Estimation Theory: Problems

Most of the problems and exercises of this collection have been taken from previous years exams on subjects related to Estimation Theory. The topics covered by each exercise are shown next to the exercise number according to:

1.1. General view of estimation problems.

1.2. Bayes' Estimation Theory.

1.3. Frequently used Bayes' estimators: MSE, MAP, MAD.

1.4. Maximum Likelihood estimation.

1.5. Estimation with Gaussian likelihoods.

1.6. Constrained estimators. Linear minimum mean square error estimation.

1.7. Bias and Variance of estimators.

1.8. Machine design of estimators.

Notation:

- $\hat{S}_{\mathrm{MSE}}$: Minimum Mean Square Error estimator.
- $\hat{S}_{\mathrm{MAD}}$: Mininimum Mean Absolute Deviation Error estimator.
- $\hat{S}_{\mathrm{MAP}}$: Maximum a posteriori estimator.
- $\hat{S}_{\mathrm{ML}}$: Maximum likelihood estimator.
- $\hat{S}_{\mathrm{LMSE}}$: Linear Minimum Mean Square Error estimator.

Exercise 1 (1.6)

We wish to design a linear minimum mean square error estimator for the estimation of random variable S based on the observation of random variables X_1 and X_2. It is known that:

$$\begin{array}{lll}
\mathbb{E}\{S\} = \frac{1}{2} & \mathbb{E}\{X_1\} = 1 & \mathbb{E}\{X_2\} = 0 \\
\mathbb{E}\{SX_1\} = 1 & \mathbb{E}\{SX_2\} = 2 & \mathbb{E}\{X_1X_2\} = \frac{1}{2} \\
\mathbb{E}\left\{S^2\right\} = 4 & \mathbb{E}\left\{X_1^2\right\} = \frac{3}{2} & \mathbb{E}\left\{X_2^2\right\} = 2
\end{array}$$

Obtain the weights of estimator $\hat{S}_{\mathrm{LMSE}} = w_0 + w_1X_1 + w_2X_2$, and calculate its mean square error $\mathbb{E}\left\{(S - \hat{S}_{\mathrm{LMSE}})^2\right\}$.

Solution: $w_0 = \frac{1}{2} \quad w_1 = 0 \quad w_2 = 1$

$\mathbb{E}\left\{(S - \hat{S}_{\mathrm{LMSE}})^2\right\} = \frac{7}{4}$

Exercise 2 (1.2; 1.3; 1.7)

Consider the estimation of a random variable S from another random variable X, where their joint probability density function (pdf) is given by:

$$p_{S,X}(s,x) = \frac{6}{7}(x+s)^2, \quad 0 \le x, s \le 1$$

(a) Obtain $p_X(x)$.

(b) Obtain $p_{S|X}(s|x)$.

(c) Calculate the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(d) Calculate the MAP estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(e) Indicate which is the bias and the variance of the MAP estimator.

Solution:

(a) $p_X(x) = \frac{2}{7}\left(3x^2+3x+1\right) \quad 0 < x < 1.$

(b) $p_{S|X}(s|x) = \dfrac{(x+s)^2}{x^2+x+\frac{1}{3}} \quad 0 < s < 1.$

(c) $\hat{S}_{\mathrm{MSE}} = \dfrac{\frac{X^2}{2}+\frac{2X}{3}+\frac{1}{4}}{X^2+X+\frac{1}{3}}.$

(d) Given that $p_{S|X}(s|x)$ is strictly increasing with s, $\hat{S}_{\mathrm{MAP}} = 1$.

(e) $p_S(s) = \frac{2}{7}\left(3s^2+3s+1\right), \ 0 < s < 1$, thus, $\mathbb{E}\{S\} = \frac{9}{14}$.
The bias is $-\frac{5}{14}$, whereas the variance is 0.

Exercise 3 (1.7)

We want to estimate the mean m of a random variable X with variance v, using a set of $K+1$ independent observations of such random variable, $\left\{X^{(k)}\right\}_{k=1}^{K+1}$. Consider the following estimators:

$$\hat{M}_1 = \frac{a}{K}\sum_{k=1}^{K} X^{(k)} \qquad \hat{M}_2 = X^{(K+1)} \qquad \hat{M}_3 = \lambda \hat{M}_1 + (1-\lambda)\hat{M}_2$$

a being a positive constant, strictly less than one, and λ another constant to be set.

(a) Compare the bias and variance of estimators $\hat{M}_1$ and $\hat{M}_2$.

(b) Find the bias, the variance, and mean square error (MSE) of estimator $\hat{M}_3$, simplifying your result for $K \longrightarrow \infty$.

Solution:

(a)
$$\mathbb{E}\left\{\hat{M}_1 - m\right\} = (a-1)\,m \qquad \mathbb{E}\left\{\hat{M}_2 - m\right\} = 0$$
$$\mathrm{Var}\left\{\hat{M}_1\right\} = \frac{a^2 v}{K} \qquad \mathrm{Var}\left\{\hat{M}_2\right\} = v$$

(b) $\mathbb{E}\left\{\hat{M}_3 - m\right\} = \lambda\,(a-1)\,m$ $\qquad$ $\mathrm{Var}\left\{\hat{M}_3\right\} = \dfrac{\lambda^2 a^2 v}{K} + v\,(1-\lambda)^2$

$\mathbb{E}\left\{\left(\hat{M}_3 - m\right)^2\right\} = \dfrac{\lambda^2 a^2 v}{K} + v\,(1-\lambda)^2 + \lambda^2\,(a-1)^2\,m^2$

If $K \longrightarrow \infty$, $\mathrm{Var}\left\{\hat{M}_3\right\} = v\,(1-\lambda)^2$ and $\mathbb{E}\left\{\left(\hat{M}_3 - m\right)^2\right\} = v\,(1-\lambda)^2 + \lambda^2\,(a-1)^2\,m^2$.

Exercise 4 (1.7)

Let S and X be two unidimensional random variables. Variable X is characterized by the following probability density function:

$$p_X(x) = \begin{cases} \frac{1}{2}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

It is known that the minimum mean square error estimator of S based on X is

$$\hat{S}_{\mathrm{MMSE}}(X) = -\frac{1}{2}\mathrm{sign}\,(X) = \begin{cases} -\frac{1}{2}, & X \geq 0 \\ \frac{1}{2}, & X < 0 \end{cases}$$

It is also known that the mean square error associated to this estimator is $\frac{1}{12}$. An alternative estimator of S is defined as

$$\hat{S}_1 = -X$$

(a) Obtain the bias of estimator $\hat{S}_1$.

(b) Calculate the following mathematical expectations: $\mathbb{E}\{SX\}$ and $\mathbb{E}\{S^2\}$.

(c) Find the mean square error incurred by $\hat{S}_1$.

Solution:

(a) The estimator is unbiased.

(b) $\mathbb{E}\{SX\} = -\dfrac{1}{4}$ and $\mathbb{E}\{S^2\} = \dfrac{1}{3}$

(c) The MSE of $\hat{S}_1$ is $\dfrac{1}{6}$.

Exercise 5 (1.6; 1.8)

We want to design a low-complexity linear regression model for a random variable S. This variable is statistically related to three other random variables X_1, X_2, and X_3, which can be observed. The following table includes four independent realizations of the random experiment.

X_1	X_2	X_3	S
3	−1	0	−1
−2	0	1	−2
0	−1	2	0
−1	2	−3	3

The objective of the problem is to evaluate two different strategies to build the aforementioned low-complexity regression model:

- An exact linear least squares regression model that uses only two of the available observable variables.
- An approximation to the linear minimum mean square error estimator that uses the three variables. The approximation consists in assuming a diagonal covariance matrix for the observations.

In order to do that:

(a) Determine which of the three variables will be included in the first design. The selection is carried out in two steps: The firstly selected variable is the one whose sample covariance with S has the largest absolute value; the second variable will then be the one with the smallest sample covariance (again, in absolute terms) with the variable chosen during the first stage.

(b) Build the least-squares linear regression model of S using the two variables selected in the previous section.

(c) Now, obtain the linear estimator specified in the second design. To do that, calculate first the diagonal entries of the covariance matrix of the observation variables, and the covariance vector between observations and variable S, again using sample averages over the available samples.

(d) Which of the two designs incurs in a smaller average quadratic error over the available data?

Solution:

(a) $\bar{v}_{X_1,S} = -0.5$, $\bar{v}_{X_2,S} = 1.75$, $\bar{v}_{X_3,S} = -2.75$. The first variable to be used is X_3.

$\bar{v}_{X_1,X_3} = 0.25$, $\bar{v}_{X_2,X_3} = -2$. Thus, the second variable is X_1.

(b) $\hat{S}_1 = -0.087X_1 - 0.7795X_3$

(c) $\hat{S}_2 = -0.1429X_1 + 1.1667X_2 - 0.7857X_3$

(d) The average square error of $\hat{S}_1$ (over the provided samples) is 1.3128. The average square error of $\hat{S}_2$ is 3.3656. The first design achieves a smaller error.

Exercise 6 (1.4; 1.7)

A random variable X follows a unilateral exponential distribution with parameter $a > 0$:

$$p_X(x) = \frac{1}{a}\exp\left(-\frac{x}{a}\right) \qquad x > 0$$

As it is known, the mean and variance of X are given by a and a^2, respectively.

(a) Obtain the maximum likelihood estimator of a, $\hat{A}_{\mathrm{ML}}$, based on a set of K independent observations of random variable X, $\left\{X^{(k)}\right\}_{k=1}^{K}$.

(b) Consider now a new estimator based on the previous one, and characterized by expression:

$$\hat{A} = c \cdot \hat{A}_{\mathrm{ML}},$$

where $0 \leq c \leq 1$ is shrinkage constant that allows re-scaling the ML estimator. Find the bias squared, the variance, and the Mean Square Error (MSE) o the new estimator, and represent them all together in the same plot as a function of c.

(c) Find the value of c which minimizes the MSE, c^*, and discuss its evolution as the number of available observations increases. Calculate the MSE of the estimator associated to c^*.

(d) Determine the range of values of c for which the MSE of $\hat{A}$ is smaller than the MSE of the ML estimator, and explain how such range changes as $K \longrightarrow \infty$. Discuss your result.

Solution:

(a) $\hat{A}_{\mathrm{ML}} = \frac{1}{K}\sum_{k=1}^{K} X^{(k)}$

(b) $\hat{A} = \frac{c}{K}\sum_{k=1}^{K} X^{(k)}$

$\mathbb{E}\left\{\hat{A} - a\right\}^2 = (c-1)^2 a^2, \qquad \mathrm{Var}\left\{\hat{A}\right\} = \frac{c^2 a^2}{K}, \qquad \mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = (c-1)^2 a^2 + \frac{c^2 a^2}{K}$

(c) $c^* = \frac{K}{K+1}, \qquad c^* \longrightarrow 1 \ (K \longrightarrow \infty), \qquad \mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = \frac{a^2}{K+1} \ (c = c^*)$

(d) The range of values is: $c \in \left[\frac{K-1}{K+1}, 1\right]$, which narrows as K increases.

