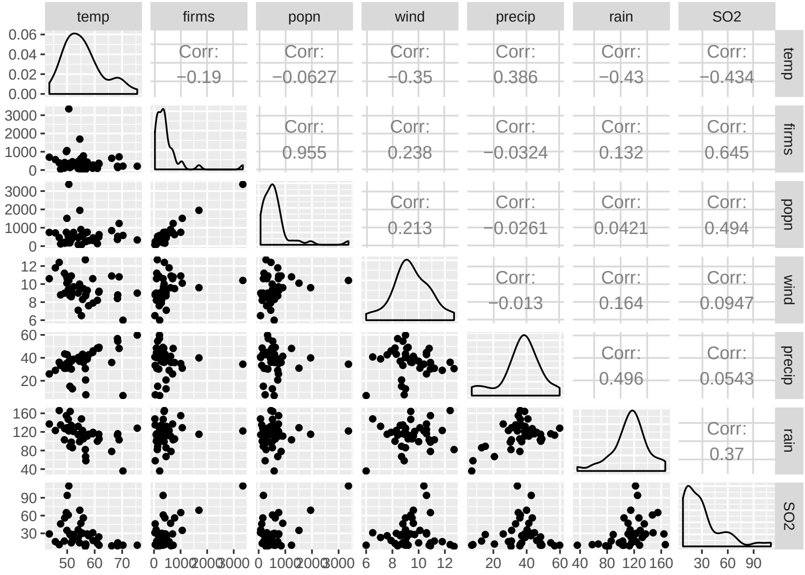
Bayesian Variable Selection

Hoff Chapter 9, Mixtures of g-Priors Liang et al JASA

March 1, 2017

US Air Example



lm summary

```
lm(formula = log(SO2) ~ temp + log(firms) + log(popn) + wi
    precip + rain, data = usair)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.7142760  1.6475086   4.075 0.000261 ***
temp        -0.0649495  0.0227711  -2.852 0.007333 **
log(firms)   0.3698588  0.1934076   1.912 0.064289 .
log(popn)   -0.1771293  0.2335520  -0.758 0.453428
wind        -0.1738606  0.0656713  -2.647 0.012204 *
precip       0.0156032  0.0132718   1.176 0.247893
rain         0.0009153  0.0057335   0.160 0.874104
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '

Residual standard error: 0.5108 on 34 degrees of freedom
Multiple R-squared:  0.5503,^^IAdjusted R-squared:  0.471
F-statistic: 6.936 on 6 and 34 DF,  p-value: 7.12e-05
```

Zellner's g-prior

Centered model:

$$\mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}^c \boldsymbol{\beta} + \epsilon$$

where $\mathbf{X}^c$ is the centered design matrix where all variables have had their mean subtracted

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- $p(\alpha \mid \phi) \propto 1$
- $\boldsymbol{\beta} \mid \alpha, \phi, \boldsymbol{\gamma} \sim \mathsf{N}(0, g\phi^{-1}(\mathbf{X}^{c\prime}\mathbf{X}^c)^{-1})$

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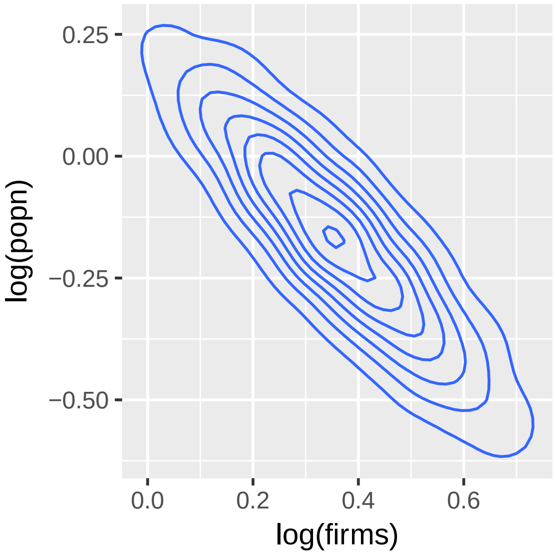
$$\boldsymbol{\beta} \mid \mathbf{Y}, \alpha, \phi \sim N\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}}, \phi^{-1}\frac{g}{1+g}(\mathbf{X}^T\mathbf{X})^{-1}\right)$$

$$\phi \mid \mathbf{Y} \sim \mathrm{Gamma}\left(\frac{n-1}{2}, \frac{\mathrm{SSE} - \frac{1}{1+g}\hat{\boldsymbol{\beta}}^T(\mathbf{X}^T\mathbf{X})\hat{\boldsymbol{\beta}}}{2}\right)$$

joint posterior draws of beta's



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- Inclusion of un-necessary terms yields less precise estimates, particularly if explanatory variables are highly correlated with each other
- reduced MSE: reduced variance but possibly higher bias
- it is too "expensive" to use all variables

