Estimation Theory: Problems

Notation:

- $\hat{S}_{\text{MSE}}$: Minimum Mean Square Error estimator.
- $\hat{S}_{\text{MAD}}$: Minimum Mean Absolute Deviation Error estimator.
- $\hat{S}_{\text{MAP}}$: Maximum a posteriori estimator.
- $\hat{S}_{\text{ML}}$: Maximum likelihood estimator.
- $\hat{S}_{\text{LMSE}}$: Linear Minimum Mean Square Error estimator.

Contents

1 Bayesian Estimation

1.1 Bayesian Estimation: MSE

ET1

The joint p.d.f. of two random variables S and X is:

$$p_{S,X}(s,x) = 6s, \qquad 0 < s < x, \qquad 0 < x < 1$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.
(b) The conditional bias
(c) The unconditional bias of estimator $\hat{S}_{\text{MSE}}$.

Solution:

(a) Noting that

$$p_X(x) = \int_0^x 6sds = 3x^2, \qquad 0 < x < 1$$

we have

$$p_{S|X}(s|x) = \frac{2s}{x^2}, \qquad 0 < s < x, \qquad 0 < x < 1$$

therefore

$$\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} = \int_0^x \frac{2s^2}{x^2} = \frac{2}{3}x$$

(b)

$$p_S(s) = \int p_{S,X}(s,x)dx = \int_s^1 6sdx = 6s(1-s), \qquad 0 < s < 1$$

we have

$$p_{X|S}(x|s) = \frac{1}{1-s}, \qquad 0 < s < x, \qquad 0 < x < 1$$

and

$$\mathbb{E}\{X|s\} = \int xp_{X|S}(x|s)dx = \frac{1}{1-s}\int_s^1 xdx = \frac{1}{2(1-s)}$$

Therefore

$$\mathbb{E}\{\hat{S}_{\text{MSE}}|s\} = \frac{2}{3}\mathbb{E}\{X|s\} = \frac{1}{3(1-s)}$$

and the conditional bias is

$$\text{bias}\{\hat{S}_{\text{MSE}}|s\} = \mathbb{E}\{\hat{S}_{\text{MSE}}|s\} - s = \frac{1}{3(1-s)} - s$$

(c) Since

$$\mathbb{E}\{S\} = \int_0^1 6s^2(1-s) = \frac{1}{2}$$
$$\mathbb{E}\{\hat{S}_{\text{MSE}}\} = \frac{2}{3}\mathbb{E}\{X\} = \frac{2}{3}\int_0^1 3x^3dx = \frac{1}{2}$$

the estimator is unbiased

ET2

Obtain the minimum mean square error estimator of random variable S based on the observation of random variable X, in the following two cases:

(a)

$$p_{X,S}(x,s) = \begin{cases} 1, & 0 \le x \le 1, \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$p_{X,S}(x,s) = \begin{cases} 2, & 0 \le s \le 1-x, \quad 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

(a) $\hat{S}_{\text{MSE}} = \frac{1}{2}$

(b) $\hat{S}_{\text{MSE}} = \frac{1-X}{2}$

ET3

The joint distribution of random variables S and X is given by:

$$p_{S,X}(s,x) = \sqrt{\frac{x(1-x)}{2\pi}} \exp\left(-\frac{(s-x(1-x))^2}{2x(1-x)}\right), \qquad 0 < x < 1, \qquad s \in \mathbb{R}$$

(a) Find the minimum mean square error (MSE) estimate of S based on the observation x.

(b) Compute the conditional MSE given observation x, $\mathbb{E}\{(S - \hat{S}_{\text{MSE}})^2 | X = x\}$, for the estimator obtained in part a).

(c) Compute the MSE of the estimator obtained in part a).

Solution:

(a) $\hat{S}_{\text{MSE}} = X(1-X)$

(b) $\mathbb{E}\{(S - \hat{S}_{\text{MSE}})^2 | X = x\} = x(1-x)$

(c) $\mathbb{E}\{(S - \hat{S}_{\text{MSE}})^2\} = \dfrac{1}{30}$

ET4

Let S and X be two unidimensional random variables. Variable X is characterized by the following probability density function:

$$p_X(x) = \begin{cases} \frac{1}{2}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

It is known that the minimum MSE estimator of S based on X is

$$\hat{S}_{\text{MSE}}(X) = -\frac{1}{2}\text{sign}(X) = \begin{cases} -\dfrac{1}{2}, & X \ge 0 \\ \dfrac{1}{2}, & X < 0 \end{cases}$$

It is also known that the MSE associated to this estimator is $\dfrac{1}{12}$. An alternative estimator of S is defined as

$$\hat{S}_1 = -X$$

(a) Obtain the unconditional bias of estimator $\hat{S}_1$.