Exercise 7 (1.8)

The current intensity through the branch of a circuit can be characterized by equation

$$i(t) = A\cos(\omega_{\mathrm{o}} t)\exp^{-\alpha t} + B\sin(\omega_{\mathrm{o}} t)\exp^{-\alpha t} + C$$

ω_{o} and α being two known constants. For determining the other model parameters, A, B, and C, we have access to a set of meaures of $i(t)$ for K different time instants, i.e., a set of pairs $\{t^{(k)}, i(t^{(k)})\}_{k=1}^{K}$.

Provide expressions that can be used to calculate the values of parameters A, B, and C, which minimize the quadratic error of the model averaged over the set of available samples.

Solution: Defining $x_1^{(k)} = \cos(\omega_{\mathrm{o}} t^{(k)})\exp^{-\alpha t^{(k)}}$ and $x_2^{(k)} = \sin(\omega_{\mathrm{o}} t^{(k)})\exp^{-\alpha t^{(k)}}$, and denoting by $\mathbf{X}_{\mathrm{e}}$ the extended observations matrix $\{x_1^{(k)}, x_2^{(k)}\}_{k=1}^{K}$ and by $\mathbf{i}$ a vector with components $\{i(t^{(k)})\}_{k=1}^{K}$, we have that:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \left(\mathbf{X}_{\mathrm{e}}^T \mathbf{X}_{\mathrm{e}}\right)^{-1} \mathbf{X}_{\mathrm{e}}^T \mathbf{i}$$

Exercise 8 (1.7)

Consider the estimation of a r.v. X from a set of K independent observations $\{X^{(k)}\}_{k=1}^{K}$, and also consider the following estimator:

$$\hat{M} = \frac{a}{K}\sum_{k=1}^{K} X^{(k)},$$

with a a constant to determine.

(a) Obtain the bias and the variance of the estimator as a function of a.

(b) Which value of a minimizes the variance? Is there any value of a that produces an unbiased estimator?

(c) Obtain the mean square error of the estimator, and find the value of a that minimizes it.

Solution:

(a) $\mathbb{E}\left\{\hat{M}-m\right\}=(a-1)m$; $\text{Var}\left\{\hat{M}\right\}=\dfrac{a^2v}{K}$.

(b) The variance is minimized by $a=0$, whereas the bias is null for $a=1$.

(c) $\mathbb{E}\left\{(\hat{M}-m)^2\right\}=(a-1)^2m^2+\dfrac{a^2v}{K}$. Thus, the value of a minimizing it is $a^*=\dfrac{m^2}{m^2+v/K}$.

Exercise 9 (1.5)

We have access to the two following observations for estimating a random variable S:

$$X_1=S+N_1$$
$$X_2=\alpha S+N_2$$

where α is a known constant, and S, N_1, and N_2 are independent Gaussian random variables, with zero mean and variances v_s, v_n, and v_n, respectively.

(a) Obtain the minimum mean square error estimator of S given X_1 and X_2, $\hat{S}_1$ and $\hat{S}_2$, respectively.

(b) Calculate the mean square error of each of the estimators from the previous section. Which of the two provides a smaller MSE? Justify your answer for the different values of parameter α.

(c) Obtain the minimum mean square error estimator of S based on the joint observation of variables X_1 and X_2, i.e., as a function of the observation vector $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\hat{S}_{\text{MMSE}}$.

Solution:

(a) $\hat{S}_1=\dfrac{v_s}{v_s+v_n}X_1$ and $\hat{S}_2=\dfrac{\alpha v_s}{\alpha^2 v_s+v_n}X_2$.

(b) $\mathbb{E}\{E_1^2\}=\dfrac{v_sv_n}{v_s+v_n}$ and $\mathbb{E}\{E_2^2\}=\dfrac{v_sv_n}{\alpha^2v_s+v_n}$. For $|\alpha|>1$ the mean square error of $\hat{S}_2$ is smaller than that of $\hat{S}_1$.

(c) $\hat{S}_{\text{MMSE}}=\left[\dfrac{1}{1+\alpha^2+v_n/v_s},\dfrac{\alpha}{1+\alpha^2+v_n/v_s}\right]\mathbf{X}$

Exercise 10 (1.6)

The joint p.d.f. of random variables X and S is given by

$$p_{X,S}(x,s)=\begin{cases} x+s & 0\le x\le 1 \text{ and } 0\le s\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}}=w_0+w_1X$.

Solution: $\hat{S}_{\text{LMSE}} = \frac{7}{11} - \frac{X}{11}$

Exercise 11 (1.2; 1.3; 1.4; 1.7)

We want to estimate the value of a positive random variable S using a random observation X, which is related with S via

$$X = R/S$$

R being a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Obtain the likelihood of S, $p_{X|S}(x|s)$.

(b) Find the maximum likelihood estimator of S given X, $\hat{S}_{\text{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The joint p.d.f. of S and X, $p_{S,X}(s,x)$, and the *a posteriori* distribution of S, $p_{S|X}(s|x)$.

(d) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\text{MAP}}$.

(e) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(f) The bias of estimators $\hat{S}_{\text{MAP}}$ and $\hat{S}_{\text{MSE}}$.

Solution:

(a) $p_{X|S}(x|s) = s \exp(-xs), \quad x > 0.$

(b) $\hat{S}_{\text{ML}} = \frac{1}{X}.$

(c) $p_{X,S}(x,s) = s \exp\left(-s(x+1)\right), \quad x, s > 0;$

$p_{S|X} = (x+1)^2 s \exp\left(-s(x+1)\right), \quad s > 0.$

(d) $\hat{S}_{\text{MAP}} = \frac{1}{X+1}.$

(e) $\hat{S}_{\text{MSE}} = \frac{2}{X+1}.$

(f) $\mathbb{E}\{S - \hat{S}_{\text{MAP}}\} = \frac{1}{2}; \quad \mathbb{E}\{S - \hat{S}_{\text{MSE}}\} = 0.$

Exercise 12 (1.6; 1.8)

We wish to build an estimator for random variable S with the following analytical shape:

$$\hat{S} = w_0 + wX^3$$

(a) Let us define r.v. $Y = X^3$. Indicate which statistics are sufficient to determine the weights of the estimation model.

(b) An analyst wants to adjust the previous model, but he does not have statistical information about the problem. Therefore, he recurs to sample estimations of the sufficient statistics, based on a set of available labelled pairs of the involved random variables:

$$\{X^{(k)}, S^{(k)}\}_{k=1}^{4} = \{(-1, -0.55), (0, 0.5), (1, 1.57), (2, 8.7)\}$$

Determine the weights w_0 and w that the analyst would obtain.

Solution:

(a) $\mathbb{E}\{X\}$, $\mathbb{E}\{Y\}$, v_y and v_{sy} (or any other set from which these can be obtained).

(b) $w = 1.0256$ and $w_0 = 0.5038$.

Exercise 13 (1.2; 1.3; 1.4)

Random variables S and X are jointly distributed according to

$$p_{S,X}(s,x) = \alpha s x^2, \qquad 0 < s < 1-x, \quad 0 < x < 1$$

α being a parameter that needs to be determined.

(a) Stablish the expressions for the marginal probability density functions $p_X(x)$ and $p_S(s)$.

(b) Obtain the MAP estimator of S given X, $\hat{S}_{\mathrm{MAP}}(X)$.

(c) Obtain the ML estimator of S given X, $\hat{S}_{\mathrm{ML}}(X)$.

(d) Obtain the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}(X)$.

(e) Compare the previous estimators according to the mean square errors given X in which they incur.

Solution:

(a) $p_X(x) = 30x^2(1-x)^2, \qquad 0 < x < 1,$

$p_S(s) = 20s(1-s)^3, \qquad 0 < s < 1,$

(b) $\hat{S}_{\mathrm{MAP}}(X) = 1 - X$

(c) $\hat{S}_{\mathrm{ML}}(X) = 1 - X$

(d) $\hat{S}_{\mathrm{MSE}}(X) = \frac{2}{3}(1 - X)$

(e) $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MAP}}(X)\right)^2 \mid x\right\} = \mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{ML}}(X)\right)^2 \mid x\right\} = \frac{1}{6}(1-x)^2$

$\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MSE}}(X)\right)^2 \mid x\right\} = \frac{1}{18}(1-x)^2$

Exercise 14 (1.8)

We want to approximate function $f(x) = 2^x$ in the interval $[0,3]$ by means of simple polynomial models, using regression techniques. In order to do so, we consider the points $x^{(k)} = k-1$, with $1 \leq k \leq 4$, and design a regression curve $y = g(x)$, where $g(x)$ is a polynomium, minimizing the average square error given by

$$E = \sum_{k=1}^{4} \left[g\left(x^{(k)}\right) - f\left(x^{(k)}\right) \right]^2$$

(a) For $g(x) = w_0 + w_1 x + w_2 x^2$, find the matrix equation that characterizes the coefficient vector $w = [w_0, w_1, w_2]^T$ which minimizes the error.

(b) For $g(x) = v x^2$, find the optimal coefficient v.

Solution:

(a) $\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 15 \\ 34 \\ 90 \end{pmatrix}$

(b) $v = \dfrac{90}{98}$

Exercise 15 (1.2; 1.3; 1.4; 1.7)

Consider the estimation of a r.v. S from another random variable X. The joint p.d.f. of the two variables is given by:

$$p_{X,S}(x,s) = \begin{cases} 6x, & 0 \le x \le s, \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(b) Obtain the maximum likelihood estimator of S given X, $\hat{S}_{\text{ML}}$.

(c) Find the probability density function of the previous estimators, $p_{\hat{S}_{\text{MSE}}}(\hat{s})$ and $p_{\hat{S}_{\text{ML}}}(\hat{s})$, and provide a plot of them.

(d) Find the mean and the variance of the error of both estimators.

Solution:

(a) $\hat{S}_{\text{MSE}}(X) = \frac{1}{2}(1+X)$

(b) $\hat{S}_{\text{ML}}(X) = X$

(c) $p_{\hat{S}_{\text{MSE}}}(\hat{s}) = 24(2\hat{s}-1)(1-\hat{s}), \qquad \frac{1}{2} \le \hat{s} \le 1$

$p_{\hat{S}_{\text{ML}}}(\hat{s}) = 6\hat{s}(1-\hat{s}), \qquad 0 \le \hat{s} \le 1$

(d) $\mathbb{E}\{S - \hat{S}_{\text{ML}}\} = \dfrac{1}{4}, \qquad \mathbb{E}\{S - \hat{S}_{\text{MSE}}\} = 0$

$\text{Var}\{S - \hat{S}_{\text{ML}}\} = \dfrac{13}{80}, \qquad Var\{S - \hat{S}_{\text{MSE}}\} = \dfrac{1}{40}$

Exercise 16 (1.6)

Consider the design of a linear minimum mean square estimator of random variable S based on the observation of random variable X_1. The following statistical information is known:

$$\begin{array}{ll} \mathbb{E}\{X_1\} = 0 & \mathbb{E}\{S\} = 1 \\ \mathbb{E}\{X_1^2\} = 1 & \mathbb{E}\{X_1 S\} = 2 \end{array}$$

(a) Which of the two following designs will incur in a smaller MSE?

$$\begin{aligned} \hat{S}_a &= w_{0a} + w_{1a} X_1 \\ \hat{S}_b &= w_{1b} X_1 \end{aligned}$$

(b) If we have access to a second random variable X_2 satisfying

$$\begin{array}{ll} \mathbb{E}\{X_2\} = 1 & \mathbb{E}\{X_2^2\} = 2 \\ \mathbb{E}\{X_1 X_2\} = \frac{1}{2} & \mathbb{E}\{S X_2\} = 2 \end{array}$$

justify if estimator $\hat{S}_c = w_{0c} + w_{1c} X_1 + w_{2c} X_2$ has a smaller mean quadratic error than the estimators considered in Section (a).

