Confidence Regions

STA 721: Lecture 15

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Outline

- Confidence Interverals from Test Statistics
- Pivotal Quantities
- Confidence intervals for parameters
- Prediction Intervals

Readings:

- Christensen Appendix C, Chapter 3

Goals

For the regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ we usually want to do more than just testing that $\boldsymbol{\beta}$ is zero

- what is a plausible range for β_j?
- what is a plausible set of values for β_j and β_k?
- what is a a plausible range of values for $\mathbf{x}\boldsymbol{\beta}$ for a particular $\mathbf{x}$?
- what is a plausible range of values for $\mathbf{Y}_{n+1}$ for a given value of $\mathbf{x}_{n+1}$?

Look at confidence intervals, confidence regions, prediction regions and Bayesian regions

Confidence Sets

For a random variable $\mathbf{Y} \sim \mathbf{P} \in \{P_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$

▼ **Definition:** Confidence Region

A set valued function C is a $(1-\alpha) \times 100\%$ confidence region for $\boldsymbol{\theta}$ if

$$P_{\boldsymbol{\theta}}(\{\boldsymbol{\theta} \in C(\mathbf{Y})\}) = 1 - \alpha \, \forall \, \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

- In this case we say $C(Y)$ is a $1-\alpha$ confidence region for the parameter $\boldsymbol{\theta}$
- there is some true value of $\boldsymbol{\theta}$, and the confidence region will cover it with probability $1-\alpha$ no matter what it is.
- the randomness is due to $\mathbf{Y}$ and $C(\mathbf{Y})$
- once we observe $\mathbf{Y}$ everything is fixed, so region may not include the true $\boldsymbol{\theta}$

Hypothesis Tests and Rejection/Acceptance Regions

Recall for a level α test of a point null hypothesis

- we reject H with probability α when H is true
- for each test we can construct:
  - a rejection region $R(\boldsymbol{\theta}) \subset \mathcal{Y}$, the Y values for which we reject H
  - an acceptance region $A(\boldsymbol{\theta}) \subset \mathcal{Y}$, the Y values for which we accept H
- these sets are complements of each other (for non-randomized tests)

$$\Pr(\mathbf{Y} \in A(\boldsymbol{\theta}) \mid \boldsymbol{\theta}) = 1 - \alpha$$

Duality of Hypothesis-Testing/Confidence Regions

Suppose we have a level α test for every possible valuse of $\boldsymbol{\theta}$

- for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, let $A(\boldsymbol{\theta})$ be the acceptance region of the test $\mathbf{Y} \sim P_{\boldsymbol{\theta}}$
- then $P(\mathbf{Y} \in A(\boldsymbol{\theta}) \mid \boldsymbol{\theta}) = 1 - \alpha$ for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- This collection of hypothesis tests can be “inverted” to construct a confidence region for θ, as follows:
- define $C(\mathbf{Y}) = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} : \mathbf{Y} \in A(\boldsymbol{\theta})\}$
- this is the set of $\boldsymbol{\theta}$ values that are not rejected when $\mathbf{Y} = \mathbf{y}$ is observed
- then C is a $1 - \alpha$ confidence region for $\boldsymbol{\theta}$

Confidence Intervals for Regression Parameters

For the linear model $\mathbf{Y} \sim \textsf{N}(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$, confidence intervals for β_j can be constructed from inverting the approriate t-test.

Acceptance Region & Confidence Interval

Confidence Intervals for Linear Functions

For a linear function of the parameters $\lambda = \mathbf{a}^T \boldsymbol{\beta}$ we can construct a confidence interval by inverting the appropriate t-test

- most important example $\mathbf{a}^T \boldsymbol{\beta} = \mathbf{x}^T \boldsymbol{\beta} = \mathsf{E}[\mathbf{Y} \mid \mathbf{x}]$
- suppose you are testing $H : \mathbf{a}^T \boldsymbol{\beta} = m$
- If H is true, $\mathbf{a}^T \boldsymbol{\beta} - m \sim \mathsf{N}(0, \sigma^2 v)$ where $v = \mathbf{a}^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{a})$
- $s^2 \sim \sigma^2 \chi^2_{n-p}/(n-p)$ independent of $\mathbf{a}^T \hat{\boldsymbol{\beta}}$
- then $t = \frac{\mathbf{a}^T \hat{\boldsymbol{\beta}} - m}{s\sqrt{v}} \sim t_{n-p}$
- a $1-\alpha$ confidence interval for $\mathbf{a}^T \boldsymbol{\beta}$ is

$$\mathbf{a}^T \hat{\boldsymbol{\beta}} \pm s\sqrt{v}\, t_{n-p,1-\alpha/2}$$

Prediction Regions and Intervals

Related to CI for $\mathsf{E}[Y \mid \mathbf{x}] = \mathbf{x}^T\boldsymbol{\beta}$, we may wish to construct a prediction interval for a new observation Y^* at $\mathbf{x}_*$

