Lecture 5: Introduction to Hierarchical Modelling, Empirical Bayes, and MCMC

Merlise Clyde

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Normal Means Model

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Hierarchical Viewpoint: Let's borrow information from other observations!

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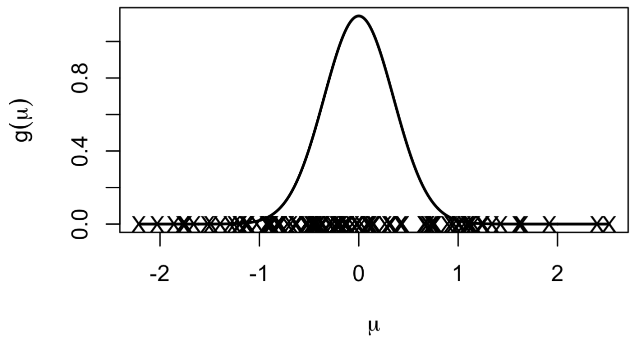

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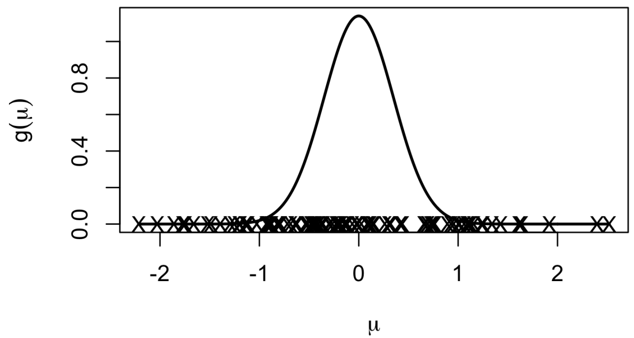

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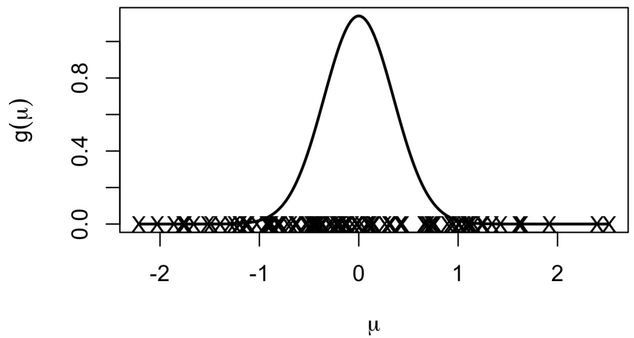

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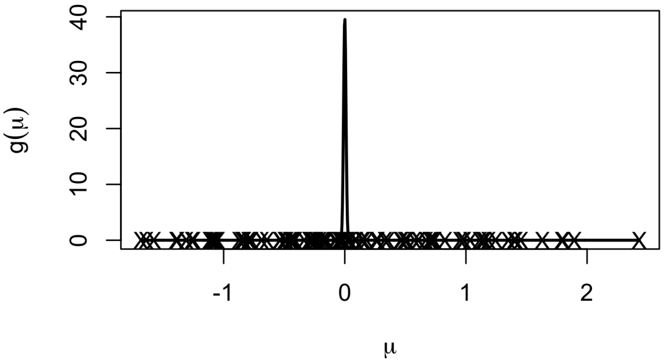

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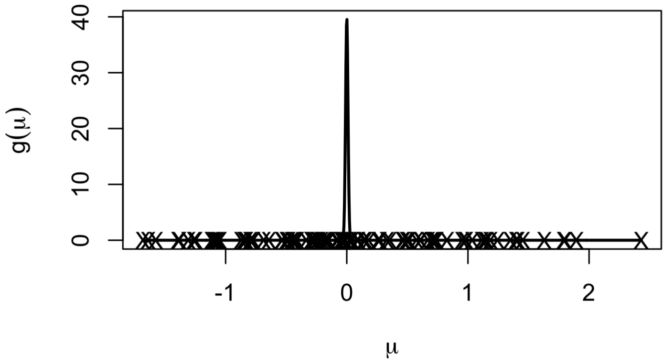

