STA 601: Lecture 4

Comparing Estimators & Prior/Posterior Checks

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Normal Model Setup from Last Class

- independent observations $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ where each $y_i \sim \mathsf{N}(\theta, 1/\tau)$ (iid)
- The likelihood for θ is proportional to the sampling model

$$
\begin{aligned}
\mathcal{L}(\theta) &\propto \tau^{\frac{n}{2}} \ \exp\left\{-\frac{1}{2}\tau \sum_{i=1}^{n}(y_i - \theta)^2\right\} \\
&\propto \tau^{\frac{n}{2}} \ \exp\left\{-\frac{1}{2}\tau \sum_{i=1}^{n}\left[(y_i - \bar{y}) - (\theta - \bar{y})\right]^2\right\} \\
&\propto \tau^{\frac{n}{2}} \ \exp\left\{-\frac{1}{2}\tau \left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + \sum_{i=1}^{n}(\theta - \bar{y})^2\right]\right\} \\
&\propto \tau^{\frac{n}{2}} \ \exp\left\{-\frac{1}{2}\tau \left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\theta - \bar{y})^2\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\tau n(\theta - \bar{y})^2\right\}
\end{aligned}
$$

Exercises for Practice

Try this

1) Use $\mathcal{L}(\theta)$ based on n observations to find $\pi(\theta \mid y_1, \ldots, y_n)$ based on the sufficient statistics and prior $\theta \sim \mathsf{N}(\theta_0, 1/\tau_0)$

2) Use $\pi(\theta \mid y_1, \ldots, y_n)$ to find the posterior predictive distribution for Y_{n+1}

After n observations

Posterior for θ

$$\theta \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0\theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau_0 + n\tau}\right)$$

Posterior Predictive Distribution for Y_{n+1}

$$Y_{n+1} \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0\theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau} + \frac{1}{\tau_0 + n\tau}\right)$$

- Shrinkage of the MLE to the prior mean

Results with Jeffreys' Prior

- What if $\tau_0 \to 0$? (or $\sigma_0^2 \to \infty$)
- Prior predictive $\mathsf{N}(\theta_0, \sigma_0^2 + \sigma^2)$ (not proper in the limit)
- Posterior for θ (formal posterior)

$$\theta \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\frac{\tau_0\theta_0 + n\tau\bar{y}}{\tau_0 + n\tau}, \frac{1}{\tau_0 + n\tau}\right)$$

$$\to \qquad \theta \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\bar{y}, \frac{1}{n\tau}\right)$$

Posterior Predictive $Y_{n+1} \mid y_1, \ldots, y_n \sim \mathsf{N}\left(\bar{y}, \sigma^2(1 + \frac{1}{n})\right)$

Comparing Estimators

Expected loss (from frequentist perspective) of using Bayes Estimator

- Posterior mean is optimal under squared error loss (min Bayes Risk) [also absolute error loss]

Compute Mean Square Error (or Expected Average Loss)

$$\mathsf{E}_{\bar{y}|\theta}\left[\left(\hat{\theta} - \theta\right)^2 \mid \theta\right]$$

$$= \mathsf{Bias}(\hat{\theta})^2 + \mathsf{Var}(\hat{\theta})$$

- For the MLE $\bar{Y}$ this is just the variance of $\bar{Y}$ or σ^2/n

MSE for Bayes

$$\mathsf{E}_{\bar{y}\mid\theta}\left[\left(\hat{\theta}-\theta\right)^2 \mid \theta\right] = \mathsf{MSE} = \mathsf{Bias}(\hat{\theta})^2 + \mathsf{Var}(\hat{\theta})$$

- Bias of Bayes Estimate

$$\mathsf{E}_{\bar{Y}\mid\theta}\left[\frac{\tau_0\theta_0 + \tau n\bar{Y}}{\tau_0 + \tau n}\right] = \frac{\tau_0(\theta_0 - \theta)}{\tau_0 + \tau n}$$

