

# Estimation Theory: Problems

Most of the problems and exercises of this collection have been taken from previous years exams on subjects related to Estimation Theory. The topics covered by each exercise are shown next to the exercise number according to:

- 1.1. General view of estimation problems.
- 1.2. Bayes' Estimation Theory.
- 1.3. Frequently used Bayes' estimators: MSE, MAP, MAD.
- 1.4. Maximum Likelihood estimation.
- 1.5. Estimation with Gaussian likelihoods.
- 1.6. Constrained estimators. Linear minimum mean square error estimation.
- 1.7. Bias and Variance of estimators.
- 1.8. Machine design of estimators.

## Notation:

- $\hat{S}_{\text{MSE}}$ : Minimum Mean Square Error estimator.
- $\hat{S}_{\text{MAD}}$ : Minimum Mean Absolute Deviation Error estimator.
- $\hat{S}_{\text{MAP}}$ : Maximum a posteriori estimator.
- $\hat{S}_{\text{ML}}$ : Maximum likelihood estimator.
- $\hat{S}_{\text{LMSE}}$ : Linear Minimum Mean Square Error estimator.

## 1. Bayesian Estimation

### Exercise 1 (Bayesian)

The joint distribution of random variables  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = 12s^2, \quad 0 < s < x, \quad 0 < x < 1$$

- (a) Determine the estimate of  $S$ ,  $\hat{S}_c$ , which minimizes the mean cost given the observation  $x$  of the following cost function:

$$c(s, \hat{s}) = \frac{(s - \hat{s})^2}{s}$$

- (b) Determine the conditional mean cost given observation  $x$ ,  $\mathbb{E}\{c(S, \hat{S}_c) | X = x\}$ , for the estimator obtained in part a).

### Solution:

(a)  $\hat{S}_c = \frac{2X}{3}$

(b)  $\mathbb{E}\{c(S, \hat{S}_c) | X = x\} = \frac{x}{12}$

## 2. Bayesian Estimation: MSE and MAD

### Exercise 2 (1.3)

Obtain the minimum mean square error estimator of random variable  $S$  based on the observation of random variable  $X$ , in the following two cases:

(a)

$$p_{X,S}(x, s) = \begin{cases} 1, & 0 \leq x \leq 1, \quad 0 \leq s \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$p_{X,S}(x, s) = \begin{cases} 2, & 0 \leq s \leq 1 - x, \quad 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

#### Solution:

(a)  $\hat{S}_{\text{MMSE}} = \frac{1}{2}$

(b)  $\hat{S}_{\text{MMSE}} = \frac{1-X}{2}$

### Exercise 3 (ML)

Let  $X$  be a measurement of the instantaneous voltage at a circuit node. In such node exists a signal component with value  $S$ , contaminated by Gaussian additive noise with mean zero and variance  $v$ , which is independent of  $S$ . A priori,  $S$  follows a Gaussian pdf with both the mean and variance equal to 1.

- Assuming  $v$  is known, obtain the maximum likelihood estimator of  $S$ ,  $\hat{s}_{\text{ML}}(x)$ .
- Calculate mean square error incurred by estimator  $\hat{S}_{\text{ML}}(x)$ .
- Obtain the likelihood of  $v$  given  $x$ , i.e.  $p_{X|v}(x|v)$ .
- Calculate the maximum likelihood estimator of  $v$ ,  $\hat{v}_{\text{ML}}(x)$ .

#### Solution:

- (a) According to the statement,

$$X = S + R$$

(where  $R$  is the noise).  $R$  zero-mean Gaussian with variance  $v$ , a  $S$  is unity-mean Gaussian with variance 1. Therefore,  $X|S$  is Gaussian, with mean

$$m_{X|s} = \mathbb{E}\{X|s\} = \mathbb{E}\{S + R|s\} = \mathbb{E}\{S|s\} + \mathbb{E}\{R|s\} = s + \mathbb{E}\{R\} = s.$$

and variance

$$\mathbb{E}\{(X - m_{X|s})^2|s\} = \mathbb{E}\{(X - s)^2|s\} = \mathbb{E}\{R^2|s\} = v$$

Thus

$$p_{X|S}(x|s) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-s)^2}{2v}\right)$$

and, maximizing with respect to  $s$ ,

$$\hat{s}_{\text{ML}}(x) = x$$

(b) The MSE of estimator  $s_{\text{ML}}$  is

$$\mathbb{E}\{(\hat{S}_{\text{ML}} - S)^2\} = \mathbb{E}\{(X - S)^2\} = \mathbb{E}\{(S + R - S)^2\} = \mathbb{E}\{R^2\} = v$$

(c)  $X$  is Gaussian with mean

$$\mathbb{E}\{X\} = \mathbb{E}\{S\} + \mathbb{E}\{R\} = 1$$

and variance

$$\mathbb{E}\{(X - 1)^2\} = \mathbb{E}\{(S - 1)^2\} + \mathbb{E}\{R^2\} = v + 1$$

Therefore,

$$p_X(x) = \frac{1}{\sqrt{2\pi(v+1)}} \exp\left(-\frac{(x-1)^2}{2(v+1)}\right)$$

( $v$  is a deterministic parameter of the distribution, so that  $p_X(x) \equiv p_{X|v}(x|v)$ )

(d)

$$\begin{aligned} \hat{v}_{\text{ML}}(x) &= \arg \max_v p_{X|v}(x|v) = \arg \max_v \log(p_{X|v}(x|v)) \\ &= \arg \max_v \left\{ -\frac{1}{2} \log(2\pi(v+1)) - \frac{(x-1)^2}{2(v+1)} \right\} \\ &= \arg \min_v \left\{ \log(v+1) + \frac{(x-1)^2}{v+1} \right\} \end{aligned}$$

Note that  $v$  is non-negative. The value of  $v$  with zero-derivative of the log-likelihood is

$$v = (x-1)^2 - 1$$

but it could be negative. In that case, the ML estimator is the closest non-negative value, that is, 0. Thus,

$$\hat{v}_{\text{ML}}(x) = \max[(x-1)^2 - 1, 0]$$

#### Exercise 4 (1.2; 1.3; 1.7)

Consider the estimation of a random variable  $S$  from another random variable  $X$ , where their joint probability density function (pdf) is given by:

$$p_{S,X}(s, x) = \frac{6}{7} (x+s)^2, \quad 0 \leq x, s \leq 1$$

(a) Obtain  $p_X(x)$ .

(b) Obtain  $p_{S|X}(s|x)$ .

(c) Calculate the minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}$ .

(d) Calculate the MAP estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .

(e) Indicate which is the bias and the variance of the MAP estimator.

**Solution:**

$$(a) \quad p_X(x) = \frac{2}{7} (3x^2 + 3x + 1) \quad 0 < x < 1.$$

$$(b) \quad p_{S|X}(s|x) = \frac{(x+s)^2}{x^2 + x + \frac{1}{3}} \quad 0 < s < 1.$$

$$(c) \quad \hat{S}_{\text{MSE}} = \frac{\frac{X^2}{2} + \frac{2X}{3} + \frac{1}{4}}{X^2 + X + \frac{1}{3}}.$$

$$(d) \quad \text{Given that } p_{S|X}(s|x) \text{ is strictly increasing with } s, \hat{S}_{\text{MAP}} = 1.$$

$$(e) \quad p_S(s) = \frac{2}{7} (3s^2 + 3s + 1), \quad 0 < s < 1, \text{ thus, } \mathbb{E}\{S\} = \frac{9}{14}.$$

The bias is  $-\frac{5}{14}$ , whereas the variance is 0.

### Exercise 5 (Bayesian, MSE, MAP, MAD)

The posterior distribution of random variable  $S$  given  $X$  is

$$p_{S|X}(s|x) = \frac{1}{s \ln(x)}, \quad 1 \leq s \leq x.$$

We want to estimate  $S$  based on the observation of  $X$ .

- (a) Compute the minimum mean square error estimate,  $\hat{s}_{\text{MMSE}}$ .
- (b) Compute the maximum *a posteriori* (MAP) estimate.
- (c) Compute the estimate with minimum absolute deviation (MAD).
- (d) Compute the Bayesian estimator,  $\hat{s}_{\text{B}}$ , for a cost  $c(s, \hat{s}) = s(s - \hat{s})^2$ .

#### Solution:

$$(a) \quad \hat{s}_{\text{MSE}} = \frac{x-1}{\ln(x)}$$

$$(b) \quad \hat{s}_{\text{MAP}} = 1$$

$$(c) \quad \hat{s}_{\text{MAD}} = \sqrt{x}$$

$$(d) \quad \hat{s}_{\text{B}} = \frac{1}{2}(1+x)$$

### Exercise 6 (1.3; 1.7)

Consider the estimation of a random variable  $S$  based on the observation of  $X$ , where the two variables satisfy the following joint pdf:

$$p_{S,X}(s, x) = \begin{cases} 6s, & 0 < s < x-1, \quad 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Obtain:

- (a) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (b) The maximum *a posteriori* estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- (c) The minimum mean absolute error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAD}}$ .

- (d) The bias of all previous estimators.

**Solution:**

A video resolution of this problem (in Spanish) can be found in

<http://decisionyestimacion.blogspot.com/2013/05/problema-6-estimacion.html>

(a)  $\hat{S}_{\text{MMSE}} = \frac{2}{3}(X - 1)$

(b)  $\hat{S}_{\text{MAP}} = X - 1$

(c)  $\hat{S}_{\text{MAD}} = \frac{1}{\sqrt{2}}(X - 1)$

(d)  $\mathbb{E}\{S - \hat{S}_{\text{MMSE}}\} = 0 \quad \mathbb{E}\{S - \hat{S}_{\text{MAP}}\} = -0.25 \quad \mathbb{E}\{S - \hat{S}_{\text{MAD}}\} = -0.03$

**Exercise 7 (1.2; 1.3; 1.4)**

Random variables  $S$  and  $X$  are jointly distributed according to

$$p_{S,X}(s, x) = \alpha s x^2, \quad 0 < s < 1 - x, \quad 0 < x < 1$$

$\alpha$  being a parameter that needs to be determined.

- Stablish the expressions for the marginal probability density functions  $p_X(x)$  and  $p_S(s)$ .
- Obtain the MAP estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}(X)$ .
- Obtain the ML estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}(X)$ .
- Obtain the minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}(X)$ .
- Compare the previous estimators according to the mean square errors given  $X$  in which they incur.

**Solution:**

(a)  $p_X(x) = 30x^2(1 - x)^2, \quad 0 < x < 1,$

$$p_S(s) = 20s(1 - s)^3, \quad 0 < s < 1,$$

(b)  $\hat{S}_{\text{MAP}}(X) = 1 - X$

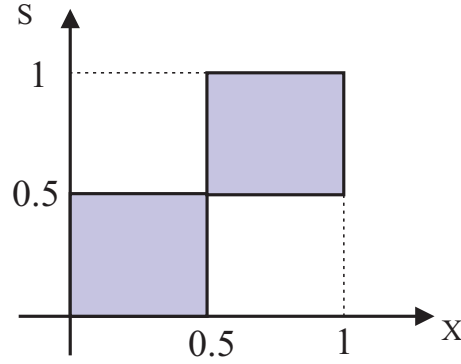
(c)  $\hat{S}_{\text{ML}}(X) = 1 - X$

(d)  $\hat{S}_{\text{MSE}}(X) = \frac{2}{3}(1 - X)$

(e)  $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MAP}}(X)\right)^2 \mid x\right\} = \mathbb{E}\left\{\left(S - \hat{S}_{\text{ML}}(X)\right)^2 \mid x\right\} = \frac{1}{6}(1 - x)^2$   
 $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2 \mid x\right\} = \frac{1}{18}(1 - x)^2$

**Exercise 8 (1.3)**

In the plot below, the shaded region shows the domain of a joint distribution of  $S$  and  $X$ , i.e., the set of points for which  $p_{X,S}(x, s) \neq 0$ .