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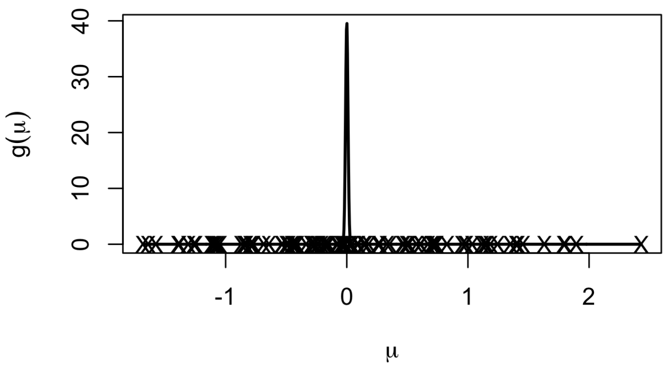

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- Not forced to choose either - what about some weighted average between y_i and $\bar{y}$?

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- not necessarily a prior!
- Now estimate μ_i (let $\phi = 1/\sigma^2$ and $\phi_\mu = 1/\sigma^2_\mu$)
- Calculate the "posterior" $\mu_i \mid y_i, \mu, \phi, \phi_\mu$

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- MSE or Bias-Variance Tradeoff

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- Examples: lasso regression, ridge regression, various kinds of hierarchical Bayesian models, etc.

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- Assume σ^2 is known (say 1)

$$\hat{\sigma}^2_\mu = \frac{\sum (y_i - \bar{y})^2}{n} - 1$$

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- Estimates of variances may be negative - constrain to 0 on the boundary)
- Fully Bayes would put a prior on the unknowns

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- Except for simple cases (conjugate models) $m(y)$ is not available analytically

Large Sample Approximations

- Appeal to BvM (Bayesian Central Limit Theorem) and approximate $\pi(\theta \mid y)$ with a Gaussian distribution centered at the posterior mode $\hat{\theta}$ and asymptotic covariance matrix

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Stochastic methods

Stochastic Integration

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- use twice for numerator and denominator

Important Sampling Estimate

Estimate of $m(y)$

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Important Sampling Estimate

Estimate of $m(y)$

$$m(y) \approx \frac{1}{T} \sum_{t=1}^{T} \frac{\pi(\theta^{(t)})\mathcal{L}(\theta^{(t)})}{q(\theta^{(t)})} \qquad \theta^{(t)} \sim q(\theta)$$

$$\mathsf{E}[h(\theta) \mid y] \approx \frac{\sum_{t=1}^{T} h(\theta^{(t)}) \frac{\pi(\theta^{(t)})\mathcal{L}(\theta^{(t)})}{q(\theta^{(t)})}}{\sum_{t=1}^{T} \frac{\pi(\theta^{(t)})\mathcal{L}(\theta^{(t)})}{q(\theta^{(t)})}} \qquad \theta^{(t)} \sim q(\theta)$$

$$\mathsf{E}[h(\theta) \mid y] \approx \sum_{t=1}^{T} h(\theta^{(t)}) w(\theta^{(t)}) \qquad \theta^{(t)} \sim q(\theta)$$

with un-normalized weights $w(\theta^{(t)}) \propto \frac{\pi(\theta^{(t)})\mathcal{L}(\theta^{(t)})}{q(\theta^{(t)})}$

(normalize to sum to 1)

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- biased samples initially but get closer to the target

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- ratio of posterior densities where normalizing constant cancels!

