Bayesian Variable Selection & Bayesian Model Averaging

Hoff Chapter 9, Liang et al 2008, Hoeting et al (1999), Clyde & George (2004)

October 31, 2022

Zellner's g-prior

Zellner's g-prior(s) $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathsf{b}_0, g(\mathsf{X}^T\mathsf{X})^{-1}/\phi)$

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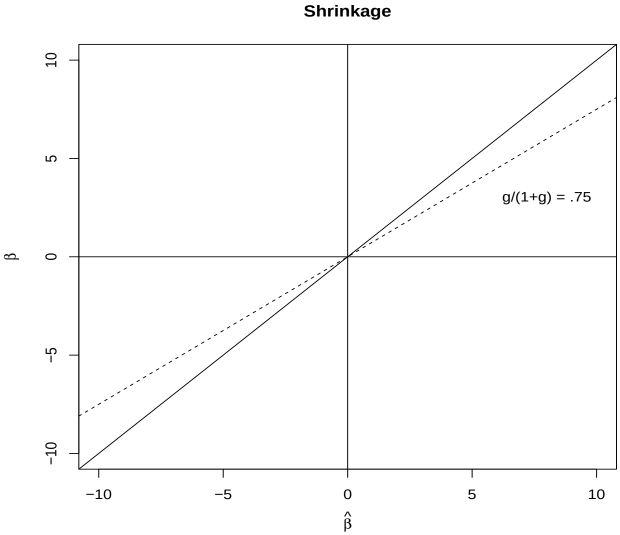
- Choice of g?
- $\frac{g}{1+g}$ weight given to the data
- Fixed g effect does not vanish as $n \to \infty$
- Use $g = n$ or place a prior diistribution on g

Shrinkage

Posterior mean under g-prior with $\mathrm{b}_0 = 0$ $\frac{g}{1+g}\hat{\beta}$



Ridge Regression

- If $X^T X$ is nearly singular, certain elements of $\boldsymbol{\beta}$ or (linear combinations of $\boldsymbol{\beta}$) may have huge variances under the g-prior (or flat prior) as the MLEs are highly unstable!
- **Ridge regression** protects against the explosion of variances and ill-conditioning with the conjugate prior:

$$\boldsymbol{\beta} \mid \phi \sim \mathrm{N}(0, \frac{1}{\phi\lambda} \mathrm{I}_p)$$

- Posterior for $\boldsymbol{\beta}$ (conjugate case)

$$\boldsymbol{\beta} \mid \phi, \lambda, \mathrm{Y} \sim \mathrm{N}\left((\lambda \mathrm{I}_p + \mathrm{X}^T\mathrm{X})^{-1}\mathrm{X}^T\mathrm{Y}, \frac{1}{\phi}(\lambda \mathrm{I}_p + \mathrm{X}^T\mathrm{X})^{-1}\right)$$

- induces shrinkage as well!

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- Some variables may not be relevant ($\beta_j = 0$)
- Can we infer a "good" model from the data?
- Expand model hierarchically to introduce another latent variable $\boldsymbol{\gamma}$ that encodes models $\mathcal{M}_\gamma$ $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots \gamma_p)^T$ where

$$
\begin{aligned}
\gamma_j = 0 &\Leftrightarrow \beta_j = 0 \\
\gamma_j = 1 &\Leftrightarrow \beta_j \neq 0
\end{aligned}
$$

- Find Bayes factors and posterior probabilities of models $\mathcal{M}_\gamma$
- 2^p models!

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where X^c is the centered design matrix where all variables have had their mean subtracted

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- $\boldsymbol{\beta}_\gamma \mid \alpha, \phi, \boldsymbol{\gamma} \sim N(0, g\phi^{-1}({X^c_\gamma}' X^c_\gamma)^{-1})$

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which leads to marginal likelihood of $\boldsymbol{\gamma}$ that is proportional to

$$p(\mathrm{Y} \mid \gamma) = C(1+g)^{\frac{n-p-1}{2}}(1 + g(1 - R^2_\gamma))^{-\frac{(n-1)}{2}}$$

where R^2 is the usual coefficient of determination for model $\mathcal{M}_\gamma$.

