STA 601: Lecture 1

Basics of Bayesian Statistics

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Ingredients

(1) **Prior Distribution** $\pi(\theta)$ for unknown θ

(2) **Likelihood Function** $\mathcal{L}(\theta \mid y) \propto p(y \mid \theta)$ (sampling model)

(3) **Posterior Distribution**

$$\pi(\theta|y) = \frac{\pi(\theta)p(y \mid \theta)}{\int_\Theta \pi(\theta)p(y \mid \theta)\mathrm{d}\theta} = \frac{\pi(\theta)p(y \mid \theta)}{p(y)}$$

(4) **Loss Function** Depends on what you want to report; estimate of θ, predict future Y_{n+1}, etc

Posterior Depends on Likelihoods

- Likelihood is defined up to a consant

$$c\,\mathcal{L}(\theta \mid Y) = p(y \mid \theta)$$

- Bayes' Rule

$$\pi(\theta|y) = \frac{\pi(\theta)p(y \mid \theta)}{\int_\Theta \pi(\theta)p(y \mid \theta)\mathrm{d}\theta} = \frac{\pi(\theta)c\mathcal{L}(\theta \mid y)}{\int_\Theta \pi(\theta)c\mathcal{L}(\theta \mid y)\mathrm{d}\theta} = \frac{\pi(\theta)\mathcal{L}(\theta \mid y)}{m(y)}$$

- $m(y)$ is proportional to the marginal distribution of data

$$m(y) = \int_\Theta \pi(\theta)\mathcal{L}(\theta \mid y)\mathrm{d}\theta$$

- marginal likelihood of this model or "evidence"

  Note: the marginal likelihood and maximized likelihood are *very* different!

Binomial Example

$$Y \mid n, \theta \sim \textsf{Binomial}(n, \theta)$$

$$p(y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$\mathcal{L}(\theta \mid y) = \theta^y (1-\theta)^{n-y}$$

MLE $\hat{\theta}$ of Binomial is $\bar{y} = y/n$ proportion of successes

Recall Derivation:

Marginal Likelihood

$$m(y) = \int_{\Theta} \mathcal{L}(\theta \mid y)\pi(\theta)\mathrm{d}\theta = \int_{\Theta} \theta^{y}(1-\theta)^{n-y}\pi(\theta)\mathrm{d}\theta$$

"Averaging" likelihood over prior

Binomial Example

- **Prior** $\theta \sim \mathsf{U}(0,1)$ or $\pi(\theta) = 1, \quad \text{for } \theta \in (0,1)$

- **Marginal**

$$m(y) = \int_0^1 \theta^y (1-\theta)^{n-y}\, 1 \, \mathrm{d}\theta$$

$$m(y) = \int_0^1 \theta^{(y+1)-1} (1-\theta)^{(n-y+1)-1}\, 1 \, \mathrm{d}\theta = B(y+1, n-y+1)$$

- Special function known as the **beta function** (see Rudin)

$$B(a,b) = \int_0^1 \theta^{a-1} (1-\theta)^{b-1} \, \mathrm{d}\theta$$

Posterior Distribution

$$\pi(\theta \mid y) = \frac{1}{B(y+1, n-y+1)} \theta^{(y+1)-1} (1-\theta)^{(n-y+1)-1} \qquad \theta \mid y \sim \mathsf{Beta}((y+1, n-y+1)$$

Beta Prior Distributions

Beta(a,b) is a probability density function (pdf) on (0,1),

$$\pi(\theta) = \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}$$

Use the "**kernel**" trick

$$\pi(\theta \mid y) \propto \mathcal{L}(\theta \mid y) \pi(\theta)$$

Prior to Posterior Updating

- **Prior** $\text{Beta}(a, b)$
- **Posterior** $\text{Beta}(a + y, b + n - y)$
- **Conjugate** prior & posterior distribution are in the same family of distributions, (Beta)
- Simple updating of information from the prior to posterior
  - $a + b$ "prior sample size" (number of trials in a hypothetical experiment)
  - a "number of successes"
  - b "number of failures"
- Should be easy to do "prior elicitation " (process of choosing the prior hyperparamters)

