# Mathematical Induction

# § 5.1-5.2: Mathematical Induction

- A powerful, rigorous technique for proving that a predicate $P(n)$ is true for all positive integers.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

$$\begin{array}{l} P(1) \\ \forall n \geq 1\ (P(n) \rightarrow P(n+1)) \\ \hline \therefore \forall n \geq 1 P(n) \end{array}$$

*"The First Principle of Mathematical Induction"*

# Outline of an Inductive Proof

- Let us say we want to prove $\forall n\, P(n)$…
  - Do the *base case* (or *basis step*): Prove $P(1)$.
  - Do the *inductive step*: Prove $\forall n\, (P(n) \rightarrow P(n+1))$
    - *E.g.* you could use a direct proof, as follows:
    - Let $n \in \mathbf{N}$, assume $P(n)$. (*inductive hypothesis*)
    - Now, under this assumption, prove $P(n+1)$.
  - The inductive inference rule then gives us $\forall n\, P(n)$.

# Generalizing Induction

- Rule can also be used to prove $\forall n \geq c\ P(n)$ for a given constant $c \in \mathbf{Z}$, where maybe $c \neq 1$.
  - In this circumstance, the base case is to prove $P(c)$ rather than $P(1)$, and the inductive step is to prove $\forall n \geq c\ (P(n) \rightarrow P(n+1))$.
- Induction can also be used to prove $\forall n \geq c\ P(a_n)$ for any arbitrary series $\{a_n\}$.

# Second Principle of Induction

A.k.a. "Strong Induction"

- Characterized by another inference rule:

$$\dfrac{\begin{array}{l} P(1) \\ \forall n\geq 1: (\overbrace{\forall 1\leq k\leq n\ P(k)}^{P \text{ is true in } all \text{ previous cases}} \rightarrow P(n+1)\ ) \end{array}}{\therefore \forall n\geq 1: P(n)}$$

- The only difference between this and the 1st principle is that:
  - the inductive step here makes use of the stronger hypothesis that $P(k)$ is true for *all* smaller numbers $k<n+1$, not just for $k=n$.

# 2nd Principle Example

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. $P(n)$="$n$ can be…"
- **Base case:** 12=3(4), 13=2(4)+1(5), 14=1(4)+2(5), 15=3(5), so $\forall 12 \le n \le 15$, $P(n)$.
- **Inductive step:** Let $n \ge 15$, assume $\forall 12 \le k \le n\ P(k)$. Note $12 \le n-3 \le n$, so $P(n-3)$, so add a 4-cent stamp to get postage for $n+1$.

# Strong Induction

- BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively. This shows that P(12), P(13), P(14), and P(15) are true.
- INDUCTIVE STEP:
  - The inductive hypothesis is the statement that P(j) is true for $12 \le j \le k$, where k is an integer with $k \ge 15$.
  - To complete the inductive step, we assume that we can form postage of j cents, where $12 \le j \le k$. We need to show that under the assumption that P(k + 1) is true, we can also form postage of k + 1 cents.

(Cont…)

# Strong Induction(Cont)

- INDUCTIVE STEP:
  - …
  - Using the inductive hypothesis, we can assume that $P(k - 3)$ is true because $k - 3 \geq 12$, that is, we can form postage of $k - 3$ cents using just 4-cent and 5-cent stamps. To form postage of $k + 1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k - 3$ cents.
  - That is, we have shown that if the inductive hypothesis is true, then $P(k + 1)$ is also true. This completes the inductive step.

Hence, $P(n)$ holds for all $n \geq 12$.

# 示例总结

- 为什么$n \geq 12$?
  - 对于$n < 12$，$P(n)$不一定为真。$n=11, 7, 6, 3, 2, 1$
  - 数学归纳法的起点（base）选择，一定是从此开始$P(n)$恒为真

# 示例总结

- 为什么选4个数作为base case?
  - 因为有4分邮资，所以对于某个k使得P(k)，必然有P(k+4)成立，即$P(k) \rightarrow P(k+4)$, 因此若存在P(k), P(k+1), P(k+2), P(k+3)均为真，则可保证对于所有$n \geq k$，P(n)为真，可以应用强数学归纳法
  - 若选择的数少于4个，仍使用强数学归纳法，会有“漏掉”的情况，比如，只证明了P(12), P(13), P(14)，则P(15)、P(19)…是没有对应的base的，如$P(11) \rightarrow P(15)$，但P(11)为假