- a $1-\alpha$ prediction interval for Y^* is a set valued function of $\mathbf{Y}$, $C(\mathbf{Y})$ such that

$$\Pr(\mathbf{Y}^* \in C(\mathbf{Y}) \mid \boldsymbol{\beta}, \sigma^2) = 1 - \alpha$$

  where the distribution is computed using the distribution of $\mathbf{Y}^*$

- this use the idea of a *pivotal quantity*: a function of the data and the parameters that has a known distribution that does not depend on any unknown parameters.
- for prediction, $Y^* = \mathbf{x}_*^T\boldsymbol{\beta} + \boldsymbol{\epsilon}^*$ where $\boldsymbol{\epsilon}^* \sim \mathsf{N}(0, \sigma^2)$ independent of $\boldsymbol{\epsilon}$

$$\begin{aligned}
\mathsf{E}[Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}}] &= \mathbf{x}_*^T\boldsymbol{\beta} - \mathbf{x}_*^T\boldsymbol{\beta} = 0 \\
\mathsf{Var}(Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}}) &= \mathsf{Var}(\boldsymbol{\epsilon}^*) + \mathsf{Var}(\mathbf{x}_*^T\hat{\boldsymbol{\beta}}) = \sigma^2 + \sigma^2\mathbf{x}_*^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_* \\
Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}} &\sim \mathsf{N}(0, \sigma^2(1 + \mathbf{x}_*^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_*))
\end{aligned}$$

Pivotal Quantity and Prediction Intervals

Since $\hat{\boldsymbol{\beta}}$ and s^2 are independent, we can construct a pivotal quantity for $Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}}$:

$$\frac{Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}}}{s\sqrt{1 + \mathbf{x}_*^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_*}} \sim t_{n-p}$$

- therefore

$$\Pr\left(\frac{|Y^* - \mathbf{x}_*^T\hat{\boldsymbol{\beta}}|}{s\sqrt{1 + \mathbf{x}_*^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_*}} < t_{n-p,1-\alpha/2}\right) = 1 - \alpha$$

- Rearranging gives a $1 - \alpha$ prediction interval for Y^*:

$$\mathbf{x}_*^T\hat{\boldsymbol{\beta}} \pm s\sqrt{1 + \mathbf{x}_*^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_*}t_{n-p,1-\alpha/2}$$

Joint Confidence Regions for $\boldsymbol{\beta}$

- we can construct a joint confidence region for $\boldsymbol{\beta}$ based on inverting a test $H : \boldsymbol{\beta} = \boldsymbol{\beta}_0$. Recall:

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} &\sim \mathsf{N}(0, \sigma^2 (\mathbf{X}^T\mathbf{X})^{-1}) \\
(\mathbf{X}^T\mathbf{X})^{-1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &\sim \mathsf{N}(0, \sigma^2 \mathbf{I}) \\
(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T (\mathbf{X}^T\mathbf{X})^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &\sim \sigma^2 \chi^2_p
\end{aligned}
$$

Bayesian Credible Regions

- In a Bayesian setting, we have a posterior distribution for $\boldsymbol{\beta}$ given the data $\mathbf{Y}$
- a set $C \in \mathbb{R}^p$ is a $1-\alpha$ posterior credible region (sometimes called a Bayesian confidence region) if $\Pr(\boldsymbol{\beta} \in C \mid \mathbf{Y}) = 1 - \alpha$
- lots of sets have this property, but we usually want the most probable values of $\boldsymbol{\beta}$ given the data
- this motivates looking at the highest posterior density (HPD) region which is a $1-\alpha$ credible set C such that the values in C have higher posterior density than those outside of C
- the HPD region is the smallest region that contains $1-\alpha$ of the posterior probability

Bayesian Credible Regions

- For a normal prior and normal likelihood, the posterior for $\boldsymbol{\beta}$ conditional on σ^2 is normal with say posterior mean $\mathbf{b}_n$ and posterior precision $\mathbf{\Phi}_n$
- the posterior density as a function of $\boldsymbol{\beta}$ for a fixed σ^2 is

$$p(\boldsymbol{\beta} \mid \mathbf{Y}) \propto \exp\left\{-(\boldsymbol{\beta} - \mathbf{b}_n)^T \mathbf{\Phi}_n (\boldsymbol{\beta} - \mathbf{b}_n)/2\right\}$$

