

Relational Product of BDDs in External Memory

Steffan Christ Sølvesten, Jaco van de Pol

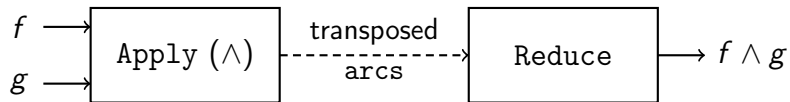
SPIN 2025

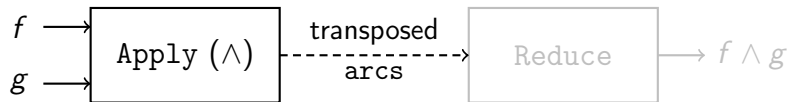


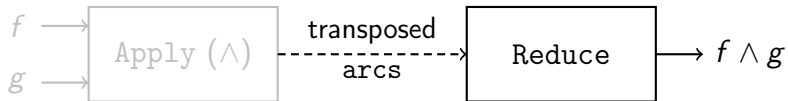
$$Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'})[\vec{x}' / \vec{x}]$$

[TACAS 22]

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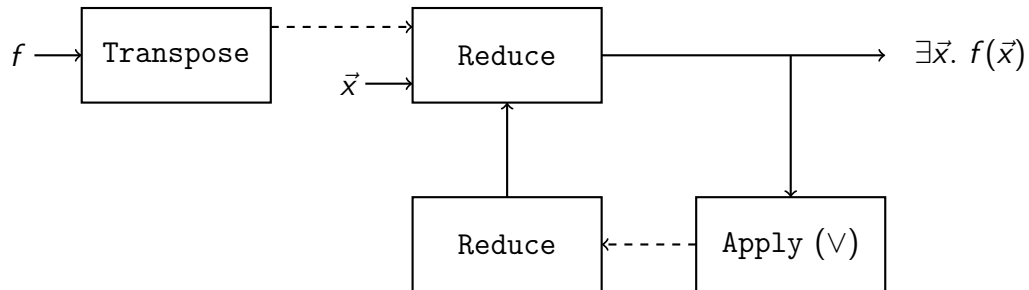




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[TACAS 25]



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[TACAS 25]

$$\begin{array}{c}
\text{[TACAS 22]} \\
| \\
\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'}) \equiv (\underbrace{\exists \vec{x}. S_{\vec{x}} \wedge T_{\vec{x}, \vec{x}'}}_{\text{[TACAS 25]}}) \underbrace{[\vec{x}' / \vec{x}]}_{\text{[SPIN 25]}}
\end{array}$$

Replace

Definition

A relabelling π is monotonic if $x_i < x_j \implies \pi(x_i) < \pi(x_j)$

Lemma

If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

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- One can apply π in a single linear scan.

$\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.

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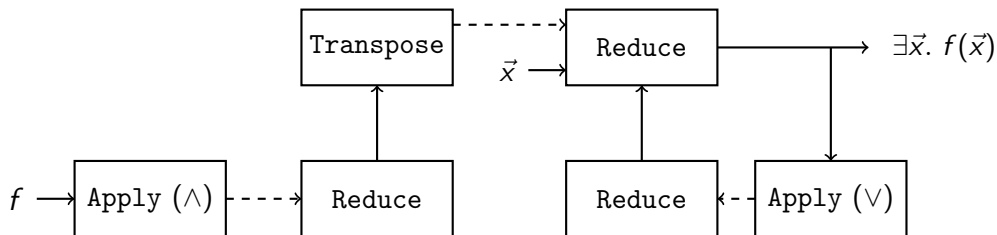
If π is monotonic, then the BDD $f(\vec{x})$ is isomorphic to $f(\pi(\vec{x}))$.

- **One can apply π in a single linear scan.**
 $\mathcal{O}(N)$ time, $2 \cdot \text{scan}(N)$ I/Os, and N external space.
- **One can incorporate π into a (succeeding) top-down Apply sweep.**
 $\mathcal{O}(N)$ time, 0 I/Os, and 0 external space.
- **One can incorporate π into a (preceeding) bottom-up Reduce sweep.**
 $\mathcal{O}(n)$ time, 0 I/Os, and 0 external space.

AndExists

Observation

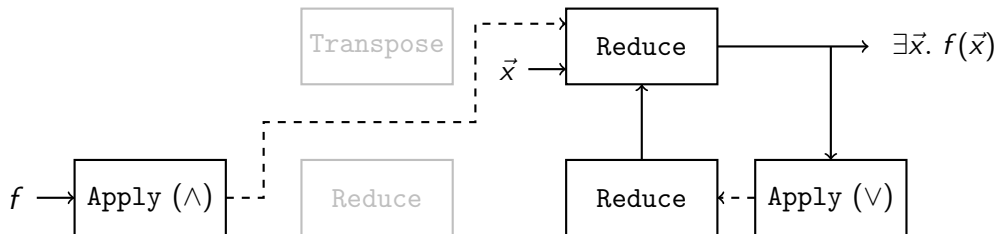
The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:



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- **The outer accumulating Reduce sweep of the Exists can do double-duty as the Reduce sweep of the preceeding And operation.**

This saves $\Theta(\text{sort}(N))$ time and I/Os.

