Outliers & Robust Bayesian Regression

Readings: Hoff Chapter 9, West JRSSB 1984, Fúquene, Pérez & Pericchi 2015

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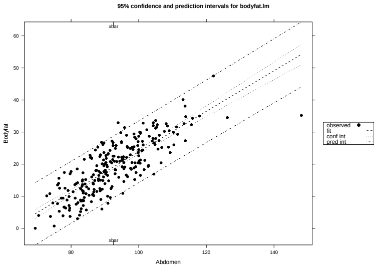
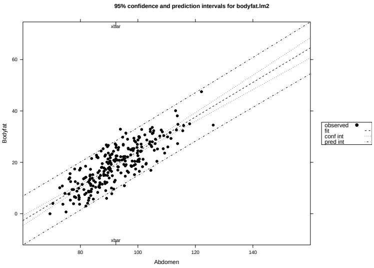
Outliers in Regression

- Hoeting, Madigan and Raftery (in various permutations) consider the problem of simultaneous variable selection and outlier identification.
- This is implemented in the package `BMA` in the function `MC3.REG`. This has the advantage that more than 2 points may be considered as outliers at the same time.
- The function uses a Markov chain to identify both important variables and potential outliers, but is coded in Fortran so should run reasonably quickly.
- Can also use BAS or other variable selection programs

Body Fat Data: Intervals w/ All Data

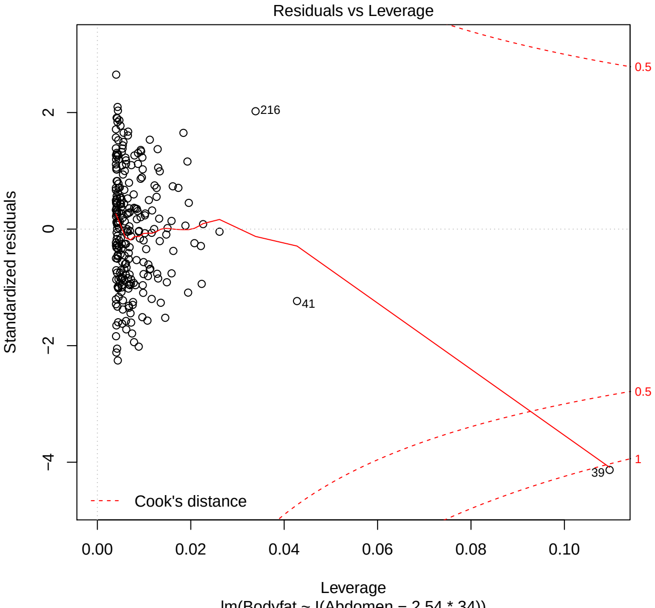
Response % Body Fat and Predictor Waist Circumference





Which analysis do we use? with Case 39 or not – or something different?

Cook's Distance



Options for Handling Influential Cases

- Are there scientific grounds for eliminating the case?
- Test if the case has a different mean than population
- Report results with and without the case
- Model Averaging to Account for Model Uncertainty?
- Full model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{I}_n \delta + \epsilon$
- 2^n submodels $\gamma_i = 0 \Leftrightarrow \delta_i = 0$
- If $\gamma_i = 1$ then case i has a different mean "mean shift" outliers.

Mean Shift = Variance Inflation

- Model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{I}_n\delta + \epsilon$
- Prior

$$\delta_i \mid \gamma_i \sim N(0, V\sigma^2\gamma_i)$$
$$\gamma_i \sim \text{Ber}(\pi)$$

Then ϵ_i given σ^2 is independent of δ_i and

$$\epsilon_i^* \equiv \epsilon_i + \delta_i \mid \sigma^2 \begin{cases} N(0, \sigma^2) & wp \quad (1-\pi) \\ N(0, \sigma^2(1+V)) & wp \quad \pi \end{cases}$$

Model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon^*$ "variance inflation"
$V + 1 = K = 7$ in the paper by Hoeting et al. package `BMA`

Simultaneous Outlier and Variable Selection

```
MC3.REG(all.y = bodyfat$Bodyfat, all.x = as.matrix(bodyfat$Abdom
       num.its = 10000, outliers = TRUE)

Model parameters: PI=0.02 K=7 nu=2.58 lambda=0.28 phi=2.85

  15  models were selected
 Best  5  models (cumulative posterior probability =  0.9939):

           prob    model 1  model 2  model 3  model 4  model 5
variables
  all.x    1          x        x        x        x        x
outliers
  39       0.94932   x        x        .        x        .
  204      0.04117   .        .        .        x        .
  207      0.10427   .        x        .        .        x

post prob          0.815    0.095    0.044    0.035    0.004
```

