STA 601: Lecture 2

Loss Functions, Bayes Risk and Posterior Summaries

Merlise Clyde

Last Time ...

- Introduction to "ingredients" of Bayesian analysis
- Illustrated a simple Beta-Binomial conjugate example
- Posterior $\pi(\theta \mid y)$ is a $\textsf{Beta}(a + y, b + n - y)$

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Today ...

- an introduction to loss functions
- Bayes Risk
- optimal decisions and estimators

Bayes estimate

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2) What if we want to produce an interval estimate (θ_L, θ_U)?

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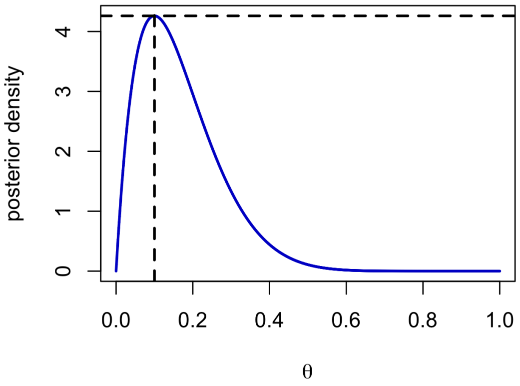
2) What if we want to produce an interval estimate (θ_L, θ_U)?

These would provide alternatives to the frequentist MLEs and confidence intervals

Heuristically



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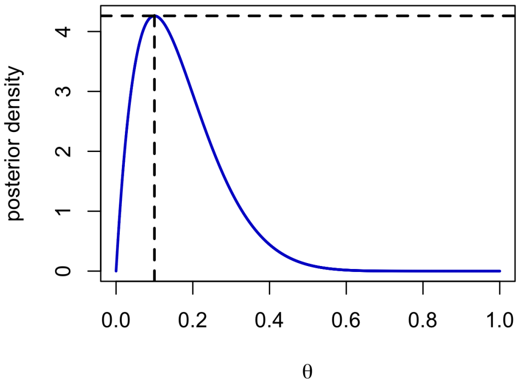

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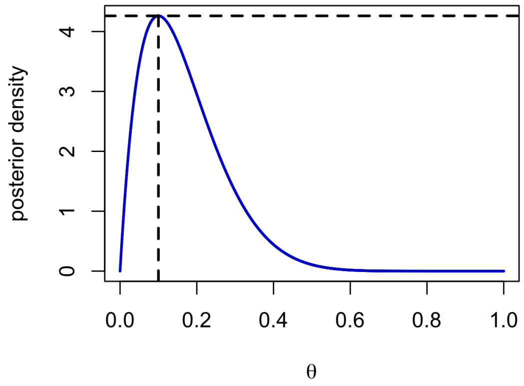

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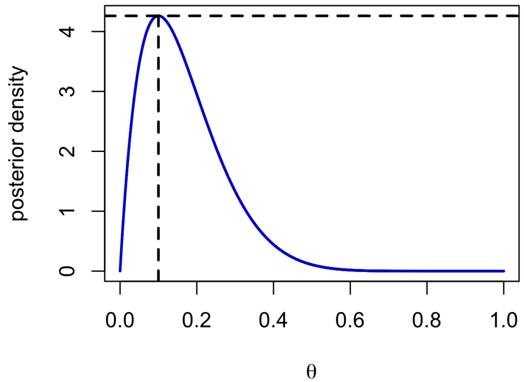

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- lots of intervals that satisfy this! which one is "best"?

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But how do we deal with the fact that we do not know θ?

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- **Frequentist risk** also exists for evaluating a given estimator under true value of θ

$$\mathsf{E}_{p(y|\theta_{\text{true}})}[l(\theta_{\text{true}}, \hat{\theta}))]$$

Squared Error Loss

A common choice for point estimation is squared error loss:

$$R(\hat{\theta}) = \mathsf{E}_{\pi(\theta|y)}[l(\theta, \hat{\theta})] = \int_{\Theta} (\hat{\theta} - \theta)^2 \pi(\theta \mid y)\, d\theta$$

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- Expand and take derivative of $R(\hat{\theta}$ with respect to $\hat{\theta}$)

Let's work it out!

