

I/O-efficient Symbolic Model Checking

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Adiar

github.com/ssoelvsten/adiar





Aggarwal and Vitter '87 : I/O-model



The I/O-model by Aggarwal and Vitter '87

Aggarwal and Vitter '87 : I/O-model

For any realistic values of N , M , and B we have that

$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

Theorem (Aggarwal and Vitter '87)

N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.

Bryant '86 : Binary Decision Diagram



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 \wedge x_2)$

Examples of (Reduced Ordered) Binary Decision Diagrams.

Bryant '86 : Binary Decision Diagram

Theorem

For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



(2) Merge duplicate nodes

Arge '95 : BDD + I/O-model



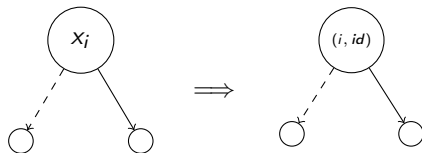
Running time for solving a problem that does not need more than 3 GiB.



Adiar v1.0 : BDD



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$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

Adiar v1.0 : BDD



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(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



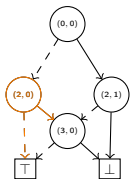
]

(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:
 $(0, 0) \xrightarrow{\top} (1, 0)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

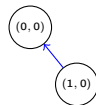
Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app}:1$:

[
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:
 $(0, 0) \xrightarrow{\perp} (2, 0)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 1))$

Priority Queue: $Q_{app}:1$:

[

$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

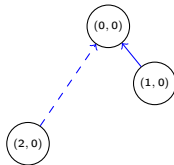
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad ((3, 0), (3, 1))$,

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:

$\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:
 $(2, 1) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

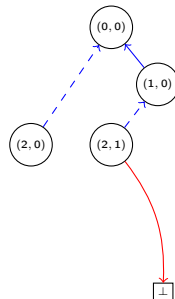
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(1, 0) \xrightarrow{\perp} (2, 1)$

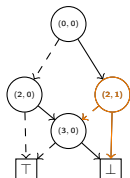


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

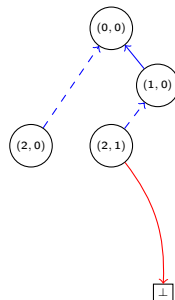
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

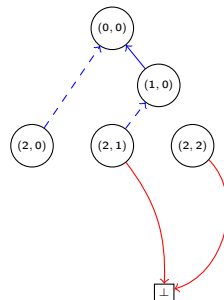
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(2, 2) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

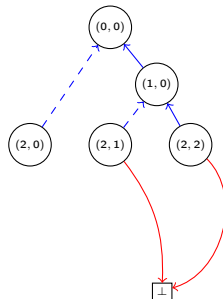
$(2, 1) \stackrel{\perp}{\rightarrow} ((3, 0), (3, 0))$,
 $(2, 0) \stackrel{\top}{\rightarrow} ((3, 1), (3, 0))$,
 $(2, 2) \stackrel{\perp}{\rightarrow} ((3, 1), (3, 0))$,
 $(2, 0) \stackrel{\perp}{\rightarrow} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(1, 0) \stackrel{\top}{\rightarrow} (2, 2)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3) \wedge (x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

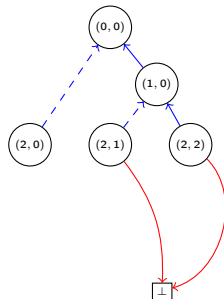
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$

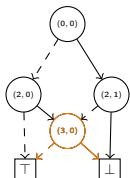


(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

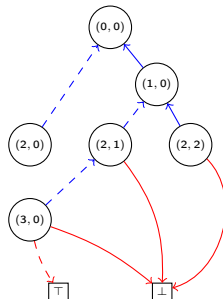
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

Output:
 $(2, 1) \xrightarrow{\perp} (3, 0)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

