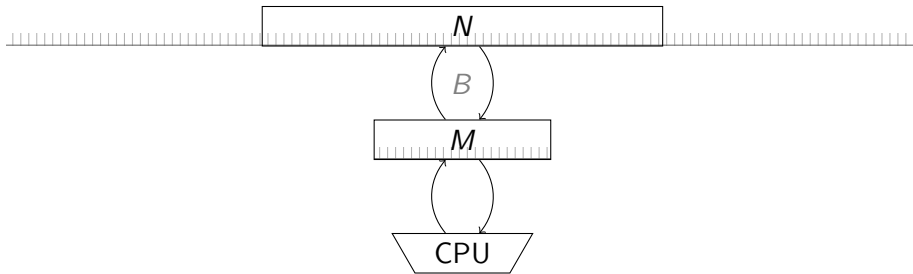


Efficient External Memory Algorithms for Binary Decision Diagram Manipulation

Steffan Christ Sølvesten, Jaco van de Pol,
Anna Blume Jakobsen, and Mathias Weller Berg Thomasen

25th of April, 2021





The I/O model by Aggarwal and Vitter '87

For any realistic values of N , M , and B we have that

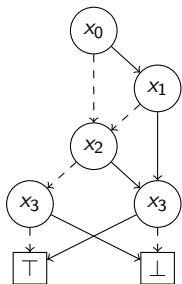
$$N/B < \text{sort}(N) \triangleq N/B \cdot \log_{M/B} N/B \ll N ,$$

Theorem (Aggarwal and Vitter '87)

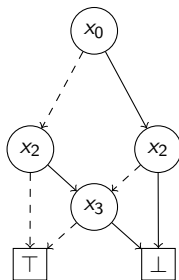
N elements can be sorted in $\Theta(\text{sort}(N))$ I/Os.

Theorem (Arge '95)

N elements can be inserted in and extracted from a Priority Queue in $\Theta(\text{sort}(N))$ I/Os.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

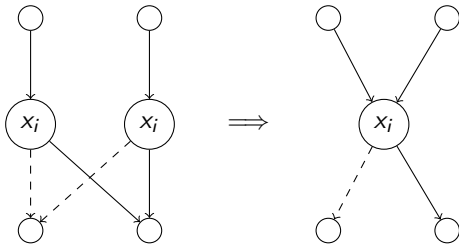
Examples of (Reduced Ordered) Binary Decision Diagrams.

Theorem (Bryant '86)

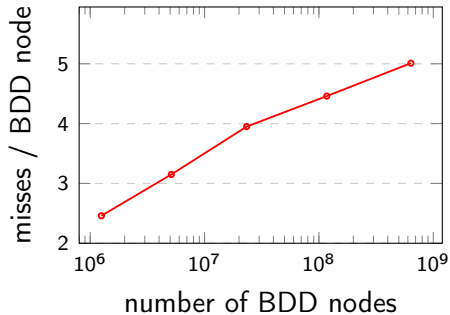
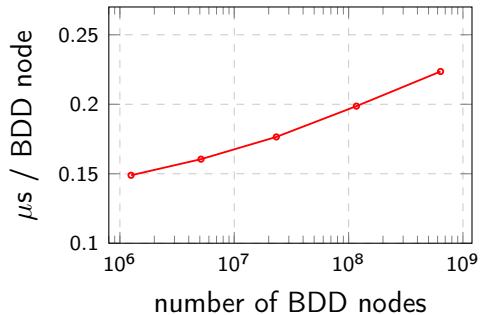
For a fixed variable order, if one exhaustively applies the two rules below, then one obtains the Reduced OBDD, which is a unique canonical form of the function.



(1) Remove redundant nodes



(2) Merge duplicate nodes

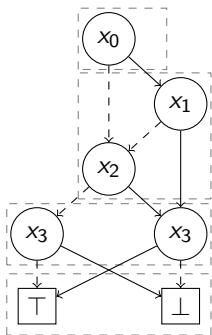


—●— BuDDy

Cache behaviour for the N -Queens problem.



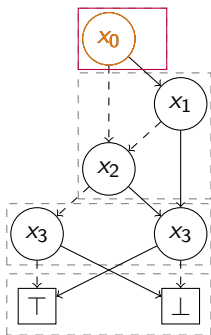
Running time for *Tic-Tac-Toe* with $N = 21$.



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

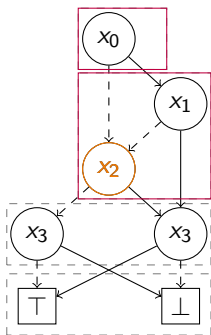
node I/Os	cache lookups
0	0



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

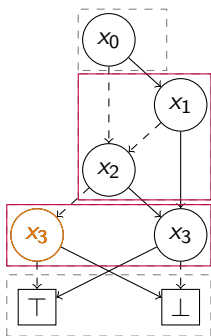
node I/Os	cache lookups
1	1



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

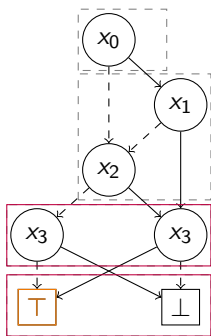
node I/Os	cache lookups
2	2



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

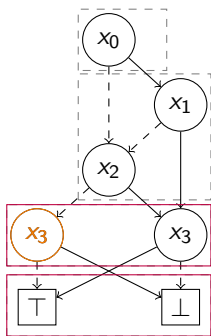
node I/Os	cache lookups
3	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

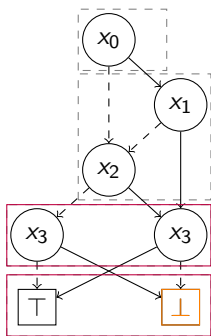
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

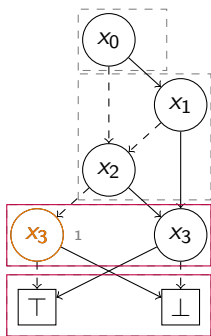
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

