Estimation Theory: Problems

Notation:

- $\hat{S}_{\text{MSE}}$: Minimum Mean Square Error estimator.
- $\hat{S}_{\text{MAD}}$: Minimum Mean Absolute Deviation Error estimator.
- $\hat{S}_{\text{MAP}}$: Maximum a posteriori estimator.
- $\hat{S}_{\text{ML}}$: Maximum likelihood estimator.
- $\hat{S}_{\text{LMSE}}$: Linear Minimum Mean Square Error estimator.

Contents

1 Problems

ET1

We wish to design a linear minimum mean square error estimator for the estimation of random variable S based on the observation of random variables X_1 and X_2. It is known that:

$$\begin{array}{lll} \mathbb{E}\{S\} = \frac{1}{2} & \mathbb{E}\{X_1\} = 1 & \mathbb{E}\{X_2\} = 0 \\ \mathbb{E}\{SX_1\} = 1 & \mathbb{E}\{SX_2\} = 2 & \mathbb{E}\{X_1X_2\} = \frac{1}{2} \\ \mathbb{E}\{S^2\} = 4 & \mathbb{E}\{X_1^2\} = \frac{3}{2} & \mathbb{E}\{X_2^2\} = 2 \end{array}$$

Obtain the weights of estimator $\hat{S}_{\text{LMSE}} = w_0 + w_1X_1 + w_2X_2$, and calculate its mean square error $\mathbb{E}\left\{(S - \hat{S}_{\text{LMSE}})^2\right\}$.

Solution: A video resolution of this problem (in Spanish) can be found in

`http://decisionyestimacion.blogspot.com/2013/05/p1-estimacion.html`

$w_0 = \frac{1}{2} \quad w_1 = 0 \quad w_2 = 1$

$\mathbb{E}\left\{(S - \hat{S}_{\text{LMSE}})^2\right\} = \frac{7}{4}$

ET2

Consider the estimation of a random variable S from another random variable X, given the joint probability density function (pdf)

$$p_{S,X}(s,x) = \frac{6}{7}(x+s)^2, \qquad 0 \le x \le 1, \quad 0 \le s \le 1$$

(a) Find $p_X(x)$.

(b) Find $p_{S|X}(s|x)$.

(c) Compute the minimum MSE estimator of S given X, $\hat{S}_{\text{MSE}}$.

(d) Compute the MAP estimator of S given X, $\hat{S}_{\text{MAP}}$.

(e) Compute the bias and the variance of the MAP estimator.

Solution:

(a)

$$\begin{aligned} p_X(x) &= \int_0^1 p_{S,X}(s,x)ds = \int_0^1 \frac{6}{7}(x+s)^2\,ds \\ &= \frac{2}{7}\left(3x^2 + 3x + 1\right), \qquad 0 \le x \le 1 \end{aligned}$$

(b)

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{(x+s)^2}{x^2 + x + \frac{1}{3}}, \qquad 0 \le x \le 1, \quad 0 \le s \le 1$$

(c)

$$\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} = \int_0^1 s p_{S|X}(s|x) ds = \frac{1}{x^2 + x + \frac{1}{3}} \int_0^1 s\,(x+s)^2\, ds$$
$$= \frac{\frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4}}{x^2 + x + \frac{1}{3}}.$$

(d) Given that $p_{S|X}(s|x)$ increases with s for $0 \leq x \leq 1$ and $0 \leq s \leq 1$, $\hat{S}_{\text{MAP}} = 1$.

(e) Since

$$p_S(s) = \int_0^1 p_{S,X}(s,x) ds = \int_0^1 \frac{6}{7}\,(x+s)^2\, dx$$
$$= \frac{2}{7}\left(3s^2 + 3s + 1\right), \quad 0 < s < 1$$

we have

$$\mathbb{E}\left\{S\right\} = \int_0^1 s p_S(s) ds = \frac{2}{7} \int_0^1 s\left(3s^2 + 3s + 1\right) ds = \frac{9}{14},$$

and, thus, the expected bias is

$$\mathbb{E}\{\hat{S}_{\text{MAP}}\} - \mathbb{E}\{S\} = 1 - \frac{9}{14} = \frac{5}{14}$$

Since $\hat{S}_{\text{MAP}} = 1$ (constant and independent of X), its variance is zero.

ET3

A random variable X follows a unilateral exponential distribution with parameter $a > 0$:

$$p_X(x) = \frac{1}{a} \exp\left(-\frac{x}{a}\right) \quad x > 0$$

As it is known, the mean and variance of X are given by a and a^2, respectively.

(a) Obtain the maximum likelihood estimator of a, $\hat{A}_{\text{ML}}$, based on a set of K independent observations of random variable X, $\{X_k\}_{k=0}^{K-1}$.

(b) Consider now a new estimator based on the previous one, and characterized by expression:

$$\hat{A} = c \cdot \hat{A}_{\text{ML}},$$

where $0 \leq c \leq 1$ is shrinkage constant that allows re-scaling the ML estimator. Find the bias squared, the variance, and the Mean Square Error (MSE) o the new estimator, and represent them all together in the same plot as a function of c.

(c) Find the value of c which minimizes the MSE, c^*, and discuss its evolution as the number of available observations increases. Calculate the MSE of the estimator associated to c^*.

(d) Determine the range of values of c for which the MSE of $\hat{A}$ is smaller than the MSE of the ML estimator, and explain how such range changes as $K \longrightarrow \infty$. Discuss your result.

Solution: A video resolution of this problem (in Spanish) can be found in
`http://decisionyestimacion.blogspot.com/2013/05/problema-6-estimacion.html`

(a) $\hat{A}_{\text{ML}} = \frac{1}{K}\sum_{k=1}^{K} X_k$

(b) $\hat{A} = \frac{c}{K}\sum_{k=1}^{K} X_k$

$\mathbb{E}\left\{\hat{A} - a\right\}^2 = (c-1)^2 a^2,$

$\text{var}\left\{\hat{A}\right\} = \frac{c^2a^2}{K},$

$\mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = (c-1)^2 a^2 + \frac{c^2a^2}{K}$

(c) $c^* = \frac{K}{K+1}$, $c^* \to 1$ $(K \to \infty)$,

$\mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = \frac{a^2}{K+1}$ $(c = c^*)$

(d) The range of values is: $c \in \left[\frac{K-1}{K+1}, 1\right]$, which narrows as K increases.

ET4

We have access to the two following observations for estimating a random variable S:

$$
\begin{aligned}
X_1 &= S + N_1 \\
X_2 &= \alpha S + N_2
\end{aligned}
$$

where α is a known constant, and S, N_1, and N_2 are independent Gaussian random variables, with zero mean and variances v_s, v_n, and v_n, respectively.

(a) Obtain the minimum mean square error estimator of S given X_1 and X_2, $\hat{S}_1$ and $\hat{S}_2$, respectively.

(b) Calculate the mean square error of each of the estimators from the previous section. Which of the two provides a smaller MSE? Justify your answer for the different values of parameter α.

(c) Obtain the minimum mean square error estimator of S based on the joint observation of variables X_1 and X_2, i.e., as a function of the observation vector $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\hat{S}_{\text{MSE}}$.

Solution:

(a) S y X_2 are jointly Gaussian, with means

$$
\begin{aligned}
m_S &= 0 \\
m_{X_2} &= \alpha m_S + \mathbb{E}\{N_2\} = 0,
\end{aligned}
$$

variances v_s and

$$
\begin{aligned}
v_{X_2} &= \mathbb{E}\{(X_2 - m_{X_2})^2\} = \mathbb{E}\{X_2^2\} = \mathbb{E}\left\{(\alpha S + N_2)^2\right\} \\
&= \alpha^2\mathbb{E}\{S^2\} + 2\alpha\mathbb{E}\{SN_2\} + \mathbb{E}\{N_2^2\} \\
&= \alpha^2 v_s + v_n
\end{aligned}
$$

respectively, and covariance

$$\begin{aligned} v_{SX_2} &= \mathbb{E}\{(S - m_S)(X_2 - m_{X_2})\} = \mathbb{E}\{SX_2\} = \mathbb{E}\{S(\alpha S + N_2)\} \\ &= \alpha v_s \end{aligned}$$

Thus, the MMSE estimate of S given X_2 is

$$\begin{aligned} \hat{s}_2 = m_{S|X_2} &= m_S + \frac{v_{SX_2}}{v_{X_2}}(x_2 - m_{X_2}) = \frac{v_{SX_2}}{v_{X_2}} x_2 \\ &= \frac{\alpha v_s}{\alpha^2 v_s + v_n} x_2 \end{aligned}$$

On the other hand, given that the relation between X_1 and S is formally equivalent to that of X_2 and S for $\alpha = 1$, it is straighforward to see that the MMSE estimate of S given X_1 es equivalent to take $\alpha = 1$ in the expression above, that is

$$\hat{s}_1 = \frac{v_s}{v_s + v_n} x_2$$

(b) The mean square error $\hat{S}_2$ can be computed as

$$\begin{aligned} \mathbb{E}\left\{\left(S - \hat{S}_2\right)^2\right\} &= \mathbb{E}\left\{\left(S - \frac{\alpha v_S}{\alpha^2 v_s + v_n} X_2\right)^2\right\} \\ &= \mathbb{E}\left\{S^2\right\} - 2\frac{\alpha v_s}{\alpha^2 v_s + v_n}\mathbb{E}\{SX_2\} + \left(\frac{\alpha v_s}{\alpha^2 v_s + v_n}\right)^2 \mathbb{E}\{X_2^2\} \\ &= v_s - 2\frac{\alpha v_s}{\alpha^2 v_s + v_n} v_{SX_2} + \left(\frac{\alpha v_s}{\alpha^2 v_s + v_n}\right)^2 v_{X_2} \\ &= v_s - \frac{\alpha^2 v_s^2}{\alpha^2 v_s + v_n} \\ &= \frac{v_s v_n}{\alpha^2 v_s + v_n} \end{aligned}$$

(alternativelly, it can be computed in a more straightforward manner taking into account that the minimum MSE must be equal to $v_{S|X_2}$).

