

STA 601: Missing data and imputation

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Introduction to missing data

- Missing data/nonresponse is fairly common in real data. For example,
 - Failure to respond to survey question
 - Subject misses some clinic visits out of all possible
 - Only subset of subjects asked certain questions
- Recall that our posterior computation usually depends on the data through $p(Y|\theta)$, which cannot be computed (at least directly) when some of the y_i values are missing.
- The most common software packages often throw away all subjects with incomplete data (can lead to bias and precision loss).
- Some individuals impute missing values with a mean or some other fixed value (ignores uncertainty).



Missing data mechanisms

- Data are said to be missing completely at random (MCAR) if the reason for missingness does not depend on the values of the observed data or missing data.
- For example, suppose
 - you handed out a double-sided survey questionnaire of 20 questions to a sample of participants;
 - questions 1-15 were on the first page but questions 16-20 were at the back; and
 - some of the participants did not respond to questions 16-20.
- Then, the values for questions 16-20 for those people who did not respond would be MCAR if they simply did not realize the pages were double-sided; they had no reason to ignore those questions.
- **This is rarely plausible in practice!**



Missing data mechanisms

- Data are said to be missing at random (MAR) if, conditional on the values of the observed data, the reason for missingness does not depend on the missing data.
- Using our previous example, suppose
 - questions 1-15 include demographic information such as age and education;
 - questions 16-20 include income related questions; and
 - once again, some participants did not respond to questions 16-20.
- Then, the values for questions 16-20 for those people who did not respond would be MAR if younger people are more likely not to respond to those income related questions than old people, where age is observed for all participants.
- **This is the most commonly assumed mechanism in practice!**



Missing data mechanisms

- Data are said to be missing not at random (MNAR or NMAR) if the reason for missingness depends on the actual values of the missing (unobserved) data.
- Continuing with our previous example, suppose again that
 - questions 1-15 include demographic information such as age and education;
 - questions 16-20 include income related questions; and
 - once again, some of the participants did not respond to questions 16-20.
- Then, the values for questions 16-20 for those people who did not respond would be MNAR if people who earn more money are less likely to respond to those income related questions than old people.



Mathematical formulation

- Consider the multivariate data scenario with $\mathbf{Y}_i = (\mathbf{Y}_{i1}, \dots, \mathbf{Y}_{in})^T$, where $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{ip})^T$, for $i = 1, \dots, n$.
- For now, we will assume the multivariate normal model as the sampling model, so that each $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{ip})^T \sim \mathcal{N}_p(\boldsymbol{\theta}, \Sigma)$.
- Suppose now that \mathbf{Y} contains missing values.
- We can separate \mathbf{Y} into the observed and missing parts, that is, $\mathbf{Y} = (\mathbf{Y}_{obs}, \mathbf{Y}_{mis})$.
- Then for each individual, $\mathbf{Y}_i = (\mathbf{Y}_{i,obs}, \mathbf{Y}_{i,mis})$.



Mathematical Formulation

- Let
 - j index variables (where i already indexes individuals),
 - $r_{ij} = 1$ when y_{ij} is missing,
 - $r_{ij} = 0$ when y_{ij} is observed.
- Here, r_{ij} is known as the missingness indicator of variable j for person i .
- Also, let
 - $\mathbf{R}_i = (r_{i1}, \dots, r_{ip})^T$ be the vector of missing indicators for person i .
 - $\mathbf{R} = (\mathbf{R}_1, \dots, \mathbf{R}_n)$ be the matrix of missing indicators for everyone.
 - ψ be the set of parameters associated with \mathbf{R} .
- Assume ψ and (θ, Σ) are distinct.



Mathematical Formulation

- MCAR:

$$p(\mathbf{R}|\mathbf{Y}, \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = p(\mathbf{R}|\boldsymbol{\psi})$$

- MAR:

$$p(\mathbf{R}|\mathbf{Y}, \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = p(\mathbf{R}|\mathbf{Y}_{obs}, \boldsymbol{\psi})$$

- MNAR:

$$p(\mathbf{R}|\mathbf{Y}, \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = p(\mathbf{R}|\mathbf{Y}_{obs}, \mathbf{Y}_{mis}, \boldsymbol{\psi})$$



Implications for likelihood function

- Each type of mechanism has a different implication on the likelihood of the observed data \mathbf{Y}_{obs} , and the missing data indicator \mathbf{R} .

- Without missingness in \mathbf{Y} , the likelihood of the observed data is

$$p(\mathbf{Y}_{obs}|\boldsymbol{\theta}, \Sigma)$$

- With missingness in \mathbf{Y} , the likelihood of the observed data is instead

$$p(\mathbf{Y}_{obs}, \mathbf{R}|\boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = \int p(\mathbf{R}|\mathbf{Y}_{obs}, \mathbf{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis}|\boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis}$$

- Since we do not actually observe \mathbf{Y}_{mis} , we would like to be able to integrate it out so we don't have to deal with it.
- That is, we would like to infer $(\boldsymbol{\theta}, \Sigma)$ (and sometimes, $\boldsymbol{\psi}$) using only the observed data.



