Lecture 9: Gibbs and Data Augmentation

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September 29

Binary Regression

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- probit link

$$p(x_i^T \beta) = \Phi(x_i^T \beta)$$

- $\Phi()$ is the Normal cdf

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seemingly no, but there is a trick!

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- complete data likelihood

Augmentation Strategy

Set

- $y_i = 1_{(Z_i > 0)}$ i.e. ($y_i = 1$ if $Z_i > 0$)
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- $Z_i = x_i^T \beta + \epsilon_i \qquad \epsilon_i \overset{iid}{\sim}$ N(0,1)
- Relationship to probit model:

$$
\begin{aligned}
\Pr(y = 1 \mid \beta) &= P(Z_i > 0 \mid \beta) \\
&= P(Z_i - x_i^T \beta > -x^T \beta) \\
&= P(\epsilon_i > -x^T \beta) \\
&= 1 - \Phi(-x_i^T \beta) \\
&= \Phi(x_i^T \beta)
\end{aligned}
$$

Augmented Posterior & Gibbs

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$$\mathsf{N}(\beta; b_0, \phi_0) \left\{ \prod_{i=1}^{n} \mathsf{N}(Z_i; x_i^T \beta, 1) \right\} \left\{ \prod_{i=1}^{n} y_i 1_{(Z_i > 0)} + (1 - y_i) 1_{(Z_i < 0)} \right\}$$

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$$\pi(Z_i \mid \beta, Z_{[-i]}, y_1, \ldots, y_n) \propto \mathsf{N}(Z_i; x_i^T \beta, 1) 1_{(Z_i < 0)} \text{ if } y_1 = 0$$

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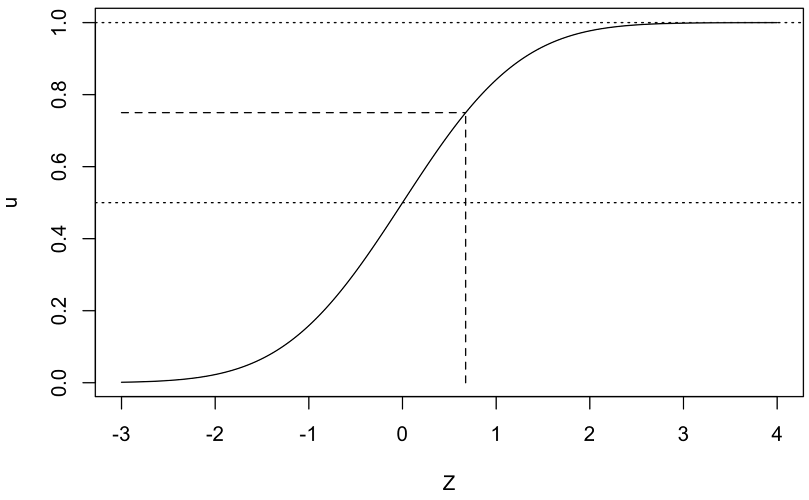
- sample from independent truncated normal distributions !
- two block Gibbs sampler $\theta_{[1]} = \beta$ and $\theta_{[2]} = (Z_1, \ldots, Z_n)^T$

Truncated Normal Sampling

- Use inverse cdf method for cdf F
- If $u \sim U(0,1)$ set $z = F^{-1}(u)$



- if $Z \in (a,b)$, Draw $u \sim U(F(a), F(b))$ and set $z = F^{-1}(u)$

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- Modeling heavy tailed distributions such as t errors in regression

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- latent variables to allow Gibbs steps but not always better!