In a simliar way, the MSE of estimate $\hat{S}_1$ is equivalent to take $\alpha = 1$ in the previous expression,

$$\mathbb{E}\left\{\left(S - \hat{S}_1\right)^2\right\} = \frac{v_s v_n}{v_s + v_n}$$

For $|\alpha| > 1$ we can see that the MSE of $\hat{S}_2$ is smaller thant that of $\hat{S}_1$.

(c) Defining vectors $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ y $\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, we can express the model equation as

$$\mathbf{X} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} S + \mathbf{N}$$

S a $\mathbf{X}$ are jointly Gaussian, with means

$$\begin{aligned} m_S &= 0 \\ \mathbf{m_X} &= \begin{bmatrix} m_{X_1} \\ m_{X_2} \end{bmatrix} = 0 \end{aligned}$$

variances v_s y

$$\begin{aligned}
\mathbf{V_X} &= \mathbb{E}\{(\mathbf{X}-\mathbf{m_X})(\mathbf{X}-\mathbf{m_X})^\intercal\} = \mathbb{E}\{\mathbf{X}\mathbf{X}^\intercal\} \\
&= \mathbb{E}\left\{\left(\begin{bmatrix}1\\ \alpha\end{bmatrix} S + \mathbf{N}\right)\left(\begin{bmatrix}1\\ \alpha\end{bmatrix} S + \mathbf{N}\right)^\intercal\right\} \\
&= \begin{bmatrix}1\\ \alpha\end{bmatrix}\begin{bmatrix}1\\ \alpha\end{bmatrix}^\intercal \mathbb{E}\{S^2\} + \begin{bmatrix}1\\ \alpha\end{bmatrix}\mathbb{E}\{S\mathbf{N}^\intercal\} + \mathbb{E}\{S\mathbf{N}\}\begin{bmatrix}1\\ \alpha\end{bmatrix}^\intercal + \mathbb{E}\{\mathbf{N}\mathbf{N}^\intercal\} \\
&= v_s \begin{bmatrix}1 & \alpha\\ \alpha & \alpha^2\end{bmatrix} + v_n\mathbf{I} \\
&= \begin{bmatrix} v_s + v_n & v_s\alpha \\ v_s\alpha & v_s\alpha^2 + v_n\end{bmatrix},
\end{aligned}$$

respectively, and covariances

$$\mathbf{V}_{S\mathbf{X}} = \begin{bmatrix} v_{SX_1} \\ v_{SX_2}\end{bmatrix}^\intercal = \begin{bmatrix} v_s \\ \alpha v_s\end{bmatrix}^\intercal$$

Thes, the MMSE estimate of S given $\mathbf{X}$ is

$$\begin{aligned}
\mathbf{m}_{S|\mathbf{X}} &= m_S + \mathbf{V}_{S\mathbf{X}}\mathbf{V_X}^{-1}(\mathbf{x}-\mathbf{m_X}) = \mathbf{V}_{S\mathbf{X}}\mathbf{V_X}^{-1}\mathbf{x} \\
&= v_s \begin{bmatrix}1\\ \alpha\end{bmatrix}^\intercal \begin{bmatrix} v_s + v_n & v_s\alpha \\ v_s\alpha & v_s\alpha^2 + v_n\end{bmatrix}^{-1}\mathbf{x} \\
&= \frac{v_s}{(1+\alpha^2)v_s + v_n}(x_1 + \alpha x_2)
\end{aligned}$$

ET5

The joint p.d.f. of random variables X and S is given by

$$p_{X,S}(x,s) = \begin{cases} x+s & 0 \le x \le 1 \text{ and } 0 \le s \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the linear minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{LMSE}} = w_0 + w_1 X$.

Solution: $\hat{S}_{\mathrm{LMSE}} = \dfrac{7}{11} - \dfrac{X}{11}$

ET6

We want to estimate the value of a positive random variable S using a random observation X, which is related with S via

$$X = R/S$$

R being a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Obtain the likelihood of S, $p_{X|S}(x|s)$.

(b) Find the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The joint p.d.f. of S and X, $p_{S,X}(s,x)$, and the *a posteriori* distribution of S, $p_{S|X}(s|x)$.

(d) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\text{MAP}}$.

(e) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(f) The bias of estimators $\hat{S}_{\text{MAP}}$ and $\hat{S}_{\text{MSE}}$.

Solution:

(a) $p_{X|S}(x|s) = s\,\exp(-xs), \quad x > 0.$

(b) $\hat{S}_{\text{ML}} = \dfrac{1}{X}.$

(c) $p_{X,S}(x,s) = s\,\exp\left(-s(x+1)\right), \quad x, s > 0;$

$p_{S|X} = (x+1)^2 s\,\exp\left(-s(x+1)\right), \quad s > 0.$

(d) $\hat{S}_{\text{MAP}} = \dfrac{1}{X+1}.$

(e) $\hat{S}_{\text{MSE}} = \dfrac{2}{X+1}.$

(f) $\mathbb{E}\{\hat{S}_{\text{MAP}} - S\} = -\dfrac{1}{2}; \quad \mathbb{E}\{\hat{S}_{\text{MSE}} - S\} = 0.$

ET7

We wish to build an estimator for random variable S with the following analytical shape:

$$\hat{S} = w_0 + wX^3$$

(a) Let us define r.v. $Y = X^3$. Indicate which statistics are sufficient to determine the weights of the estimation model.

(b) An analyst wants to adjust the previous model, but he does not have statistical information about the problem. Therefore, he recurs to sample estimations of the sufficient statistics, based on a set of available labelled pairs of the involved random variables:

$$\{X^{(k)}, S^{(k)}\}_{k=1}^{4} = \{(-1, -0.55), (0, 0.5), (1, 1.57), (2, 8.7)\}$$

Determine the weights w_0 and w that the analyst would obtain.

Solution:

(a) $\mathbb{E}\{X\}$, $\mathbb{E}\{Y\}$, v_y and v_{sy} (or any other set from which these can be obtained).

(b) $w = 1.0256$ and $w_0 = 0.5038$.

ET8

Random variables S and X are jointly distributed according to

$$p_{S,X}(s,x) = \alpha s x^2, \qquad 0 \le s \le 1 - x, \quad 0 \le x \le 1$$

α being a parameter that needs to be determined.

(a) Find the expressions for the marginal probability density functions $p_X(x)$ and $p_S(s)$.

(b) Obtain the MAP estimator of S given X, $\hat{S}_{\text{MAP}}(X)$.

(c) Obtain the ML estimator of S given X, $\hat{S}_{\text{ML}}(X)$.

(d) Obtain the minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}(X)$.

(e) Compare the previous estimators according to the mean square errors given X in which they incur.

Solution:

(a) Parameter α must take the value that makes the integral of the distribution a unity. Since

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{S,X}(s,x)dsdx &= \int_0^1\int_0^{1-x} \alpha s x^2 dsdx = \alpha\int_0^1 x^2\int_0^{1-x} sdsdx\\ &= \alpha\int_0^1 x^2\left[\frac{1}{2}s^2\right]_0^{1-x}dx = \frac{\alpha}{2}\int_0^1 x^2(1-x)^2dx\\ &= \frac{\alpha}{60}\end{aligned}$$

we have $\alpha = 60$ and, thus

$$\begin{aligned}p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^{1-x} 60sx^2 ds = 60x^2\int_0^{1-x} sds\\ &= 30x^2(1-x)^2, \qquad 0\le x\le 1\end{aligned}$$

$$\begin{aligned}p_S(s) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)dx = \int_0^{1-s} 60sx^2 ds = 60s\int_0^{1-s} x^2 ds\\ &= 20s(1-s)^3, \qquad 0\le s\le 1\end{aligned}$$

(b)

$$\begin{aligned}\hat{s}_{\text{MAP}} &= \operatorname*{argmax}_s p_{S|X}(s|x) = \operatorname*{argmax}_s \frac{p_{S,X}(s,x)}{p_X(x)} = \operatorname*{argmax}_s p_{S,X}(s,x)\\ &= \operatorname*{argmax}_{s\in[0,1-x]} 60sx^2 = \operatorname*{argmax}_{s\in[0,1-x]} s\\ &= 1-x\end{aligned}$$

(c) Since the likelihood function is

$$p_{X|S}(x|s) = \frac{p_{S,X}(s,x)}{p_S(s)} = \frac{60sx^2}{20s(1-s)^3} = \frac{3x^2}{(1-s)^3}, \qquad 0\le s\le 1-x, \quad 0\le x\le 1$$

the ML estimator is

$$\begin{aligned}\hat{s}_{\text{ML}} &= \operatorname*{argmax}_s p_{X|S}(x|s) = \operatorname*{argmax}_{s\in[0,1-x]} \frac{3x^2}{(1-s)^3} = \operatorname*{argmax}_{s\in[0,1-x]} \frac{1}{(1-s)^3} = \operatorname*{argmin}_{s\in[0,1-x]} (1-s)^3\\ &= 1-x\end{aligned}$$

