Bayesian Variable Selection & Bayesian Model Averaging

Hoff Chapter 9, Liang et al 2008, Hoeting et al (1999), Clyde & George (2004)

November 2, 2022

Prior & Posterior Recap

$$\mathsf{Y} \mid \alpha, \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \phi, \boldsymbol{\gamma} \sim \mathsf{N}(1\alpha + \mathsf{X}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}, \mathsf{I}_n/\phi)$$

$$p(\alpha, \phi) \propto 1/\phi \quad \boldsymbol{\beta}_{\boldsymbol{\gamma}} \mid \boldsymbol{\gamma}, \phi \sim \mathsf{N}(0, \frac{g}{\phi}(\mathsf{X}_{\boldsymbol{\gamma}}^T\mathsf{X}_{\boldsymbol{\gamma}})^{-1}) \quad \boldsymbol{\gamma} \sim p(\boldsymbol{\gamma})$$

$$\alpha \mid \phi, y \sim \mathsf{N}\left(\bar{y}, \frac{1}{n\phi}\right)$$

$$\boldsymbol{\beta}_{\gamma} \mid \gamma, \phi, g, y \sim \mathsf{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}}_{\gamma}, \frac{g}{1+g}\frac{1}{\phi}\left[\mathsf{X}_{\gamma}{}^T\mathsf{X}_{\gamma}\right]^{-1}\right)$$

$$\phi \mid \gamma, y \sim \mathsf{Gamma}\left(\frac{n-1}{2}, \frac{\mathsf{TotalSS} - \frac{g}{1+g}\mathsf{RegSS}}{2}\right)$$

$$p(\boldsymbol{\gamma} \mid \mathsf{Y}) = \frac{BF(\boldsymbol{\gamma} : \boldsymbol{\gamma}0)p(\boldsymbol{\gamma})/p(\boldsymbol{\gamma}0)}{\sum_{\boldsymbol{\gamma}' \in \Gamma} BF(\boldsymbol{\gamma}' : \boldsymbol{\gamma}0)p(\boldsymbol{\gamma}')/p(\boldsymbol{\gamma}0)}$$

$$BF(\boldsymbol{\gamma} : \boldsymbol{\gamma}0) = (1+g)^{(n-1-p_{\boldsymbol{\gamma}})/2}(1+g(1-R^2_{\boldsymbol{\gamma}}))^{-(n-1)/2}$$

Choice of g: Bartlett's Paradox

The Bayes factor for comparing γ to the null model:

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- What happens to BF as $g \to \infty$?

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- What happens to BF as $g \to \infty$?
- Why is this a paradox?

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- BF converges to a fixed constant $(1+g)^{n-1-p_\gamma/2}$ (does not go to infinity

"Information Inconsistency" see Liang et al JASA 2008

Mixtures of g priors & Information consistency

- Need $BF \to \infty$ if $R^2_\gamma \to 1$
- Put a prior on g

$$BF(\gamma : \gamma_0) = \frac{C \int (1+g)^{(n-1-p_\gamma)/2}(1+g(1-R^2_\gamma))^{-(n-1)/2}\pi(g)dg}{C}$$

- interchange limit and integration as $R^2 \to 1$ want

$$E_g[(1+g)^{(n-1-p_\gamma)/2}]$$

  to diverge
- hyper-g prior (Liang et al JASA 2008)

$$p(g) = \frac{a-2}{2}(1+g)^{-a/2}$$

  or $g/(1+g) \sim Beta(1,(a-2)/2)$
- prior expectation converges if $a > n + 1 - p_\gamma$
- Consider minimal model $p_\gamma = 1$ and $n = 3$ (can estimate intercept, one coefficient, and σ^2, then $a > 3$ integral exists
- For $2 < a \leq 3$ integral diverges and resolves the information paradox!

