# Stochastic Processes: Problems

10 de marzo de 2022

## 1. Probability densities

**Exercise P1 ()**

Given random variables $S$ and $X$ with

$$p_{S,X}(s,x) = se^{-s-sx}, \qquad s \geq 0, \quad x \geq 0 \tag{1}$$

Compute

(a) The marginal distribution of $S$, $p_S(s)$

(b) The marginal distribution of $X$, $p_X(x)$

(c) The conditional probability density function (pdf) of $S$ given $X$, $p_{S|X}(s|x)$.

(d) The conditional pdf of $X$ given $S$, $p_{X|S}(x|s)$.

(e) The cumulative distribution function of $S$, $F_S(s)$

(f) The cumulative distribution function of $X$, $F_X(x)$

**Solution:**

(a)

$$\begin{aligned} p_S(s) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)dx = \int_0^{\infty} se^{-s-sx}dx \\ &= e^{-s}\int_0^{\infty} se^{-sx}dx = e^{-s}. \qquad s \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^{\infty} se^{-s(x+1)}ds \\ &= \frac{1}{(x+1)^2}\int_0^{\infty} se^{-s}ds = \frac{1}{(x+1)^2}, \qquad x \geq 0 \end{aligned}$$

(c)

$$F_S(s) = \int_{-\infty}^{s} p_S(s')ds' = \int_0^{s} e^{-s'}ds' = 1 - e^{-s}$$

(d)

$$F_X(x) = \int_{-\infty}^{x} p_X(x')dx' = \int_0^{x} \frac{1}{(x'+1)^2}dx' = 1 - \frac{1}{x+1} = \frac{x}{x+1}$$

**Exercise P2 ()**

Given random variables $S$ and $X$ with

$$p_{S,X}(s,x) = 2, \qquad 0 \leq s \leq x \leq 1 \tag{2}$$

Compute

(a) The marginal distribution of $S$, $p_S(s)$

(b) The marginal distribution of $X$, $p_X(x)$

(c) The conditional distribution of $S$ given $X$, $p_{S|X}(s|x)$.

(d) The conditional distribution of $X$ given $S$, $p_{X|S}(x|s)$.

**Solution:**

(a)

$$\begin{aligned} p_S(s) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)dx = \int_s^1 2dx \\ &= 2 \cdot (1-s), \qquad 0 \leq s \leq 1 \end{aligned}$$

(b)

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^x 2ds \\ &= 2x, \qquad 0 \leq x \leq 1 \end{aligned}$$

(c)

$$p_{X|S}(x|s) = \frac{p_{X,S}(x,s)}{p_S(s)} = \frac{1}{1-s}, \qquad 0 \leq s \leq x \leq 1$$

(d)

$$p_{S|X}(s|x) = \frac{p_{X,S}(x,s)}{p_S(s)} = \frac{1}{x}, \qquad 0 \leq s \leq x \leq 1$$

**Exercise P3 ()**

Random variables $R$ follows a uniform distribution in $[0,1]$, i.e.

$$p_R(r) = 1, \qquad 0 \leq s \leq x \geq 1 \tag{3}$$

Compute the pdf of $X = -\log(R)$.

**Solution:**

(a)

$$p_X(x) = ...$$