Lecture 5: Basics of Bayesian Hypothesis Testing

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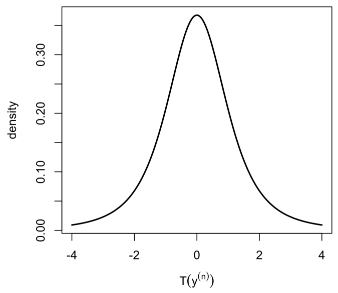

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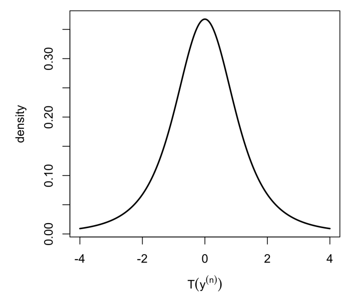

- **p-value**: Calculate the probability of seeing a dataset/test statistics as extreme or more extreme than the oberved data with repeated sampling under the null hypothesis

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Note: we *never* conclude in favor of $\mathcal{H}_0$. We are looking for enough evidence to reject $\mathcal{H}_0$. But if we fail to reject we do not conclude that it is true!

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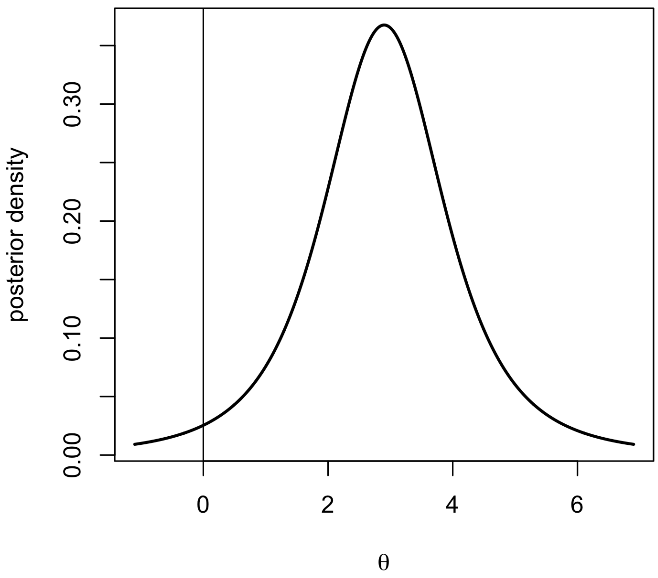

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Tail Areas:

1. Compute $\Pr(\theta > 0 \mid y^{(n)})$ and $\Pr(\theta < 0 \mid y^{(n)})$
2. Report minimum of these probabilities as a "Bayesian p-value"

Note: Tail probability is not the same as $\Pr(\mathcal{H}_0 \mid y^{(n)})$

Formal Bayesian Hypothesis Testing

Unknowns are $\mathcal{H}_0$ and $\mathcal{H}_1$

Put a prior on the actual hypotheses/models, that is, on $\pi(\mathcal{H}_0) = \Pr(\mathcal{H}_0 = \mathrm{True})$ and $\pi(\mathcal{H}_1) = \Pr(\mathcal{H}_1 = \mathrm{True})$.

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$$p(y^{(n)} \mid \mathcal{H}_1) = \int_{\Theta} p(y^{(n)} \mid \mathcal{H}_1, \theta) p(\theta \mid \mathcal{H}_1) \, d\theta$$

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- Compute marginal likelihoods for each hypothesis, that is, $\mathcal{L}(\mathcal{H}_0)$ and $\mathcal{L}(\mathcal{H}_1)$.
- Obtain posterior probabilities of $\mathcal{H}_0$ and $\mathcal{H}_1$ via Bayes Theorem.