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Need $BF \to \infty$ if $R^2 \to 1 \Leftrightarrow E_g[(1+g)^{(n-1-p_\gamma)/2}]$ diverges (proof in Liang et al)

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- robust prior (Bayarri et al Annals of Statistics 2012
- Intrinsic prior (Womack et al JASA 2015)

All have prior tails for β that behave like a Cauchy distribution and (the latter 4) marginal likelihoods that can be computed using special hypergeometric functions (${}_2F_1$, Appell F_1)

USair Data

```
> library(BAS)
> data(usair, package="HH")
> poll.bma = bas.lm(log(SO2) ~ temp + log(mfgfirms) +
+                                log(popn) + wind +
+                                precip + raindays,
+                   data=usair,
+                   prior="JZS",  #Jeffrey-Zellner-Siow
+                   alpha=nrow(usair), # n
+                   n.models=2^6,
+                   modelprior = uniform(),
+                   method="deterministic")
```

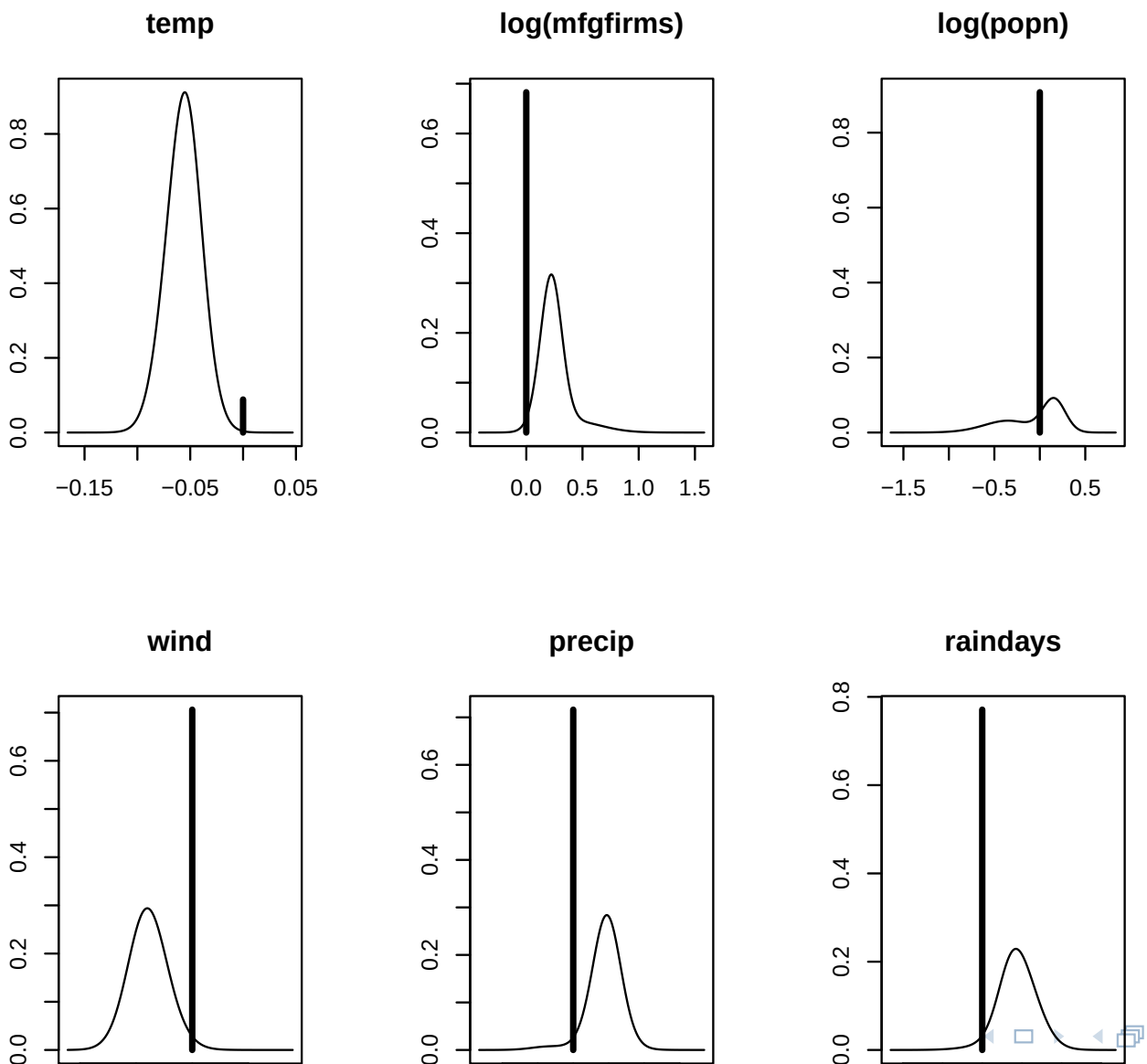
Summary

```
> summary(poll.bma)
```

	P(B != 0 | Y)	model 1	model 2	model 3	model
Intercept	1.00000000	1.00000	1.0000000	1.0000000	1.000000
temp	0.91158530	1.00000	1.0000000	1.0000000	1.000000
log(mfgfirms)	0.31718916	0.00000	0.0000000	0.0000000	1.000000
log(popn)	0.09223957	0.00000	0.0000000	0.0000000	0.000000
wind	0.29394451	0.00000	0.0000000	0.0000000	1.000000
precip	0.28384942	0.00000	1.0000000	0.0000000	1.000000
raindays	0.22903262	0.00000	0.0000000	1.0000000	0.000000
BF	NA	1.00000	0.3286643	0.2697945	0.265587
PostProbs	NA	0.29410	0.0967000	0.0794000	0.078100
R2	NA	0.29860	0.3775000	0.3714000	0.542700
dim	NA	2.00000	3.0000000	3.0000000	5.000000
logmarg	NA	3.14406	2.0313422	1.8339656	1.818248

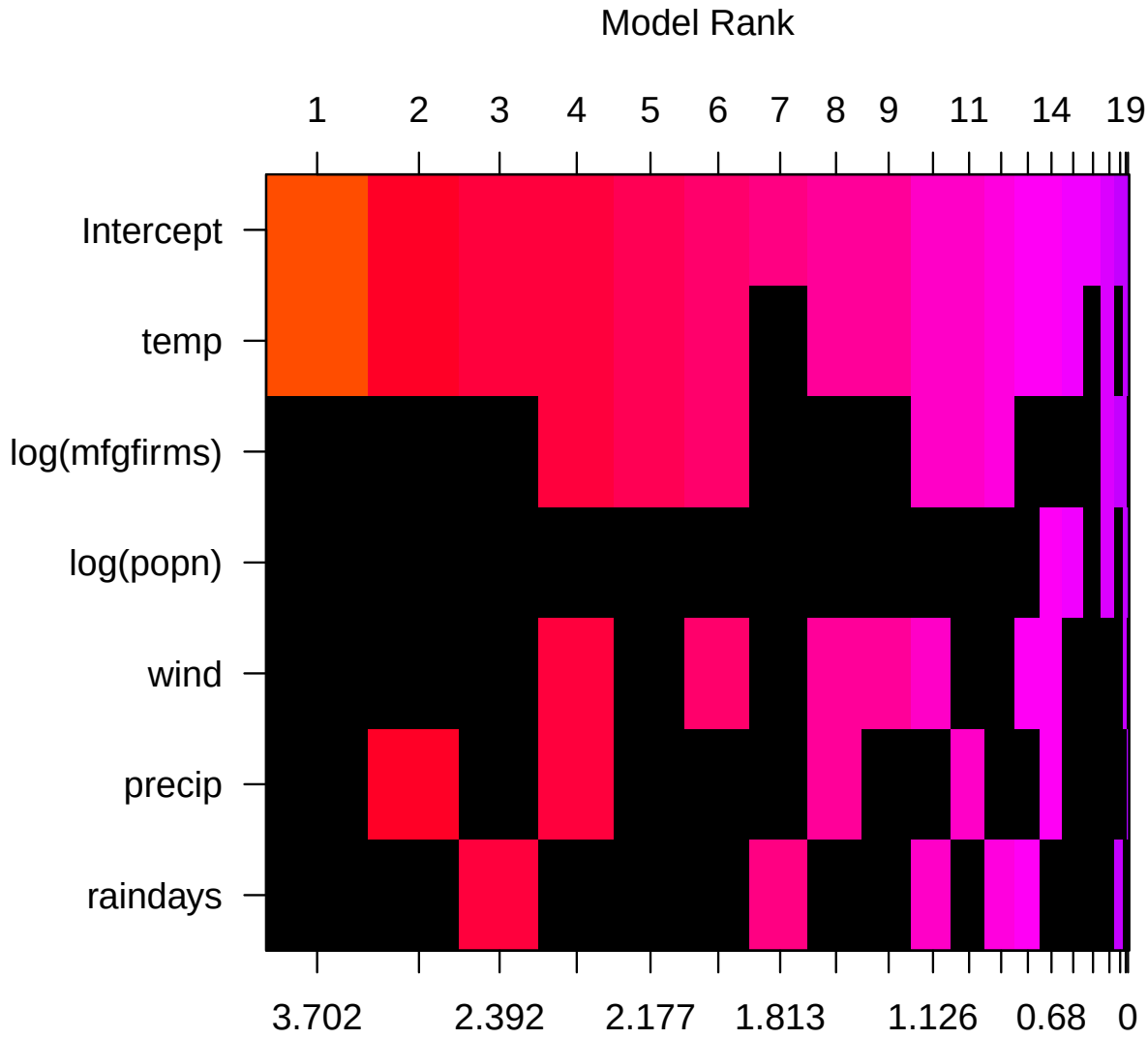
Plots

```
> beta = coef(poll.bma)
> par(mfrow=c(2,3)); plot(beta, subset=2:7,ask=F)
```



