

STA 601: Lecture 3

The Normal Model & Prior/Posterior Predictive Distributions

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Outline

- Normal Model



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- Predictive Distributions



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 - Predicting/forecasting future events



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- Comparing Estimators



Normal Model Setup

- Suppose we have independent observations

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- for simplicity we will treat τ as known initially.



Marginal Distribution

- Recall that the **marginal distribution** is

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- Need to specify a prior for θ on \mathbb{R}



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- Natural choice is a Normal/Gaussian distribution (Conjugate prior)

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- parameterization invariant and invariant to shift changes in the data (group invariance)



Prior Predictive for a Single Case

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- 3) Read off **posterior precision** and **posterior mean**
- 4) **Complete the square**
- 5) **Integrate** out θ to obtain marginal!



Try it!

$$p(y) \propto \int_{\mathbb{R}} \exp \left\{ -\frac{1}{2} [\tau(y - \theta)^2 + \tau_0(\theta - \theta_0)^2] \right\} d\theta$$



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$$Y \sim N\left(\theta_0, \frac{1}{\tau_0} + \frac{1}{\tau}\right) \text{ or } N(\theta_0, \sigma^2 + \sigma_0^2)$$



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- two sources of variability: variability from the model for the data and prior variability



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- takes into account uncertainty about θ and variability in y !



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- prior variance is now $\sigma_1^2 = 1/\tau_1$



Posterior Predictive for y_2 given y_1

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- Find the mean and variance using iterated expectations:
- mean

$$\mathbb{E}[Y_2 \mid y_1] = \mathbb{E}_{\theta \mid y_1} [\mathbb{E}_{Y_2 \mid y_1, \theta} [Y_2 \mid y_1, \theta] \mid y_1]$$



Variance via Iterated Expectations

$$\text{Var}[Y_2 \mid y_1] =$$

$$\mathbb{E}_{\theta|y_1}[\text{Var}_{Y_2|y_1,\theta}[Y_2 \mid y_1, \theta] \mid y_1] + \text{Var}_{\theta|y_1}[\mathbb{E}_{Y_2|y_1,\theta}[Y_2 \mid y_1, \theta] \mid y_1]$$



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- new precision

$$\tau_2 = \tau_1 + \tau = \tau_0 + 2\tau$$



After n observations

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- Cannot reduce the error for prediction Y_{n+1} due to σ^2
- predictive distribution for a next observation given *everything* we know - prior and likelihood



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- Posterior variance of $\theta = \sigma^2/n$ (same as variance of the MLE)



Posterior Predictive Distribution

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Under Jeffreys' prior

$$Y_{n+1} \mid y_1, \dots, y_n \sim N\left(\bar{y}, \sigma^2\left(1 + \frac{1}{n}\right)\right)$$



Posterior Predictive Distribution

Posterior predictive distribution for Y_{n+1}

$$Y_{n+1} \mid y_1, \dots, y_n \sim N\left(\frac{\tau_0\theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau} + \frac{1}{\tau_0 + n\tau}\right)$$

Under Jeffreys' prior

$$Y_{n+1} \mid y_1, \dots, y_n \sim N\left(\bar{y}, \sigma^2\left(1 + \frac{1}{n}\right)\right)$$

Captures extra uncertainty due to not knowing θ (compared to plug-in approach where we plug in MLE in sampling model!)



Comparing Estimators

Expected loss (from frequentist perspective) of using Bayes Estimator



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- Posterior mean is optimal under squared error loss (min Bayes Risk)
[also absolute error loss]



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Compute Mean Square Error (or Expected Average Loss)

$$= \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta})$$



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Expected loss (from frequentist perspective) of using Bayes Estimator

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Compute Mean Square Error (or Expected Average Loss)

$$E_{\bar{y}|\theta} \left[\left(\hat{\theta} - \theta \right)^2 | \theta \right]$$

- For the MLE \bar{Y} this is just the variance of \bar{Y} or σ^2/n



MSE for Bayes

$$\mathbb{E}_{\bar{y}|\theta} \left[(\hat{\theta} - \theta)^2 \mid \theta \right] = \text{MSE} = \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta})$$

- Bias of Bayes Estimate

$$\mathbb{E}_{\bar{Y}|\theta} \left[\frac{\tau_0 \theta_0 + \tau n \bar{Y}}{\tau_0 + \tau n} \right] = \frac{\tau_0 (\theta_0 - \theta)}{\tau_0 + \tau n}$$



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- Variance

$$\text{Var} \left(\frac{\tau_0 \theta_0 + \tau n \bar{Y}}{\tau_0 + \tau n} - \theta \mid \theta \right) = \frac{\tau n}{(\tau_0 + \tau n)^2}$$



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(Frequentist) expected Loss when truth is θ

$$\text{MSE} = \frac{\tau_0^2 (\theta - \theta_0)^2 + \tau n}{(\tau_0 + \tau n)^2}$$



MSE for Bayes

$$\mathbb{E}_{\bar{y}|\theta} \left[\left(\hat{\theta} - \theta \right)^2 \mid \theta \right] = \text{MSE} = \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta})$$

- ## ■ Bias of Bayes Estimate

$$\mathsf{E}_{\bar{Y}|\theta} \left[\frac{\tau_0 \theta_0 + \tau n \bar{Y}}{\tau_0 + \tau n} \right] = \frac{\tau_0 (\theta_0 - \theta)}{\tau_0 + \tau n}$$