- Variance

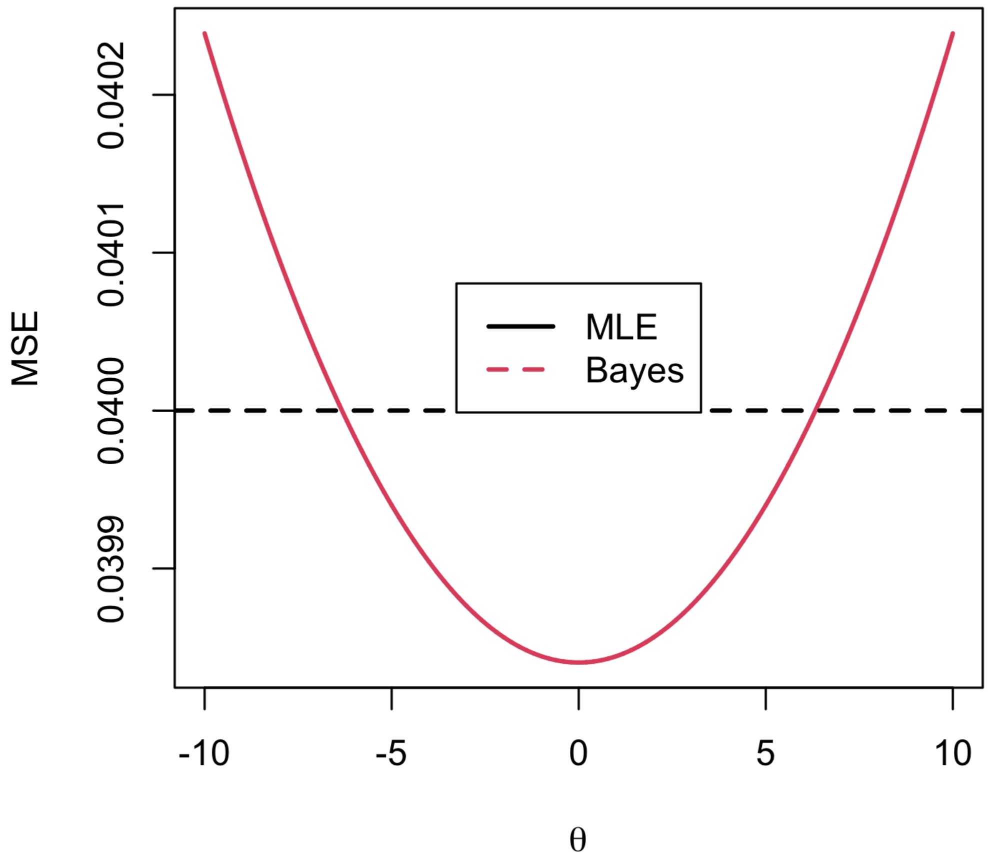
$$\mathsf{Var}\left(\frac{\tau_0\theta_0 + \tau n\bar{Y}}{\tau_0 + \tau n} - \theta \mid \theta\right) = \frac{\tau n}{(\tau_0 + \tau n)^2}$$

(Frequentist) expected Loss when truth is θ

$$\mathsf{MSE} = \frac{\tau_0^2(\theta - \theta_0)^2 + \tau n}{(\tau_0 + \tau n)^2}$$

Behavior ?

Plot



Exercise

Repeat this for estimating a future Y under squared error loss using a proper prior and Jeffreys' prior

$$\mathsf{E}_{Y_{n+1}\mid\theta}\left[(Y_{n+1} - \mathsf{E}[Y_{n+1} \mid y_1, \ldots, n])^2]\right]$$

Uses of Posterior Predictive

- Plot the entire density or summarize
- Available analytically for conjugate families
- Monte Carlo Approximation

$$p(y_{n+1} \mid y_1, \dots y_n) \approx \frac{1}{T} \sum_{t=1}^{T} p(y_{n+t} \mid \theta^{(t)})$$

where $\theta^{(t)} \sim \pi(\theta \mid y_1, \dots y_n)$ for $t = 1, \dots, T$

- T samples from the posterior distribution
- Empirical Estimates & Quantiles from Monte Carlo Samples

Model Diagnostics

- Need an accurate specification of likelihood function (and reasonable prior)

- George Box: *All models are wrong but some are useful*

- "Useful" $\rightarrow$ model provides a good approximation; there aren't clear aspects of the data that are ignored or misspecified

Example

$$Y_i \sim \text{Poisson}(\theta) \qquad i = 1, \ldots, n$$

How might our model be misspecified?

- Poisson assumes that $\text{E}(Y_i) = \text{Var}(Y_i) = \theta$
- it's *very* common for data to be **over-dispersed** $\text{E}(Y_i) < \text{Var}(Y_i)$
- **zero-inflation** many more zero values than consistent with the poisson model
- Can we use the Posterior Predictive to diagnose whether these are issues with our observed data?

