STA 601: Bayesian Model Choice in Linear Regression

STA 601 Fall 2021

Merlise Clyde

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5. Compute marginal posterior probabilities $\Pr[\gamma|Y]$ for each model, and select a model based on the posterior probabilities or use the full posterior over all models!

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- Let $p_\gamma(Y)$ denote the marginal likelihood of the data under model γ, that is, $p[Y|\gamma]$. As before,

$$\hat{\Pi}_\gamma = \Pr[\gamma|Y] = \frac{p[Y|\gamma] \cdot p[\gamma]}{\sum_{\gamma^\star \in \Gamma} p[Y|\gamma^\star] \cdot p[\gamma^\star]} = \frac{p_\gamma(Y)\Pi_\gamma}{\sum_{\gamma^\star \in \Gamma} p_{\gamma^\star}(Y)\Pi_{\gamma^\star}}$$

$$= \frac{\Pi_\gamma \cdot \left[\int_{\Theta_\gamma} p_\gamma(Y|\theta_\gamma) \cdot \pi(\theta_\gamma)\mathrm{d}\theta_\gamma\right]}{\sum_{\gamma^\star \in \Gamma} \Pi_{\gamma^\star} \cdot \left[\int_{\Theta_{\gamma^\star}} p_{\gamma^\star}(Y|\theta_{\gamma^\star}) \cdot \pi(\theta_{\gamma^\star})\mathrm{d}\theta_{\gamma^\star}\right]}.$$

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- If we assume a uniform prior on Γ, that is, $\Pi_\gamma = \frac{1}{\#\Gamma}$, for all $\gamma \in \Gamma$, then

$$\hat{\Pi}_\gamma = \frac{p_\gamma(Y)}{\sum_{\gamma^\star \in \Gamma} p_{\gamma^\star}(Y)} = \frac{\left[\int_{\Theta_\gamma} p_\gamma(Y|\theta_\gamma) \cdot \pi(\theta_\gamma) \mathrm{d}\theta_\gamma\right]}{\sum_{\gamma^\star \in \Gamma} \left[\int_{\Theta_{\gamma^\star}} p_{\gamma^\star}(Y|\theta_{\gamma^\star}) \cdot \pi(\theta_{\gamma^\star}) \mathrm{d}\theta_{\gamma^\star}\right]}$$

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that is,

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$$R(\hat{\gamma}) = \sum_{\gamma \in \Gamma} \mathbf{1}(\hat{\boldsymbol{\gamma}} \neq \boldsymbol{\gamma}) \cdot \hat{\Pi}_{\gamma} = 0 \cdot \hat{\Pi}_{\gamma_{\text{true}}} + \sum_{\gamma \neq \gamma_{\text{true}}} \hat{\Pi}_{\gamma} = \sum_{\gamma \neq \hat{\gamma}} \hat{\Pi}_{\gamma} = 1 - \hat{\Pi}_{\hat{\gamma}}$$

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- To minimize $R(\hat{\gamma})$, choose $\hat{\gamma}$ such that $\hat{\Pi}_{\hat{\gamma}}$ is the largest! That is, select the model with the largest posterior probability.

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- For predictions, we care about the posterior predictive distribution, that is

$$\begin{aligned}
p(y_{n+1}|Y=(y_1,\ldots,y_n)) &= \int_\Gamma \int_{\Theta_\gamma} p(y_{n+1}|\gamma,\theta_\gamma)\cdot\pi(\gamma,\theta_\gamma|Y)\;\mathrm{d}\theta_\gamma\mathrm{d}\gamma \\
&= \int_\Gamma \int_{\Theta_\gamma} p(y_{n+1}|\gamma,\theta_\gamma)\cdot\pi(\theta_\gamma|Y,\gamma)\cdot\Pr[\gamma|Y]\;\mathrm{d}\theta_\gamma\mathrm{d}\gamma \\
&= \sum_{\gamma\in\Gamma} \int_{\Theta_\gamma} p(y_{n+1}|\gamma,\theta_\gamma)\cdot\pi(\theta_\gamma|Y,\gamma)\cdot\hat{\Pi}_\gamma\;\mathrm{d}\theta_\gamma \\
&= \sum_{\gamma\in\Gamma} \hat{\Pi}_\gamma\cdot\int_{\Theta_\gamma} p(y_{n+1}|\gamma,\theta_\gamma)\cdot\pi(\theta_\gamma|Y,\gamma)\;\mathrm{d}\theta_\gamma \\
&= \sum_{\gamma\in\Gamma} \hat{\Pi}_\gamma\cdot p(y_{n+1}|Y,\gamma),
\end{aligned}$$

which is just averaging out the predictions from each model, over all possible models in Γ, with the posterior probability of each model, and this is known as Bayesian model averaging (BMA).