(d) Since the posterior distribution is

$$p_{S|x}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{60sx^2}{30x^2(1-x)^2} = \frac{2s}{(1-x)^2}, \qquad 0\le s\le 1-x, \quad 0\le x\le 1$$

the minimum MSE estimator will be

$$\begin{aligned}\hat{s}_{\text{MSE}} &= \mathbb{E}\{S|x\} = \int_{-\infty}^{\infty} sp_{S|X}(s|x)ds = \frac{2}{(1-x)^2}\int_0^{1-x} s^2 ds\\ &= \frac{2}{3}(1-x)\end{aligned}$$

(e)

$$\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{MAP}}\right)^2 \mid x\right\} = \mathbb{E}\left\{(S-(1-x))^2 \mid x\right\} = \int_{-\infty}^{\infty} (s-(1-x))^2\, p_{S|X}(s|x) ds$$

$$= \frac{2}{(1-x)^2}\int_0^{1-x} s\,(s-(1-x))^2\, ds$$

$$= \frac{1}{6}(1-x)^2$$

Since $\hat{S}_{\mathrm{ML}} = \hat{S}_{\mathrm{MAP}}$, its MSE will be identycal,

$$\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{ML}}\right)^2 \mid x\right\} = \frac{1}{6}(1-x)^2$$

Finally,

$$\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{MSE}}(X)\right)^2 \mid x\right\} = \mathbb{E}\left\{\left(S-\frac{2}{3}(1-x)\right)^2 \mid x\right\} = \int_0^{1-x} \frac{2s\left(s-\frac{2}{3}(1-x)\right)^2}{(1-x)^2} ds$$

$$= \frac{2}{(1-x)^2}\int_0^{1-x} s\left(s-\frac{2}{3}(1-x)\right)^2 ds$$

$$= \frac{1}{18}(1-x)^2$$

ET9

Consider the estimation of a r.v. S from another random variable X. The joint p.d.f. of the two variables is given by:

$$p_{X,S}(x,s) = \begin{cases} 6x, & 0 \le x \le s, \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(b) Obtain the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

(c) Find the probability density function of the previous estimators, $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s})$ and $p_{\hat{S}_{\mathrm{ML}}}(\hat{s})$, and provide a plot of them.

(d) Find the mean and the variance of the error of both estimators.

Solution:

(a) $\hat{S}_{\mathrm{MSE}}(X) = \frac{1}{2}(1+X)$

(b) $\hat{S}_{\mathrm{ML}}(X) = X$

(c) $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s}) = 24(2\hat{s}-1)(1-\hat{s}), \qquad \frac{1}{2} \le \hat{s} \le 1$

$p_{\hat{S}_{\mathrm{ML}}}(\hat{s}) = 6\hat{s}(1-\hat{s}), \qquad 0 \le \hat{s} \le 1$

(d) $\mathbb{E}\{S-\hat{S}_{\mathrm{ML}}\} = \dfrac{1}{4}, \qquad \mathbb{E}\{S-\hat{S}_{\mathrm{MSE}}\} = 0$

$\mathrm{Var}\{S-\hat{S}_{\mathrm{ML}}\} = \dfrac{13}{80}, \qquad Var\{S-\hat{S}_{\mathrm{MSE}}\} = \dfrac{1}{40}$

ET10

Consider the design of a linear minimum mean square estimator of random variable S based on the observation of random variable X_1. The following statistical information is known:

$$\begin{array}{ll} \mathbb{E}\{X_1\} = 0 & \mathbb{E}\{S\} = 1 \\ \mathbb{E}\{X_1^2\} = 1 & \mathbb{E}\{X_1 S\} = 2 \end{array}$$

(a) Which of the two following designs will incur in a smaller MSE?

$$\begin{aligned} \hat{S}_a &= w_{0a} + w_{1a} X_1 \\ \hat{S}_b &= w_{1b} X_1 \end{aligned}$$

(b) If we have access to a second random variable X_2 satisfying

$$\begin{array}{ll} \mathbb{E}\{X_2\} = 1 & \mathbb{E}\{X_2^2\} = 2 \\ \mathbb{E}\{X_1 X_2\} = \frac{1}{2} & \mathbb{E}\{S X_2\} = 2 \end{array}$$

justify if estimator $\hat{S}_c = w_{0c} + w_{1c} X_1 + w_{2c} X_2$ has a smaller mean quadratic error than the estimators considered in Section (a).

Solution:

(a) Let $\hat{S}_a^* = w_{0a}^* + w_{1a}^* X_1$ and $\hat{S}_b^* = w_{1b}^* X_1$ be the minimum MSE estimates for each one of the designs. Given that $\hat{S}_b^*$ can be expressed as an estimate in the form $w_{0a} + w_{1a} X_1$ (by taking $w_{0a} = 0$ y $w_{1a} = w_{1b}^*$), we can say that

$$\text{MSE}\{\hat{S}_a^*\} \leq \text{MSE}\{\hat{S}_b^*\}$$

To determine if the MSE of $\hat{S}_a^*$ is strictly less than that of $\hat{S}_b^*$, we will compute the weigst of estimate $\hat{S}_a^*$. Defining

$$\mathbf{Z} = \begin{bmatrix} 1 \\ X_1 \end{bmatrix}$$

we get

$$\mathbf{w}_a^* = \mathbf{R}_{\mathbf{Z}}^{-1} \mathbf{r}_{S\mathbf{Z}} = \begin{bmatrix} 1 & \mathbb{E}\{X_1\} \\ \mathbb{E}\{X_1\} & \mathbb{E}\{X_1^2\} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\{S\} \\ \mathbb{E}\{S X_1\} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Given that this minimum is unique and $\mathbf{w}_a^* \neq \begin{bmatrix} 0 \\ w_{1b}^* \end{bmatrix}$, the relation

$$\text{MSE}\{\hat{S}_a^*\} < \text{MSE}\{\hat{S}_b^*\}$$

holds necessarily.

(b) Defining

$$\mathbf{Z} = \begin{bmatrix} 1 \\ X_1 \\ X_2 \end{bmatrix}$$

the estimate $\hat{S}_c^*$ with minimum MSE will be given by the weight vector

$$\begin{aligned} \mathbf{w}_c^* = \mathbf{R}_{\mathbf{Z}}^{-1} \mathbf{r}_{S\mathbf{Z}} &= \begin{bmatrix} 1 & \mathbb{E}\{X_1\} & \mathbb{E}\{X_2\} \\ \mathbb{E}\{X_1\} & \mathbb{E}\{X_1^2\} & \mathbb{E}\{X_1 X_2\} \\ \mathbb{E}\{X_2\} & \mathbb{E}\{X_1 X_2\} & \mathbb{E}\{X_2^2\} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\{S\} \\ \mathbb{E}\{S X_1\} \\ \mathbb{E}\{S X_2\} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

Thus, $\hat{S}_c^* = 1 + 2X_1 = \hat{S}_a^*$ and, consequently,

$$\text{MSE}\{\hat{S}_a^*\} = \text{MSE}\{\hat{S}_c^*\}$$

ET11

The joint p.d.f. of two random variables S and X is:

$$p_{S,X}(s,x) = 6s, \qquad 0 < s < x, \qquad 0 < x < 1$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.
(b) The conditional bias
(c) The unconditional bias of estimator $\hat{S}_{\text{MSE}}$.

Solution:

(a) Noting that

$$p_X(x) = \int_0^x 6sds = 3x^2, \qquad 0 < x < 1$$

we have

$$p_{S|X}(s|x) = \frac{2s}{x^2}, \qquad 0 < s < x, \qquad 0 < x < 1$$

therefore

$$\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} = \int_0^x \frac{2s^2}{x^2} = \frac{2}{3}x$$

(b)

$$p_S(s) = \int p_{S,X}(s,x)dx = \int_s^1 6sdx = 6s(1-s), \qquad 0 < s < 1$$

we have

$$p_{X|S}(x|s) = \frac{1}{1-s}, \qquad 0 < s < x, \qquad 0 < x < 1$$

and

$$\mathbb{E}\{X|s\} = \int x p_{X|S}(x|s)dx = \frac{1}{1-s}\int_s^1 xdx = \frac{1}{2(1-s)}$$

Therefore

$$\mathbb{E}\{\hat{S}_{\text{MSE}}|s\} = \frac{2}{3}\mathbb{E}\{X|s\} = \frac{1}{3(1-s)}$$

and the conditional bias is

$$\text{bias}\{\hat{S}_{\text{MSE}}|s\} = \mathbb{E}\{\hat{S}_{\text{MSE}}|s\} - s = \frac{1}{3(1-s)} - s$$

(c) Since

$$\mathbb{E}\{S\} = \int_0^1 6s^2(1-s) = \frac{1}{2}$$

$$\mathbb{E}\{\hat{S}_{\text{MSE}}\} = \frac{2}{3}\mathbb{E}\{X\} = \frac{2}{3}\int_0^1 3x^3dx = \frac{1}{2}$$

the estimator is unbiased

ET12

The joint p.d.f. of two random variables S and X is given by:

$$p_{S,X}(s,x) = \alpha, \quad -1 < x < 1, \quad 0 \le s \le |x|$$

(a) Obtain the marginal p.d.f. of X, $p_X(x)$, specifying the value of α.

(b) Find the estimators of S based on variable X that minimize the mean square error (MSE), ($\overline{C}_{\text{MSE}} = \mathbb{E}\left\{(S-\hat{S})^2\right\}$) and mean absolute deviation (MAD) ($\overline{C}_{\text{MAD}} = \mathbb{E}\left\{|S-\hat{S}|\right\}$), $\hat{S}_{\text{MSE}}$ and $\hat{S}_{\text{MAD}}$, respectively.