Summaries & Properties

Recall that for $\theta \sim \textsf{Beta}(a, b)$ $a + b = n_0$

$$\textsf{E}[\theta] = \frac{a}{a+b} \equiv \theta_0$$

Posterior mean

$$\textsf{E}[\theta \mid y] = \frac{a+y}{a+b+n} \equiv \tilde{\theta}$$

Rewrite with MLE $\hat{\theta} = \bar{y} = \frac{y}{n}$ and prior mean

$$\textsf{E}[\theta \mid y] = \frac{a+y}{a+b+n} = \frac{n_0}{n_0+n}\theta_0 + \frac{n}{n_0+n}\hat{\theta}$$

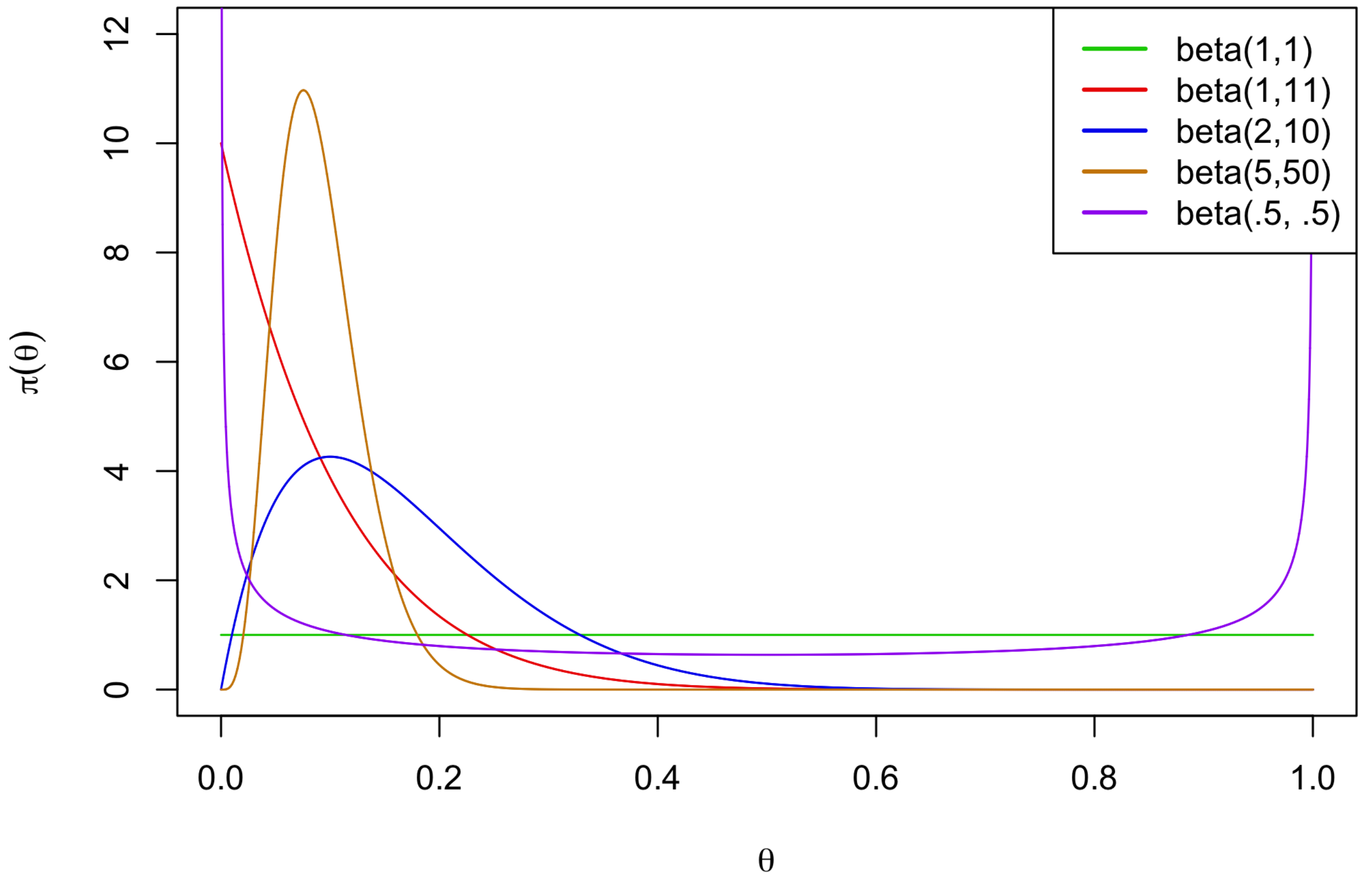
Weighted average of prior mean and MLE where weight for $\theta_0 \propto n_0$ and weight for $\hat{\theta} \propto n$