Solution:

(a) $w_{0,a}$ is different from 0; therefore, the MSE of $\hat{S}_a$ is smaller than the MSE of $\hat{S}_b$.

(b) The optimal weights of $\hat{S}_c$ are

$$w_{1,c} = 2 \qquad w_{2,c} = 0$$

Since $\hat{S}_a = \hat{S}_c$ both estimators incur in the same MSE, which is smaller than that of $\hat{S}_b$.

Exercise 17 (1.7)

The variance v of a zero-mean r.v. X is estimated from K independent observations of the variable, $\left\{X^{(k)}\right\}_{k=1}^{K}$, using the following estimator:

$$\hat{V} = \frac{1}{K}\left[\sum_{k=1}^{K} X^{(k)}\right]^2$$

(a) Find the bias of such an estimator.

(b) For $K = 2$, and assuming known $\mathbb{E}\left\{X^4\right\} = \alpha$, obtain the variance of the estimator.

Solution:

(a) The estimator is unbiased.

(b) $\mathrm{Var}\left\{\hat{V}\right\} = \frac{1}{2}\left[\alpha + v^2\right]$

Exercise 18 (1.2; 1.3; 1.7)

The joint p.d.f. of two random variables S and X is:

$$p_{S,X}(s,x) = \begin{cases} 6s, & 0 < s < x \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MMSE}}$.

(b) The bias of such estimator.

Solution:

(a) $\hat{S}_{\mathrm{MMSE}} = \frac{2}{3}X$.

(b) The estimator is unbiased.

Exercise 19 (1.2; 1.3)

The joint p.d.f. of two random variables S and X is given by:

$$p_{S,X}(s,x) = \alpha, \quad -1 < x < 1, \quad 0 < s < |x|$$

(a) Obtain the marginal p.d.f. of X, $p_X(x)$, specifying the value of α.

(b) Find the estimators of S based on variable X that minimize the mean square error (MSE), ($\overline{C}_{\mathrm{MSE}} = \mathbb{E}\left\{(S - \hat{S})^2\right\}$) and mean absolute deviation (MAD) ($\overline{C}_{\mathrm{MAD}} = \mathbb{E}\left\{|S - \hat{S}|\right\}$), $\hat{S}_{\mathrm{MMSE}}$ and $\hat{S}_{\mathrm{MAD}}$, respectively.

(c) If the estimators analytical shape is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to previously considered costs: MSE and MAD, i.e., $\hat{S}_{q,\mathrm{MMSE}} = w_1 X^2$ and $\hat{S}_{q,\mathrm{MAD}} = w_2 X^2$, respectively.

Solution:

(a) $p_X(x) = |x|$, $-1 < x < 1$

(b) $\hat{S}_{\mathrm{MMSE}}(X) = \hat{S}_{\mathrm{MAD}}(X) = |X|/2$

(c) $\hat{S}_{q,\mathrm{MMSE}}(X) = 3X^2/5$; $\hat{S}_{q,\mathrm{MAD}}(X) = 5X^2/8$

Exercise 20 (1.4)

Consider a random variable X with p.d.f.

$$p_X(x) = a \exp\left[-a(x-d)\right] u(x-d)$$

where $a > 0$ and d are two parameters.
Find the maximum likelihood estimators of both parameters, $\hat{A}_{\mathrm{ML}}$ and $\hat{D}_{\mathrm{ML}}$, as a function of K samples of X independently drawn, $\left\{X^{(k)}\right\}_{k=1}^{K}$.

Solution: $\hat{A}_{\mathrm{ML}} = \left[\frac{1}{K}\sum_{k=1}^{K}\left(X^{(k)} - \text{mín}\{X^{(k)}\}\right)\right]^{-1}$, $\qquad \hat{D}_{\mathrm{ML}} = \text{mín}_k\{X^{(k)}\}$

Exercise 21 (1.2; 1.7)

Random variables S and X have a joint probability density function given by

$$p_{S,X}(s,x) = \begin{cases} 10s, & 0 < s < x^2 \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider the estimation of S based on the observation of X, with the objective to minimize the following cost function:

$$c(S,\hat{S}) = S^2\left(S - \hat{S}\right)^2$$

Find:

(a) The optimal estimator of S, $\hat{S}_C$, which minimizes the mean cost given X, $\mathbb{E}\left\{c(S,\hat{S})|x\right\}$.

(b) The analytically constrained estimator $\hat{S}_L = wX$ which minimizes the mean cost $\mathbb{E}\left\{c(S,\hat{S})\right\}$.

(c) The mean cost of both estimators: $\mathbb{E}\left\{c(S,\hat{S}_C)\right\}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\}$.

(d) The bias of both estimators: $\mathbb{E}\left\{S - \hat{S}_C\right\}$ and $\mathbb{E}\left\{S - \hat{S}_L\right\}$.

(e) The variance of both estimators: $\mathrm{Var}\left\{S - \hat{S}_C\right\}$ and $\mathrm{Var}\left\{S - \hat{S}_L\right\}$.

Solution:

(a) $\hat{S}_C = \frac{4}{5}X^2$

(b) $\hat{S}_L = \frac{11}{15}X$

(c) $\mathbb{E}\left\{c(S,\hat{S}_C)\right\} = \frac{1}{195}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\} = \frac{7}{1170}$

(d) $\mathbb{E}\left\{S-\hat{S}_C\right\} = -\frac{2}{21}$ and $\mathbb{E}\left\{S-\hat{S}_L\right\} = -0.1349$

(e) $\text{Var}\left\{S-\hat{S}_C\right\} = 0.03163$ and $\text{Var}\left\{S-\hat{S}_L\right\} = 0.0326$

Exercise 22 (1.2; 1.3; 1.6)

Random variables S and X are characterized by the following joint distribution:

$$p_{S,X}(s,x) = c, \quad 0 < s < 1, \;\; s < x < 2s$$

with c a constant.

(a) Plot the support of the p.d.f., and use it to calculate the value of c.

(b) Give the expressions for the marginal p.d.f. of the random variables: $p_S(s)$ and $p_X(x)$.

(c) Find the minimum mean square error estimator of S based on the observation of X, $\hat{S}_{\text{MSE}}(X)$. Plot the estimator on the same plot as the support of $p_{S,X}(s,x)$, and discuss whether it would had been possible to obtain the estimator without analytical derivations.

(d) Calculate the mean square error $\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}(X)\right)^2\right\}$ incurred by the previous estimator.

(e) Now, find the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}}(X)$. Again, plot the estimator together with the support of $p_{S,X}(s,x)$. Discuss your result.

(f) Obtain the mean square error $\mathbb{E}\left\{\left(S-\hat{S}_{\text{LMSE}}(X)\right)^2\right\}$ of the linear estimator, and compare it with $\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}(X)\right)^2\right\}$.

(g) It is perceived (e.g., visualizing several samples of (X,S)) that there exist different statistical behaviors for $0 < X < 1$ and $1 < X < 2$. What would occur if, based on this, different optimal linear estimators where designed for each of the intervals ($\hat{S}_{A,\text{LMSE}}(X)$ y $\hat{S}_{B,\text{LMSE}}(X)$, respectively)? Verify analytically the proposed solution.

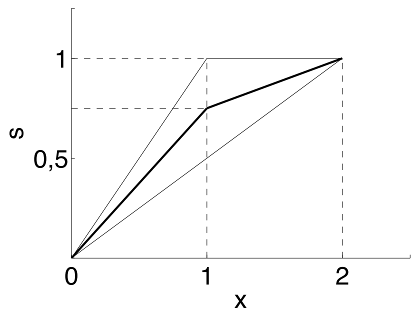
Solution:

(a) Since the area of the support of $p_{S,X}(s,x)$ is $1/2$, $c = 2$.

(b) $p_S(s) = 2s,\; 0 < s < 1;\qquad p_X(x) = \begin{cases} x & ,0 < x < 1 \\ 2-x & ,1 < x < 2 \end{cases}$

(c)

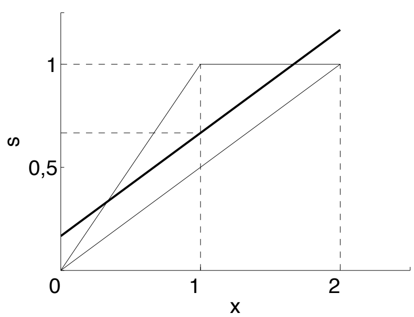
$$\hat{S}_{\text{MSE}}(X) = \begin{cases} \dfrac{3X}{4}, & 0 < X < 1 \\[2ex] \dfrac{1}{2}\left(\dfrac{X}{2}+1\right), & 1 < X < 2 \end{cases}$$



Since for every value X we have a uniform *a posteriori distribution* $p_{S|X}(s|x)$, the MSE estimator is given as the average between the minimum and maximum values of S (for each X).

(d) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\} = \dfrac{1}{96}$

(e) $\hat{S}_{\text{LMSE}}(X) = \dfrac{X}{2} + \dfrac{1}{6}$



(f) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}(X)\right)^2\right\} = \dfrac{11}{24}$, which is larger than $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$

(g) $\hat{S}_{A,\text{LMSE}}(X) = \dfrac{3X}{4}$ and $\hat{S}_{B,\text{LMSE}}(X) = \dfrac{1}{2}\left(\dfrac{X}{2} + 1\right)$. When jointly considered, these estimators compose $\hat{S}_{\text{MSE}}(X)$.
$p_A(s,x)$ and $p_B(s,x)$ are uniform, and now the linear estimators will also be optimal.

Exercise 23 (1.5; 1.7)

Consider the estimation of a random vector $\mathbf{S}$ from a statistically related observation vector $\mathbf{X}$:

$$\mathbf{X} = H\mathbf{S} + \mathbf{R}$$

H being a known matrix, $\mathbf{R}$ a noise vector with distribution $G(\mathbf{0}, v_r I)$, and $\mathbf{S}$ the random vector to be estimated, whose distribution is $G(\mathbb{E}\{\mathbf{S}\}, V_s)$. It is also known that $\mathbf{S}$ and $\mathbf{R}$ are independent random vectors:

(a) Find the ML estimator of $\mathbf{S}$, $\hat{\mathbf{S}}_{\text{ML}}$.

(b) Is the ML estimator unbiased? Justify your answer.

(c) As it is known, the MSE estimator of $\mathbf{S}$ is given by:

$$\hat{\mathbf{S}}_{\text{MSE}} = \left(H^T H + v_r V_s^{-1}\right)^{-1} H^T \mathbf{X}$$

Obtain the bias of $\hat{\mathbf{S}}_{\text{MSE}}$ and indicate under which conditions such bias vanishes.

