Lecture 9: Gibbs Sampling and Data Augmentation

STA702

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https://sta702-F23.github.io/website/

Normal Linear Regression Example

- Model

$$
\begin{aligned}
Y_i \mid \beta, \phi &\overset{ind}{\sim} \mathsf{N}(x_i^T \beta, 1/\phi) \\
Y \mid \beta, \phi &\sim \mathsf{N}(X\beta, \phi^{-1} I_n) \\
\beta &\sim \mathsf{N}(b_0, \Phi_0^{-1}) \\
\phi &\sim \mathsf{Gamma}(v_0/2, s_0/2)
\end{aligned}
$$

- x_i is a $p \times 1$ vector of predictors and X is $n \times p$ matrix
- β is a $p \times 1$ vector of coefficients
- Φ_0 is a $p \times p$ prior precision matrix
- Multivariate Normal density for β

$$
\pi(\beta \mid b_0, \Phi_0) = \frac{|\Phi_0|^{1/2}}{(2\pi)^{p/2}} \exp\left\{ -\frac{1}{2} (\beta - b_0)^T \Phi_0 (\beta - b_0) \right\}
$$

Full Conditional for β

$$
\begin{aligned}
\beta \mid \phi, y_1, \ldots, y_n &\sim \mathsf{N}(b_n, \Phi_n^{-1}) \\
b_n &= (\Phi_0 + \phi X^T X)^{-1}(\Phi_0 b_0 + \phi X^T X \hat{\beta}) \\
\Phi_n &= \Phi_0 + \phi X^T X
\end{aligned}
$$

Derivation continued

Full Conditional for ϕ

$$\phi \mid \beta, y_1, \ldots, y_n \sim \textsf{Gamma}\left(\frac{v_0 + n}{2}, \frac{s_0 + \sum_i (y_i - x_i^T \beta)^2}{2}\right)$$

Choice of Prior Precision

- Non-Informative $\Phi_0 \to 0$
- Formal Posterior given ϕ

$$\beta \mid \phi, y_1, \ldots, y_n \sim \mathsf{N}(\hat{\beta}, \phi^{-1}(X^T X)^{-1})$$

- needs $X^T X$ to be full rank for distribution to be unique!

Binary Regression

$$Y_i \mid \beta \sim \mathsf{Ber}(p(x_i^T \beta))$$

where $\Pr(Y_i = 1 \mid \beta) = p(x_i^T \beta))$ and linear predictor $x_i^T \beta = \lambda_i$

- link function for binary regression is any 1-1 function g that will map $(0, 1) \to \mathbb{R}$, i.e. $g(p(\lambda)) = \lambda$
- logistic regression uses the logit link

$$\log \left(\frac{p(\lambda_i)}{1 - p(\lambda_i)} \right) = x_i^T \beta = \lambda_i$$

- probit link

$$p(x_i^T \beta) = \Phi(x_i^T \beta)$$

- $\Phi()$ is the Normal cdf

Likelihood and Posterior

Likelihood:

$$\mathcal{L}(\beta) \propto \prod_{i=1}^{n} \Phi(x_i^T \beta)^{y_i} (1 - \Phi(x_i^T \beta))^{1-y_i}$$

- prior $\beta \sim \mathsf{N}_p(b_0, \Phi_0)$
- posterior $\pi(\beta) \propto \pi(\beta)\mathcal{L}(\beta)$
- How to approximate the posterior?
  - asymptotic Normal approximation
  - MH with Independence chain or adaptive Metropolis
  - stan (Hamiltonian Monte Carlo)
  - Gibbs ?
- seemingly no, but there is a trick!

Data Augmentation

- Consider an **augmented** posterior

$$\pi(\beta, Z \mid y) \propto \pi(\beta)\pi(Z \mid \beta)\pi(y \mid Z, \theta)$$

- IF we choose $\pi(Z \mid \beta)$ and $\pi(y \mid Z, \theta)$ carefully, we can carry out Gibbs and get samples of $\pi(\beta \mid y)$!
- desired marginal of joint augmented posterior

$$\pi(\beta \mid y) = \int_{\mathcal{Z}} \pi(\beta, z \mid y)\, dz$$

- We have to choose latent prior and sampling model such that

$$p(y \mid \beta) = \int_{\mathcal{Z}} \pi(z \mid \beta)\pi(y \mid \beta, z)\, dz$$

- complete data likelihood $\pi(z \mid \beta)\pi(y \mid \beta, z)$

Augmentation Strategy

Set

- $y_i = 1_{(Z_i > 0)}$ i.e. ($y_i = 1$ if $Z_i > 0$)
- $y_i = 1_{(Z_i < 0)}$ i.e. ($y_i = 0$ if $Z_i < 0$)
- $Z_i = x_i^T \beta + \epsilon_i \qquad \epsilon_i \overset{iid}{\sim}$ N(0,1)
- Relationship to probit model:

$$\begin{aligned}
\Pr(y = 1 \mid \beta) &= P(Z_i > 0 \mid \beta) \\
&= P(Z_i - x_i^T \beta > -x^T \beta) \\
&= P(\epsilon_i > -x^T \beta) \\
&= 1 - \Phi(-x_i^T \beta) \\
&= \Phi(x_i^T \beta)
\end{aligned}$$

