STA 601: Missing data and imputation

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Introduction to missing data

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- **This is rarely plausible in practice!**

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- **This is the most commonly assumed mechanism in practice!**

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Mathematical formulation

- Consider the multivariate data scenario with $\boldsymbol{Y}_i = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)^T$, where $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ip})^T$, for $i = 1, \ldots, n$.

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- We can separate $\boldsymbol{Y}$ into the observed and missing parts, that is, $\boldsymbol{Y} = (\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis})$.
- Then for each individual, $\boldsymbol{Y}_i = (\boldsymbol{Y}_{i,obs}, \boldsymbol{Y}_{i,mis})$.

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  - j index variables (where i already indexes individuals),
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  - $\boldsymbol{R}_i = (r_{i1}, \ldots, r_{ip})^T$ be the vector of missing indicators for person i.
  - $\boldsymbol{R} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n)$ be the matrix of missing indicators for everyone.
  - $\boldsymbol{\psi}$ be the set of parameters associated with $\boldsymbol{R}$.

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  - $\boldsymbol{\psi}$ be the set of parameters associated with $\boldsymbol{R}$.
- Assume $\boldsymbol{\psi}$ and $(\boldsymbol{\theta}, \Sigma)$ are distinct.

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- MNAR:

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- With missingness in $\boldsymbol{Y}$, the likelihood of the observed data is instead

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- Since we do not actually observe $\boldsymbol{Y}_{mis}$, we would like to be able to integrate it out so we don't have to deal with it.

- That is, we would like to infer $(\boldsymbol{\theta}, \Sigma)$ (and sometimes, $\boldsymbol{\psi}$) using only the observed data.

Likelihood function: MCAR

- For MCAR, we have:

$$
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- For inference on $(\boldsymbol{\theta}, \Sigma)$, we can simply focus on $p(\boldsymbol{Y}_{obs}|\boldsymbol{\theta}, \Sigma)$ in the likelihood function, since $(\boldsymbol{R}|\boldsymbol{\psi})$ does not include any $\boldsymbol{Y}$.

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- Also, if we want to infer the missingness mechanism through $\boldsymbol{\psi}$, we would need to deal with $p(\boldsymbol{R}|\boldsymbol{Y}_{obs}, \boldsymbol{\psi})$ anyway.

Likelihood function: MNAR

- For MNAR, we have:

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- We must include the model for $\boldsymbol{R}$ and also infer the missing data $\boldsymbol{Y}_{mis}$.

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- So how can we tell the type of mechanism we are dealing with?
- In general, we don't know!!!
- Rare that data are MCAR (unless planned beforehand); more likely that data are MNAR.
- **Compromise**: assume data are MAR if we include enough variables in model for the missing data indicator R.
- Whenever we talk about missing data in this course, we will do so in the context of MCAR and MAR.

Bayesian inference with missing data

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- By thinking of the missing data as **another set of parameters**, we can sample them from the "posterior predictive" distribution of the missing data conditional on the observed data and parameters:

$$p(\boldsymbol{Y}_{mis}|\boldsymbol{Y}_{obs}, \boldsymbol{\theta}, \Sigma) \propto \prod_{i=1}^{n} p(\boldsymbol{Y}_{i,mis}|\boldsymbol{Y}_{i,obs}, \boldsymbol{\theta}, \Sigma).$$

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Gibbs sampler with missing data

At iteration $s+1$, do the following

1. Sample $\boldsymbol{\theta}^{(s+1)}$ from its multivariate normal full conditional

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3. For each $i = 1, \ldots, n$, with at least one "1" value in the missingness indicator vector $\boldsymbol{R}_i$, sample $\boldsymbol{Y}_{i,mis}^{(s+1)}$ from the full conditional

$$p(\boldsymbol{Y}_{i,mis}^{(s+1)}|\boldsymbol{Y}_{i,obs}, \boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)}),$$

   which is also multivariate normal, with its form derived from the original sampling model but with the updated parameters, that is, $\boldsymbol{Y}_i^{(s+1)} = (\boldsymbol{Y}_{i,obs}, \boldsymbol{Y}_{i,mis}^{(s+1)})^T \sim \mathcal{N}_p(\boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)})$.

