UNIVERSITY OF BRITISH COLUMBIA
Department of Statistics
Stat 460/560: Statistical Inference I
Assignment 1

The assignment is due on Wednesday, **September 26, 2018**, in class.

1. Suppose that $X_1, X_2, ..., X_n$ are iid random variables with mean μ and variance $\sigma^2 < \infty$. Let $\overline{X}$ be the sample mean and S^2 be the sample variance.

   (a) Find the limiting distribution for the following three test statistics:

   $$T_1 = \sqrt{n}\left(\frac{\overline{X} - \mu}{S}\right)$$

   $$T_2 = n\left(\frac{\overline{X} - \mu}{S}\right)^2$$

   and

   $$T_3 = \sqrt{n}\left(\frac{1}{\overline{X}} - \frac{1}{\mu}\right),$$

   assuming that $\mu \neq 0$. Justify all the steps in your derivation of the limiting distributions.

   (b) This part of the question concerns confidence intervals, which can be constructed using statistics from part (a). Suppose that $n = 36$, $\overline{X} = 0.5$ and $S^2 = 0.1$. For parts i.-iii., include both analytical expressions in terms of n, μ, S, as well as the corresponding numerical values.

      i. Construct an approximate 95% confidence interval for $1/\mu$ based on T_1.

      ii. Construct an approximate 95% confidence interval for $1/\mu$ based on T_3.

      iii. Which confidence interval is shorter?

      iv. (**Grads**) Conduct a Monte Carlo simulation study to determine whether the conclusion in part iii. is to be expected, assuming the data are normally distributed with mean 0.5 and variance 0.1. In particular, use the CLT to determine what number of replicates are needed to accurately estimate the mean difference between the confidence intervals lengths. Fully justify your reasoning, include a pseudo-code for the steps of your simulation study and the corresponding R code.

2. Let $X_1, X_2, ..., X_n$ be i.i.d. exponential random variables with rate parameter $\lambda > 0$.

   (a) Find the distribution of the sample maximum

   $$M_n = \max\{X_1, X_2, ..., X_n\}.$$

   (b) Show that $M_n \xrightarrow{p} \infty$. (Note that the relation $X_n \xrightarrow{p} \infty$ is interpreted as $1/X_n \xrightarrow{p} 0$.)

   (c) Find the limiting distribution for

   $$\widetilde{M_n} = M_n - \log n.$$

   (d) Propose an estimator for parameter λ and derive its approximate distribution for large values of n.

3. Suppose$(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are i.i.d. with common joint distribution $F(x, y)$. Suppose that $\mathbb{E}(X^4) < \infty$ and $\mathbb{E}(Y^4) < \infty$. Let

   $$\sigma_{XY} = \mathbb{E}\{(X - \mu_X)(Y - \mu_Y)\}$$

   and

   $$\widehat{\sigma}_{XY} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right).$$

   (a) Show that $\widehat{\sigma}_{XY} \xrightarrow{p} \sigma_{XY}$.

   (b) Find the limiting distribution, as $n \to \infty$, for

   $$\sqrt{n}\left(\widehat{\sigma}_{XY} - \sigma_{XY}\right).$$