Bayesian Linear Regression

Practical Issues: the posterior probability that the model is true

$$\hat{\Pi}_\gamma = \frac{\Pi_\gamma p_\gamma(Y)}{\sum_{\gamma^\star \in \Gamma} \Pi_{\gamma^\star} p_{\gamma^\star}(Y)}.$$

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- Can put priors on hyperparameters cases and integrate or use numerical approximations!
- May not be able to enumerate! Gibbs or MCMC for more flexibility!

Bayesian Variable Selection (BVS)

- Rewrite each model $\gamma \in \Gamma$ as

$$\boldsymbol{Y} \mid \alpha, \boldsymbol{\beta}_\gamma, \gamma, \phi \sim \mathcal{N}_n(\mathbf{1}_n \alpha + \boldsymbol{X}_\gamma \boldsymbol{\beta}_\gamma, \phi^{-1} \boldsymbol{I}_{n \times n})$$

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- $\gamma = (\gamma_1, \ldots, \gamma_p) \in \{0, 1\}^p$, so that the cardinality of Γ is 2^p, the number of models in Γ.

$$\gamma_j = \begin{cases} 1 & \text{if the j'th predictor is included in the model} \\ 0 & \text{if it is not} \end{cases}$$

- $p_\gamma \equiv \sum_{j=1}^p \gamma_j$, so that p_γ is the number of predictors included in model γ

- $\boldsymbol{X}_\gamma$ ($n \times p_\gamma$) is the matrix of predictors with $\gamma_j = 1$ (wolg design matrix with centered columns)

- $\boldsymbol{\beta}_\gamma$ ($p_\gamma \times 1$)is the corresponding vector of predictors with $\gamma_j = 1$

BVS

- Recall that we can also write each model as

$$Y_i = 1\alpha + \boldsymbol{\beta}_\gamma^T \boldsymbol{x}_{i\gamma} + \epsilon_i; \quad \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \phi^{-1}).$$

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- As an example, suppose we had data with 5 potential predictors including the intercept, so that each $\boldsymbol{x}_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5})$, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$.

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- Then for model with $\gamma = (1, 0, 0, 0, 0)$, $Y_i = \boldsymbol{\beta}_\gamma^T \boldsymbol{x}_{i\gamma} + \epsilon_i$

$$\implies Y_i = \alpha + \beta_1 x_{i1} + \epsilon_i; \quad \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, 1/\phi),$$

  with $p_\gamma = 1$.

- Whereas for model with $\gamma = (0, 0, 1, 1, 0)$, $Y_i = \alpha + \boldsymbol{\beta}_\gamma^T \boldsymbol{x}_{i\gamma} + \epsilon_i$

$$\implies Y_i = \alpha + \beta_3 x_{i3} + \beta_4 x_{i4} + \epsilon_i; \quad \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2 1/\phi),$$

  with $p_\gamma = 2$.

Steps

The outline for variable selection would be as follows:

1) Write down likelihood under model γ. That is,

$$p(\boldsymbol{y}|\boldsymbol{X}, \gamma, \alpha, \boldsymbol{\beta}_\gamma, \phi) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{\phi}{2}(\boldsymbol{y} - \mathbf{1}\alpha - \boldsymbol{X}_\gamma\boldsymbol{\beta}_\gamma)^T(\boldsymbol{y} - \mathbf{1}\alpha - \boldsymbol{X}_\gamma\boldsymbol{\beta}_\gamma)\right\}$$

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2) Define a prior for γ, $\Pi_\gamma = \Pr[\gamma]$.

- $p(\gamma_j = 1) = .5 \Rightarrow p(\gamma) = .5^p$ Uniform on space of models and $p_\gamma \sim \mathsf{Bin}(p, .5)$
- $\gamma_j \mid \pi \overset{iid}{\sim} \mathsf{Ber}(\pi)$ and $\pi \sim \mathsf{Beta}(a, b)$ then $p_\gamma \sim \mathsf{Beta\text{-}Binomial}(a, b)$

$$p(p_\gamma \mid p, a, b) = \frac{\Gamma(p+1)\Gamma(p_\gamma + a)\Gamma(p - p_\gamma + b)\Gamma(a+b)}{\Gamma(p_\gamma + 1)\Gamma(p - p_\gamma + 1)\Gamma(p + a + b)\Gamma(a)\Gamma(b)}$$

$$p_\gamma \sim \mathsf{Beta\text{-}Binomial}(1, 1) \sim \mathsf{Unif}(0, p)$$

Prior on model specific parameters

3) Using independent Jeffrey's priors on common parameters and the g-prior we have

$$\pi(\alpha, \phi) = \phi^{-1}$$
$$\pi(\boldsymbol{\beta}_\gamma | \phi) = \mathsf{N}_p\left(\boldsymbol{\beta}_{0\gamma} = \mathbf{0}, \Sigma_{0\gamma} = \frac{g}{\phi}\left[\boldsymbol{X}_\gamma{}^T \boldsymbol{X}_\gamma\right]^{-1}\right)$$