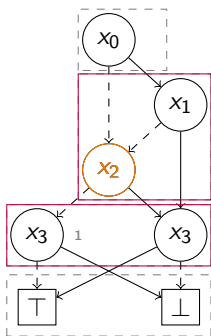
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

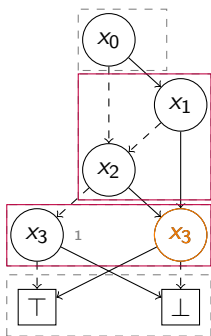
node I/Os	cache lookups
4	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

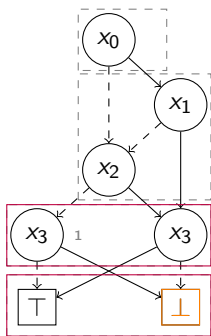
node I/Os	cache lookups
5	3



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

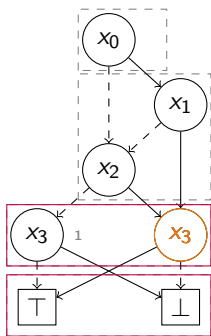
node I/Os	cache lookups
5	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

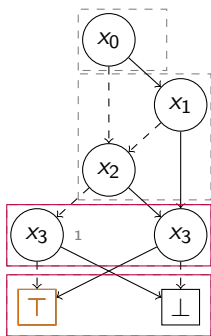
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

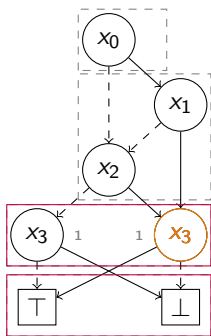
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

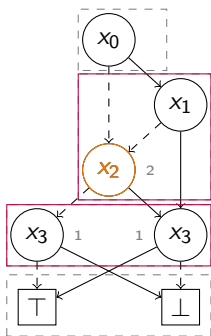
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

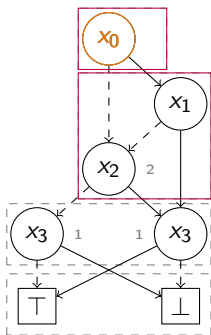
node I/Os	cache lookups
6	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

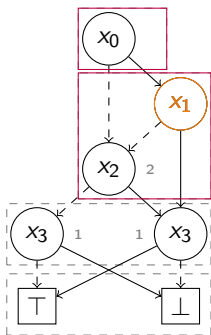
node I/Os	cache lookups
7	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

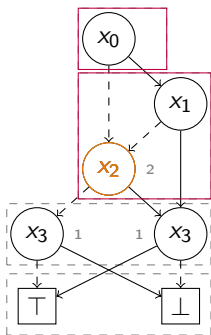
node I/Os	cache lookups
8	4



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

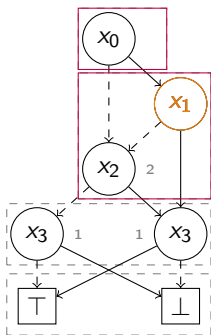
node I/Os	cache lookups
8	5



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

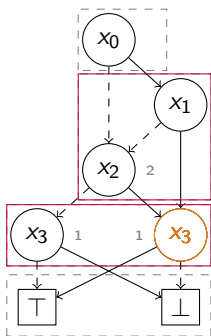
node I/Os	cache lookups
8	6



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

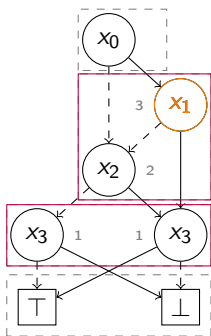
node I/Os	cache lookups
8	6



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

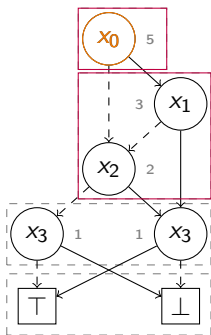
node I/Os	cache lookups
9	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

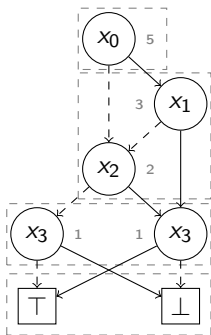
node I/Os	cache lookups
9	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

$$M = 4, B = 2$$

node I/Os	cache lookups
10	7



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

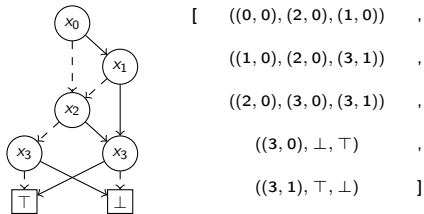
$$M = 4, B = 2$$

node I/Os	cache lookups
10	7

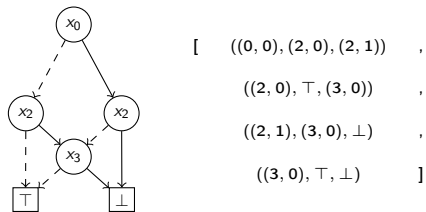




$$(i_1, id_1) < (i_2, id_2) \equiv i_1 < i_2 \vee (i_1 = i_2 \wedge id_i < id_j)$$



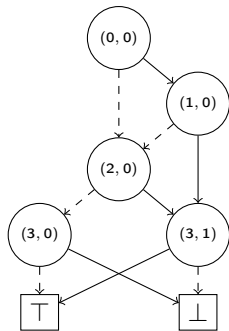
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

Node-based representation of prior shown BDDs

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

CountPaths Example



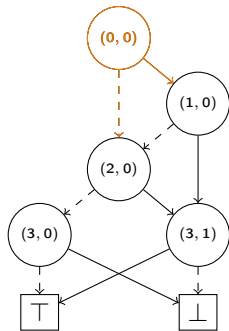
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example



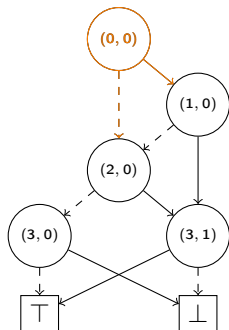
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[