Posterior Distribution with Uniform Prior on Model Space

```
> image(poll.bma, rotate=FALSE)
```

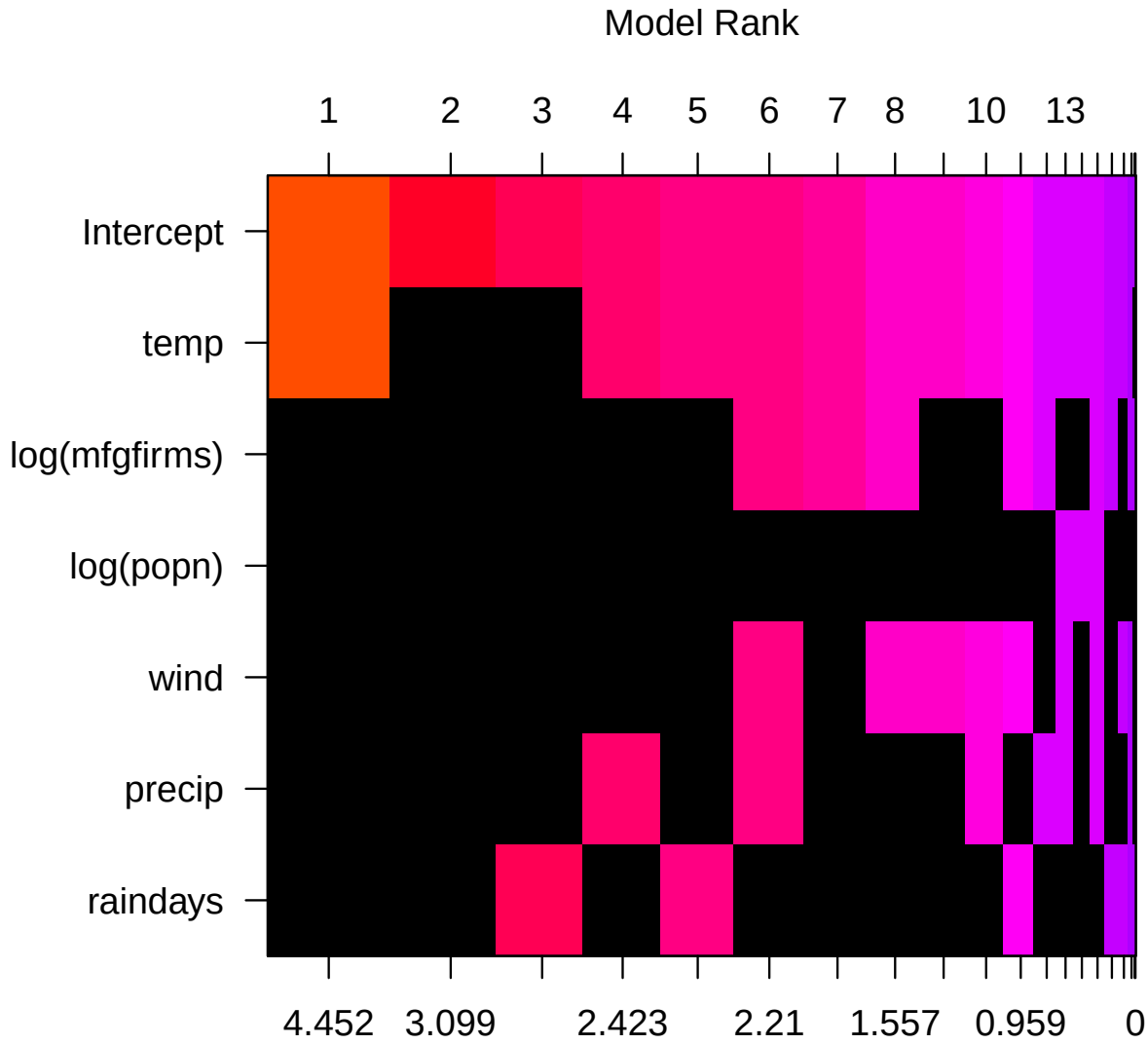


Posterior Distribution with BB(1,1) Prior on Model Space

```
> poll.bb.bma = bas.lm(log(SO2) ~ temp + log(mfgfirms) +
+                                   log(popn) + wind +
+                                   precip + raindays,
+                      data=usair,
+                      prior="JZS",
+                      alpha=nrow(usair),
+                      n.models=2^6,  #enumerate
+                      modelprior=beta.binomial(1,1))
```

BB(1,1) Prior on Model Space

```
> image(poll.bb.bma, rotate=FALSE)
```



Summary

- Choice of prior on $\boldsymbol{\beta}_\gamma$
- g-priors or mixtures of g (sensitivity)
- priors on the models (sensitivity)
- posterior summaries - select a model or "average" over all models

Diabetes Example from Hoff $p = 64$

