Inference Using t-Distributions

DISTRIBUTION OF THE SAMPLE AVERAGE

USING THE T-DISTRIBUTION FOR ONE SAMPLE INFERENCE

STARTING TO EXPLORE T-DISTRIBUTION FOR TWO SAMPLE INFERENCE

Study Analysis

MONDAY, Sept. 5, 2016 (HealthDay News) -- Scientists say they've identified a brain circuit in mice that plays a key role in the sleep-wake cycle.

(HEALTHDAY)

The circuit is a key component of the brain's reward system, according to researchers from Stanford University in Palo Alto, Calif.

The investigators saw that as the mice ramped down for sleep, activity in this brain circuit decreased. The researchers also saw that activating this circuit could rouse the animals from sleep.

These findings could potentially lead to new treatments for sleep problems, the researchers said.

"This has potential huge clinical relevance," senior author Luis de Lecea, professor of psychiatry and behavioral sciences, said in a university news release.

"Insomnia, a multibillion-dollar market for pharmaceutical companies, has traditionally been treated with drugs such as benzodiazepines that nonspecifically shut down the entire brain," he explained.

Study Analysis

"Now we see the possibility of developing therapies that, by narrowly targeting this newly identified circuit, could induce much higher-quality sleep," de Lecea said.

Research on animals often fails to produce the same results in humans, though the researchers said the brain circuitry involving the reward system is similar in all vertebrates.

The study was published online Sept. 5 in the journal *Nature Neuroscience*.

Between 25 percent and 30 percent of Americans have sleep problems, according to the U.S. National Institutes of Health.

Also, sleep-wake cycle disturbances are common among people with conditions such as schizophrenia and bipolar disorder, the researchers said.

One Sample Inference With the t-Distribution

Z-ratio

- Facts about $\bar{Y}$:
  1. $\bar{Y}$ "approx. equals" μ
  2. Variance($\bar{Y}$) $= \frac{\sigma^2}{n}$
  3. $\bar{Y}$ "approx. distributed" normal if *n* is larger than 30

$\bar{Y}$ approx. equals μ with standard error $\frac{\sigma}{\sqrt{n}}$

- Therefore, if we center and rescale…

$Z = \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}$ is "approx. distributed" standard normal

Distribution of Sample Average

- This last fact is due to the **CENTRAL LIMIT THEOREM (CLT)**

- **Take away message (when σ is known):**
  - "WE CAN EVALUATE HOW RARE/EXTREME AN OBSERVED SAMPLE AVERAGE WOULD BE RELATIVE TO A NORMAL DISTRIBUTION"

About that known σ …

So far, we have treated the standard deviation, σ, as known

While this can happen in practice, often we have to **ESTIMATE** σ using the same data we use to estimate μ

ESTIMATE σ:

$$s = \frac{\sqrt{(y_1 - \bar{y})^2 + (y_2 - \bar{y})^2 + \ \dots + (y_n - \bar{y})^2}}{\sqrt{n-1}}$$

Example: If we have data 79, 83, 84, 89, 90 mm for digitus tertius. What is an estimate of the standard deviation?

Answer: $\bar{y} = 85$ mm. Therefore,

$$s = \frac{\sqrt{(79-85)^2+(83-85)^2+(84-85)^2+(89-85)^2+(90-85)^2}}{\sqrt{5-1}} = \frac{\sqrt{6^2+2^2+1^2+4^2+5^2}}{\sqrt{4}} = 4.527\text{mm}$$

t-Ratio

- Facts about $\bar{Y}$:
  1. $\bar{Y}$ "approx. equals" μ
  2. Variance$(\bar{Y}) = \frac{\sigma^2}{n}$
  3. $\bar{Y}$ "approx. distributed" normal if n is larger than 30
- Additionally, we use s as an estimate of σ

- **THEN:**

T$= \frac{\bar{Y}-\mu}{s/\sqrt{n}}$ is "approx. distributed" t with *(n-1)* degrees of freedom

Note the Difference…

- $s = \frac{\sqrt{(y_1 - \bar{y})^2 + (y_2 - \bar{y})^2 + \ \ldots + (y_n - \bar{y})^2}}{\sqrt{n-1}}$

Versus

- T$= \frac{\bar{Y} - \mu}{s/\sqrt{n}}$

- This is an easy spot for confusion!

t-Confidence Interval

$$\bar{Y} - E < \mu < \bar{Y} + E$$

$$E = t_{\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}}$$

How does $z_{\alpha/2}$ compare with $t_{\alpha/2,(n-1)}$?

