Bayesian Estimation in Linear Models & Choice of Prior Distributions

STA521 Predicitve Models Duke University

Merlise Clyde

February 27, 2017

Bayesian Estimation

Model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with $\boldsymbol{\epsilon} \sim \mathsf{N}(\mathbf{0}_n, \sigma^2 \mathbf{I}_n)$ is equivalent to

$$\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n/\phi)$$

$\phi = 1/\sigma^2$ is the *precision*.

In the Bayesian paradigm describe uncertainty about unknown parameters using probability distributions

- Prior Distribution $p(\boldsymbol{\beta}, \phi)$ describes uncertainty about parameters prior to seeing the data
- Posterior Distribution $p(\boldsymbol{\beta}, \phi \mid \mathbf{Y})$ describes uncertainty about the parameters after updating beliefs given the observed data
- updating rule is based on Bayes Theorem

$$p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \propto \mathcal{L}(\boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi)$$

reweight prior beliefs by likelihood of parameters under observed data

Prior Distributions

Factor joint prior distribution

$$p(\boldsymbol{\beta}, \phi) = p(\boldsymbol{\beta} \mid \phi) p(\phi)$$

Convenient choice is to take "Conjugate" family

- $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(b_0, \Phi_0^{-1}/\phi)$ where b_0 is the prior mean and Φ^{-1}/ϕ is the prior covariance of $\boldsymbol{\beta}$

$$p(\boldsymbol{\beta} \mid \phi) = \frac{1}{(2\pi)^{p/2}} |\phi \Phi_0|^{1/2} e^{-\frac{1}{2}\phi(\boldsymbol{\beta}-\mathbf{b}_0)^T \Phi_0 (\boldsymbol{\beta}-\mathbf{b}_0)}$$

- $\phi \sim \mathbf{G}(\nu_0/2, \mathsf{SS}_0/2)$ with $\mathsf{E}(\sigma^2) = \mathsf{SS}_0/(\nu_0 - 2)$

$$p(\phi) = \frac{1}{\Gamma(\nu_0/2)} \left(\frac{\mathsf{SS}_0}{2}\right)^{\nu_0/2} \phi^{\nu_0/2-1} e^{-\phi \mathsf{SS}_0/2}$$

- $(\boldsymbol{\beta}, \phi)^T \sim \mathsf{NG}(\mathbf{b}_0, \Phi_0, \nu_0, \mathsf{SS}_0)$
- Conjugate "Normal-Gamma" family implies

$$(\boldsymbol{\beta}, \phi)^T \mid \mathbf{Y} \sim \mathsf{NG}(\mathbf{b}_n, \Phi_n, \nu_n, \mathsf{SS}_n)$$

Finding the Posterior Distribution

Express Likelihood: $\mathcal{L}(\beta, \phi) \propto \phi^{n/2} e^{-\phi \frac{\text{SSE}}{2}} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}$

$$
\begin{aligned}
p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad & \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\text{SSE}+\text{SS}_0)} \\
& e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_0)^T\Phi(\boldsymbol{\beta}-\mathbf{b}_0)}
\end{aligned}
$$

Finding the Posterior Distribution

$$p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0)}$$
$$e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_0)^T\Phi(\boldsymbol{\beta}-\mathbf{b}_0)}$$

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta} - 2\boldsymbol{\beta}^T\Phi\mathbf{b} + \mathbf{b}^T\Phi\mathbf{b})\right\}$$

- Expand quadratics and regroup terms
- Read off posterior precision from Quadratic in $\boldsymbol{\beta}$
- Read off posterior mean from Linear term in $\boldsymbol{\beta}$
- will need to complete the quadratic in the posterior mean