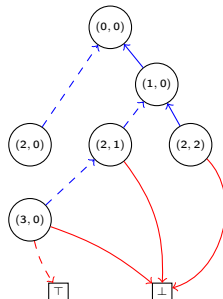
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \text{ ? } x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

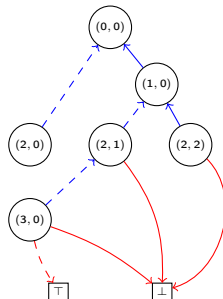
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \text{ ? } x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

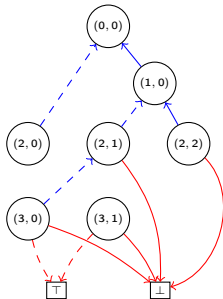
$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:

$(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

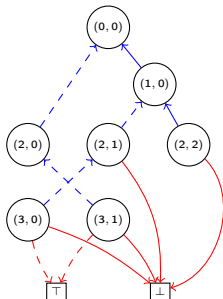
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(2, 0) \xrightarrow{\perp} (3, 1)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

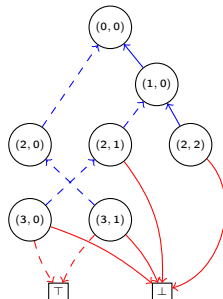
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

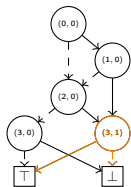
]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

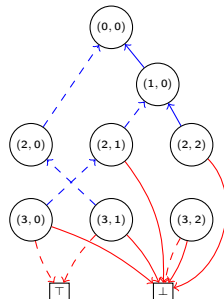
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

]

Output:

$(2, 0) \xrightarrow{T} (3, 2), (2, 2) \xrightarrow{F} (3, 2)$



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

Priority Queue: $Q_{app:2}$:

[

]

]

Output:



(c) $(a) \wedge (b)$

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD

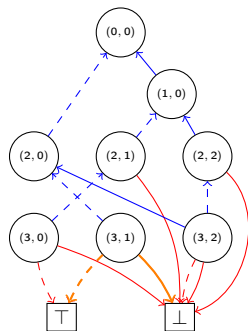


(c) $(a) \wedge (b)$

[Level: 3
 $[(3, 2) \mapsto \perp]$]

(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

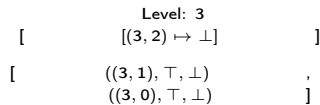
Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad]$

(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad [(3, 1) \mapsto (3, \max)] \quad , \quad]$
 $\quad \quad \quad ((3, 0), \top, \perp)$

Output:
 $((3, \max), \top, \perp)$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad [(3, 1) \mapsto (3, \max)] \quad , \quad [(3, 0) \mapsto (3, \max)] \quad]$

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

]

Level: 3

[

[$(3, 1) \mapsto (3, \max)$] ,

[$(3, 0) \mapsto (3, \max)$]]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{T} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

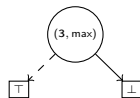
Level: 3

[

[$(3, 0) \mapsto (3, \max)$] ,

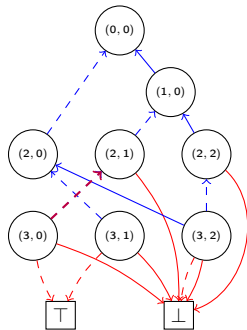
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

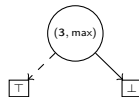
Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

Level: 3

[
 [
 ,
]
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



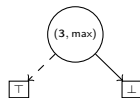
(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{T} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

[
 Level: 2
 $[(2, 2) \mapsto \perp]$
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

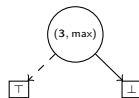
[

$((2, 1), (3, \max), \perp)$

,

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

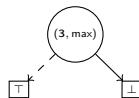
$((2, 1), (3, \max), \perp)$

,

$((2, 0), (3, \max), \perp)$

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

,

$((2, 0), (3, \max), \perp)$

]