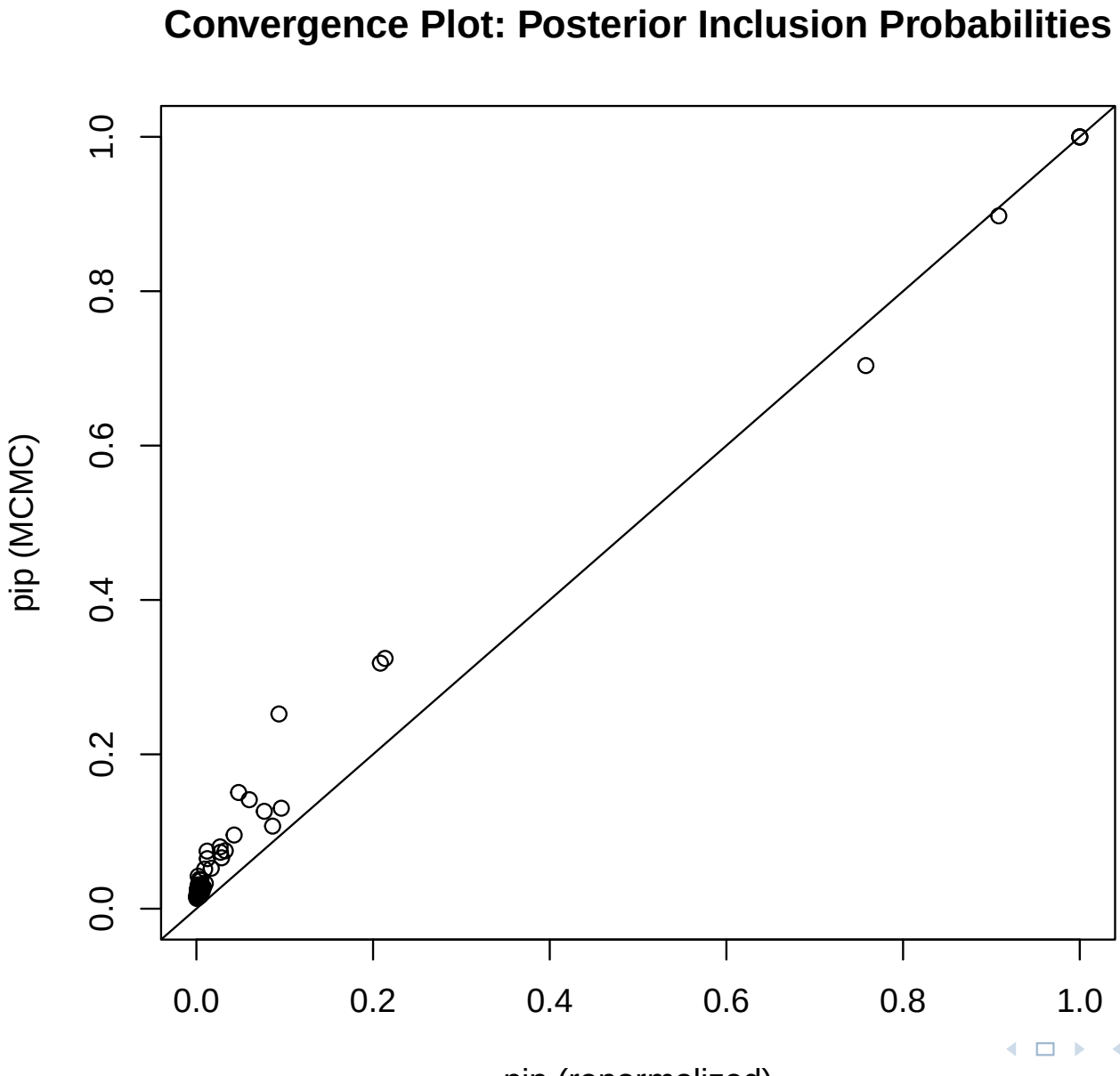
```
> set.seed(8675309)
> source("yX.diabetes.train.txt")
> diabetes.train = as.data.frame(diabetes.train)
> source("yX.diabetes.test.txt")
> diabetes.test = as.data.frame(diabetes.test)
> colnames(diabetes.test)[1] = "y"
> str(diabetes.train)
'data.frame':        342 obs. of  65 variables:
 $ y       : num  -0.0147 -1.0005 -0.1444 0.6987 -0.2222 ...
 $ age     : num  0.7996 -0.0395 1.7913 -1.8703 0.113 ...
 $ sex     : num  1.064 -0.937 1.064 -0.937 -0.937 ...
 $ bmi     : num  1.296 -1.081 0.933 -0.243 -0.764 ...
 $ map     : num  0.459 -0.553 -0.119 -0.77 0.459 ...
 $ tc      : num  -0.9287 -0.1774 -0.9576 0.256 0.0826 ...
 $ ldl     : num  -0.731 -0.402 -0.718 0.525 0.328 ...
 $ hdl     : num  -0.911 1.563 -0.679 -0.757 0.171 ...
 $ tch     : num  -0.0544 -0.8294 -0.0544 0.7205 -0.0544 ...
 $ ltg     : num  0.4181 -1.4349 0.0601 0.4765 -0.6718 ...
```

