Bayesian Estimation in Linear Models

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Bayesian Estimation

Model $Y_i \mid \boldsymbol{\beta}, \phi \overset{iid}{\sim} \mathsf{N}(\mathbf{x}_i^T \boldsymbol{\beta}, 1/\phi)$ is equivalent to

$$\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\beta, \mathbf{I}_n/\phi)$$

$\phi = 1/\sigma^2$ is the *precision*.

- x_i is a $p \times 1$ vector of predictors and X is $n \times p$ matrix
- $\boldsymbol{\beta}$ is $p \times 1$ vector of regression coefficients
- $\phi = 1/\sigma^2$ is the precision in the data
- Likelihood

$$\mathcal{L}(\beta, \phi) \propto \frac{|\phi \mathbf{I}_n|^{1/2}}{(2\pi)^{n/2}} \exp\left\{ -\frac{\phi}{2}(\mathbf{Y} - \mathbf{X}\beta)^T I_n (\mathbf{Y} - \mathbf{X}\beta) \right\}$$

Prior Distributions

Factor joint prior distribution $p(\boldsymbol{\beta}, \phi) = p(\boldsymbol{\beta} \mid \phi) p(\phi)$
Convenient choice is to take

- $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(b_0, \Phi_0^{-1}/\phi)$ where b_0 is the prior mean and Φ_0^{-1}/ϕ is the prior covariance of $\boldsymbol{\beta}$

$$\pi(\boldsymbol{\beta} \mid \phi = \frac{|\phi \Phi_0|^{1/2}}{(2\pi)^{p/2}} \exp\left\{ -\frac{\phi}{2} (\boldsymbol{\beta} - \mathbf{b}_0)^T \Phi_0 (\boldsymbol{\beta} - \mathbf{b}_0) \right\}$$

- $\phi \sim \mathbf{G}(\nu_0/2, \mathsf{SS}_0/2)$ with $\mathsf{E}(\sigma^2) = \mathsf{SS}_0/(\nu_0 - 2)$

$$p(\phi) = \frac{1}{\Gamma(\nu_0/2)} \left(\frac{\mathsf{SS}_0}{2} \right)^{\nu_0/2} \phi^{\nu_0/2 - 1} e^{-\phi \mathsf{SS}_0/2}$$

- $(\boldsymbol{\beta}, \phi)^T \sim \mathsf{NG}(\mathbf{b}_0, \Phi_0, \nu_0, \mathsf{SS}_0)$
- Conjugate "Normal-Gamma" family implies

$$(\boldsymbol{\beta}, \phi)^T \mid \mathbf{Y} \sim \mathsf{NG}(\mathbf{b}_n, \Phi_n, \nu_n, \mathsf{SS}_n)$$

Finding the Posterior Distribution

Express Likelihood: $\mathcal{L}(\beta, \phi) \propto \phi^{n/2} e^{-\phi \frac{\text{SSE}}{2}} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}$

$$p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\text{SSE}+\text{SS}_0)} \times \\ e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_0)^T\Phi(\boldsymbol{\beta}-\mathbf{b}_0)}$$

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta} - 2\boldsymbol{\beta}^T\Phi\mathbf{b} + \mathbf{b}^T\Phi\mathbf{b})\right\}$$

- Expand quadratics and regroup terms
- Read off posterior precision from Quadratic in $\boldsymbol{\beta}$
- Read off posterior mean from Linear term in $\boldsymbol{\beta}$
- will need to complete the quadratic in the posterior mean

Expand and Regroup

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta}-2\boldsymbol{\beta}^T\Phi\mathbf{b}+\mathbf{b}^T\Phi\mathbf{b})\right\}$$

$$\begin{aligned}
p(\boldsymbol{\beta},\phi \mid \mathbf{Y}) &\propto \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0)} \times \\
&\quad e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_0)^T\Phi_0(\boldsymbol{\beta}-\mathbf{b}_0)} \\
&= \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0)} \times \\
&\quad e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}+\Phi_0)\boldsymbol{\beta}\right)} \times \\
&\quad e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \times \\
&\quad e^{-\frac{\phi}{2}(\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0)}
\end{aligned}$$