Output:
 $((2, \max), (3, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

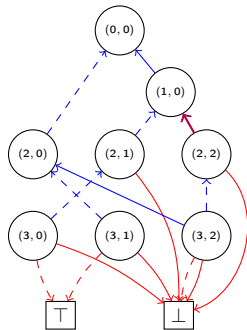
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{\top} \perp$,

]

Level: 2

[

]

[

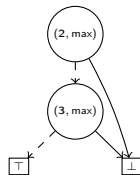
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{\top} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

]

[

$[(2, 0) \mapsto (2, \max)]$

,

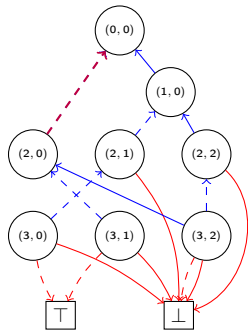
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{\top} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

[

[

]

,
]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

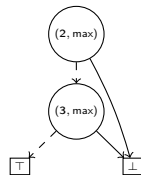
]

[

$((1, 0), (2, \max), \perp)$

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

]

[

$[(1, 0) \mapsto (1, \max)]$

]

Output:
 $((1, \max), (2, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\top} (1, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

]

[

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

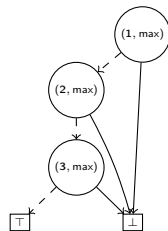
Level: 0

[

]

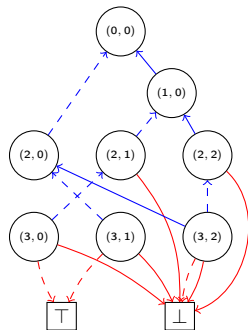
[$((0, 0), (2, \max), (1, \max))$]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

[

]

$[(0, 0) \mapsto (0, \max)]$

]

Output:
 $((0, \max), (2, \max), (1, \max))$



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

[

]

[

]

Output:



(d) $(a) \wedge (b)$ reduced

Adiar v1.0 : BDD



—●— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimal running time for the *Queens* problems.

Adiar v1.0 : BDD



—●— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimal running time for the *Queens* problems.

Adiar v1.0 : BDD



—●— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimal running time for the *Queens* problems.

Adiar v1.0 : BDD



Minimal running time for the *Queens* problems.

Adiar v1.2 : Internal Memory



Minimal running time for the *Queens* problems.

Adiar v1.2 : Internal Memory



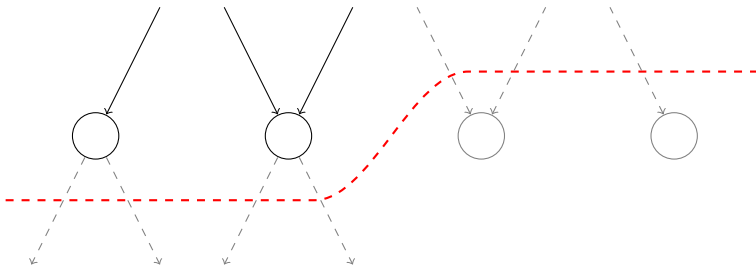
Adiar v1.2 : Internal Memory



Adiar v1.2 : Internal Memory



Adiar v1.2 : Internal Memory



Adiar v1.2 : Internal Memory



Adiar v1.2 : Internal Memory



Adiar v1.2 : Internal Memory

Definition (i-level cut)

\mathcal{L}



Adiar v1.2 : Internal Memory

Definition (i-level cut)

\mathcal{L}



Lemma

The maximum i -level cut problem is in P for $i \in \{1, 2\}$.

Adiar v1.2 : Internal Memory

Definition (i-level cut)

\mathcal{L}



Lemma

The maximum i -level cut problem is in P for $i \in \{1, 2\}$.

Theorem (Lampis, Kaouri, Mitsou 2011)

The maximum i -level cut problem is NP-complete for $i \geq 4$.

Adiar v1.2 : Internal Memory

Theorem

The maximum (i -level) cut of a BDD with N internal nodes is $N + 1$.