Bayesian Approach - Decisions

Loss function for hypothesis testing

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Loss function:

$$L(\hat{\mathcal{H}}, \mathcal{H}) = w_1 \, 1(\hat{\mathcal{H}} = 1, \mathcal{H} = 0) + w_2 \, 1(\hat{\mathcal{H}} = 0, \mathcal{H} = 1)$$

- w_1 weights how bad making a Type I error
- w_2 weights how bad making a Type II error

Loss Function Functions and Decisions

- Relative weights

$$L(\hat{\mathcal{H}}, \mathcal{H}) = 1(\hat{\mathcal{H}} = 1, \mathcal{H} = 0) + w\, 1(\hat{\mathcal{H}} = 0, \mathcal{H} = 1)$$

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- Minimize loss by picking hypothesis with the highest posterior probability

Bayesian hypothesis testing

- Using Bayes theorem,

$$\pi(\mathcal{H}_1 \mid Y) = \frac{p(y^{(n)} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)}{p(y^{(n)} \mid \mathcal{H}_0)\pi(\mathcal{H}_0) + p(y^{(n)} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)},$$

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- The ratio $\frac{p(y^{(n)}|\mathcal{H}_0)}{p(y^{(n)}|\mathcal{H}_1)}$ is known as the **Bayes factor** in favor of $\mathcal{H}_0$, and often written as $\mathcal{BF}_{01}$. Similarly, we can compute $\mathcal{BF}_{10}$.

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- Notice that for our example,

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- That is, here, as $\mathcal{BF}_{01} \uparrow$, $\pi(\mathcal{H}_1 \mid Y) \downarrow$.

Bayes factors

- Let's look at another way to think of Bayes factors. First, recall that

$$\pi(\mathcal{H}_1 \mid Y) = \frac{p(y^{(n)} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)}{p(y^{(n)} \mid \mathcal{H}_0)\pi(\mathcal{H}_0) + p(y^{(n)} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)},$$

so that

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\end{aligned}$$

$$\therefore \underbrace{\frac{\pi(\mathcal{H}_0 \mid Y)}{\pi(\mathcal{H}_1 \mid Y)}}_{\text{posterior odds}} = \underbrace{\frac{\pi(\mathcal{H}_0)}{\pi(\mathcal{H}_1)}}_{\text{prior odds}} \times \underbrace{\frac{p(y^{(n)} \mid \mathcal{H}_0)}{p(y^{(n)} \mid \mathcal{H}_1)}}_{\text{Bayes factor } \mathcal{BF}_{01}}$$

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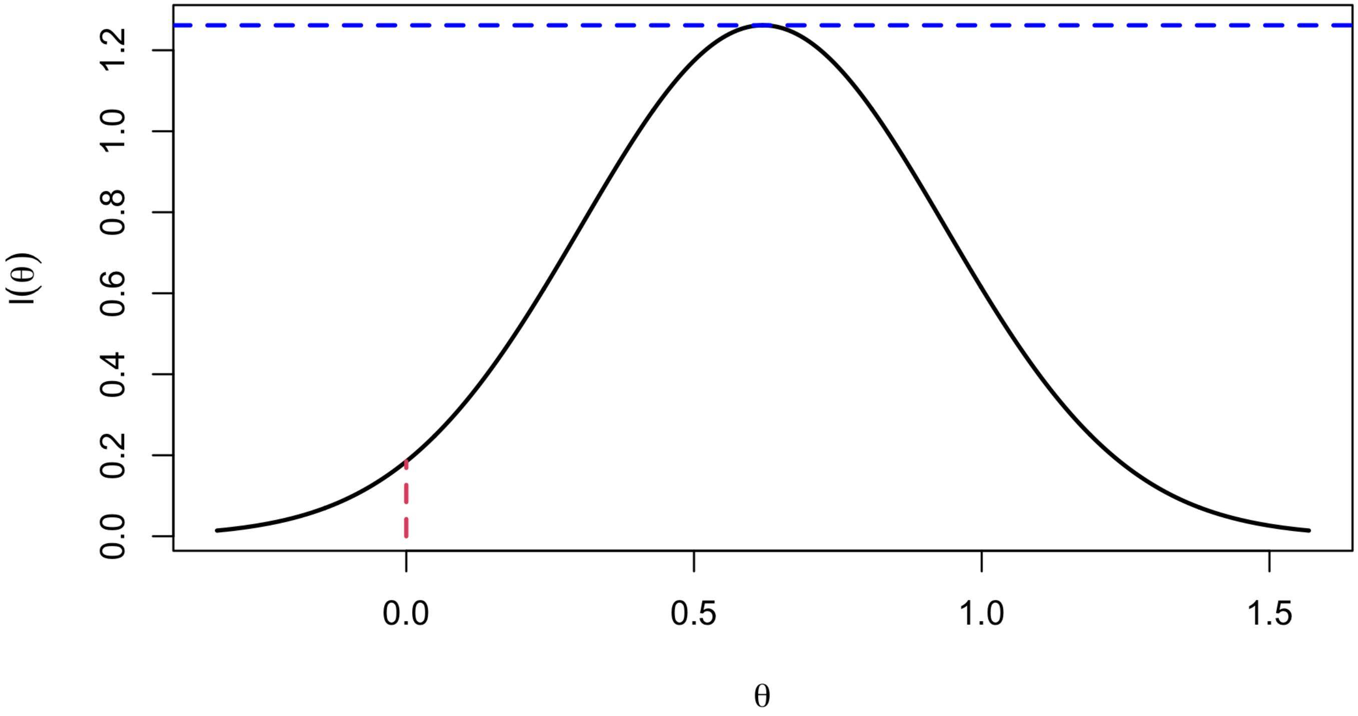
- Therefore, the Bayes factor can be thought of as the factor by which our prior odds change (towards the posterior odds) in the light of the data.