(c) If the estimator is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to cost MSE, i.e., $\hat{S}_{q,\text{MSE}} = w_1 X^2$.

(d) (*Hard*) If the estimator is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to MAD, i.e., $\hat{S}_{q,\text{MAD}} = w_2 X^2$.

Solution:

(a)

$$p_X(x) = \int_0^{|x|} \alpha dx = \alpha|x|, \quad -1 < x < 1$$

Since the area of the pdf must be unity,

$$\int_{-1}^{1} p_X(x)dx = \int_{-1}^{1} \alpha|x|dx = \alpha = 1$$

therefore $\alpha = 1$ and

$$p_X(x) = |x|, \quad -1 < x < 1$$

(b) The posterior distribution is

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{1}{|x|}, \quad 0 \le s \le |x|$$

which is a uniform distribution. Therefore, both the mean and the median are in the middle point:

$$\hat{S}_{\text{MMSE}} = \hat{S}_{\text{MAD}} = |X|/2$$

(c)

$$\begin{aligned}
w_1 &= \frac{\mathbb{E}\{SX^2\}}{\mathbb{E}\{X^4\}} = \frac{\int_{-1}^{1} \mathbb{E}\{SX^2|x\}|x|dx}{\int_{-1}^{1} x^4|x|dx} = \frac{\int_{-1}^{1} x^2\mathbb{E}\{S|x\}|x|dx}{\int_{-1}^{1} x^4|x|dx} \\
&= \frac{2\int_0^1 \frac{1}{2}x^4 dx}{2\int_0^1 x^5 dx} = \frac{3}{5}
\end{aligned}$$

Therefore

$$\hat{S}_{q,\text{MSE}}(X) = 3X^2/5$$

(d) The MAD for any estimator in the form w_2X^2 is given by

$$\begin{aligned}
\overline{C}_{\text{MAD}} = \mathbb{E}\left\{|S-\hat{S}|\right\} &= \int_{-1}^{1}\int_0^{|x|} |s - w_2x^2| ds dx \\
&= 2\int_0^1\int_0^x |s - w_2x^2| ds dx
\end{aligned}$$

For $w_2 \leq 0$ we have

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= 2\int_0^1 \int_0^x (s - w_2x^2)dsdx = \int_0^1 \left[(x - w_2x^2)^2 - w_2^2x^4\right] dx \\
&= \int_0^1 \left[x^2 - 2w_2x^3\right] dx = \frac{1}{3} - \frac{1}{2}w_2
\end{aligned}
$$

and, for $w_2 > 0$,

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= 2\left(\int_0^1 \int_0^{\min(x,w_2x^2)} (w_2x^2 - s)dsdx + \int_0^1 \int_{\min(x,w_2x^2)}^x (s - w_2x^2)dsdx\right) \\
&= \int_0^1 \left[-(w_2x^2 - s)^2\right]_0^{\min(x,w_2x^2)} dx + \int_0^1 \left[(s - w_2x^2)^2\right]_{\min(x,w_2x^2)}^x dx \\
&= \int_0^1 \left[w_2^2x^4 - (w_2x^2 - \min(x, w_2x^2))^2\right] dx + \int_0^1 \left[(x - w_2x^2)^2 - (\min(x, w_2x^2) - w_2x^2)^2\right] dx
\end{aligned}
$$

Now, for $0 \leq w_2 \leq 1$, since $0 \leq x \leq 1$ we have $\min(x, w_2x^2) = w_2x^2$, so that

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= \int_0^1 w_2^2x^4dx + \int_0^1 (x - w_2x^2)^2dx \\
&= \frac{1}{5}w_2^2 + \frac{1}{3} - \frac{1}{2}w_2 + \frac{1}{5}w_2^2 = \frac{2}{5}w_2^2 - \frac{1}{2}w_2 + \frac{1}{3}
\end{aligned}
$$

Finally, for $w_2 > 1$, we get

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= \int_0^{\frac{1}{w_2}} w_2^2x^4dx + \int_0^{\frac{1}{w_2}} (x - w_2x^2)^2dx + \int_{\frac{1}{w_2}}^1 \left[w_2^2x^4 - (w_2x^2 - x)^2\right] dx \\
&= \int_0^1 w_2^2x^4dx + \int_0^{\frac{1}{w_2}} (x - w_2x^2)^2dx - \int_{\frac{1}{w_2}}^1 (w_2x^2 - x)^2dx \\
&= \frac{1}{5}w_2^2 + \int_0^{\frac{1}{w_2}} \left[x^2 - 2w_2x^3 + w_2^2x^4\right] dx - \int_{\frac{1}{w_2}}^1 \left[x^2 - 2w_2x^3 + w_2^2x^4\right] dx \\
&= \frac{1}{5}w_2^2 + 2\left[\frac{1}{3w_2^3} - \frac{2w_2}{4w_2^4} + \frac{w_2^2}{5w_2^5}\right] - \left[\frac{1}{3} - \frac{w_2}{2} + \frac{1}{5}w_2^2\right] \\
&= \frac{2}{3w_2^3} - \frac{1}{w_2^3} + \frac{2}{5w_2^3} - \frac{1}{3} + \frac{w_2}{2} = \frac{w_2}{2} - \frac{1}{3} + \frac{1}{15w_2^3}
\end{aligned}
$$

Since, for $w_2 > 1$,

$$
\frac{d\overline{C}_{\text{MAD}}}{dw_2} = \frac{1}{2} - \frac{1}{5w_2^4} > 0
$$

the risk grows for $w_2 > 1$. Since it is also decreasing for $w_2 < 0$, the minimum is in $[0, 1]$. Therefore,

$$
w_2^* = \underset{w_2\in[0,1]}{\text{argmin}} \left\{\frac{2}{5}w_2^2 - \frac{1}{2}w_2 + \frac{1}{3}\right\} = \frac{5}{8}
$$

Consider a random variable X with p.d.f.

$$p_X(x) = a \exp\left[-a(x-d)\right], \qquad x \geq d$$

where $a > 0$ and d are two parameters.

Find the maximum likelihood estimators of both parameters, $\hat{a}_{\mathrm{ML}}$ and $\hat{d}_{\mathrm{ML}}$, as a function of K samples of X independently drawn, $\{x_k\}_{k=0}^{K-1}$.

Solution: The ML estimates a and d are given by

$$(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) = \underset{a,d}{\operatorname{argmax}} \prod_{k=1}^{K} \left(a \exp\left(-a(x_k - d)\right) u(x_k - d)\right)$$

Note that if $d > x_k$ for some sample x_k, we have $u(x_k - d) = 0$ and, thus, the total likelihood is 0. Therefore, $\hat{d}_{\mathrm{ML}} \leq x_k$, for all k, or, equivalently, $\hat{d}_{\mathrm{ML}} \leq \min_k \{x_k\}$, and we can write

$$(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) = \underset{a,d|d\geq x_{\min}}{\operatorname{argmax}} \prod_{k=1}^{K} \left(a \exp\left(-a(x_k - d)\right)\right)$$

where $x_{\min} = \max_k \{x_k\}$.

Minimizing the logarithm, we can write

$$\begin{aligned}
(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) &= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \sum_{k=1}^{K} \left(\log(a) - a(x_k - d)\right) \\
&= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \left(K\log(a) - a\left(\sum_{k=1}^{K} x_k - Kd\right)\right) \\
&= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \left(K\log(a) + Kad - a\sum_{k=1}^{K} x_k\right)
\end{aligned}$$

Given that the function to maximize increases with d, $\hat{d}_{\mathrm{ML}}$ will be the highest values of d in the feasible interval, that is,

$$\hat{d}_{\mathrm{ML}} = x_{\min} = \min_k \{x_k\}$$

and, thus,

$$\begin{aligned}
\hat{a}_{\mathrm{ML}} &= \underset{a}{\operatorname{argmax}} \left(K\log(a) + Ka \cdot \hat{d}_{\mathrm{ML}} - a\sum_{k=1}^{K} x_k\right) \\
&= \frac{K}{\sum_{k=1}^{K} \left(x_k - \min_k\{x_k\}\right)}
\end{aligned}$$

(where the maximum has been computed by differentiation)

ET14

Random variables S and X have a joint probability density function given by

$$p_{S,X}(s,x) = \begin{cases} 10s, & 0 < s < x^2 \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider the estimation of S based on the observation of X, with the objective to minimize the following cost function:

$$c(S,\hat{S}) = S^2\left(S-\hat{S}\right)^2$$

Find:

(a) The Bayesian estimator, $\hat{S}_C$, for the given cost.

(b) The linear estimator $\hat{S}_L = wX$ which minimizes the risk $\mathbb{E}\left\{c(S,\hat{S})\right\}$.

(c) The risk of both estimators: $\mathbb{E}\left\{c(S,\hat{S}_C)\right\}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\}$.

(d) The unconditional bias of both estimators.

(e) The variance of the error both estimators: $\operatorname{var}\left\{S-\hat{S}_C\right\}$ and $\operatorname{var}\left\{S-\hat{S}_L\right\}$.