Adiar v1.2 : Internal Memory

Theorem

The maximum (i -level) cut of a BDD with N internal nodes is $N + 1$.

Theorem

For $i \in \{1, 2\}$, the maximum i -level cut of the (unreduced) output of Apply is upper bounded by the product of the inputs' corresponding i -level cuts.

Adiar v1.2 : Internal Memory

Theorem

The maximum (i -level) cut of a BDD with N internal nodes is $N + 1$.

Theorem

For $i \in \{1, 2\}$, the maximum i -level cut of the (unreduced) output of Apply is upper bounded by the product of the inputs' corresponding i -level cuts.

Lemma

The maximum 2-level cut of a BDD is upper bounded by $\frac{3}{2}$ its 1-level cut.



$$RelProd(S, T) \equiv (\exists \vec{x}. S(\vec{x}) \wedge T(\vec{x}, \vec{x}')) [\vec{x}' / \vec{x}]$$

$$RelProd(S, T) \equiv (\exists \vec{x}. S(\vec{x}) \wedge T(\vec{x}, \vec{x}')) [\vec{x}' / \vec{x}]$$

Adiar v2.0 : Multi-variable Quantification

```
1 exists(f, V)
2   if f =  $\perp$   $\vee$  f =  $\top$ 
3       then f
4   else if  $V \cap \{i \in \mathbb{N} \mid i \geq \text{top}(f)\} = \emptyset$ 
5       then f
6   else if top(f)  $\notin V$ 
7       then Node { top(f), exists(f.low, V), exists(f.high, V) }
8   else let low  = exists(f.low, V)
9         high = exists(f.high, V)
10      in or(low, high)
```

A recursive multi-variable **exists** operation.