Posterior Predictive (PP) Checks

- $y^{(n)}$ is observed & fixed training data
- $p(y_{n+1} \mid y^{(n)})$ is PP distributoin
- $\tilde{y}_t^{(n)}$ is t^{th} new dataset sampled from the PP of size n (same as training)
- $p(\tilde{y}_t^{(n)} \mid y^{(n)})$ is PP of new data sets
- compare some feature of the observed data to the datasets simulated from the PP

Formally

- choose a "test statistic" $t(\cdot)$ that captures some summary of the data, e.g. $\text{Var}(y^{(n)})$ for over-dispersion
- $t(y^{(n)}) \equiv t_{\text{obs}}$ value of test statistic in observed data
- $t(\tilde{y}^{(n)}) \equiv t_{\text{pred}}$ value of test statistic for a random dataset drawn from the posterior predictive
- plot posterior predictive distribution of $t(\tilde{y}^{(n)})$
- add t_{obs} to plot
- How *extreme* is t_{obs} compared to the distribution of $t(\tilde{y}^{(n)})$

Example Over Dispersion

Posterior Predictive p-values (PPPs)

- p-value is probability of seeing something as extreme or more so under a hypothetical "null" model & are uniformally distributed under the "null" model

- PPPs advocated by Gelman & Rubin in papers and BDA are not **valid** p-values. They are do not have a uniform distribution under the hypothesis that the model is correctly specified

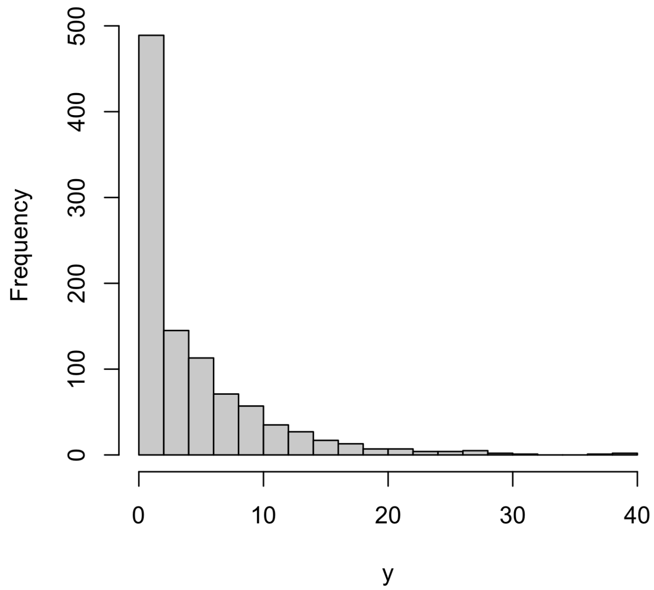
- the PPPs tend to be concentrated around 0.5, tends not to reject (conservative)

- theoretical reason for the incorrect distribution is due to double use of the data

DO NOT USE as a formal test! use as a diagnostic plot to see how model might fall flat

Better approach is to split the data use one piece to learn θ and the other to calculate t_{obs}

Zero Inflated Distribution



- Let the $t()$ be the proportion of zeros

$$t(y) = \frac{\sum_{i=1}^{n} 1(y_i = 0)}{n}$$

Posterior Predictive Distribution

Modeling Over-Dispersion

- Original Model $Y_i \mid \theta \sim \textsf{Poisson}(\theta)$
- cause of overdispersion is variation in the rate

$$Y_i \mid \theta \sim \textsf{Poisson}(\theta_i)$$

$$\theta_i \sim \pi_\theta()$$

- $\pi_\theta()$ characterizes variation in the rate parameter across inviduals
- Simple Two Stage Hierarchical Model

Example

$$\theta_i \sim \mathsf{Gamma}(\phi\mu, \phi)$$

- Find pmf for $Y_i \mid \mu, \phi$
- Find $\mathsf{E}[Y_i \mid \mu, \phi]$ and $\mathsf{Var}[Y_i \mid \mu, \phi]$
- Homework:

$$\theta_i \sim \mathsf{Gamma}(\phi, \phi/\mu)$$

- Can either of these model zero-inflation?
- See Bayarri & Berger (2000) for more discussion about why PPP should not be used as a test