

# Stochastic Processes: Problems

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## 1. Markov Processes

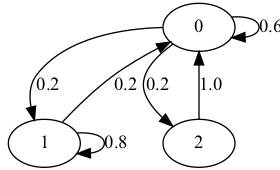
### MP1

Let  $X_k, k \geq 0$  be a Markov chain with state space  $\mathcal{Z} = \{0, 1\}$  and transition probabilities  $P\{X_k = 1 | X_{k-1} = 0\} = 0.8$  and  $P\{X_k = 0 | X_{k-1} = 1\} = 0.4$ .

- (a) Draw the corresponding transition graph
- (b) Assume that the initial state is  $X_0 = 1$ . Compute  $P\{X_2 = 1\}$
- (c) Compute the stationary distribution.

### MP2

Let  $X_k, k \geq 0$  be a Markov chain with state space  $\mathcal{Z} = \{0, 1, 2\}$  and the transition graph shown in the figure.



- (a) Show the transition matrix
- (b) Compute  $P\{X_{22} = 1 | X_{20} = 2\}$
- (c) Compute the stationary distribution.

### MP3

Let  $X_k, k \geq 0$  be a Markov chain with state space  $\mathcal{Z} = \{0, 1, 2, 3\}$ . The initial state is 0, that is,  $P\{X_0 = 0\} = 1$ . If, at time  $n$ , the process is in state  $i < 3$ , at time  $n + 1$  it will remain in the same state with probability  $1 - p$  or jump to state  $i + 1$  with probability  $p$ .

$$P\{X_{n+1} = i + 1 | X_n = i\} = p$$

$$P\{X_{n+1} = i | X_n = i\} = 1 - p$$

If the process is in state 3, it will remain in the same state with probability 1.

- (a) Find the transition matrix
- (b) Show the transition graph
- (c) Compute  $P\{X_2 = 1\}$
- (d) Compute  $P\{X_n = 0\}$ , for any  $n > 0$
- (e) Find a stationary distribution for this process

**MP4**

A video game consists of  $N$  consecutive levels,  $0, 1, \dots, N - 1$ . The player starts at level 0. If a player passes level  $i$ , she enters level  $i + 1$ , if not, she returns back to level 0. It is known that all phases have the same difficulty, so, if a player is at level  $i$ , she reaches level  $i + 1$  with probability  $q$ , and returns back to 0 with probability  $1 - q$ .

When the player reaches stage  $N - 1$ , she gets a medal, returns to level 0 and the game continues.

Let  $X_k$  be the stochastic process that represents the sequence of levels during a game, such that  $X_k = i$  means that the player was at level  $i$  at time  $k$ . The game begins at  $X_0 = 0$ .

- Formulate the problem as a stationary Markov process, and draw the transition graph for  $N = 6$ .
- Assuming  $N \geq 2$ , compute  $P\{X_2 = 1\}$ .
- Assuming  $N \geq 2$ , determine the probability of obtaining a medal exactly at time  $k$ , that is  $P\{X_k = N - 1\}$ , for  $k = 0, 1, \dots, N$ .
- For  $N = 2$ , determine the stationary distribution.
- For  $N = \infty$ , determine the stationary distribution

**MP5**

A game has three players, named  $G_0$ ,  $G_1$  and  $G_2$  and consists of a sequence of rounds. At each round, only one of the players enters the game.

The result of each round can be win or lose. If the active player wins a round, she can play the next one. If she loses, she must pass the turn to one of the other players, which is chosen at random with equal probabilities.

Based on the game skills of the players, it is known that the winning probabilities are 0.4 (for  $G_0$ ), 0.6 ( $G_1$ ) and 0.8 ( $G_2$ ).

Let  $X_k$  be the one-sided stochastic process that represents the sequence of active players during a game, such that  $X_k = i$  means that the active player at time  $k$  is  $G_i$ .

The game always starts with player 0, that is,  $X_0 = 0$ .

- Formulate the problem as a stationary Markov chain: compute the transition matrix and draw the transition graph.
- Compute the probability that the first players in the sequence are 0, 1, 2, 1 (i.e.,  $P\{X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 1\}$ ).
- Compute  $P\{X_2 = 1\}$ .
- Compute the stationary distribution
- Players  $G_0$  and  $G_1$  are unsatisfied because  $G_2$ , with better skills, plays most rounds. They decide to make the game fairer, in the sense that, in the long term, everyone plays with the same frequency. To do so, they proceed as follows:
  - If  $G_0$  is the active player and loses, she passes the turn to player  $G_1$  with probability  $q$  and to  $G_2$  with probability  $1 - q$ .
  - If  $G_1$  is the active player and loses, she passes the turn to player  $G_0$  with probability  $r$  and to  $G_2$  with probability  $1 - r$ .