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- Models for the variable selection problem are based on a subset of the $\mathbf{X}_1, \ldots \mathbf{X}_p$ variables

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- Each value of $\boldsymbol{\gamma}$ represents one of the 2^p models.
- Under model $\mathcal{M}_\gamma$:

$$\mathbf{Y} \mid \alpha, \boldsymbol{\beta}, \sigma^2, \boldsymbol{\gamma} \sim \mathsf{N}(\mathbf{1}\alpha + \mathbf{X}_{\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \sigma^2 \mathbf{I})$$

  Where $\mathbf{X}_{\boldsymbol{\gamma}}$ is design matrix using the columns in $\mathbf{X}$ where $\gamma_j = 1$ and $\boldsymbol{\beta}_{\boldsymbol{\gamma}}$ is the subset of $\boldsymbol{\beta}$ that are non-zero.

Posterior Probabilities of Models

- Posterior model probabilities

$$p(\mathcal{M}_j \mid \mathbf{Y}) = \frac{p(\mathbf{Y} \mid \mathcal{M}_j)p(\mathcal{M}_j)}{\sum_j p(\mathbf{Y} \mid \mathcal{M}_j)p(\mathcal{M}_j)}$$

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Marginal likelihod of a model is proportional to

$$p(\mathbf{Y} \mid \mathcal{M}_\gamma) = \iint p(\mathbf{Y} \mid \boldsymbol{\beta}_\gamma, \sigma^2) p(\boldsymbol{\beta}_\gamma \mid \boldsymbol{\gamma}, \sigma^2) p(\sigma^2 \mid \boldsymbol{\gamma}) d\boldsymbol{\beta}\, d\sigma^2$$

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- Probability $\beta_j \neq 0$: $\sum_{\mathcal{M}_j : \beta_j \neq 0} p(\mathcal{M}_j \mid \mathbf{Y})$ (marginal posterior inclusion probability)

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Prior Distributions

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- Bayesian Model choice requires proper prior distributions on regression coefficients (exception parameters that are included in all models)
- Vague but proper priors may lead to paradoxes!
- Conjugate Normal-Gammas lead to closed form expressions for marginal likelihoods, Zellner's g-prior is one of the most popular.

Zellner's g-prior within Models

Centered model:

$$\mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}^c_\gamma \boldsymbol{\beta}_\gamma + \epsilon$$

- Common parameters

$$p(\alpha, \phi) \propto \phi^{-1}$$

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- Model Specific parameters

$$\boldsymbol{\beta}_\gamma \mid \alpha, \phi, \boldsymbol{\gamma} \sim \mathrm{N}(0, g\phi^{-1}(\mathbf{X}^{c\,\prime}_\gamma \mathbf{X}^c_\gamma)^{-1})$$

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$$p(\alpha, \phi) \propto \phi^{-1}$$

- Model Specific parameters

$$\boldsymbol{\beta}_\gamma \mid \alpha, \phi, \boldsymbol{\gamma} \sim \mathsf{N}(0, g\phi^{-1}(\mathbf{X}_\gamma^{c\,\prime}\mathbf{X}_\gamma^c)^{-1})$$

- Marginal likelihood of $\mathcal{M}_\gamma$ is proportional to

$$p(\mathbf{Y} \mid \mathcal{M}_\gamma) = C(1+g)^{\frac{n-p-1}{2}}(1+g(1-R_\gamma^2))^{-\frac{(n-1)}{2}}$$

where R_γ^2 is the usual R^2 for model $\mathcal{M}_\gamma$ and C is a constant that is $p(\mathbf{Y} \mid \mathcal{M}_0)$ (model with intercept alone)

Zellner's g-prior within Models

Centered model:

$$\mathbf{Y} = \mathbf{1}_n \alpha + \mathbf{X}^c_{\boldsymbol{\gamma}} \boldsymbol{\beta}_\gamma + \epsilon$$

