Advanced Measure Theory

Based by Prof.Dr.Dümbgen Lutz
ZnV
The life is a function of the value you bring to others.

May 28, 2024

Contents

Chapter 1

Signed Measures

Start the coming trip from finite signed measures.
Assume $(\Omega, \mathcal{A})$ be a measurable space, where Ω is a nonempty set equipped with a $\sigma-$algebra $\mathcal{A}$ over Ω.

1.1 The Hahn-Jordan Decomposition

Definition 1.1.1: Finite signed measure

A function $\nu : \mathcal{A} \to \mathbb{R}$ is called a finite signed measure on $(\Omega, \mathcal{A})$, if

1. $\nu(\emptyset) = 0$,
2. ν is $\sigma-$additive.

Note:-

$\sigma-$additive: for arbitrary disjoint sets $A_1, A_2, A_3, \cdots \in \mathcal{A}$,

$$\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n).$$

Question 1

What is the difference to a finite measure?

$\nu(A)$ may be negative for some sets $A \in \mathcal{A}$.

Question 2

Why has the series $\sum_{n=1}^{\infty} \nu(A_n)$ converge absolutely?

Because the set $\bigcup_{n=1} A_n$ is invariant w.r.t.reorderings of the sets $A_1, A_2, \cdots$.

Example 1.1.1 (Finite signed measure 1)

$$\nu := Q - P \tag{1.1}$$

with finite measures P, Q on $(\Omega, \mathcal{A})$.

Proof: To prove that $\nu := Q - P$ with finite measures P and Q on $(\Omega, \mathcal{A})$ is a finite signed measure, we need to verify that ν satisfies the two conditions.

(1): $\nu(\emptyset) = 0$

Since both P and Q are finite measures, they satisfy $P(\emptyset) = 0$ and $Q(\emptyset) = 0$. Thus,

$$\nu(\emptyset) = Q(\emptyset) - P(\emptyset) = 0 - 0 = 0.$$

Hence, $\nu(\emptyset) = 0$, satisfying condition (1).
(2):ν is σ-additive. To show that ν is σ-additive, we need to demonstrate that for any countable collection of disjoint sets $\{A_n\}$ in $\mathcal{A}$,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n).$$

Since P and Q are finite measures, they are σ-additive. Therefore, we have

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

and

$$Q\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Q(A_n).$$

Now, consider ν applied to the union of the disjoint sets $\{A_n\}$:

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = Q\left(\bigcup_{n=1}^{\infty} A_n\right) - P\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Using the σ-additivity of Q and P, this becomes

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Q(A_n) - \sum_{n=1}^{\infty} P(A_n).$$

Since $\nu(A_n) = Q(A_n) - P(A_n)$ for each n, we have

$$\sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} (Q(A_n) - P(A_n)).$$

Thus,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Q(A_n) - \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} (Q(A_n) - P(A_n)) = \sum_{n=1}^{\infty} \nu(A_n).$$

This shows that ν is σ-additive, satisfying condition (2).

We have verified that $\nu := Q - P$ satisfies both conditions confirming that ν is indeed a finite signed measure. 😄

Example 1.1.2 (Finite signed measure 2)

$$\nu(A) := \int_A f \, d\mu = \int \mathbf{1}_A f \, d\mu \tag{1.2}$$

with a measure μ on $(\Omega, \mathcal{A})$ and a function $f \in \mathcal{L}^1(\mu)$.i.e.,$f : \Omega \to \bar{\mathbb{R}}$ is $\mathcal{A}-$measurable with $\int |f| \, d\mu < \infty$.

Proof: Main idea: By means of linearity of integrals and dominated convergence.
Given:

- A measure μ on $(\Omega, \mathcal{A})$.
- A function $f \in \mathcal{L}^1(\mu)$, i.e., $f : \Omega \to \bar{\mathbb{R}}$ is $\mathcal{A}$-measurable with $\int |f| \, d\mu < \infty$.

Check Condition (1):$\nu(\emptyset) = 0$.
By definition,

$$\nu(\emptyset) = \int_{\emptyset} f \, d\mu.$$

Since the integral over the empty set is zero,

$$\nu(\emptyset) = 0.$$

Check Condition (2): ν is σ-additive.
By definition,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu.$$

Since the sets A_n are disjoint, we can apply the linearity and σ-additivity of the integral:

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \int \mathbf{1}_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \int \left(\sum_{n=1}^{\infty} \mathbf{1}_{A_n}\right) f \, d\mu.$$

By the Monotone Convergence Theorem (or Dominated Convergence Theorem), we have:

$$\int \left(\sum_{n=1}^{\infty} \mathbf{1}_{A_n}\right) f \, d\mu = \sum_{n=1}^{\infty} \int \mathbf{1}_{A_n} f \, d\mu = \sum_{n=1}^{\infty} \nu(A_n).$$

Thus,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n).$$

This shows that ν is σ-additive, satisfying condition (2).

Since ν satisfies both (1) and (2), we have shown that $\nu(A) := \int_A f \, d\mu$ defines a finite signed measure on $(\Omega, \mathcal{A})$.

$$\boxed{\nu(A) = \int_A f \, d\mu \text{ is a finite signed measure.}}$$

😄

1.2 Radon-Nikodym Derivatives

1.2.1 Finite measures

1.2.2 Absolute continuity and $\sigma-$finite measures

Chapter 2

Abstract Integrals

2.1 Lattices and Stone Lattices

2.2 Abstract and Usual Integrals

2.3 Representations of Dual Spaces

2.4 Prohorov's Theorem