Please, provide justified answers to the following questions:

- If it is known that  $p_{X,S}(x, s)$  is constant in its domain, which is the MSE estimator of  $S$  given  $X$ ? Provide a graphical representation of this estimator.
- Is there any  $p_{X,S}(x, s)$  with the previous domain for which the MSE estimator of  $S$  given  $X$  is  $\hat{S}_{\text{MMSE}} = X/2$ ?
- Justify if there exists any  $p_{X,S}(x, s)$  with the previous domain, so that  $\hat{S} = 0.5$  is:
  - The minimum mean square error estimator of  $S$  given  $X$ .
  - The minimum mean absolute deviation estimator of  $S$  given  $X$ .
  - The maximum *a posteriori* estimator of  $S$  given  $X$ .

**Solution:**

- $\hat{S}_{\text{MMSE}} = 0.25$  for  $0 < x < 0.5$  and  $\hat{S}_{\text{MMSE}} = 0.75$  for  $0.5 < x < 1$
- When  $0.5 < x < 1$ ,  $p_{S|X}(s|x)$  is non-zero for  $0.5 < s < 1$ , thus  $X/2$  can never be the mean of  $p_{S|X}(s|x)$  for that range of  $X$ .
- $\hat{S} = 0.5$  cannot be the mean or the median of  $p_{S|X}(s|x)$ , but it can be its maximum. Therefore,  $\hat{S} = 0.5$  can just be  $\hat{S}_{\text{MAP}}$  (but not  $\hat{S}_{\text{MMSE}}$  or  $\hat{S}_{\text{MAD}}$ ).

**Exercise 9 (MSE)**

The joint distribution of random variables  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = \sqrt{\frac{x(1-x)}{2\pi}} \exp\left(-\frac{(s-x(1-x))^2}{2x(1-x)}\right), \quad 0 < x < 1, \quad s \in \mathbb{R}$$

- Determine the minimum mean square error estimate of  $S$  based on the observation  $x$ .
- Determine the conditional mean square error given observation  $x$ ,  $\mathbb{E}\{(S - \hat{S}_{\text{MMSE}})^2 | X = x\}$ , for the estimator obtained in part a).
- Determine the mean square error of the estimator obtained in part a).

**Solution:**

- $\hat{S}_{\text{MMSE}} = X(1 - X)$
- $\mathbb{E}\{(S - \hat{S}_{\text{MMSE}})^2 | X = x\} = x(1 - x)$
- $\mathbb{E}\{(S - \hat{S}_{\text{MMSE}})^2\} = \frac{1}{30}$

**Exercise 10 (MSE, MAD)**

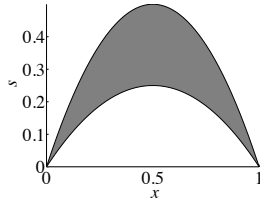
The joint pdf of two random variables  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = \begin{cases} \frac{4s}{x(1-x)}, & x(1-x) < s < 2x(1-x), \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Provide an approximate representation of the support of the joint distribution.
- Obtain the estimator  $\hat{S}_{\text{MMSE}}$ .
- Obtain the estimator  $\hat{S}_{\text{MAD}}$ .

**Solution:**

- The support of the pdf is the shaded region in the following figure:



- $\hat{S}_{\text{MMSE}} = \frac{14}{9}x(1-x).$

- $\hat{S}_{\text{MAD}} = \sqrt{\frac{5}{2}}x(1-x)$

**Exercise 11 (LMSE)**

The joint distribution of  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = \frac{4}{(x+s)^3}, \quad s \geq 1, \quad x \geq 1.$$

- Find  $p_{S|X}(s|x)$ .
- Find the MAP estimator of  $S$  given  $X$ .
- Find the minimum absolute deviation (MAD) estimator of  $S$  given  $X$ .

**Solution:**

- $p_{S|X}(s|x) = \frac{2(x+1)^2}{(x+s)^3}, \quad x \geq 1.$

- $\hat{S}_{\text{MAP}} = 1.$

- $\hat{S}_{\text{MAD}} = (\sqrt{2} - 1)X + \sqrt{2}.$

**Exercise 12 (1.2; 1.3; 1.7)**

The joint p.d.f. of two random variables  $S$  and  $X$  is:

$$p_{S,X}(s, x) = \begin{cases} 6s, & 0 < s < x \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find:

- (a) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (b) The bias of such estimator.

**Solution:**

- (a)  $\hat{S}_{\text{MMSE}} = \frac{2}{3}X$ .
- (b) The estimator is unbiased.

**Exercise 13 (1.3; 1.7)**

Consider three independent random variables,  $S_1$ ,  $S_2$  and  $S_3$ , with the same *a priori* pdf. We wish to estimate  $S_1$  from a single observation of  $X = S_1 + S_2 + S_3$ .

- (a) Justify briefly why the minimum mean square error estimator of  $S_1$  given  $x$  is  $\hat{s}_1 = \frac{x}{3}$ .
- (b) Find the bias of the estimator given in (a). Is it biased or unbiased?

Assuming in the following that  $S_1$ ,  $S_2$  and  $S_3$  follow a uniform distribution between  $-1$  and  $1$ :

- (c) Obtain the pdf of  $X$  given  $s_1$ .
- (d) Find the variance of estimator  $\hat{S}_1$ .

**Solution:**

- (a) Using symmetry arguments, all  $\mathbb{E}\{S_i|x\}$  should be the same and sum up to  $x$ .
- (b) The estimator is unbiased.
- (c)  $p_{X|S_1}(x|s_1) = 1/2 - |x - s_1|/4$  for  $-2 < x < 2$ , and 0 otherwise.
- (d)  $\text{Var}(S_1) = 1/9$

**Exercise 14 (1.7)**

Let  $S$  and  $X$  be two unidimensional random variables. Variable  $X$  is characterized by the following probability density function:

$$p_X(x) = \begin{cases} \frac{1}{2}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

It is known that the minimum mean square error estimator of  $S$  based on  $X$  is

$$\hat{S}_{\text{MMSE}}(X) = -\frac{1}{2}\text{sign}(X) = \begin{cases} -\frac{1}{2}, & X \geq 0 \\ \frac{1}{2}, & X < 0 \end{cases}$$

It is also known that the mean square error associated to this estimator is  $\frac{1}{12}$ . An alternative estimator of  $S$  is defined as

$$\hat{S}_1 = -X$$

- (a) Obtain the bias of estimator  $\hat{S}_1$ .
- (b) Calculate the following mathematical expectations:  $\mathbb{E}\{SX\}$  and  $\mathbb{E}\{S^2\}$ .
- (c) Find the mean square error incurred by  $\hat{S}_1$ .



**Solution:**

- (a) The estimator is unbiased.
- (b)  $\mathbb{E}\{SX\} = -\frac{1}{4}$  and  $\mathbb{E}\{S^2\} = \frac{1}{3}$
- (c) The MSE of  $\hat{S}_1$  is  $\frac{1}{6}$ .

**Exercise 15 (1.2; 1.3; 1.4; 1.7)**

Consider the estimation of a r.v.  $S$  from another random variable  $X$ . The joint p.d.f. of the two variables is given by:

$$p_{X,S}(x, s) = \begin{cases} 6x, & 0 \leq x \leq s, \quad 0 \leq s \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Find the minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}$ .
- (b) Obtain the maximum likelihood estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}$ .
- (c) Find the probability density function of the previous estimators,  $p_{\hat{S}_{\text{MSE}}}(\hat{s})$  and  $p_{\hat{S}_{\text{ML}}}(\hat{s})$ , and provide a plot of them.
- (d) Find the mean and the variance of the error of both estimators.

**Solution:**

- (a)  $\hat{S}_{\text{MSE}}(X) = \frac{1}{2}(1 + X)$
- (b)  $\hat{S}_{\text{ML}}(X) = X$
- (c)  $p_{\hat{S}_{\text{MSE}}}(\hat{s}) = 24(2\hat{s} - 1)(1 - \hat{s}), \quad \frac{1}{2} \leq \hat{s} \leq 1$
- $$p_{\hat{S}_{\text{ML}}}(\hat{s}) = 6\hat{s}(1 - \hat{s}), \quad 0 \leq \hat{s} \leq 1$$
- (d)  $\mathbb{E}\{S - \hat{S}_{\text{ML}}\} = \frac{1}{4}, \quad \mathbb{E}\{S - \hat{S}_{\text{MSE}}\} = 0$
- $$\text{Var}\{S - \hat{S}_{\text{ML}}\} = \frac{13}{80}, \quad \text{Var}\{S - \hat{S}_{\text{MSE}}\} = \frac{1}{40}$$

### 3. Transformations of random variables

**Exercise 16 (1.2; 1.3; 1.4; 1.7)**

We want to estimate the value of a positive random variable  $S$  using a random observation  $X$ , which is related with  $S$  via

$$X = R/S$$

$R$  being a r.v. independent of  $S$  with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

- (a) Obtain the likelihood of  $S$ ,  $p_{X|S}(x|s)$ .
- (b) Find the maximum likelihood estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}$ .

Knowing also that the p.d.f. of  $S$  is  $p_S(s) = \exp(-s)$ ,  $s > 0$ , obtain:

- (c) The joint p.d.f. of  $S$  and  $X$ ,  $p_{S,X}(s, x)$ , and the *a posteriori* distribution of  $S$ ,  $p_{S|X}(s|x)$ .
- (d) The maximum *a posteriori* estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- (e) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}$ .
- (f) The bias of estimators  $\hat{S}_{\text{MAP}}$  and  $\hat{S}_{\text{MSE}}$ .

**Solution:**

(a)  $p_{X|S}(x|s) = s \exp(-xs), \quad x > 0.$

(b)  $\hat{S}_{\text{ML}} = \frac{1}{X}.$

(c)  $p_{X,S}(x, s) = s \exp(-s(x+1)), \quad x, s > 0;$

$$p_{S|X} = (x+1)^2 s \exp(-s(x+1)), \quad s > 0.$$

(d)  $\hat{S}_{\text{MAP}} = \frac{1}{X+1}.$

(e)  $\hat{S}_{\text{MSE}} = \frac{2}{X+1}.$

(f)  $\mathbb{E}\{S - \hat{S}_{\text{MAP}}\} = \frac{1}{2}; \quad \mathbb{E}\{S - \hat{S}_{\text{MSE}}\} = 0.$

**Exercise 17 (1.3; 1.4; 1.7)**

Consider an estimation problem where the goal is to estimate a random variable  $S$  using an observation of another random variable  $X$  characterized by:

$$X = S + N$$

where the prior pdf of  $S$  is

$$p_S(s) = s \exp(-s) \quad s > 0$$

and where  $N$  is an additive noise, independent of  $S$ , with the following distribution

$$p_N(n) = \exp(-n) \quad n > 0$$

Find:

- (a) The maximum likelihood estimator of  $S$ ,  $\hat{S}_{\text{ML}}$ .
- (b) The joint pdf of  $X$  and  $S$ ,  $p_{X,S}(x, s)$ , and the posterior pdf of  $S$  given  $X$ ,  $p_{S|X}(s|x)$ .
- (c) The maximum a posteriori estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- (d) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (e) The bias of all previous estimators,  $\hat{S}_{\text{ML}}$ ,  $\hat{S}_{\text{MAP}}$  and  $\hat{S}_{\text{MMSE}}$ .
- (f) Which of the previous estimators has a minimum variance? Justify your answer without calculating the variances of the estimators.