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$$p(\alpha, \phi) \propto \phi^{-1}$$

- Model Specific parameters

$$\boldsymbol{\beta}_\gamma \mid \alpha, \phi, \boldsymbol{\gamma} \sim \mathsf{N}(0, g\phi^{-1}(\mathbf{X}^{c\,\prime}_{\boldsymbol{\gamma}} \mathbf{X}^c_{\boldsymbol{\gamma}})^{-1})$$

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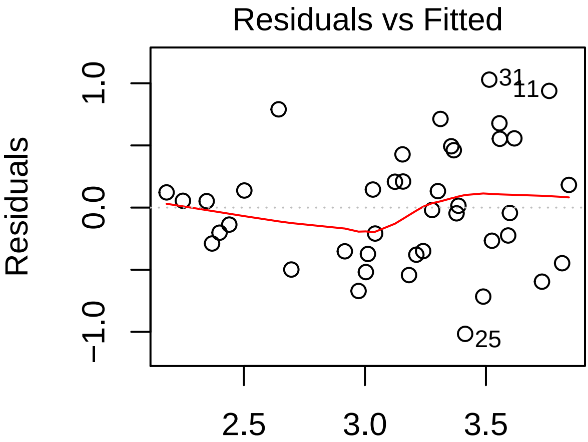
  where R^2_γ is the usual R^2 for model $\mathcal{M}_\gamma$ and C is a constant that is $p(\mathbf{Y} \mid \mathcal{M}_0)$ (model with intercept alone)
- uniform distribution over space of models $p(\mathcal{M}_\gamma) = 1/(2^p)$

USair Data: Enumeration of All Models

```
library(devtools)
suppressMessages(install_github("merliseclyde/BAS"))  # cu
library(BAS)
poll.bma = bas.lm(log(SO2) ~ temp + log(firms) +
                             log(popn) + wind +
                             precip+ rain,
                  data=usair,
                  prior="g-prior",
                  alpha=41,     # g = n
                  n.models=2^7,# enumerate (can omit)
                  method="deterministic")   # fast enumera
```

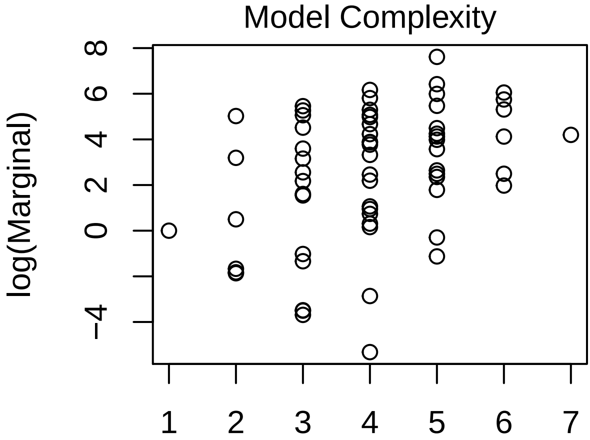
residual plot)

```
plot(poll.bma, which=1)
```



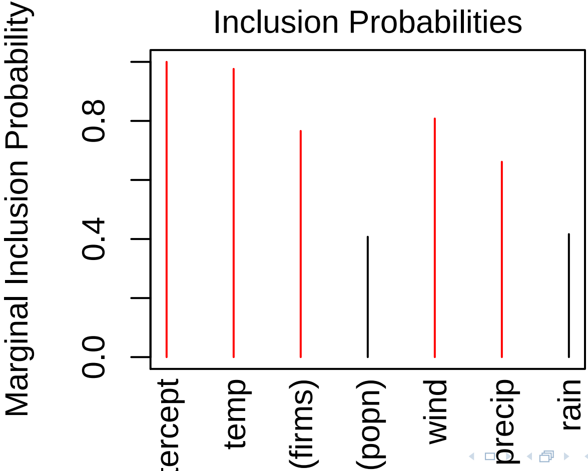
Model Complexity)

```
plot(poll.bma, which=3)
```