Identify Hyperparameters and Complete the Quadratic

Quadratic in Normal

$$\exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b})^T\Phi(\boldsymbol{\beta}-\mathbf{b})\right\} = \exp\left\{-\frac{\phi}{2}(\boldsymbol{\beta}^T\Phi\boldsymbol{\beta} - 2\boldsymbol{\beta}^T\Phi\mathbf{b} + \mathbf{b}^T\Phi\mathbf{b})\right\}$$

Let $\Phi_n = \mathbf{X}^T\mathbf{X} + \Phi_0$

$$\begin{aligned}
p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad & \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0)} \\
& e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\mathbf{X}^T\mathbf{X}+\Phi_0)\boldsymbol{\beta}\right)} \\
& e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T\Phi_n\Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \\
& e^{-\frac{\phi}{2}(\mathbf{b}_n^T\Phi_n\mathbf{b}_n-\mathbf{b}_n^T\Phi_0\mathbf{b}_n)} \\
& e^{-\frac{\phi}{2}(\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0)} \\
= \quad & \phi^{\frac{n+p+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0+\hat{\boldsymbol{\beta}}^T\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T\Phi_0\mathbf{b}_0-\mathbf{b}_n^T\Phi_n\mathbf{b}_n)} \\
& e^{-\frac{\phi}{2}\left(\boldsymbol{\beta}^T(\Phi_n)\boldsymbol{\beta}\right)} \\
& e^{-\frac{\phi}{2}\left(-2\boldsymbol{\beta}^T\Phi_n\Phi_n^{-1}(\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}}+\Phi_0\mathbf{b}_0)\right)} \\
& e^{-\frac{\phi}{2}(\mathbf{b}_n^T\Phi_n\mathbf{b}_n)}
\end{aligned}$$

Posterior Distribution

$$
\begin{aligned}
p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \quad \propto \quad & \phi^{\frac{n+\nu_0}{2}-1} e^{-\frac{\phi}{2}(\mathsf{SSE}+\mathsf{SS}_0+\hat{\boldsymbol{\beta}}^T \mathbf{X}^T \mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{b}_0^T \Phi_0 \mathbf{b}_0-\mathbf{b}_n^T \Phi_n \mathbf{b}_n)} \\
& \phi^{\frac{p}{2}} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\mathbf{b}_n)^T \Phi_n (\boldsymbol{\beta}-\mathbf{b}_n)}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_n &= \mathbf{X}^T \mathbf{X} + \Phi_0 \\
\mathbf{b}_n &= \Phi_n^{-1}(\mathbf{X}^T \mathbf{X}\hat{\boldsymbol{\beta}} + \Phi_0 \mathbf{b}_0)
\end{aligned}
$$

Posterior Distribution

$$
\begin{aligned}
\boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{b}_n, (\phi \Phi_n)^{-1}) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}\left(\frac{n+\nu_0}{2}, \frac{\mathsf{SSE} + \mathsf{SS}_0 + \hat{\boldsymbol{\beta}}^T \mathbf{X}^T \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{b}_0^T \Phi_0 \mathbf{b}_0 - \mathbf{b}_n^T \Phi_n \mathbf{b}_n}{2}\right)
\end{aligned}
$$

Marginal Distribution from Normal–Gamma

Theorem

Let $\boldsymbol{\theta} \mid \phi \sim N(m, \frac{1}{\phi}\Sigma)$ *and* $\phi \sim \mathbf{G}(\nu/2, \nu\hat{\sigma}^2/2)$. *Then* $\boldsymbol{\theta}$ *(*$p \times 1$*) has a* p *dimensional multivariate* t *distribution*

$$\boldsymbol{\theta} \sim t_\nu(m, \hat{\sigma}^2\Sigma)$$

with density

$$p(\boldsymbol{\theta}) \propto \left[1 + \frac{1}{\nu}\frac{(\boldsymbol{\theta} - m)^T\Sigma^{-1}(\boldsymbol{\theta} - m)}{\hat{\sigma}^2}\right]^{-\frac{p+\nu}{2}}$$

