# STA 702: Missing data and imputation

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- Most software packages often throw away all subjects with incomplete data (can lead to bias and precision loss).
- Some individuals impute missing values with a mean or some other fixed value (ignores uncertainty).
- Imputing missing data is actually quite natural in the Bayesian context.

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  - questions 1-15 were on the first page but questions 16-20 were at the back; and
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- **This is rarely plausible in practice!**

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# Multivariate Formulation

- Consider the multivariate data scenario with $\boldsymbol{Y}_i = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)^T$, where $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ip})^T$, for $i = 1, \ldots, n$.

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- For now, we will assume the multivariate normal model as the sampling model, so that each $p$ dimensional $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ip})^T \sim \mathcal{N}_p(\boldsymbol{\theta}, \Sigma)$.

$$p(\boldsymbol{Y}_i \mid \boldsymbol{\theta}, \Sigma) = \frac{|\Sigma|^{-1/2}}{(2\pi)^{p/2}} \exp\left\{ -\frac{1}{2} (\boldsymbol{Y} - \boldsymbol{\theta})^T \Sigma^{-1} (\boldsymbol{Y} - \boldsymbol{\theta}) \right\}$$

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- Also, let
  - $\boldsymbol{R}_i = (r_{i1}, \ldots, r_{ip})^T$ be the vector of missing indicators for person $i$.
  - $\boldsymbol{R} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_n)$ be the matrix of missing indicators for everyone.
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  - $\boldsymbol{\psi}$ be the set of parameters associated with $\boldsymbol{R}$.
- Assume $\boldsymbol{\psi}$ and $(\boldsymbol{\theta}, \Sigma)$ are distinct.

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# Implications for likelihood function

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- With missingness in $\boldsymbol{Y}$, the likelihood of the observed data is instead

$$p(\boldsymbol{Y}_{obs}, \boldsymbol{R}|\boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = \int p(\boldsymbol{R}|\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis}|\boldsymbol{\theta}, \Sigma) \mathrm{d}\boldsymbol{Y}_{mis}$$

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- Since we do not actually observe $\boldsymbol{Y}_{mis}$, we would like to be able to integrate it out so we don't have to deal with it and infer $(\boldsymbol{\theta}, \Sigma)$ using only the observed data.

# Likelihood function: MAR

- Focus on MAR

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- Hard!

# Bayesian inference with missing data

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- Think of the missing data as **latent variables** and sample from the "posterior predictive" distribution of the missing data conditional on the observed data and parameters:

$$p(\boldsymbol{Y}_{mis}|\boldsymbol{Y}_{obs}, \boldsymbol{\theta}, \Sigma) \propto \prod_{i=1}^{n} p(\boldsymbol{Y}_{i,mis}|\boldsymbol{Y}_{i,obs}, \boldsymbol{\theta}, \Sigma).$$

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- In the case of the multivariate normal model, each $p(\boldsymbol{Y}_{i,mis}|\boldsymbol{Y}_{i,obs},\boldsymbol{\theta},\Sigma)$ is just a normal distribution, and we can leverage results on conditional distributions for normal models.

# Model for Missing Data

- Rewrite as $\boldsymbol{Y}_i$ in block form

$$\boldsymbol{Y}_i = \begin{pmatrix} \boldsymbol{Y}_{i,mis} \\ \boldsymbol{Y}_{i,obs} \end{pmatrix} \sim \mathcal{N}_p \left[ \begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right],$$

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- This sampling technique actually encodes MAR since the imputations for $\boldsymbol{Y}_{mis}$ depend on the $\boldsymbol{Y}_{obs}$.

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where

- $\eta_0 > p - 1$ is the "degrees of freedom", and
- $\boldsymbol{S}_0$ is a $p \times p$ positive definite matrix.

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- Jeffreys prior (improper limiting case)

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- The **Wishart distribution** provides a conditionally-conjugate prior for the precision matrix $\Sigma^{-1}$ in a multivariate normal model.

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- Here, $E[\Phi] = \eta_0 \boldsymbol{S}_0$.