Likelihood function: MCAR

- For MCAR, we have:

$$\begin{aligned} p(\mathbf{Y}_{obs}, \mathbf{R} | \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) &= \int p(\mathbf{R} | \mathbf{Y}_{obs}, \mathbf{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= \int p(\mathbf{R} | \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= p(\mathbf{R} | \boldsymbol{\psi}) \cdot \int p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= p(\mathbf{R} | \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs} | \boldsymbol{\theta}, \Sigma). \end{aligned}$$

- For inference on $(\boldsymbol{\theta}, \Sigma)$, we can simply focus on $p(\mathbf{Y}_{obs} | \boldsymbol{\theta}, \Sigma)$ in the likelihood function, since $(\mathbf{R} | \boldsymbol{\psi})$ does not include any \mathbf{Y} .



Likelihood function: MAR

- For MAR, we have:

$$\begin{aligned} p(\mathbf{Y}_{obs}, \mathbf{R} | \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) &= \int p(\mathbf{R} | \mathbf{Y}_{obs}, \mathbf{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= \int p(\mathbf{R} | \mathbf{Y}_{obs}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= p(\mathbf{R} | \mathbf{Y}_{obs}, \boldsymbol{\psi}) \cdot \int p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis} \\ &= p(\mathbf{R} | \mathbf{Y}_{obs}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs} | \boldsymbol{\theta}, \Sigma). \end{aligned}$$

- For inference on $(\boldsymbol{\theta}, \Sigma)$, we can once again focus on $p(\mathbf{Y}_{obs} | \boldsymbol{\theta}, \Sigma)$ in the likelihood function
- Also, if we want to infer the missingness mechanism through $\boldsymbol{\psi}$, we would need to deal with $p(\mathbf{R} | \mathbf{Y}_{obs}, \boldsymbol{\psi})$ anyway.



Likelihood function: MNAR

- For MNAR, we have:

$$p(\mathbf{Y}_{obs}, \mathbf{R} | \boldsymbol{\theta}, \Sigma, \boldsymbol{\psi}) = \int p(\mathbf{R} | \mathbf{Y}_{obs}, \mathbf{Y}_{mis}, \boldsymbol{\psi}) \cdot p(\mathbf{Y}_{obs}, \mathbf{Y}_{mis} | \boldsymbol{\theta}, \Sigma) d\mathbf{Y}_{mis}$$

- The likelihood under MNAR cannot simplify any further.
- In this case, we cannot ignore the missing data when making inferences about $(\boldsymbol{\theta}, \Sigma)$.
- We must include the model for \mathbf{R} and also infer the missing data \mathbf{Y}_{mis} .



How to tell in practice?

- So how can we tell the type of mechanism we are dealing with?
- In general, we don't know!!!
- Rare that data are MCAR (unless planned beforehand); more likely that data are MNAR.
- **Compromise:** assume data are MAR if we include enough variables in model for the missing data indicator R .
- Whenever we talk about missing data in this course, we will do so in the context of MCAR and MAR.



Bayesian inference with missing data

- As we have seen, for MCAR and MAR, we can focus on $p(\mathbf{Y}_{obs}|\boldsymbol{\theta}, \Sigma)$ in the likelihood function, when inferring $(\boldsymbol{\theta}, \Sigma)$.
- While this is great, for posterior sampling under most models (especially multivariate models), we actually do need all the \mathbf{y} 's to update the parameters.
- In addition, we may actually want to learn about the missing values, in addition to inferring $(\boldsymbol{\theta}, \Sigma)$.
- By thinking of the missing data as **another set of parameters**, we can sample them from the "posterior predictive" distribution of the missing data conditional on the observed data and parameters:

$$p(\mathbf{Y}_{mis}|\mathbf{Y}_{obs}, \boldsymbol{\theta}, \Sigma) \propto \prod_{i=1}^n p(\mathbf{Y}_{i,mis}|\mathbf{Y}_{i,obs}, \boldsymbol{\theta}, \Sigma).$$



Gibbs sampler with missing data

At iteration $s + 1$, do the following

1. Sample $\boldsymbol{\theta}^{(s+1)}$ from its multivariate normal full conditional

$$p(\boldsymbol{\theta}^{(s+1)} | \mathbf{Y}_{obs}, \mathbf{Y}_{mis}^{(s)}, \Sigma^{(s)}).$$

2. Sample $\Sigma^{(s+1)}$ from its inverse-Wishart full conditional

$$p(\Sigma^{(s+1)} | \mathbf{Y}_{obs}, \mathbf{Y}_{mis}^{(s)}, \boldsymbol{\theta}^{(s+1)}).$$

