Lecture 10: More MCMC: Blocked Metropolis-Hastings and Gibbs

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September 27



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 convenient to break problems in to K blocks and update them separately

$$lacksquare heta = (heta_{[1]}, \dots, heta_{[K]}) = (heta_1, \dots, heta_p)$$

At iteration s, for k = 1, ..., K Cycle thru blocks: (fixed order or random order)

- lacksquare propose $heta^*_{[k]} \sim q_k(heta_{[k]} \mid heta^{(s)}_{[< k]}, heta^{(s-1)}_{[> k]})$
- set $\theta_{[k]}^{(s)} = \theta_{[k]}^*$ with probability

$$\min \left\{1, \frac{\pi(\theta_{[< k]}^{(s)}, \theta_{[k]}^*, \theta_{[> k]}^{(s-1)}) \mathcal{L}(\theta_{[< k]}^{(s)}, \theta_{[k]}^*, \theta_{[> k]}^{(s-1)}) / q_k(\theta_{[k]}^* \mid \theta_{[< k]}^{(s)}, \theta_{[> k]}^{(s-1)})}{\pi(\theta_{[< k]}^{(s)}, \theta_{[k]}^{(s-1)}, \theta_{[> k]}^{(s-1)}) \mathcal{L}(\theta_{[< k]}^{(s)}, \theta_{[k]}^{(s-1)}, \theta_{[> k]}^{(s-1)}) / q_k(\theta_{[k]}^{(s-1)} \mid \theta_{[< k]}^{(s)}, \theta_{[> k]}^{(s-1)})}\right\}$$



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$$\pi(heta_{[k]} \mid heta_{[-k]}, y) = rac{\pi(heta_{[k]}, heta_{[-k]} \mid y)}{\pi(heta_{[-k]} \mid y))} \propto \pi(heta_{[k]}, heta_{[-k]} \mid y)$$



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ight\} \end{aligned}$$



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• proposal distribution q_k for the kth block is the **full conditional** distribution for $\theta_{[k]}$

$$\begin{split} \pi(\theta_{[k]} \mid \theta_{[-k]}, y) &= \frac{\pi(\theta_{[k]}, \theta_{[-k]} \mid y)}{\pi(\theta_{[-k]} \mid y))} \propto \pi(\theta_{[k]}, \theta_{[-k]} \mid y) \\ \pi(\theta_{[k]} \mid \theta_{[-k]}, y) &\propto \mathcal{L}(\theta_{[k]}, \theta_{[-k]}) \pi(\theta_{[k]}, \theta_{[-k]}) \\ \min \left\{ 1, \frac{\pi(\theta_{[< k]}^{(s)}, \theta_{[k]}^*, \theta_{[> k]}^{(s-1)}) \mathcal{L}(\theta_{[< k]}^{(s)}, \theta_{[k]}^*, \theta_{[> k]}^{(s-1)}) / q_k(\theta_{[k]}^* \mid \theta_{[< k]}^{(s)}, \theta_{[> k]}^{(s-1)})}{\pi(\theta_{[< k]}^{(s)}, \theta_{[k]}^{(s-1)}, \theta_{[> k]}^{(s-1)}) \mathcal{L}(\theta_{[< k]}^{(s)}, \theta_{[k]}^{(s-1)}, \theta_{[> k]}^{(s-1)}) / q_k(\theta_{[k]}^{(s-1)} \mid \theta_{[< k]}^{(s)}, \theta_{[> k]}^{(s-1)})} \right\} \end{split}$$

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- acceptance probability is always 1!
- even though joint distribution is messy, full conditionals may be (conditionally) conjugate and easy to sample from!



Univariate Normal Example

Model

$$Y_i \mid \mu, \sigma^2 \stackrel{iid}{\sim} \mathsf{N}(\mu, 1/\phi) \ \mu \sim \mathsf{N}(\mu_0, 1/ au_0) \ \phi \sim \mathsf{Gamma}(a/2, b/2)$$



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- Joint prior is a product of independent Normal-Gamma
- Is $\pi(\mu, \phi \mid y_1, \dots, y_n)$ also a Normal-Gamma family?



Full Conditional for the Mean

The full conditional distributions $\mu \mid \phi, y_1, \dots, y_n$

$$egin{aligned} \mu \mid \phi, y_1, \dots, y_n &\sim \mathsf{N}(\hat{\mu}, 1/ au_n) \ \hat{\mu} &= rac{ au_0 \mu_0 + n \phi ar{y}}{ au_0 + n \phi} \ au_n &= au_0 + n \phi \end{aligned}$$



$$\phi \mid \mu, y_1, \dots, y_n \sim \mathsf{Gamma}(a_n/2, b_n/2) \ a_n = a + n \ b_n = b + \sum_i (y_i - \mu)^2$$



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 x_i is a $p \times 1$ vector of predictors and X is $n \times p$ matrix



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Multivariate Normal density for β

$$\pi(eta \mid b_0, \Phi_0) = rac{\left|\Phi_0
ight|^{1/2}}{(2\pi)^{p/2}} \mathrm{exp}igg\{ -rac{1}{2}(eta - b_0)^T \Phi_0(eta - b_0) igg\}$$



Full Conditional for β

$$eta \mid \phi, y_1, \dots, y_n \sim \mathsf{N}(b_n, \Phi_n^{-1}) \ b_n = (\Phi_0 + \phi X^T X)^{-1} (\Phi_0 b_0 + \phi X^T X \hat{eta}) \ \Phi_n = \Phi_0 + \phi X^T X$$



Derivation continued



Full Conditional for ϕ

$$\phi \mid eta, y_1, \dots, y_n \sim \mathsf{Gamma}((v_0 + n)/2, (s_0 + \sum_i (y_i - x_i^T eta)))$$



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• Formal Posterior given ϕ

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• needs X^TX to be full rank for distribution to be unique



the model in vector form

$$Y \sim \mathsf{N}_n(Xeta,\phi^{-1}I_n)$$



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- What if we transform the X matrix by $\tilde{X} = XH$ where H is $p \times p$ and invertible
- obtain the posterior for $\tilde{\beta}$ using Y and \tilde{X}

$$Y \sim \mathsf{N}_n(ilde{X} ilde{eta},\phi^{-1}I_n)$$

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• with some linear algebra we can show that this is true if $b_0 = 0$ and Φ_0 is kX^TX for some k (show!)

Zellner's g-prior

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Conjugate so we could skip Gibbs sampling and sample directly from gamma and then conditional normal!



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Posterior for β (conjugate case)

$$eta \mid \phi, \lambda, y_1, \dots, y_n \sim \mathsf{N}\left((\lambda I_p + X^T X)^{-1} X^T Y, rac{1}{\phi}(\lambda I_p + X^T X)^{-1}
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Posterior mean (or mode) given λ is biased, but can show that there always is a value of λ where the frequentist's expected squared error loss is smaller for the Ridge estimator than MLE!



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- Bayes Regression and choice of Φ_0 in general is a very important problem and provides the foundation for many variations on shrinkage estimators, variable selection, hierarchical models, nonparameteric regression and more!
- Be sure that you can derive the full conditional posteriors for β and ϕ as well as the joint posterior in the conjugate case!



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- Introduce latent variables (data augmentation) to allow Gibbs steps (Next class)