Adiar v2.0 : Multi-variable Quantification

```
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2   if f =  $\perp$   $\vee$  f =  $\top$ 
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5     then f
6   else if  $\text{top}(f) \notin V$ 
7     then Node {  $\text{top}(f)$ , exists(f.low, V), exists(f.high, V) }
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A recursive multi-variable **exists** operation.

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```

A recursive multi-variable **exists** operation.

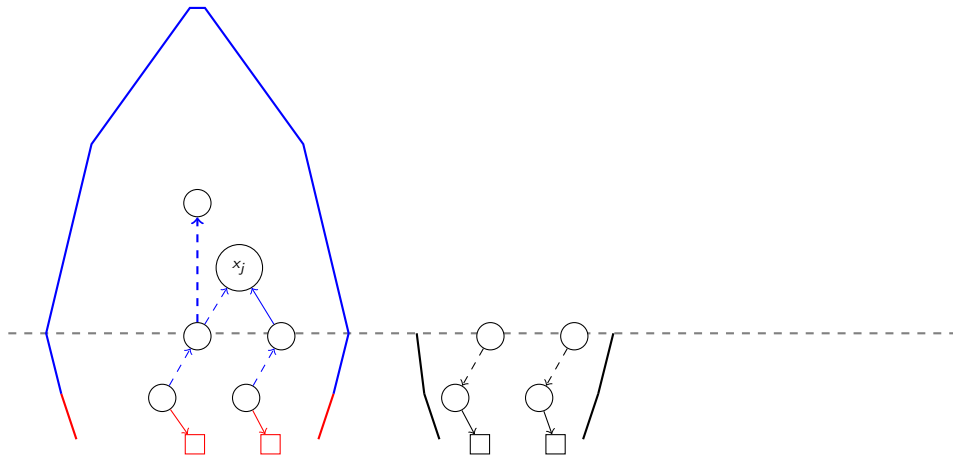
Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



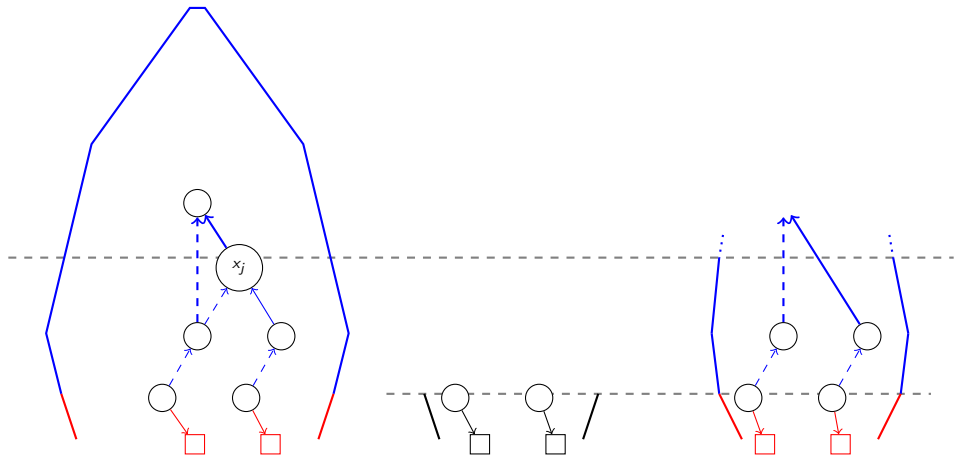
Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



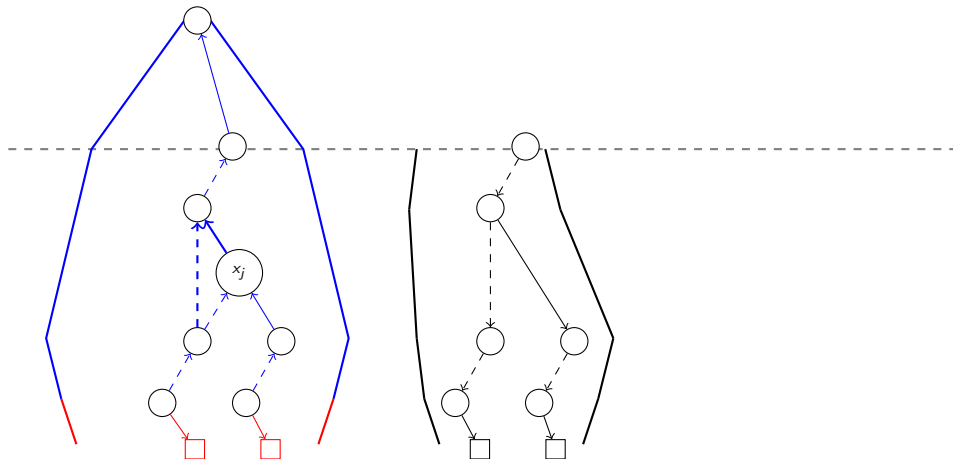
Adiar v2.0 : Multi-variable Quantification



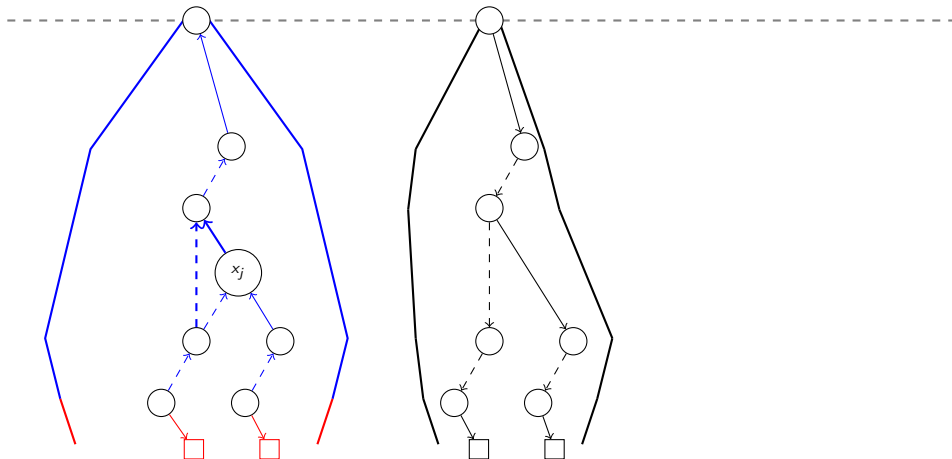
Adiar v2.0 : Multi-variable Quantification



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Adiar v2.0 : Multi-variable Quantification



Adiar v2.0 : Multi-variable Quantification



$$RelProd(S, T) \equiv (\exists \vec{x}. S(\vec{x}) \wedge T(\vec{x}, \vec{x}')) [\vec{x}' / \vec{x}]$$

Adiar v2.1 : Variable Reordering

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3     high = substitute(f.high)
4     i'    = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$



$\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Adiar v2.1 : Variable Reordering

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
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Adiar v2.1 : Variable Reordering

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$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$



$\neg(x_0 ? x_2 \vee x_1 : x_2 \wedge x_1)$

Adiar v2.1 : Variable Reordering

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A recursive **substitute** operation.

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$



$\neg(x_0 ? x_2 \vee x_1 : x_2 \wedge x_1)$

Adiar v2.1 : Variable Reordering

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$

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3     high = substitute(f.high)
4     i'    = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.



...

Adiar v2.1 : Variable Reordering

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3     high = substitute(f.high)
4     i'    = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.

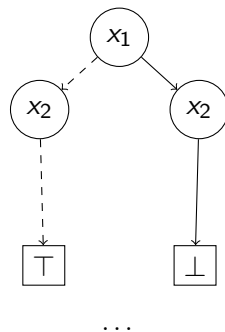


Adiar v2.1 : Variable Reordering

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3     high = substitute(f.high)
4     i'    = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.



Adiar v2.1 : Variable Reordering

$[0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 1]$

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3     high = substitute(f.high)
4     i'    = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.



$\neg(x_3 \text{ ? } x_2 \vee x_1 : x_2 \wedge x_1)$

Adiar v2.1 : Variable Reordering

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3       high = substitute(f.high)
4       i'   = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.

Time	Space	I/O
$O(NT)$	$O(NT)$	$O(NT)$

Complexity of depth-first **substitute**

Adiar v2.1 : Variable Reordering