- ## ■ Variance

$$\text{Var} \left(\frac{\tau_0 \theta_0 + \tau n \bar{Y}}{\tau_0 + \tau n} - \theta \mid \theta \right) = \frac{\tau n}{(\tau_0 + \tau n)^2}$$

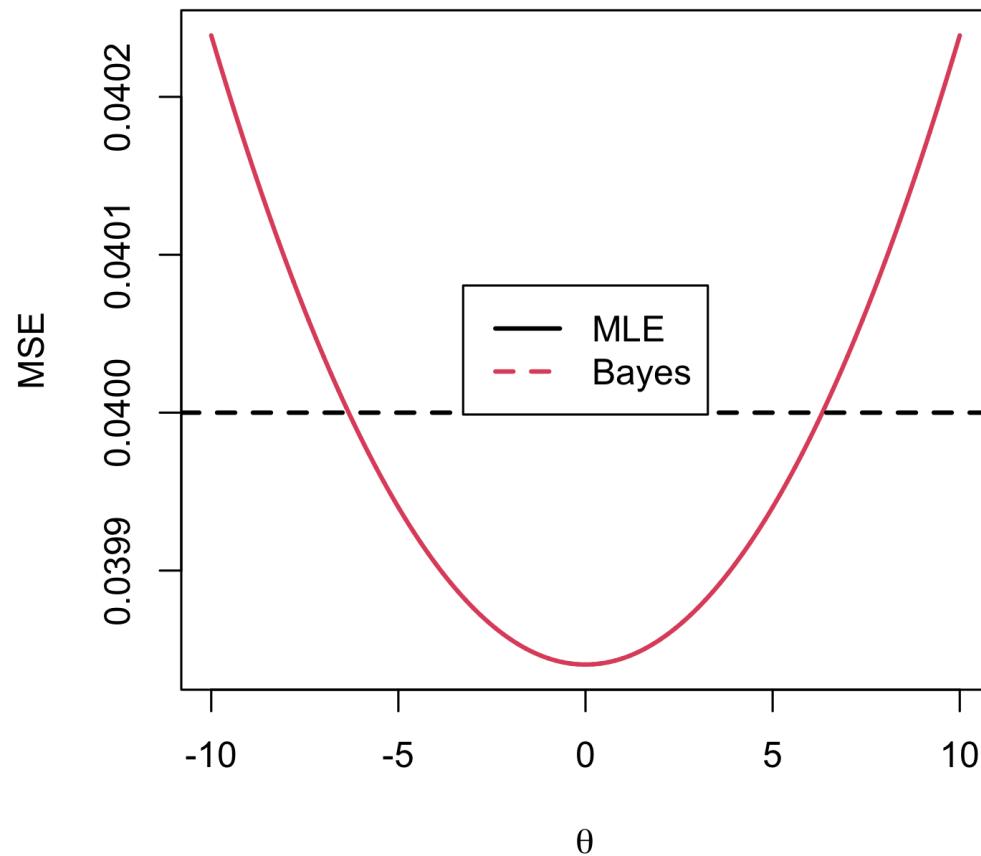
(Frequentist) expected Loss when truth is θ

$$\text{MSE} = \frac{\tau_0^2(\theta - \theta_0)^2 + \tau n}{(\tau_0 + \tau n)^2}$$

Behavior?



Plot



Updating with n Observations

- We can use the $\mathcal{L}(\theta)$ based on n observations and repeat completing the square with the original prior $\theta \sim N(\theta_0, 1/\tau_0)$



Likelihood Function

- The likelihood for θ is proportional to the sampling model

$$p(y | \theta, \tau) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \tau^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \tau (y_i - \theta)^2 \right\}$$



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Rewrite in terms of sufficient statistics!



Simplification

$$\begin{aligned}\mathcal{L}(\theta) &\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau \sum_{i=1}^n (y_i - \theta)^2\right\} \\ &\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau \sum_{i=1}^n [(y_i - \bar{y}) - (\theta - \bar{y})]^2\right\} \\ &\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau \left[\sum_{i=1}^n (y_i - \bar{y})^2 + \sum_{i=1}^n (\theta - \bar{y})^2\right]\right\} \\ &\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau \left[\sum_{i=1}^n (y_i - \bar{y})^2 + n(\theta - \bar{y})^2\right]\right\} \\ &\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau s^2(n-1)\right\} \exp\left\{-\frac{1}{2}\tau n(\theta - \bar{y})^2\right\}. \\ &\propto \exp\left\{-\frac{1}{2}\tau n(\theta - \bar{y})^2\right\}\end{aligned}$$



Exercises for Practice

Try this

- 1) Use $\mathcal{L}(\theta)$ based on n observations and $\pi(\theta)$ to find $\pi(\theta \mid y_1, \dots, y_n)$ based on the sufficient statistics



Exercises for Practice

Try this

- 1) Use $\mathcal{L}(\theta)$ based on n observations and $\pi(\theta)$ to find $\pi(\theta | y_1, \dots, y_n)$ based on the sufficient statistics
- 2) Use $\pi(\theta | y_1, \dots, y_n)$ to find the posterior predictive distribution for y_{n+1}