Expand and Regroup

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta}-2\boldsymbol{\beta}^T\Phi\mathbf{b}+\mathbf{b}^T\Phi\mathbf{b})\right\}$$

$$\begin{aligned}
p(\boldsymbol{\beta},\phi \mid \mathbf{Y}) &\propto \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\textsf{SSE}+\textsf{SS}_0)} \\
&\quad e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_0)^T\boldsymbol{\Phi}_0(\boldsymbol{\beta}-\mathbf{b}_0)} \\
&= \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\textsf{SSE}+\textsf{SS}_0)} \\
&\quad e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}+\Phi_0)\boldsymbol{\beta}\right)} \\
&\quad e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \\
&\quad e^{-\frac{\phi}{2}(\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0)}
\end{aligned}$$

Identify Hyperparameters and Complete the Quadratic

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta} - 2\boldsymbol{\beta}^T\Phi\mathbf{b} + \mathbf{b}^T\Phi\mathbf{b})\right\}$$

Let $\Phi_n = \mathbf{X}^T\mathbf{X} + \Phi_0$

$$\begin{aligned}
p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad & \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathrm{SSE}+\mathrm{SS}_0)} \\
& e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}+\Phi_0)\boldsymbol{\beta}\right)} \\
& e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T\Phi_n\Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \\
& e^{-\frac{\phi}{2}(\mathbf{b}_n^T\Phi_n\mathbf{b}_n - \mathbf{b}_n^T\Phi_0\mathbf{b}_n)} \\
& e^{-\frac{\phi}{2}(\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0)} \\
= \quad & \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathrm{SSE}+\mathrm{SS}_0+\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0-\mathbf{b}_n^T\Phi_n\mathbf{b}_n)} \\
& e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\Phi_n)\boldsymbol{\beta}\right)} \\
& e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T\Phi_n\Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \\
& e^{-\frac{\phi}{2}(\mathbf{b}_n^T\Phi_n\mathbf{b}_n)}
\end{aligned}$$

Posterior Distribution

$$
\begin{aligned}
p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad &\propto \quad \phi^{\frac{n+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0+\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0-\mathbf{b}_n^T\Phi_n\mathbf{b}_n)} \\
& \quad\quad \phi^{\frac{p}{2}} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_n)^T\Phi_n(\boldsymbol{\beta}-\mathbf{b}_n)}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_n &= \mathbf{X}^T\mathbf{X} + \Phi_0 \\
\mathbf{b}_n &= \Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} + \Phi_0\mathbf{b}_0)
\end{aligned}
$$

Posterior Distribution

$$
\begin{aligned}
\boldsymbol{\beta} \mid \phi, \mathbf{Y} \quad &\sim \quad \mathsf{N}(\mathbf{b}_n, (\phi\Phi_n)^{-1}) \\
\phi \mid \mathbf{Y} \quad &\sim \quad \mathbf{G}\left(\frac{n+\nu_0}{2}, \frac{\mathsf{SSE}+\mathsf{SS}_0+\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0-\mathbf{b}_n^T\Phi_n\mathbf{b}_n}{2}\right)
\end{aligned}
$$

Predictive Distribution

Suppose $\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi \sim \mathsf{N}(\mathbf{X}^*\boldsymbol{\beta}, \mathbf{I}/\phi)$ and is conditionally independent of $\mathbf{Y}$ given $\boldsymbol{\beta}$ and ϕ

What is the predictive distribution of $\mathbf{Y}^* \mid \mathbf{Y}$?

$\mathbf{Y}^* = \mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^*$ is independent of $\mathbf{Y}$ given ϕ

$$
\begin{aligned}
\mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\epsilon}^* \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{X}^*\mathbf{b}_n, (\mathbf{X}^*\Phi_n^{-1}\mathbf{X}^{*T} + \mathbf{I})/\phi) \\
\mathbf{Y}^* \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{X}^*\mathbf{b}_n, (\mathbf{X}^*\Phi_n^{-1}\mathbf{X}^{*T} + \mathbf{I})/\phi) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}\left(\frac{\nu_n}{2}, \frac{\hat{\sigma}^2\nu_n}{2}\right) \\
\mathbf{Y}^* \mid \mathbf{Y} &\sim t_{\nu_n}(\mathbf{X}^*\mathbf{b}_n, \hat{\sigma}^2(\mathbf{I} + \mathbf{X}^*\Phi_n^{-1}\mathbf{X}^T))
\end{aligned}
$$