Derivation

Marginal density $p(\boldsymbol{\theta}) = \int p(\boldsymbol{\theta} \mid \phi) p(\phi)\, d\phi$

$$
\begin{aligned}
p(\boldsymbol{\theta}) &\propto \int |\Sigma/\phi|^{-1/2} e^{-\frac{\phi}{2}(\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)} \phi^{\nu/2-1} e^{-\phi\frac{\nu\hat{\sigma}^2}{2}}\, d\phi \\
&\propto \int \phi^{p/2}\phi^{\nu/2-1} e^{-\phi\frac{(\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)+\nu\hat{\sigma}^2}{2}}\, d\phi \\
&\propto \int \phi^{\frac{p+\nu}{2}-1} e^{-\phi\frac{(\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)+\nu\hat{\sigma}^2}{2}}\, d\phi \\
&= \Gamma((p+\nu)/2)\left(\frac{(\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)+\nu\hat{\sigma}^2}{2}\right)^{-\frac{p+\nu}{2}} \\
&\propto \left((\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)+\nu\hat{\sigma}^2\right)^{-\frac{p+\nu}{2}} \\
&\propto \left(1+\frac{1}{\nu}\frac{(\boldsymbol{\theta}-m)^T\Sigma^{-1}(\boldsymbol{\theta}-m)}{\hat{\sigma}^2}\right)^{-\frac{p+\nu}{2}}
\end{aligned}
$$

Marginal Posterior Distribution of $\boldsymbol{\beta}$

$$\begin{aligned} \boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{b}_n, \phi^{-1}\Phi_n^{-1}) \\ \phi \mid \mathbf{Y} &\sim \mathbf{G}\left(\frac{\nu_n}{2}, \frac{\mathsf{SS}_n}{2}\right) \end{aligned}$$

Let $\hat{\sigma}^2 = \mathsf{SS}_n/\nu_n$ (Bayesian MSE)
Then the marginal posterior distribution of $\boldsymbol{\beta}$ is

$$\boldsymbol{\beta} \mid \mathbf{Y} \sim t_{\nu_n}(\mathbf{b}_n, \hat{\sigma}^2\Phi_n^{-1})$$

Any linear combination $\lambda^T\boldsymbol{\beta}$

$$\lambda^T\boldsymbol{\beta} \mid \mathbf{Y} \sim t_{\nu_n}(\lambda^T\mathbf{b}_n, \hat{\sigma}^2\lambda^T\Phi_n^{-1}\lambda)$$

has a univariate t distribution with $\mathbf{v}_n$ degrees of freedom

Predictive Distribution

Suppose $\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi \sim \mathsf{N}(\mathbf{X}^*\boldsymbol{\beta}, \mathbf{I}/\phi)$ and is conditionally independent of $\mathbf{Y}$ given $\boldsymbol{\beta}$ and ϕ

What is the predictive distribution of $\mathbf{Y}^* \mid \mathbf{Y}$?

$\mathbf{Y}^* = \mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\epsilon}^*$ and $\boldsymbol{\epsilon}^*$ is independent of $\mathbf{Y}$ given ϕ

$$
\begin{aligned}
\mathbf{X}^*\boldsymbol{\beta} + \boldsymbol{\epsilon}^* \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{X}^*\mathbf{b}_n, (\mathbf{X}^*\Phi_n^{-1}\mathbf{X}^{*T} + \mathbf{I})/\phi) \\
\mathbf{Y}^* \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{X}^*\mathbf{b}_n, (\mathbf{X}^*\Phi_n^{-1}\mathbf{X}^{*T} + \mathbf{I})/\phi) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}\left(\frac{\nu_n}{2}, \frac{\hat{\sigma}^2\nu_n}{2}\right) \\
\mathbf{Y}^* \mid \mathbf{Y} &\sim t_{\nu_n}(\mathbf{X}^*\mathbf{b}_n, \hat{\sigma}^2(\mathbf{I} + \mathbf{X}^*\Phi_n^{-1}\mathbf{X}^T))
\end{aligned}
$$