Likelihoods & Evidence

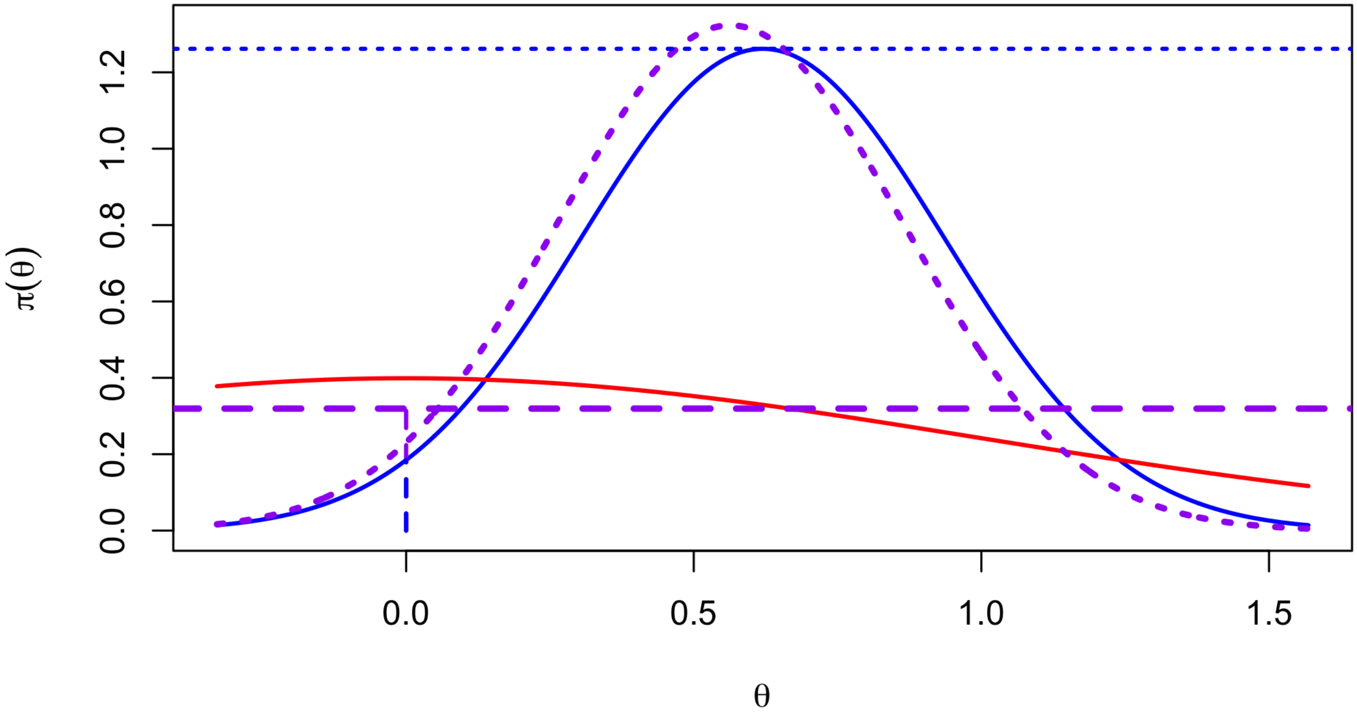
Maximized Likelihood



p-value = 0.05

Marginal Likelihoods & Evidence

Maximized Likelihood



$\mathcal{BF}_{10} = 1.73$

Candidate's Formula (Besag 1989)

