

Lecture 7: Introduction to Hierarchical Modelling, Empirical Bayes, and MCMC

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September 16



Normal Means Model

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Hierarchical Viewpoint: Let's borrow information from other observations!



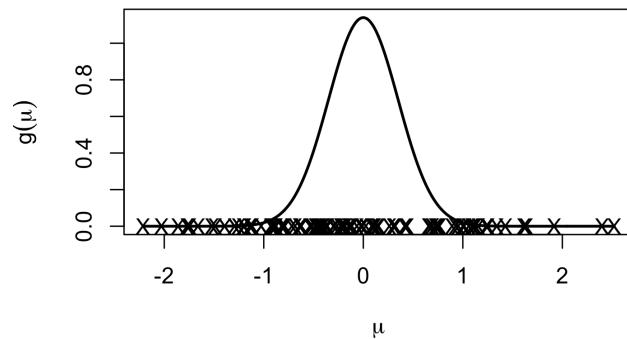
Motivation

- Example y_i is difference in gene expression for the i^{th} gene between cancer and control lines



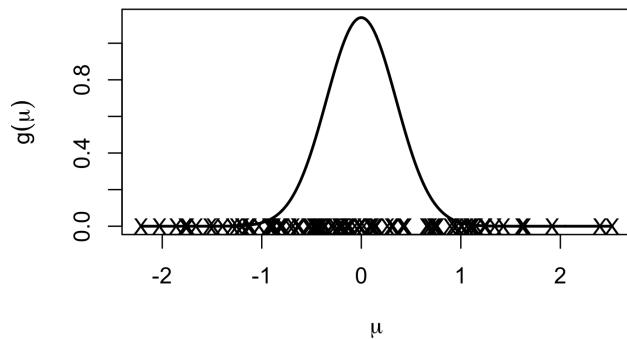
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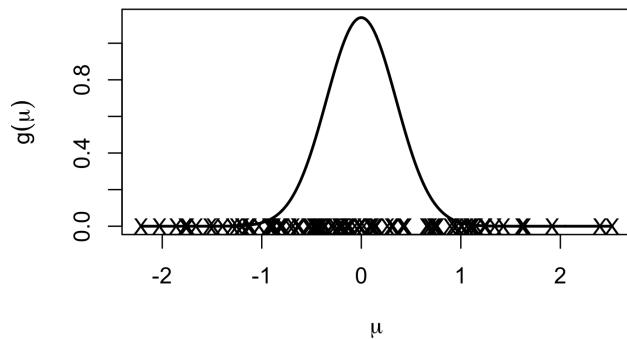


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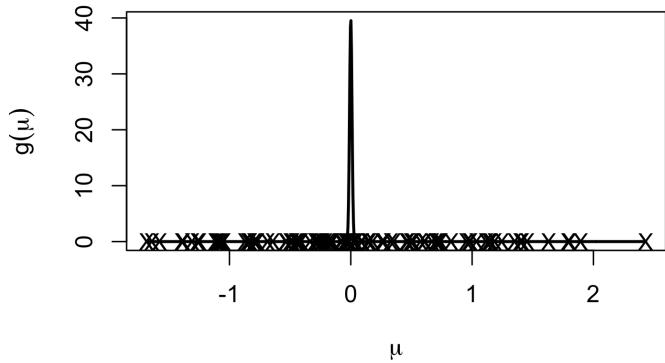
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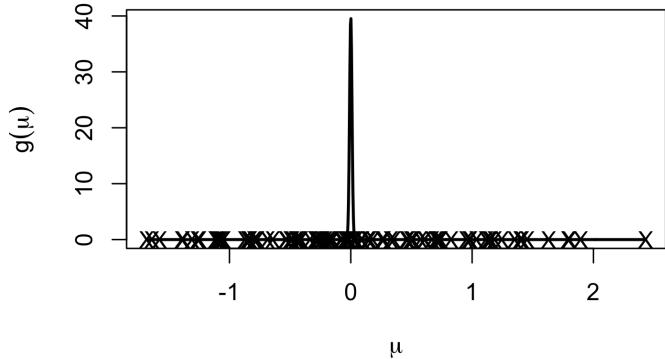
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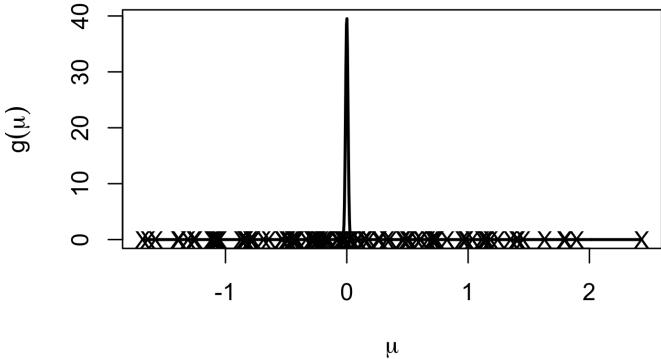