Alternative Derivation

Conditional Distribution:

$$
\begin{aligned}
f(\mathbf{Y}^* \mid \mathbf{Y}) &= \frac{f(\mathbf{Y}^*, \mathbf{Y})}{f(\mathbf{Y})} \\
&= \frac{\iint f(\mathbf{Y}^*, \mathbf{Y} \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi) d\boldsymbol{\beta}\, d\phi}{f(\mathbf{Y})} \\
&= \frac{\iint f(\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi) f(\mathbf{Y} \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi) d\boldsymbol{\beta}\, d\phi}{f(\mathbf{Y})} \\
&= \iint f(\mathbf{Y}^* \mid \boldsymbol{\beta}, \phi) p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) d\boldsymbol{\beta}\, d\phi
\end{aligned}
$$

$$
\mathbf{Y}^* = \mathbf{X}^* \boldsymbol{\beta} + \boldsymbol{\epsilon}^* \mid \mathbf{Y}, \phi \sim \mathsf{N}(\mathbf{X}^* \mathbf{b}_n, \phi^{-1}(\mathbf{I} + \mathbf{X}^* \Phi_n \mathbf{X}^{*T}))
$$

Use result about Marginals of Normal-Gamma family to integrate out ϕ

Conjugate Priors

Definition

A class of prior distributions $\mathcal{P}$ for $\boldsymbol{\theta}$ is conjugate for a sampling model $p(y \mid \boldsymbol{\theta})$ if for every $p(\boldsymbol{\theta}) \in \mathcal{P}$, $p(\boldsymbol{\theta} \mid \mathbf{Y}) \in \mathcal{P}$.

Advantages:

- Closed form distributions for most quantities; bypass MCMC for calculations
- Simple updating in terms of sufficient statistics "weighted average"
- Interpretation as prior samples - prior sample size
- Elicitation of prior through imaginary or historical data
- limiting "non-proper" form recovers MLEs

Choice of conjugate prior?

Unit Information Prior

Unit information prior $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\hat{\boldsymbol{\beta}}, n(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$

- Fisher Information is $\phi\mathbf{X}^T\mathbf{X}$ based on a sample of n observations
- Inverse Fisher information is covariance matrix of MLE
- "average information" in one observation is $\phi\mathbf{X}^T\mathbf{X}/n$
- center prior at MLE and base covariance on the information in "1" observation
- Posterior mean

$$\frac{n}{1+n}\hat{\boldsymbol{\beta}} + \frac{1}{1+n}\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$$

- Posterior Distribution

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi \sim \mathsf{N}\left(\hat{\boldsymbol{\beta}}, \frac{n}{1+n}(\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}\right)$$

Cannot represent real prior beliefs; double use of data

Zellner's g-prior

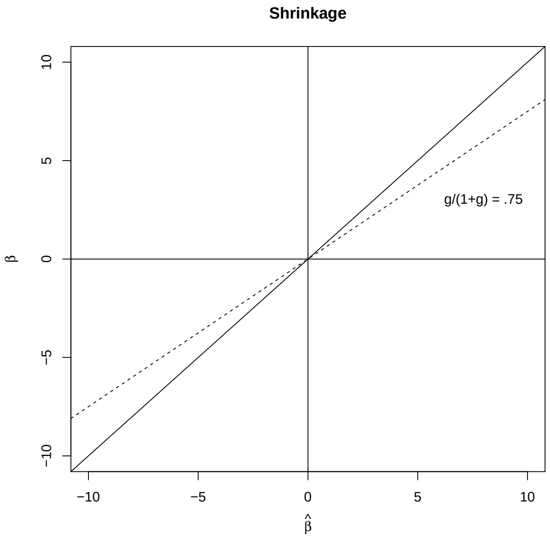
Zellner's g-prior(s) $\boldsymbol{\beta} \mid \phi \sim \mathrm{N}(\mathbf{b}_0, g(\mathbf{X}^T\mathbf{X})^{-1}/\phi)$