]

CountPaths Example

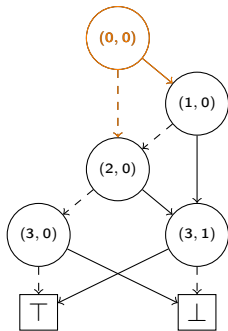


(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	0	0

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{\top} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

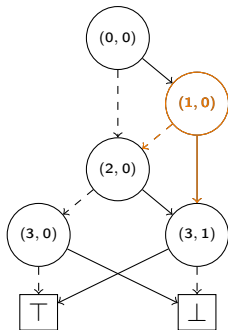
Seek	Sum	Result
(1, 0)	0	0

Priority Queue: Q_{count} :

[$((0, 0) \xrightarrow{T} (1, 0), 1)$,
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(1, 0)

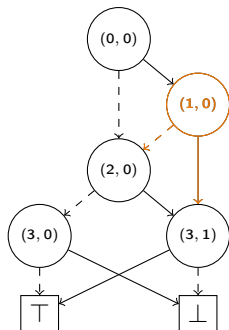
Sum
1

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(1, 0)	1	0

Priority Queue: Q_{count} :

[

$((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	0	0

Priority Queue: Q_{count} :

```
[
  ((0, 0)  $\xrightarrow{\perp}$  (2, 0), 1) ,
  ((1, 0)  $\xrightarrow{\perp}$  (2, 0), 1) ,
  ((1, 0)  $\xrightarrow{\top}$  (3, 1), 1) ,
]
```

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[
 $((0, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\perp} (2, 0), 1)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	1	0

Priority Queue: Q_{count} :

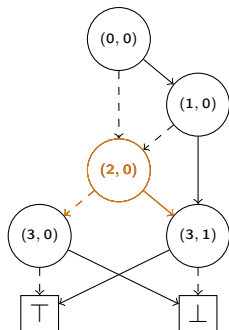
[

$((1, 0) \xrightarrow{\perp} (2, 0), 1)$,

$((1, 0) \xrightarrow{\top} (3, 1), 1)$,

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(2, 0)	2	0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(2, 0)

Sum
2

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
0

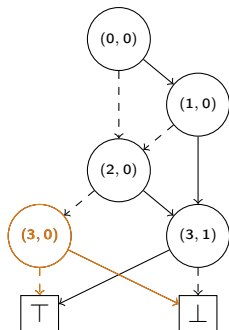
Result
0

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\perp} (3, 0), 2)$,
 $((1, 0) \xrightarrow{\top} (3, 1), 1)$,
 $((2, 0) \xrightarrow{\top} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 0)

Sum
2

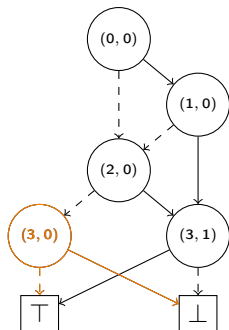
Result
0

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek	Sum	Result
(3, 0)	2	2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
0

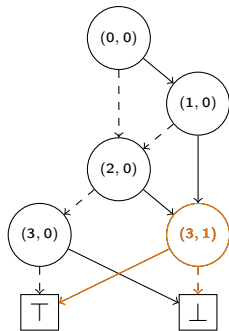
Result
2

Priority Queue: Q_{count} :

[

$((1, 0) \xrightarrow{T} (3, 1), 1)$,
 $((2, 0) \xrightarrow{T} (3, 1), 2)$]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
1

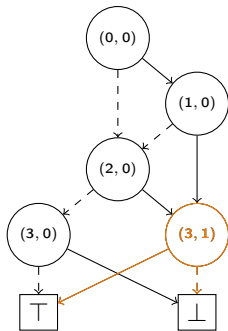
Result
2

Priority Queue: Q_{count} :

[

$((2, 0) \xrightarrow{\top} (3, 1), \quad 2) \quad]$

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

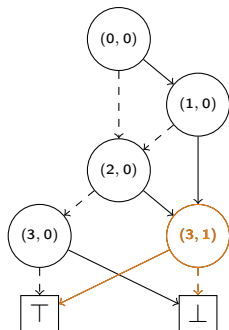
Result
2

Priority Queue: Q_{count} :

[

]

CountPaths Example



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Seek
(3, 1)

Sum
3

Result
5

Priority Queue: Q_{count} :

[

]

CountPaths Example



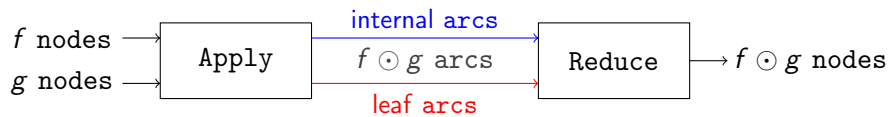
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Result
5

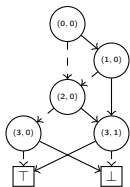
Priority Queue: Q_{count} :

[

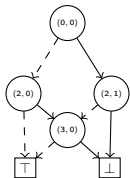
]



Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

Apply Example (\wedge)



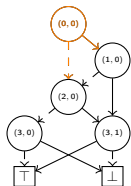
(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 \oplus x_2 \vee x_3 : x_2 \wedge x_3)$

(c) $(a) \wedge (b)$

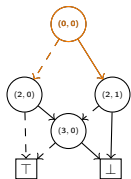
Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$

Priority Queue: $Q_{app:1}$:

[$(0,0) \xrightarrow{\top} ((1,0), (2,1))$,
 $(0,0) \xrightarrow{\perp} ((2,0), (2,0))$,

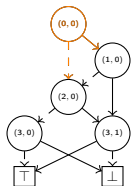


(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

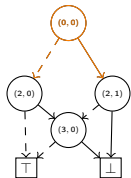
]

(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

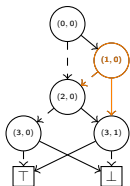
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,



]

(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

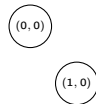
Seek:

$\min((1, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

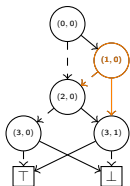
[$(0, 0) \xrightarrow{\top} ((1, 0), (2, 1))$,
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((1, 0), (2, 1))$

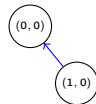
Priority Queue: $Q_{app:1}$:

[

$(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

]

Output:
 $(0, 0) \xrightarrow{\top} (1, 0)$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



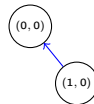
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



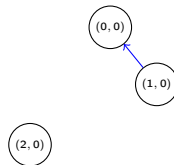
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

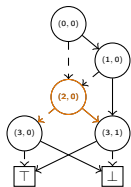
[
 $(0, 0) \xrightarrow{\perp} ((2, 0), (2, 0))$,
 $(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:

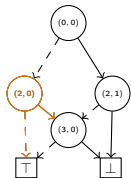


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 0))$

Priority Queue: $Q_{app:1}$:

[

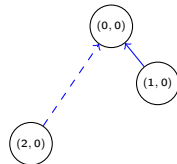
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

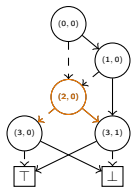
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:
 $(0, 0) \xrightarrow{\perp} (2, 0)$

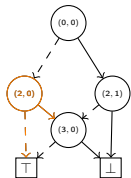


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

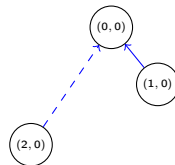
Seek:
 $\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

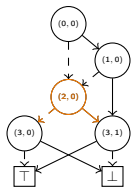
$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$,
 $(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

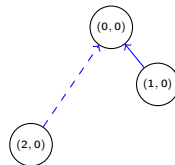
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$
 Priority Queue: $Q_{app:1}$:
 [

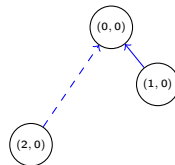
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:
 [$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1)) \quad ((3, 0), (3, 1))$,
]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

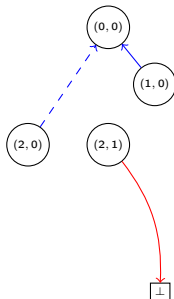
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[$(1, 0) \xrightarrow{\perp} ((2, 0), (2, 1))$ $((3, 0), (3, 1))$,

]

Output:
 $(2, 1) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((2, 0), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,

$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

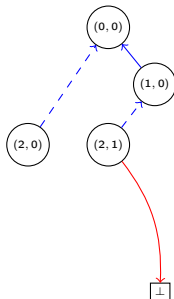
$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(1, 0) \xrightarrow{\perp} (2, 1)$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

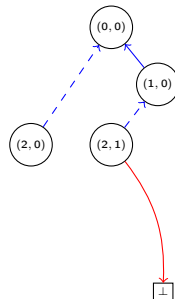
[

$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

Output:

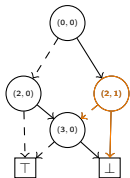


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

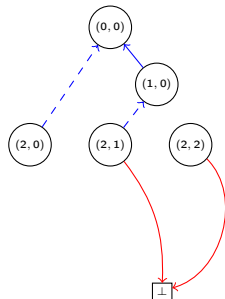
$(1, 0) \xrightarrow{\top} ((3, 1), (2, 1))$,
 $(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(2, 2) \xrightarrow{\top} \perp$

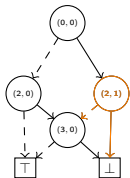


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (2, 1))$

Priority Queue: $Q_{app:1}$:

[

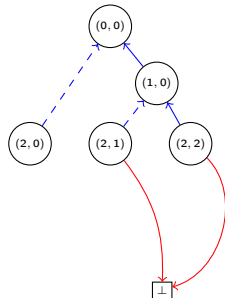
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(1, 0) \xrightarrow{\top} (2, 2)$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

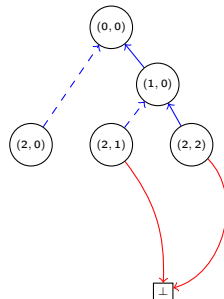
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:

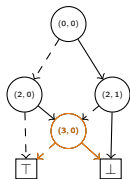


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

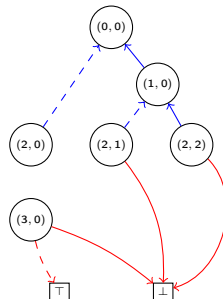
$(2, 1) \xrightarrow{\perp} ((3, 0), (3, 0))$,
 $(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

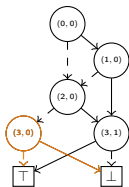
]

Output:
 $(3, 0) \xrightarrow{\perp} \top, (3, 0) \xrightarrow{\top} \perp$

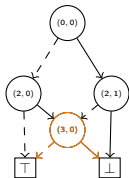


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

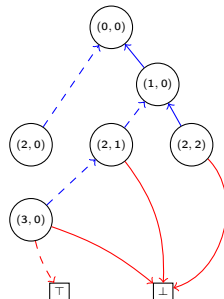
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

]

Output:
 $(2, 1) \xrightarrow{\perp} (3, 0)$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



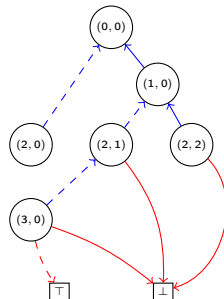
(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$
 Priority Queue: $Q_{app:1}$:
 [

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0))$,
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0))$,
 $(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

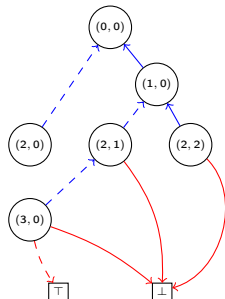
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

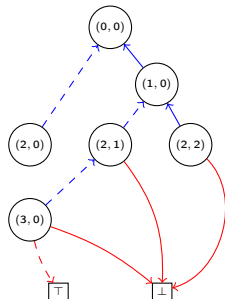
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