Gibbs sampler with missing data

- Rewrite $\boldsymbol{Y}_i^{(s+1)} = (\boldsymbol{Y}_{i,mis}, \boldsymbol{Y}_{i,obs}^{(s+1)})^T \sim \mathcal{N}_p(\boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)})$ as

$$\boldsymbol{Y}_i = \begin{pmatrix} \boldsymbol{Y}_{i,mis} \\ \boldsymbol{Y}_{i,obs} \end{pmatrix} \sim \mathcal{N}_p \left[\begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right],$$

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  so that we can take advantage of the conditional normal results.

- That is, we have

$$\boldsymbol{Y}_{i,mis} | \boldsymbol{Y}_{i,obs} = \boldsymbol{y}_{i,obs} \sim \mathcal{N}\left(\boldsymbol{\theta}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{y}_{i,obs} - \boldsymbol{\theta}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right).$$

  as the multivariate normal distribution (or univariate normal distribution if $\boldsymbol{Y}_i$ only has one missing entry) we need in step 3 of the Gibbs sampler.

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- Now let's look at the reading comprehension example in Hoff. We will add missing values to the original data and refit the model.

Reading example with missing data

```r
Y <- as.matrix(dget("http://www2.stat.duke.edu/~pdh10/FCBS/Inline/

#Add 20% missing data; MCAR
set.seed(1234)
Y_WithMiss <- Y #So we can keep the full data
Miss_frac <- 0.20
R <- matrix(rbinom(nrow(Y_WithMiss)*ncol(Y_WithMiss),1,Miss_frac),
            nrow(Y_WithMiss),ncol(Y_WithMiss))
Y_WithMiss[R==1]<-NA
Y_WithMiss[1:12,]
```

```
##       pretest posttest
##  [1,]      59       77
##  [2,]      43       39
##  [3,]      34       46
##  [4,]      32       NA
##  [5,]      NA       38
##  [6,]      38       NA
##  [7,]      55       NA
##  [8,]      67       86
##  [9,]      64       77
## [10,]      45       60
## [11,]      49       50
```

Compare to inference from full data

With missing data:

```
apply(THETA_WithMiss,2,summary)
```

```
##          theta_1  theta_2
## Min.    30.45459 38.29322
## 1st Qu. 43.65988 51.96991
## Median  45.60829 54.19592
## Mean    45.63192 54.20408
## 3rd Qu. 47.61896 56.48918
## Max.    58.81206 70.49105
```

Based on true data:

```
apply(THETA,2,summary)
```

```
##          theta_1  theta_2
## Min.    34.88365 37.80999
## 1st Qu. 45.29473 51.47834
## Median  47.28229 53.65172
## Mean    47.26301 53.64100
## 3rd Qu. 49.21423 55.81819
```