Hint: You can use the following expression to solve the exercise:

$$\int_0^\infty x^N \exp(-x) dx = N!$$

**Solution:**

- (a)  $\hat{S}_{\text{ML}} = X$
- (b)  $p_{X,S}(x, s) = s \exp(-x) \quad x > s, s > 0$   
 $p_{S|X}(s|x) = \frac{2s}{x^2} \quad 0 < s < x, x > 0$
- (c)  $\hat{S}_{\text{MAP}} = X$
- (d)  $\hat{S}_{\text{MMSE}} = \frac{2}{3}X$
- (e)  $\mathbb{E}\{S - \hat{S}_{\text{ML}}\} = \mathbb{E}\{S - \hat{S}_{\text{MAP}}\} = -1$   
 $\mathbb{E}\{S - \hat{S}_{\text{MMSE}}\} = 0$   
 $\text{Var}\{\hat{S}_{\text{MMSE}}\} < \text{Var}\{\hat{S}_{\text{MAP}}\} = \text{Var}\{\hat{S}_{\text{ML}}\}$

**Exercise 18 (ML)**

Random variable  $X$  is driven by the likelihood function

$$p_{X|S}(x|s) = \ln\left(\frac{1}{s}\right) s^x, \quad x \geq 0, \quad 0 \leq s \leq 1$$

where  $s$  is an unknown parameter.

- (a) Compute the ML estimate of  $s$  given  $x$ .
- (b) Compute the value of the maximum likelihood for  $x = 1$ .
- (c) Compute the ML estimate of  $s$  given  $K$  a set  $\mathcal{C} = \{(x^{(k)}), k = 1, \dots, K\}$  de  $K$  of independent realizations of  $X$ .

**Solution:**

- (a)  $\hat{S}_{\text{ML}} = \exp\left(-\frac{1}{X}\right)$
- (b)  $p_{X|S}(1|s_{\text{ML}}) = \frac{1}{e}$
- (c)  $\hat{S}_{\text{ML}} = \exp\left(-\frac{K}{\sum_{k=1}^K x^{(k)}}\right)$

**Exercise 19 (ML, MAP, MSE)**

We want to estimate the value of a random variable  $S$  using a random observation  $X$ , which is related to  $S$  via

$$X = S - R$$

where  $R$  is a r.v. independent of  $S$  with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

- (a) Find the maximum likelihood estimator of  $S$  given  $X, \hat{S}_{\text{ML}}$ .
- Knowing also that the p.d.f. of  $S$  is  $p_S(s) = \exp(-s), s > 0$ , obtain:
- (c) The maximum *a posteriori* estimator of  $S$  given  $X, \hat{S}_{\text{MAP}}$ .

- (d) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .

**Solution:**

(a)  $p_{X|S}(x|s) = \exp(x - s), \quad x < s.$

$$\hat{S}_{\text{ML}} = X.$$

(b)  $p_{X,S}(x, s) = \exp(-x - 2s), \quad s > 0; x < s;$

$$\hat{S}_{\text{MAP}} = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$$

(c)  $p_{S|X} = \begin{cases} 2\exp(-2s), & s > 0 & \text{if } x < 0 \\ 2\exp(2x)\exp(-2s), & s > x & \text{if } x > 0 \end{cases}$

$$\hat{S}_{\text{MMSE}} = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ \frac{1}{2} + x & \text{if } x > 0 \end{cases}$$

## 4. Bayesian estimation with constraints

### Exercise 20 (1.2; 1.7)

Random variables  $S$  and  $X$  have a joint probability density function given by

$$p_{S,X}(s, x) = \begin{cases} 10s, & 0 < s < x^2 \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider the estimation of  $S$  based on the observation of  $X$ , with the objective to minimize the following cost function:

$$c(S, \hat{S}) = S^2 (S - \hat{S})^2$$

Find:

- The optimal estimator of  $S$ ,  $\hat{S}_C$ , which minimizes the mean cost given  $X$ ,  $\mathbb{E}\{c(S, \hat{S})|x\}$ .
- The analytically constrained estimator  $\hat{S}_L = wX$  which minimizes the mean cost  $\mathbb{E}\{c(S, \hat{S})\}$ .
- The mean cost of both estimators:  $\mathbb{E}\{c(S, \hat{S}_C)\}$  and  $\mathbb{E}\{c(S, \hat{S}_L)\}$ .
- The bias of both estimators:  $\mathbb{E}\{S - \hat{S}_C\}$  and  $\mathbb{E}\{S - \hat{S}_L\}$ .
- The variance of both estimators:  $\text{Var}\{S - \hat{S}_C\}$  and  $\text{Var}\{S - \hat{S}_L\}$ .

**Solution:**

(a)  $\hat{S}_C = \frac{4}{5}X^2$

(b)  $\hat{S}_L = \frac{11}{15}X$

(c)  $\mathbb{E}\{c(S, \hat{S}_C)\} = \frac{1}{195}$  and  $\mathbb{E}\{c(S, \hat{S}_L)\} = \frac{7}{1170}$

(d)  $\mathbb{E}\{S - \hat{S}_C\} = -\frac{2}{21}$  and  $\mathbb{E}\{S - \hat{S}_L\} = -0.1349$

(e)  $\text{Var}\{S - \hat{S}_C\} = 0.03163$  and  $\text{Var}\{S - \hat{S}_L\} = 0.0326$

**Exercise 21 (1.2)**

Consider the family of cost functions given by

$$c(S, \hat{S}) = \frac{1}{N+1} \hat{S}^{N+1} + \frac{1}{N(N+1)} S^{N+1} - \frac{1}{N} S \hat{S}^N$$

where  $N$  is a non-negative and odd integer.

(a) Assuming that

$$p_{S,X}(s, x) = \frac{1}{\lambda x} \exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right) u(s)u(x) \quad \lambda > 0$$

find the minimum cost estimator of  $S$  given  $X$ .

(b) Obtain the minimum mean cost.

(c) Determine the coefficient  $w$  that minimizes the mean cost of an estimator with analytical shape

$$\hat{S}_L = wX^m$$

$m$  being a positive integer.

Hint:  $\int_0^\infty x^N \exp(-x) dx = N!$

**Solution:**

(a)  $\hat{S} = X$

(b)  $\mathbb{E}\{c(S, \hat{S})\} = ((N+1)! - 1)(N-1)!\lambda^{N+1}$

(c)  $w = \frac{(Nm+1)!}{(Nm+m)!\lambda^{m-1}}$

**Exercise 22 (1.2; 1.3)**

The joint p.d.f. of two random variables  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = \alpha, \quad -1 < x < 1, \quad 0 < s < |x|$$

(a) Obtain the marginal p.d.f. of  $X$ ,  $p_X(x)$ , specifying the value of  $\alpha$ .

(b) Find the estimators of  $S$  based on variable  $X$  that minimize the mean square error (MSE),  $(\bar{C}_{\text{MSE}} = \mathbb{E}\{(S - \hat{S})^2\})$  and mean absolute deviation (MAD)  $(\bar{C}_{\text{MAD}} = \mathbb{E}\{|S - \hat{S}|\})$ ,  $\hat{S}_{\text{MMSE}}$  and  $\hat{S}_{\text{MAD}}$ , respectively.

(c) If the estimators analytical shape is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to previously considered costs: MSE and MAD, i.e.,  $\hat{S}_{q,\text{MMSE}} = w_1 X^2$  and  $\hat{S}_{q,\text{MAD}} = w_2 X^2$ , respectively.

**Solution:**

(a)  $p_X(x) = |x|, \quad -1 < x < 1$

(b)  $\hat{S}_{\text{MMSE}}(X) = \hat{S}_{\text{MAD}}(X) = |X|/2$

(c)  $\hat{S}_{q,\text{MMSE}}(X) = 3X^2/5$ ;  $\hat{S}_{q,\text{MAD}}(X) = 5X^2/8$

#### 4.1. Minimum MSE estimation with linear constraints (in the parameters)

##### Exercise 23 (1.2;1.3)

We wish to estimate a random variable  $S$  based on the observation of  $X$ . Their joint pdf is given by:

$$p_{S,X}(s, x) = 24xs, \quad 0 \leq s \leq 1 - x, \quad 0 < x < 1$$

Find:

- The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- The MAP estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- The MAD estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAD}}$ .
- The estimator with analytical shape  $\hat{S}_{\text{q,MSE}} = wX^2$  that minimizes the mean square error.

**Solution:**

(a)  $\hat{S}_{\text{MMSE}} = \frac{2}{3}(1 - X)$ .

(b) Since  $p_{S|X}(s|x)$  is strictly increasing with respect to  $s$ ,  $\hat{S}_{\text{MAP}} = 1 - X$ .

(c)  $\hat{S}_{\text{MAD}} = \frac{1 - X}{\sqrt{2}}$ .

(d)  $w = 0.8$

##### Exercise 24 (1.6)

We wish to design a linear minimum mean square error estimator for the estimation of random variable  $S$  based on the observation of random variables  $X_1$  and  $X_2$ . It is known that:

$$\begin{aligned} \mathbb{E}\{S\} &= \frac{1}{2} & \mathbb{E}\{X_1\} &= 1 & \mathbb{E}\{X_2\} &= 0 \\ \mathbb{E}\{SX_1\} &= 1 & \mathbb{E}\{SX_2\} &= 2 & \mathbb{E}\{X_1X_2\} &= \frac{1}{2} \\ \mathbb{E}\{S^2\} &= 4 & \mathbb{E}\{X_1^2\} &= \frac{3}{2} & \mathbb{E}\{X_2^2\} &= 2 \end{aligned}$$

Obtain the weights of estimator  $\hat{S}_{\text{LMSE}} = w_0 + w_1X_1 + w_2X_2$ , and calculate its mean square error  $\mathbb{E}\{(S - \hat{S}_{\text{LMSE}})^2\}$ .

**Solution:** A video resolution of this problem (in Spanish) can be found in

<http://decisionyestimacion.blogspot.com/2013/05/p1-estimacion.html>

$$w_0 = \frac{1}{2} \quad w_1 = 0 \quad w_2 = 1$$

$$\mathbb{E}\{(S - \hat{S}_{\text{LMSE}})^2\} = \frac{7}{4}$$

##### Exercise 25 (ML, MAP, MSE)

We want to estimate the value of a random variable  $S$  using a random observation  $X$ , from which the joint probability distribution is known

$$p_{S,X}(s, x) = 4x, \quad 0 < s < x^2, \quad 0 < x < 1$$

Obtain:

- (a) The maximum likelihood estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}$ .
- (b) The maximum *a posteriori* estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- (c) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (d) The linear estimator of  $S$ , with minimum mean square error, given  $X$ ,  $\hat{S}_{\text{LMMSE}} = w_0 + w_1 x$ .

**Solution:**

- (a)  $\hat{S}_{\text{ML}} = X^2$ .
- (b)  $\hat{S}_{\text{MAP}}$  is given by any value of  $S \in (0, X^2)$
- (c)  $\hat{S}_{\text{MMSE}} = \frac{X^2}{2}$
- (d)  $\hat{S}_{\text{LMMSE}} = X - \frac{7}{15}$

### Exercise 26 (LMSE)

The joint pdf of two random variables  $S$  and  $X$  is given by:

$$p_{S,X}(s, x) = \frac{1}{3}(x + s) \quad 0 < x < 2, \quad 0 < s < 1.$$

- (a) Obtain the Linear Minimum Mean Square Error estimator,  $\hat{S}_{\text{LMSE}} = w_0 + w_1 X$ .
- (b) Calculate the mean square error of the estimator.