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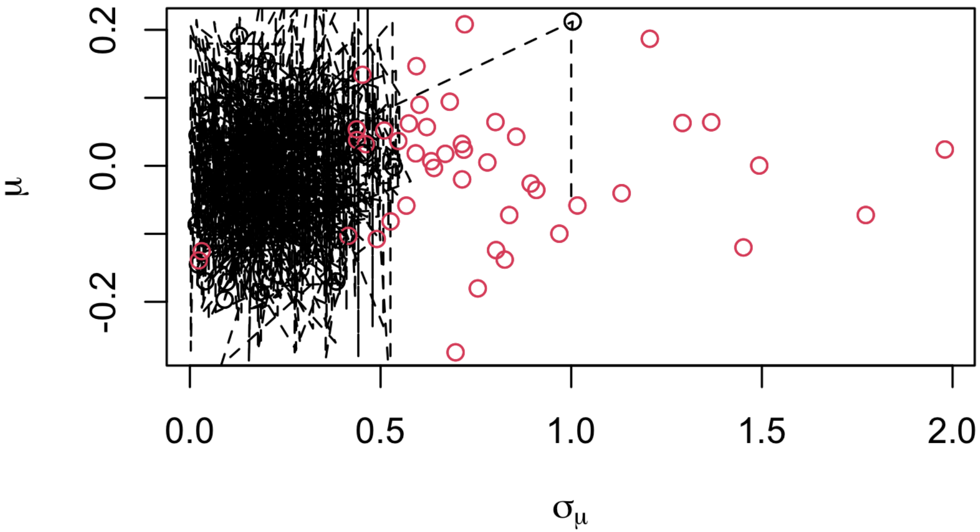
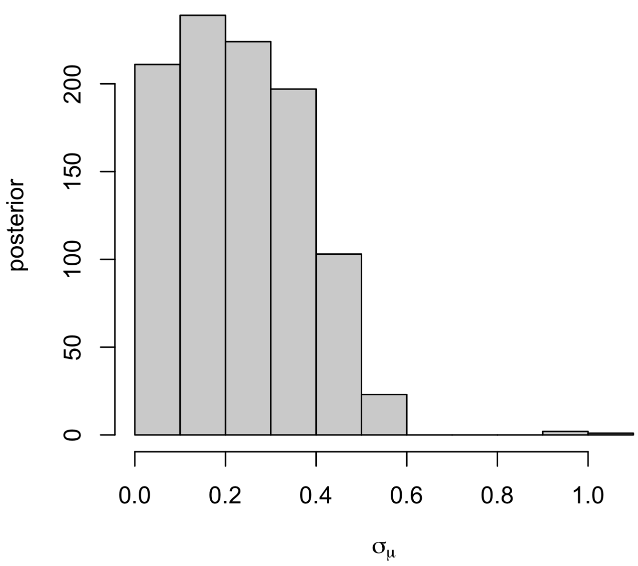
- Use a $\textsf{Cauchy}(0,1)$ prior on σ_μ
- Symmetric proposal for σ_τ? Try a normal with variance $\frac{2.4^2}{d}\textsf{var}(\sigma_\mu)$ where d is the dimension of θ (d = 1)