MCMC with BAS

```
> library(BAS)
> diabetes.bas = bas.lm(y ~ ., data=diabetes.train,
+                       prior = "JZS",
+                       method="MCMC",
+                       n.models = 10000,
+                       MCMC.iterations=150000,
+                       thin = 10,
+                       initprobs="eplogp",
+                       force.heredity=FALSE)
> system.time(bas.lm(y ~ ., data=diabetes.train,
+                      prior = "JZS",
+                      method="MCMC", n.models = 10000,
+                      MCMC.iterations=150000,
+                      thin = 10,  initprobs="eplogp",
+                      force.heredity=FALSE))
   user  system elapsed
  6.852   0.294   7.157
>
```

Diagnostics

```
> diagnostics(diabetes.bas, type="pip")
```

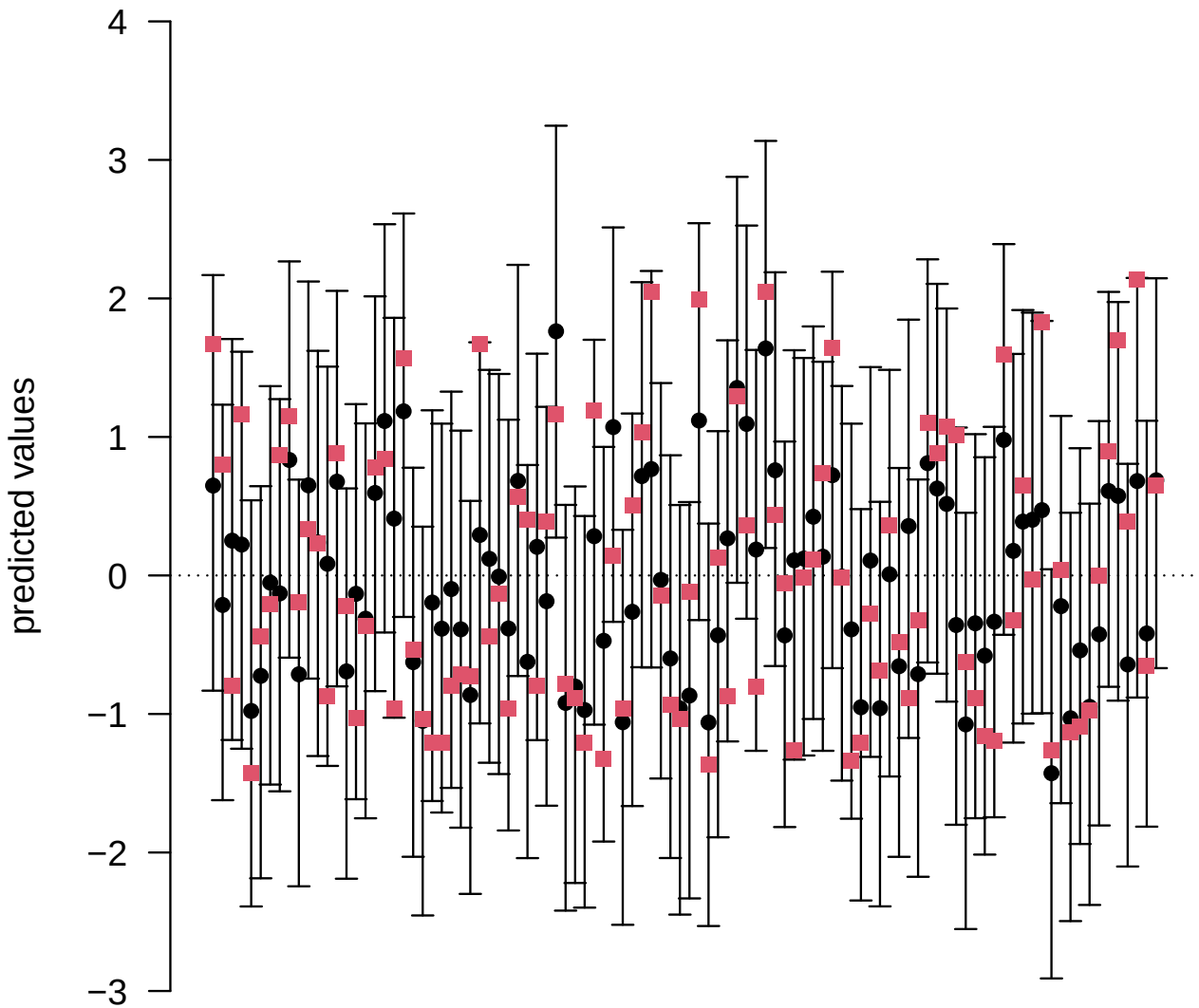


Prediction

```
> pred.bas = predict(diabetes.bas,
+                    newdata=diabetes.test,
+                    estimator="BMA",
+                    se=TRUE)
> mean((pred.bas$fit- diabetes.test$y)^2)
[1] 0.4552798
> ci.bas = confint(pred.bas);
> coverage = mean(diabetes.test$y > ci.bas[,1] & diabetes.te
> coverage
[1] 1
```

95% prediction intervals

```
> plot(ci.bas); points(diabetes.test$y, col=2, pch=15)
NULL
```



Selection and Prediction

- BMA - optimal for squared error loss Bayes
- HPM: Highest Posterior Probability model (not optimal for prediction) but for selection