Solution:

(a)

$$\begin{aligned}
\hat{s}_c &= \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{c(S,\hat{s})|x\} = \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{S^2\left(S-\hat{s}\right)^2|x\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{S^4 - 2S^3\hat{s} + S^2\hat{s}^2|x\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\, \left\{\mathbb{E}\{S^4|x\} - 2\mathbb{E}\{S^3|x\}\hat{s} + \mathbb{E}\{S^2|x\}\hat{s}^2\right\} \\
&= \frac{\mathbb{E}\{S^3|x\}}{\mathbb{E}\{S^2|x\}}
\end{aligned}$$

Noting that

$$p_X(x) = \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = 10\int_0^{x^2} sds = \frac{5x^4}{2}$$

and, thus,

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{4s}{x^4}, \qquad 0 \le s \le x^2, \quad 0 \le x \le 1$$

therefore

$$\hat{s}_c = \frac{\mathbb{E}\{S^3|x\}}{\mathbb{E}\{S^2|x\}} = \frac{\int_{-\infty}^{\infty} s^3 p_{S|X}(s|x)ds}{\int_{-\infty}^{\infty} s^2 p_{S|X}(s|x)ds} = \frac{\frac{4}{x^4}\int_0^{x^2} s^4 ds}{\frac{4}{x^4}\int_0^{x^2} s^3 ds} = \frac{\frac{x^{10}}{5}}{\frac{x^8}{4}} = \frac{4}{5}x^2$$

(b)

$$\begin{aligned}
w &= \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{c(S,wX)\} = \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{S^2\left(S-wX\right)^2\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\, \mathbb{E}\{S^4 - 2S^3Xw + S^2X^2w\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\, \left\{\mathbb{E}\{S^4\} - 2\mathbb{E}\{S^3X\}w + \mathbb{E}\{S^2X^2\}w^2\right\} \\
&= \frac{\mathbb{E}\{S^3X\}}{\mathbb{E}\{S^2X^2\}}
\end{aligned}$$

Noting that, for any $m \leq 0$, $n \leq 0$

$$\begin{aligned}
\mathbb{E}\{S^m X^n\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^m x^n p_{S,X}(s,x) ds dx \\
&= 10\int_0^1 x^n \int_0^{x^2} s^{m+1} ds dx = \frac{10}{m+2}\int_0^1 x^{2m+n+4} dx \\
&= \frac{10}{(m+2)(2m+n+5)}
\end{aligned}$$

we can write

$$w = \frac{\mathbb{E}\{S^3 X\}}{\mathbb{E}\{S^2 X^2\}} = \frac{\frac{1}{6}}{\frac{5}{22}} = \frac{11}{15}$$

Therefore, the linear estimation minimizing the overall risk is

$$\hat{S}_L = \frac{11}{15} X$$

(c) For any estimator $\hat{S}$, the overall risk is

$$\mathbb{E}\left\{c(S,\hat{S})\right\} = \mathbb{E}\{S^2\left(S-\hat{S}\right)^2\} = \mathbb{E}\{S^4\} - 2\mathbb{E}\{S^3\hat{S}\} + \mathbb{E}\{S^2\hat{S}^2\} = \frac{5}{39} - 2\mathbb{E}\{S^3\hat{S}\} + \mathbb{E}\{S^2\hat{S}^2\}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left\{c(S,\hat{S}_C)\right\} &= \frac{5}{39} - \frac{8}{5}\mathbb{E}\{S^3X^2\} + \frac{16}{25}\mathbb{E}\{S^2X^4\} \\
&= \frac{5}{39} - \frac{8}{5}\cdot\frac{2}{13} + \frac{16}{25}\cdot\frac{5}{26} = \frac{1}{195} \\
\mathbb{E}\left\{c(S,\hat{S}_L)\right\} &= \frac{5}{39} - \frac{22}{15}\mathbb{E}\{S^3X\} + \frac{11^2}{15^2}\mathbb{E}\{S^2X^2\} \\
&= \frac{5}{39} - \frac{22}{15}\cdot\frac{1}{6} + \frac{11^2}{15^2}\cdot\frac{5}{22} = \frac{7}{1170}
\end{aligned}$$

(d) The bias is

$$\begin{aligned}
B_C &= \mathbb{E}\left\{\hat{S}_C - S\right\} = \frac{4}{5}\mathbb{E}\left\{X^2\right\} - \mathbb{E}\{S\} = \frac{4}{7} - \frac{10}{21} = \frac{2}{21} \\
B_L &= \mathbb{E}\left\{\hat{S}_L - S\right\} = \frac{11}{15}\mathbb{E}\left\{X\right\} - \mathbb{E}\{S\} = \frac{11}{18} - \frac{10}{21} = \frac{17}{126}
\end{aligned}$$

(e) Using the bias-variance decomposition,

$$\begin{aligned}
\mathrm{Var}\left\{S - \hat{S}_C\right\} &= \mathbb{E}\left\{\left(S-\hat{S}_C\right)^2\right\} - B_C^2 \\
&= \mathbb{E}\left\{S^2\right\} + \mathbb{E}\left\{\hat{S}_C^2\right\} - 2\mathbb{E}\left\{S\hat{S}_C\right\} - B_C^2 \\
&= \mathbb{E}\left\{S^2\right\} + \frac{16}{25}\mathbb{E}\left\{X^4\right\} - \frac{8}{5}\mathbb{E}\left\{SX^2\right\} - B_C^2 \\
&= \frac{5}{18} + \frac{16}{25}\cdot\frac{5}{9} - \frac{8}{5}\frac{10}{27} - \frac{4}{441} = \frac{419}{13230} \approx 0.03167
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\mathrm{Var}\left\{S - \hat{S}_L\right\} &= \mathbb{E}\left\{S^2\right\} + \mathbb{E}\left\{\hat{S}_L^2\right\} - 2\mathbb{E}\left\{S\hat{S}_L\right\} - B_L^2 \\
&= \mathbb{E}\left\{S^2\right\} + \frac{121}{225}\mathbb{E}\left\{X^2\right\} - \frac{22}{15}\mathbb{E}\left\{SX\right\} - B_L^2 \\
&= \frac{5}{18} + \frac{121}{225}\cdot\frac{5}{7} - \frac{22}{15}\frac{5}{12} - \frac{17^2}{126^2} = \frac{2587}{79380} \approx 0.03259
\end{aligned}$$

ET15

Random variables S and X are characterized by the following joint distribution:

$$p_{S,X}(s,x) = c, \quad 0 < s < 1, \;\; s < x < 2s$$

with c a constant.

(a) Plot the support of the p.d.f., and use it to calculate the value of c.

(b) Give the expressions for the marginal p.d.f. of the random variables: $p_S(s)$ and $p_X(x)$.

(c) Find the minimum mean square error estimator of S based on the observation of X, $\hat{S}_{\text{MSE}}(X)$. Plot the estimator on the same plot as the support of $p_{S,X}(s,x)$, and discuss whether it would had been possible to obtain the estimator without analytical derivations.

(d) Calculate the mean square error $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$ incurred by the previous estimator.

(e) Now, find the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}}(X)$. Again, plot the estimator together with the support of $p_{S,X}(s,x)$. Discuss your result.

(f) Obtain the mean square error $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}(X)\right)^2\right\}$ of the linear estimator, and compare it with $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$.

(g) It is perceived (e.g., visualizing several samples of (X,S)) that there exist different statistical behaviors for $0 < X < 1$ and $1 < X < 2$. What would occur if, based on this, different optimal linear estimators where designed for each of the intervals ($\hat{S}_{A,\text{LMSE}}(X)$ y $\hat{S}_{B,\text{LMSE}}(X)$, respectively)? Verify analytically the proposed solution.

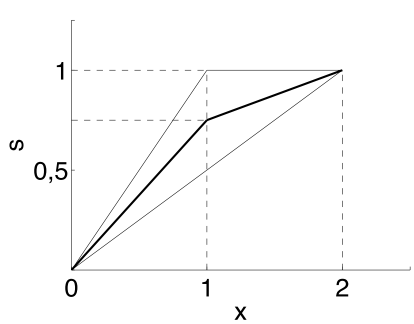
Solution:

(a) Since the area of the support of $p_{S,X}(s,x)$ is $1/2$, $c = 2$.

(b) $p_S(s) = 2s, \; 0 < s < 1;$ $\qquad p_X(x) = \begin{cases} x & ,0 < x < 1 \\ 2 - x & ,1 < x < 2 \end{cases}$

(c)

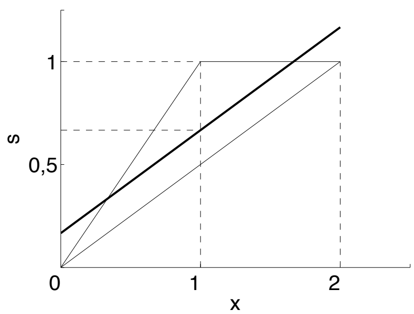
$$\hat{S}_{\text{MSE}}(X) = \begin{cases} \dfrac{3X}{4}, & 0 < X < 1 \\[2ex] \dfrac{1}{2}\left(\dfrac{X}{2} + 1\right), & 1 < X < 2 \end{cases}$$



Since for every value X we have a uniform *a posteriori distribution* $p_{S|X}(s|x)$, the MSE estimator is given as the average between the minimum and maximum values of S (for each X).

(d) $\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}(X)\right)^2\right\}=\dfrac{1}{96}$

(e) $\hat{S}_{\text{LMSE}}(X)=\dfrac{X}{2}+\dfrac{1}{6}$



(f) $\mathbb{E}\left\{\left(S-\hat{S}_{\text{LMSE}}(X)\right)^2\right\}=\dfrac{11}{24}$, which is larger than $\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}(X)\right)^2\right\}$

(g) $\hat{S}_{A,\text{LMSE}}(X)=\dfrac{3X}{4}$ and $\hat{S}_{B,\text{LMSE}}(X)=\dfrac{1}{2}\left(\dfrac{X}{2}+1\right)$. When jointly considered, these estimators compose $\hat{S}_{\text{MSE}}(X)$.
$p_A(s,x)$ and $p_B(s,x)$ are uniform, and now the linear estimators will also be optimal.