Change Error Assumptions

$$
\begin{aligned}
Y_i &\overset{\text{ind}}{\sim} t(\nu, \alpha + \beta x_i, 1/\phi) \\
L(\alpha, \beta, \phi) &\propto \prod_{i=1}^{n} \phi^{1/2} \left(1 + \frac{\phi(y_i - \alpha - \beta x_i)^2}{\nu}\right)^{-\frac{(\nu+1)}{2}}
\end{aligned}
$$

Use Prior $p(\alpha, \beta, \phi) \propto 1/\phi$

Posterior distribution

$$
p(\alpha, \beta, \phi \mid Y) \propto \phi^{n/2-1} \prod_{i=1}^{n} \left(1 + \frac{\phi(y_i - \alpha - \beta x_i)^2}{\nu}\right)^{-\frac{(\nu+1)}{2}}
$$

Bounded Influence - West 1984 (and references within)

Treat σ^2 as given, then *influence* of individual observations on the posterior distribution of $\boldsymbol{\beta}$ in the model where $\mathsf{E}[\mathsf{Y}_i] = \mathsf{x}_i^T \boldsymbol{\beta}$ is investigated through the score function:

$$\frac{d}{d\boldsymbol{\beta}} \log p(\boldsymbol{\beta} \mid \mathsf{Y}) = \frac{d}{d\boldsymbol{\beta}} \log p(\boldsymbol{\beta}) + \sum_{i=1}^{n} \mathsf{x}_i g(y_i - \mathsf{x}_i^T \boldsymbol{\beta})$$

where

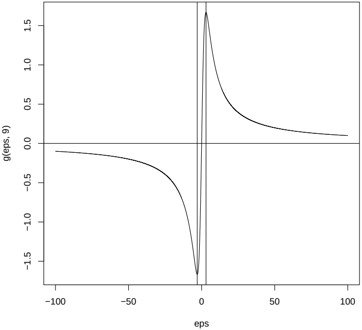
$$g(\boldsymbol{\epsilon}) = -\frac{d}{d\boldsymbol{\epsilon}} \log p(\boldsymbol{\epsilon})$$

is the influence function of the error distribution (unimodal, continuous, differentiable, symmetric)

An outlying observation y_j is accommodated if the posterior distribution for $p(\boldsymbol{\beta} \mid \mathsf{Y}_{(i)})$ converges to $p(\boldsymbol{\beta} \mid \mathsf{Y})$ for all $\boldsymbol{\beta}$ as $|\mathsf{Y}_i| \to \infty$. Requires error models with influence functions that go to zero such as the Student t (O'Hagan, 1979)

Choice of df

- Score function for t with α degrees of freedom has turning points at $\pm\sqrt{\alpha}$



- $g'(\epsilon)$ is negative when $\epsilon^2 > \alpha$ (standardized errors)
- Contribution of observation to information matrix is negative and the observation is doubtful
- Suggest taking $\alpha = 8$ or $\alpha = 9$ to reject errors larger than $\sqrt{8}$ or 3 sd.

Scale-Mixtures of Normal Representation

$$Z_i \overset{\text{iid}}{\sim} t(\nu, 0, \sigma^2) \Leftrightarrow$$

$$Z_i \mid \lambda_i \overset{\text{ind}}{\sim} N(0, \sigma^2/\lambda_i)$$
$$\lambda_i \overset{\text{iid}}{\sim} G(\nu/2, \nu/2)$$

Integrate out "latent" λ's to obtain marginal distribution.

Latent Variable Model

$$
\begin{aligned}
Y_i \mid \alpha, \beta, \phi, \lambda &\overset{\text{ind}}{\sim} N(\alpha + \beta x_i, \frac{1}{\phi \lambda_i}) \\
\lambda_i &\overset{\text{iid}}{\sim} G(\nu/2, \nu/2) \\
p(\alpha, \beta, \phi) &\propto 1/\phi
\end{aligned}
$$

Joint Posterior Distribution:

$$
\begin{aligned}
p((\alpha, \beta, \phi, \lambda_1, \ldots, \lambda_n \mid Y) \propto \quad & \phi^{n/2} \exp\left\{ -\frac{\phi}{2} \sum \lambda_i (y_i - \alpha - \beta x_i)^2 \right\} \times \\
& \phi^{-1} \\
& \prod_{i=1}^{n} \lambda_i^{\nu/2 - 1} \exp(-\lambda_i \nu/2)
\end{aligned}
$$