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Trade-off of model complexity versus goodness of fit

Lastly, assign distribution to space of models

Sketch

- Integrate out β_γ using sums of normals

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- Integrate out ϕ (gamma)

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- Find determinant of $\phi(\mathbf{I}_n + g\mathbf{P}_{\mathbf{X}_\gamma})$
- Integrate out intercept (normal)
- Integrate out ϕ (gamma)
- algebra to simplify quadratic forms to R^2_γ

Or integrate α, $\boldsymbol{\beta}_\gamma$ and ϕ (complete the square!)

Posteriors

$$\alpha \mid \phi, y \sim \mathsf{N}\left(\bar{y}, \frac{1}{n\phi}\right)$$

$$\boldsymbol{\beta}_\gamma \mid \gamma, \phi, g, y \sim \mathsf{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}}_\gamma, \frac{g}{1+g}\frac{1}{\phi}\left[\mathbf{X}_\gamma{}^T\mathbf{X}_\gamma\right]^{-1}\right)$$

$$\phi \mid \gamma, y \sim \mathsf{Gamma}\left(\frac{n-1}{2}, \frac{\mathsf{TotalSS} - \frac{g}{1+g}\mathsf{RegSS}}{2}\right)$$

$$p(\gamma \mid y) \propto p(y \mid \gamma)p(\gamma)$$

$$\mathsf{TotalSS} \equiv \sum_i (y_i - \bar{y})^2 \qquad \mathsf{RegSS} \equiv \hat{\boldsymbol{\beta}}_\gamma^T \mathbf{X}_\gamma^T \mathbf{X}_\gamma \hat{\beta}\gamma$$

$$R^2_\gamma = \frac{\mathsf{RegSS}}{\mathsf{TotalSS}} = 1 - \frac{\mathsf{ErrorSS}}{\mathsf{TotalSS}}$$

Priors on Model Space

$p(\mathcal{M}_\gamma) \Leftrightarrow p(\gamma)$

- $p(\gamma_j = 1) = .5 \Rightarrow P(\mathcal{M}_\gamma) = .5^p$ Uniform on space of models

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- $p(\gamma_j = 1) = .5 \Rightarrow P(\mathcal{M}_\gamma) = .5^p$ Uniform on space of models
  $p_\gamma \sim \text{Bin}(p, .5)$
- $\gamma_j \mid \pi \overset{\text{iid}}{\sim} \text{Ber}(\pi)$ and $\pi \sim \text{Beta}(a, b)$ then $p_\gamma \sim \text{BB}_p(a, b)$

  $$p(p_\gamma \mid p, a, b) = \frac{\Gamma(p+1)\Gamma(p_\gamma + a)\Gamma(p - p_\gamma + b)\Gamma(a+b)}{\Gamma(p_\gamma + 1)\Gamma(p - p_\gamma + 1)\Gamma(p + a + b)\Gamma(a)\Gamma(b)}$$
- $p_\gamma \sim \text{BB}_p(1, 1) \sim \text{Unif}(0, p)$

Posterior Probabilities of Models

- Calculate analytically under enumeration

$$p(\mathcal{M}_\gamma \mid \mathrm{Y}) = \frac{p(\mathrm{Y} \mid \gamma)p(\gamma)}{\sum_{\gamma' \in \Gamma} p(\mathrm{Y} \mid \gamma')p(\gamma')}$$

  Express as a function of Bayes factors and prior odds!
- Use MCMC over Γ - Gibbs, Metropolis Hastings if p is large
- slow convergence/poor mixing with high correlations

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$$p(\mathcal{M}_\gamma \mid Y) = \frac{p(Y \mid \gamma)p(\gamma)}{\sum_{\gamma' \in \Gamma} p(Y \mid \gamma')p(\gamma')}$$

  Express as a function of Bayes factors and prior odds!
- Use MCMC over Γ - Gibbs, Metropolis Hastings if p is large
- slow convergence/poor mixing with high correlations
- Metropolis Hastings algorithms more flexibility (swap pairs of variables)
- Do we need to run MCMC over γ, $\boldsymbol{\beta}_\gamma$, α, and ϕ?