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- Posterior mean is the **Bayes optimal estimator** for θ under squared error loss
- In the beta-binomial case for example, the optimal Bayes estimate under squared error loss is

$$\hat{\theta} = \frac{a+y}{a+b+n},$$

What about other loss functions?

- Clearly, squared error is only one possible loss function. An alternative is **absolute loss**, which has

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- Recall that for a continuous random variable Y with cdf F, the median of the distribution is the value z, which satisfies

$$F(z) = \Pr(Y \leq z) = \frac{1}{2} = \Pr(Y \geq z) = 1 - F(z).$$

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- As long as we know how to evaluate the CDF of the distribution we have, we can solve for z.

Beta-Binomial

- For the beta-binomial model, the CDF of the beta posterior can be written as

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- For other popular distributions, switch out the beta.

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- Bayes optimal estimator or action is the estimator/action that minimizes the expected posterior loss marginalizing out any unknowns over posterior/predictive distribution.

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- Is this a reasonable loss function?

Interval Estimates

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- In any given sample, you don't know whether you're in the lucky 95% or the unlucky 5%. You just know that either the interval covers the parameter, or it doesn't (useful, but not too helpful clearly).
- Often based on aysmptotics i.e use a Wald or other type of frequentist asymptotic interval $\hat{\theta} \pm 1.96\,\mathrm{se}(\hat{\theta})$

Bayesian Intervals

- We want a Bayesian alternative to confidence intervals

for some pre-specified value of α

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How to choose $[l(y),\ u(y)]$?

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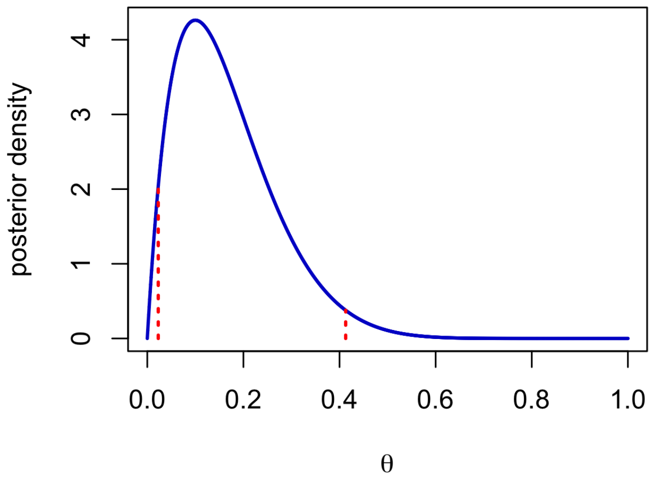
Convenient conceptually and easy as we just take the $\alpha/2$ and $1 - \alpha/2$ quantiles of $\pi(\theta \mid y)$ as $l(y)$ and $u(y)$, respectively.

Beta-Binomial Equal-tailed Interval

```
a = 1; b= 1; y = 1; n = 10
ly = qbeta(0.025, a + y, b + n - y)
uy = qbeta(0.975, a + y, b + n - y)
c(ly, uy)
```

```
## [1] 0.0228312 0.4127799
```