**Solution:**

- (a)  $\hat{S}_{\text{LMSE}} = \frac{14}{23} - \frac{1}{23}x$ .
- (b)  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{LMSE}} \right)^2 \right\} = \frac{11}{138}$

### Exercise 27 (1.6)

The joint p.d.f. of random variables  $X$  and  $S$  is given by

$$p_{X,S}(x, s) = \begin{cases} x + s & 0 \leq x \leq 1 \text{ and } 0 \leq s \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the linear minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{LMSE}} = w_0 + w_1 X$ .

**Solution:**  $\hat{S}_{\text{LMSE}} = \frac{7}{11} - \frac{X}{11}$

### Exercise 28 (ML, LMSE)

We wish to estimate random variable  $S$  from random variable  $X$ . They are related as

$$X = S \cdot T$$

where  $S$  and  $T$  are independent random variables, both uniformly distributed between 0 and 1.

- (a) Obtain the mean and the variance of  $S$ .
- (b) Obtain the maximum likelihood estimator of  $S$  as a function of  $X$ ,  $\hat{S}_{\text{ML}}$ .
- (c) Plot the support of the joint distribution of  $S$  and  $X$ . Calculate also the joint probability density function of  $S$  and  $X$ ,  $p_{S,X}(s, x)$ .
- (d) Obtain the mean of  $X$  and its mean quadratic value. (Hint: It may be convenient not to compute  $p_X(x)$  as an intermediate result).
- (e) Design the linear minimum mean square error estimator,  $\hat{S}_{\text{LMSE}} = w_0^* + w^* X$ .
- (f) Plot the estimators that have been designed in this problem on top of the same coordinate axis  $X$ - $S$ , and discuss which of them incurs in the smallest mean square error.

**Solution:**

(a)  $\mathbb{E}\{S\} = \frac{1}{2}$  y  $v_x = \frac{1}{12}$

(b)  $\hat{S}_{\text{ML}} = X$

(c)  $p_{S,X}(s, x) = \frac{1}{s}$  en  $0 < x < s < 1$

(d)  $\mathbb{E}\{X\} = \frac{1}{4}$  y  $\mathbb{E}\{X^2\} = \frac{1}{9}$

(e)  $w_0^* = \frac{83}{168}$  y  $w^* = \frac{1}{42}$

(f) Dado que los tres son lineales, necesariamente  $\hat{S}_{\text{LMSE}}$  es el de menor error cuadrático medio.

**Exercise 29 (LMSE)**

An electric company owns two wind farms. The total generated power can be modeled as

$$S = 10(2X_1 + X_2),$$

where  $S$  is the generated power, and  $X_i$ ,  $i = 1, 2$ , represents wind speed at each farm. It is further known that the joint distribution of  $X_1$  and  $X_2$  is given by

$$p_{X_1, X_2}(x_1, x_2) = \frac{1}{a b} \exp\left(-\frac{x_1}{a}\right), \quad \text{for } 0 < x_1 < \infty \quad \text{and} \quad x_1 < x_2 < x_1 + b \quad (1)$$

with  $b = a \ln 2$ .

We wish to design a linear minimum mean square error estimator of  $S$  based just on observation  $X_1$ . In order to do so:

- (a) Sketch the support of the joint distribution of  $X_1$  and  $X_2$ .
- (b) Obtain the marginal distribution of  $X_1$ . Calculate the mean and mean square value of such variable.
- (c) Find the mean of  $X_2$  and the correlation between  $X_1$  and  $X_2$ :  $\mathbb{E}\{X_1 X_2\}$ .
- (d) Find the optimum weights of the desired LMSE estimator.  $\hat{S}_{\text{LMSE}} = w_0^* + w_1^* X_1$

**Solution:**

(b)  $p_{X_1}(x_1) = \frac{1}{a} \exp\left(-\frac{x_1}{a}\right)$ , for  $x_1 > 0$

$\mathbb{E}\{X_1\} = a$  y  $\mathbb{E}\{X_1^2\} = 2a^2$

(c)  $\mathbb{E}\{X_2\} = a + \frac{b}{2}$

$\mathbb{E}\{X_1 X_2\} = 2a^2 + \frac{ba}{2}$



**Exercise 30 (MSE, MAP, MSE)**

The joint distribution of random variables  $S$  and  $X$  is given by

$$p_{S,X}(s, x) = 15s, \quad 0 \leq x \leq 1, \quad x^2 \leq s \leq x$$

We want to estimate  $S$  after observing  $X$ .

- Compute the minimum mean square error estimate.
- Compute the maximum a posteriori estimate.
- Compute the minimum mean square error estimate in the form  $\hat{S} = wX$

**Solution:**

$$(a) \hat{s}_{\text{MSE}} = \frac{2}{3} \frac{x - x^4}{1 - x^2}$$

$$(b) \hat{s}_{\text{MAP}} = x$$

$$(c) w = \frac{7}{8}$$

**Exercise 31 (1.3; 1.6)**

Let  $X$  and  $S$  be two random variables with joint pdf

$$p_{X,S}(x, s) \begin{cases} \alpha & ; \quad 0 < x < 1, \quad 0 < s < 2(1-x) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

with  $\alpha$  a constant.

- Plot the support of the pdf, and use it to determine the value of  $\alpha$ .
- Obtain the posterior pdf of  $S$  given  $X$ ,  $p_{S|X}(s|x)$ .
- Find the minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}$
- Find the linear minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{LMSE}}$ .

**Solution:**

$$(a) \alpha = 1$$

$$(b) p_{S|X}(s|x) = \frac{1}{2(1-x)}$$

$$(c) \hat{S}_{\text{MSE}} = 1 - X$$

$$(d) \hat{S}_{\text{LMSE}} = 1 - X$$

**Exercise 32 (Change of variable, LMSE)**

Two random iid variables  $X$  and  $Y$  follow a uniform pdf between 0 and 1. Two new random variables are generated as  $U = \max(X, Y)$  and  $V = \min(X, Y)$ , i.e., they are defined as the maximum and minimum, respectively, of the original uniform iid variables.

- Compute  $p_{U|X}(u|x)$  and  $p_{V|X}(v|x)$ .
- Compute  $p_U(u)$  and  $p_V(v)$ .
- Compute  $\mathbb{E}[U]$ ,  $\mathbb{E}[U^2]$  and  $\mathbb{E}[V]$ .
- Compute the LMMSE estimator of  $V$  given  $U$ ,  $\hat{v}_{\text{LMMSE}}(u)$ .

Hint: In case you didn't notice, in part (a) you are performing a change of variable. You might find it helpful to draw  $\min(x, y)$  and  $\max(x, y)$  as functions of  $y$  for a fixed value of  $x$ . The flat parts of these two functions will produce Dirac deltas in the corresponding results.

**Solution:**

- (a)  $p_{U|X}(u|x) = x\delta(u - x) + 1$  for  $x \leq u \leq 1$ , and 0 otherwise  
 $p_{V|X}(v|x) = (1 - x)\delta(v - x) + 1$  for  $0 \leq v \leq x$ , and 0 otherwise
- (b)  $p_U(u) = 2u$  for  $0 \leq u \leq 1$ , and 0 otherwise  
 $p_V(v) = 2(1 - v)$  for  $0 \leq v \leq 1$ , and 0 otherwise
- (c)  $\mathbb{E}[U] = 2/3$ ,  $\mathbb{E}[U^2] = 1/2$  and  $\mathbb{E}[V] = 1/3$
- (d)  $\hat{v}_{\text{LMMSE}}(u) = u/2$ .

**Exercise 33 (1.2; 1.3; 1.6)**

Random variables  $S$  and  $X$  are characterized by the following joint distribution:

$$p_{S,X}(s, x) = c, \quad 0 < s < 1, \quad s < x < 2s$$

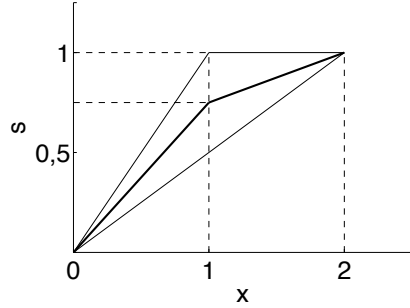
with  $c$  a constant.

- (a) Plot the support of the p.d.f., and use it to calculate the value of  $c$ .
- (b) Give the expressions for the marginal p.d.f. of the random variables:  $p_S(s)$  and  $p_X(x)$ .
- (c) Find the minimum mean square error estimator of  $S$  based on the observation of  $X$ ,  $\hat{S}_{\text{MSE}}(X)$ . Plot the estimator on the same plot as the support of  $p_{S,X}(s, x)$ , and discuss whether it would have been possible to obtain the estimator without analytical derivations.
- (d) Calculate the mean square error  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{MSE}}(X) \right)^2 \right\}$  incurred by the previous estimator.
- (e) Now, find the linear minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{LMSE}}(X)$ . Again, plot the estimator together with the support of  $p_{S,X}(s, x)$ . Discuss your result.
- (f) Obtain the mean square error  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{LMSE}}(X) \right)^2 \right\}$  of the linear estimator, and compare it with  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{MSE}}(X) \right)^2 \right\}$ .
- (g) It is perceived (e.g., visualizing several samples of  $(X, S)$ ) that there exist different statistical behaviors for  $0 < X < 1$  and  $1 < X < 2$ . What would occur if, based on this, different optimal linear estimators were designed for each of the intervals ( $\hat{S}_{A, \text{LMSE}}(X)$  y  $\hat{S}_{B, \text{LMSE}}(X)$ , respectively)? Verify analytically the proposed solution.

**Solution:**

- (a) Since the area of the support of  $p_{S,X}(s, x)$  is  $1/2$ ,  $c = 2$ .
- (b)  $p_S(s) = 2s$ ,  $0 < s < 1$ ;  $p_X(x) = \begin{cases} x & , 0 < x < 1 \\ 2 - x & , 1 < x < 2 \end{cases}$
- (c)

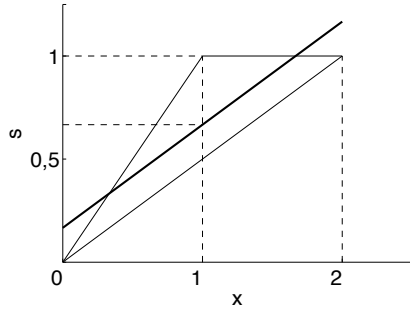
$$\hat{S}_{\text{MSE}}(X) = \begin{cases} \frac{3X}{4}, & 0 < X < 1 \\ \frac{1}{2} \left( \frac{X}{2} + 1 \right), & 1 < X < 2 \end{cases}$$



Since for every value  $X$  we have a uniform *a posteriori* distribution  $p_{S|X}(s|x)$ , the MSE estimator is given as the average between the minimum and maximum values of  $S$  (for each  $X$ ).

(d)  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{MSE}}(X) \right)^2 \right\} = \frac{1}{96}$

(e)  $\hat{S}_{\text{LMSE}}(X) = \frac{X}{2} + \frac{1}{6}$



(f)  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{LMSE}}(X) \right)^2 \right\} = \frac{11}{24}$ , which is larger than  $\mathbb{E} \left\{ \left( S - \hat{S}_{\text{MSE}}(X) \right)^2 \right\}$

(g)  $\hat{S}_{A,\text{LMSE}}(X) = \frac{3X}{4}$  and  $\hat{S}_{B,\text{LMSE}}(X) = \frac{1}{2} \left( \frac{X}{2} + 1 \right)$ . When jointly considered, these estimators compose  $\hat{S}_{\text{MSE}}(X)$ .  
 $p_A(s, x)$  and  $p_B(s, x)$  are uniform, and now the linear estimators will also be optimal.