Choice of g: Bartlett's Paradox

The Bayes factor for comparing γ to the null model:

$$BF(\gamma : \gamma 0) = (1+g)^{(n-1-p_\gamma)/2}(1+g(1-R^2_\gamma))^{-(n-1)/2}$$

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- What happens to BF as $g \to \infty$?

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- What happens to BF as $g \to \infty$?
- why is this a paradox?

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- Let g be a fixed constant and take n fixed.

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- Let $F = \frac{R^2_\gamma / p_\gamma}{(1-R^2_\gamma)/(n-1-p_\gamma)}$

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- As $R^2_\gamma \to 1$, $F \to \infty$ LR test would reject γ_0 where F is the usual F statistic for comparing model γ to γ_0
- BF converges to a fixed constant $(1+g)^{n-1-p_\gamma/2}$ (does not go to infinity

"Information Inconsistency" see Liang et al JASA 2008

Mixtures of g priors & Information consistency

- Need $BF \to \infty$ if $\mathrm{R}^2_\gamma \to 1$
- Put a prior on g

$$BF(\gamma : \gamma_0) = \frac{C \int (1+g)^{(n-1-p_\gamma)/2}(1+g(1-R^2_\gamma))^{-(n-1)/2}\pi(g)dg}{C}$$

- interchange limit and integration as $R^2 \to 1$ want

$$\mathrm{E}_g[(1+g)^{(n-1-p_\gamma)/2}]$$

  to diverge
- hyper-g prior (Liang et al JASA 2008)

$$p(g) = \frac{a-2}{2}(1+g)^{-a/2}$$

  or $g/(1+g) \sim Beta(1, (a-2)/2)$
- prior expectation converges if $a > n+1-p_\gamma$
- Consider minimal model $p_\gamma = 1$ and $n = 3$ (can estimate intercept, one coefficient, and σ^2, then $a > 3$ integral exists
- For $2 < a \leq 3$ integral diverges and resolves the information paradox!

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Need $BF \rightarrow \infty$ if $\mathrm{R}^2 \rightarrow 1 \Leftrightarrow \mathrm{E}_g[(1+g)^{(n-1-p_\gamma)/2}]$ diverges (proof in Liang et al)

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- robust prior (Bayarri et al Annals of Statistics 2012

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- Zellner-Siow Cauchy prior $1/g \sim G(1/2, n/2)$
- robust prior (Bayarri et al Annals of Statistics 2012
- Intrinsic prior (Womack et al JASA 2015)

All have prior tails for β that behave like a Cauchy distribution and (the latter 4) marginal likelihoods that can be computed using special hypergeometric functions (${}_2F_1$, Appell F_1)

USair Data

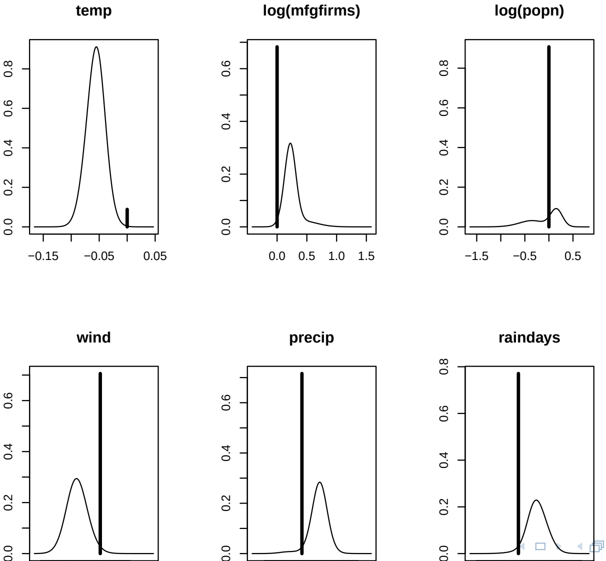
```
> library(BAS)
> data(usair, package="HH")
> poll.bma = bas.lm(log(SO2) ~ temp + log(mfgfirms) +
+                               log(popn) + wind +
+                               precip + raindays,
+                      data=usair,
+                      prior="JZS",  #Jeffrey-Zellner-Siow
+                      alpha=nrow(usair), # n
+                      n.models=2^6,
+                      modelprior = uniform(),
+                      method="deterministic")
```