Alternative expression for Bayes Factor

$$\frac{p(y^{(n)} \mid \mathcal{H}_1)}{p(y^{(n)} \mid \mathcal{H}_0)} = \frac{\pi_\theta(0 \mid \mathcal{H}_1)}{\pi_\theta(0 \mid y^{(n)}, \mathcal{H}_1)}$$

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- Savage-Dickey Ratio

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- centered at value for θ under $\mathcal{H}_0$ (goes back to Jeffreys)

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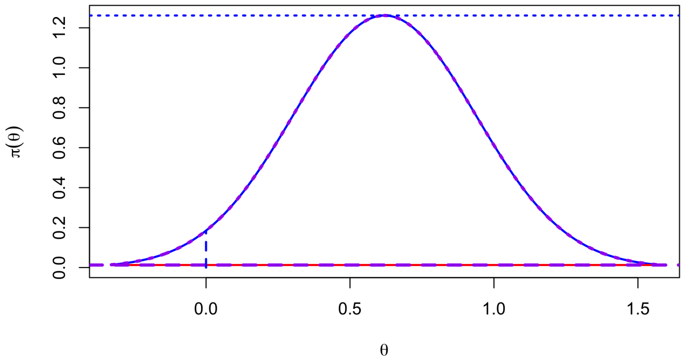
- What happens of $\tau_0 \to 0$?

Precision



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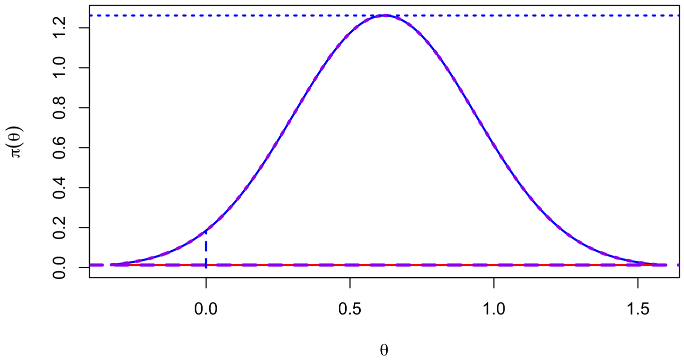

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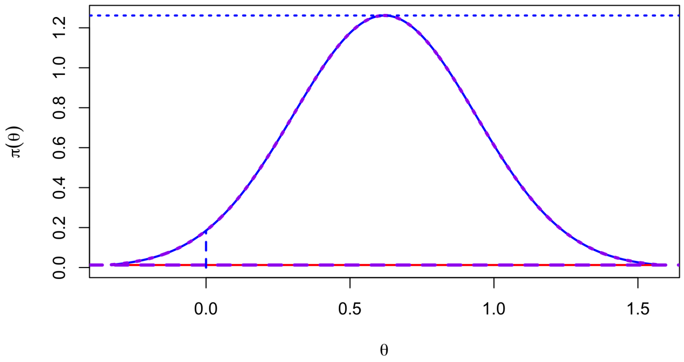

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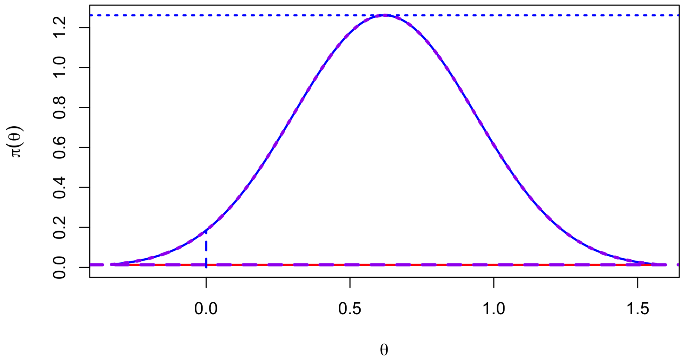

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Bartlett's Paradox - the paradox is that a seemingly non-informative prior for θ is very informative about $\mathcal{H}$!