Alternative Derivation

Conditional Distribution:

$$
\begin{aligned}
f(\mathbf{Y}^* \mid \mathbf{Y}) &= \frac{f(\mathbf{Y}^*, \mathbf{Y})}{f(\mathbf{Y})} \\
&= \frac{\iint f(\mathbf{Y}^*, \mathbf{Y} \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi) d\boldsymbol{\beta}\, d\phi}{f(\mathbf{Y})} \\
&= \frac{\iint f(\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi) f(\mathbf{Y} \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi) d\boldsymbol{\beta}\, d\phi}{f(\mathbf{Y})} \\
&= \iint f(\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) d\boldsymbol{\beta}\, d\phi
\end{aligned}
$$

Conjugate Priors

Definition

A class of prior distributions $\mathcal{P}$ for $\boldsymbol{\theta}$ is conjugate for a sampling model $p(y \mid \boldsymbol{\theta})$ if for every $p(\boldsymbol{\theta}) \in \mathcal{P}$, $p(\boldsymbol{\theta} \mid \mathbf{Y}) \in \mathcal{P}$.

Advantages:

- Closed form distributions for most quantities; bypass Monte Carlo for calculations
- Simple updating in terms of sufficient statistics "weighted average" - useful with big data
- Interpretation as prior samples - prior sample size
- Elicitation of prior through imaginary or historical data
- limiting "non-proper" form recovers MLEs

Choice of conjugate prior?

Unit Information Prior

Unit information prior $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\hat{\boldsymbol{\beta}}, n(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$

- Fisher Information is $\phi\mathbf{X}^T\mathbf{X}$ based on a sample of n observations
- Inverse Fisher information is covariance matrix of MLE
- "average information" in one observation is $\phi\mathbf{X}^T\mathbf{X}/n$
- center prior at MLE and base covariance on the information in "1" observation
- Posterior mean

$$\frac{n}{1+n}\hat{\boldsymbol{\beta}} + \frac{1}{1+n}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$$

- Posterior Distribution

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi \sim \mathsf{N}\left(\hat{\boldsymbol{\beta}}, \frac{n}{1+n}(\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}\right)$$

Cannot represent real prior beliefs; double use of data

Zellner's g-prior

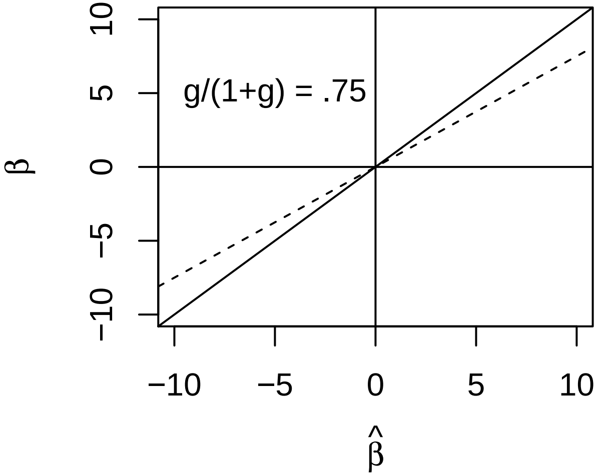
Zellner's g-prior(s) $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathbf{b}_0, g(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi \sim \mathsf{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}} + \frac{1}{1+g}\mathbf{b}_0, \frac{g}{1+g}(\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}\right)$$

- Zellner proposed informative choice for the prior mean
- Invariance under linear transformations of X and Y
- Avoids extra inverses beyond those in obtaining OLS estimates (computational)
- Choice of g?
- $\frac{g}{1+g}$ weight given to the data