Posteriors

- With those pieces, the conditional posteriors are straightforward

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\begin{aligned}
\alpha \mid \phi, y &\sim \mathsf{N}\left(\bar{y}, \frac{1}{n\phi}\right) \\
\boldsymbol{\beta}_\gamma \mid \gamma, \phi, g, y &\sim \mathsf{N}\left(\frac{g}{1+g}\hat{\boldsymbol{\beta}}_\gamma, \frac{g}{1+g}\frac{1}{\phi}\left[{\boldsymbol{X}_\gamma}^T\boldsymbol{X}_\gamma\right]^{-1}\right) \\
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- due to conjugacy, the marginal likelihood of γ is proportional to

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p(Y \mid \gamma) = C(1+g)^{\frac{n-p\gamma-1}{2}}(1+g(1-R^2_\gamma))^{-\frac{(n-1)}{2}}
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- R^2_γ is the usual coefficient of determination for model γ,

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R^2_\gamma = 1 - \frac{(y-\hat{y}_\gamma)^T(y-\hat{y}_\gamma)}{(y-\mathbf{1}\bar{y})^T(y-\mathbf{1}\bar{y})}
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- we can run a collapsed Gibbs or MH sampler over just Γ!

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- we will focus on using R packages for implementing

Examples with BAS

```r
library(BAS)
data(usair, package="HH")
poll.bma = bas.lm(log(SO2) ~ temp + log(mfgfirms) +
                             log(popn) + wind +
                             precip + raindays,
                  data=usair,
                  prior="g-prior",
                  alpha=nrow(usair), # g = n
                  n.models=2^6,
                  modelprior = uniform(),
                  method="deterministic")
```

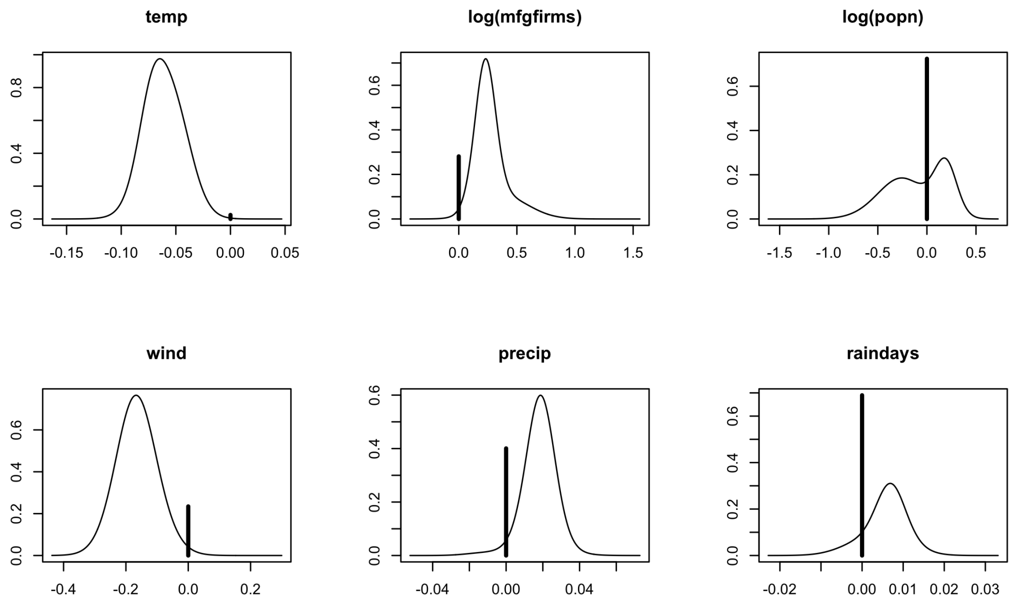
Summaries

```
poll.bma
```

```
##
## Call:
## bas.lm(formula = log(SO2) ~ temp + log(mfgfirms) + log(popn) +
##     wind + precip + raindays, data = usair, n.models = 2^6, prior = "g-p
##     alpha = nrow(usair), modelprior = uniform(), method = "deterministic
##
##
##  Marginal Posterior Inclusion Probabilities:
##     Intercept          temp  log(mfgfirms)      log(popn)          win
##        1.0000        0.9755         0.7190         0.2757         0.765
##        precip      raindays
##        0.5994        0.3104
```

Plots of Coefficients

```
beta = coef(poll.bma)
par(mfrow=c(2,3));  plot(beta, subset=2:7,ask=F)
```



Summary of Coefficients

```
beta
```

```
##
##  Marginal Posterior Summaries of Coefficients:
##
##  Using  BMA
##
##  Based on the top  64 models
##               post mean  post SD   post p(B != 0)
## Intercept      3.153004   0.082872   1.000000
## temp          -0.059724   0.020675   0.975504
## log(mfgfirms)  0.195716   0.177190   0.719031
## log(popn)     -0.026093   0.164277   0.275681
## wind          -0.126379   0.090777   0.765449
## precip         0.010821   0.011497   0.599380
## raindays       0.001803   0.004023   0.310357
```

Iterated Expectations!

Model Space Visualization

```
image(poll.bma, rotate=FALSE)
```