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- posterior sample size $\eta_0 + n$

- posterior sum of squares $\boldsymbol{S}_0 + \sum_{i=1}^{n}(\boldsymbol{Y}_i - \boldsymbol{\theta})(\boldsymbol{Y}_i - \boldsymbol{\theta}$

# Gibbs sampler with missing data

At iteration $s+1$, do the following

1. Sample $\boldsymbol{\theta}^{(s+1)}$ from its multivariate normal full conditional

$$p(\boldsymbol{\theta}^{(s+1)}|\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis}^{(s)}, \Sigma^{(s)}).$$

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3. For each $i = 1, \ldots, n$, with at least one "1" value in the missingness indicator vector $\boldsymbol{R}_i$, sample $\boldsymbol{Y}_{i,mis}^{(s+1)}$ from the full conditional

$$\boldsymbol{Y}_{i,mis}^{(s+1)} | \boldsymbol{Y}_{i,obs}, \boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)} \sim \mathcal{N}\left(\boldsymbol{\theta}_1^{(s+1)} + \Sigma_{12}^{(s+1)} {\Sigma_{22}^{(s+1)}}^{-1} (\boldsymbol{Y}_{i,obs} - \boldsymbol{\theta}_2^{(s+1)}), \Sigma_{11}^{(s+1)} - \Sigma_{12}^{(s+1)} {\Sigma_{22}^{(s+1)}}^{-1} \Sigma_{21}^{(s+1)}\right)$$

derived from the original sampling model but with the updated parameters, $\boldsymbol{Y}_i^{(s+1)} = (\boldsymbol{Y}_{i,obs}, \boldsymbol{Y}_{i,mis}^{(s+1)})^T \sim \mathcal{N}_p(\boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)})$.

# Reading example from Hoff with missing data

```
Y <- as.matrix(dget("http://www2.stat.duke.edu/~pdh10/FCBS/Inline/

#Add 20% missing data; MCAR
set.seed(1234)
Y_WithMiss <- Y #So we can keep the full data
Miss_frac <- 0.20
R <- matrix(rbinom(nrow(Y_WithMiss)*ncol(Y_WithMiss),1,Miss_frac),
            nrow(Y_WithMiss),ncol(Y_WithMiss))
Y_WithMiss[R==1]<-NA
Y_WithMiss[1:12,]
```

```
##       pretest posttest
##  [1,]      59       77
##  [2,]      43       39
##  [3,]      34       46
##  [4,]      32       NA
##  [5,]      NA       38
##  [6,]      38       NA
##  [7,]      55       NA
##  [8,]      67       86
##  [9,]      64       77
```

# Compare to inference from full data

With missing data:

```
apply(THETA_WithMiss,2,summary)
```

```
##          theta_1  theta_2
## Min.    30.45459 38.29322
## 1st Qu. 43.65988 51.96991
## Median  45.60829 54.19592
## Mean    45.63192 54.20408
## 3rd Qu. 47.61896 56.48918
## Max.    58.81206 70.49105
```

Based on true data:

```
apply(THETA,2,summary)
```

```
##          theta_1  theta_2
## Min.    34.88365 37.80999
## 1st Qu. 45.29473 51.47834
## Median  47.28229 53.65172
## Mean    47.26301 53.64100
## 3rd Qu. 49.21423 55.81819
```