Model Specification via R2jags

```
rr.model = function() {
  for (i in 1:n) {
    mu[i] <- alpha0 + alpha1*(X[i] - Xbar)
    lambda[i] ~ dgamma(9/2, 9/2)
    prec[i] <- phi*lambda[i]
    Y[i] ~ dnorm(mu[i], prec[i])
  }
  phi ~ dgamma(1.0E-6, 1.0E-6)
  alpha0 ~ dnorm(0, 1.0E-6)
  alpha1 ~ dnorm(0,1.0E-6)
}
```

Specifying which Parameters to Save

The parameters to be monitored and returned to R are specified with the variable `parameters`

```
parameters = c("beta0", "beta1", "sigma",
               "mu34", "y34", "lambda[39]")
```

- All of the above (except lambda) are calculated from the other parameters. (See R-code for definitions of these parameters.)
- `mu34` and `y34` are the mean functions and predictions for a man with a 34 in waist.
- `lambda[39]` saves only the 39th case of λ
- To save a whole vector (for example all lambdas, just give the vector name)

Output

	mean	sd	2.5%	50%	97.5%
beta0	-41.70	2.75	-46.91	-41.67	-36.40
beta1	0.66	0.03	0.60	0.66	0.71
sigma	4.48	0.23	4.05	4.46	4.96
mu34	15.10	0.35	14.43	15.10	15.82
y34	14.94	5.15	4.37	15.21	24.65
lambda[39]	0.33	0.16	0.11	0.30	0.72

95% HPD interval for expected bodyfat $(14.5, 15.8)$

95% HPD interval for bodyfat $(5.1, 25.3)$

Comparison

- 95% Probability Interval for β is $(0.60, 0.71)$ with t_9 errors
- 95% Confidence Interval for β is $(0.58, 0.69)$ (all data normal model)
- 95% Confidence Interval for β is $(0.61, 0.73)$ (normal model without case 39)

Results intermediate without having to remove any observations
Case 39 down weighted by λ_{39}

Full Conditional for λ_j

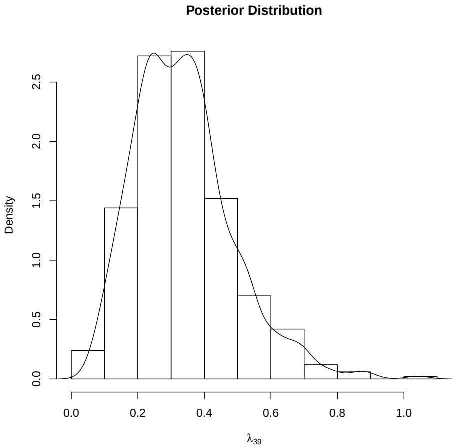
$$\begin{aligned} p(\lambda_j \mid \text{rest}, Y) &\propto p(\alpha, \beta, \phi, \lambda_1, \ldots, \lambda_n \mid Y) \\ &\propto \phi^{n/2-1} \prod_{i=1}^{n} \exp\left\{-\frac{\phi}{2}\lambda_i (y_i - \alpha - \beta x_i)^2\right\} \times \\ &\quad \prod_{i=1}^{n} \lambda_i^{\frac{\nu+1}{2}-1} \exp(-\lambda_i \frac{\nu}{2}) \end{aligned}$$

Ignore all terms except those that involve λ_j

$$\lambda_j \mid \text{rest}, Y \sim G\left(\frac{\nu+1}{2}, \frac{\phi(y_j - \alpha - \beta x_j)^2 + \nu}{2}\right)$$

Weights

Under prior $E[\lambda_i] = 1$
Under posterior, large residuals are down-weighted (approximately those bigger than $\sqrt{\nu}$)



Prior Distributions on Parameter

As a general recommendation, the prior distribution should have "heavier" tails than the likelihood

- with t_9 errors use a t_α with $\alpha < 9$
- also represent via scale mixture of normals
- Horseshoe, Double Pareto, Cauchy all have heavier tails

Sumary

- Classical diagnostics useful for EDA (checking data, potential outliers/influential points) or posterior predictive checks
- BMA/BVS and Bayesian robust regression avoid interactive decision making about outliers
- Robust Regression (Bayes) can still identify outliers through distribution on weights
- continuous versus mixture distribution on scale parameters
- Other mixtures (sub populations?) on scales and β?