Determine if  $G_0$  and  $G_1$  will succeed in making a fair game, that is, if there exist values  $q$  and  $r$  so that the stationary distribution is uniform, i.e.,  $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . If so, compute them.

**MP6**

Consider the markov chain  $X_k$  given by state space  $\mathcal{S} = \{0, 1, 2, 3\}$  and transition probability

matrix

$$P = \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}$$

It is known that the initial state is  $X_0 = 0$  with probability 1.

- (5 %) (a) Draw the transition graph.
- (7 %) (b) Compute the state probability distribution at time  $k = 2$ .
- (8 %) (c) Find the stationary distribution.
- (5 %) (d) Find the expected value of the time to reach state 1 for the first time.

## 2. Stationary Processes

### SP1

Let  $X_n$  be i.i.d. stochastic process with probability density function

$$p_X(x) = x \exp(-x), \quad x \geq 0$$

Assume that  $X_n$  is the input to a linear system with impulse response

$$h_n = \delta[n] + 0.5\delta[n-1]$$

the system output  $Y_n$ , is corrupted by a Gaussian i.i.d noise  $E_n$  (independent of  $X_n$ ) with mean zero and unit variance, to produce the final process

$$Z_n = Y_n + E_n$$

- (a) Compute the autocorrelation functions  $r_X[n]$  and  $r_E[n]$  of  $X_n$  and  $E_n$ , respectively.
- (b) Compute the autocorrelation function of  $Y_n$ ,  $r_Y[n]$
- (c) Compute the autocorrelation function of  $Z_n$ ,  $r_Z[n]$
- (d) Compute the power spectrum of  $Z_n$ ,  $S_Z(\omega)$ .

### SP2

The stochastic process  $X_n$  is given by,

$$X_n = \exp(-S_n)$$

where  $S_n$  is an i.i.d. process with probability density function

$$p_S(s) = \lambda \exp(-\lambda s), \quad s \geq 0, \quad \lambda > 0$$

Assume that  $X_n$  is the input to a linear and time-invariant system with impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

with output  $Y_n$

- (a) Compute the mean of the process,  $\mu_X = \mathbb{E}\{X_n\}$ .
- (b) Compute the autocorrelation function,  $r_X[n]$ .
- (c) Compute the power spectrum of the process  $Y_n$  for  $\lambda = 1$

**SP3**

Let  $S_n$  be a two-sided IID process with

$$p_{S_n}(s) = 6s(1-s), \quad 0 \leq s \leq 1.$$

Also, let  $X_n$  be the process given by

$$X_n = S_n(1 - S_n),$$

and let  $Y_n$  be the stochastic process given by

$$Y_n = X_n - \frac{1}{2}X_{n-1} - \frac{1}{2}X_{n+1}$$

- Compute the autocorrelation function of  $X_n$ ,  $r_X[n]$ .
- Compute the power spectrum of  $Y_n$ ,  $S_Y(e^{j\omega})$ .

**SP4**

The stochastic process  $X_n$  is given by the pair of equations

$$X_n = S_n \cdot R_n$$

$$S_n = W_n - \frac{1}{2}W_{n-1}$$

where  $W_n$  is a Gaussian i.i.d. process with mean 0 and variance  $v$ , and  $R_n$  is stationary processes with autocorrelation function

$$r_R[n] = 2^{-|n|}$$

Processes  $W_n$  and  $R_n$  are mutually independent.

- Compute the autocorrelation function of  $W_n$ ,  $r_W[n]$
- Compute and draw the autocorrelation function of  $S_n$ ,  $r_S[n]$
- Compute and draw the autocorrelation function of  $X_n$ ,  $r_X[n]$
- Compute the power spectrum of the process  $Z_n = \sum_{k=0}^{\infty} 2^{-k} X_{n-k}$

**SP5**

The stochastic process  $X_n$  is the sum of two i.i.d. stochastic processes  $S_n$  and  $R_n$ ,

$$X_n = S_n + R_n$$

with probability density functions

$$p_S(s) = s \exp(-s), \quad s \geq 0$$

and

$$p_R(r) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right)$$

respectively. The processes  $S_n$  and  $R_n$  are mutually independent. Assume that  $X_n$  is the input to a linear and time-invariant system with impulse response

$$h_n = \frac{1}{2^n} u[n]$$

with output  $Y_n$

- Compute the autocorrelation function,  $r_X[n]$ , and the power spectrum,  $S_X(\omega)$ , of  $X_n$
- Compute the autocorrelation function,  $r_Y[n]$ , and the power spectrum,  $S_Y(\omega)$ , of  $Y_n$

**SP6**

Let  $X_n$  be a discrete two-sided IID process with probability density function

$$p_X(x) = \frac{1}{2}, \quad -1 \leq x \leq 1$$

Let  $Y_n$  be the process defined by

$$Y_n = X_n^3$$

Let  $Z_n$  be the output of a linear time-invariant filter with impulse response:

$$h[n] = \frac{u[n]}{3^n}$$

when the input is  $Y_n$ .

