

Portfolio Optimization Problem

Consider a scenario where an investor has £20,000 to invest in a combination of the following:

- Stock ABC sells today at £20 per share.
- A European call option to buy 100 shares of stock ABC at £15 per share exactly six months from today sells for £1,000.
- The investor can also raise additional funds which can be immediately invested, if desired, by selling call options with the above characteristics.
- In addition, a 6-month riskless zero-coupon bond with a £100 face value sells for £90.
- The investor has decided to limit the number of call options that they buy or sell to at most 50.

The investor considers three scenarios for the price of stock ABC six months from today:

- The price remains the same at £20.
- The price goes up to £40.
- The price drops to £12.

The investor's best estimate is that each of these scenarios is equally likely. Suppose that the investor wants a profit of at least £2,000 in any of the three scenarios for the price of ABC six months from today.

Objective

The goal is to **maximize** the expected profit from investing in a combination of stocks, bonds, and options, subject to certain constraints.

Decision Variables:

- x : Number of shares of stock ABC to buy.
- y : Number of call options to buy.
- z : Number of call options to sell.
- a : Auxiliary variable for bonds

Parameters:

- Stock price per share (ABC): £20
- Call option price for 100 shares: £1,000
- Strike price of call option: £15
- Zero-coupon bond face value: £100
- Zero-coupon bond price: £90
- Initial investment: £20,000
- Maximum number of call options to buy/sell: 50
- Minimum desired profit: £2,000
- Stock price scenarios: (equally likely)
 - Remains the same at £20
 - Increase to £40
 - Decrease to £12

Formulation

Investment Constraint: The total investment in stocks, call options, and bonds should not exceed the initial investment of £20,000. It is formulated as follows:

$$20x + 1000y - 1000z + 90a \leq 20000$$

Here, $20x$ represents the cost of buying x shares of stock, $10y$ and $-10z$ represent the cost (or revenue) from buying and selling call options (each option is for 100 shares and thus scaled by 10), and $90a$ represents the investment in zero-coupon bonds.

Profit Constraints: Ensure a profit of at least £2,000 in each of the three scenarios.

- Scenario 1 (Stock price remains at £20):

$$X(20 - 20) + (MAX(0, 20 - 15) \times 100y) - (MAX(0, 20 - 15) \times 100z) - (100 - 90)a \geq 2000$$

$$\text{Simplified:} \quad 500y - 500z + 10a \geq 2000$$

- Scenario 2 (Stock price rises to £40):

$$X(40 - 20) + (MAX(0, 40 - 15) \times 100y) - (MAX(0, 40 - 15) \times 100z) - (100 - 90)a \geq 2000$$

$$\text{Simplified:} \quad 20x + 2500y - 2500z + 10a \geq 2000$$

- Scenario 3 (Stock price falls to £12):

$$X(12 - 20) + (MAX(0, 12 - 15) \times 100y) - (MAX(0, 12 - 15) \times 100z) - (100 - 90)a \geq 2000$$

$$\text{Simplified:} \quad -8x + 10a \geq 2000$$

Option Constraints:

- Limit the number of call options bought or sold to at most 50:

$$y, z \leq 50$$

- Ensure non-negative amounts for buying and selling options:

$$x, y, z, a \geq 0$$

- Ensure that all decision variables are integers:

$$x, y, z, a \in \mathbb{Z}$$

Objective Function:

Maximize the profit across all scenarios, given that each scenario is equally likely (probability of $\frac{1}{3}$):

$$\text{Expected Profit} = \frac{1}{3}(\text{Profit in Scenario 1} + \text{Profit in Scenario 2} + \text{Profit in Scenario 3})$$

Putting it all together, we will have the following linear program. The initial equation for Expected Profit is given by:

$$\text{Expected Profit} = \frac{1}{3}[(500y - 500z + 10a) + (20x + 2500y - 2500z + 10a) + (-8x + 10a)]$$