Shrinkage

Posterior mean under g-prior with $\mathbf{b}_0 = 0$ $\tilde{\boldsymbol{\beta}} = \frac{g}{1+g}\hat{\boldsymbol{\beta}}$



Jeffreys Prior

Jeffreys proposed a default procedure so that resulting prior would be invariant to model parameterization

$$p(\boldsymbol{\theta}) \propto |\mathcal{I}(\boldsymbol{\theta})|^{1/2}$$

where $\mathcal{I}(\boldsymbol{\theta})$ is the Expected Fisher Information matrix

$$\mathcal{I}(\theta) = -\mathsf{E}[\left[\frac{\partial^2 \log(\mathcal{L}(\boldsymbol{\theta}))}{\partial\theta_i \partial\theta_j}\right]]$$

Fisher Information Matrix

Log Likelihood

$$\log(\mathcal{L}(\boldsymbol{\beta}, \phi)) = \frac{n}{2}\log(\phi) - \frac{\phi}{2}\|(\mathbf{I} - \mathsf{P}_{\mathbf{x}})\mathbf{Y}\|^2 - \frac{\phi}{2}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$$

$$\begin{aligned}
\frac{\partial^2 \log \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} &= \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & -(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \\ -(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X}) & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix} \\
\mathsf{E}[\frac{\partial^2 \log \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}] &= \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix} \\
\mathcal{I}((\boldsymbol{\beta}, \phi)^T) &= \begin{bmatrix} \phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & \frac{n}{2}\frac{1}{\phi^2} \end{bmatrix}
\end{aligned}$$

Jeffreys Prior

Jeffreys Prior

$$
\begin{aligned}
p_J(\boldsymbol{\beta}, \phi) &\propto |\mathcal{I}((\boldsymbol{\beta}, \phi)^T)|^{1/2} \\
&= |\phi(\mathbf{X}^T\mathbf{X}|^{1/2} \left(\frac{n}{2}\frac{1}{\phi^2}\right)^{1/2} \\
&\propto \phi^{p/2-1}|\mathbf{X}^T\mathbf{X}|^{1/2} \\
&\propto \phi^{p/2-1}
\end{aligned}
$$

Improper prior $\iint p_J(\boldsymbol{\beta}, \phi)\, d\boldsymbol{\beta}\, d\phi$ not finite

Formal Bayes Posterior

$$p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \propto p(\mathbf{Y} \mid \boldsymbol{\beta}, \phi)\phi^{p/2-1}$$

if this is integrable, then renormalize to obtain formal posterior distribution

$$\begin{aligned} \boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\hat{\boldsymbol{\beta}}, (\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}) \\ \phi \mid \mathbf{Y} &\sim \mathbf{G}(n/2, \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|^2/2) \end{aligned}$$

Limiting case of Conjugate prior with $\mathbf{b}_0 = 0$, $\Phi = \mathbf{0}$, $\nu_0 = 0$ and $\mathsf{SS}_0 = 0$

Posterior for ϕ does not depend on dimension p;

Jeffreys did not recommend using this

Independent Jeffreys Prior "Reference Prior"

- Treat $\boldsymbol{\beta}$ and ϕ separately ("orthogonal parameterization")
- $p_{IJ}(\boldsymbol{\beta}) \propto |\mathcal{I}(\boldsymbol{\beta})|^{1/2}$
- $p_{IJ}(\phi) \propto |\mathcal{I}(\phi)\|^{1/2}$

$$\mathcal{I}((\boldsymbol{\beta}, \phi)^T) = \left[\begin{array}{cc} \phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & \frac{n}{2}\frac{1}{\phi^2} \end{array} \right]$$

$$p_{IJ}(\boldsymbol{\beta}) \propto |\phi \mathbf{X}^T\mathbf{X}|^{1/2} \propto 1$$

$$p_{IJ}(\phi) \propto \phi^{-1}$$

Independent Jeffreys Prior is

$$p_{IJ}(\beta, \phi) \propto p_{IJ}(\boldsymbol{\beta}) p_{IJ}(\phi) = \phi^{-1}$$