Augmented Posterior & Gibbs

- two block Gibbs sampler $\theta_{[1]} = \beta$ and $\theta_{[2]} = (Z_1, \ldots, Z_n)^T$

$$\pi(Z_1, \ldots, Z_n,\, \beta \mid y) \propto$$
$$\mathsf{N}(\beta; b_0, \phi_0) \left\{ \prod_{i=1}^{n} \mathsf{N}(Z_i; x_i^T \beta, 1) \right\} \left\{ \prod_{i=1}^{n} y_i 1_{(Z_i > 0)} + (1 - y_i) 1_{(Z_i < 0)} \right\}$$

- full conditional for β given Z_i's based on Normal-Normal regression

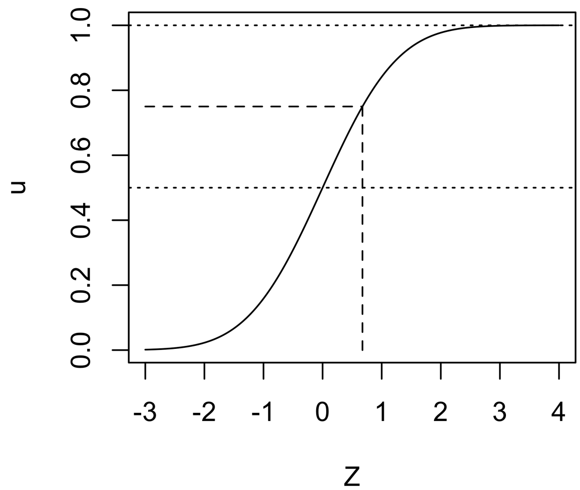
$$\beta \mid Z_1, \ldots, Z_n, y_1, \ldots, y_n \sim \mathsf{N}(b_n, \Phi_n)$$

- Full conditional for latent Z_i (product of independent dist's)

$$\pi(Z_i \mid \beta, Z_{[-i]}, y_1, \ldots, y_n) \propto \mathsf{N}(Z_i; x_i^T \beta, 1) 1_{(Z_i > 0)} \text{ if } y_1 = 1$$
$$\pi(Z_i \mid \beta, Z_{[-i]}, y_1, \ldots, y_n) \propto \mathsf{N}(Z_i; x_i^T \beta, 1) 1_{(Z_i < 0)} \text{ if } y_1 = 0$$

Truncated Sampling

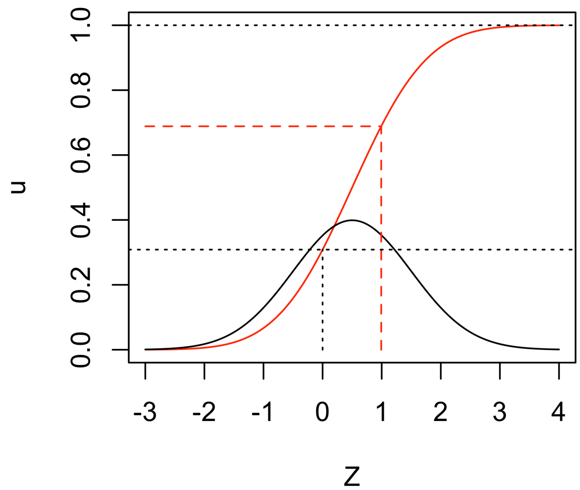
- Use inverse cdf method for cdf F
- If $U \sim U(0,1)$ set $X = F^{-1}(U)$
- if $X \in (a,b)$, Draw $X \sim U(F(a), F(b))$ and set $X = F^{-1}(u)$



Truncated Normal Sampling

- sample from independent truncated normal distributions for full conditional for Z_i
- if $Y_i = 1$ then $Z_i \sim \text{Normal}(x_i^T\beta, 1)I(0, \infty)$
- standard truncated normal $\tilde{Z} = Z_i - x_i^T\beta \in (-x_i^T\beta, \infty)$

1. Generate $U \sim \text{Uniform}(\Phi(-x_i^T\beta), \Phi(\infty))$
2. Set $\tilde{z} = \Phi^{-1}(U)$ (Standard truncated normal)
3. Shift $Z_i = x_i^T\beta + \tilde{z}$



- U = 0.69, $Z_i = x_i^T\beta + \Phi^{-1}(U)$ = 0.99

Comments on Gibbs

- Why don't we treat each individual θ_j as a separate block?
- Gibbs always accepts, but can mix slowly if parameters in different blocks are highly correlated!
- Use block sizes in Gibbs that are as big as possible to improve mixing (proven faster convergence)
- Collapse the sampler by integrating out as many parameters as possible (as long as resulting sampler has good mixing)
- can use Gibbs steps and (adaptive) Metropolis Hastings steps together
- latent variables may allow Gibbs steps, but not always better compared to MH!

Data Augmentation in General

DA is a broader than a computational trick allowing Gibbs sampling

- random effects or latent variable modeling i.e we introduce latent variables to simplify dependence structure modelling
- Modeling heavy tailed distributions for priors or errors in robust regression as mixtures of normals
- outliesr
- variable selection
- missing data
- Next class:
  - Multivariate Normal data
  - Wishart and inverse-Wishart distributions
  - missing data