### Exercise 34 (1.6)

Random vector  $X = [X_1, X_2, X_3]^T$  follows a p.d.f. with mean  $\mathbf{m} = \mathbf{0}$  and covariance matrix

$$V_{XX} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) Obtain the coefficients ( $w_0$ ,  $w_1$  and  $w_2$ ) of the linear minimum mean square error estimator of  $X_3$  given  $X_1$  and  $X_2$ ,

$$\hat{X}_{3,\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

- (b) Calculate the mean square error of the estimator  $\mathbb{E} \left\{ \left( X_3 - \hat{X}_{3,\text{LMSE}} \right)^2 \right\}$ .

**Solution:**

$$(a) \hat{X}_{3,\text{LMSE}} = -\frac{1}{5}X_1 + \frac{4}{5}X_2$$

$$(b) \mathbb{E} \left\{ \left( X_3 - \hat{X}_{3,\text{LMSE}} \right)^2 \right\} = \frac{8}{5}$$

## 5. Estimation with the Gaussian distribution

### Exercise 35 (1.5)

We have access to the two following observations for estimating a random variable  $S$ :

$$X_1 = S + N_1$$

$$X_2 = \alpha S + N_2$$

where  $\alpha$  is a known constant, and  $S$ ,  $N_1$ , and  $N_2$  are independent Gaussian random variables, with zero mean and variances  $v_s$ ,  $v_n$ , and  $v_n$ , respectively.

- Obtain the minimum mean square error estimator of  $S$  given  $X_1$  and  $X_2$ ,  $\hat{S}_1$  and  $\hat{S}_2$ , respectively.
- Calculate the mean square error of each of the estimators from the previous section. Which of the two provides a smaller MSE? Justify your answer for the different values of parameter  $\alpha$ .
- Obtain the minimum mean square error estimator of  $S$  based on the joint observation of variables  $X_1$  and  $X_2$ , i.e., as a function of the observation vector  $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ ,  $\hat{S}_{\text{MMSE}}$ .

**Solution:**

$$(a) \hat{S}_1 = \frac{v_s}{v_s + v_n} X_1 \text{ and } \hat{S}_2 = \frac{\alpha v_s}{\alpha^2 v_s + v_n} X_2.$$

$$(b) \mathbb{E}\{E_1^2\} = \frac{v_s v_n}{v_s + v_n} \text{ and } \mathbb{E}\{E_2^2\} = \frac{v_s v_n}{\alpha^2 v_s + v_n}. \text{ For } |\alpha| > 1 \text{ the mean square error of } \hat{S}_2 \text{ is smaller than that of } \hat{S}_1.$$

$$(c) \hat{S}_{\text{MMSE}} = \left[ \frac{1}{1 + \alpha^2 + v_n/v_s}, \frac{\alpha}{1 + \alpha^2 + v_n/v_s} \right] \mathbf{X}$$

### Exercise 36 (1.5; 1.7)

We wish to estimate a random variable  $S$  from the observation of another random variable  $X$  given by:

$$X = S + N_1 + N_2$$

where  $S$  is Gaussian-distributed, with mean and variance  $m_s$  and  $v_s$ , respectively, and where  $N_1$  and  $N_2$  are two noise random variables, independent of  $S$ , and with joint p.d.f.

$$p_{N_1, N_2}(n_1, n_2) \sim G \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{bmatrix} \right)$$

Obtain:

- (a) The minimum mean square error estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (b) The maximum a posteriori estimator of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .
- (c) The bias and variance of both estimators.

**Solution:**

$$(a) \hat{S}_{\text{MMSE}} = \frac{v_s}{v_s + v_1 + v_2 + 2v_{12}} X + m_s \left( 1 - \frac{v_s}{v_s + v_1 + v_2 + 2v_{12}} \right)$$

$$(b) \hat{S}_{\text{MAP}} = \hat{S}_{\text{MMSE}}$$

$$(c) \mathbb{E} \{ S - \hat{S}_{\text{MMSE}} \} = \mathbb{E} \{ S - \hat{S}_{\text{MAP}} \} = 0$$

$$\text{Var} \{ \hat{S}_{\text{MMSE}} \} = \text{Var} \{ \hat{S}_{\text{MAP}} \} = \frac{v_s^2}{v_s + v_1 + v_2 + 2v_{12}}$$

**Exercise 37 (MSE, MSE)**

We have taken two measurements  $X_1$  and  $X_2$  about the value of some unknown variable  $S$ . We know that  $X_1$  and  $X_2$  are related to  $S$  by means of

$$X_1 = S + T_1$$

$$X_2 = S + T_2$$

where  $T_1$  and  $T_2$  are independent random variables (and independent from  $S$ ), with zero mean and variances 0.1 y 0.3, respectively

Also, we know that  $S$  es una Gaussian random variable with mena 4 y and variance 0.9.

- (a) Compute the MMSE estimator of  $S$  given  $X_1$ . Denote it as  $\hat{S}_1$ .
- (b) Compute the Mean Square Error for estimator  $\hat{S}_1$ .
- (c) Compute the MMSE estimator of  $S$  given  $Z = \frac{1}{2}(X_1 + X_2)$ . Denote it as  $\hat{S}_z$ .
- (d) Compute the probability of  $\hat{S}_1$  being higher than  $\hat{S}_z$ , i.e.,  $P\{\hat{S}_1 > \hat{S}_z\}$ . In case you cannot compute the solution analytically, express it by means of the function

$$F(x) = \int_{-\infty}^x \frac{1}{2\pi} \exp\left(-\frac{z^2}{2}\right) dz$$

**Solution:**

$$(a) \hat{S}_1 = 0.9X_1 + 0.4.$$

$$(b) \text{MSE} = 0.09$$

$$(c) \hat{S}_z = 0.9Z + 0.4.$$

$$(d) P\{\hat{S}_1 > \hat{S}_z\} = \frac{1}{2}.$$

**Exercise 38 (Gauss, MSE)**

Random variables  $S$  and  $X$  are defined by the equation

$$X = \mathbf{v}^\top \mathbf{R}$$

$$S = \mathbf{w}^\top \mathbf{R}$$

where  $\mathbf{v} = (v_0, v_1)^\top$  and  $\mathbf{w} = (w_0, w_1)^\top$  are known parameter vectors and  $\mathbf{R} = (R_0, R_1)^\top$  is a random vector. Random variables  $R_0$  and  $R_1$  are independent Gaussian random variables with zero mean and variance  $\sigma_R^2$ .

- (a) Compute  $\mathbb{E}\{X\}$ ,  $\mathbb{E}\{S\}$ ,  $\mathbb{E}\{X^2\}$ ,  $\mathbb{E}\{S^2\}$  and  $\mathbb{E}\{SX\}$ .
- (b) Compute the minimum MSE estimate of  $S$  given  $X$ ,  $\hat{S}_{\text{MSE}}$ .
- (c) Compute the MSE of estimate  $\hat{S}_{\text{MSE}}$ .
- (d) The minimum MSE may depend on parameter vectors  $\mathbf{v}$  and  $\mathbf{w}$ . Compute the value of  $\mathbf{v}$  minimizing the minimum MSE.
- (e) Compute the minimum MSE estimate of  $S$  given  $R_0$ ,  $\hat{S}'_{\text{MSE}}$ .

**Solution:**

(a)

$$\begin{aligned}
\mathbb{E}\{X\} &= \mathbf{v}^\top \mathbb{E}\{\mathbf{R}\} = 0 \\
\mathbb{E}\{S\} &= \mathbf{w}^\top \mathbb{E}\{\mathbf{R}\} = 0 \\
\mathbb{E}\{X^2\} &= \mathbf{v}^\top \mathbb{E}\{\mathbf{R}\mathbf{R}^\top\} \mathbf{v} = \sigma_R^2 \|\mathbf{v}\|^2 \\
\mathbb{E}\{S^2\} &= \mathbf{w}^\top \mathbb{E}\{\mathbf{R}\mathbf{R}^\top\} \mathbf{w} = \sigma_R^2 \|\mathbf{w}\|^2 \\
\mathbb{E}\{SX\} &= \mathbf{v}^\top \mathbb{E}\{\mathbf{R}\mathbf{R}^\top\} \mathbf{w} = \sigma_R^2 \mathbf{v}^\top \mathbf{w}
\end{aligned}$$

(b) Since

$$\begin{pmatrix} X \\ S \end{pmatrix} = \begin{pmatrix} \mathbf{v}^\top \\ \mathbf{w}^\top \end{pmatrix} \mathbf{R} \quad (2)$$

we can see that  $(X, S)$  is a linear function of two Gaussian random variables ( $R_0$  and  $R_1$ ). Thus,  $X$  and  $S$  are jointly Gaussian, and the minimum MSE estimate is given by

$$\hat{S}_{\text{MSE}} = m_S + \frac{V_{SX}}{V_X} (X - m_X) = \frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2} X$$

(c)

$$\begin{aligned}
MSE &= \mathbb{E}\{(S - \hat{S}_{\text{MSE}})^2\} \\
&= \mathbb{E}\left\{\left(S - \frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2} X\right)^2\right\} \\
&= \mathbb{E}\{S^2\} - 2 \frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2} \mathbb{E}\{SX\} + \left(\frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2}\right)^2 \mathbb{E}\{X^2\} \\
&= \sigma_R^2 \|\mathbf{w}\|^2 - 2 \frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2} \sigma_R^2 \mathbf{v}^\top \mathbf{w} + \left(\frac{\mathbf{v}^\top \mathbf{w}}{\|\mathbf{v}\|^2}\right)^2 \sigma_R^2 \|\mathbf{v}\|^2 \\
&= \sigma_R^2 \|\mathbf{w}\|^2 \left(1 - \frac{(\mathbf{v}^\top \mathbf{w})^2}{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2}\right)
\end{aligned}$$

(d) From the relations  $X = \mathbf{v}^\top \mathbf{R}$  and  $S = \mathbf{w}^\top \mathbf{R}$ , we can see that, if  $\mathbf{v} = \mathbf{w}$ , we have  $S = X$ . In that case the minimum MSE estimate is  $\hat{S}_{\text{MSE}} = X$  with  $MSE = 0$ . Therefore, the minimum value of the minimum MSE is 0.

(e)

$$\begin{aligned}
\hat{S}'_{\text{MSE}} &= \mathbb{E}\{S|R_0\} = \\
&= \mathbb{E}\{w_0 R_0 + w_1 R_1 | R_0\} \\
&= w_0 \mathbb{E}\{R_0 | R_0\} + w_1 \mathbb{E}\{R_1 | R_0\} \\
&= w_0 R_0
\end{aligned}$$

**Exercise 39 (1.5; 1.7)**

Consider the estimation of a random vector  $\mathbf{S}$  from a statistically related observation vector  $\mathbf{X}$ :

$$\mathbf{X} = H\mathbf{S} + \mathbf{R}$$

$H$  being a known matrix,  $\mathbf{R}$  a noise vector with distribution  $G(\mathbf{0}, v_r I)$ , and  $\mathbf{S}$  the random vector to be estimated, whose distribution is  $G(\mathbb{E}\{\mathbf{S}\}, V_s)$ . It is also known that  $\mathbf{S}$  and  $\mathbf{R}$  are independent random vectors:

- Find the ML estimator of  $\mathbf{S}$ ,  $\hat{\mathbf{S}}_{\text{ML}}$ .
- Is the ML estimator unbiased? Justify your answer.
- As it is known, the MSE estimator of  $\mathbf{S}$  is given by:

$$\hat{\mathbf{S}}_{\text{MSE}} = (H^T H + v_r V_s^{-1})^{-1} H^T \mathbf{X}$$

Obtain the bias of  $\hat{\mathbf{S}}_{\text{MSE}}$  and indicate under which conditions such bias vanishes.