Properties

$$\tilde{\theta} = \frac{n_0}{n_0 + n}\theta_0 + \frac{n}{n_0 + n}\hat{\theta}$$

- in finite samples we get **shrinkage**: posterior mean pulls the MLE toward the prior mean; amount depends on prior sample size n_0 and data sample size n
- **regularization** effect to reduce Mean Squared Error for estimation with small sample sizes and noisy data
  - introduces some bias (in the frequentist sense) due to prior mean θ_0
  - reduces variance (bias-variance trade-off)
- helpful in the Binomial case, when sample size is small or $\theta_{\text{true}} \approx 0$ (rare events) and $\hat{\theta} = 0$ (inbalanced categorical data)
- as we get more information from the data $n \to \infty$ we have $\tilde{\theta} \to \hat{\theta}$ and **consistency** ! As $n \to \infty$, $\mathsf{E}[\tilde{\theta}] \to \theta_{\text{true}}$

Some possible prior densities



Prior Choice

- Is the uniform prior $\textsf{Beta}(1,1)$ non-informative?
  - No- if $y = 0$ (or n) sparse/rare events saying that we have a prior "historical" sample with 1 success and 1 failure ($a = 1$ and $b = 1$) can be very informative
- What about a uniform prior on the log odds? $\eta \equiv \log\left(\frac{\theta}{1-\theta}\right)$?

$$\pi(\eta) \propto 1, \qquad \eta \in \mathbb{R}$$

  - Is this a **proper** prior distribution?
  - what would be induced measure for θ?
  - Find Jacobian

$$\pi(\theta) \propto \theta^{-1}(1-\theta)^{-1}, \qquad \theta \in (0,1)$$

- limiting case of a Beta $a \to 0$ and $b \to 0$ (Haldane's prior)

Formal Bayes

- use of improper prior and turn the Bayesian crank
- calculate $m(y)$ and renormalize likelihood times "improper prior" if $m(y)$ is finite
- formal posterior is $\textsf{Beta}(y, n-y)$ and reasonable only if $y \neq 0$ or $y \neq n$ as $B(0, -)$ and $B(-, 0)$ (normalizing constant) are undefined!
- no shrinkage $\textsf{E}[\theta \mid y] = \frac{y}{n} = \tilde{\theta} = \hat{\theta}$

Invariance

Jeffreys argues that priors should be invariant to transformations to be non-informative

i.e. if we reparameterize with $\theta = h(\rho)$ then the rule should be that

$$\pi_\theta(\theta) = \left| \frac{d\rho}{d\theta} \right| \pi_\rho(h^{-1}(\theta))$$

Jefferys' rule is to pick $\pi(\rho) \propto (I(\rho))^{1/2}$

Expected Fisher Information for ρ

$$I(\rho) = -\mathsf{E}\left[\frac{d^2 \log(\mathcal{L}(\rho))}{d^2\rho} \right]$$

For the Binomial example $\pi(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$

Thus Jefferys' prior is a $\textsf{Beta}(1/2, 1/2)$

Why ?

Chain Rule!

Find Jefferys' prior for θ

Find information matrix for ρ from $I(\theta)$

Show that the prior satisfies the invariance property that