t-Confidence Interval

How does $z_{\alpha/2}$ compare with $t_{\alpha/2,(n-1)}$?

```
DATA quant;
    t2   = QUANTILE('T',.975,2);
    t10  = QUANTILE('T',.975,10);
    t20  = QUANTILE('T',.975,20);
    t30  = QUANTILE('T',.975,30);
    t80  = QUANTILE('T',.975,80);
    t800 = QUANTILE('T',.975,800);
    z    = QUANTILE('NORMAL',.975,0,1);
RUN;

PROC PRINT DATA=quant;
RUN;
```

Obs	t2	t10	t20	t30	t80	t800	z
1	4.30265	2.22814	2.08596	2.04227	1.99006	1.96293	1.95996

Revisiting…

Length of Time to Earn a Bachelor's Degree: In a study of the length of time that students require to earn bachelors degrees, 80 students are randomly selected and found to have a sample mean of 4.8 years (National Center for Education Statistics). We compute the sample standard deviation, *s* = 2.2 years. Let's construct a 98.2% confidence interval.

As we are estimating the standard deviation, we use the T-Statistic:

$$E = t_{\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}} = t_{\alpha/2,(n-1)} \cdot 0.246$$

Revisiting...

```
/* Get quantiles */
DATA quant;
     t   = QUANTILE('T',1-0.018/2,80-1);
RUN;

PROC PRINT DATA=quant;
RUN;
```

Obs	t
1	2.41600

As we are estimating the standard deviation, we use the T-Statistic:

$$E = t_{\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}} = 2.416 \cdot 0.246 = 0.594$$

Revisiting…

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$$E = t_{\alpha/2,(n-1)} \cdot \frac{s}{\sqrt{n}} = 2.416 \cdot 0.246 = 0.594$$

We are 98.2% confident that the mean graduation time is between

$4.8 \pm E =$ [4.026, 5.394] years

”This range of population means is plausible based on this data”

Revisiting…

(CONFIDENCE INTERVAL)

Let's suppose instead of asking for a range of plausible values, we want to know if $\mu_0 = 4$ years is a plausible value

(NULL HYPOTHESIS)

$$H_{0:}\,\mu = 4$$
$$H_{A:}\,\mu \neq 4$$

How much evidence does this provide that 4 years isn't plausible?

(TEST STATISTIC)

$$T = \frac{\bar{y}-\mu_0}{s/\sqrt{n}} = \frac{4.8-4}{2.2/\sqrt{80}} = 3.252$$

Revisiting…

Let's suppose instead of asking for a range of plausible values, we want to know if $\mu_0 = 4$ years is a plausible value

$$H_{0:}\,\mu = 4$$
$$H_{A:}\,\mu \neq 4$$

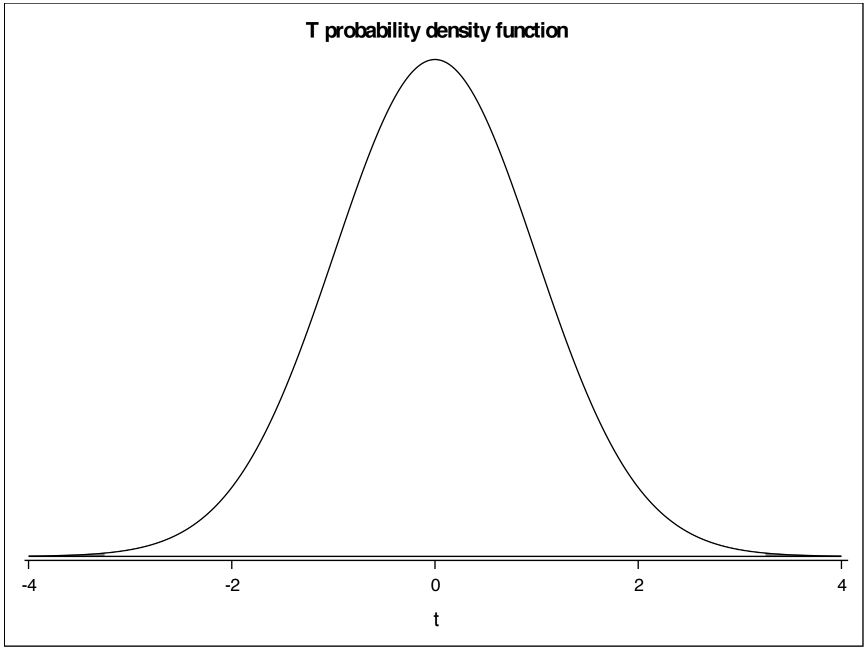
```
/* Get probabilities */
DATA prob;
    p = 2*(1-CDF('T',3.252,80-1));
RUN;

PROC PRINT DATA=prob;
RUN;
```

$$T = \frac{\bar{Y}-\mu_0}{s/\sqrt{n}} = \frac{4.8-4}{2.2/\sqrt{80}} = 3.252$$

Obs	p
1	.001686240

(P-VALUE)