**Solution:**

- $\hat{\mathbf{S}}_{\text{ML}} = (H^T H)^{-1} H^T \mathbf{X}$
- The estimator is unbiased.
- $\mathbb{E}\{\hat{\mathbf{S}}_{\text{MSE}} - \mathbf{S}\} = (H^T H + v_r V_s^{-1})^{-1} H^T H \mathbb{E}\{\mathbf{S}\} - \mathbb{E}\{\mathbf{S}\}$ . The bias goes to zero as the noise power decreases towards 0.

**Exercise 40 (MSE)**

The random variables  $S$ ,  $X_1$  and  $X_2$  are jointly Gaussian. The parameters of its joint distribution are unknown, but it is known that:

- The marginal distribution of  $X_1$  and  $X_2$  is

$$p_{X_1, X_2}(x_1, x_2) \sim G\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\right)$$

- The minimum mean square error (MSE) estimator of  $S$  based on  $X_1$  only is:

$$\hat{S}_{\text{MMSE},1} = \frac{1}{2}X_1,$$

- The minimum MSE estimator of  $S$  based on  $X_2$  only is:

$$\hat{S}_{\text{MMSE},2} = X_2.$$

We want to estimate  $S$  given a new random variable  $X_3$  given by the following linear combination of  $X_1$  and  $X_2$

$$X_3 = X_1 + X_2.$$

- Compute the mean value of  $X_3$ ,  $\mathbb{E}[X_3]$ , its variance,  $v_{X_3}$ , and its covariance with  $S$ ,  $v_{S, X_3}$ .
- Compute the minimum MSE estimator of  $S$  based on  $X_3$  only,  $\hat{S}_{\text{MMSE},3}$ .
- Given that  $\mathbb{E}[S^2] = 1$ , compute the MSE of the estimator computed in the previous section,  $\mathbb{E}[(S - \hat{S}_{\text{MMSE},3})^2]$ .

**Solution:**

- (a)  $\mathbb{E}[X_3] = 0, v_{X_3} = 3, v_{S, X_3} = 3/2.$
- (b)  $\hat{S}_{\text{MMSE},3} = 1/2 X_3.$
- (c)  $\mathbb{E}[(S - \hat{S}_{\text{MMSE},3})^2] = 1/4.$

**Exercise 41 (1.4)**

An unknown deterministic parameter  $s$ ,  $s > 0$  is measured using two different systems, which provide observations

$$X_i = A_i s + N_i, \quad i = 1, 2$$

where  $\{A_i\}, \{N_i\}$ , are independent Gaussian random vectors, with means  $\mathbb{E}\{A_i\} = 1, \mathbb{E}\{N_i\} = 0$ , and variances  $\{v_{A_i}\}, \{v_{N_i}\}$ , respectively ( $i = 1, 2$ ).

- (a) Establish the expression that defines the ML estimator of  $s$ ,  $\hat{S}_{\text{ML}}$ .
- (b) Obtain  $\hat{S}_{\text{ML}}$  for the particular case  $v_{A_i} = 0, i = 1, 2$ .
- (c) Obtain  $\hat{S}_{\text{ML}}$  for the particular case  $v_{N_i} = 0, i = 1, 2$ .

**Solution:**

- (a)  $\hat{S}_{\text{ML}} = \arg \min_s \left\{ \ln [(s^2 v_{A1} + v_{N1})(s^2 v_{A2} + v_{N2})] + \frac{(s - X_1)^2}{s^2 v_{A1} + v_{N1}} + \frac{(s - X_2)^2}{s^2 v_{A2} + v_{N2}} \right\}$
- (b)  $\hat{S}_{\text{ML}} = \frac{v_{N2} X_1 + v_{N1} X_2}{v_{N1} + v_{N2}}$
- (c)  $\hat{S}_{\text{ML}} = \frac{1}{4} \sqrt{\left( \frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}} \right)^2 + 8 \left( \frac{X_1^2}{v_{A1}} + \frac{X_2^2}{v_{A2}} \right)} - \left( \frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}} \right)$

**Exercise 42 (1.5)**

Two independent Gaussian variables  $Z_1$  and  $Z_2$  have means 2 and 1, respectively. Both variables have unit variance. We wish to estimate the difference  $S = Z_1 - Z_2$ .

- (a) Obtain  $p_S(s)$ , the MMSE estimator of  $S$ , and the mean square error of such estimator if no other information is available.
- (b) Consider now we can observe  $X = Z_1 + Z_2 = 3$ . Find  $p_{S|X}(s|x)$ , the MMSE estimator of  $S$  given  $X$ , and the mean square error of such estimator. Discuss your result in relation to your answers to the previous subsection.

**Solution:**

- (a)  $\hat{S}_{\text{MMSE}} = 1; \mathbb{E} \left\{ \left( S - \hat{S}_{\text{MMSE}} \right)^2 \right\} = 2.$
- (b)  $X$  is independent of  $S$ , thus the answers are the same as in subsection (a).

## 6. Estimation from IID samples

**Exercise 43 (ML)**

Random variable  $X$  is characterized by the following density:



$$p_X(x) = a^2 x \exp(-ax), \quad x \geq 0$$

where  $a$  is an unknown parameter.

- Obtain the expression of the ML estimator of  $a$  for a given set  $\mathcal{C} = \{x^{(k)}, k = 1, \dots, K\}$  of  $K$  independent realizations of  $X$ .
- Given the set of observations  $\mathcal{C} = \{0.2, 0.5, 0.8, 1\}$ , find the value of  $\hat{A}_{\text{ML}}$ .

**Solution:**

$$(a) \quad \hat{A}_{\text{ML}} = \frac{2K}{\sum_{k=1}^K x^{(k)}}$$

$$(b) \quad \hat{A}_{\text{ML}} = 3.2$$

**Exercise 44 (1.4; 1.7)**

An order- $N$  Erlang probability density is characterized by the following expression:

$$p_X(x) = \frac{a^N x^{N-1} \exp(-ax)}{(N-1)!} \quad x > 0, \quad a > 0$$

Assume that  $N$  is known. Considering that the mean of the distribution is given by  $m = N/a$ , obtain:

- The ML estimator of the mean using  $K$  independent observations of the variable,  $\hat{M}_{\text{ML}}$ .
- The bias of  $\hat{M}_{\text{ML}}$ .
- Is  $\hat{M}_{\text{ML}}$  variance-consistent?

**Solution:**

$$(a) \quad \hat{M}_{\text{ML}} = \frac{1}{K} \sum_{k=1}^K X^{(k)}$$

(b) The estimator is unbiased.

(c)  $\text{Var} \left\{ \hat{M}_{\text{ML}} \right\} = \frac{v_x}{K}$ ; therefore, the estimator is variance-consistent.

**Exercise 45 (1.4; 1.7)**

We have access to a collection of observations  $\{x^{(k)}, k = 1, \dots, K\}$  independently drawn from a Rayleigh distribution with parameter  $b$ , i.e.,

$$p_{X|b}(x|b) = \frac{2}{b} x \exp\left(-\frac{x^2}{b}\right), \quad x \geq 0$$

with  $b > 0$ .

- Determine the ML estimator of  $b$ ,  $\hat{b}_{\text{ML}}$ .
- If the observations are  $\{2, 0, 1, 1\}$ , what would be the most likelihood value of  $b$ ?
- Find the bias of the estimator.

**Solution:**

- (a)  $\hat{b}_{\text{ML}} = \frac{1}{K} \sum_{k=1}^K \left(x^{(k)}\right)^2$
- (b)  $\hat{b}_{\text{ML}} = 1.5$
- (c) The estimator is unbiased.

**Exercise 46 (ML)**

The joint distribution of random variables  $S$  and  $X$  is known to be:

$$p_{X|s}(x|s) = s \exp(-s \exp(-x) + x), \quad , \quad x \in \mathbb{R}.$$

- (a) Compute the ML estimate of  $S$  given  $x$ .
- (b) Compute the ML estimate of  $S$  given  $K$  independent and identically distributed observations,  $\{X^{(k)}, k = 1, \dots, K\}$ .
- (c) Let  $K = 2$  and assume that  $x^{(1)} = 0$ ,  $x^{(2)} = -\ln 2$ . Compute the value of the maximum likelihood for these observations.

**Solution:**

- (a)  $\hat{S}_{\text{ML}} = \exp(X)$ .
- (b)  $\hat{S}_{\text{ML}} = \frac{K}{\sum_{k=1}^K \exp(-X^{(k)})}$ .
- (c)  $p_{X^{(1)}, X^{(2)}|s}(0, -\ln(2)|\hat{s}_{\text{ML}}) = \frac{2 \exp(-2)}{9}$ .

**Exercise 47 (1.4; 1.7)**

We have access to a set of observations  $\{x^{(k)}, k = 1, \dots, K\}$ , independently drawn from a Pareto distribution with deterministic parameters  $\alpha$  and  $\beta$ , i.e.,

$$p_{X|\alpha, \beta}(x|\alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, \quad x \geq \beta$$

with  $\alpha > 1$  and  $\beta > 0$ .

- (a) Assuming the value of  $\beta$  is known, find the ML estimator of  $\alpha$ ,  $\hat{\alpha}_{\text{ML}}$ .
- (b) For  $K = 1$  (i.e., just one observation), find the ML estimator of  $\beta$ ,  $\hat{\beta}_{\text{ML}}$ .
- (c) For  $K = 1$ , determine the bias of  $\hat{\beta}_{\text{ML}}$ .

**Solution:**

- (a)  $\hat{\alpha}_{\text{ML}} = \frac{1}{\frac{1}{K} \sum_{k=1}^K \ln \frac{x^{(k)}}{\beta}}$
- (b)  $\hat{\beta}_{\text{ML}} = x^{(1)}$
- (c)  $\text{Sesgo} \left\{ \hat{\beta}_{\text{ML}} \right\} = -\frac{1}{\alpha - 1} \beta$

**Exercise 48 (1.4)**

Consider a random variable  $X$  with p.d.f.

$$p_X(x) = a \exp[-a(x-d)]u(x-d)$$

where  $a > 0$  and  $d$  are two parameters.

Find the maximum likelihood estimators of both parameters,  $\hat{A}_{\text{ML}}$  and  $\hat{D}_{\text{ML}}$ , as a function of  $K$  samples of  $X$  independently drawn,  $\{X^{(k)}\}_{k=1}^K$ .

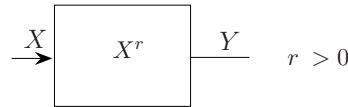
$$\textbf{Solution: } \hat{A}_{\text{ML}} = \left[ \frac{1}{K} \sum_{k=1}^K (X^{(k)} - \min\{X^{(k)}\}) \right]^{-1}, \quad \hat{D}_{\text{ML}} = \min_k \{X^{(k)}\}$$

**Exercise 49 (1.4)**

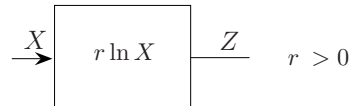
A random variable  $X$  with p.d.f.

$$p_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

is transformed as indicated in the figure, producing a random observation  $Y$ .



- Obtain the maximum likelihood estimator of  $r$ ,  $\hat{r}_{\text{ML}}$ , based on  $K$  independently drawn observations of  $Y$ .
- Now, consider the following situation



and obtain  $\hat{r}_{\text{ML}}$  using  $K$  independent observations of random variable  $Z$ . Discuss your result.

**Solution:**

- $\hat{r}_{\text{ML}} = -\frac{1}{K} \sum_{k=1}^K \ln Y^{(k)}$ . The unknown parameter of the transformation is being identified.
- $\hat{r}_{\text{ML}} = -\frac{1}{K} \sum_{k=1}^K Z^{(k)}$ . It is coherent with the previous estimator since  $Z = \ln Y$ , which is a deterministic (and invertible) transformation of  $Y$ .