ET16

Consider the estimation of a random vector $\mathbf{S}$ from a statistically related observation vector $\mathbf{X}$:

$$\mathbf{X}=\mathbf{HS}+\mathbf{R}$$

where $\mathbf{H}$ is a known matrix, $\mathbf{R}$ a noise vector with distribution $\mathcal{N}(\mathbf{0},v_r\mathbf{I})$, and $\mathbf{S}$ the random vector to be estimated, whose distribution is $\mathcal{N}(\mathbf{m_S},\mathbf{V_S})$. It is also known that $\mathbf{S}$ and $\mathbf{R}$ are independent random vectors:

(a) Find the ML estimator of $\mathbf{S}$, $\hat{\mathbf{S}}_{\text{ML}}$.

(b) Is the ML estimator unbiased? Justify your answer.

(c) As it is known, the MSE estimator of $\mathbf{S}$ is given by:

$$\hat{\mathbf{S}}_{\text{MSE}}=\left(\mathbf{H}^\top\mathbf{H}+v_r\mathbf{V_S}^{-1}\right)^{-1}\mathbf{H}^\top\mathbf{X}$$

Obtain the bias of $\hat{\mathbf{S}}_{\text{MSE}}$ and indicate under which conditions such bias vanishes.

Solution:

(a) $\hat{\mathbf{S}}_{\text{ML}}=\left(\mathbf{H}^\top\mathbf{H}\right)^{-1}\mathbf{H}^\top\mathbf{X}$

(b) The estimator is unbiased.

(c) $\mathbb{E}\left\{\hat{\mathbf{S}}_{\text{MSE}}-\mathbf{S}\right\}=\left(\mathbf{H}^\top\mathbf{H}+v_r\mathbf{V_S}^{-1}\right)^{-1}\mathbf{H}^\top\mathbf{H}\mathbf{m_S}-\mathbf{m_S}$. The bias goes to zero as the noise power decreases towards 0.

ET17

We have access to a set of K samples, $\{X_k\}_{k=0}^{K-1}$, independently drawn from a random variable

X with p.d.f.

$$p_X(x) = \frac{1}{bx^2} \exp\left(-\frac{1}{bx}\right), \qquad x \geq 0$$

with $b > 0$ a constant.

(a) Find the ML estimator of b as a function of the available samples, $\hat{B}_{\text{ML}}$.

(b) Verify that random variable $Y = 1/X$ is characterized by a unilateral exponential p.d.f. $p_Y(y)$, and obtain the value of the mean of such distribution.

(c) Considering your answers to the previous sections, is $\hat{B}_{\text{ML}}$ an unbiased estimator?

Solution:

(a) Maximizing the log-likelihood, we can write (assuming that, according to the probability model, all samples are non-negative)

$$\begin{aligned}
\hat{b}_{\text{ML}} &= \underset{b}{\operatorname{argmax}} \sum_{k=0}^{K-1} \log\left(p_X(x_k)\right) \\
&= \underset{b}{\operatorname{argmax}} \sum_{k=0}^{K-1} \log\left(\frac{1}{bx_k^2} \exp\left(-\frac{1}{bx_k}\right)\right) \\
&= \underset{b}{\operatorname{argmax}} \left(-K\log(b) - 2\sum_{k=0}^{K-1} \log\left(x_k\right) - \frac{1}{b}\sum_{k=0}^{K-1} \frac{1}{x_k}\right) \\
&= \underset{b}{\operatorname{argmax}} \left(-K\log(b) - \frac{1}{b}\sum_{k=0}^{K-1} \frac{1}{x_k}\right) \\
&= \frac{1}{K}\sum_{k=0}^{K-1} \frac{1}{x_k}
\end{aligned}$$

where the last step has been solved for derivation.

(b)

$$\begin{aligned}
p_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{d}{dy}P\{Y \leq y\} = \frac{d}{dy}P\left\{\frac{1}{X} \leq y\right\} \\
&= \frac{d}{dy}P\left\{X \geq \frac{1}{y}\right\} = \frac{d}{dy}\left(1 - F_X\left(\frac{1}{y}\right)\right) = \frac{1}{y^2}p_X\left(\frac{1}{y}\right) \\
&= \frac{1}{b}\exp\left(-\frac{y}{b}\right), \qquad y \geq 0
\end{aligned}$$

(c) Given that

$$\hat{B}_{\text{ML}} = \frac{1}{K}\sum_{k=0}^{K-1} \frac{1}{X_k}$$

the mean of the estimator is

$$\begin{aligned}
\mathbb{E}\{\hat{B}_{\text{ML}}\} &= \frac{1}{K}\sum_{k=0}^{K-1} \mathbb{E}\left\{\frac{1}{X_k}\right\} = \mathbb{E}\left\{\frac{1}{X}\right\} \\
&= \mathbb{E}\left\{Y\right\} = \int_0^\infty y\frac{1}{b}\exp\left(-\frac{y}{b}\right) = b
\end{aligned}$$

Thus $\hat{B}_{\text{ML}}$ is unbiased

ET18

Consider the family of cost functions given by

$$c(S,\hat{S}) = \frac{1}{N+1}\hat{S}^{N+1} + \frac{1}{N(N+1)}S^{N+1} - \frac{1}{N}S\hat{S}^N$$

where N is a non-negative and odd integer, and assume that

$$p_{S,X}(s,x) = \frac{1}{\lambda x}\exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right) \qquad s \geq 0, \qquad x \geq 0, \qquad \lambda > 0$$

(a) Find the Bayesian estimator of S given X for the given costs.

(b) Obtain the minimum risk.

(c) Determine the coefficient w that minimizes the risk of an estimator with analytical shape

$$\hat{S}_L = wX^m$$

m being a positive integer.

Hint: $\int_0^\infty x^N \exp(-x)\, dx = N!$

Solution:

(a) The conditional risk is given by

$$\begin{aligned} R_{\hat{s}} &= \mathbb{E}\{c(S,\hat{s})|\mathbf{x}\} \\ &= \mathbb{E}\left\{\frac{1}{N+1}\hat{s}^{N+1} + \frac{1}{N(N+1)}S^{N+1} - \frac{1}{N}S\hat{s}^N \mid x\right\} \\ &= \frac{1}{N+1}\hat{s}^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}\mathbb{E}\left\{S \mid x\right\}\hat{s}^N \end{aligned}$$

Since this risk is a differentiable function of $\hat{s}$, the minimum must be at a stationary point

$$\begin{aligned} \frac{\partial R_{\hat{s}}}{\partial \hat{s}} = 0 \quad &\Leftrightarrow \quad \hat{s}^N - \mathbb{E}\left\{S \mid x\right\}\hat{s}^{N-1} = 0 \\ &\Leftrightarrow \quad \hat{s}^{N-1}\left(\hat{s} - \mathbb{E}\left\{S \mid x\right\}\right) = 0 \end{aligned}$$

Thus the minimizer of the conditional risk is

$$\hat{s}^* = \mathbb{E}\{S \mid x\}.$$

To compute the conditional mean, we need the posterior distribution of S. Noting that

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^\infty \frac{1}{\lambda x}\exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right)ds \\ &= \frac{1}{\lambda x}\exp\left(-\frac{x}{\lambda}\right)\int_0^\infty \exp\left(-\frac{s}{x}\right)ds = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right) \end{aligned}$$

we have

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{1}{x}\exp\left(-\frac{s}{x}\right)$$

so that

$$\hat{s}^* = \mathbb{E}\{S \mid x\} = \int_{-\infty}^{\infty} s p_{S|X}(s|x)ds = \int_0^\infty \frac{s}{x}\exp\left(-\frac{s}{x}\right)ds = x$$

(b) Since the minimum conditional risk is

$$
\begin{aligned}
R_{\hat{s}} &= \frac{1}{N+1}\left(\hat{s}^*\right)^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}\mathbb{E}\left\{S \mid x\right\}\left(\hat{s}^*\right)^{N} \\
&= \frac{1}{N+1}x^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}x^{N+1} \\
&= \frac{1}{N(N+1)}\left(\int_0^\infty \frac{s^{N+1}}{x}\exp\left(-\frac{s}{x}\right)ds - x^{N+1}\right) \\
&= \frac{(N+1)!-1}{N(N+1)}x^{N+1}
\end{aligned}
$$

the minimum risk can be computed as

$$
\begin{aligned}
\mathbb{E}\{c(S,\hat{S}\} &= \int_{-\infty}^{\infty}\mathbb{E}\{c(S,\hat{s})|\mathbf{x}\}p_X(x)dx \\
&= \frac{(N+1)!-1}{\lambda N(N+1)}\int_0^\infty x^{N+1}\exp\left(-\frac{x}{\lambda}\right)dx \\
&= \frac{(N+1)!-1}{N(N+1)}(N+1)!\lambda^{N+1} \\
&= (N+1)!-1)(N-1)!\lambda^{N+1}
\end{aligned}
$$

(c) If $\hat{S} = wX^m$, the risk is given by

$$
\begin{aligned}
R &= \mathbb{E}\{c(S,\hat{s})\} \\
&= \frac{1}{N+1}\mathbb{E}\left\{\hat{S}^{N+1}\right\} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1}\right\} - \frac{1}{N}\mathbb{E}\left\{S\hat{S}^{N}\right\} \\
&= \frac{1}{N+1}\mathbb{E}\left\{X^{m(N+1)}\right\}w^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1}\right\} - \frac{1}{N}\mathbb{E}\left\{SX^{mN}\right\}w^{N}
\end{aligned}
$$