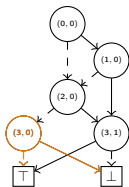
]

Output:

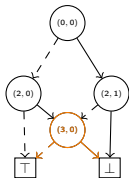


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

$(2, 0) \xrightarrow{\perp} ((3, 0), \top)$]

Priority Queue: $Q_{app:2}$:

[

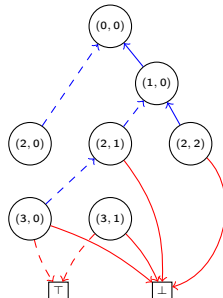
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(3, 1) \xrightarrow{\perp} \top, (3, 1) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\min((3, 0), \top)$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

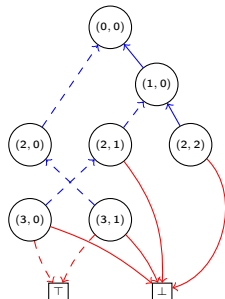
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

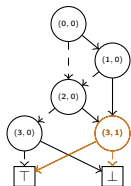
]

Output:
 $(2, 0) \xrightarrow{\perp} (3, 1)$

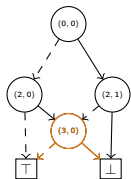


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

[

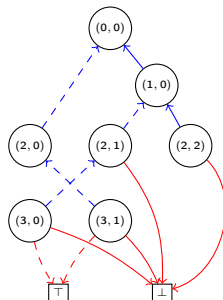
$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$

$(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

Priority Queue: $Q_{app:2}$:

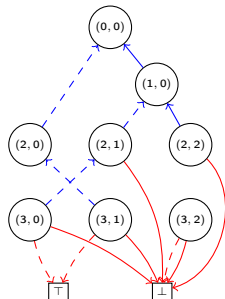
[

$(2, 0) \xrightarrow{\top} ((3, 1), (3, 0)) \quad (\top, \perp)$
 $(2, 2) \xrightarrow{\perp} ((3, 1), (3, 0)) \quad (\top, \perp)$

,

]

Output:
 $(3, 2) \xrightarrow{\perp} \perp, (3, 2) \xrightarrow{\top} \perp$



(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Seek:
 $\max((3, 1), (3, 0))$

Priority Queue: $Q_{app:1}$:

[

]

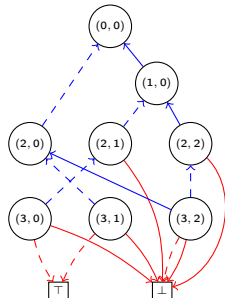
Priority Queue: $Q_{app:2}$:

[

]

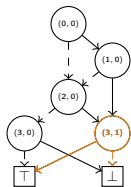
Output:

$(2, 0) \xrightarrow{\top} (3, 2), (2, 2) \xrightarrow{\perp} (3, 2)$

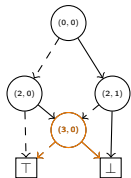


(c) $(a) \wedge (b)$

Apply Example (\wedge)



(a) $(x_0 \wedge x_1 \wedge x_3) \vee (x_2 \oplus x_3)$



(b) $\neg(x_0 ? x_2 \vee x_3 : x_2 \wedge x_3)$

Priority Queue: $Q_{app:1}$:

[

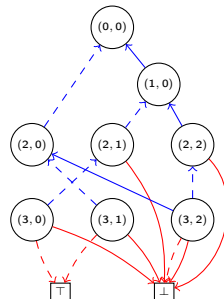
Priority Queue: $Q_{app:2}$:

[

]

]

Output:



(c) $(a) \wedge (b)$

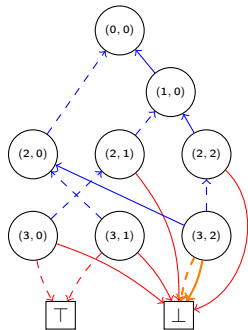
Reduce Example



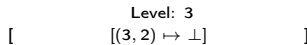
(c) $(a) \wedge (b)$

(d) $(a) \wedge (b)$ reduced

Reduce Example

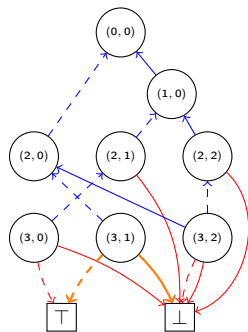


(c) $(a) \wedge (b)$



(d) $(a) \wedge (b)$ reduced

Reduce Example

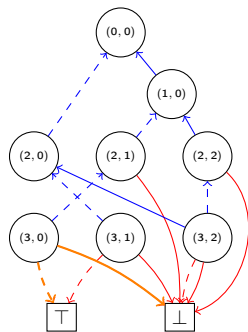


(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad]$

(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad ((3, 1), \top, \perp) \quad , \quad$
 $\quad ((3, 0), \top, \perp) \quad]$

(d) $(a) \wedge (b)$ reduced

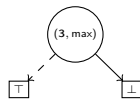
Reduce Example



(c) $(a) \wedge (b)$

Level: 3
 $[\quad [(3, 2) \mapsto \perp] \quad]$
 $[\quad [(3, 1) \mapsto (3, \max)] \quad , \quad]$
 $\quad \quad \quad ((3, 0), \top, \perp)$

Output:
 $((3, \max), \top, \perp)$



(d) $(a) \wedge (b)$ reduced

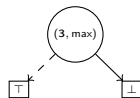
Reduce Example



(c) $(a) \wedge (b)$

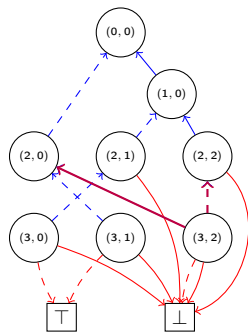
Level: 3
 $\left[\begin{array}{c} [(3, 2) \mapsto \perp] \end{array} \right]$
 $\left[\begin{array}{c} [(3, 1) \mapsto (3, \max)] \\ [(3, 0) \mapsto (3, \max)] \end{array} \right],$

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{\top} \perp$,

]