Solution:

(a) $\hat{\mathbf{S}}_{\text{ML}} = \left(H^T H\right)^{-1} H^T \mathbf{X}$

(b) The estimator is unbiased.

(c) $\mathbb{E}\left\{\hat{\mathbf{S}}_{\text{MSE}} - \mathbf{S}\right\} = \left(H^T H + v_r V_s^{-1}\right)^{-1} H^T H \mathbb{E}\{\mathbf{S}\} - \mathbb{E}\{\mathbf{S}\}$. The bias goes to zero as the noise power decreases towards 0.

Exercise 24 (1.4; 1.7)

We have access to a set of K samples, $\left\{X^{(k)}\right\}_{k=1}^{K}$, independently drawn from a random variable X with p.d.f.

$$p_X(x) = \frac{1}{bx^2} \exp\left(-\frac{1}{bx}\right) u(x)$$

with $b > 0$ a constant.

(a) Find the ML estimator of b as a function of the available samples, $\hat{B}_{\text{ML}}$.

(b) Verify that random variable $Y = 1/X$ is characterized by a unilateral exponential p.d.f. $p_Y(y)$, and obtain the value of the mean of such distribution.

(c) Considering your answers to the previous sections, is $\hat{B}_{\text{ML}}$ an unbiased estimator?

Solution:

(a) $\hat{B}_{\text{ML}} = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{X^{(k)}}$

(b) $p_Y(y) = \frac{1}{b} \exp\left(-\frac{y}{b}\right) u(y)$

(c) The estimator is unbiased.

Exercise 25 (1.2)

Consider the family of cost functions given by

$$c(S, \hat{S}) = \frac{1}{N+1} \hat{S}^{N+1} + \frac{1}{N(N+1)} S^{N+1} - \frac{1}{N} S \hat{S}^N$$

where N is a non-negative and odd integer.

(a) Assuming that

$$p_{S,X}(s,x) = \frac{1}{\lambda x} \exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right) u(s) u(x) \quad \lambda > 0$$

find the minimum cost estimator of S given X.

(b) Obtain the minimum mean cost.

(c) Determine the coefficient w that minimizes the mean cost of an estimator with analytical shape

$$\hat{S}_L = w X^m$$

m being a positive integer.

Hint: $\int_0^\infty x^N \exp(-x)\, dx = N!$

Solution:

(a) $\hat{S} = X$

(b) $\mathbb{E}\left\{c(S,\hat{S})\right\} = ((N+1)! - 1)\,(N-1)!\lambda^{N+1}$

(c) $w = \dfrac{(Nm+1)!}{(Nm+m)!\lambda^{m-1}}$

Exercise 26 (1.4; 1.7)

An order-N Erlang probability density is characterized by the following expresion:

$$p_X(x) = \frac{a^N x^{N-1}\exp(-ax)}{(N-1)!} \qquad x > 0, \quad a > 0$$

Assume that N is known. Considering that the mean of the distribution is given by $m = N/a$, obtain:

(a) The ML estimator of the mean using K independent observations of the variable, $\hat{M}_{\mathrm{ML}}$.

(b) The bias of $\hat{M}_{\mathrm{ML}}$.

(c) Is $\hat{M}_{\mathrm{ML}}$ variance-consistent?

Solution:

(a) $\hat{M}_{\mathrm{ML}} = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} X^{(k)}$

(b) The estimator is unbiased.

(c) $\mathrm{Var}\left\{\hat{M}_{\mathrm{ML}}\right\} = \dfrac{v_x}{K}$; therefore, the estimator is variance-consistent.

Exercise 27 (1.6)

Random vector $X = [X_1, X_2, X_3]^T$ follows a p.d.f. with mean $\mathbf{m} = \mathbf{0}$ and covariance matrix

$$V_{XX} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) Obtain the coefficients (w_0, w_1 and w_2) of the linear minimum mean square error estimator of X_3 given X_1 and X_2,

$$\hat{X}_{3,\mathrm{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Calculate the mean square error of the estimator $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\mathrm{LMSE}}\right)^2\right\}$.

Solution:

(a) $\hat{X}_{3,\mathrm{LMSE}} = -\dfrac{1}{5}X_1 + \dfrac{4}{5}X_2$

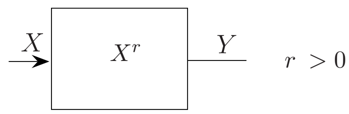
(b) $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\mathrm{LMSE}}\right)^2\right\} = \dfrac{8}{5}$

Exercise 28 (1.4)

A random variable X with p.d.f.

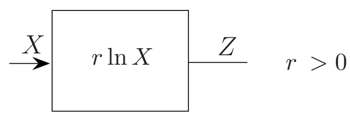
$$p_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

is transformed as indicated in the figure, producing a random observation Y.



(a) Obtain the maximum likelihood estimator of r, $\hat{R}_{\mathrm{ML}}$, based on K independently drawn observations of Y.

(b) Now, consider the following situation



and obtain $\hat{R}_{\mathrm{ML}}$ using K independent observations of random variable Z. Discuss your result.

Solution:

(a) $\hat{R}_{\mathrm{ML}} = -\frac{1}{K}\sum_{k=1}^{K} \ln Y^{(k)}$. The unknown parameter of the transformation is being identified.

(b) $\hat{R}_{\mathrm{ML}} = -\frac{1}{K}\sum_{k=1}^{K} Z^{(k)}$. It is coherent with the previous estimator since $Z = \ln Y$, which is a deterministic (and invertible) transformation of Y.

Exercise 29 (1.4)

An unknown deterministic parameter s, $s > 0$ is measured using two different systems, which provide observations

$$X_i = A_i s + N_i, \quad i = 1, 2$$

where $\{A_i\}, \{N_i\}$, are independent Gaussian random vectors, with means $\mathbb{E}\{A_i\} = 1, \mathbb{E}\{N_i\} = 0$, and variances $\{v_{Ai}\}$, $\{v_{Ni}\}$, respectively $(i = 1, 2)$.

(a) Stablish the expression that defines the ML estimator of s, $\hat{S}_{\mathrm{ML}}$.

(b) Obtain $\hat{S}_{\mathrm{ML}}$ for the particular case $v_{Ai} = 0, i = 1, 2$.

(c) Obtain $\hat{S}_{\mathrm{ML}}$ for the particular case $v_{Ni} = 0, i = 1, 2$.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = \arg\ \text{mín}_s \left\{ \ln\left[(s^2 v_{A1} + v_{N1})(s^2 v_{A2} + v_{N2})\right] + \frac{(s - X_1)^2}{s^2 v_{A1} + v_{N1}} + \frac{(s - X_2)^2}{s^2 v_{A2} + v_{N2}} \right\}$

(b) $\hat{S}_{\mathrm{ML}} = \frac{v_{N2} X_1 + v_{N1} X_2}{v_{N1} + v_{N2}}$

(c) $\hat{S}_{\mathrm{ML}} = \frac{1}{4}\sqrt{\left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)^2 + 8\left(\frac{X_1^2}{v_{A1}} + \frac{X_2^2}{v_{A2}}\right)} - \left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)$

Exercise 30 (1.3; 1.6)

Let X and S be two random variables with joint pdf

$$p_{X,S}(x,s) \begin{cases} \alpha & ; \quad 0 < x < 1,\ 0 < s < 2(1-x) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

with α a constant.

(a) Plot the support of the pdf, and use it to determine the value of α.

(b) Obtain the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) Find the minimum mean square error estimator of S given X, $\widehat{S}_{\text{MSE}}$

(d) Find the linear minimum mean square error estimator of S given X, $\widehat{S}_{\text{LMSE}}$.

Solution:

(a) $\alpha = 1$

(b) $p_{S|X}(s|x) = \dfrac{1}{2\,(1-x)}$

(c) $\widehat{S}_{\text{MSE}} = 1 - X$

(d) $\widehat{S}_{\text{LMSE}} = 1 - X$

Exercise 31 (1.3; 1.4; 1.7)

Consider an estimation problem where the goal is to estimate a random variable S using an observation of another random variable X characterized by:

$$X = S + N$$

where the prior pdf of S is

$$p_S(s) = s\exp(-s) \quad s > 0$$

and where N is an additive noise, independent of S, with the following distribution

$$p_N(n) = \exp(-n) \quad n > 0$$

Find:

(a) The maximum likelihood estimator of S, $\widehat{S}_{\text{ML}}$.

(b) The joint pdf of X and S, $p_{X,S}(x,s)$, and the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) The maximum a posteriori estimator of S given X, $\widehat{S}_{\text{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\widehat{S}_{\text{MMSE}}$.

(e) The bias of all previous estimators, $\widehat{S}_{\text{ML}}$, $\widehat{S}_{\text{MAP}}$ and $\widehat{S}_{\text{MMSE}}$.

(f) Which of the previous estimators has a minimum variance? Justify your answer without calculating the variances of the estimators.

Hint: You can use the following expression to solve the exercise:

$$\int_0^{\infty} x^N \exp(-x)\, dx = N!$$

Solution:

(a) $\widehat{S}_{\text{ML}} = X$

(b) $p_{X,S}(x,s) = s\exp(-x) \quad x > s,\ s > 0$

$p_{S|X}(s|x) = \dfrac{2s}{x^2} \quad 0 < s < x,\ x > 0$

(c) $\widehat{S}_{\text{MAP}} = X$

(d) $\widehat{S}_{\text{MMSE}} = \dfrac{2}{3}X$

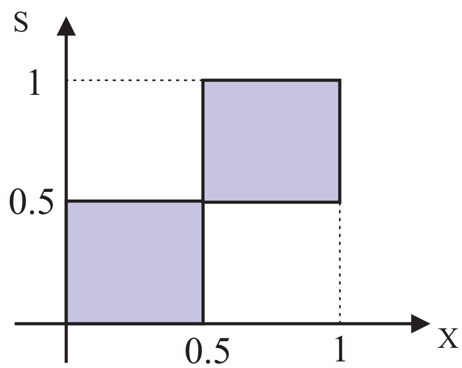
(e) $\mathbb{E}\left\{S - \widehat{S}_{\text{ML}}\right\} = \mathbb{E}\left\{S - \widehat{S}_{\text{MAP}}\right\} = -1$

$\mathbb{E}\left\{S - \widehat{S}_{\text{MMSE}}\right\} = 0$

$\text{Var}\left\{\widehat{S}_{\text{MMSE}}\right\} < \text{Var}\left\{\widehat{S}_{\text{MAP}}\right\} = \text{Var}\left\{\widehat{S}_{\text{ML}}\right\}$

Exercise 32 (1.3)

In the plot below, the shaded region shows the domain of a joint distribution of S and X, i.e., the set of points for which $p_{X,S}(x,s) \neq 0$.



Please, provide justified answers to the following questions:

(a) If it is known that $p_{X,S}(x,s)$ is constant in its domain, which is the MSE estimator of S given X? Provide a graphical representation of this estimator.

(b) Is there any $p_{X,S}(x,s)$ with the previous domain for which the MSE estimator of S given X is $\widehat{S}_{\text{MMSE}} = X/2$?