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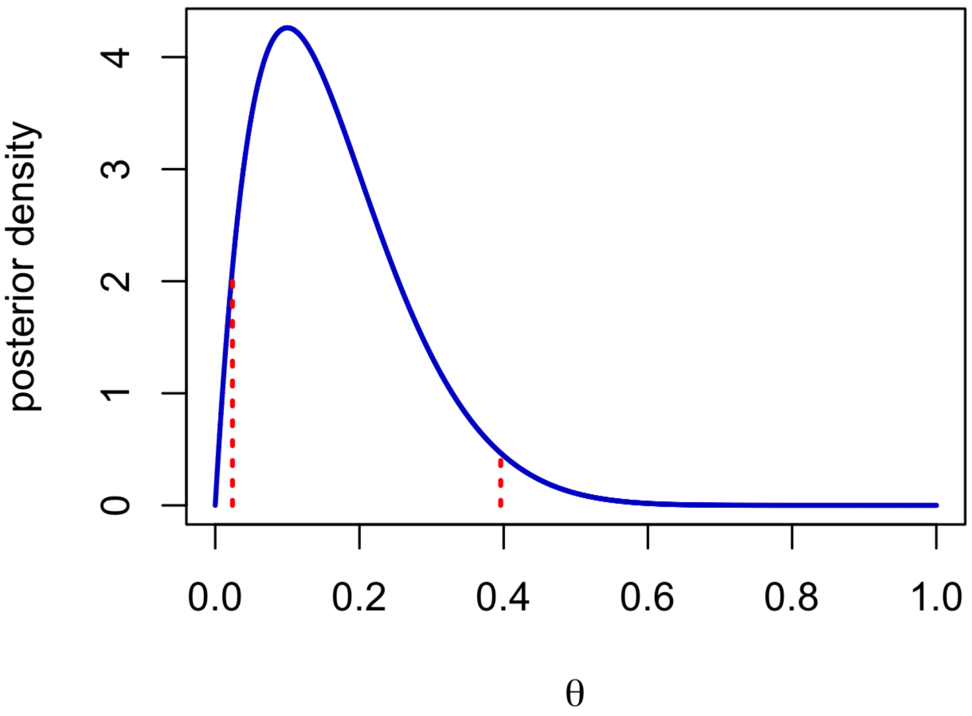
- what about MC quantile estimates?

- Find the 2.5th and 97.5th percentile from the empirical distribution

```
theta = rbeta(1000, a + y, b + n - y)
quantile(theta, c(0.025, 0.975))
```

```
##       2.5%      97.5%
## 0.02388901 0.39587384
```

Equal-Tail Interval



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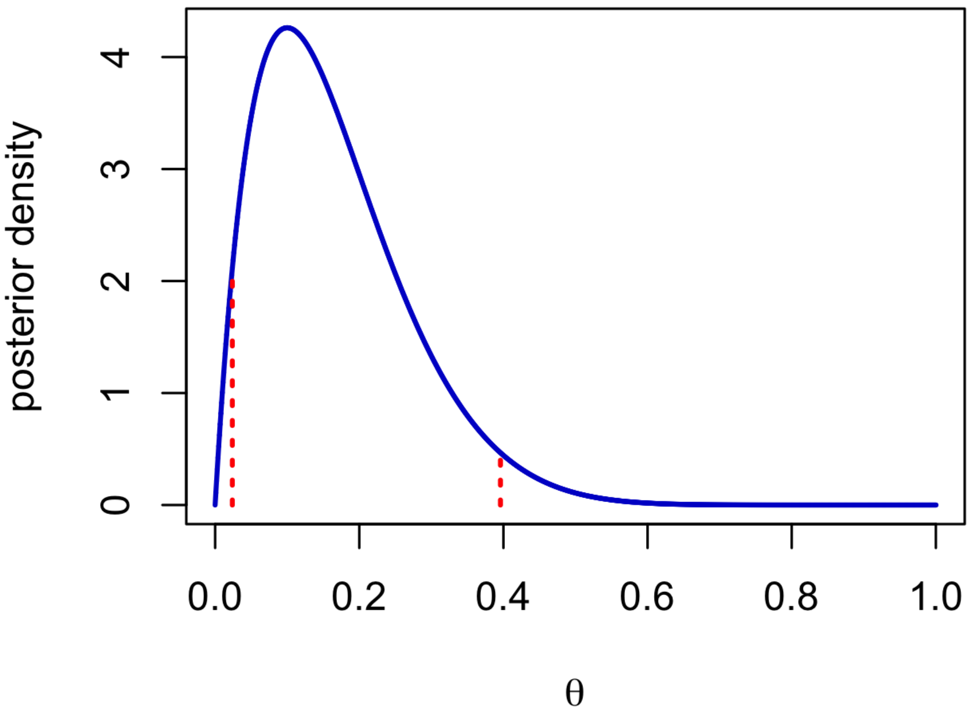

Note there are values of θ outside the quantile-based credible interval, with higher density than some values inside the interval.