```
1 substitute(f, i_map)
2   let low  = substitute(f.low)
3       high = substitute(f.high)
4       i'   = i_map[top(f)]
5   in bubble(i', low, high)
```

A recursive **substitute** operation.

Time	Space	I/O
$O(NT)$	$O(NT)$	$O(NT)$

Complexity of depth-first **substitute**

Time	Space	I/O
$O(NT \log T)$	$O(N + T)$	$O(N \cdot \text{sort}(T))$

Complexity of level-by-level **substitute**

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)

Given BDD f_π with variable ordering π and remapping of variables $m : \mathbb{N} \rightarrow \mathbb{N}$, construct $f'_\pi \equiv f_\pi[x/m(x)]$.

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)

Given BDD f_π with variable ordering π and remapping of variables $m : \mathbb{N} \rightarrow \mathbb{N}$, construct $f'_\pi \equiv f_\pi[x/m(x)]$.

Problem (Static Variable Reordering)

Given BDD f_π with variable ordering π and another variable ordering π' , construct $f_{\pi'} \equiv f_\pi$.

Adiar v2.1 : Variable Reordering

Problem (Variable Replacement)

Given BDD f_π with variable ordering π and remapping of variables $m : \mathbb{N} \rightarrow \mathbb{N}$,
construct $f'_\pi \equiv f_\pi[x/m(x)]$.

Problem (Static Variable Reordering)

Given BDD f_π with variable ordering π and another variable ordering π' ,
construct $f_{\pi'} \equiv f_\pi$.

Problem (Dynamic Variable Reordering)

Given BDD f_π with variable ordering π ,
find π' and construct $f_{\pi'} \equiv f_\pi$ such that $|f_{\pi'}|$ is minimal.