(c) Justify if there exists any $p_{X,S}(x,s)$ with the previous domain, so that $\widehat{S} = 0.5$ is:
- The minimum mean square error estimator of S given X.
- The minimum mean absolute deviation estimator of S given X.
- The maximum *a posteriori* estimator of S given X.

Solution:

(a) $\widehat{S}_{\text{MMSE}} = 0.25$ for $0 < x < 0.5$ and $\widehat{S}_{\text{MMSE}} = 0.75$ for $0.5 < x < 1$

(b) When $0.5 < x < 1$, $p_{S|X}(s|x)$ is non-zero for $0.5 < s < 1$, thus $X/2$ can never be the mean of $p_{S|X}(s|x)$ for that range of X.

(c) $\widehat{S} = 0.5$ cannot be the mean or the median of $p_{S|X}(s|x)$, but it can be its maximum. Therefore, $\widehat{S} = 0.5$ can just be $\widehat{S}_{\text{MAP}}$ (but not $\widehat{S}_{\text{MMSE}}$ or $\widehat{S}_{\text{MAD}}$).

Exercise 33 (1.3; 1.4)

A random variable S follows an exponential pdf

$$p_S(s) = \lambda e^{-\lambda s}, \qquad s > 0$$

with $\lambda > 0$. Consider now a discrete random variable X related to S via a Poisson distribution, i.e.,

$$P_{X|S}(x|s) = \frac{s^x e^{-s}}{x!}, \qquad x = 0, 1, 2, \ldots$$

(a) Determine the ML estimator of S given x.

(b) Assume now that we have access to K independent realizations $\{(x^{(k)}, s^{(k)}), k = 1, \ldots, K\}$ of (X, S). Find the ML estimator of λ based on these observations.

(c) Find the MAP estimation of S for $x = 1$.

Solution:

(a) $\widehat{S}_{\mathrm{ML}} = X$

(b) $\widehat{\lambda}_{\mathrm{ML}} = \dfrac{1}{\frac{1}{K}\sum_{k=1}^{K} s^{(k)}}$

(c) $\widehat{S}_{\mathrm{MAP}} = \dfrac{X}{1+\lambda}$

Exercise 34 (1.5; 1.7)

We wish to estimate a random variable S from the observation of another random variable X given by:

$$X = S + N_1 + N_2$$

where S is Gaussian-distributed, with mean and variance m_s and v_s, respectively, and where N_1 and N_2 are two noise random variables, independent of S, and with joint p.d.f.

$$p_{N_1,N_2}(n_1, n_2) \sim G\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{bmatrix}\right)$$

Obtain:

(a) The minimum mean square error estimator of S given X, $\widehat{S}_{\mathrm{MMSE}}$.

(b) The maximum a posteriori estimator of S given X, $\widehat{S}_{\mathrm{MAP}}$.

(c) The bias and variance of both estimators.

Solution:

(a) $\widehat{S}_{\mathrm{MMSE}} = \dfrac{v_s}{v_s + v_1 + v_2 + 2v_{12}} X + m_s\left(1 - \dfrac{v_s}{v_s + v_1 + v_2 + 2v_{12}}\right)$

(b) $\widehat{S}_{\mathrm{MAP}} = \widehat{S}_{\mathrm{MMSE}}$

(c) $\mathbb{E}\left\{S - \widehat{S}_{\mathrm{MMSE}}\right\} = \quad \mathbb{E}\left\{S - \widehat{S}_{\mathrm{MAP}}\right\} = 0$

$\mathrm{Var}\left\{\widehat{S}_{\mathrm{MMSE}}\right\} = \mathrm{Var}\left\{\widehat{S}_{\mathrm{MAP}}\right\} = \dfrac{v_s^2}{v_s + v_1 + v_2 + 2v_{12}}$

Exercise 35 (1.4; 1.7)

We have access to a set of observations $\{x^{(k)}, k = 1, \ldots, K\}$, independently drawn from a Pareto distribution with deterministic parameters α and β, i.e.,

$$p_{X|\alpha,\beta}(x|\alpha,\beta) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}, \qquad x \geq \beta$$

with $\alpha > 1$ and $\beta > 0$.

(a) Assuming the value of β is known, find the ML estimator of α, $\widehat{\alpha}_{\mathrm{ML}}$.

(b) For $K = 1$ (i.e., just one observation), find the ML estimator of β, $\widehat{\beta}_{\mathrm{ML}}$.

(c) For $K = 1$, determine the bias of $\widehat{\beta}_{\mathrm{ML}}$.

Solution:

(a) $\widehat{\alpha}_{\mathrm{ML}} = \dfrac{1}{\frac{1}{K}\sum_{k=1}^{K} \ln \frac{x^{(k)}}{\beta}}$

(b) $\widehat{\beta}_{\mathrm{ML}} = x^{(1)}$

(c) Sesgo $\left\{\widehat{\beta}_{\mathrm{ML}}\right\} = -\dfrac{1}{\alpha - 1}\beta$

Exercise 36 (1.3)

Obtain the minimum mean square error estimator of random variable S based on the observation of random variable X, in the following two cases:

(a)

$$p_{X,S}(x,s) = \begin{cases} 1, & 0 \leq x \leq 1, \quad 0 \leq s \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$p_{X,S}(x,s) = \begin{cases} 2, & 0 \leq s \leq 1 - x, \quad 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

(a) $\widehat{S}_{\mathrm{MMSE}} = \frac{1}{2}$

(b) $\widehat{S}_{\mathrm{MMSE}} = \frac{1-X}{2}$

Exercise 37 (1.3; 1.7)

Consider the estimation of a random variable S based on the observation of X, where the two variables satisfy the following joint pdf:

$$p_{S,X}(s,x) = \begin{cases} 6s, & 0 < s < x - 1, \quad 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Obtain:

(a) The minimum mean square error estimator of S given X, $\widehat{S}_{\mathrm{MMSE}}$.

(b) The maximum *a posteriori* estimator of S given X, $\widehat{S}_{\mathrm{MAP}}$.

(c) The minimum mean absolute error estimator of S given X, $\widehat{S}_{\mathrm{MAD}}$.

(d) The bias of all previous estimators.

Solution:

(a) $\widehat{S}_{\text{MMSE}} = \frac{2}{3}(X-1)$

(b) $\widehat{S}_{\text{MAP}} = X - 1$

(c) $\widehat{S}_{\text{MAD}} = \frac{1}{\sqrt{2}}(X-1)$

(d) $\mathbb{E}\left\{S - \widehat{S}_{\text{MMSE}}\right\} = 0 \quad \mathbb{E}\left\{S - \widehat{S}_{\text{MAP}}\right\} = -0.25 \quad \mathbb{E}\left\{S - \widehat{S}_{\text{MAD}}\right\} = -0.03$

Exercise 38 (1.4; 1.7)

We have access to a collection of observations $\{x^{(k)}, k = 1, \ldots, K\}$ independently drawn from a Rayleigh distribution with parameter b, i.e.,

$$p_{X|b}(x|b) = \frac{2}{b} x \exp\left(-\frac{x^2}{b}\right), \qquad x \geq 0$$

with $b > 0$.

(a) Determine the ML estimator of b, $\widehat{b}_{\text{ML}}$.

(b) If the observations are $\{2, 0, 1, 1\}$, what would be the most likelihood value of b?

(c) Find the bias of the estimator.

Solution:

(a) $\widehat{b}_{\text{ML}} = \dfrac{1}{K} \displaystyle\sum_{k=1}^{K} \left(x^{(k)}\right)^2$

(b) $\widehat{b}_{\text{ML}} = 1.5$

(c) The estimator is unbiased.

Exercise 39 (1.3; 1.7)

Consider three independent random variables, S_1, S_2 and S_3, with the same *a priori* pdf. We wish to estimate S_1 from a single observation of $X = S_1 + S_2 + S_3$.

(a) Justify briefly why the minimum mean square error estimator of S_1 given x is $\hat{s}_1 = \dfrac{x}{3}$.

(b) Find the bias of the estimator given in (a). Is it biased or unbiased?

Assuming in the following that S_1, S_2 and S_3 follow a uniform distribution between -1 and 1:

(c) Obtain the pdf of X given s_1.

(d) Find the variance of estimator $\hat{S}_1$.

Solution:

(a) Using symmetry arguments, all $\mathbb{E}\{S_i|x\}$ should be the same and sum up to x.

(b) The estimator is unbiased.

(c) $p_{X|S_1}(x|s_1) = 1/2 - |x - s_1|/4$ for $-2 < x < 2$, and 0 otherwise.

(d) $\text{Var}(S_1) = 1/9$

Exercise 40 (1.5)

Two independent Gaussian variables Z_1 and Z_2 have means 2 and 1, respectively. Both variables have unit variance. We wish to estimate the difference $S = Z_1 - Z_2$.

(a) Obtain $p_S(s)$, the MMSE estimator of S, and the mean square error of such estimator if no other information is available.

(b) Consider now we can observe $X = Z_1 + Z_2 = 3$. Find $p_{S|X}(s|x)$, the MMSE estimator of S given X, and the mean square error of such estimator. Discuss your result in relation to your answers to the previous subsection.

Solution:

(a) $\widehat{S}_{\text{MMSE}} = 1$; $\mathbb{E}\left\{\left(S - \widehat{S}_{\text{MMSE}}\right)^2\right\} = 2$.

(b) X is independent of S, thus the answers are the same as in subsection (a).

Exercise 41 (1.2;1.3)

We wish to estimate a random variable S based on the observation of X. Their joint pdf is given by:

$$p_{S,X}(s,x) = 24xs, \quad 0 \leq s \leq 1 - x, \; 0 < x < 1$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MMSE}}$.

(b) The MAP estimator of S given X, $\hat{S}_{\text{MAP}}$.

(c) The MAD estimator of S given X, $\hat{S}_{\text{MAD}}$.

(d) The estimator with analytical shape $\hat{S}_{\text{q,MSE}} = wX^2$ that minimizes the mean square error.

Solution:

(a) $\hat{S}_{\text{MMSE}} = \dfrac{2}{3}(1 - X)$.

(b) Since $p_{S|X}(s|x)$ is strictly increasing with respect to s, $\hat{S}_{\text{MAP}} = 1 - X$.

(c) $\hat{S}_{\text{MAD}} = \dfrac{1 - X}{\sqrt{2}}$.

(d) $w = 0.8$

Exercise 42 (LMSE)

The joint pdf of two random variables S and X is given by:

$$p_{S,X}(s,x) = \frac{1}{3}(x + s) \quad 0 < x < 2, \;\; 0 < s < 1.$$

(a) Obtain the Linear Minimum Mean Square Error estimator, $\widehat{S}_{\text{LMSE}} = w_0 + w_1 X$.

(b) Calculate the mean square error of the estimator.

Solution:

(a) $\hat{s}_{\text{LMSE}} = \frac{14}{23} - \frac{1}{23}x.$

(b) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}\right)^2\right\} = \frac{11}{138}$

Exercise 43 (ML, MSE)

The manager of an IT company intends to analyze the productivity of his employees by estimating the time S they need to implement a certain computer program. With this goal, at 12:00 a.m. the manager requests the implementation of the program; instead of directly starting the coding task, the employees need to finish first whatever task they are currently carrying out, what requires an additional time N. As a consequence, the total elapsed time between the request of the program implementation and each employee's notification indicating the conclusion of the task is $X = S + N$.