Simplifying this, we get:

$$\text{Expected Profit} = 4x + 1000y - 1000z + 10a$$

Optimization Result

Based on the given problem and the solution obtained using Excel Solver and verified with pulp in Python, we can draw the following conclusions about the investor's portfolio optimization:

Decision Variables:

- Number of shares of stock ABC (x): 0
- Number of call options bought (y): 2
- Number of call options sold (z): 0
- Auxiliary variable for bonds (a): 200

Investment Allocation:

- Shares of stock ABC: £0
- European call options: £2,000
- Zero-coupon bonds: £18,000
- Total Investment: £20,000

Profit Scenarios:

- Scenario 1: Stock price remains at £20
 - Profit: £3,000
- Scenario 2: Stock price rises to £40
 - Profit: £7,000
- Scenario 3: Stock price falls to £12
 - Profit: £2,000

Expected Profit: £4,000

Analysis

Investment Strategy:

- The investor chooses to invest £18,000 in zero-coupon bonds and £2,000 in European call options, without purchasing any shares of stock ABC.
- This allocation maximizes the expected profit while ensuring a minimum profit of £2,000 in each scenario.

Risk Management:

- The choice to buy 2 call options and avoid selling call options reduces potential losses in case the stock price falls.
- Investing heavily in zero-coupon bonds ensures a stable return, reducing the overall risk of the portfolio.

Profit Stability:

- The strategy ensures that the investor makes a profit in all three scenarios, with profits ranging from £2,000 to £7,000.
- The expected profit of £4,000 is achieved by equally weighting the possible outcomes.

Role of Zero-Coupon Bonds:

- The zero-coupon bonds will still return their face value without being affected by changes in the stock price.
- To meet the minimum profit of £2,000 under all the given conditions, the investor needs to invest significantly in zero-coupon bonds to achieve that stability.

Further Analysis

Comprehensive Analysis of Varying Initial Investments

In our analysis, we examined the impact of varying initial investments on expected profit and investment decisions using both Python and Excel VBA. This combined approach offers a holistic view of how different investment amounts influence the optimal strategy, expected returns, and Return on Investment (ROI).

Analysis Insights: Expected Profit, Investment Decisions, and ROI

Based on the provided charts and analysis, we can draw several insights regarding the expected profit, investment decisions, and ROI as the initial investment varies from £18,000 to £100,000.

