# Lecture 14: Basics of Bayesian Hypothesis Testing

STA702

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# Feature Selection via Shrinkage

- modal estimates in regression models under certain shrinkage priors will set a subset of coefficients to zero
- not true with posterior mean
- multi-modal posterior
- no prior probability that coefficient is zero
- how should we approach selection/hypothesis testing?
- Bayesian Hypothesis Testing

# Basics of Bayesian Hypothesis Testing

Suppose we have univariate data $Y_i \overset{iid}{\sim} \mathcal{N}(\theta, 1)$, $\mathbf{Y} = (y_i, \ldots, y_n)^T$

- goal is to test $\mathcal{H}_0 : \theta = 0; \;\; \text{vs}\; \mathcal{H}_1 : \theta \neq 0$
- Additional unknowns are $\mathcal{H}_0$ and $\mathcal{H}_1$
- Put a prior on the actual hypotheses/models, that is, on $\pi(\mathcal{H}_0) = \Pr(\mathcal{H}_0 = \text{True})$ and $\pi(\mathcal{H}_1) = \Pr(\mathcal{H}_1 = \text{True})$.
- (Marginal) Likelihood of the hypotheses: $\mathcal{L}(\mathcal{H}_i) \propto p(\mathbf{y} \mid \mathcal{H}_i)$

$$p(\mathbf{y} \mid \mathcal{H}_0) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp -\frac{1}{2}(y_i - 0)^2$$

$$p(\mathbf{y} \mid \mathcal{H}_1) = \int_\Theta p(\mathbf{y} \mid \mathcal{H}_1, \theta) p(\theta \mid \mathcal{H}_1)\, d\theta$$

# Bayesian Approach

- Need priors distributions on parameters under each hypothesis
  - in our simple normal model, the only additional unknown parameter is $\theta$
  - under $\mathcal{H}_0, \theta = 0$ with probability 1
  - under $\mathcal{H}_0, \theta \in \mathbb{R}$ we could take $\pi(\theta) = \mathcal{N}(\theta_0, 1/\tau_0^2)$.
- Compute marginal likelihoods for each hypothesis, that is, $\mathcal{L}(\mathcal{H}_0)$ and $\mathcal{L}(\mathcal{H}_1)$.
- Obtain posterior probabilities of $\mathcal{H}_0$ and $\mathcal{H}_1$ via Bayes Theorem.

$$\pi(\mathcal{H}_1 \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)}{p(\mathbf{y} \mid \mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)}$$

- Provides a joint posterior distribution for $\theta$ and $\mathcal{H}_i$: $p(\theta \mid \mathcal{H}_i, \mathbf{y})$ and $\pi(\mathcal{H}_i \mid \mathbf{y})$

# Hypothesis Tests via Decision Theory

- Loss function for hypothesis testing
  - $\hat{\mathcal{H}}$ is the chosen hypothesis
  - $\mathcal{H}_{true}$ is the true hypothesis, $\mathcal{H}$ for short
- Two types of errors:
  - Type I error: $\hat{\mathcal{H}} = 1$ and $\mathcal{H} = 0$
  - Type II error: $\hat{\mathcal{H}} = 0$ and $\mathcal{H} = 1$
- Loss function:

$$L(\hat{\mathcal{H}}, \mathcal{H}) = w_1\, 1(\hat{\mathcal{H}} = 1, \mathcal{H} = 0) + w_2\, 1(\hat{\mathcal{H}} = 0, \mathcal{H} = 1)$$

  - $w_1$ weights how bad it is to make a Type I error
  - $w_2$ weights how bad it is to make a Type II error

# Loss Function Functions and Decisions

- Relative weights $w = w_2/w_1$

$$L(\hat{\mathcal{H}}, \mathcal{H}) = 1(\hat{\mathcal{H}} = 1, \mathcal{H} = 0) + w\, 1(\hat{\mathcal{H}} = 0, \mathcal{H} = 1)$$

- Special case $w = 1$

$$L(\hat{\mathcal{H}}, \mathcal{H}) = 1(\hat{\mathcal{H}} \neq \mathcal{H})$$

- known as 0-1 loss (most common)
- Bayes Risk (Posterior Expected Loss)

$$\mathsf{E}_{\mathcal{H}|\mathbf{y}}[L(\hat{\mathcal{H}}, \mathcal{H})] = 1(\hat{\mathcal{H}} = 1)\pi(\mathcal{H}_0 \mid \mathbf{y}) + 1(\hat{\mathcal{H}} = 0)\pi(\mathcal{H}_1 \mid \mathbf{y})$$

- Minimize loss by picking hypothesis with the highest posterior probability

# Bayesian hypothesis testing

- Using Bayes theorem,

$$\pi(\mathcal{H}_1 \mid \mathbf{y}) = \frac{p(\mathbf{y} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)}{p(\mathbf{y} \mid \mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y} \mid \mathcal{H}_1)\pi(\mathcal{H}_1)},$$