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- Now estimate μ_i (let $\phi = 1/\sigma^2$ and $\phi_\mu = 1/\sigma_\mu^2$)



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- not necessarily a prior!
- Now estimate μ_i (let $\phi = 1/\sigma^2$ and $\phi_\mu = 1/\sigma_\mu^2$)
- Calculate the "posterior" $\mu_i | y_i, \mu, \phi, \phi_\mu$



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- MSE or Bias-Variance Tradeoff



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- Examples: lasso regression, ridge regression, various kinds of hierarchical Bayesian models, etc.



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- Assume σ^2 is known (say 1)

$$\hat{\sigma}_\mu^2 = \frac{\sum(y_i - \bar{y})^2}{n} - 1$$



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- Fully Bayes would put a prior on the unknowns



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- Except for simple cases (conjugate models) $m(y)$ is not available analytically



Large Sample Approximations

- Appeal to BvM (Bayesian Central Limit Theorem) and approximate $\pi(\theta | y)$ with a Gaussian distribution centered at the posterior mode $\hat{\theta}$ and asymptotic covariance matrix

$$V_\theta = \left[-\frac{\partial^2}{\partial \theta \partial \theta^T} \{ \log(\pi(\theta)) + \log(\mathcal{L}(\theta)) \} \right]^{-1}$$



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Stochastic methods



Stochastic Integration

$$\mathbb{E}[h(\theta) \mid y] = \int_{\Theta} h(\theta) \pi(\theta \mid y) d\theta \approx \frac{1}{T} \sum_{t=1}^T h(\theta^{(t)}) \quad \theta^{(t)} \sim \pi(\theta \mid y)$$



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- use twice for numerator and denominator



Important Sampling Estimate

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with un-normalized weights $w(\theta^{(t)}) \propto \frac{\pi(\theta^{(t)}) \mathcal{L}(\theta^{(t)})}{q(\theta^{(t)})}$

(normalize to sum to 1)



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- biased samples initially but get closer to the target



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- ratio of posterior densities where normalizing constant cancels!



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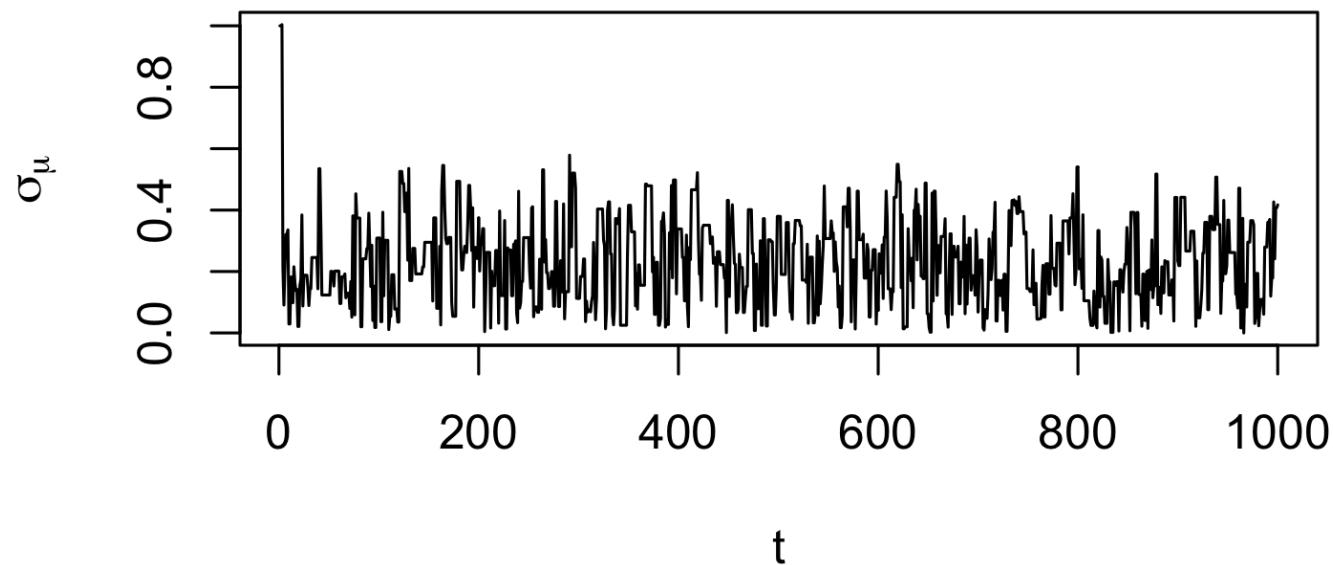
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- Take $\sigma^2 = 1$
- Use a Cauchy(0, 1) prior on σ_μ
- Symmetric proposal for σ_τ ? Try a normal with variance $\frac{2.4^2}{d} \text{var}(\sigma_\mu)$ where d is the dimension of θ ($d = 1$)