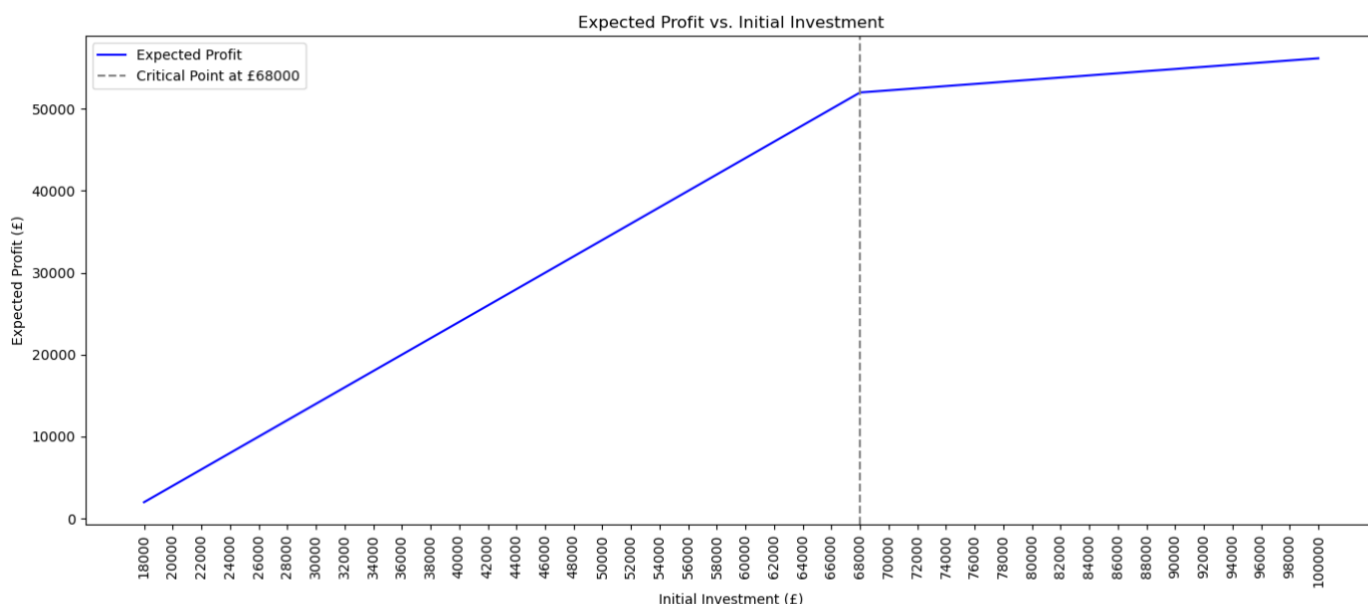
Expected Profit vs. Initial Investment

Linear Growth:

- The expected profit increases linearly with the initial investment up to a certain point, demonstrating that higher initial investments generally yield higher expected profits. This is expected, as more capital allows for more substantial investments in profitable assets.

Critical Point at £68,000:

- There is a noticeable change in the slope of the expected profit at the £68,000 mark. Up to £68,000, the expected profit increases rapidly. Beyond this point, the rate of increase in expected profit slows down.
- This suggests that the investment strategy reaches a point of diminishing returns at £68,000, indicating that additional investment beyond this amount does not yield as high an incremental profit as the investments up to £68,000.



Investment Decisions vs. Initial Investment

Shares to Buy (x):

- The number of shares to buy increases gradually with the initial investment. This reflects a balanced strategy where a portion of the increased investment is allocated to purchasing more shares of stock ABC.

Call Options to Buy (y):

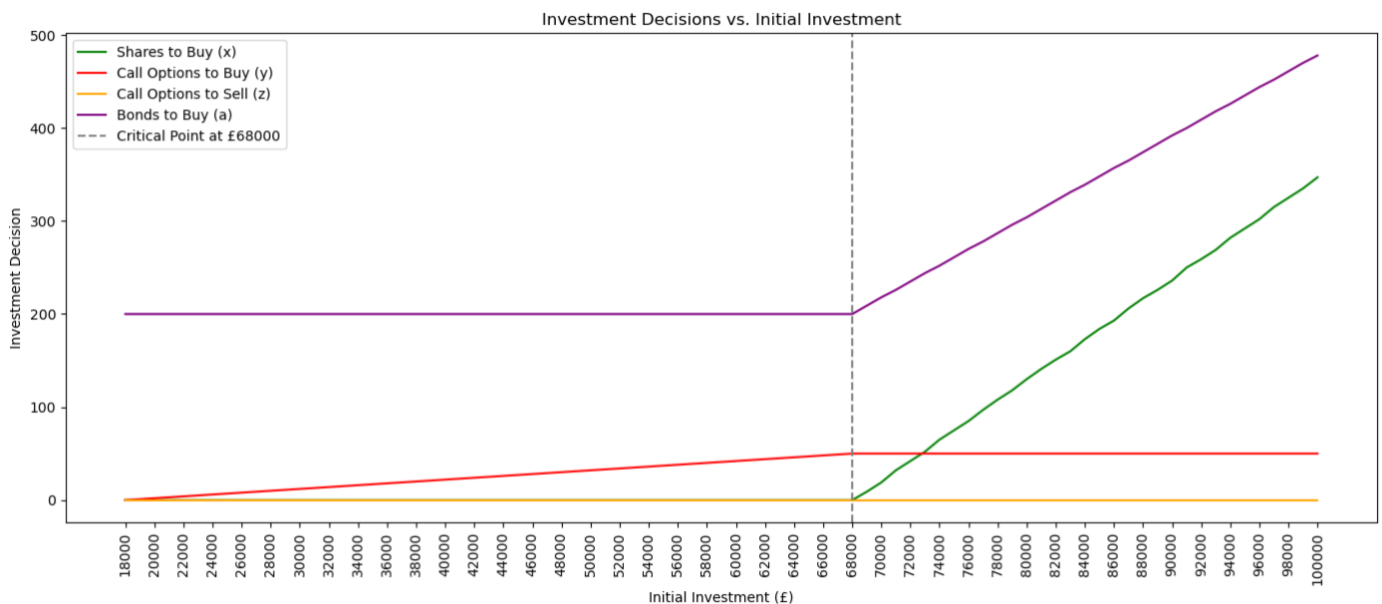
- The number of call options to buy remains constant at a lower level until the critical point (£68,000). This indicates a strategy where call options are utilized up to a certain extent and then capped.
- Beyond the critical point, there is no significant increase in call options to buy, suggesting that call options are not a preferred investment beyond a certain initial investment level

