Log Transformations in Linear Regression

This document provides details about the model interpretion when the predictor and/or response variables are log-transformed. For simplicity, we will discuss transformations for the simple linear regression model:

$$y = \beta_0 + \beta_1 x \tag{1}$$

All results and interpretations can be easily extended to transformations in multiple regression models.

Note: *log* refers to the natural logarithm.

Log-transformation on the response variable

Suppose we fit a linear regression model with $\log(y)$, the log-transformed y, as the response variable. Under this model, we assume a linear relationship exists between x and $\log(y)$, such that $\log(y) \sim N(\beta_0 + \beta_1 x, \sigma^2)$ for some β_0, β_1 and σ^2. In other words, we can model the relationship between x and $\log(y)$ using the model in (2).

$$\log(y) = \beta_0 + \beta_1 x \tag{2}$$

If we interpret the model in terms of $\log(y)$, then we can use the usual interpretations for slope and intercept. When reporting results, however, it is best to give all interpretations in terms of the original response variable y, since interpretations using log-transformed variables are often more difficult to truly understand.

In order to get back on the original scale, we need to use the exponential function (also known as the anti-log), $\exp\{x\} = e^x$. Therefore, we use the model in (2) for interpretations and predictions, we will use (3) to state our conclusions in terms of y.

$$\begin{aligned}
\exp\{\log(y)\} &= \exp\{\beta_0 + \beta_1 x\} \\
\Rightarrow y &= \exp\{\beta_0 + \beta_1 x\} \qquad (3) \\
\Rightarrow y &= \exp\{\beta_0\}\exp\{\beta_1 x\}
\end{aligned}$$

In order to interpret the slope and intercept, we need to first understand the relationship between the mean, median and log transformations.

Mean, Median, and Log Transformations

Suppose we have a dataset `y` that contains the following observations:

```r
y <- c(3,5,6,7,8)
y
```

```
## [1] 3 5 6 7 8
```

If we log-transform the values of `y` then calculate the mean and median, we have

```r
log_y <- tibble(log_y = log(y))
summary <- log_y %>%
  summarise(mean_log_y = mean(log_y), median_log_y = median(log_y))
kable(summary,digits=5)
```

mean_log_y	median_log_y
1.70503	1.79176

If we calculate the mean and median of `y`, then log-transform the mean and median, we have

```
centers <- tibble(y) %>% summarise(mean_y = mean(y), median_y = median(y))
summary2 <- centers %>%
  summarise(log_mean = log(mean_y), log_median = log(median_y))
kable(summary2,digits=5)
```

log_mean	log_median
1.75786	1.79176

This is a simple illustration to show

1. $\text{Mean}[\log(y)] \neq \log[\text{Mean}(y)]$ - the mean and log are not commutable
2. $\text{Median}[\log(y)] = \log[\text{Median}(y)]$ - the median and log are commutable

Interpretaton of model coefficients

Using (2), the mean $\log(y)$ for any given value of x is $\beta_0 + \beta_1 x$; however, this does **not** indicate that the mean of $y = \exp\{\beta_0 + \beta_1 x\}$ (see previous section). From the assumptions of linear regression, we assume that for any given value of x, the distribution of $\log(y)$ is Normal, and therefore symmetric. Thus the median of $\log(y)$ is equal to the mean of $\log(y)$, i.e $\text{Median}(\log(y)) = \beta_0 + \beta_1 x$.

Since the log and the median are commutable, $\text{Median}(\log(y)) = \beta_0 + \beta_1 x \Rightarrow \text{Median}(y) = \exp\{\beta_0 + \beta_1 x\}$. Thus, when we log-transform the response variable, the interpretation of the intercept and slope are in terms of the effect on the **median** of y.

Intercept: The intercept is expected median of y when the predictor variable equals 0. Therefore, when $x = 0$,

$$\begin{aligned} \log(y) &= \beta_0 + \beta_1 \times 0 = \beta_0 \\ \Rightarrow y &= \exp\{\beta_0\} \end{aligned} \tag{4}$$

Interpretation: When $x = 0$, the median of y is expected to be $\exp\{\beta_0\}$.