# Compare to inference from full data

With missing data:

```
apply(SIGMA_WithMiss,2,summary)
```

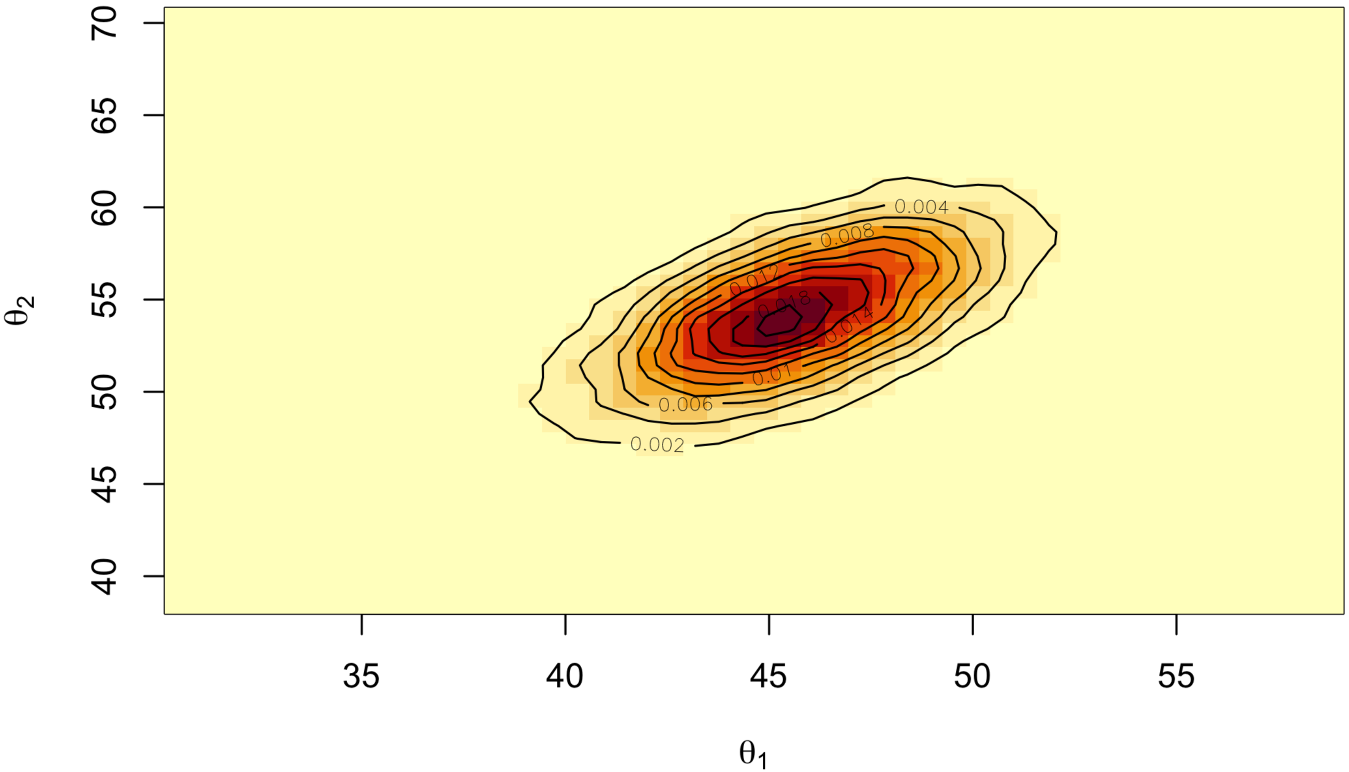
```
##          sigma_11  sigma_12  sigma_21  sigma_22
## Min.      64.0883 -20.39204 -20.39204  82.55346
## 1st Qu. 149.8338 109.84218 109.84218 190.25962
## Median  182.4496 142.34686 142.34686 233.43312
## Mean    193.9803 152.12898 152.12898 248.67527
## 3rd Qu. 224.0994 182.75082 182.75082 289.47663
## Max.    734.8704 668.77332 668.77332 981.99916
```

Based on true data:

```
apply(SIGMA,2,summary)
```

```
##          sigma_11  sigma_12  sigma_21  sigma_22
## Min.      76.4661 -38.75561 -38.75561  93.65776
## 1st Qu. 157.5870 113.32529 113.32529 203.69192
## Median  190.6578 145.08962 145.08962 246.08696
## Mean    201.9547 155.20374 155.20374 260.11361
## 3rd Qu. 233.5809 186.36991 186.36991 300.70840
```

# Posterior distribution of the mean



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- How can we predict $y_{i,1}^\star$ given $y_{i,2}^\star$, for $i = 1, \ldots, n^\star$?
- Well, we can view this as a "train $\rightarrow$ test" prediction problem rather than a missing data problem on an original data.

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- That is, given the posterior samples of the parameters, and the test values for $y_{i2}^{\star}$, draw from the posterior predictive distribution of $(y_{i,1}^{\star}|y_{i,2}^{\star},\{(y_{1,1},y_{1,2}),\ldots,(y_{n,1},y_{n,2})\})$.

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- No need to incorporate the prediction problem into your original Gibbs sampler!

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- For MNAR, we have:

$$p(\boldsymbol{Y}_{obs}, \boldsymbol{R}|\boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = \int p(\boldsymbol{R}|\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\boldsymbol{Y}_{obs}, \boldsymbol{Y}_{mis}|\boldsymbol{\theta}, \Sigma)\mathrm{d}\boldsymbol{Y}_{mis}$$

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- In general, we don't know!!!
- Rare that data are MCAR (unless planned beforehand); more likely that data are MNAR or MNAR.