- so a highest posterior density region has the form

$$C = \{\boldsymbol{\beta} : (\boldsymbol{\beta} - \mathbf{b}_n)^T \mathbf{\Phi}_n^{-1} (\boldsymbol{\beta} - \mathbf{b}_n) < q\}$$

$$\begin{aligned}
\boldsymbol{\beta} - \boldsymbol{\beta}_n \mid \sigma^2 &\sim \textsf{N}(0, \mathbf{\Phi}_n^{-1}) \\
\mathbf{\Phi}_n^{1/2}(\boldsymbol{\beta} - \boldsymbol{\beta}_n) \mid \sigma^2 &\sim \textsf{N}(0, \mathbf{I}) \\
(\boldsymbol{\beta} - \boldsymbol{\beta}_n)^T \mathbf{\Phi}_n (\boldsymbol{\beta} - \boldsymbol{\beta}_n) \mid \sigma^2 &\sim \chi^2_p
\end{aligned}$$

Bayesian HPD Regions For Unknown σ^2

- For unknown σ^2 we need to integrate out σ^2 to get the marginal posterior for $\boldsymbol{\beta}$
- for conjugate priors, $\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathbf{b}_0, (\phi\boldsymbol{\Phi}_0)^{-1})$ and $\phi \mid \mathbf{Y} \sim \mathbf{G}(a_n/2, b_n/2)$, then

$$\begin{aligned} \boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\mathbf{b}_n, (\phi\boldsymbol{\Phi}_n)^{-1}) \\ \phi \mid \mathbf{Y} &\sim \mathbf{G}(a_n/2, b_n/2) \\ \boldsymbol{\beta} \mid \mathbf{Y} &\sim \mathsf{St}(a_n, \mathbf{b}_n, \hat{\sigma}^2\boldsymbol{\Phi}_n^{-1}) \end{aligned}$$

  where $\mathsf{St}(a_n, \mathbf{b}_n, (\hat{\sigma}^2\boldsymbol{\Phi}_n)^{-1})$ is a multivariate Student-t distribution with a_n degrees of freedom location $\mathbf{b}_n$ and scale matrix $\hat{\sigma}^2\boldsymbol{\Phi}_n^{-1}$ with $\hat{\sigma}^2 = b_n/a_n$

- density of $\boldsymbol{\beta}$ is

$$p(\boldsymbol{\beta} \mid \mathbf{Y}) \propto \left(1 + \frac{(\boldsymbol{\beta} - \mathbf{b}_n)^T\boldsymbol{\Phi}_n^{-1}(\boldsymbol{\beta} - \mathbf{b}_n)}{a_n\hat{\sigma}^2}\right)^{-(a_n+p)/2}$$

Reference Posterior Distribution

For the reference prior $\pi(\boldsymbol{\beta}, \phi) \propto 1/\phi$ and the likelihood $p(\mathbf{Y} \mid \boldsymbol{\beta})$, the posterior is proportional to the likelihood times ϕ^{-1}

- (generalized) posterior distribution:

$$
\begin{aligned}
\boldsymbol{\beta} \mid \phi, \mathbf{Y} &\sim \mathsf{N}(\hat{\boldsymbol{\beta}}, (\phi \mathbf{X}^T \mathbf{X})^{-1}) \\
\phi \mid \mathbf{Y} &\sim \mathbf{G}((n-p)/2, \mathsf{SSE}/2)
\end{aligned}
$$

  if $n > p$

- marginal posterior distribution for $\boldsymbol{\beta}$ is multivariate Student-t with $n-p$ degrees of freedom, location $\hat{\boldsymbol{\beta}}$ and scale matrix $\hat{\sigma}^2 \mathbf{X}^T \mathbf{X}^{-1}$

Duality

- the posterior density $\boldsymbol{\beta}$ is a monotonically decreasing function of $Q(\boldsymbol{\beta}) \equiv (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^T \mathbf{X}^T \mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$ so contours of $p(\boldsymbol{\beta} \mid \mathbf{Y})$ are ellipsoidal in the parameter space of $\boldsymbol{\beta}$
- the quantity $Q(\boldsymbol{\beta})/p\hat{\sigma}^2$ is distributed *a posteriori*

$$Q(\boldsymbol{\beta})/p\hat{\sigma}^2 \sim F(p, n-p)$$

  and the ellipsoidal contour of $p(\boldsymbol{\beta} \mid \mathbf{Y})$ is defined as $\frac{Q(\boldsymbol{\beta})}{p\hat{\sigma}^2} = F(p, n-p, \alpha)$. (Box & Tiao 1973)
- then HPD regions for $\boldsymbol{\beta}$ are the same as confidence regions for $\boldsymbol{\beta}$ based on the F-distribution
- marginals of $\beta_j, \mathbf{x}^T\boldsymbol{\beta}$ and Y^* are also univariate Student-t with $n-p$ degrees of freedom
- difference is in the interpretation of the regions i.e posterior probability that $\boldsymbol{\beta}$ is in the given the data vs the probability a priori that the region covers the true $\boldsymbol{\beta}$