Formal Posterior Distribution

With Independent Jeffreys Prior

$$p_{IJ}(\beta, \phi) \propto p_{IJ}(\boldsymbol{\beta}) p_{IJ}(\phi) = \phi^{-1}$$

Formal Posterior Distribution (Show!)

$$\begin{aligned}
\boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathrm{N}(\hat{\boldsymbol{\beta}}, (\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}((n-p)/2, \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|^2/2) \\
\boldsymbol{\beta} \mid \mathbf{Y} &\sim t_{n-p}(\hat{\boldsymbol{\beta}}, \hat{\sigma}^2(\mathbf{X}^T\mathbf{X})^{-1})
\end{aligned}$$

Bayesian Credible Sets $p(\boldsymbol{\beta} \in C_\alpha) = 1 - \alpha$ correspond to frequentist Confidence Regions

$$\frac{\boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}\hat{\boldsymbol{\beta}}}{\sqrt{\hat{\sigma}^2\boldsymbol{\lambda}^T(\mathbf{X}^T\mathbf{X})^{-1}\boldsymbol{\lambda}}} \sim t_{n-p}$$

Disadvantages of Conjugate Priors

Disadvantages:

- Results may have be sensitive to the prior mean which may appear as an "outlier"

- Cannot capture all possible prior beliefs
- Mixtures of Conjugate Priors

Mixtures of Conjugate Priors

Theorem (Diaconis & Ylivisaker 1985)

Given a sampling model $p(y \mid \boldsymbol{\theta})$ from an exponential family, any prior distribution can be expressed as a mixture of conjugate prior distributions

- Prior $p(\boldsymbol{\theta}) = \int p(\boldsymbol{\theta} \mid \omega) p(\omega)\, d\omega$
- Posterior

$$
\begin{aligned}
p(\boldsymbol{\theta} \mid \mathbf{Y}) &\propto \int p(\mathbf{Y} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \omega) p(\omega)\, d\omega \\
&\propto \int \frac{p(\mathbf{Y} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \omega)}{p(\mathbf{Y} \mid \omega)} p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega \\
&\propto \int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega \\
p(\boldsymbol{\theta} \mid \mathbf{Y}) &= \frac{\int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}{\int p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}
\end{aligned}
$$

Zellner-Siow Cauchy Prior

- Model $\mathbf{Y} \sim \mathrm{N}(\mathbf{1}\beta_0 + \mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n\phi)$ and assume $\mathbf{X}^T\mathbf{1} = 0$ case where X is centered and has mean 0
- Conditional Zellner g prior on $\boldsymbol{\beta}$

$$\boldsymbol{\beta} \mid g, \phi, \beta_0 \sim \mathrm{N}(0, g(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$$

- Independent Jeffrey's prior on β_0, ϕ

$$p(\beta_0, \phi \mid g) \propto 1/\phi$$

- Gamma prior: $1/g \sim G(1/2, 1/2) \Rightarrow$ Cauchy prior distribution

$$\boldsymbol{\beta} \mid \phi, \beta_0 \sim C(0, g(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$$

- Posterior Distributions

$$\beta_0 \mid \mathbf{Y}, \boldsymbol{\beta}, \phi, g \sim N(\bar{Y}, \frac{1}{n\phi})$$

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi, g \sim N(\frac{g}{1+g}\hat{\boldsymbol{\beta}}, \frac{g}{1+g}\frac{1}{\phi}(\mathbf{X}^T\mathbf{X})^{-1})$$

$$\phi \mid \mathbf{Y}, g \sim G(,)$$

$$g \mid \mathbf{Y} \sim ?$$