Compare to inference from full data

With missing data:

```
apply(SIGMA_WithMiss,2,summary)
```

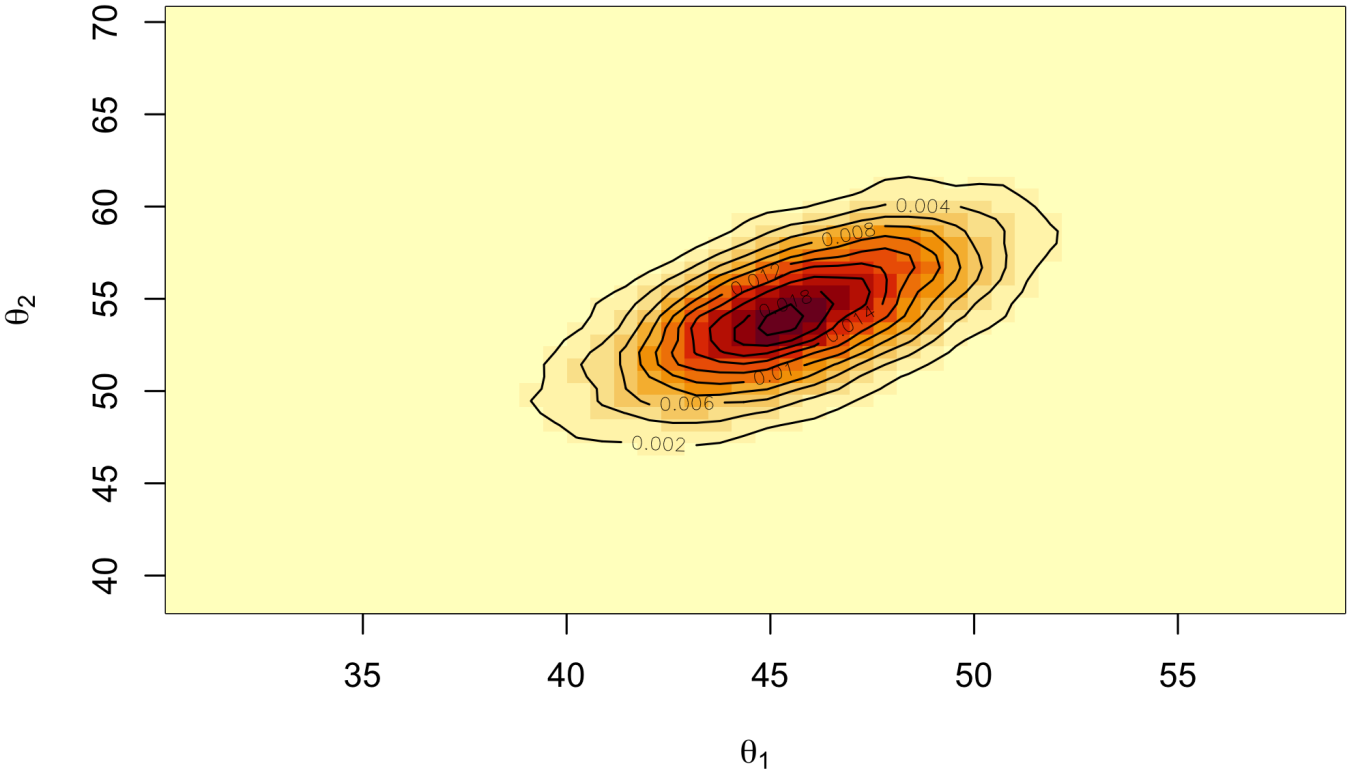
```
##         sigma_11  sigma_12  sigma_21  sigma_22
## Min.     64.0883 -20.39204 -20.39204  82.55346
## 1st Qu. 149.8338 109.84218 109.84218 190.25962
## Median  182.4496 142.34686 142.34686 233.43312
## Mean    193.9803 152.12898 152.12898 248.67527
## 3rd Qu. 224.0994 182.75082 182.75082 289.47663
## Max.    734.8704 668.77332 668.77332 981.99916
```

Based on true data:

```
apply(SIGMA,2,summary)
```

```
##         sigma_11  sigma_12  sigma_21  sigma_22
## Min.     76.4661 -38.75561 -38.75561  93.65776
## 1st Qu. 157.5870 113.32529 113.32529 203.69192
## Median  190.6578 145.08962 145.08962 246.08696
## Mean    201.9547 155.20374 155.20374 260.11361
## 3rd Qu. 233.5809 186.36991 186.36991 300.70840
```

Posterior distribution of the mean



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- Well, we can view this as a "train $\rightarrow$ test" prediction problem rather than a missing data problem on an original data.

Missing data vs predictions for new observations

- That is, given the posterior samples of the parameters, and the test values for $y^{\star}_{i2}$, draw from the posterior predictive distribution of $(y^{\star}_{i,1}|y^{\star}_{i,2}, \{(y_{1,1}, y_{1,2}), \ldots, (y_{n,1}, y_{n,2})\})$.

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- No need to incorporate the prediction problem into your original Gibbs sampler!