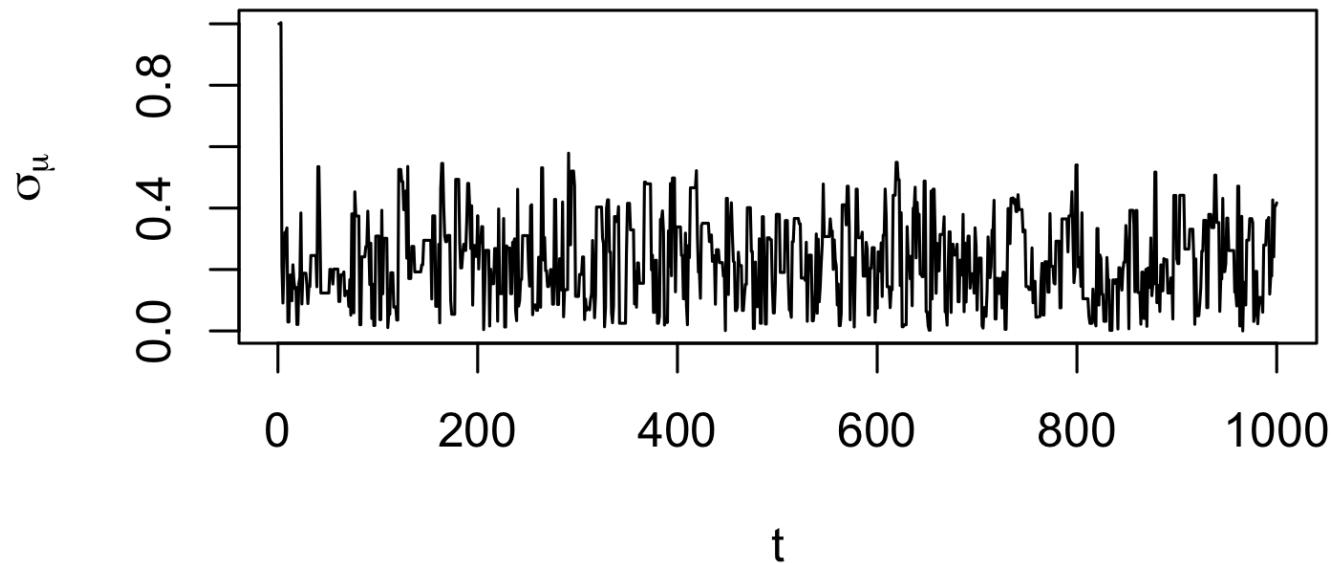
Trace Plots



- Acceptance probability is 0.57



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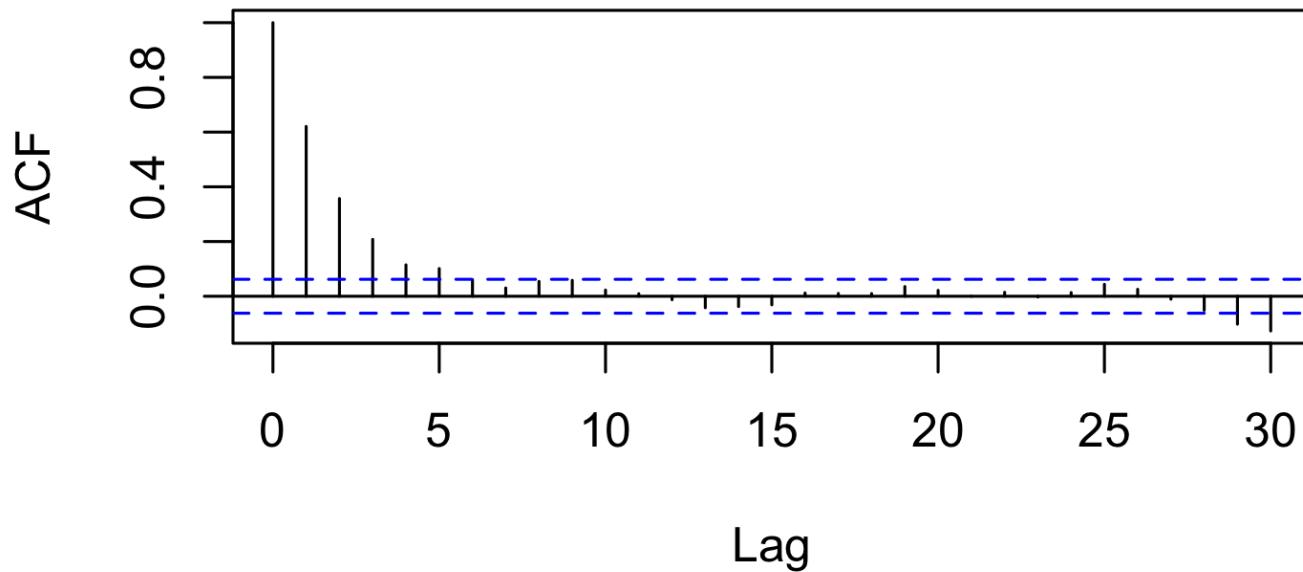


- Acceptance probability is 0.57