Call Options to Sell (z):

- The number of call options to sell remains at a consistent level, indicating that selling call options is part of a steady strategy regardless of the initial investment size.

Bonds to Buy (a):

- The amount invested in zero-coupon bonds increases significantly and consistently with the initial investment. Bonds form a substantial part of the investment strategy, particularly as the initial investment increases.
- This heavy allocation towards bonds reflects a conservative approach to secure risk-free returns, especially as the investment amount grows.



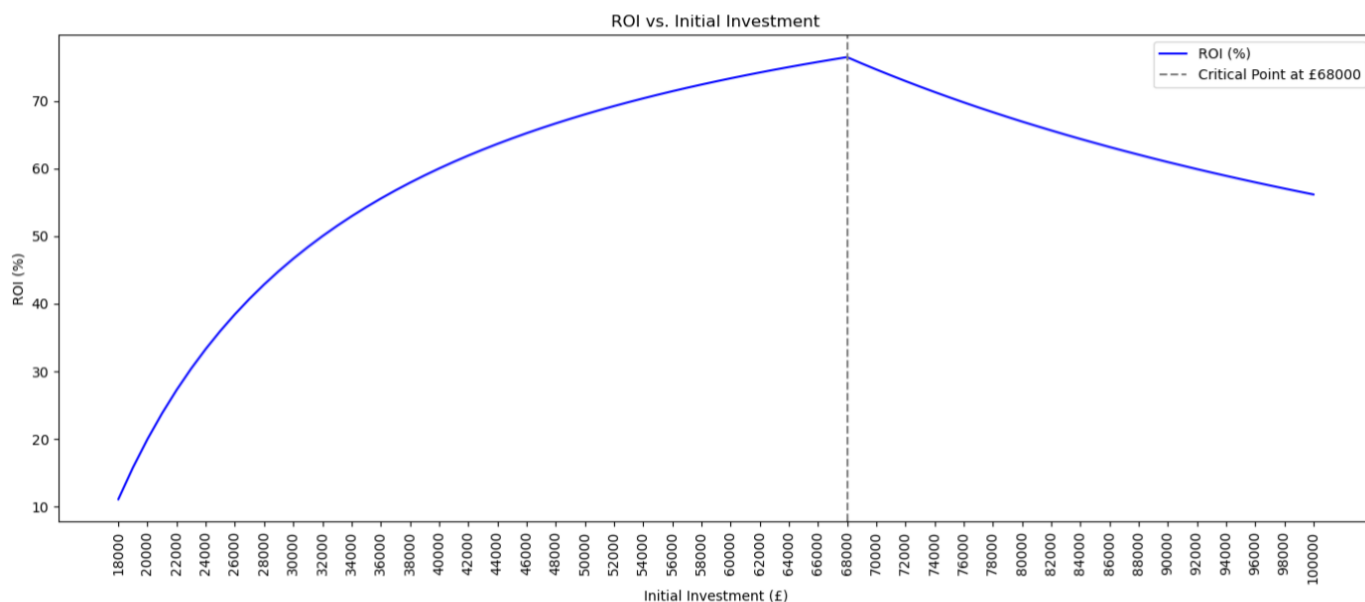
ROI vs. Initial Investment

Parabolic Growth and Decline:

- The ROI increases rapidly with the initial investment, peaking at around the £68,000 mark. This peak indicates the point at which the investment strategy is most efficient in generating returns relative to the amount invested.
- Beyond £68,000, the ROI begins to decline, suggesting that the efficiency of additional investments decreases, leading to diminishing returns.