- If $\pi(\mathcal{H}_0) = 0.5$ and $\pi(\mathcal{H}_1) = 0.5$ *a priori*, then

$$\pi(\mathcal{H}_1 \mid \mathbf{y}) = \frac{0.5p(\mathbf{y} \mid \mathcal{H}_1)}{0.5p(\mathbf{y} \mid \mathcal{H}_0) + 0.5p(\mathbf{y} \mid \mathcal{H}_1)}$$

$$= \frac{p(\mathbf{y} \mid \mathcal{H}_1)}{p(\mathbf{y} \mid \mathcal{H}_0) + p(\mathbf{y} \mid \mathcal{H}_1)} = \frac{1}{\frac{p(\mathbf{y}|\mathcal{H}_0)}{p(\mathbf{y}|\mathcal{H}_1)} + 1}$$

# Bayes factors

- The ratio $\frac{p(\mathbf{y}|\mathcal{H}_0)}{p(\mathbf{y}|\mathcal{H}_1)}$ is a ratio of marginal likelihoods and is known as the **Bayes factor** in favor of $\mathcal{H}_0$, written as $\mathcal{BF}_{01}$. Similarly, we can compute $\mathcal{BF}_{10}$ via the inverse ratio.
- Bayes factors provide a weight of evidence in the data in favor of one model over another. and are used as an alternative to the frequentist p-value.
- **Rule of Thumb**: $\mathcal{BF}_{01} > 10$ is strong evidence for $\mathcal{H}_0$; $\mathcal{BF}_{01} > 100$ is decisive evidence for $\mathcal{H}_0$.
- In the example (with equal prior probabilities),

$$\pi(\mathcal{H}_1 \mid \mathbf{y}) = \frac{1}{\frac{p(\mathbf{y}|\mathcal{H}_0)}{p(\mathbf{y}|\mathcal{H}_1)} + 1} = \frac{1}{\mathcal{BF}_{01} + 1}$$

- the higher the value of $\mathcal{BF}_{01}$, that is, the weight of evidence in the data in favor of $\mathcal{H}_0$, the lower the marginal posterior probability that $\mathcal{H}_1$ is true.
- $\mathcal{BF}_{01} \uparrow, \pi(\mathcal{H}_1 \mid \mathbf{y}) \downarrow$.

# Posterior Odds and Bayes Factors

- Posterior odds $\frac{\pi(\mathcal{H}_0|\mathbf{y})}{\pi(\mathcal{H}_1|\mathbf{y})}$

$$\frac{\pi(\mathcal{H}_0|\mathbf{y})}{\pi(\mathcal{H}_1|\mathbf{y})} = \frac{p(\mathbf{y}|\mathcal{H}_0)\pi(\mathcal{H}_0)}{p(\mathbf{y}|\mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y}|\mathcal{H}_1)\pi(\mathcal{H}_1)} \div \frac{p(\mathbf{y}|\mathcal{H}_1)\pi(\mathcal{H}_1)}{p(\mathbf{y}\mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y}|\mathcal{H}_1)\pi(}$$

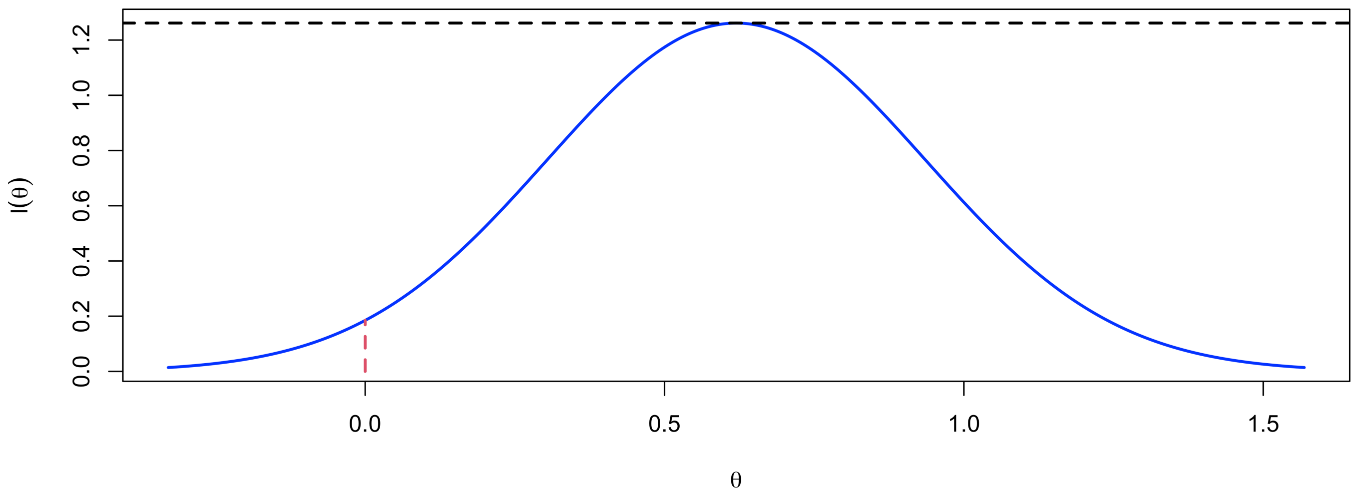
$$= \frac{p(\mathbf{y}|\mathcal{H}_0)\pi(\mathcal{H}_0)}{p(\mathbf{y}|\mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y}|\mathcal{H}_1)\pi(\mathcal{H}_1)} \times \frac{p(\mathbf{y}|\mathcal{H}_0)\pi(\mathcal{H}_0) + p(\mathbf{y}|\mathcal{H}_1)\pi}{p(\mathbf{y}|\mathcal{H}_1)\pi(\mathcal{H}_1)}$$