$$\boldsymbol{\beta} \mid \mathbf{Y}, \phi \sim \mathrm{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}} + \frac{1}{1+g}\mathbf{b}_0, \frac{g}{1+g}(\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}\right)$$

- Invariance: Require posterior of $\mathbf{X}\beta$ equal the posterior of $\mathbf{XH}\alpha$ ($\mathbf{a}_0 = \mathbf{H}^{-1}\mathbf{b}_0$) (take $\mathbf{b}_0 = \mathbf{0}$)
- Choice of g?
- $\frac{g}{1+g}$ weight given to the data
- Fixed g effect does not vanish as $n \to \infty$
- Use $g = n$ or place a prior diistribution on g

Shrinkage

Posterior mean under g-prior with $\mathbf{b}_0 = 0$ $\frac{g}{1+g}\hat{\boldsymbol{\beta}}$



Ridge Regression

- If $\mathbf{X}^T\mathbf{X}$ is nearly singular, certain elements of $\boldsymbol{\beta}$ or (linear combinations of $\boldsymbol{\beta}$) may have huge variances under the g-prior (or flat prior) as the MLEs are highly unstable!
- **Ridge regression** protects against the explosion of variances and ill-conditioning with the conjugate prior:

$$\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(0, \frac{1}{\phi\lambda}\mathbf{I}_p)$$

- Posterior for $\boldsymbol{\beta}$ (conjugate case)

$$\boldsymbol{\beta} \mid \phi, \lambda, \mathbf{Y} \sim \mathsf{N}\left((\lambda\mathbf{I}_p + \mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}, \frac{1}{\phi}(\lambda\mathbf{I}_p + \mathbf{X}^T\mathbf{X})^{-1}\right)$$

- induces shrinkage as well!

Jeffreys Prior

Jeffreys proposed a default procedure so that resulting prior would be invariant to model parameterization

$$p(\boldsymbol{\theta}) \propto |\mathcal{I}(\boldsymbol{\theta})|^{1/2}$$

where $\mathcal{I}(\boldsymbol{\theta})$ is the Expected Fisher Information matrix

$$\mathcal{I}(\theta) = -\mathsf{E}[\left[\frac{\partial^2 \log(\mathcal{L}(\boldsymbol{\theta}))}{\partial\theta_i \partial\theta_j}\right]]$$

Fisher Information Matrix

Log Likelihood

$$\log(\mathcal{L}(\boldsymbol{\beta}, \phi)) = \frac{n}{2}\log(\phi) - \frac{\phi}{2}\|(\mathbf{I} - \mathbf{P_x})\mathbf{Y}\|^2 - \frac{\phi}{2}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$$

$$\begin{aligned}
\frac{\partial^2 \log \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} &= \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & -(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \\ -(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X}) & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix} \\
\mathrm{E}[\frac{\partial^2 \log \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}] &= \begin{bmatrix} -\phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & -\frac{n}{2}\frac{1}{\phi^2} \end{bmatrix} \\
\mathcal{I}((\boldsymbol{\beta}, \phi)^T) &= \begin{bmatrix} \phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & \frac{n}{2}\frac{1}{\phi^2} \end{bmatrix}
\end{aligned}$$