**Exercise 50 (ML, MAP, MSE)**

Random variables  $S$  and  $X$  are related by the likelihood function

$$p_{X|S}(x|s) = xs^2 \exp(-sx), \quad x \geq 0, s \geq 0$$

and prior distribution

$$p_S(s) = \exp(-s), \quad s \geq 0$$

- (a) Compute the ML estimate of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}$ .  
 (b) Compute the MAP estimate of  $S$  given  $X$ ,  $\hat{S}_{\text{MAP}}$ .  
 (c) Compute the minimum MSE estimate,  $\hat{S}_{\text{MSE}}$ .  
 (d) Compute the ML estimate based on the set of i.i.d. observations  $\{x^{(0)}, x^{(1)}, \dots, x^{(K-1)}\}$ .

**Solution:**

(a)

$$\begin{aligned}\hat{s}_{\text{ML}} &= \operatorname{argmax}_s p_{X|S}(x|s) \\ &= \operatorname{argmax}_s x s^2 \exp(-sx) \\ &= \operatorname{argmax}_s \{2 \ln s - sx\} = \frac{2}{x}\end{aligned}$$

$$\begin{aligned}\hat{s}_{\text{MAP}} &= \operatorname{argmax}_s p_{X|S}(x|s) p_S(s) \\ &= \operatorname{argmax}_s x s^2 \exp(-s(1+x)) \\ &= \operatorname{argmax}_s \{2 \ln s - s(1+x)\} = \frac{2}{1+x}\end{aligned}$$

(b) Noting that

$$\begin{aligned}p_X(x) &= \int_0^\infty p_{X|S}(x|s) p_S(s) ds \\ &= \int_0^\infty x s^2 \exp(-s(1+x)) ds\end{aligned}$$

Applying the variable change  $u = (1+x)s$ , we get

$$\begin{aligned}p_X(x) &= \frac{x}{(1+x)^3} \int_0^\infty u^2 \exp(-u) du \\ &= \frac{2x}{(1+x)^3}\end{aligned}$$

Therefore

$$\begin{aligned}p_{S|X}(s|x) &= \frac{p_{X|S}(x|s) p_S(s)}{p_X(x)} \\ &= \frac{x s^2 \exp(-s(1+x))}{\frac{2x}{(1+x)^3}} \\ &= \frac{1}{2} (1+x)^3 s^2 \exp(-s(1+x))\end{aligned}$$

and, thus

$$\begin{aligned}
 \hat{s}_{\text{MMSE}} &= \mathbb{E}\{S|x\} \\
 &= \int_0^\infty \frac{1}{2}(1+x)^3 s^3 \exp(-s(1+x)) ds \\
 &= \frac{1}{2(1+x)} \int_0^\infty s^3 \exp(-u) du \\
 &= \frac{3!}{2(1+x)} = \frac{3}{1+x}
 \end{aligned}$$

(c)

$$\begin{aligned}
 \hat{s}_{\text{ML}} &= \underset{s}{\operatorname{argmax}} \prod_{k=0}^{K-1} p_{X|S}(x^{(k)}|s) \\
 &= \underset{s}{\operatorname{argmax}} \left( \prod_{k=0}^{K-1} x^{(k)} \right) s^{2K} \exp \left( -s \sum_{k=0}^{K-1} x^{(k)} \right) \\
 &= \underset{s}{\operatorname{argmax}} \left\{ 2 \ln s - s \frac{1}{K} \sum_{k=0}^{K-1} x^{(k)} \right\} \\
 &= \frac{2K}{\sum_{k=0}^{K-1} x^{(k)}}
 \end{aligned}$$

### Exercise 51 (1.4; 1.7)

We have access to a set of  $K$  samples,  $\{X^{(k)}\}_{k=1}^K$ , independently drawn from a random variable  $X$  with p.d.f.

$$p_X(x) = \frac{1}{bx^2} \exp\left(-\frac{1}{bx}\right) u(x)$$

with  $b > 0$  a constant.

- Find the ML estimator of  $b$  as a function of the available samples,  $\hat{B}_{\text{ML}}$ .
- Verify that random variable  $Y = 1/X$  is characterized by a unilateral exponential p.d.f.  $p_Y(y)$ , and obtain the value of the mean of such distribution.
- Considering your answers to the previous sections, is  $\hat{B}_{\text{ML}}$  an unbiased estimator?

**Solution:**

$$(a) \quad \hat{B}_{\text{ML}} = \frac{1}{K} \sum_{k=1}^K \frac{1}{X^{(k)}}$$

$$(b) \quad p_Y(y) = \frac{1}{b} \exp\left(-\frac{y}{b}\right) u(y)$$

(c) The estimator is unbiased.

### Exercise 52 (1.3; 1.4)

A random variable  $S$  follows an exponential pdf

$$p_S(s) = \lambda e^{-\lambda s}, \quad s > 0$$

with  $\lambda > 0$ . Consider now a discrete random variable  $X$  related to  $S$  via a Poisson distribution, i.e.,

$$P_{X|S}(x|s) = \frac{s^x e^{-s}}{x!}, \quad x = 0, 1, 2, \dots$$

- Determine the ML estimator of  $S$  given  $x$ .
- Assume now that we have access to  $K$  independent realizations  $\{(x^{(k)}, s^{(k)}), k = 1, \dots, K\}$  of  $(X, S)$ . Find the ML estimator of  $\lambda$  based on these observations.
- Find the MAP estimation of  $S$  for  $x = 1$ .

**Solution:**

$$(a) \hat{S}_{\text{ML}} = X$$

$$(b) \hat{\lambda}_{\text{ML}} = \frac{1}{\frac{1}{K} \sum_{k=1}^K s^{(k)}}$$

$$(c) \hat{S}_{\text{MAP}} = \frac{X}{1 + \lambda}$$

**Exercise 53 (ML, MSE)**

The manager of an IT company intends to analyze the productivity of his employees by estimating the time  $S$  they need to implement a certain computer program. With this goal, at 12:00 a.m. the manager requests the implementation of the program; instead of directly starting the coding task, the employees need to finish first whatever task they are currently carrying out, what requires an additional time  $N$ . As a consequence, the total elapsed time between the request of the program implementation and each employee's notification indicating the conclusion of the task is  $X = S + N$ .

It is known that the time  $N$  that the employees need to finish the tasks and start the program implementation can be modeled as the following exponential distribution

$$p_N(n) = a \exp(-an) \quad n > 0,$$

whereas the time for coding the program,  $S$ , follows also an exponential distribution, in this case characterized by the expression

$$p_S(s) = b \exp(-b(s - c)) \quad s > c.$$

- Before the described process, a simulation has been carried out using a control group, and measuring directly the times  $N$  and  $S$  required by the members of this group. As a result of the test, four independent observations were obtained for each variable. Concretely, the four observations for  $N$  were 6, 10, 12, and 20 minutes, whereas the observations for  $S$  were 6, 12, 18, and 36 minutes. Based on these observations, estimate using maximum likelihood the values of constants  $a$ ,  $b$ , and  $c$ .

Consider in the following  $a = 10$  minutes,  $b = 10$  minutes, and  $c = 5$  minutes.

- For the actual productivity test, the manager receives notifications from three different employees indicating that they have finished the implementation of the program at 12:25, 12:30, and 12:40 a.m. Estimate using maximum likelihood the time that each employee needed for the implementation of the program.
- Repeat the estimation of the previous subsection if a minimum mean square error estimator were used.

**Solution:**

- (a)  $\hat{a}_{\text{ML}} = \frac{K}{\sum_{k=1}^K n^{(k)}} = \frac{1}{12} \text{ minutes}^{-1}$ ;  
 $\hat{c}_{\text{ML}} = \min_k \{s^{(k)}\} = 6 \text{ minutes}$ ;  
 $\hat{b}_{\text{ML}} = \frac{K}{\sum_{k=1}^K (s^{(k)} - \hat{c}_{\text{ML}})} = \frac{1}{12} \text{ minutes}^{-1}$ .
- (b)  $\hat{s}_{\text{ML}} = x$ .  $\hat{s}_{\text{ML}}(x = 25) = 25$ ,  $\hat{s}_{\text{ML}}(x = 30) = 30$ ,  $\hat{s}_{\text{ML}}(x = 40) = 40$ .
- (c)  $\hat{s}_{\text{MMSE}} = \frac{x+5}{2}$ .  $\hat{s}_{\text{MMSE}}(x = 25) = 15$ ,  $\hat{s}_{\text{MMSE}}(x = 30) = 17.5$ ,  $\hat{s}_{\text{MMSE}}(x = 40) = 22.5$ .

**Exercise 54 (ML)**

An energetic student gets up early every morning and reaches the bus stop exactly at 8.00 am, the scheduled arrival time of the only bus that can take him to the university. The bus is usually late and never arrives before its scheduled arrival time. The pdf of the delay of the bus

$$p_{T_B|\Lambda_B}(t_B|\lambda_B) = \lambda_B \exp(-\lambda_B t_B), \quad 0 < t_B \text{ min}$$

where  $t_B$  are the minutes of delay. The delays are iid for each day.

A second, lazy student makes use of the same bus, but isn't as punctual as the energetic one. The delay in the arrival of the lazy student to the bus stop follows the pdf

$$p_{T_L|\Lambda_L}(t_L|\lambda_L) = \lambda_L \exp(-\lambda_L t_L), \quad 0 < t_L \text{ min}$$

where  $T_L$  is the delay wrt to the energetic student in reaching the bus stop. These delays are iid for each day. Finally,  $T_B$  and  $T_E$  are independent.

- (a) Modeling: The first five days of the course the bus arrived to the bus stop 0, 6, 15, 20 and 24 minutes late, whereas the lazy student arrived to the bus stop 15, 10, 12, 5 and 3 minutes late. Estimate  $\lambda_B$  and  $\lambda_L$  using ML. Specify units.  
 Consider this ML estimates as the true values for  $\lambda_B$  and  $\lambda_L$  for the remainder of the exercise.
- (b) Compute the expected waiting time for the energetic student at the bus stop.
- (c) The sixth day, the lazy student arrives to the bus stop at 8.05 am. He meets there the energetic student and asks him how much longer are both expected to wait (the expected time) until the bus comes. Compute this quantity and contrast it with your answer to b).  
 Hint: You are being asked to compute  $\mathbb{E}[t_B - 5 \text{ min} | t_B > 5 \text{ min}]$ .
- (d) If the lazy student misses the bus, he won't attend university that day. Assuming this arrival process for the bus and the lazy student is repeated for the rest of the course, compute the *expected* percentage of days that each student attends university.

**Solution:**

- (a)  $\lambda_B = \frac{1}{13} \text{ min}^{-1}$ ,  $\lambda_V = \frac{1}{9} \text{ min}^{-1}$  y  $\mathbb{E}[t_B] = 13 \text{ min}$ .
- (b)  $\mathbb{E}[t_B - 5 \text{ min} | t_B > 5 \text{ min}] = 13 \text{ min}$ . Arriving 5 min late doesn't save waiting time.  
 On average, he has to wait as much as the willful student the remaining days. This apparent paradox arises from the fact that there is an additional information in the sixth day: the bus will be more than 5 minutes late.
- (c)  $p(t_V < t_B) = \frac{\lambda_V}{\lambda_V + \lambda_B}$ , in percentage  $\frac{100\lambda_V}{\lambda_V + \lambda_B} \%$ . The willful student always goes.

**Exercise 55 (MSE)**

A research company is working on a new communication prototype able to modify the noise distribution before being mixed with the signal. In this way, the receiver observes the following signal:

$$X = S + N_{\text{mod}},$$

where  $S$  is a Gaussian r.v. with zero mean and variance  $v_s$ , and  $N_{\text{mod}}$  is the new noise whose value is given by the following expression:

$$N_{\text{mod}} = \lambda N_1 + (1 - \lambda) N_2$$

$N_1$  and  $N_2$  being two independent Gaussian random variables, independent from  $S$ , with zero mean and variance  $v$ ; whereas  $\lambda$  is a control parameter which takes values from 0 to 1.