- Goal is around 0.44 in 1 dimension to 0.23 in higher dimensions

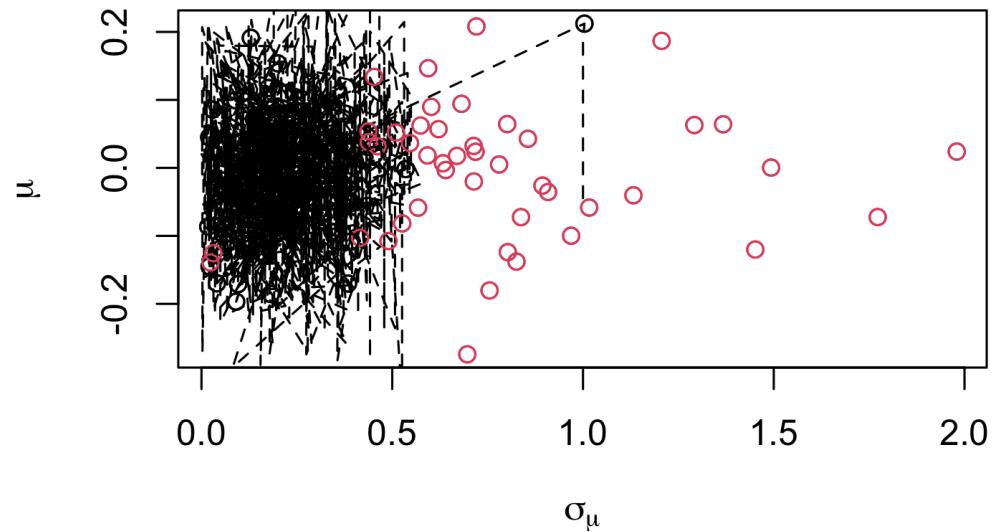


AutoCorrelation Function

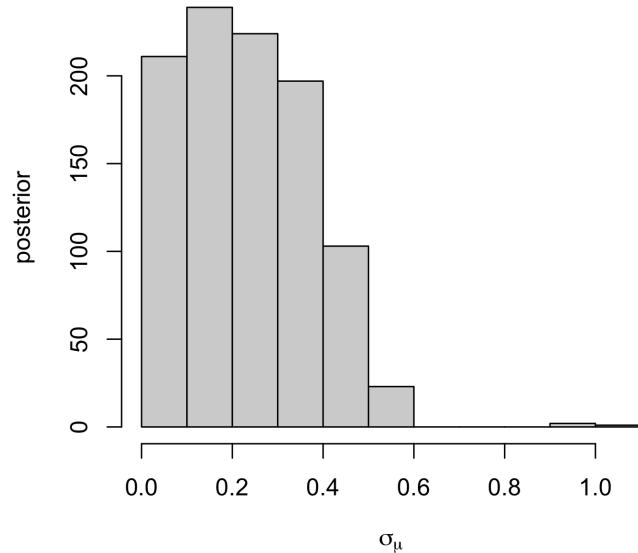
$$\sigma_{\mu}$$



Joint Posterior



Marginal Posterior



MLE of σ_μ is 0.11