HPD region

- A $100 \times (1-\alpha)$ highest posterior density (HPD) region is a subset $s(y)$ of the parameter space Θ such that

    1. $\Pr(\theta \in s(y) \mid y) = 1 - \alpha$; and

    2. If $\theta_a \in s(y)$ and $\theta_b \notin s(y)$, then $p(\theta_a \mid y) > p(\theta_b \mid y)$ (highest density set)

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- $\Rightarrow$ **All** points in a HPD region have higher posterior density than points outside the region.

HPD region

- A $100 \times (1 - \alpha)$ highest posterior density (HPD) region is a subset $s(y)$ of the parameter space Θ such that
    1. $\Pr(\theta \in s(y) \mid y) = 1 - \alpha$; and
    2. If $\theta_a \in s(y)$ and $\theta_b \notin s(y)$, then $p(\theta_a \mid y) > p(\theta_b \mid y)$ (highest density set)
- $\Rightarrow$ **All** points in a HPD region have higher posterior density than points outside the region.
- The basic idea is to gradually move a horizontal line down across the density, including in the HPD region all values of θ with a density above the horizontal line.

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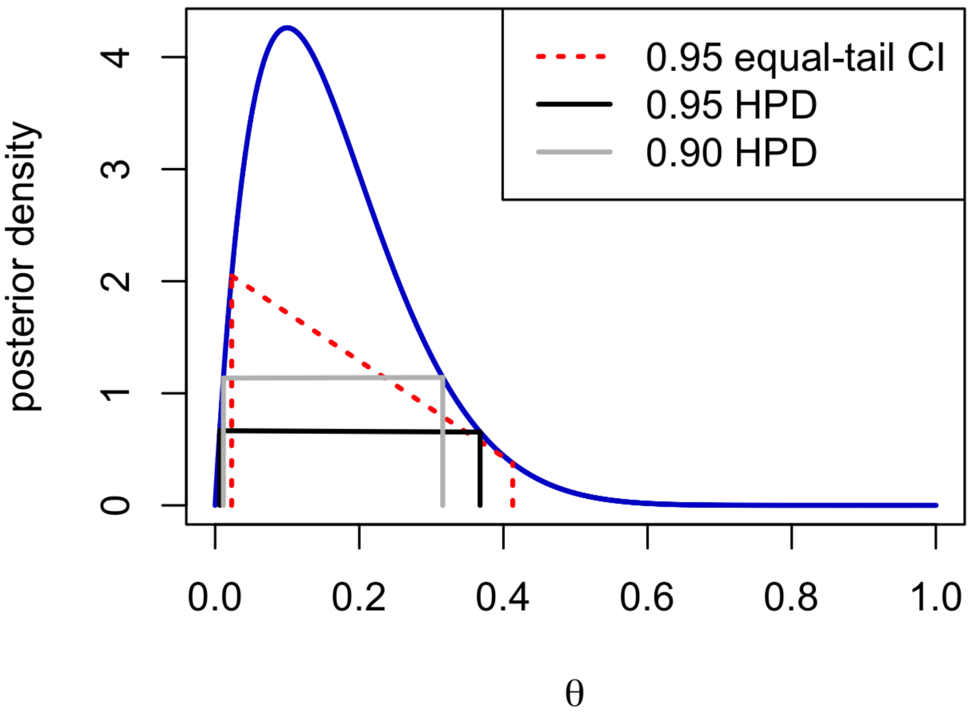
- Stop moving the line down when the posterior probability of the values of θ in the region reaches $1-\alpha$.

Simulation Based

```
suppressMessages(library(rjags))
HPDinterval(as.mcmc(theta))
```

```
##            lower     upper
## var1 0.004605827 0.3539385
## attr(,"Probability")
## [1] 0.95
```

HPD Intervals



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- More computationally intensive to solve!

Loss Functions for Interval Estimation

See "The Bayesian Choice" by Christian Robert Section 5.5.5

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- parametric model is "close" to the true data-generating process
- model diagnostics & changing the model can reduce the gap between model we are using and the true data generating process