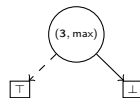
Level: 3

[

[$(3, 1) \mapsto (3, \max)$] ,

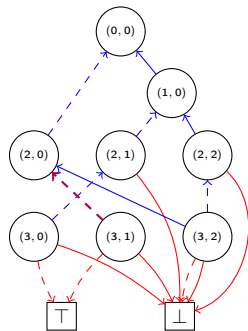
[$(3, 0) \mapsto (3, \max)$]]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,

$(2, 0) \xrightarrow{T} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

]

Level: 3

[

]

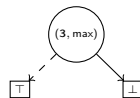
[

$[(3, 0) \mapsto (3, \max)]$

,

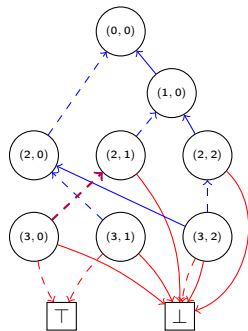
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

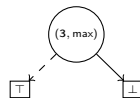
Priority Queue: Q_{red} :

[$(2, 2) \xrightarrow{\perp} \perp$,
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,

Level: 3

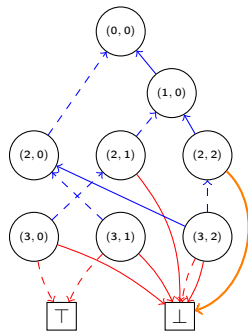
[
 [
 ,
]
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



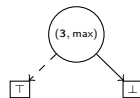
(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[
 $(2, 1) \xrightarrow{\perp} (3, \max)$,
 $(2, 0) \xrightarrow{\top} \perp$,
 $(2, 0) \xrightarrow{\perp} (3, \max)$,
]

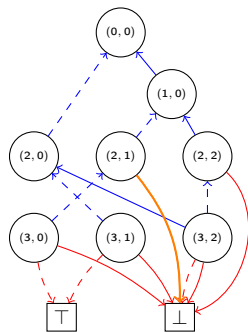
[
 Level: 2
 $[(2, 2) \mapsto \perp]$
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(2, 0) \xrightarrow{\top} \perp$,

$(2, 0) \xrightarrow{\perp} (3, \max)$,

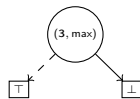
]

Level: 2

[$[(2, 2) \mapsto \perp]$]

[$((2, 1), (3, \max), \perp)$,]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

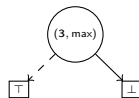
$((2, 1), (3, \max), \perp)$

,

$((2, 0), (3, \max), \perp)$

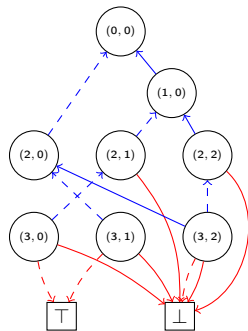
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

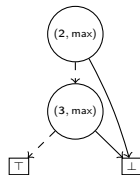
$[(2, 1) \mapsto (2, \max)]$

,

$((2, 0), (3, \max), \perp)$

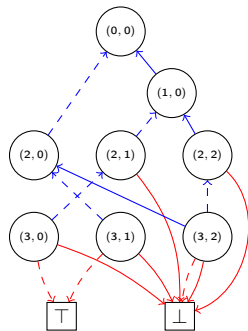
]

Output:
 $((2, \max), (3, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 2

[

$[(2, 2) \mapsto \perp]$

]

[

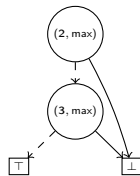
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

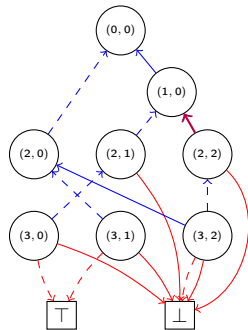
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{\top} \perp$,

]

Level: 2

[

]

[

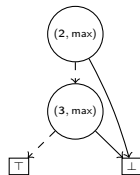
$[(2, 1) \mapsto (2, \max)]$

,

$[(2, 0) \mapsto (2, \max)]$

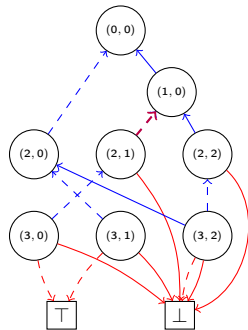
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{\top} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

]

Level: 2

[

[

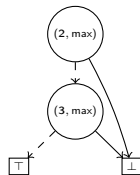
$[(2, 0) \mapsto (2, \max)]$

]

,

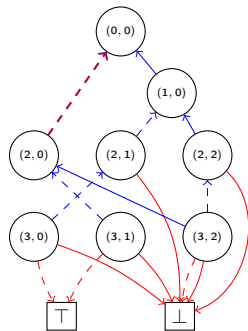
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(1, 0) \xrightarrow{T} \perp$,

$(1, 0) \xrightarrow{\perp} (2, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 2

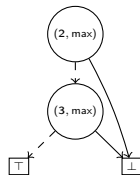
[

[

]

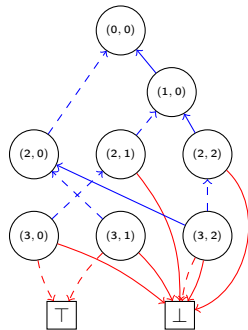
,
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

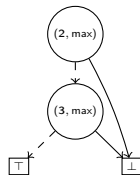
]

[

$((1, 0), (2, \max), \perp)$

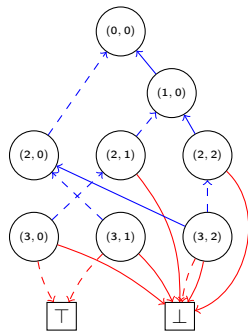
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

[

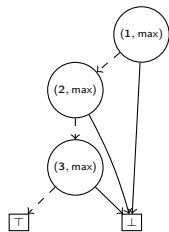
]