It is known that the time N that the employees need to finish the tasks and start the program implementation can be modeled as the following exponential distribution

$$p_N(n) = a \exp\left(-an\right) \quad n > 0,$$

whereas the time for coding the program, S, follows also an exponential distribution, in this case characterized by the expression

$$p_S(s) = b \exp\left(-b\left(s - c\right)\right) \quad s > c.$$

(a) Before the described process, a simulation has been carried out using a control group, and measuring directly the times N and S required by the members of this group. As a result of the test, four independent observations were obtained for each variable. Concretely, the four observations for N were 6, 10, 12, and 20 minutes, whereas the observations for S were 6, 12, 18, and 36 minutes. Based on these observations, estimate using maximum likelihood the values of constants a, b, and c.

Consider in the following $a = 10$ minutes, $b = 10$ minutes, and $c = 5$ minutes.

(b) For the actual productivity test, the manager receives notifications from three different employees indicating that they have finished the implementation of the program at 12:25, 12:30, and 12:40 a.m. Estimate using maximum likelihood the time that each employee needed for the implementation of the program.

(c) Repeat the estimation of the previous subsection if a minimum mean square error estimator were used.

Solution:

(a) $\hat{a}_{\text{ML}} = \frac{K}{\sum_{k=1}^{K} n^{(k)}} = \frac{1}{12}$ minutes^{-1};
$\hat{c}_{\text{ML}} = \text{mín}_k \left\{s^{(k)}\right\} = 6$ minutes;
$\hat{b}_{\text{ML}} = \frac{K}{\sum_{k=1}^{K}\left(s^{(k)} - \hat{c}_{\text{ML}}\right)} = \frac{1}{12}$ minutes^{-1}.

(b) $\hat{s}_{\text{ML}} = x$. $\hat{s}_{\text{ML}}(x = 25) = 25$, $\hat{s}_{\text{ML}}(x = 30) = 30$,$\hat{s}_{\text{ML}}(x = 40) = 40$.

(c) $\hat{s}_{\text{MMSE}} = \frac{x+5}{2}$. $\hat{s}_{\text{MMSE}}(x = 25) = 15$, $\hat{s}_{\text{MMSE}}(x = 30) = 17.5$,$\hat{s}_{\text{MMSE}}(x = 40) = 22.5$.

Exercise 44 (MSE, MAD)

The joint pdf of two random variables S and X is given by:

$$p_{S,X}(s,x) = \left[\begin{array}{lll} \frac{4s}{x(1-x)}, & x(1-x) < s < 2x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} & \end{array}\right.$$

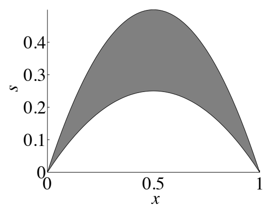
(a) Provide an approximate representation of the support of the joint distribution.

(b) Obtain the estimator $\widehat{S}_{\text{MMSE}}$.

(c) Obtain the estimator $\widehat{S}_{\text{MAD}}$.

Solution:

(a) The support of the pdf is the shaded region in the following figure:



(b) $\hat{s}_{\text{MMSE}} = \dfrac{14}{9}x(1-x)$.

(c) $\hat{s}_{\text{MAD}} = \sqrt{\dfrac{5}{2}}x(1-x)$

Exercise 45 (ML)

Let x be a measurement of the instantaneous voltage at a circuit node. In such node exists a signal component with value s, contaminated by Gaussian aditive noise with mean zero and variance v. A priori, the value of s follows a Gaussian pdf with both the mean and variance equal to 1.

(a) Assuming v is known, obtain the maximum likelihood estimator of s, $\hat{s}_{\text{ML}}(x)$.

(b) Calculate mean square error incurred by estimator $\hat{s}_{\text{ML}}(x)$.

(c) Obtain the likelihood of v given x, i.e. $p_{X|v}(x|v)$.

(d) Calculate the maximum likelihood estimator of v, $\hat{v}_{\text{ML}}(x)$.

Solution:

(a) $\hat{s}_{\text{ML}}(x) = x$

(b) $\mathbb{E}[(x-s)^2] = v$

(c) $p(x|v) = G(x|1, v+1)$

(d) $\hat{v}_{\text{ML}}(x) = \text{máx}[(x-2)x, 0]$

Exercise 46 (LMSE)

The joint distribution of S and X is given by:

$$p_{S,X}(s,x) = \frac{4}{(x+s)^3}, \qquad s \geq 1, \quad x \geq 1.$$

(a) Find $p_{S|X}(s|x)$.

(b) Find the MAP estimator of S given X.

(c) Find the minimum absolute deviation (MAD) estimator of S given X.

Solution:

(a) $p_{S|X}(s|x) = \dfrac{2(x+1)^2}{(x+s)^3}, \qquad , x \geq 1.$

(b) $\widehat{S}_{\text{MAP}} = 1.$

(c) $\widehat{S}_{\text{MAD}} = (\sqrt{2}-1)X + \sqrt{2}.$

Exercise 47 (ML)

An energetic student gets up early every morning and reaches the bus stop exactly at 8.00 am, the scheduled arrival time of the only bus that can take take him to the university. The bus is usually late and never arrives before its scheduled arrival time. The pdf of the delay of the bus

$$p_{T_{\text{B}}|\Lambda_{\text{B}}}(t_{\text{B}}|\lambda_{\text{B}}) = \lambda_{\text{B}} \exp(-\lambda_{\text{B}} t_{\text{B}}), \quad 0 < t_{\text{B}} \text{ min}$$

where t_{B} are the minutes of delay. The delays are iid for each day.

A second, lazy student makes use of the same bus, but isn't as punctual as the energetic one. The delay in the arrival of the lazy student to the bus stop follows the pdf

$$p_{T_{\text{L}}|\Lambda_{\text{L}}}(t_{\text{L}}|\lambda_{\text{L}}) = \lambda_{\text{L}} \exp(-\lambda_{\text{L}} t_{\text{L}}), \quad 0 < t_{\text{L}} \text{ min}$$

where T_{L} is the delay wrt to the energetic student in reaching the bus stop. These delays are iid for each day. Finally, T_{B} and T_{E} are independent.

(a) Modeling: The first five days of the course the bus arrived to the bus stop 0, 6, 15, 20 and 24 minutes late, whereas the lazy student arrived to the bus stop 15, 10, 12, 5 and 3 minutes late. Estimate λ_{B} and λ_{L} using ML. Specify units.

Consider this ML estimates as the true values for λ_{B} and λ_{L} for the remainder of the exercise.

(b) Compute the expected waiting time for the energetic student at the bus stop.

(c) The sixth day, the lazy student arrives to the bus stop at 8.05 am. He meets there the energetic student and asks him how much longer are both expected to wait (the expected time) until the bus comes. Compute this quantity and contrast it with your answer to b).

Hint: You are being asked to compute $\mathbb{E}[t_{\text{B}} - 5 \text{ min}|t_{\text{B}} > 5 \text{ min}]$.

(d) If the lazy student misses the bus, he won't attend university that day. Assuming this arrival process for the bus and the lazy student is repeated for the rest of the course, compute the *expected* percentage of days that each student attends university.

Solution:

(a) $\lambda_{\text{B}} = \frac{1}{13} \text{ min}^{-1}$, $\lambda_{\text{V}} = \frac{1}{9} \text{ min}^{-1}$ y $\mathbb{E}[t_{\text{B}}] = 13$ min.

(b) $\mathbb{E}[t_{\text{B}} - 5 \text{ min}|t_{\text{B}} > 5 \text{ min}] = 13$ min. Arriving 5 min late doesn't save waiting time. On average, he has to wait as much as the willful student the remaining days. This apparent paradox arises from the fact that there is an additional information in the sixth day: the bus will be more than 5 minutes late.

(c) $p(t_{\text{V}} < t_{\text{B}}) = \frac{\lambda_{\text{v}}}{\lambda_{\text{v}}+\lambda_{\text{B}}}$, in percentage $\frac{100\lambda_{\text{v}}}{\lambda_{\text{v}}+\lambda_{\text{B}}}$ %. The willful student always goes.

Exercise 48 (MSE)

The random variables S, X_1 and X_2 are jointly Gaussian. The parameters of its joint distribution are unknown, but it is known that:

- The marginal distribution of X_1 and X_2 is

$$p_{X_1,X_2}(x_1,x_2) \sim G\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}\right]\right)$$

- The minimum mean square error (MSE) estimator of S based on X_1 only is:

$$\widehat{S}_{\text{MMSE},1} = \frac{1}{2}X_1,$$

- The minimum MSE estimator of S based on X_2 only is:

$$\widehat{S}_{\text{MMSE},2} = X_2.$$

We want to estimate S given a new random variable X_3 given by the following linear combination of X_1 and X_2

$$X_3 = X_1 + X_2.$$

(a) Compute the mean value of X_3, $\mathbb{E}[X_3]$, its variance, v_{X_3}, and its covariance with S, v_{S,X_3}.

(b) Compute the minimum MSE estimator os S based on X_3 only, $\widehat{S}_{\text{MMSE},3}$.

(c) Given that $\mathbb{E}[S^2] = 1$, compute the MSE of the estimator computed in the previous section, $\mathbb{E}[(S - \widehat{S}_{\text{MMSE},3})^2]$.

Solution:

(a) $\mathbb{E}[X_3] = 0, v_{X_3} = 3$, $v_{S,X_3} = 3/2$.

(b) $\widehat{S}_{\text{MMSE},3} = 1/2X_3$.

(c) $\mathbb{E}[(S - \widehat{S}_{\text{MMSE},3})^2] = 1/4$.

Exercise 49 (Change of variable, LMSE)

Two random iid variables X and Y follow a uniform pdf between 0 and 1. Two new random variables are generated as $U = \text{máx}(X, Y)$ and $V = \text{mín}(X, Y)$, i.e., they are defined as the maximum and minimum, respectively, of the original uniform iid variables.

(a) Compute $p_{U|X}(u|x)$ and $p_{V|X}(v|x)$.

(b) Compute $p_U(u)$ and $p_V(v)$.

(c) Compute $\mathbb{E}[U]$, $\mathbb{E}[U^2]$ and $\mathbb{E}[V]$.

(d) Compute the LMMSE estimator of V given U, $\hat{v}_{\text{LMMSE}}(u)$.

Hint: In case you didn't notice, in part (a) you are performing a change of variable. You might find it helpful to draw $\text{mín}(x, y)$ and $\text{máx}(x, y)$ as functions of y for a fixed value of x. The flat parts of these two functions will produce Dirac deltas in the corresponding results.

Solution:

(a) $p_{U|X}(u|x) = x\delta(u - x) + 1$ for $x \leq u \leq 1$, and 0 otherwise

$p_{V|X}(v|x) = (1 - x)\delta(v - x) + 1$ for $0 \leq v \leq x$, and 0 otherwise

(b) $p_U(u) = 2u$ for $0 \leq u \leq 1$, and 0 otherwise

$p_V(v) = 2(1 - v)$ for $0 \leq v \leq 1$, and 0 otherwise

(c) $\mathbb{E}[U] = 2/3$, $\mathbb{E}[U^2] = 1/2$ and $\mathbb{E}[V] = 1/3$

(d) $\hat{v}_{\text{LMMSE}}(u) = u/2$.

