for games on a grid (extended version)*”. arXiv (2023).

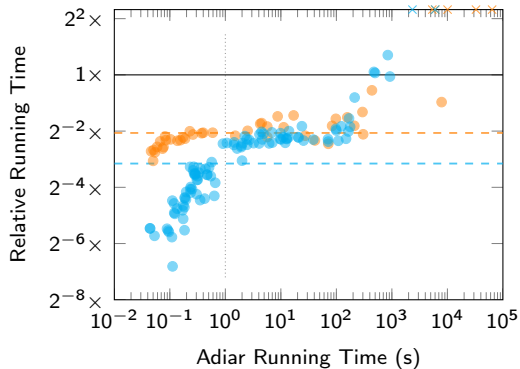
## Benchmarks : Single vs. Nested Quantification



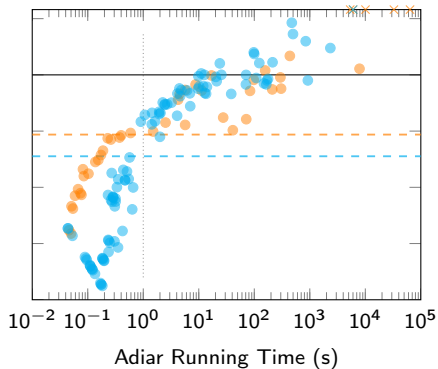
## Benchmarks : Comparison to CAL



# Benchmarks : Comparison to BuDDy and CUDD



(a) BuDDy



(b) CUDD

● GoE    ● QBF

## Steffan Christ Sølvesten

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🌐 [ssoelvsten.github.io](https://ssoelvsten.github.io)

## Adiar

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📄 [github.com/ssoelvsten/adiar](https://github.com/ssoelvsten/adiar)

📖 [ssoelvsten.github.io/adiar](https://ssoelvsten.github.io/adiar)



## Nested Sweeping Framework

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New BDD algorithms:

- ✓ Multi-variable Quantification
- ✓ Relational Product
- Functional Composition
- Variable Reordering

Other Decision Diagrams:

- Quantum Multi-valued Decision Diagrams
- Polynomial Boolean Rings





## Better Transposition: Deepest Variable Quantification

$$bdd\_exists(f, \max(\vec{x}))$$

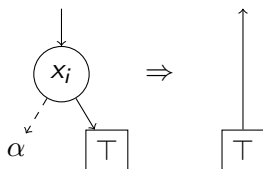
### Motivation

Includes the first nested sweep inside of the transposition step.

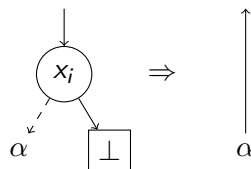
### In Practice

Slows down computation time on average by 4.7%.

## Better Transposition: Pruning $\top$ Siblings



(a) Pruning due to  $\top$ .

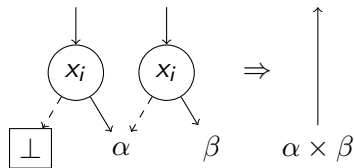


(b) Pruning due to  $\perp$ .

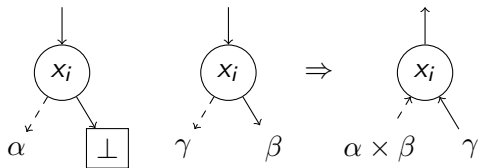
### In Practice

If it prunes subtrees, running time can improve up to 21%. Otherwise, it adds an overhead of up to 2%.

## Better Transposition: Partial Quantification



(a) Fully quantified pair of nodes.



(b) Partially quantified pair of nodes.

### In Practice

In some instances, it improves performance by a factor of  $\sim 2\times$ .

In others, it slows down by  $\sim 2\times$ .

### Observation

Instances improved by  $\top$  Pruning were disjoint from *Partial Quantification*.



# Optimisations for Nested Sweeping

- **Leave terminal arcs out of nested sweeps.**

Required to satisfy invariants in nested algorithms from [TACAS 22].

- **Bail-out of Inner Sweep if all edges are subtree-preserving.**

In practice, 75.6% of all levels are skippable (median of 81.9% for each benchmark).

- **Use a sorted list as a bridge from the outer to the inner sweep.**

Postpones initialising data structures for the nested sweep until it is invoked.

- More memory available for the outer Reduce sweep.
- Sorting once and then merging on-the-fly with a priority queue can be faster than maintaining a larger priority queue.
- This enables the *levelised priority queues* [TACAS\* 22], *levelised cuts* [ATVA 23], and *levelised random access* [SPIN 24] optimisations.



	Single Quantification		Nested Sweeping	
	LOC	# Tests	LOC	# Tests
nested_sweeping.h	–	–	1287	104
quantify	548	84	1152	152
core/...	326	–	904	–
bdd/...	122	64	157	108
zdd/...	100	20	20	44