[

$[(1, 0) \mapsto (1, \max)]$

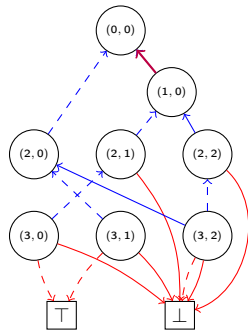
]

Output:
 $((1, \max), (2, \max), \perp)$



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

$(0, 0) \xrightarrow{\top} (1, \max)$,

$(0, 0) \xrightarrow{\perp} (2, \max)$]

Level: 1

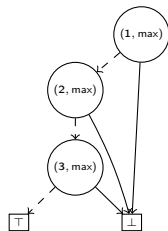
[

]

[

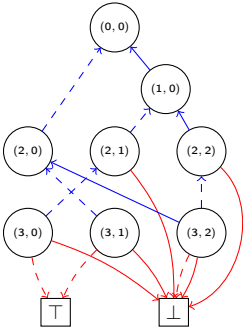
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

[

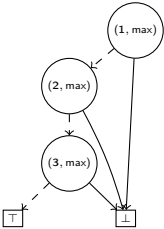
]

[

$((0,0), (2,max), (1,max))$

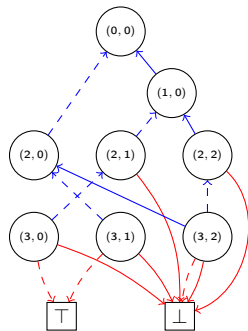
]

Output:



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

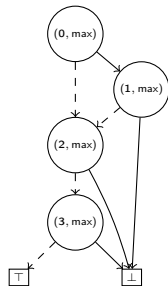
[

]

$[(0, 0) \mapsto (0, \max)]$

]

Output:
 $((0, \max), (2, \max), (1, \max))$



(d) $(a) \wedge (b)$ reduced

Reduce Example



(c) $(a) \wedge (b)$

Priority Queue: Q_{red} :

[

]

Level: 0

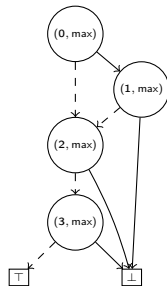
[

]

[

]

Output:

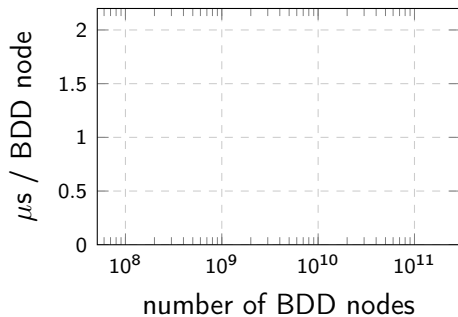
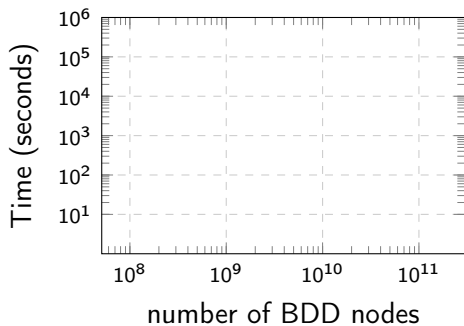


(d) $(a) \wedge (b)$ reduced

Algorithm		Depth-first	Time-forwarded
Reduce		$O(N)$	$O(\text{sort}(N))$
BDD Manipulation			
Apply	$f \odot g$	$O(N_f \cdot N_g)$	$O(\text{sort}(N_f \cdot N_g))$
If-Then-Else	$f ? g : h$	$O(N_f \cdot N_g \cdot N_h)$	$O(\text{sort}(N_f \cdot N_g \cdot N_h))$
Restrict	$f _{x_i=v}$	$O(N)$	$O(\text{sort}(N))$
Negation	$\neg f$	$O(1)$	$O(1)$
Quantification	$\exists/\forall v : f _{x_i=v}$	$O(N^2)$	$O(\text{sort}(N^2))$
Counting			
Count Paths	#paths in f to \top	$O(N)$	$O(\text{sort}(N))$
Count SAT	$\#x : f(x)$	$O(N)$	$O(\text{sort}(N))$
Other			
Equality	$f \equiv g$	$O(1)$	$O(\text{sort}(N))$
Evaluate	$f(x)$	$O(L)$	$O(N/B)$
Min/Max SAT	$\min / \max\{x \mid f(x)\}$	$O(L)$	$O(N/B)$

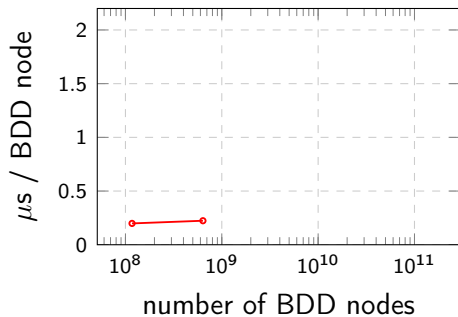
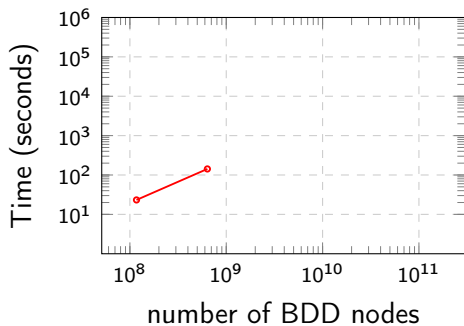
Adiar

github.com/ssoelvsten/adiar



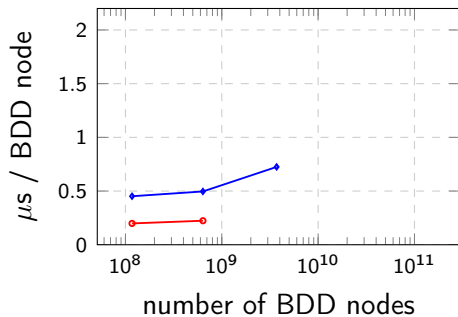
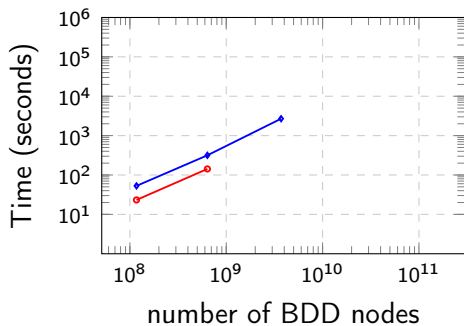
—○— BuDDy —●— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