- Is  $X_n$  wide-sense-stationary (WSS)? Is it strict-sense stationary (SSS)?
- Compute the mean  $\mu_Y[n]$  and the autocorrelation function,  $r_Y[n]$ , of  $Y_n$ .
- Compute the power spectrum of  $Z_n$ ,  $S_Z(e^{j\omega})$
- Find the impulse response  $g[n]$  of a linear-time invariant system such that, if  $V_n$  is the output of the system for input  $Z_n$ , the autocorrelation function of  $V_n$  is

$$r_V[n] = \delta[n]$$

**SP7**

Suppose that  $X_n$  is a two-sided binary Bernoulli( $p$ ) process, that is, an IID process given by

$$P_{X_n}(k) = \begin{bmatrix} p, & k = 1 \\ 1 - p, & k = 0 \end{bmatrix}, \quad n \in \mathbb{Z}$$

Using  $X_n$ , we define the following random processes

$$\begin{aligned} T_n &= X_n \cdot X_{n-1} \\ U_n &= X_n \cdot X_{n-1} \cdots X_{n-\ell}, \quad \ell \geq 1 \end{aligned}$$

where operator  $\oplus$  denotes mod 2 addition

- Compute the probability mass function of  $T_n$ ,  $P_{T_n}(k)$ ,  $k \in \{0, 1\}$ .
- Compute the autocorrelation function,  $r_T[n]$ , of  $T_n$ .
- Compute the power spectrum of  $T_n$ ,  $S_T(\omega)$ .
- Compute the probability mass function of  $U_n$ ,  $P_{U_n}(k)$ ,  $k \in \{0, 1\}$ .
- Compute the autocorrelation function,  $r_U[n]$ , of  $U_n$ .

**SP8**

Suppose that  $X_n$  is a two-sided binary Bernoulli( $p$ ) process, that is, an IID process given by

$$P_{X_n}(k) = \begin{bmatrix} p, & k = 1 \\ 1 - p, & k = 0 \end{bmatrix}, \quad n \in \mathbb{Z}$$

Suppose that  $W_n$  is another binary Bernoulli( $\alpha$ ) process, statistically independent of process  $X_n$  (that is, any collection of samples from  $X_n$  is independent from any collection of samples from  $W_n$ ).

Using  $X_n$  and  $W_n$ , we define the following random processes

$$\begin{aligned} Y_n &= X_n \oplus W_n, \\ Z_n &= X_n \oplus X_{n-1} \end{aligned}$$

where operator  $\oplus$  denotes mod 2 addition

- (a) Compute the probability mass function of  $Y_n$ , that is,  $P_{Y_n}(k) = P\{Y_n = k\}$ ,  $k \in \{0, 1\}$ .
- (b) Compute the autocorrelation function,  $r_Y[n]$ , of  $Y_n$ .
- (c) Compute the power spectrum of  $Y_n$ ,  $S_Y(\omega)$ .
- (d) Compute the probability mass function of  $Z_n$ ,  $P_{Z_n}(k)$ ,  $k \in \{0, 1\}$ . To simplify some expressions, you can express your results as a function of variable  $h = p(1 - p)$ .
- (e) Compute the autocorrelation function,  $r_Z[n]$ , of  $Z_n$ .
- (f) Compute the power spectrum of  $Z_n$ ,  $S_Z(\omega)$ .

**SP9**

The IID stochastic process  $X_n$  is given by a probability density function

$$p_X(x) = \frac{1}{2}, \quad -1 \leq x \leq 1$$

The stochastic process given by  $Y_n = X_n^p$ , where  $p$  is an arbitrary positive and odd integer, is the input to a linear and time invariant system with impulse response

$$h[n] = \delta[n] - \delta[n - 2]$$

Let  $Z_n$  be the output of this filter.

- (a) Compute the autocorrelation function and the power spectrum of  $X_n$ .
- (b) Compute the autocorrelation function and the power spectrum of  $Y_n$ .
- (c) Compute the autocorrelation function and the power spectrum of  $Z_n$ .