Exercise 50 (ML)

The joint distribution of random variables S and X is known to be:

$$p_{X|s}(x|s) = s\exp(-s\exp(-x) + x), \qquad , \quad x \in \mathbb{R}.$$

(a) Compute the ML estimate of S given x.

(b) Compute the ML estimate of S given K independent and identically distributed observations, $\{X^{(k)}, k = 1, \ldots, K\}$.

(c) Let $K = 2$ and assume that $x^{(1)} = 0$, $x^{(2)} = -\ln 2$. Compute the value of the maximum likelihood for these observations.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = \exp(X)$.

(b) $\hat{S}_{\mathrm{ML}} = \dfrac{K}{\sum_{k=1}^{K} \exp(-X^{(k)})}$.

(c) $p_{X^{(1)},X^{(2)}|s}(0, -\ln(2)|\hat{s}_{\mathrm{ML}}) = \dfrac{2\exp(-2)}{9}$.

Exercise 51 (ML, MAP, MSE)

We want to estimate the value of a random variable S using a random observation X, which is related to S via

$$X = S - R$$

where R is a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Find the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MMSE}}$.

Solution:

(a) $p_{X|S}(x|s) = \exp(x - s), \quad x < s$.
$\hat{S}_{\mathrm{ML}} = X$.

(b) $p_{X,S}(x, s) = \exp(-x - 2s), \quad s > 0;\ x < s;$

$$\hat{S}_{\mathrm{MAP}} = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$$

(c) $$p_{S|X} = \begin{cases} 2\exp(-2s), \quad s > 0 & \text{si } x < 0 \\ 2\exp(2x)\exp(-2s), \quad s > x & \text{if } x > 0 \end{cases}$$

$$\hat{S}_{\mathrm{MMSE}} = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ \frac{1}{2} + x & \text{if } x > 0 \end{cases}$$

Exercise 52 (ML)

Random variable X is driven by the likelihood function

$$p_{X|S}(x|s) = \ln\left(\frac{1}{s}\right) s^x, \qquad x \geq 0, \qquad 0 \leq s \leq 1$$

where s is an unknown parameter.

(a) Compute the ML estimate of s given x.

(b) Compute the value of the maximum likelihood for $x = 1$.

(c) Compute the ML estimate of s given K a set $\mathcal{C} = \{(x^{(k)}), k = 1, \ldots, K\}$ de K of independent realizations of X.

Solution:

(a) $\widehat{S}_{\mathrm{ML}} = \exp\left(-\frac{1}{X}\right)$

(b) $p_{X|S}(1|s_{\mathrm{ML}}) = \frac{1}{e}$

(c) $\widehat{S}_{\mathrm{ML}} = \exp\left(-\frac{K}{\sum_{k=1}^{K} x^{(k)}}\right)$

Exercise 53 (ML, MAP, MSE)

We want to estimate the value of a random variable S using a random observation X, from which the joint probability distribution is known

$$p_{S,X}(s,x) = 4x, \quad 0 < s < x^2, \quad 0 < x < 1$$

Obtain:

(a) The maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

(b) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(c) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MMSE}}$.

(d) The linear estimator of S, with minimum mean square error, given X, $\hat{S}_{\mathrm{LMMSE}} = w_0 + w_1 x$.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = X^2$.

(b) $\hat{S}_{\mathrm{MAP}}$ is given by any value of $S \in (0, X^2)$

(c) $\hat{S}_{\mathrm{MMSE}} = \frac{X^2}{2}$

(d) $\hat{S}_{\mathrm{LMMSE}} = X - \frac{7}{15}$

Exercise 54 (ML)

Random variable X is characterized by the following density:

$$p_X(x) = a^2 x \exp(-ax), \qquad x \geq 0$$

where a is an unknown parameter.

(a) Obtain the expression of the ML estimator of a for a given set $\mathcal{C} = \{(x^{(k)}), k = 1, \ldots, K\}$ of K independent realizations of X.

(b) Given the set of observations $\mathcal{C} = \{0.2, 0.5, 0.8, 1\}$, find the value of $\widehat{A}_{\mathrm{ML}}$.

Solution:

(a) $\widehat{A}_{\mathrm{ML}} = \frac{2K}{\sum_{k=1}^{K} x^{(k)}}$

(b) $\widehat{A}_{\mathrm{ML}} = 3.2$

Exercise 55 (LMSE)

An electric company owns two wind farms. The total generated power can be modeled as

$$S = 10(2X_1 + X_2),$$

where S is the generated power, and $X_i,\ i = 1, 2$, represents wind speed at each farm. It is further known that the joint distribution of X_1 and X_2 is given by

$$p_{X_1,X_2}(x_1, x_2) = \frac{1}{a\,b} \exp\left(-\frac{x_1}{a}\right), \quad \text{for } 0 < x_1 < \infty \quad \text{and} \quad x_1 < x_2 < x_1 + b \tag{1}$$

with $b = a \ln 2$.

We wish to design a linear minimum mean square error estimator of S based just on observation X_1. In order to do so:

(a) Sketch the support of the joint distribution of X_1 and X_2.

(b) Obtain the marginal distribution of X_1. Calculate the mean and mean square value of such variable.

(c) Find the mean of X_2 and the correlation between X_1 and X_2: $\mathbb{E}\{X_1 X_2\}$.

(d) Find the optimum weights of the desired LMSE estimator. $\hat{S}_{\text{LMSE}} = w_0^* + w_1^* X_1$

Solution:

(b) $p_{X_1}(x_1) = \frac{1}{a} \exp\left(-\frac{x_1}{a}\right)$, for $x_1 > 0$
$\mathbb{E}\{X_1\} = a$ y $\mathbb{E}\{X_1^2\} = 2a^2$

(c) $\mathbb{E}\{X_2\} = a + \frac{b}{2}$
$\mathbb{E}\{X_1 X_2\} = 2a^2 + \frac{ba}{2}$

Exercise 56 (MSE)

The joint distribution of random variables S and X is given by:

$$p_{S,X}(s, x) = \sqrt{\frac{x(1-x)}{2\pi}} \exp\left(-\frac{(s - x(1-x))^2}{2x(1-x)}\right), \qquad 0 < x < 1, \qquad s \in \mathbb{R}$$

(a) Determine the minimum mean square error estimate of S based on the observation x.

(b) Determine the conditional mean square error given observation x, $\mathbb{E}\{(S - \widehat{S}_{\text{MMSE}})^2 | X = x\}$, for the estimator obtained in part a).

(c) Determine the mean square error of the estimator obtained in part a).

Solution:

(a) $\widehat{S}_{\text{MMSE}} = X(1 - X)$

(b) $\mathbb{E}\{(S - \widehat{S}_{\text{MMSE}})^2 | X = x\} = x(1 - x)$

(c) $\mathbb{E}\{(S - \widehat{S}_{\text{MMSE}})^2\} = \dfrac{1}{30}$

Exercise 57 (MSE)

A research company is working on a new communication prototype able to modify the noise distribution before being mixed with the signal. In this way, the receiver observes the following signal:

$$X = S + N_{mod},$$

where S is a Gaussian r.v. with zero mean and variance v_s, and N_{mod} is the new noise whose value is given by the following expression:

$$N_{\text{mod}} = \lambda N_1 + (1-\lambda) N_2$$

N_1 and N_2 being two independent Gaussian random variables, independent from S, with zero mean and variance v; whereas λ is a control parameter which takes values from 0 to 1.

(a) Find the distribution of N_{mod} as a function of λ.

(b) Obtain the MMSE estimator of S given X for the new system and compute the mean square error of this estimator as a function of λ.

(c) Compute the value of λ, λ_{opt}, which provides the minimum mean square error.

(d) Obtain, in terms of reduction of the mean square error, the advantage provided by this model when λ is set to λ_{opt} with respect to a model using $\lambda = 0$ or $\lambda = 1$. Compute this error reduction when $v_s = v = 1$.

The technician in charge of designing the system generating N_{mod} has gone on vacation, leaving the system online but without specifying what value of λ is being used. The new intern is tasked with obtaining a set of independent observations of the noise at the system's output $\{N_{\text{mod}}^{(k)}\}_{k=1}^{K}$, and computing a maximum likelihood estimation of the value of λ, $\hat{\lambda}_{ML}$.

(e) Considering $v = 3$ and the observation set $\{-1, 0, 2, 1\}$, obtain the value of $\hat{\lambda}_{ML}$.

Solution:

(a) N_{mod} is Gaussian with zero mean and variance $v_{MOD} = \lambda^2 v + (1-\lambda)^2 v$.

(b) $\hat{s}_{MSE} = \frac{v_s}{v_s + v_{MOD}} x \qquad MSE_{MOD} = \frac{(\lambda^2 + (1-\lambda)^2) v v_s}{v_s + (\lambda^2 + (1-\lambda)^2) v}$

(c) $\lambda_{\text{opt}} = 0.5$

(d) $\Delta MSE = \frac{1}{6}$

(e) $\hat{\lambda}_{ML} = 0.5$

Exercise 58 (MSE, MAP, MSE)

The joint distribution of random variables S and X is given by

$$p_{S,X}(s,x) = 15s, \qquad 0 \le x \le 1, \quad x^2 \le s \le x$$

We want to estimate S after observing X.

(a) Compute the minimun mean square error estimate.

(b) Compute the maximum a posteriori estimate.

(c) Compute the minimun mean square error estimate in the form $\widehat{S} = wX$

Solution:

(a) $\hat{s}_{\text{MSE}} = \dfrac{2}{3} \dfrac{x - x^4}{1 - x^2}$

(b) $\hat{s}_{\text{MAP}} = x$

(c) $w = \dfrac{7}{8}$

Exercise 59 (ML, LMSE)

We wish to estimate random variable S from random variable X. They are related as

$$X = S \cdot T$$

where S and T are independent random variables, both uniformly distributed between 0 and 1.

(a) Obtain the mean and the variance of S.

(b) Obtain the maximum likelihood estimator of S as a function of X, $\hat{S}_{\mathrm{ML}}$.

(c) Plot the support of the joint distribution of S and X. Calculate also the joint probability density function of S and X, $p_{S,X}(s,x)$.

(d) Obtain the mean of X and its mean quadratic value. (Hint: It may be convenient not to compute $p_X(x)$ as an intermediate result).

(e) Design the linear minimum mean square error estimator, $\hat{S}_{\mathrm{LMSE}} = w_0^* + w^* X$.

(f) Plot the estimators that have been designed in this problem on top of the same coordinate axis X-S, and discuss which of them incurs in the smallest mean square error.