Critical Point at £68,000:

- The critical point at £68,000 is evident in the ROI chart as well, where the ROI reaches its maximum. This reinforces the observation from the expected profit chart that £68,000 is a significant threshold for the investment strategy.



Investment Return Analysis with Different Initial Investments (Excel VBA)

Investment Return Analysis										
Initial investment (£)	20000	30000	40000	50000	60000	70000	80000	90000	100000	110000
number of shares of stock ABC (x)	0	0	0	0	0	19	130	236	347	456
number of call options bought (y)	2	12	50	32	42	50	50	50	50	50
number of call options sold (z)	0	0	28	0	0	0	0	0	0	0
auxiliary variable for bonds (a)	200	200	200	200	200	218	304	392	478	565
Shares of stock ABC	0	0	0	0	0	380	2600	4720	6940	9120
European call option	2000	12000	22000	32000	42000	50000	50000	50000	50000	50000
Zero-coupon bonds	18000	18000	18000	18000	18000	19620	27360	35280	43020	50850
Total Investment	20000	30000	40000	50000	60000	70000	79960	90000	99960	109970
Profit : Scenario 1	3000	8000	13000	18000	23000	27180	28040	28920	29780	30650
Profit : Scenario 2	7000	32000	57000	82000	107000	127560	130640	133640	136720	139770
Profit : Scenario 3	2000	2000	2000	2000	2000	2028	2000	2032	2004	2002
Expected Profit	4000	14000	24000	34000	44000	52256	53560	54864	56168	57474
ROI : Return on Investment (%)	20.00%	46.67%	60.00%	68.00%	73.33%	74.65%	66.95%	60.96%	56.17%	52.25%
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Optimal Investment Analysis

From this further analysis, we determined that an initial investment of £68,000 yields the highest ROI of 76.47%.

Optimal Investment Allocation:

- Shares of stock ABC (k): £0
- European call options (m): £50,000
- Zero-coupon bonds (n): £18,000
- Total Investment: £68,000

Profit Scenarios:

- Scenario 1: Stock price remains at £20
- Profit: £27,000
- Scenario 2: Stock price rises to £40
- Profit: £127,000
- Scenario 3: Stock price falls to £12
- Profit: £2,000

Expected Profit: £52,000

Return on Investment (ROI%): 76.47%

Conclusion

The combined analysis from Python and Excel VBA highlights several key findings:

Optimal Allocation:

- The optimal investment strategy involves a mix of stocks, call options, and bonds, with a notable emphasis on bonds as the initial investment increases.
- Stocks and call options play a crucial role up to the critical investment point (£68,000), beyond which bonds dominate the investment strategy.

Diminishing Returns:

- The diminishing returns beyond the £68,000 investment highlight the importance of finding a balance between risk and return. Investors should consider this critical point when deciding on the amount to invest.

Balanced Strategy:

- The consistent allocation towards call options and the gradual increase in shares bought suggest a balanced and diversified investment approach, optimizing for both risk and return.

Efficiency of Investments:

- The ROI analysis shows that the investment strategy is most efficient up to the £68,000 mark. Beyond this point, additional investments yield lower incremental returns, indicating diminishing efficiency.

Role of Zero-Coupon Bonds:

- Investing in zero-coupon bonds is crucial for achieving the minimum required profit of £2,000 under all conditions. This ensures profit stability and risk management.

By understanding these trends, investors can tailor their strategies based on their available capital and desired outcomes, ensuring an optimized and informed investment decision-making process. This comprehensive analysis demonstrates that strategic investment in call options and zero-coupon bonds can significantly enhance returns while managing risk effectively.

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