Summary

```
> summary(poll.bma)
               P(B != 0 | Y) model 1   model 2   model 3   model
Intercept         1.00000000 1.00000 1.0000000 1.0000000 1.000000
temp              0.91158530 1.00000 1.0000000 1.0000000 1.000000
log(mfgfirms)     0.31718916 0.00000 0.0000000 0.0000000 1.000000
log(popn)         0.09223957 0.00000 0.0000000 0.0000000 0.000000
wind              0.29394451 0.00000 0.0000000 0.0000000 1.000000
precip            0.28384942 0.00000 1.0000000 0.0000000 1.000000
raindays          0.22903262 0.00000 0.0000000 1.0000000 0.000000
BF                        NA 1.00000 0.3286643 0.2697945 0.265587
PostProbs                 NA 0.29410 0.0967000 0.0794000 0.078100
R2                        NA 0.29860 0.3775000 0.3714000 0.542700
dim                       NA 2.00000 3.0000000 3.0000000 5.000000
logmarg                   NA 3.14406 2.0313422 1.8339656 1.818248
```

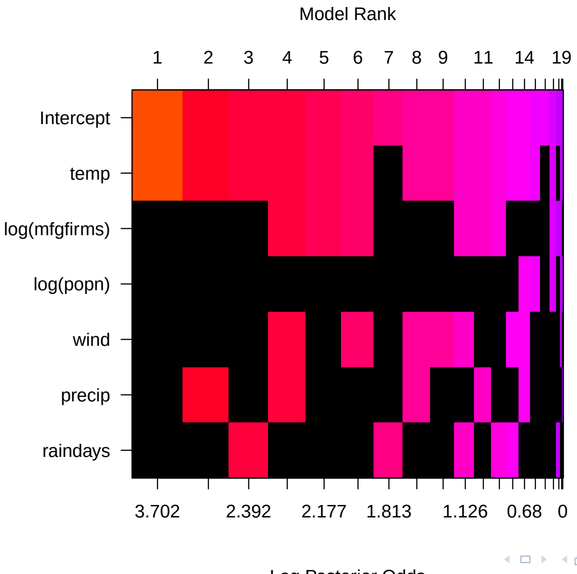
Plots

```
> beta = coef(poll.bma)
> par(mfrow=c(2,3));  plot(beta, subset=2:7,ask=F)
```



Posterior Distribution with Uniform Prior on Model Space

```
> image(poll.bma, rotate=FALSE)
```

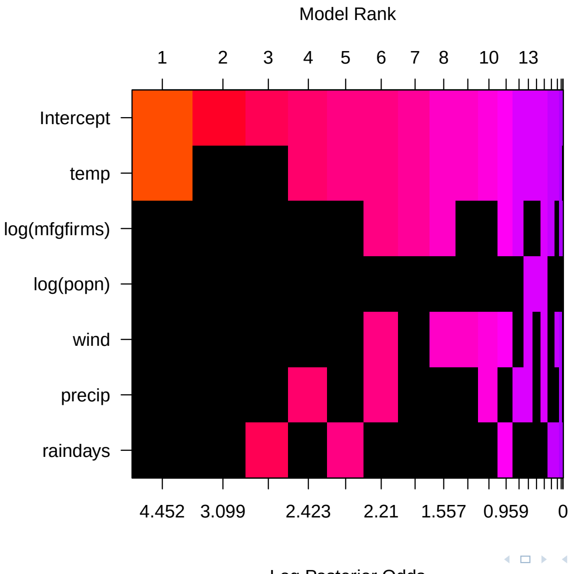


Posterior Distribution with BB(1,1) Prior on Model Space

```
> poll.bb.bma = bas.lm(log(SO2) ~ temp + log(mfgfirms) +
+                                 log(popn) + wind +
+                                 precip + raindays,
+                      data=usair,
+                      prior="JZS",
+                      alpha=nrow(usair),
+                      n.models=2^6,  #enumerate
+                      modelprior=beta.binomial(1,1))
```

BB(1,1) Prior on Model Space

```
> image(poll.bb.bma, rotate=FALSE)
```



Summary

- Choice of prior on β_γ
- g-priors or mixtures of g (sensitivity)
- priors on the models (sensitivity)
- posterior summaries - select a model or "average" over all models