Spatial Statistics: Point-referenced Data

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STA 440L, Duke University

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(Slides were adapted from notes by Joshua Warren, PhD)

Goals of a Point-referenced Analysis

1. Estimation and explanation

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   - Typical regression parameter estimation

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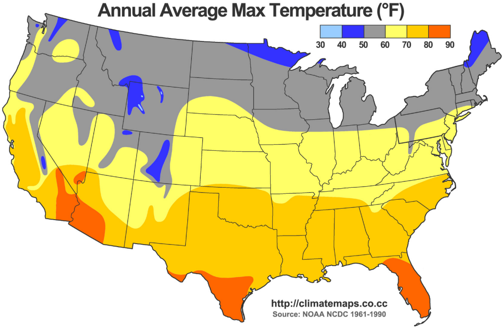
3. Design issues
   - Where to put a new air pollution monitor to optimize future prediction criteria?

Goals of a Point-referenced Analysis

- Estimation and Explanation



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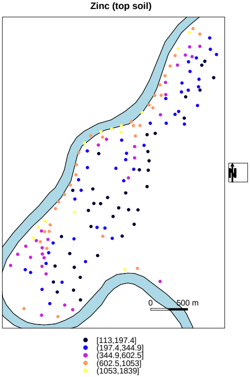

Figure: Observed Data

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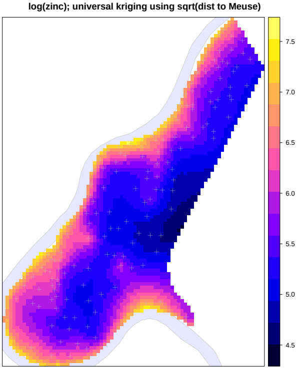

Figure: Predictions

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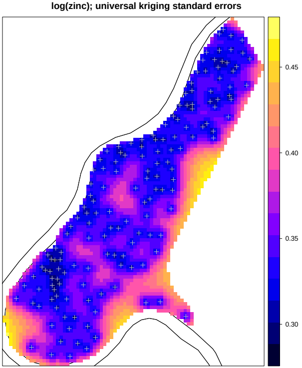

Figure: Standard Errors

Point-referenced Modeling

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- Advanced models built on similar ideas
  - Latent processes often used

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  - Inference based on this partial realization of the spatial process

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- Goal: Remove trend without overfitting

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  - Zero mean random process with spatial covariance function
- Choice of spatial covariance/semivariogram function based on sample semivariogram analysis

Point-referenced Modeling

Common Modeling Assumptions for $w(\boldsymbol{s})$

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  - **Most common assumption in applied spatial modeling!**

Point-referenced Modeling

Common Covariance Functions (Isotropic)

- Exponential:
  - $C(\|\boldsymbol{h}\|) = \begin{cases} \sigma^2 \exp\{-\phi \|\boldsymbol{h}\|\} & \|\boldsymbol{h}\| > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$
- $\text{Cov}\{w(\boldsymbol{s}), w(\boldsymbol{s}+\boldsymbol{h})\} \to 0$ as $\|h\|$ gets large
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Point-referenced Modeling

Common Covariance Functions (Isotropic)

- Gaussian:
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Common Covariance Functions (Isotropic)

- Spherical:
  - $C\left(\|\boldsymbol{h}\|\right) = \begin{cases} 0 & \|\boldsymbol{h}\| > \frac{1}{\phi} \\ \sigma^2\left\{1 - \frac{3\phi\|\boldsymbol{h}\|}{2} + \frac{1}{2}\left(\phi\|\boldsymbol{h}\|\right)^3\right\} & 0 < \|\boldsymbol{h}\| \le \frac{1}{\phi} \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$
- $\text{Cov}\left\{w\left(\boldsymbol{s}\right), w\left(\boldsymbol{s} + \boldsymbol{h}\right)\right\} = 0$ for $\|h\| \ge \frac{1}{\phi}$
- $\text{Cov}\left\{w\left(\boldsymbol{s}\right), w\left(\boldsymbol{s} + \boldsymbol{h}\right)\right\} = \tau^2 + \sigma^2$ when $\|h\| = 0$
- Independence after certain distance

Point-referenced Modeling

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  - Range: Distance at which the correlation is zero (spherical)
  - Effective Range: Distance at which the correlation is 0.05 (most other structures)

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- How to handle nonstationarity?
  - Flexible Bayesian models are ideal in these settings

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- Note: Proc Mixed and Glimmix can handle frequentist spatial models using MLE (can be very slow)

Bayesian Model Fitting

$$Y(\boldsymbol{s}_i) = \mathbf{x}(\boldsymbol{s}_i)^T \beta + w(\boldsymbol{s}_i) + \epsilon(\boldsymbol{s}_i);$$

$$\boldsymbol{s}_i \in \boldsymbol{D}; i = 1, \ldots, n$$

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- $w(\boldsymbol{s}_i)$: spatially correlated "random effect" (in frequentist terms)

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$$\boldsymbol{w} = \{w(\boldsymbol{s}_1), \ldots, w(\boldsymbol{s}_n)\}^T \sim \text{MVN}\left(\boldsymbol{0}_n, \sigma_w^2 \Sigma(\phi)\right)$$

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- $\boldsymbol{\beta}$: flat prior $\propto 1$ or $\text{MVN}\left(\mathbf{0}_{p+1}, \sigma_\beta^2 I_{p+1}\right)$; σ_β^2 fixed, large
- ϕ: typically only a single parameter; Uniform or Gamma $(\phi > 0)$

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  2. Unconditionally, marginalizing over the spatial effects
- Pros and cons to implementing each option

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- We must update the $\boldsymbol{w}$ vector each iteration since we condition on it

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- Interest in $f\left(\boldsymbol{Y}_0 | \boldsymbol{Y}\right)$ where $\boldsymbol{Y}_0 = \left\{Y\left(\boldsymbol{s}_{0,1}\right), \ldots, Y\left(\boldsymbol{s}_{0,m}\right)\right\}^T$