3. For each $i = 1, \dots, n$, with at least one "1" value in the missingness indicator vector \mathbf{R}_i , sample $\mathbf{Y}_{i,mis}^{(s+1)}$ from the full conditional

$$p(\mathbf{Y}_{i,mis}^{(s+1)} | \mathbf{Y}_{i,obs}, \boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)}),$$

which is also multivariate normal, with its form derived from the original sampling model but with the updated parameters, that is,

$$\mathbf{Y}_i^{(s+1)} = (\mathbf{Y}_{i,obs}, \mathbf{Y}_{i,mis}^{(s+1)})^T \sim \mathcal{N}_p(\boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)}).$$



Gibbs sampler with missing data

- Rewrite $\mathbf{Y}_i^{(s+1)} = (\mathbf{Y}_{i,mis}, \mathbf{Y}_{i,obs}^{(s+1)})^T \sim \mathcal{N}_p(\boldsymbol{\theta}^{(s+1)}, \Sigma^{(s+1)})$ as

$$\mathbf{Y}_i = \begin{pmatrix} \mathbf{Y}_{i,mis} \\ \mathbf{Y}_{i,obs} \end{pmatrix} \sim \mathcal{N}_p \left[\begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right],$$

so that we can take advantage of the conditional normal results.

- That is, we have

$$\mathbf{Y}_{i,mis} | \mathbf{Y}_{i,obs} = \mathbf{y}_{i,obs} \sim \mathcal{N}(\boldsymbol{\theta}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{y}_{i,obs} - \boldsymbol{\theta}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

as the multivariate normal distribution (or univariate normal distribution if \mathbf{Y}_i only has one missing entry) we need in step 3 of the Gibbs sampler.

- This sampling technique actually encodes MAR since the imputations for \mathbf{Y}_{mis} depend on the \mathbf{Y}_{obs} .
- Now let's look at the reading comprehension example in Hoff. We will add missing values to the original data and refit the model.



Reading example with missing data

```
Y <- as.matrix(dget("http://www2.stat.duke.edu/~pdh10/FCBS/Inline,  
  
#Add 20% missing data; MCAR  
set.seed(1234)  
Y_WithMiss <- Y #So we can keep the full data  
Miss_frac <- 0.20  
R <- matrix(rbinom(nrow(Y_WithMiss)*ncol(Y_WithMiss),1,Miss_frac),  
            nrow(Y_WithMiss),ncol(Y_WithMiss))  
Y_WithMiss[R==1]<-NA  
Y_WithMiss[1:12,]
```

##		pretest	posttest
##	[1,]	59	77
##	[2,]	43	39
##	[3,]	34	46
##	[4,]	32	NA
##	[5,]	NA	38
##	[6,]	38	NA
##	[7,]	55	NA
##	[8,]	67	86
##	[9,]	64	77
##	[10,]	45	60
##	[11,]	49	50



Compare to inference from full data

With missing data:

```
apply(THETA_WithMiss,2,summary)
```

```
##           theta_1  theta_2
## Min.      30.45459 38.29322
## 1st Qu.   43.65988 51.96991
## Median    45.60829 54.19592
## Mean      45.63192 54.20408
## 3rd Qu.   47.61896 56.48918
## Max.      58.81206 70.49105
```

Based on true data:

```
apply(THETA,2,summary)
```

```
##           theta_1  theta_2
## Min.      34.88365 37.80999
## 1st Qu.   45.29473 51.47834
## Median    47.28229 53.65172
## Mean      47.26301 53.64100
## 3rd Qu.   49.21423 55.81819
```



Compare to inference from full data

With missing data:

```
apply(SIGMA_WithMiss,2,summary)
```

##		sigma_11	sigma_12	sigma_21	sigma_22
##	Min.	64.0883	-20.39204	-20.39204	82.55346
##	1st Qu.	149.8338	109.84218	109.84218	190.25962
##	Median	182.4496	142.34686	142.34686	233.43312
##	Mean	193.9803	152.12898	152.12898	248.67527
##	3rd Qu.	224.0994	182.75082	182.75082	289.47663
##	Max.	734.8704	668.77332	668.77332	981.99916