Joint Posterior

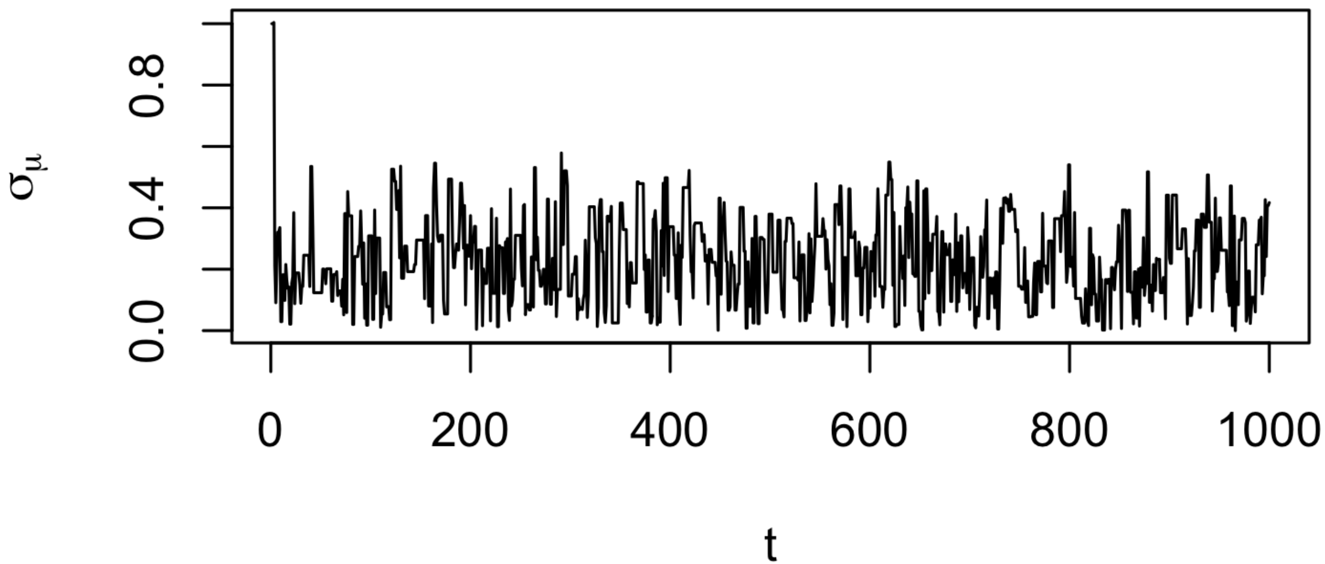


Marginal Posterior



MLE of σ_μ is 0.11

Trace Plots



- Acceptance probability is 0.57

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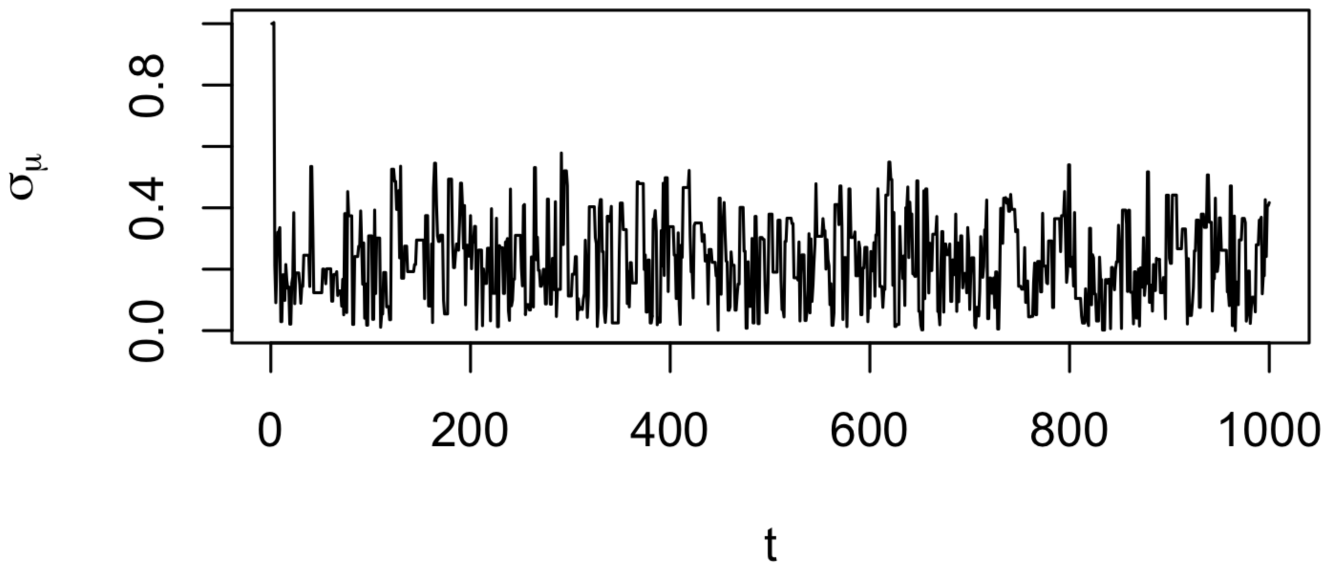

- Acceptance probability is 0.57
- Goal is around 0.44 in 1 dimension to 0.23 in higher dimensions

AutoCorrelation Function