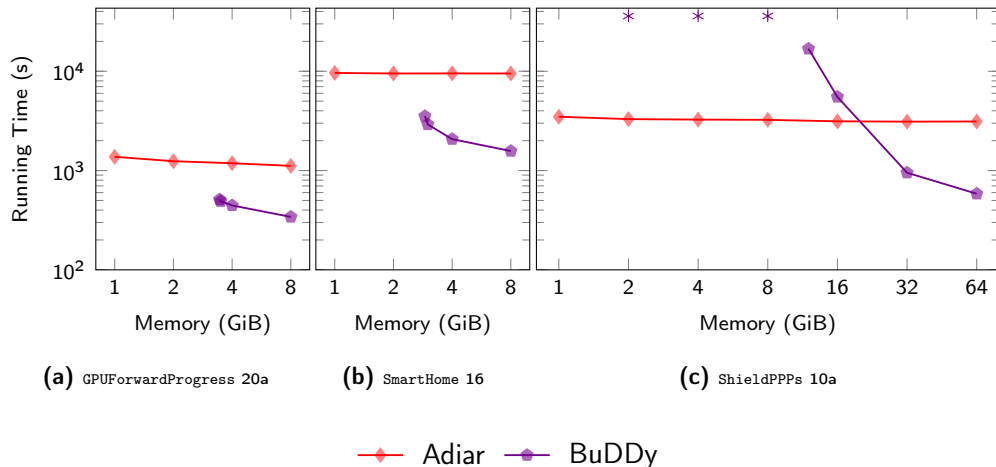
AndExists

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The I/O-efficient And [TACAS 22] and Exists [TACAS 25] operations can be merged:

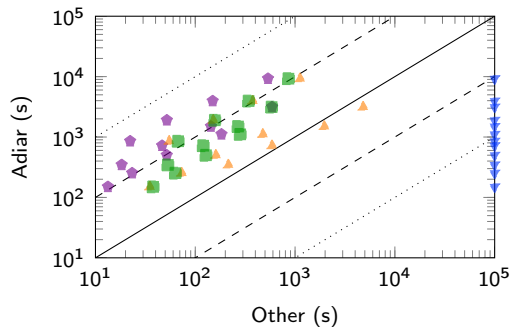
- **The outer accumulating Reduce sweep of the Exists can do double-duty as the Reduce sweep of the preceeding And operation.**
This saves $\Theta(\text{sort}(N))$ time and I/Os.
- **The And operation can prune subtrees that trivially will become redundant during the succeeding Exists.**
This can save up to $\mathcal{O}(\text{sort}(N^{2^k}))$ time and I/Os.
In practice, this only saves up to $\mathcal{O}(\text{sort}(N))$ time and I/Os.

Experiment: $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$

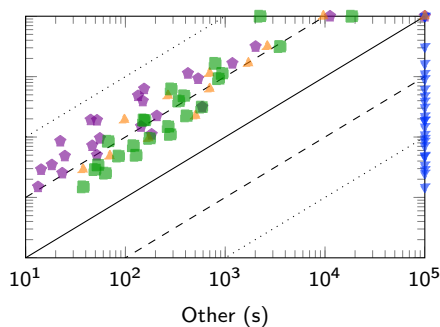


Relational Product for MCC models with a 2^{25} state space BDD. Timeouts are marked as stars.

Experiment: $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$ & $\text{Prev}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$



(a) $\text{Next}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$

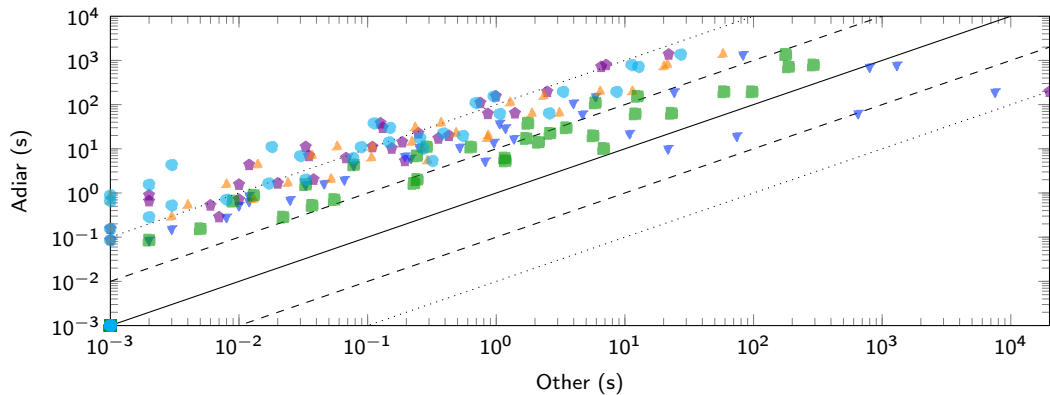


(b) $\text{Prev}(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$

◆ BuDDy ▼ CAL ▲ CUDD ■ LibBDD

Relational Product for MCC models, 384 GiB of memory, and $2^{22}, \dots, 2^{25}$ state space BDDs.

