Week Two: Inference for Proportions

This Week (and next)

- Binomial & Multinomial Distributions
- Maximum Likelihood Estimation
- Intervals and Testing

Next Week: Bayesian Inference for Proportions

Tuesday:

- Watch Week 3 videos and submit HW 2 (video notes)
- Week 3 activity

Thursday:

- Lab 1

Binomial Distribution

Recall that 7 out of 16 of us selected Sweet Peaks as our favorite ice cream shop in Bozeman (or Montana).

This data can be modeled with a Binomial distribution, where

$$P(Y = y) = \frac{n!}{y!(n-y)!}\pi^y(1-\pi)^{n-y}, \quad \text{where } y = 0, 1, 2, ...$$

Q: Why do we care?

Given a binomial distribution with specified n and π, we can estimate to probability of observing a number of successes. This is my 10th year at MSU and their have been 8 Cat-Griz football games.

Exercise: Assume that there is no difference in team ability ($\pi = \frac{1}{2}$).

- 1. Estimate the probability that MSU wins the 2025 game.
- 2. Estimate the probability that MSU won 6 of 8 games, hint: `dbinom`.
- 3. Create a figure to show the probability that MSU won 0, 1, …, 8 games.

Maximum Likelihood Estimation

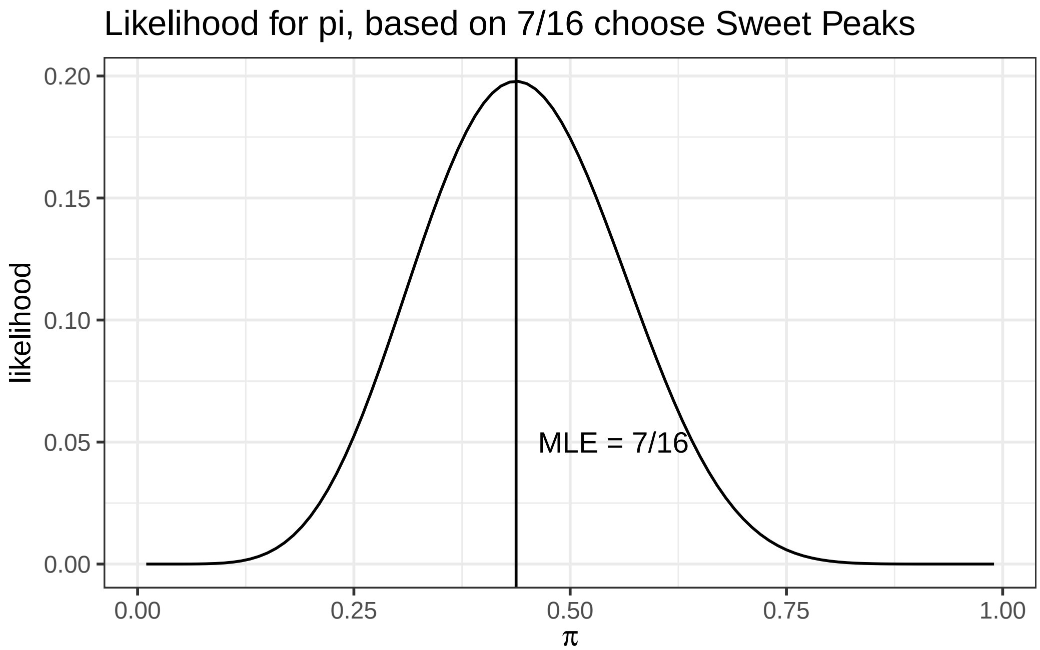
Generally our goal is to estimate π from a set of binary responses, as opposed to estimating the number of successes given π.

On the first day of class, we intuitively estimated π (or p) to be $\frac{7}{16}$. It turns out that this is also the maximum likelihood estimator for π.

```
library(tidyverse)
```

```
-- Attaching core tidyverse packages ------------------------ tidyverse 2.0.0 --
v dplyr     1.1.4     v readr     2.1.5
v forcats   1.0.0     v stringr   1.5.1
v ggplot2   3.5.2     v tibble    3.3.0
v lubridate 1.9.4     v tidyr     1.3.1
v purrr     1.1.0
-- Conflicts ------------------------------------------ tidyverse_conflicts() --
x dplyr::filter() masks stats::filter()
x dplyr::lag()    masks stats::lag()
i Use the conflicted package (<http://conflicted.r-lib.org/>) to force all conflicts to becom
```

```
pi_vals <- seq(.01, .99, by = .01)
like_vals <- dbinom(7, 16, pi_vals)
tibble(pi = pi_vals,
       likelihood = like_vals) |>
  ggplot(aes(y = likelihood, x = pi)) +
  geom_line() +
  theme_bw() +
  xlab(expression(pi)) +
  ggtitle("Likelihood for pi, based on 7/16 choose Sweet Peaks") +
  geom_vline(xintercept = 7/16) +
  annotate('text', x = .55, y = .05, label = 'MLE = 7/16')
```



Likelihood for pi, based on 7/16 choose Sweet Peaks
likelihood
0.20
0.15
0.10
0.05
0.00
MLE = 7/16
0.00
0.25
0.50
0.75
1.00
π

Construct a likelihood profile for the Cat-Griz setting.

Testing

Testing and estimation are two related, but different things.

Figure 1: Testing: Yes or No

Figure 2: Estimation: What values are reasonable

- Traditionally much of applied statistics was focused on testing and associated p-values.
- In this class, we will talk about both but testing and estimation but emphasis will be on estimation.

Testing

Given the record of the Bobcats (6 wins in 8 games) over the last 10 years, we may question whether the ability level of the teams is actually the same.

This could be formulated as a testing problem in which we ask how unusual would it be for the bobcasts to win 6 out of 8 games if the teams had the same ability level. Note the similarity to the previously created figure.

The `binom.test()` function can be used for this purpose, interpret the results.

```r
binom.test(6, 8, p = .5, alternative = "greater")
```

```
	Exact binomial test

data:  6 and 8
number of successes = 6, number of trials = 8, p-value = 0.1445
alternative hypothesis: true probability of success is greater than 0.5
95 percent confidence interval:
 0.4003106 1.0000000
sample estimates:
probability of success 
                  0.75 
```

Estimation: Uncertainty intervals

A common way to construct confidence intervals uses asymptotic theory (large samples and CLT) such that

$\hat{\pi} \pm z_{\alpha/2}(SE)$.

With a binomial distribution, the standard error (SE) can be calculated as $\sqrt{\pi(1-\pi)/n}$. We don't know π so the estimator can be used.

Use this framework to construct a confidence interval for π in our ice cream example.

There are some known issues with this procedure that we will explore in the future, but as a thought exercise

- What do the intervals look like when Y = 0?

- What happens when n is small?

Multicategory Outcomes

Thus far, we have simplified the ice cream example to a binary situation (Sweet Peaks or not).

However, the data was collected as a multicategory setting with many possible answers.

An extension to the binomial distribution, known as the multinomial distribution is appropriate in this case.

$$P(Y_1 = y_1, Y_2 = y_2, ..., Y_k = y_k) = \frac{n!}{y_1!y_2!...y_k!}\pi_1^{y_1}\pi_2^{y_2}...\pi_k^{y_k}, \quad \text{where } y_i = 0, 1, 2, ...$$