Revisiting…

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/* Get probabilities */
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PROC PRINT DATA=prob;
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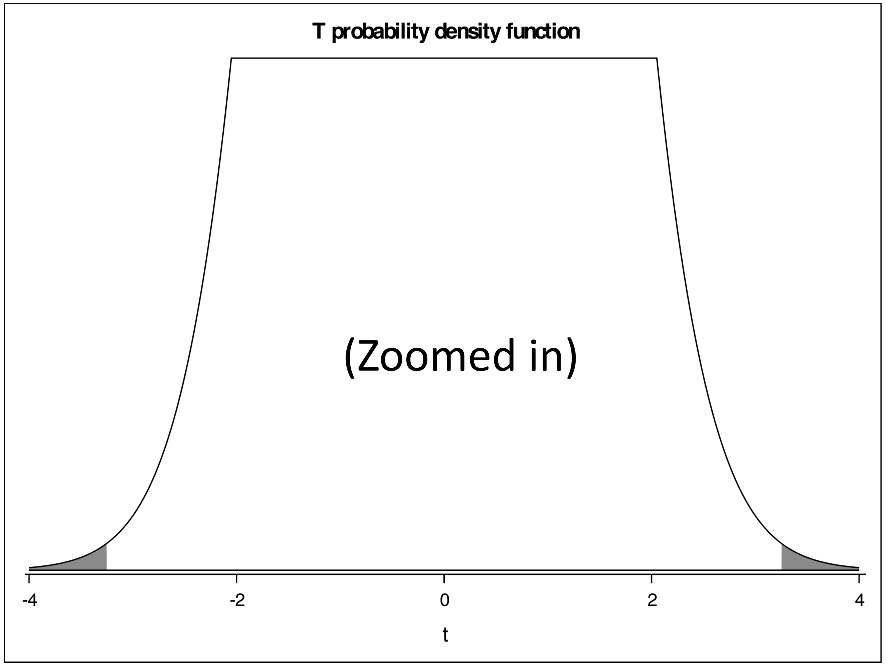
Revisiting…

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Revisiting…

Let's suppose instead of asking for a range of plausible values, we want to know if $\mu_0 = 4$ years is a plausible value

$$H_{0:}\, \mu = 4$$
$$H_{A:}\, \mu \neq 4$$

$$T = \frac{\bar{Y} - \mu_0}{s/\sqrt{n}} = \frac{4.8 - 4}{2.2/\sqrt{80}} = 3.252$$

(STATISTICAL CONCLUSION)

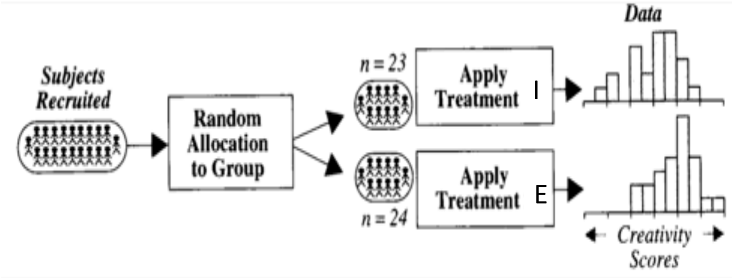
We have strong evidence that 4 years is an implausible value for mean graduation time (p-value of approx. 0.0017 from a one-sample t-test).

(SCOPE OF INFERENCE)

These graduates were randomly sampled from the National Center for Education Statistics (NCES) and hence we can extend our inference to the population that the NCES draws from.