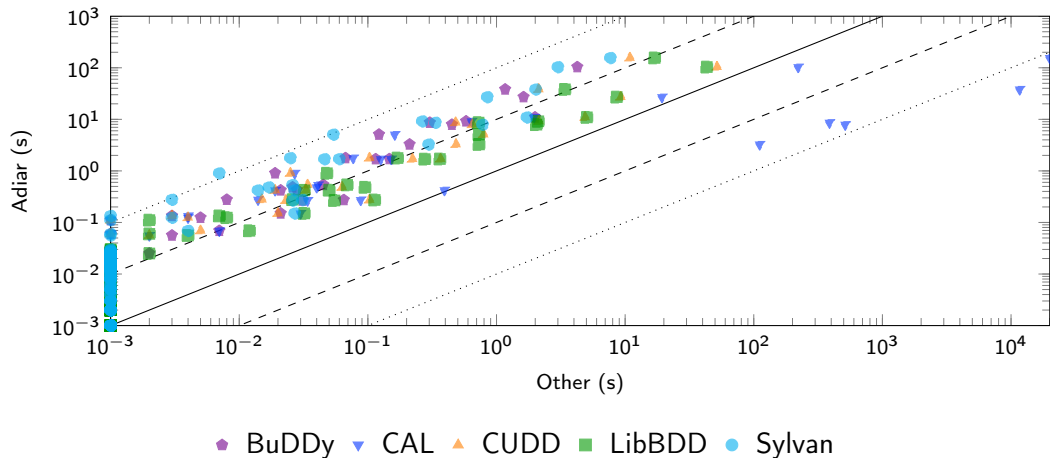
Experiment: Reachability



◆ BuDDy ▼ CAL ▲ CUDD ■ LibBDD ● Sylvan

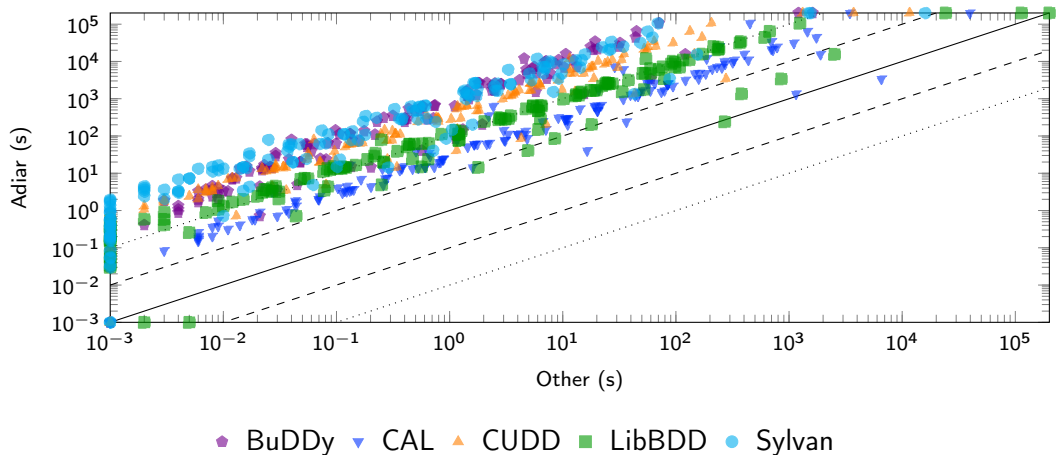
16 Petri Nets [MCC 21–23] with 384 GiB of memory.

Experiment: Deadlock Detection



16 Petri Nets [MCC 21–23] and 59 Boolean Networks [AEON, PyBoolNet] with 384 GiB RAM.

Experiment: SCC Decomposition



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Conclusions and Future Work

- To improve the I/O-efficient $Next(S_{\vec{x}}, T_{\vec{x}, \vec{x}'})$, focus on *AndExists*.
 - Factor of $\sim 2\times$ by using a *AndExists* instead of *And* and *Exists* for conventional depth-first implementations [1]. This may explain the sudden performance gap.
 - For larger instances, less than $\frac{1}{10}$ th of the time is spent on the *And*.

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- [1] Van Dijk et al.: *A Comparative Study of BDD packages for Probabilistic Symbolic Model Checking*. (2015)
 - [2] Van Dijk: *Sylvan – Multi-core Decision Diagrams*. (2016)
 - [3] Van Dijk et al.: *Multi-core on-the-fly saturation*. (2019)
 - [4] Brand et al.: *A Decision Diagram Operation for Reachability*. (2023).
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- Design a $Replace(\pi)$ for Non-monotone Variable Substitutions.

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Adiar

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