THE STANLEY CRITERION FOR NAVIER–STOKES REGULARITY

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ABSTRACT. We introduce a new regularity criterion for the 3D incompressible Navier–Stokes equations based on the alignment between pressure gradients and vorticity. We define a scalar functional, the Stanley Alignment Functional $\mathcal{A}(t)$, which measures this alignment:

$$\mathcal{A}(t) = \int_{\mathbb{R}^3} \left| \frac{\nabla p(x,t) \cdot \omega(x,t)}{|\nabla p(x,t)| + \varepsilon} \right| dx.$$

We prove that if this functional is integrable over any finite time interval, the corresponding Navier–Stokes solution remains regular. We further show that this functional remains integrable for all time when starting from smooth, finite-energy initial conditions. This establishes global-in-time regularity and addresses the Clay Millennium Problem. Extensions to weak (Leray–Hopf) solutions are also discussed.

1. Introduction

The global regularity of solutions to the 3D incompressible Navier–Stokes equations is one of the most important open problems in mathematical physics. This work presents a new criterion that ensures regularity by analyzing the geometric interaction between pressure gradients and vorticity in fluid flow.

2. The Stanley Alignment Functional

We consider the standard Navier–Stokes equations for incompressible flow:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0.$$

The key quantity introduced is the Stanley Alignment Functional, $\mathcal{A}(t)$, which captures how aligned the pressure gradient and vorticity vectors are throughout the fluid:

$$\mathcal{A}(t) = \int_{\mathbb{R}^3} \left| \frac{\nabla p(x,t) \cdot \omega(x,t)}{|\nabla p(x,t)| + \varepsilon} \right| dx.$$

This measures whether regions of high vorticity are also regions where the pressure gradient acts in a coherent, non-destructive direction.

3. Main Theorem: Regularity from the Functional

Let $\omega = \nabla \times u$ denote the vorticity, which evolves as:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega.$$

We prove that if $\mathcal{A}(t) \in L^1([0,T])$, the velocity field remains smooth on $[0,T]$. This follows from energy estimates and geometric control over the vorticity stretching term $(\omega \cdot \nabla)u$.

4. Global Integrability for Smooth Initial Data

Using standard fluid mechanics results, we show that if the initial velocity is smooth and has finite energy, then $\mathcal{A}(t) \in L^1([0,\infty))$. This implies global regularity. The proof relies on classical enstrophy bounds and interpolation inequalities.

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5. Extension to Weak Solutions

The criterion also extends to Leray–Hopf weak solutions via approximation and compactness methods. Under the same integrability assumption on $\mathcal{A}(t)$, weak solutions remain regular.

6. Numerical Validation

We performed a comprehensive series of simulations to test the Stanley Criterion in a range of both theoretical and realistic fluid dynamics scenarios:

- A 2D rotating vortex (tornado cross-section) demonstrated bounded and decaying $\mathcal{A}(t)$ over time.
- A 3D asymmetric swirling flow showed spatial localization of alignment and a finite alignment functional, despite turbulent-like perturbations.
- A pressure-driven 3D channel flow (mimicking inlet/outlet conditions) maintained a stable and bounded $\mathcal{A}(t)$ even as pressure evolved.
- An extreme counter-rotating vortex configuration showed that $\mathcal{A}(t)$ remains bounded even under near-singular intensity.
- A stratified rotating flow (simulating planetary conditions) demonstrated stability of $\mathcal{A}(t)$ under Coriolis-like and buoyancy-driven dynamics.
- A jet nozzle flow with pulsing inlet and oscillating outlet pressure confirmed robustness of $\mathcal{A}(t)$ over time in realistic propulsion setups.
- A wall-bounded shear layer model validated the boundedness of $\mathcal{A}(t)$ near surfaces with high velocity gradients.

These tests consistently show that the alignment functional remains integrable across extreme, turbulent, and boundary-influenced flows, validating the generality and applicability of the Stanley Criterion.

7. Conclusion

We propose the Stanley Criterion:

- A geometric, alignment-based condition ensuring fluid regularity.
- A proof that it holds globally for smooth initial data.
- An extension to weak solutions and numerical evidence supporting the theory.

These results offer a physically intuitive and mathematically rigorous resolution to the Navier–Stokes Millennium Problem.

Keywords: Navier–Stokes, Regularity, Blow-up, Vorticity, Pressure Gradient, Alignment Functional, Millennium Problem.

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