Inclusion Probabilities)

```
plot(poll.bma, which=4)
```



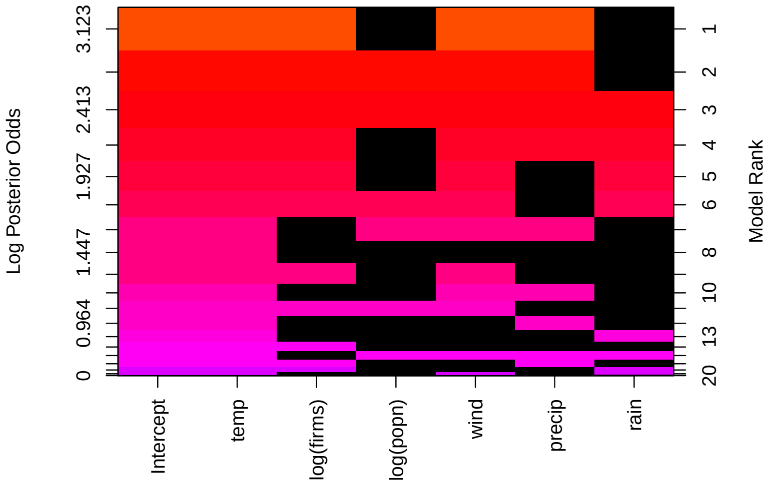
Model Space

```
summary(poll.bma)

##                P(B != 0 | Y)  model 1   model 2    model 3
## Intercept          1.0000000 1.000000 1.0000000 1.00000000 1
## temp               0.9762282 1.000000 1.0000000 1.00000000 1
## log(firms)         0.7659857 1.000000 1.0000000 1.00000000 1
## log(popn)          0.4075393 0.000000 1.0000000 1.00000000 0
## wind               0.8080832 1.000000 1.0000000 1.00000000 1
## precip             0.6615960 1.000000 1.0000000 1.00000000 1
## rain               0.4166394 0.000000 0.0000000 1.00000000 1
## BF                        NA 1.000000 0.2093987 0.03277353 0
## PostProbs                 NA 0.209600 0.1097000 0.10300000 0
## R2                        NA 0.542700 0.5500000 0.55030000 0
## dim                       NA 5.000000 6.0000000 7.00000000 6
## logmarg                   NA 7.616228 6.0527128 4.19809382 5
```

Summary

```
image(poll.bma)
```



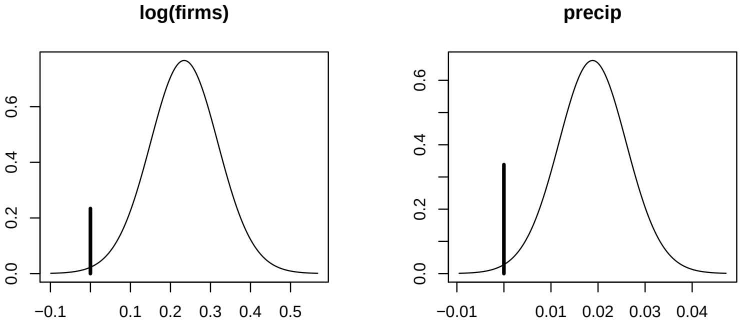
Coefficients

```
beta = coef(poll.bma, n.models=1)
beta

##
##  Marginal Posterior Summaries of Coefficients:
##
##  Based on the top  1 models
##               post mean  post SD   post p(B != 0)
## Intercept     3.15300    0.07818   1.00000
## temp         -0.07130    0.01268   0.97623
## log(firms)    0.23428    0.08573   0.76599
## log(popn)     0.00000    0.00000   0.40754
## wind         -0.17998    0.06128   0.80808
## precip        0.01884    0.00729   0.66160
## rain          0.00000    0.00000   0.41664
```

Coefficients

```
par(mfrow=c(2,2));  plot(beta, subset=c(3, 6))
```



Bayesian Confidence Intervals

```
confint(beta)

##                    2.5  %      97.5  %         beta
## Intercept   2.994993257  3.31101398  3.15300362
## temp       -0.096926645 -0.04567203 -0.07129934
## log(firms)  0.061014518  0.40753936  0.23427694
## log(popn)   0.000000000  0.00000000  0.00000000
## wind       -0.303835463 -0.05612195 -0.17997871
## precip      0.004105874  0.03357242  0.01883915
## rain        0.000000000  0.00000000  0.00000000
## attr(,"Probability")
## [1] 0.95
## attr(,"class")
## [1] "confint.bas"
```