Slope: The slope is the expected change in the median of y when x increases by 1 unit. The change in the median of y is

$$\exp\{[\beta_0 + \beta_1(x+1)] - [\beta_0 + \beta_1 x]\} = \frac{\exp\{\beta_0 + \beta_1(x+1)\}}{\exp\{\beta_0 + \beta_1 x\}} = \frac{\exp\{\beta_0\}\exp\{\beta_1 x\}\exp\{\beta_1\}}{\exp\{\beta_0\}\exp\{\beta_1 x\}} = \exp\{\beta_1\} \tag{5}$$

Thus, the median of y for $x + 1$ is $\exp\{\beta_1\}$ times the median of y for x.

Interpretation: When x increases by one unit, the median of y is expected to multiply by a factor of $\exp\{\beta_1\}$.

Log-transformation on the predictor variable

Suppose we fit a linear regression model with $\log(x)$, the log-transformed x, as the predictor variable. Under this model, we assume a linear relationship exists between $\log(x)$ and y, such that $y \sim N(\beta_0 + \beta_1 \log(x), \sigma^2)$ for some β_0, β_1 and σ^2. In other words, we can model the relationship between $\log(x)$ and y using the model in (6).

$$y = \beta_0 + \beta_1 \log(x) \tag{6}$$

Intercept: The intercept is the mean of y when $\log(x) = 0$, i.e. $x = 1$.

Interpretation: When $x = 1$ ($\log(x) = 0$), the mean of y is expected to be β_0.

Slope: The slope is interpreted in terms of the change in the mean of y when x is mutliplied by a factor of C, since $\log(Cx) = \log(x) + \log(C)$. Thus, when x is multiplied by a factor of C, the change in the mean of y is

$$\begin{aligned} -[\beta_0 + \beta_1 \log(x)] &= \beta_1[\log(Cx) - \log(x)] \\ &= \beta_1[\log(C) + \log(x) - \log(x)] \\ &= \beta_1 \log(C) \end{aligned} \tag{7}$$

Thus the mean of y changes by $\beta_1 \log(C)$ units.

Interpretation: When x is multiplied by a factor of C, the mean of y is expected to change by $\beta_1 \log(C)$ units. For example, if x is doubled, then the mean of y is expected to change by $\beta_1 \log(2)$ units.

Log-transformation on the the response and predictor variable

Suppose we fit a linear regression model with $\log(x)$, the log-transformed x, as the predictor variable and $\log(y)$, the log-transformed y, as the response variable. Under this model, we assume a linear relationship exists between $\log(x)$ and $\log(y)$, such that $\log(y) \sim N(\beta_0 + \beta_1 \log(x), \sigma^2)$ for some β_0, β_1 and σ^2. In other words, we can model the relationship between $\log(x)$ and $\log(y)$ using the model in (8).

$$\log(y) = \beta_0 + \beta_1 \log(x) \tag{8}$$

Because the response variable is log-transformed, the interpretations on the original scale will be in terms of the median of y (see the section on the log-transformed response variable for more detail).

Intercept: The intercept is the mean of y when $\log(x) = 0$, i.e. $x = 1$. Therefore, when $\log(x) = 0$,

$$\begin{aligned} \log(y) &= \beta_0 + \beta_1 \times 0 = \beta_0 \\ \Rightarrow y &= \exp\{\beta_0\} \end{aligned} \tag{9}$$

Interpretation: When $x = 1$ ($\log(x) = 0$), the median of y is expected to be $\exp\{\beta_0\}$.

Slope: The slope is interpreted in terms of the change in the median y when x is mutliplied by a factor of C, since $\log(Cx) = \log(x) + \log(C)$. Thus, when x is multiplied by a factor of C, the change in the median of y is

$$
\begin{aligned}
\exp\{[\beta_0 + \beta_1 \log(Cx)] - [\beta_0 + \beta_1 \log(x)]\} &= \exp\{\beta_1[\log(Cx) - \log(x)]\} \\
&= \exp\{\beta_1[\log(C) + \log(x) - \log(x)]\} \qquad (10) \\
&= \exp\{\beta_1 \log(C)\} = C^{\beta_1}
\end{aligned}
$$

Thus, the median of y for Cx is C^{β_1} times the median of y for x.

Interpretation: When x is multiplied by a factor of C, the median of y is expected to multiple by a factor of C^{β_1}. For example, if x is doubled, then the median of y is expected to multiply by 2^{β_1}.