By differentiation, this is minimum when

$$
\mathbb{E}\left\{X^{m(N+1)}\right\}w^{N} - \mathbb{E}\left\{SX^{mN}\right\}w^{N-1} = 0
$$

that is

$$
w = \frac{\mathbb{E}\left\{SX^{mN}\right\}}{\mathbb{E}\left\{X^{m(N+1)}\right\}}
$$

The numerator can be computed as

$$
\begin{aligned}
\mathbb{E}\left\{SX^{mN}\right\} &= \int_0^\infty \mathbb{E}\left\{SX^{mN} \mid x\right\}p_X(x)dx \\
&= \int_0^\infty x^{mN}\mathbb{E}\left\{S \mid x\right\}p_X(x)dx \\
&= \frac{1}{\lambda}\int_0^\infty x^{mN+1}\exp\left(-\frac{x}{\lambda}\right)dx = \lambda^{mN+1}(mN+1)!
\end{aligned}
$$

and the denominator is

$$
\begin{aligned}
\mathbb{E}\left\{X^{m(N+1)}\right\} &= \int_0^\infty x^{m(N+1)}p_X(x)dx \\
&= \frac{1}{\lambda}\int_0^\infty x^{m(N+1)}\exp\left(-\frac{x}{\lambda}\right)dx = \lambda^{m(N+1)}(m(N+1))!
\end{aligned}
$$

Therefore

$$
w = \frac{(Nm+1)!}{(Nm+m)!\lambda^{m-1}}
$$

ET19

An order-N Erlang probability density is characterized by the following expression:

$$p_X(x) = \frac{a^N x^{N-1} \exp(-ax)}{(N-1)!} \qquad x > 0, \quad a > 0$$

Assume that N is known. Considering that the mean of the distribution is given by $m = N/a$, obtain:

(a) The ML estimator of the mean using K independent observations of the variable, $\hat{M}_{\mathrm{ML}}$.

(b) The conditional bias of $\hat{M}_{\mathrm{ML}}$.

(c) Is $\hat{M}_{\mathrm{ML}}$ MSE-consistent?

Solution:

(a) $\hat{M}_{\mathrm{ML}} = \dfrac{1}{K} \sum_{k=1}^{K} X_k$

(b) The estimator is unbiased.

(c) $\operatorname{var}\left\{\hat{M}_{\mathrm{ML}}\right\} = \dfrac{v_x}{K}$; therefore, the estimator is MSE-consistent.

ET20

Random vector $\mathbf{X} = [X_1, X_2, X_3]^T$ follows a p.d.f. with mean $\mathbf{m} = \mathbf{0}$ and covariance matrix

$$\mathbf{V_{XX}} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) Obtain the coefficients (w_0, w_1 and w_2) of the linear minimum mean square error estimator of X_3 given X_1 and X_2,

$$\hat{X}_{3,\mathrm{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Calculate the mean square error of the estimator $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\mathrm{LMSE}}\right)^2\right\}$.

Solution:

(a) Defining

$$\mathbf{Z} = \begin{bmatrix} 1 \\ X_1 \\ X_2 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

the LMSE estimator will be given by coefficients

$$\begin{aligned}
\mathbf{w} &= \mathbf{R}_{\mathbf{Z}}^{-1} \mathbf{r}_{X_3\mathbf{Z}} = \mathbb{E}\{\mathbf{Z}\mathbf{Z}^\intercal\}^{-1} \mathbb{E}\{X_3 \mathbf{Z}\} \\
&= \begin{bmatrix} 1 & \mathbb{E}\{X_1\} & \mathbb{E}\{X_2\} \\ \mathbb{E}\{X_1\} & \mathbb{E}\{X_1^2\} & \mathbb{E}\{X_1 X_2\} \\ \mathbb{E}\{X_2\} & \mathbb{E}\{X_1 X_2\} & \mathbb{E}\{X_2^2\} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\{X_3\} \\ \mathbb{E}\{X_1 X_3\} \\ \mathbb{E}\{X_2 X_3\} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{4}{5} \end{bmatrix}
\end{aligned}$$

therefore

$$\hat{X}_{3,\mathrm{LMSE}} = -\frac{1}{5} X_1 + \frac{4}{5} X_2$$

(b)

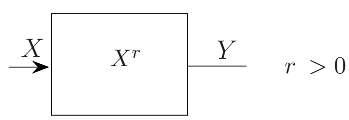
$$
\begin{aligned}
\mathbb{E}\left\{\left(X_3-\hat{X}_{3,\mathrm{LMSE}}\right)^2\right\} &= \mathbb{E}\left\{\left(X_3+\frac{1}{5}X_1-\frac{4}{5}X_2\right)^2\right\}\\
&= \mathbb{E}\left\{X_3^2\right\}+\frac{1}{25}\mathbb{E}\left\{X_1^2\right\}+\frac{16}{25}\mathbb{E}\left\{X_2^2\right\}+\frac{2}{5}\mathbb{E}\left\{X_3X_1\right\}-\frac{8}{5}\mathbb{E}\left\{X_3X_2\right\}-\frac{8}{25}\mathbb{E}\left\{X_1X_2\right\}\\
&= 3+\frac{3}{25}+\frac{16}{25}\cdot 3+\frac{2}{5}\cdot 1-\frac{8}{5}\cdot 2-\frac{8}{25}\cdot 2\\
&= \frac{8}{5}
\end{aligned}
$$

ET21

A random variable X with p.d.f.

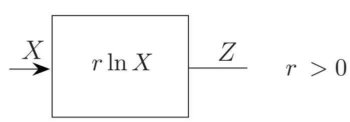
$$
p_X(x)=\begin{cases}1, & 0<x<1\\ 0, & \text{otherwise}\end{cases}
$$

is transformed as indicated in the figure, producing a random observation Y.



(a) Obtain the maximum likelihood estimator of r, $\hat{R}_{\mathrm{ML}}$, based on K independently drawn observations of Y.

(b) Now, consider the following situation



and obtain $\hat{R}_{\mathrm{ML}}$ using K independent observations of random variable Z. Discuss your result.

Solution:

(a) $\hat{R}_{\mathrm{ML}} = -\frac{1}{K}\sum_{k=0}^{K-1}\ln Y_k$. The unknown parameter of the transformation is being identified.

(b) $\hat{R}_{\mathrm{ML}} = -\frac{1}{K}\sum_{k=0}^{K-1} Z_k$. It is coherent with the previous estimator since $Z=\ln Y$, which is a deterministic (and invertible) transformation of Y.

ET22

An unknown deterministic parameter s, $s>0$ is measured using two different systems, which provide observations

$$
X_i = A_i s + N_i, \quad i=1,2
$$

where $\{A_i\},\{N_i\}$, are independent Gaussian random vectors, with means $\mathbb{E}\{A_i\}=1$, $\mathbb{E}\{N_i\}=0$, and variances $\{v_{Ai}\}$, $\{v_{Ni}\}$, respectively ($i=1,2$).

(a) State the expression that defines the ML estimator of s, $\hat{S}_{\mathrm{ML}}$.

(b) Obtain $\hat{S}_{\mathrm{ML}}$ for the particular case $v_{Ai}=0, i=1,2$.

(c) Obtain $\hat{S}_{\mathrm{ML}}$ for the particular case $v_{Ni} = 0, i = 1, 2$.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = \operatorname{argmin}_s \left\{ \ln\left[(s^2 v_{A1} + v_{N1})(s^2 v_{A2} + v_{N2})\right] + \frac{(s - X_1)^2}{s^2 v_{A1} + v_{N1}} + \frac{(s - X_2)^2}{s^2 v_{A2} + v_{N2}} \right\}$

(b) $\hat{S}_{\mathrm{ML}} = \frac{v_{N2} X_1 + v_{N1} X_2}{v_{N1} + v_{N2}}$

(c) $\hat{S}_{\mathrm{ML}} = \frac{1}{4} \sqrt{\left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)^2 + 8\left(\frac{X_1^2}{v_{A1}} + \frac{X_2^2}{v_{A2}}\right)} - \left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)$

ET23

Let X and S be two random variables with joint pdf

$$p_{X,S}(x, s) = \begin{cases} \alpha & ; \quad 0 < x < 1,\ 0 < s < 2(1 - x) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

with α a constant.

(a) Plot the support of the pdf, and use it to determine the value of α.

(b) Obtain the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) Find the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$

(d) Find the linear minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{LMSE}}$.

Solution:

(a) $\alpha = 1$

(b) $p_{S|X}(s|x) = \frac{1}{2(1-x)}$

(c) $\hat{S}_{\mathrm{MSE}} = 1 - X$

(d) $\hat{S}_{\mathrm{LMSE}} = 1 - X$

ET24

Random variables S and X are related through the stochastic equation:

$$X = S + N$$

where the prior pdf of S is

$$p_S(s) = s \exp(-s) \qquad s \geq 0$$

and where N is an additive noise, independent of S, with distribution

$$p_N(n) = \exp(-n) \qquad n \geq 0$$

Find:

(a) The maximum likelihood estimator of S, $\hat{S}_{\mathrm{ML}}$.

(b) The joint pdf of X and S, $p_{X,S}(x, s)$, and the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) The maximum a posteriori estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(e) The bias of all previous estimators, $\hat{S}_{\text{ML}}$, $\hat{S}_{\text{MAP}}$ and $\hat{S}_{\text{MSE}}$.

(f) Which of the previous estimators has a minimum variance? Justify your answer without calculating the variances of the estimators.