**44.** Prove that if $A_1, A_2, \ldots, A_n$ and $B$ are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

当n=1时. 显然成立

当n=2时 $(A_1-B)\cup(A_2-B) = (A_1\cap\overline{B})\cup(A_2\cap\overline{B})$

$= (A_1\cup A_2)\cap\overline{B} = (A_1\cup A_2) - B$

当n≥2时. 我们有

$(A_1-B)\cup(A_2-B)\cup\cdots\cup(A_n-B)\cup(A_{n+1}-B)$

$= \big((A_1\cup A_2\cup\cdots\cup A_n)-B\big)\cup(A_{n+1}-B)$

$= (A_1\cup A_2\cup\cdots\cup A_n\cup A_{n+1})-B$

故命题成立

44. i) n=1时. $A_1-B=A_1-B$ 显然成立.

ii) 假设n=k时. $(A_1-B)\cup(A_2-B)\cdots\cup(A_k-B)=(A_1\cup A_2\cup\cdots\cup A_k)-B$ 成立.

当n=k+1时. $(A_1-B)\cup(A_2-B)\cup\cdots\cup(A_k-B)\cup(A_{k+1}-B)$

$=[(A_1\cup A_2\cup\cdots\cup A_k)-B]\cup(A_{k+1}-B)$.

$=[(A_1\cup A_2\cup\cdots\cup A_k)\cap\overline{B}]\cup(A_{k+1}\cap\overline{B})$

$=[(A_1\cup A_2\cup\cdots\cup A_k\cup A_{k+1})\cap\overline{B}$

$=(A_1\cup A_2\cup\cdots\cup A_{k+1})-B$. 即n=k+1时成立.

由数学归纳法可知. 结论对 $\forall n\in N^+$ 成立.

44. 1) $n=1$，$A_1 - B = A_1 - B$ 显然

2) $n \geq 1$，

假设 $(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B$ 成立

则 $(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B$

$= (\bigcup_{i=1}^{n} A_i \cup A_{n+1}) - B$

设 $P = \bigcup_{i=1}^{n} A_i$

$$
\begin{aligned}
(P \cup A_{n+1}) - B &= \{x \mid (x \in P \vee x \in A_{n+1}) \wedge x \notin B\} \\
&= \{x \mid (x \in P \wedge x \notin B) \vee (x \in A_{n+1} \wedge x \notin B)\} \\
&= (P - B) \cup (A_{n+1} - B) \\
&= (\bigcup_{i=1}^{n} A_i - B) \cup (A_{n+1} - B) \\
&= (A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B)
\end{aligned}
$$

证毕

44. BASIS STEP: If $n=1$, then it's obviously that $A_1 - B = A_1 - B$.

If $n=2$, then $(A_1 - B) \cup (A_2 - B) = (A_1 \cap \overline{B}) \cup (A_2 \cap \overline{B}) = (A_1 \cup A_2) \cap \overline{B} = (A_1 \cup A_2) - B$

INDUCTIVE STEP: Assume that $(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B$ is true.

We know $(A_1 - B)(A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B)$

$$
\begin{aligned}
&= [(A_1 \cup A_2 \cup \cdots \cup A_n) - B] \cup (A_{n+1} - B) \\
&= [(A_1 \cup A_2 \cup \cdots \cup A_n) \cap \overline{B}] \cup (A_{n+1} \cap \overline{B}) \\
&= (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) \cap \overline{B} \\
&= (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B
\end{aligned}
$$

Hence, $(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B$, for every positive integar $n$.

**4.** Let $P(n)$ be the statement that a postage of $n$ cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

**a)** Show that the statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$.

**b)** What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?

**c)** What do you need to prove in the inductive step of a proof that $P(n)$ is true for all integers $n \geq 18$?

**d)** Complete the inductive step for $k \geq 21$.

**e)** Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.

4. a) BASIS STEP: We can form postage of 18, 19, 20 and 21 cents using one 4-cent stamp and two 7-cent stamps, three 4-cent stamps and one 7-cent stamp, five 4-cent stamps, three 7-cent stamps, respectively. This shows that $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true

INDUCTIVE STEP:

(b) The inductive hypothesis is the statement that $P(j)$ is true for $18 \leq j \leq k$, where $k$ is an integar with $k \geq 21$.

(c) To complete the inductive step, we assume that we can form postage of $j$ cents where $18 \leq j \leq k$, We need to show that under the assumption that $P(k+1)$ is true, we can also form postage of $k+1$ cents.

(d) Using the inductive hypothes's, we can assume that $P(k\ 3)$ is true because $k-3 \geq 18$, that is, we can form postage of $k-3$ cents using just 4-cent and 7-cent stamps. To form postage of $k+1$ cents, we need only add another 4 cent stamp to the stamps we used to form postage of $k\ 3$ cents.

(e) That is, we have shown that if the inductive hypothesis is true, then $P(k+1)$ is also true. This completes the inductive step.

Hence, $P(n)$ holds for all $n \geq 18$.