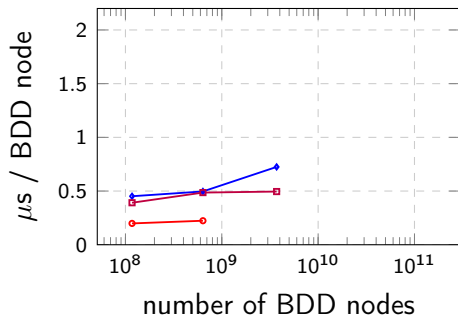
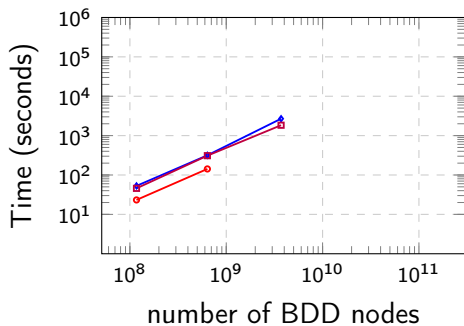
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Minimum running times for the N -Queens problem.



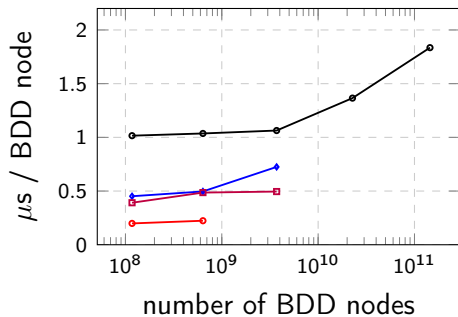
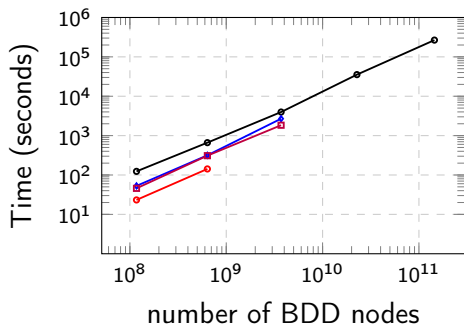
—○— BuDDy —◇— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.



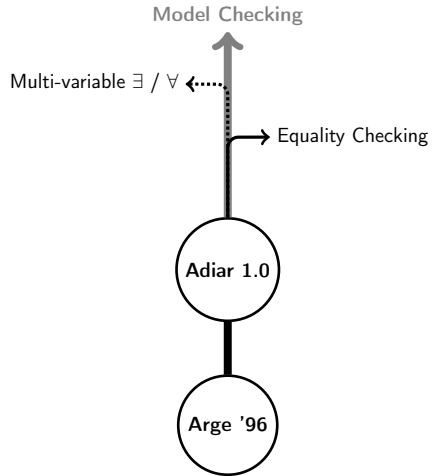
—○— BuDDy —◇— CUDD —□— Sylvan —●— Adiar

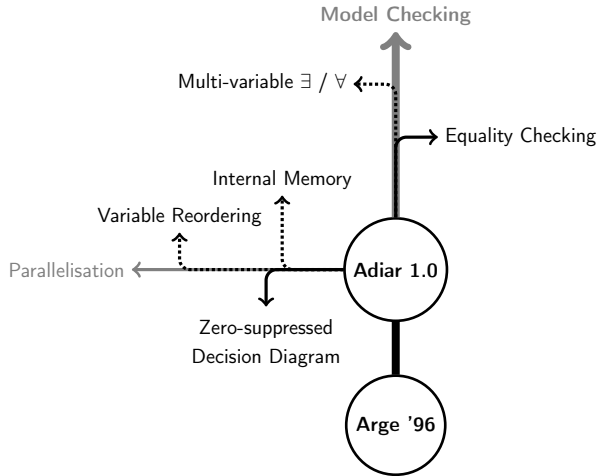
Minimum running times for the N -Queens problem.

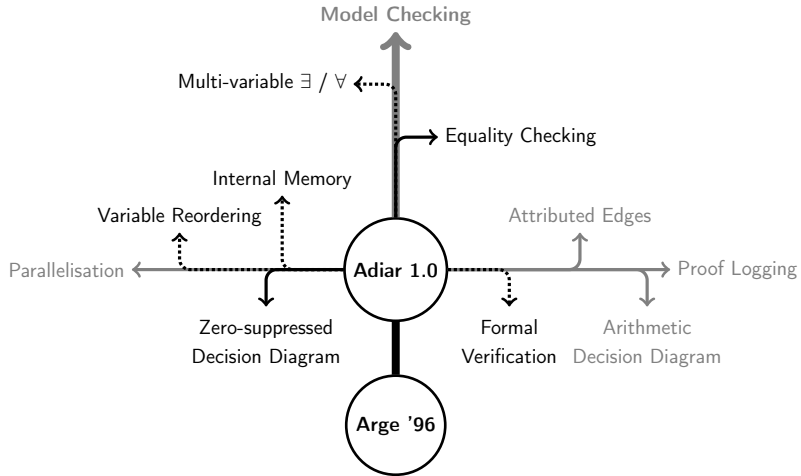


—○— BuDDy —◆— CUDD —■— Sylvan —●— Adiar

Minimum running times for the N -Queens problem.







Steffan Christ Sølvsten

✉ soelvsten@cs.au.dk

🐦 [@ssoelvsten](https://twitter.com/ssoelvsten)

Adiar

📄 github.com/ssoelvsten/adiar

📖 ssoelvsten.github.io/adiar