$$\therefore \underbrace{\frac{\pi(\mathcal{H}_0 \mid \mathbf{y})}{\pi(\mathcal{H}_1 \mid \mathbf{y})}}_{\text{posterior odds}} = \underbrace{\frac{\pi(\mathcal{H}_0)}{\pi(\mathcal{H}_1)}}_{\text{prior odds}} \times \underbrace{\frac{p(\mathbf{y} \mid \mathcal{H}_0)}{p(\mathbf{y} \mid \mathcal{H}_1)}}_{\text{Bayes factor } \mathcal{BF}_{01}}$$

- The Bayes factor can be thought of as the factor by which our prior odds change (towards posterior odds) in the light of the data.

# Likelihoods & Evidence

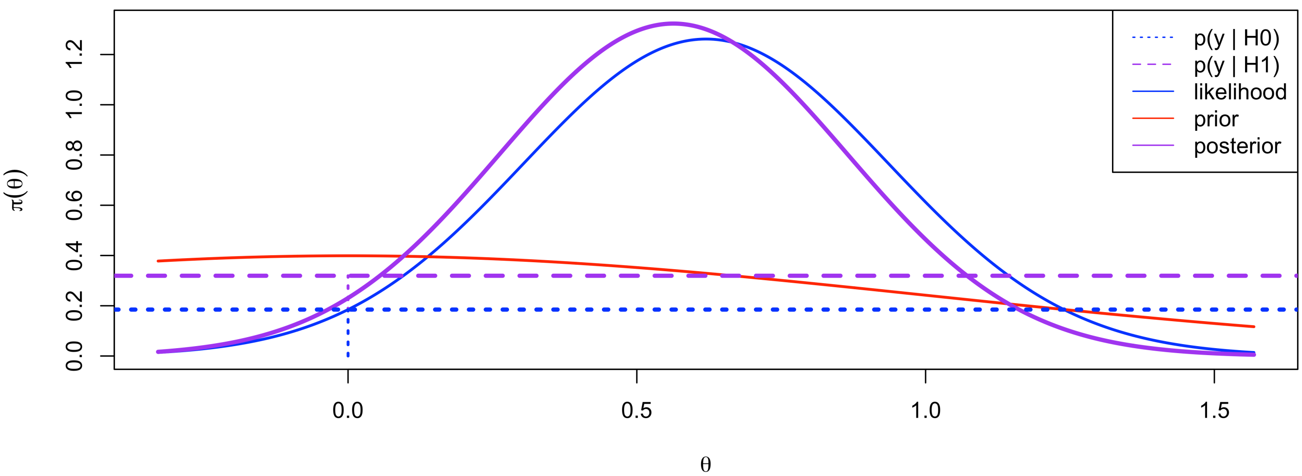
Maximized Likelihood. $n = 10$




p-value = 0.05

# Marginal Likelihoods & Evidence

Maximized & Marginal Likelihoods




$\mathcal{BF}_{10}$ = 1.73 or $\mathcal{BF}_{01}$ = 0.58

Posterior Probability of $\mathcal{H}_0$ = 0.3665

# Candidate's Formula (Besag 1989)

Alternative expression for BF based on Candidate's Formula or Savage-Dickey ratio

$$\mathcal{BF}_{01} = \frac{p(\mathbf{y} \mid \mathcal{H}_0)}{p(\mathbf{y} \mid \mathcal{H}_1)} = \frac{\pi_\theta(0 \mid \mathcal{H}_1, \mathbf{y})}{\pi_\theta(0 \mid \mathcal{H}_1)}$$