- MPM: Median Probabilty model (select model where PIP $>$ 0.5) (optimal under certain conditions; nested models)
- BPM: Best Probability Model - Model closest to BMA under loss (usually includes more predictors than HPM or MPM)

Selection

```
> pred.bas = predict(diabetes.bas,
+                    newdata=diabetes.test,
+                    estimator="BPM",
+                    se=TRUE)
> #MSE
> mean((pred.bas$fit- diabetes.test$y)^2)
[1] 0.4740667
> #Coverage
> ci.bas = confint(pred.bas)
> mean(diabetes.test$y > ci.bas[,1] &
+      diabetes.test$y < ci.bas[,2])
[1] 0.98
```

Alternatives to MCMC

- "Stochastic Search" (no guarantee samples represent posterior)

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-
- If $p > n$, can use a generalized inverse, but requires care for prior on γ!

Model averaging versus Model Selection – what are objectives?

Effect Estimation

- Coefficients in each model are adjusted for other variables in the model
- OLS: leave out a predictor with a non-zero coefficient then estimates are biased!
- Model Selection in the presence of high correlation, may leave out "redundant" variables;
- improved MSE for prediction (Bias-variance tradeoff)
- in BMA all variables are included, but coefficients are shrunk to 0
- Care needed for "causal" questions and confounder adjustment! With confounding, should not use plain BMA. Need to change prior to include potential confounders (advanced topic)