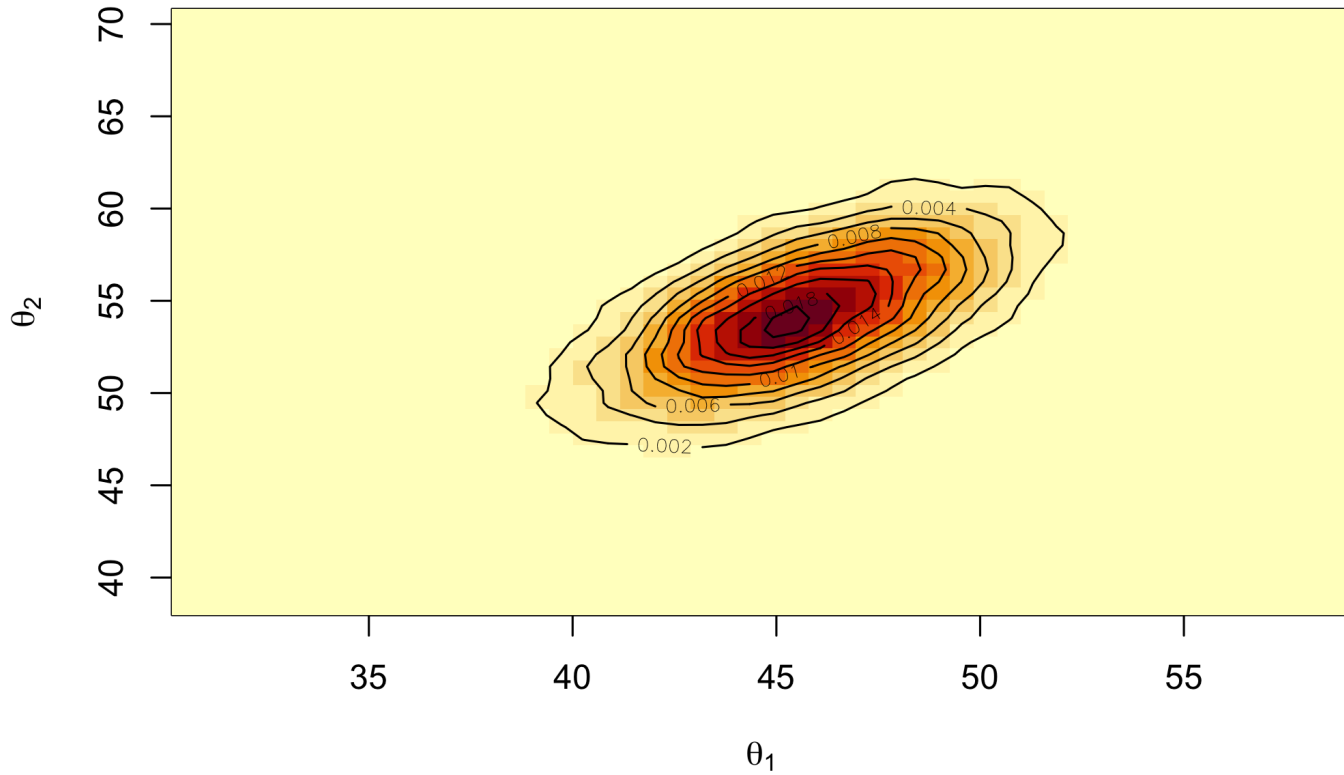
Based on true data:

```
apply(SIGMA,2,summary)
```

##		sigma_11	sigma_12	sigma_21	sigma_22
##	Min.	76.4661	-38.75561	-38.75561	93.65776
##	1st Qu.	157.5870	113.32529	113.32529	203.69192
##	Median	190.6578	145.08962	145.08962	246.08696
##	Mean	201.9547	155.20374	155.20374	260.11361
##	3rd Qu.	233.5809	186.36991	186.36991	300.70840



Posterior distribution of the mean



Missing data vs predictions for new observations

- How about predictions for completely new observations?
- That is, suppose your original dataset plus sampling model is $\mathbf{y}_i = (y_{i,1}, y_{i,2})^T \sim \mathcal{N}_2(\boldsymbol{\theta}, \Sigma), i = 1, \dots, n$.
- Suppose now you have n^* new observations with y_2^* values but no y_1^* .
- How can we predict $y_{i,1}^*$ given $y_{i,2}^*$, for $i = 1, \dots, n^*$?
- Well, we can view this as a "train \rightarrow test" prediction problem rather than a missing data problem on an original data.



Missing data vs predictions for new observations

- That is, given the posterior samples of the parameters, and the test values for y_{i2}^* , draw from the posterior predictive distribution of $(y_{i,1}^* | y_{i,2}^*, \{(y_{1,1}, y_{1,2}), \dots, (y_{n,1}, y_{n,2})\})$.
- To sample from this predictive distribution, think of compositional sampling.
- That is, for each posterior sample of (θ, Σ) , sample from $(y_{i,1} | y_{i,2}, \theta, \Sigma)$, which is just from the form of the sampling distribution.
- In this case, $(y_{i,1} | y_{i,2}, \theta, \Sigma)$ is just a normal distribution derived from $(y_{i,1}, y_{i,2} | \theta, \Sigma)$, based on the conditional normal formula.
- No need to incorporate the prediction problem into your original Gibbs sampler!