$$\pi_\theta(\theta \mid \mathcal{H}_i, \mathbf{y}) = \frac{p(\mathbf{y} \mid \theta, \mathcal{H}_i)\pi(\theta \mid \mathcal{H}_i)}{p(\mathbf{y} \mid \mathcal{H}_i)} \Rightarrow p(\mathbf{y} \mid \mathcal{H}_i) = \frac{p(\mathbf{y} \mid \theta, \mathcal{H}_i)\pi(\theta \mid \mathcal{H}_i)}{\pi_\theta(\theta \mid \mathcal{H}_i, \mathbf{y})}$$

$$\mathcal{BF}_{01} = \frac{\frac{p(\mathbf{y}|\theta,\mathcal{H}_0)\pi(\theta|\mathcal{H}_0)}{\pi_\theta(\theta|\mathcal{H}_0,\mathbf{y})}}{\frac{p(\mathbf{y}|\theta,\mathcal{H}_1)\pi(\theta|\mathcal{H}_1)}{\pi_\theta(\theta|\mathcal{H}_1,\mathbf{y})}} = \frac{\frac{p(\mathbf{y}|\theta=0)\delta_0(\theta)}{\delta_0(\theta)}}{\frac{p(\mathbf{y}|\theta,\mathcal{H}_1)\pi(\theta|\mathcal{H}_1)}{\pi_\theta(\theta|\mathcal{H}_1,\mathbf{y})}} = \frac{p(\mathbf{y} \mid \theta = 0)}{p(\mathbf{y} \mid \theta, \mathcal{H}_1)} \frac{\delta_0(\theta)}{\delta_0(\theta)} \frac{\pi_\theta(\theta \mid \mathcal{H}_1, \mathbf{y})}{\pi(\theta \mid \mathcal{H}_1)}$$

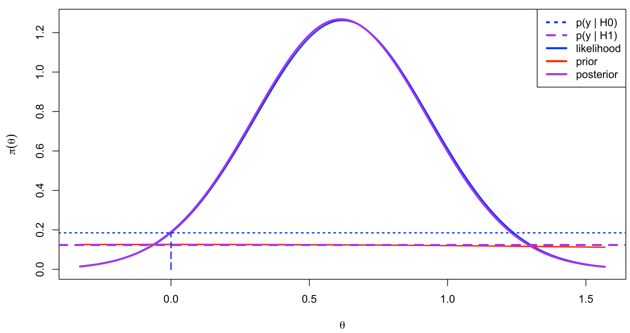
- Simplifies to the ratio of the posterior to prior densities when evaluated $\theta$ at zero

# Prior

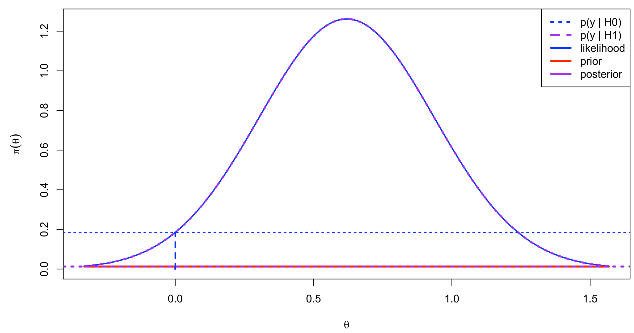
Plots were based on a $\theta \mid \mathcal{H}_1 \sim \textsf{N}(0, 1)$

- centered at value for $\theta$ under $\mathcal{H}_0$ (goes back to Jeffreys)
- "unit information prior" equivalent to a prior sample size is 1
- is this a "reasonable prior"?
  - What happens if $n \to \infty$?
  - What happens of $\tau_0 \to 0$ ? (less informative)

# Choice of Precision




- $\tau_0 = 1/10$
- Bayes Factor for $\mathcal{H}_0$ to $\mathcal{H}_1$ is $1.5$
- Posterior Probability of $\mathcal{H}_0$ = 0.6001




- $\tau_0 = 1/1000$
- Bayes Factor for $\mathcal{H}_0$ to $\mathcal{H}_1$ is $14.65$
- Posterior Probability of $\mathcal{H}_0$ = 0.9361

# Vague Priors & Hypothesis Testing

- As $\tau_0 \to 0$ the $\mathcal{BF}_{01} \to \infty$ and $\Pr(\mathcal{H}_0 \mid \mathbf{y} \to 1$!
- As we use a less & less informative prior for $\theta$ under $\mathcal{H}_1$ we obtain more & more evidence for $\mathcal{H}_0$ over $\mathcal{H}_1$!
- Known as **Bartlett's Paradox** - the paradox is that a seemingly non-informative prior for $\theta$ is very informative about $\mathcal{H}$!
- General problem with nested sequence of models. If we choose vague priors on the additional parameter in the larger model we will be favoring the smaller models under consideration!
- Similar phenomenon with increasing sample size (**Lindley's Paradox**)

> **Bottom Line** Don't use vague priors!

What should we use then?