Solution:

(a) $\hat{S}_{\text{ML}} = X$

(b) $p_{X,S}(x,s) = s\exp(-x), \qquad 0 \le s \le x$

$p_{S|X}(s|x) = \dfrac{2s}{x^2}, \qquad 0 \le s \le x$

(c) $\hat{S}_{\text{MAP}} = X$

(d) $\hat{S}_{\text{MSE}} = \dfrac{2}{3}X$

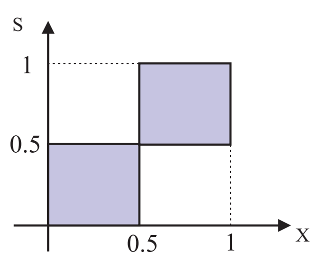
(e) $\mathbb{E}\left\{\hat{S}_{\text{ML}} - S\right\} = \mathbb{E}\left\{\hat{S}_{\text{MAP}} - S\right\} = 1$

$\mathbb{E}\left\{\hat{S}_{\text{MSE}} - S\right\} = 0$

$\text{var}\left\{\hat{S}_{\text{MSE}}\right\} < \text{var}\left\{\hat{S}_{\text{MAP}}\right\} = \text{var}\left\{\hat{S}_{\text{ML}}\right\}$

ET25

In the plot below, the shaded region shows the domain of a joint distribution of S and X, i.e., the set of points for which $p_{X,S}(x,s) \neq 0$.



Please, provide justified answers to the following questions:

(a) If it is known that $p_{X,S}(x,s)$ is constant in its domain, which is the MSE estimator of S given X? Provide a graphical representation of this estimator.

(b) Is there any $p_{X,S}(x,s)$ with the previous domain for which the MSE estimator of S given X is $\hat{S}_{\text{MSE}} = X/2$?

(c) Justify if there exists any $p_{X,S}(x,s)$ with the previous domain, so that $\hat{S} = 0.5$ is:

- The minimum mean square error estimator of S given X.
- The minimum mean absolute deviation estimator of S given X.
- The maximum *a posteriori* estimator of S given X.

Solution:

(a) $\hat{S}_{\text{MSE}} = 0.25$ for $0 < x < 0.5$ and $\hat{S}_{\text{MSE}} = 0.75$ for $0.5 < x < 1$

(b) When $0.5 < x < 1$, $p_{S|X}(s|x)$ is non-zero for $0.5 < s < 1$, thus $X/2$ can never be the mean of $p_{S|X}(s|x)$ for that range of X.

(c) $\hat{S} = 0.5$ cannot be the mean or the median of $p_{S|X}(s|x)$, but it can be its maximum. Therefore, $\hat{S} = 0.5$ can just be $\hat{S}_{\text{MAP}}$ (but not $\hat{S}_{\text{MSE}}$ or $\hat{S}_{\text{MAD}}$).

ET26

A random variable S follows an exponential pdf

$$p_S(s) = \lambda e^{-\lambda s}, \qquad s > 0$$

with $\lambda > 0$. Consider now a discrete random variable X related to S via a Poisson distribution, i.e.,

$$P_{X|S}(x|s) = \frac{s^x e^{-s}}{x!}, \qquad x = 0, 1, 2, \ldots$$

(a) Determine the ML estimator of S given x.

(b) Assume now that we have access to K independent realizations $\{(x_k, s_k), k = 0, \ldots, K-1\}$ of (X, S). Find the ML estimator of λ based on these observations.

(c) Find the MAP estimation of S for $x = 1$.

Solution:

(a) $\hat{S}_{\text{ML}} = X$

(b) $\hat{\lambda}_{\text{ML}} = K \left(\sum_{k=0}^{K-1} s_k \right)^{-1}$

(c) $\hat{S}_{\text{MAP}} = \dfrac{X}{1 + \lambda}$

ET27

N.A.

ET28

N.A.

ET29

N.A.

ET30

N.A.

ET31

N.A.

2 Additional Problems

ET32

Consider an observation

$$X = S + N$$

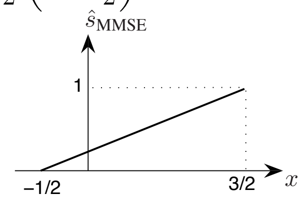
where S is a signal contaminated by additive noise N, and where S and N are independent of each other, and with probability density functions given by:

$$p_S(s) = \begin{cases} 1, & 0 < s < 1 \\ 0, & \text{otherwise} \end{cases} = \Pi(s - 1/2)$$

$$p_N(n) = \begin{cases} 1, & -1/2 < n < 1/2 \\ 0, & \text{otherwise} \end{cases} = \Pi(n)$$

Find the minimum mean square error estimator of S, $\hat{S}_{\text{MMSE}}$. Discuss your result.

Solution: $\hat{S}_{\text{MMSE}} = \frac{1}{2}\left(X + \frac{1}{2}\right) \qquad (-1/2 < x < 1/2)$



The linear change of the estimator between its minimum and maximum values ($\hat{s}_{\text{MMSE}}(-1/2) = 0, \hat{s}_{\text{MMSE}}(3/2) = 1$) are due to the addition of uniform noise.

ET33

We have access to K samples independently drawn from a random variable X which follows a Laplace distribution $L(m, v)$

$$p_X(x) = \frac{1}{\sqrt{2v}} \exp\left(-\sqrt{\frac{2}{v}}|x - m|\right)$$

Find the joint ML estimators of m, v.

Solution:

$$\begin{aligned} \hat{M}_{\text{ML}} &= \text{med}_K\{X^{(k)}\} \quad \text{(sample median)} \\ \hat{V}_{\text{ML}} &= \frac{2}{K^2}\left(\sum_k |X^{(k)} - \hat{M}_{\text{ML}}|\right)^2 \end{aligned}$$

ET34

Unidimensional random variables S and R are characterized by the following joint distribution.

$$G\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right),$$

The observable variable is given by $X = S + R$.

(a) Obtain the estimator $\hat{S}_{\text{MSE}}$.

(b) Was this result to be expected? (Consider the existing relationship between $\mathbb{E}\{R|x\}$ and $\mathbb{E}\{S|x\}$).

(c) Obtain the MSE.

Solution:

(a) $\hat{S}_{\text{MSE}} = X/2$

(b) $\mathbb{E}\{R|x\} = \mathbb{E}\{S|x\}$ (since both variables distribute identically given X)
$\mathbb{E}\{X|x\} = x = \mathbb{E}\{S + R|x\} = \mathbb{E}\{S|x\} + \mathbb{E}\{R|x\}$

(c) $\mathbb{E}\left\{\left(S - \hat{S}\right)^2\right\} = \frac{1}{2} - \frac{1}{2}\rho$

ET35

Let S, X_1, and X_2 be three zero-mean random variables satisfying:

- The covariance matrix of X_1 and X_2 is:

$$\mathbf{V}_{xx} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- the cross-covariance between S and observation vector $X = [X_1, X_2]^T$ is:

$$\mathbf{v}_{sx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Obtain the coefficients of the linear minimum mean square error estimator

$$\hat{S}_{\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Find the expected quadratic value of the estimation error, $\hat{E} = S - \hat{S}_{\text{LMSE}}$.

(c) Explain which is the role of variable X_2, which, as can be seen, is uncorrelated with the variable to be estimated (S).

Solution:

(a) $w_1 = \dfrac{1}{1-\rho^2}; \qquad w_2 = -\dfrac{\rho}{1-\rho^2}; \qquad w_0 = 0$

(b) $\mathbb{E}\{\hat{E}^2\} = \mathbb{E}\{S^2\} - \dfrac{1}{1-\rho^2}$

(c) X_2 is combined with X_1 allowing a better approximation of S.

ET36

We want to design a linear minimum mean square error estimator of a random variable S based on the observation of random variables X_1 and X_2:

$$\hat{S}_{\text{LMSE}}(X_1, X_2) = w_0 + w_1 X_1 + w_2 X_2$$

The means of the random variables are $\mathbb{E}\{S\} = 1$, $\mathbb{E}\{X_1\} = 1$, and $\mathbb{E}\{X_2\} = 0$, whereas the correlations are given by $\mathbb{E}\{S^2\} = 4$, $\mathbb{E}\{X_1^2\} = 3$, $\mathbb{E}\{X_2^2\} = 2$, $\mathbb{E}\{SX_1\} = 2$, $\mathbb{E}\{SX_2\} = 0$, and $\mathbb{E}\{X_1 X_2\} = 1$.

(a) Obtain the optimal coefficients $\{w_i\}, i = 0, 1, 2$ of $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(b) Check that $v_{SX_2} = 0$. Why can still be $w_2 \neq 0$?

(c) Calculate the mean square error incurred by the application of estimator $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(d) How does the mean square error changes if the estimator $\hat{S}'_{\text{LMSE}}(X_1) = w'_0 + w'_1 X_1$, based on the sole observation of X_1, is used instead of $\hat{S}_{\text{LMSE}}(X_1, X_2)$?

Solution:

(a) $w_0 = 1/3, \ w_1 = 2/3, \ w_2 = -1/3$

(b) Combining X_1 and X_2 is better than just using X_1 (using the geometric analogy of the Orthogonality Principle, the projection space spanned by X_1 and X_2 is larger than the one spanned by X_1 alone).

(c) $\mathbb{E}\{E^2\} = 7/3$

(d) ($w'_0 = 1/2; \ w'_1 = 1/2$). $\mathbb{E}\{E'^2\} = 3$. It increases by $2/3$ (confirming our answer to the previous subquestion).

ET37

N.A.