Solution:

(a) $\mathbb{E}\{S\} = \frac{1}{2}$ y $v_x = \frac{1}{12}$

(b) $\hat{S}_{\mathrm{ML}} = X$

(c) $p_{S,X}(s,x) = \frac{1}{s}$ en $0 < x < s < 1$

(d) $\mathbb{E}\{X\} = \frac{1}{4}$ y $\mathbb{E}\{X^2\} = \frac{1}{9}$

(e) $w_0^* = \frac{83}{168}$ y $w^* = \frac{1}{42}$

(f) Dado que los tres son lineales, necesariamente $\hat{S}_{\mathrm{LMSE}}$ es el de menor error cuadrático medio.

Exercise 60 (Bayesian)

The joint distribution of random variables S and X is given by:

$$p_{S,X}(s,x) = 12s^2, \qquad 0 < s < x, \qquad 0 < x < 1$$

(a) Determine the estimate of S, $\widehat{S}_{\mathrm{c}}$, which minimizes the mean cost given the observation x of the following cost function:

$$c(s,\widehat{s}) = \frac{(s-\widehat{s})^2}{s}$$

(b) Determine the conditional mean cost given observation x, $\mathbb{E}\{c(S,\widehat{S}_{\mathrm{c}})|X = x\}$, for the estimator obtained in part a).

Solution:

(a) $\widehat{S}_{\mathrm{c}} = \frac{2X}{3}$

(b) $\mathbb{E}\{c(S,\widehat{S}_{\mathrm{c}})|X = x\} = \frac{x}{12}$

Exercise 61 (MSE, MSE)

We have taken two measurements X_1 and X_2 about the value of some unknown variable S. We know that X_1 and X_2 are related to S by means of

$$X_1 = S + T_1$$
$$X_2 = S + T_2$$

where T_1 and T_2 are independent random variables (and independent from S), with zero mean and variances 0.1 y 0.3, respectively

Also, we know that S es una Gaussian random variable with mena 4 y and variance 0.9.

(a) Compute the MMSE estimator of S given X_1. Denote it as $\hat{S}_1$.

(b) Compute the Mean Square Error for estimator $\hat{S}_1$.

(c) Compute the MMSE estimator of S given $Z = \frac{1}{2}(X_1 + X_2)$. Denote it as $\hat{S}_z$.

(d) Compute the probability of $\hat{S}_1$ being higher than $\hat{S}_z$, i.e., $P\{\hat{S}_1 > \hat{S}_z\}$. In case you cannot compute the solution analytically, express it by means of the function

$$F(x) = \int_{-\infty}^{x} \frac{1}{2\pi} \exp\left(-\frac{z^2}{2}\right) dz$$

Solution:

(a) $\hat{S}_1 = 0.9X_1 + 0.4$.

(b) $\mathrm{MSE} = 0.09$

(c) $\hat{S}_z = 0.9Z + 0.4$.

(d) $P\{\hat{S}_1 > \hat{S}_z\} = \dfrac{1}{2}$.

Exercise 62 (Bayesian, MSE, MAP, MAD)

The posterior distribution of random variable S given X is

$$p_{S|X}(s|x) = \frac{1}{s\ln(x)}, \qquad 1 \le s \le x.$$

We want to estimate S based on the observation of X.

(a) Compute the minimun mean square error estimate, $\hat{s}_{\mathrm{MMSE}}$.

(b) Compute the maximum *a posteriori* (MAP) estimate.

(c) Compute the estimate with minimum absolute deviation (MAD).

(d) Compute the Bayesian estimator, $\hat{s}_{\mathrm{B}}$, for a cost $c(s, \hat{s}) = s(s - \hat{s})^2$.

Solution:

(a) $\hat{s}_{\mathrm{MSE}} = \dfrac{x-1}{\ln(x)}$

(b) $\hat{s}_{\mathrm{MAP}} = 1$

(c) $\hat{s}_{\mathrm{MAD}} = \sqrt{x}$

(d) $\hat{s}_{\mathrm{B}} = \dfrac{1}{2}(1 + x)$

Exercise 63 (Gauss, MSE, ML)

Consider a communication system in which the transmitted symbol S is sent through two different channels, generating observations X_1 and X_2:

$$X_1 = \alpha(S + N_1)$$

$$X_2 = 2\alpha(S + N_2)$$

with α a constant associated to the attenuation of the channels.

It is known that the noise values N_1 and N_2, which are independent of each other, can be modeled as Gaussian random variables with mean zero and variance 0.5. Furthermore, the transmitted symbol, S, is also a Gaussian random variable with mean zero and variance one, and is independent of N_1 and N_2. At the receiver, just the sum of both outputs can be observed, i.e.,

$$X = X_1 + X_2$$

(a) Obtain the minimum mean square error estimate of S given X, $\hat{S}_{\text{MMSE}}$.

(b) Compute the mean square error of estimator $\hat{S}_{\text{MMSE}}$.

(c) Obtain the maximum likelihood estimate of S given X, $\hat{S}_{\text{ML}}$.

(d) Find the marginal distribution, $p_X(x)$.

(e) The marginal distribution depends on the unknown attenuation parameter, α. Find the ML estimate of α, $\hat{\alpha}_{\text{ML}}$, based on K values, $\{x^{(k)}\}_{k=1}^{K}$, independently drawn from the marginal distribution.

Solution:

(a) $\hat{S}_{\text{ML}} = \dfrac{x}{3\alpha}$

(b) $\hat{S}_{\text{MMSE}} = \dfrac{3\alpha}{11.5\alpha^2}x$

(c) $\hat{\alpha}_{\text{ML}} = \sqrt{\frac{2}{23}\sum_k x^{(k)2}}$

(d) $\hat{\alpha}_{\text{ML}} = 2$

A. Additional Problems

Exercise 3.E1 (1.2; 1.3)

Consider an observation

$$X = S + N$$

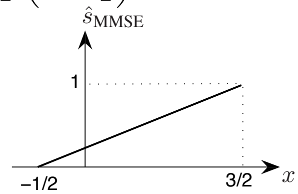
where S is a signal contaminated by additive noise N, and where S and N are independent of each other, and with probability density functions given by:

$$p_S(s) = \begin{cases} 1, & 0 < s < 1 \\ 0, & \text{otherwise} \end{cases} = \Pi(s - 1/2)$$

$$p_N(n) = \begin{cases} 1, & -1/2 < n < 1/2 \\ 0, & \text{otherwise} \end{cases} = \Pi(n)$$

Find the minimum mean square error estimator of S, $\hat{S}_{\text{MMSE}}$. Discuss your result.

Solution: $\hat{S}_{\text{MMSE}} = \frac{1}{2}\left(X + \frac{1}{2}\right) \qquad (-1/2 < x < 1/2)$



The linear change of the estimator between its minimum and maximum values ($\hat{s}_{\text{MMSE}}(-1/2) = 0, \hat{s}_{\text{MMSE}}(3/2) = 1$) are due to the addition of uniform noise.

Exercise 3.E2 (1.4)

We have access to K samples independently drawn from a random variable X which follows a Laplace distribution $L(m, v)$

$$p_X(x) = \frac{1}{\sqrt{2v}} \exp\left(-\sqrt{\frac{2}{v}}|x - m|\right)$$

Find the joint ML estimators of m, v.

Solution:

$$\begin{aligned} \hat{M}_{\text{ML}} &= \text{med}_K\{X^{(k)}\} \quad \text{(sample median)} \\ \hat{V}_{\text{ML}} &= \frac{2}{K^2}\left(\sum_k |X^{(k)} - \hat{M}_{\text{ML}}|\right)^2 \end{aligned}$$

Exercise 4.Q1 (1.5)

Unidimensional random variables S and R are characterized by the following joint distribution.

$$G\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right),$$

The observable variable is given by $X = S + R$.

(a) Obtain the estimator $\hat{S}_{\text{MSE}}$.

(b) Was this result to be expected? (Consider the existing relationship between $\mathbb{E}\{R|x\}$ and $\mathbb{E}\{S|x\}$).

(c) Obtain the MSE.

Solution:

(a) $\hat{S}_{\text{MSE}} = X/2$

(b) $\mathbb{E}\{R|x\} = \mathbb{E}\{S|x\}$ (since both variables distribute identically given X)
$\mathbb{E}\{X|x\} = x = \mathbb{E}\{S + R|x\} = \mathbb{E}\{S|x\} + \mathbb{E}\{R|x\}$

(c) $\mathbb{E}\left\{\left(S - \hat{S}\right)^2\right\} = \frac{1}{2} - \frac{1}{2}\rho$

Exercise 6.E1 (1.6)

Let S, X_1, and X_2 be three zero-mean random variables satisfying:

- The covariance matrix of X_1 and X_2 is:

$$\mathbf{V}_{xx} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- the cross-covariance between S and observation vector $X = [X_1, X_2]^T$ is:

$$\mathbf{v}_{sx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Obtain the coefficients of the linear minimum mean square error estimator

$$\hat{S}_{\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Find the expected quadratic value of the estimation error, $\hat{E} = S - \hat{S}_{\text{LMSE}}$.

(c) Explain which is the role of variable X_2, which, as can be seen, is uncorrelated with the variable to be estimated (S).

Solution:

(a) $w_1 = \dfrac{1}{1-\rho^2}$; $\quad w_2 = -\dfrac{\rho}{1-\rho^2}$; $\quad w_0 = 0$

(b) $\mathbb{E}\{\hat{E}^2\} = \mathbb{E}\{S^2\} - \dfrac{1}{1-\rho^2}$

(c) X_2 is combined with X_1 allowing a better approximation of S.

Exercise 6.E5 (1.6)

We want to design a linear minimum mean square error estimator of a random variable S based on the observation of random variables X_1 and X_2:

$$\hat{S}_{\text{LMSE}}(X_1, X_2) = w_0 + w_1 X_1 + w_2 X_2$$

The means of the random variables are $\mathbb{E}\{S\} = 1$, $\mathbb{E}\{X_1\} = 1$, and $\mathbb{E}\{X_2\} = 0$, whereas the correlations are given by $\mathbb{E}\{S^2\} = 4$, $\mathbb{E}\{X_1^2\} = 3$, $\mathbb{E}\{X_2^2\} = 2$, $\mathbb{E}\{SX_1\} = 2$, $\mathbb{E}\{SX_2\} = 0$, and $\mathbb{E}\{X_1 X_2\} = 1$.

(a) Obtain the optimal coefficients $\{w_i\}, i = 0, 1, 2$ of $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(b) Check that $v_{SX_2} = 0$. Why can still be $w_2 \neq 0$?

(c) Calculate the mean square error incurred by the application of estimator $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(d) How does the mean square error changes if the estimator $\hat{S}'_{\text{LMSE}}(X_1) = w'_0 + w'_1 X_1$, based on the sole observation of X_1, is used instead of $\hat{S}_{\text{LMSE}}(X_1, X_2)$?

Solution:

(a) $w_0 = 1/3,\ w_1 = 2/3,\ w_2 = -1/3$

(b) Combining X_1 and X_2 is better than just using X_1 (using the geometric analogy of the Orthogonality Principle, the projection space spanned by X_1 and X_2 is larger than the one spanned by X_1 alone).

(c) $\mathbb{E}\{E^2\} = 7/3$

(d) ($w'_0 = 1/2$; $w'_1 = 1/2$). $\mathbb{E}\{E'^2\} = 3$. It increases by $2/3$ (confirming our answer to the previous subquestion).