- Find the distribution of  $N_{\text{mod}}$  as a function of  $\lambda$ .
- Obtain the MMSE estimator of  $S$  given  $X$  for the new system and compute the mean square error of this estimator as a function of  $\lambda$ .
- Compute the value of  $\lambda$ ,  $\lambda_{\text{opt}}$ , which provides the minimum mean square error.
- Obtain, in terms of reduction of the mean square error, the advantage provided by this model when  $\lambda$  is set to  $\lambda_{\text{opt}}$  with respect to a model using  $\lambda = 0$  or  $\lambda = 1$ . Compute this error reduction when  $v_s = v = 1$ .

The technician in charge of designing the system generating  $N_{\text{mod}}$  has gone on vacation, leaving the system online but without specifying what value of  $\lambda$  is being used. The new intern is tasked with obtaining a set of independent observations of the noise at the system's output  $\{N_{\text{mod}}^{(k)}\}_{k=1}^K$ , and computing a maximum likelihood estimation of the value of  $\lambda$ ,  $\hat{\lambda}_{ML}$ .

- Considering  $v = 3$  and the observation set  $\{-1, 0, 2, 1\}$ , obtain the value of  $\hat{\lambda}_{ML}$ .

**Solution:**

- $N_{\text{mod}}$  is Gaussian with zero mean and variance  $v_{MOD} = \lambda^2 v + (1 - \lambda)^2 v$ .

$$(b) \quad \hat{S}_{MSE} = \frac{v_s}{v_s + v_{MOD}} x \quad MSE_{MOD} = \frac{(\lambda^2 + (1 - \lambda)^2) v v_s}{v_s + (\lambda^2 + (1 - \lambda)^2) v}$$

- $\lambda_{\text{opt}} = 0.5$

$$(d) \quad \Delta MSE = \frac{1}{6}$$

- $\hat{\lambda}_{ML} = 0.5$

**Exercise 56 (Gauss, MSE, ML)**

Consider a communication system in which the transmitted symbol  $S$  is sent through two different channels, generating observations  $X_1$  and  $X_2$ :

$$X_1 = \alpha(S + N_1)$$

$$X_2 = 2\alpha(S + N_2)$$

with  $\alpha$  a constant associated to the attenuation of the channels.

It is known that the noise values  $N_1$  and  $N_2$ , which are independent of each other, can be modeled as Gaussian random variables with mean zero and variance 0.5. Furthermore, the transmitted symbol,  $S$ , is also a Gaussian random variable with mean zero and variance one, and is independent of  $N_1$  and  $N_2$ . At the receiver, just the sum of both outputs can be observed, i.e.,

$$X = X_1 + X_2$$



- (a) Obtain the minimum mean square error estimate of  $S$  given  $X$ ,  $\hat{S}_{\text{MMSE}}$ .
- (b) Compute the mean square error of estimator  $\hat{S}_{\text{MMSE}}$ .
- (c) Obtain the maximum likelihood estimate of  $S$  given  $X$ ,  $\hat{S}_{\text{ML}}$ .
- (d) Find the marginal distribution,  $p_X(x)$ .
- (e) The marginal distribution depends on the unknown attenuation parameter,  $\alpha$ . Find the ML estimate of  $\alpha$ ,  $\hat{\alpha}_{\text{ML}}$ , based on  $K$  values,  $\{x^{(k)}\}_{k=1}^K$ , independently drawn from the marginal distribution.

**Solution:**

$$(a) \hat{S}_{\text{ML}} = \frac{x}{3\alpha}$$

$$(b) \hat{S}_{\text{MMSE}} = \frac{3\alpha}{11.5\alpha^2}x$$

$$(c) \hat{\alpha}_{\text{ML}} = \sqrt{\frac{2}{23} \sum_k x^{(k)^2}}$$

$$(d) \hat{\alpha}_{\text{ML}} = 2$$

**Exercise 57 (1.7)**

Consider the estimation of a r.v.  $X$  from a set of  $K$  independent observations  $\{X^{(k)}\}_{k=1}^K$ , and also consider the following estimator:

$$\hat{M} = \frac{a}{K} \sum_{k=1}^K X^{(k)},$$

with  $a$  a constant to determine.

- (a) Obtain the bias and the variance of the estimator as a function of  $a$ .
- (b) Which value of  $a$  minimizes the variance? Is there any value of  $a$  that produces an unbiased estimator?
- (c) Obtain the mean square error of the estimator, and find the value of  $a$  that minimizes it.

**Solution:**

$$(a) \mathbb{E}\{\hat{M} - m\} = (a - 1)m; \text{Var}\{\hat{M}\} = \frac{a^2 v}{K}.$$

(b) The variance is minimized by  $a = 0$ , whereas the bias is null for  $a = 1$ .

$$(c) \mathbb{E}\{(\hat{M} - m)^2\} = (a - 1)^2 m^2 + \frac{a^2 v}{K}. \text{ Thus, the value of } a \text{ minimizing it is } a^* = \frac{m^2}{m^2 + v/K}.$$

**Exercise 58 (1.4; 1.7)**

A random variable  $X$  follows a unilateral exponential distribution with parameter  $a > 0$ :

$$p_X(x) = \frac{1}{a} \exp\left(-\frac{x}{a}\right) \quad x > 0$$

As it is known, the mean and variance of  $X$  are given by  $a$  and  $a^2$ , respectively.

- (a) Obtain the maximum likelihood estimator of  $a$ ,  $\hat{A}_{\text{ML}}$ , based on a set of  $K$  independent observations of random variable  $X$ ,  $\{X^{(k)}\}_{k=1}^K$ .
- (b) Consider now a new estimator based on the previous one, and characterized by expression:

$$\hat{A} = c \cdot \hat{A}_{\text{ML}},$$

where  $0 \leq c \leq 1$  is shrinkage constant that allows re-scaling the ML estimator. Find the bias squared, the variance, and the Mean Square Error (MSE) of the new estimator, and represent them all together in the same plot as a function of  $c$ .

- (c) Find the value of  $c$  which minimizes the MSE,  $c^*$ , and discuss its evolution as the number of available observations increases. Calculate the MSE of the estimator associated to  $c^*$ .
- (d) Determine the range of values of  $c$  for which the MSE of  $\hat{A}$  is smaller than the MSE of the ML estimator, and explain how such range changes as  $K \rightarrow \infty$ . Discuss your result.

**Solution:**

$$(a) \hat{A}_{\text{ML}} = \frac{1}{K} \sum_{k=1}^K X^{(k)}$$

$$(b) \hat{A} = \frac{c}{K} \sum_{k=1}^K X^{(k)}$$

$$\mathbb{E} \left\{ \hat{A} - a \right\}^2 = (c-1)^2 a^2, \quad \text{Var} \left\{ \hat{A} \right\} = \frac{c^2 a^2}{K}, \quad \mathbb{E} \left\{ \left( \hat{A} - a \right)^2 \right\} = (c-1)^2 a^2 + \frac{c^2 a^2}{K}$$

$$(c) c^* = \frac{K}{K+1}, \quad c^* \rightarrow 1 \quad (K \rightarrow \infty), \quad \mathbb{E} \left\{ \left( \hat{A} - a \right)^2 \right\} = \frac{a^2}{K+1} \quad (c = c^*)$$

$$(d) \text{ The range of values is: } c \in \left[ \frac{K-1}{K+1}, 1 \right], \text{ which narrows as } K \text{ increases.}$$

**Exercise 59 (1.7)**

The variance  $v$  of a zero-mean r.v.  $X$  is estimated from  $K$  independent observations of the variable,  $\{X^{(k)}\}_{k=1}^K$ , using the following estimator:

$$\hat{V} = \frac{1}{K} \left[ \sum_{k=1}^K X^{(k)} \right]^2$$

- (a) Find the bias of such an estimator.
- (b) For  $K = 2$ , and assuming known  $\mathbb{E} \{X^4\} = \alpha$ , obtain the variance of the estimator.

**Solution:**

(a) The estimator is unbiased.

$$(b) \text{Var} \left\{ \hat{V} \right\} = \frac{1}{2} [\alpha + v^2]$$

**Exercise 60 (1.8)**

The current intensity through the branch of a circuit can be characterized by equation

$$i(t) = A \cos(\omega_0 t) \exp^{-\alpha t} + B \sin(\omega_0 t) \exp^{-\alpha t} + C$$

$\omega_o$  and  $\alpha$  being two known constants. For determining the other model parameters,  $A$ ,  $B$ , and  $C$ , we have access to a set of measures of  $i(t)$  for  $K$  different time instants, i.e., a set of pairs  $\{t^{(k)}, i(t^{(k)})\}_{k=1}^K$ .

Provide expressions that can be used to calculate the values of parameters  $A$ ,  $B$ , and  $C$ , which minimize the quadratic error of the model averaged over the set of available samples.

**Solution:** Defining  $x_1^{(k)} = \cos(\omega_o t^{(k)}) \exp^{-\alpha t^{(k)}}$  and  $x_2^{(k)} = \sin(\omega_o t^{(k)}) \exp^{-\alpha t^{(k)}}$ , and denoting by  $\mathbf{X}_e$  the extended observations matrix  $\{x_1^{(k)}, x_2^{(k)}\}_{k=1}^K$  and by  $\mathbf{i}$  a vector with components  $\{i(t^{(k)})\}_{k=1}^K$ , we have that:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = (\mathbf{X}_e^T \mathbf{X}_e)^{-1} \mathbf{X}_e^T \mathbf{i}$$

### Exercise 61 (1.7)

We want to estimate the mean  $m$  of a random variable  $X$  with variance  $v$ , using a set of  $K + 1$  independent observations of such random variable,  $\{X^{(k)}\}_{k=1}^{K+1}$ . Consider the following estimators:

$$\hat{M}_1 = \frac{a}{K} \sum_{k=1}^K X^{(k)} \quad \hat{M}_2 = X^{(K+1)} \quad \hat{M}_3 = \lambda \hat{M}_1 + (1 - \lambda) \hat{M}_2$$

$a$  being a positive constant, strictly less than one, and  $\lambda$  another constant to be set.

- Compare the bias and variance of estimators  $\hat{M}_1$  and  $\hat{M}_2$ .
- Find the bias, the variance, and mean square error (MSE) of estimator  $\hat{M}_3$ , simplifying your result for  $K \rightarrow \infty$ .

**Solution:**

$$\begin{aligned} \mathbb{E} \{ \hat{M}_1 - m \} &= (a - 1) m & \mathbb{E} \{ \hat{M}_2 - m \} &= 0 \\ \text{Var} \{ \hat{M}_1 \} &= \frac{a^2 v}{K} & \text{Var} \{ \hat{M}_2 \} &= v \end{aligned}$$

$$\begin{aligned} \mathbb{E} \{ \hat{M}_3 - m \} &= \lambda (a - 1) m & \text{Var} \{ \hat{M}_3 \} &= \frac{\lambda^2 a^2 v}{K} + v (1 - \lambda)^2 \\ \mathbb{E} \{ (\hat{M}_3 - m)^2 \} &= \frac{\lambda^2 a^2 v}{K} + v (1 - \lambda)^2 + \lambda^2 (a - 1)^2 m^2 \end{aligned}$$

$$\text{If } K \rightarrow \infty, \text{Var} \{ \hat{M}_3 \} = v (1 - \lambda)^2 \text{ and } \mathbb{E} \{ (\hat{M}_3 - m)^2 \} = v (1 - \lambda)^2 + \lambda^2 (a - 1)^2 m^2.$$