Two Sample Inference With the t-Distribution

Creativity Study



→ Population mean: μ_I

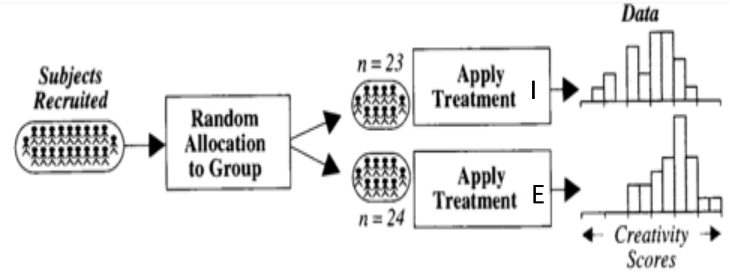
→ Population mean: μ_E

- If the questionnaires had no effect, then we would expect:

$$\mu_I = \mu_E \leftrightarrow \mu_I - \mu_E = 0$$ **(NULL HYPOTHESIS)**

- We have discussed that the sample means $\bar{Y}_I$ and $\bar{Y}_E$ are good estimates of $\mu_I, \ \mu_E$
- → $\bar{Y}_I - \bar{Y}_E$ is a reasonable estimate of $\mu_I - \mu_E$ **(TEST STATISTIC)**
- We can compute this **OBSERVED DIFFERENCE** in sample means: 4.14420
- Is 4.14420 large enough for us to conclude that $\mu_I \neq \mu_E$? **(ALTERNATIVE HYPOTHESIS)**

Creativity Study



→ Population mean: μ_I
→ Population sd: σ_I

→ Population mean: μ_E
→ Population sd: σ_E

- To quantify "large", we can re-randomly allocate units to two groups and re-compute the difference in sample means many times

(RANDOMIZATION TEST)

OR

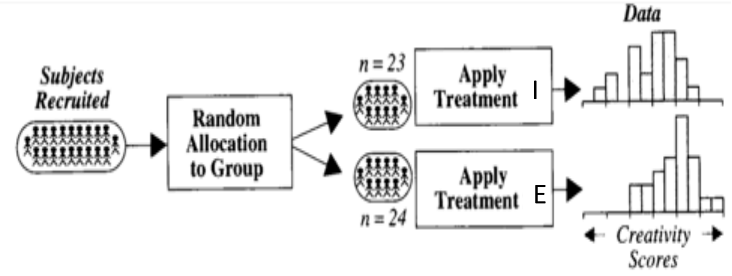
- We can appeal to the t-distribution

(TWO SAMPLE T TEST)

- We additionally need to know/estimate the standard deviation of $\bar{Y}_I - \bar{Y}_E$

- $SD(\bar{Y}_I - \bar{Y}_E) = \sqrt{\frac{\sigma_I^2}{n_I} + \frac{\sigma_E^2}{n_E}}$ ← (We need to assume the intrinsic/extrinsic groups are independent)

Creativity Study



→ Population mean: μ_I
→ Population sd: σ_I

→ Population mean: μ_E
→ Population sd: σ_E

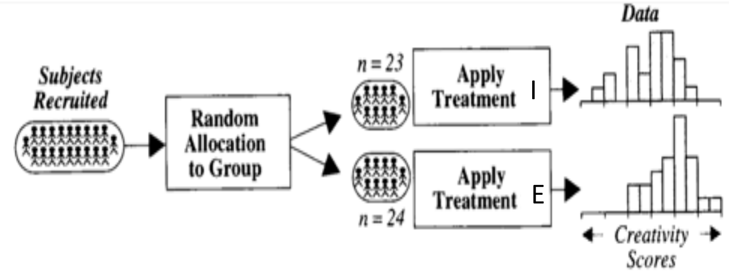
- There are two ways to estimate the standard deviation of $\bar{Y}_I - \bar{Y}_E$
  1. Pooled SD ← (Focus on this one for now)
  2. Welch's SD

- To create the pooled SD, we need to assume that $\sigma_I = \sigma_E = \sigma$

- We form an estimate of σ via: $s_p = \sqrt{\dfrac{(n_I-1)\, s_I^2 + (n_E-1)\, s_E^2}{n_I+n_E-2}}$

- $SE(\bar{Y}_I - \bar{Y}_E) = \sqrt{\dfrac{\sigma_I^2}{n_I} + \dfrac{\sigma_E^2}{n_E}} \leftrightarrow SE(\bar{Y}_I - \bar{Y}_E) = s_p\sqrt{\dfrac{1}{n_I} + \dfrac{1}{n_E}}$

Creativity Study: Next time



→ Population mean: μ_I
→ Population sd: σ_I

→ Population mean: μ_E
→ Population sd: σ_E

- Now, we can return to the usual frame work:

$$\bar{Y}_I - \bar{Y}_E \pm t_{\alpha/2,(n-2)} SE(\bar{Y}_I - \bar{Y}_E)$$

Here: the degrees of freedom is $n - 2 = n_I - 1 + n_E - 1$