Jeffreys Prior

Jeffreys Prior

$$\begin{aligned} p_J(\boldsymbol{\beta}, \phi) &\propto |\mathcal{I}((\boldsymbol{\beta}, \phi)^T)|^{1/2} \\ &= |\phi(\mathbf{X}^T\mathbf{X}|^{1/2} \left(\frac{n}{2}\frac{1}{\phi^2}\right)^{1/2} \\ &\propto \phi^{p/2-1}|\mathbf{X}^T\mathbf{X}|^{1/2} \\ &\propto \phi^{p/2-1} \end{aligned}$$

Improper prior $\iint p_J(\boldsymbol{\beta}, \phi)\, d\boldsymbol{\beta}\, d\phi$ not finite

Formal Bayes Posterior

$$p(\boldsymbol{\beta}, \phi \mid \mathbf{Y}) \propto p(\mathbf{Y} \mid \boldsymbol{\beta}, \phi)\phi^{p/2-1}$$

if this is integrable, then renormalize to obtain formal posterior distribution

$$\begin{aligned} \boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathrm{N}(\hat{\boldsymbol{\beta}}, (\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}) \\ \phi \mid \mathbf{Y} &\sim \mathbf{G}(n/2, \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|^2/2) \end{aligned}$$

Limiting case of Conjugate prior with $\mathbf{b}_0 = 0$, $\Phi = \mathbf{0}$, $\nu_0 = 0$ and $\mathrm{SS}_0 = 0$

Jeffreys did not recommend using this Posterior does not depend on dimension p

Independent Jeffreys Prior

- Treat $\boldsymbol{\beta}$ and ϕ separately ("orthogonal parameterization")
- $p_{IJ}(\boldsymbol{\beta}) \propto |\mathcal{I}(\boldsymbol{\beta})|^{1/2}$
- $p_{IJ}(\phi) \propto |\mathcal{I}(\phi)||^{1/2}$

$$\mathcal{I}((\boldsymbol{\beta}, \phi)^T) = \begin{bmatrix} \phi(\mathbf{X}^T\mathbf{X}) & \mathbf{0}_p \\ \mathbf{0}_p^T & \frac{n}{2}\frac{1}{\phi^2} \end{bmatrix}$$

$$p_{IJ}(\boldsymbol{\beta}) \propto |\phi \mathbf{X}^T\mathbf{X}|^{1/2} \propto 1$$

$$p_{IJ}(\phi) \propto \phi^{-1}$$

Independent Jeffreys Prior is

$$p_{IJ}(\beta, \phi) \propto p_{IJ}(\boldsymbol{\beta}) p_{IJ}(\phi) = \phi^{-1}$$

Formal Posterior Distribution

With Independent Jeffreys Prior

$$p_{IJ}(\beta, \phi) \propto p_{IJ}(\boldsymbol{\beta}) p_{IJ}(\phi) = \phi^{-1}$$

Formal Posterior Distribution

$$\begin{aligned}
\boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathrm{N}(\hat{\boldsymbol{\beta}}, (\mathbf{X}^T\mathbf{X})^{-1}\phi^{-1}) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}((n-p)/2, \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|^2/2) \\
\boldsymbol{\beta} \mid \mathbf{Y} &\sim t_{n-p}(\hat{\boldsymbol{\beta}}, \hat{\sigma}^2(\mathbf{X}^T\mathbf{X})^{-1})
\end{aligned}$$

Bayesian Credible Sets $p(\boldsymbol{\beta} \in C_\alpha) = 1 - \alpha$ correspond to frequentist Confidence Regions

$$\frac{\boldsymbol{\lambda}^T\boldsymbol{\beta} - \boldsymbol{\lambda}\hat{\boldsymbol{\beta}}}{\sqrt{\hat{\sigma}^2\boldsymbol{\lambda}^T(\mathbf{X}^T\mathbf{X})^{-1}\boldsymbol{\lambda}}} \sim t_{n-p}$$

Summary

- Bayes Regression with Conjugate Priors provides foundation for many hierarchical models
- Know how to complete the square/quadratic
- Prediction Distributions
- Next Bayes Factors!