Lecture 8: More MCMC: Metropolis-Hastings, Gibbs and Blocking

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September 26

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- adjustment for asymmetry in acceptance ratio is key to ensuring convergence to stationary distribution!

Special cases

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- combinations of the above!

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- proposal distribution q_k for the kth block is the **full conditional** distribution for $\theta_{[k]}$

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- even though joint distribution is messy, full conditionals may be (conditionally) conjugate and easy to sample from!

Univariate Normal Example

Model

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\begin{aligned}
Y_i \mid \mu, \sigma^2 &\overset{iid}{\sim} \mathsf{N}(\mu, 1/\phi) \\
\mu &\sim \mathsf{N}(\mu_0, 1/\tau_0) \\
\phi &\sim \mathsf{Gamma}(a/2, b/2)
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- Joint prior is a product of independent Normal-Gamma
- Is $\pi(\mu, \phi \mid y_1, \ldots, y_n)$ also a Normal-Gamma family?

Full Conditional for the Mean

The full conditional distributions $\mu \mid \phi, y_1, \ldots, y_n$

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\begin{aligned}
\mu \mid \phi, y_1, \ldots, y_n &\sim \mathsf{N}(\hat{\mu}, 1/\tau_n) \\
\hat{\mu} &= \frac{\tau_0 \mu_0 + n\phi\bar{y}}{\tau_0 + n\phi} \\
\tau_n &= \tau_0 + n\phi
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Full Conditional for the Precision

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\begin{aligned}
\phi \mid \mu, y_1, \ldots, y_n &\sim \textsf{Gamma}(a_n/2, b_n/2) \\
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b_n &= b + \sum_i (y_i - \mu)^2
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What happens with a non-informative prior i.e

$a = b = \epsilon$ as $\epsilon \to 0$?

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Y_i \mid \beta, \phi &\overset{iid}{\sim} \mathsf{N}(x_i^T \beta, 1/\phi) \\
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\beta &\sim \mathsf{N}(b_0, \Phi_0^{-1}) \\
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Multivariate Normal density for β

$$
\pi(\beta \mid b_0, \Phi_0) = \frac{|\Phi_0|^{1/2}}{(2\pi)^{p/2}} \exp\left\{ -\frac{1}{2}(\beta - b_0)^T \Phi_0 (\beta - b_0) \right\}
$$

Full Conditional for β

$$
\begin{aligned}
\beta \mid \phi, y_1, \ldots, y_n &\sim \mathsf{N}(b_n, \Phi_n^{-1}) \\
b_n &= (\Phi_0 + \phi X^T X)^{-1}(\Phi_0 b_0 + \phi X^T X \hat{\beta}) \\
\Phi_n &= \Phi_0 + \phi X^T X
\end{aligned}
$$

Derivation continued

Full Conditional for ϕ

$$\phi \mid \beta, y_1, \ldots, y_n \sim \textsf{Gamma}((v_0 + n)/2, (s_0 + \sum_i (y_i - x_i^T \beta)))$$

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- needs $X^T X$ to be full rank for distribution to be unique

Invariance and Choice of Mean/Precision

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- obtain the posterior for $\tilde{\beta}$ using Y and $\tilde{X}$

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- with some linear algebra we can show that this is true if $b_0 = 0$ and Φ_0 is kX^TX for some k (show!)

Zellner's g-prior

Popular choice is to take $k = \phi/g$ which is a special case of Zellner's g-prior

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Conjugate so we could skip Gibbs sampling and sample directly from gamma and then conditional normal!

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If $X^T X$ is nearly singular, certain elements of β or (linear combinations of β) may have huge variances under the g-prior (or flat prior) as the MLEs are highly unstable!

Ridge regression protects against the explosion of variances and ill-conditioning with the conjugate priors:

$$\beta \mid \phi \sim \mathsf{N}(0, \frac{1}{\phi\lambda} I_p)$$

Ridge Regression

If $X^T X$ is nearly singular, certain elements of β or (linear combinations of β) may have huge variances under the g-prior (or flat prior) as the MLEs are highly unstable!

Ridge regression protects against the explosion of variances and ill-conditioning with the conjugate priors:

$$\beta \mid \phi \sim \mathsf{N}(0, \frac{1}{\phi\lambda} I_p)$$

Posterior for β (conjugate case)

$$\beta \mid \phi, \lambda, y_1, \ldots, y_n \sim \mathsf{N}\left((\lambda I_p + X^T X)^{-1} X^T Y, \frac{1}{\phi}(\lambda I_p + X^T X)^{-1}\right)$$

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- Be sure that you can derive the full conditional posteriors for β and ϕ as well as the joint posterior in the conjugate case!

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- Introduce latent variables (data augmentation) to allow Gibbs steps (Next class)