

STA 721 HW 6

1. If $\mathbf{Y} \sim \mathbf{N}(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}_n)$, show that the likelihood function for $\boldsymbol{\beta}, \phi$ where $\phi = 1/\sigma^2$ can be written as

$$\mathcal{L}(\boldsymbol{\beta}, \phi) \propto \phi^{n/2} e^{-\phi \frac{\text{SSE}}{2}} e^{-\frac{\phi}{2} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T (\mathbf{X}^T \mathbf{X}) (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}.$$

Do you need to consider a Jacobian term for a change of variables? (explain)

2. Consider the prior data dependent prior $\boldsymbol{\beta} \mid \phi \sim N(\hat{\boldsymbol{\beta}}, \sigma^2 n (\mathbf{X}^T \mathbf{X})^{-1})$ and $\phi \sim G((n+2)/(2n), \text{SSE}/(2n))$ where $\hat{\boldsymbol{\beta}}$ is the MLE of $\boldsymbol{\beta}$, \mathbf{X} is $n \times p$ and rank p and SSE is the residual sum of squares.
 - (a) Find the prior mean of σ and σ^2 . (For those who like abstraction, find $E(\sigma^j)$ - when will this exist?) Please do not use the inverse gamma distribution, but use the distribution of ϕ .
 - (b) Using the likelihood above, find the conditional posterior distribution of $\boldsymbol{\beta}$ given ϕ and the marginal posterior distribution for ϕ , simplifying as much as possible. What is the posterior mean for $\boldsymbol{\beta}$ and σ^2 ?
 - (c) Find the marginal posterior distribution of β_j .
 - (d) Suppose that $\beta_j \mid \mathbf{Y}, \phi$ are independent (and uncorrelated). What does that imply about \mathbf{X} ? Will the β_j be independent after marginalizing over ϕ ? Are they uncorrelated? (show your work)
3. Consider the simple linear regression model to predict swimming times for a 50 yard practice as a function of time during the season (in weeks). Suppose that the only information that you are given is that you expect 95% of the swimming times to be between 20 and 24 seconds/50 yards and that the athletes generally improve over time.
 - (a) Under a conjugate normal-gamma prior, find the prior predictive distribution for swimming times for a given week?
 - (b) Construct an informative normal-gamma prior for parameters in the simple linear regression that satisfies the above probability statement and describe how you arrived at your choice.
 - (c) Simulate from the predictive distribution of swimming times for 6 weeks and construct a plot of the prediction intervals over time for each week. (you may need to iterate your hyper-parameter selection in order for your prior to agree with the information above). Add exact intervals based on the Student t prior predictive distribution.