(b) Compute $\mathbb{E}\{SX\}$ and $\mathbb{E}\{S^2\}$.

(c) Find the MSE incurred by $\hat{S}_1$.

Solution:

(a) The estimator is unbiased.

(b) $\mathbb{E}\{SX\} = -\frac{1}{4}$ and $\mathbb{E}\{S^2\} = \frac{1}{3}$

(c) $\mathrm{MSE}\{\hat{S}_1\} = \dfrac{1}{6}$.

ET5

Consider three independent random variables, S_1, S_2 and S_3, with the same *a priori* pdf. We wish to estimate S_1 from a single observation of $X = S_1 + S_2 + S_3$.

(a) Justify briefly why the minimum mean square error estimator of S_1 given x is $\hat{s}_1 = \dfrac{x}{3}$.

(b) Find the bias of the estimator given in (a). Is it biased or unbiased?

Assuming in the following that S_1, S_2 and S_3 follow a uniform distribution between -1 and 1:

(c) Obtain the pdf of X given s_1.

(d) Find the variance of estimator $\hat{S}_1$.

Solution:

(a) Using symmetry arguments, all $\mathbb{E}\{S_i|x\}$ should be the same and sum up to x.

(b) The estimator is unbiased.

(c) $p_{X|S_1}(x|s_1) = 1/2 - |x - s_1|/4$ for $-2 < x < 2$, and 0 otherwise.

(d) $\mathrm{var}(S_1) = 1/9$

1.2 Bayesian Estimation: MAP, MSE, MAD

ET6

The joint pdf of two random variables S and X is given by:

$$p_{S,X}(s,x) = \frac{4s}{x(1-x)}, \qquad x(1-x) < s < 2x(1-x), \qquad 0 < x < 1$$

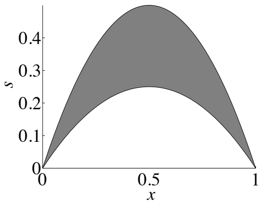
(a) Provide an approximate representation of the support of the joint distribution.

(b) Find $\hat{S}_{\mathrm{MSE}}$.

(c) Find $\hat{S}_{\mathrm{MAD}}$.

Solution:

(a) The support of the pdf is the shaded region in the figure:



(b) $\hat{s}_{\text{MSE}} = \frac{14}{9}x(1-x)$.

(c) $\hat{s}_{\text{MAD}} = \sqrt{\frac{5}{2}x(1-x)}$

ET7

The random variables S and X are related through the posterior distribution:

$$p_{S|X}(s|x) = \frac{x + 4s - s^2}{x + \frac{5}{3}}, \qquad 0 \le s \le 1, \qquad x \ge 0$$

(a) Determine the least mean square error estimator, $\hat{S}_{\text{MSE}}$.

(b) Determine the maximum posterior estimator, $\hat{S}_{\text{MAP}}$.

Solution:

(a)

$$\hat{S}_{\text{MSE}} = \mathbb{E}\{S|X = x\} = \int_0^1 s\frac{x + 4s - s^2}{x + \frac{5}{3}}ds = \frac{6x + 13}{12x + 20}$$

(b)

$$\begin{aligned}\hat{S}_{\text{MAP}} &= \underset{s}{\operatorname{argmax}}\{p_{S|X}(s|x)\} = \underset{s\in[0,1]}{\operatorname{argmax}}\left\{\frac{x + 4s - s^2}{x + \frac{5}{3}}\right\} \\ &= \underset{s\in[0,1]}{\operatorname{argmax}}\left\{x + 4s - s^2\right\}\end{aligned}$$

The parabola $x + 4s - s^2$ has a derivative

$$-2s + 4$$

which vanishes at $s = 2$, which falls outside the domain of $p_{S|X}(s|x)$. Therefore, $\hat{s}_{\text{MAP}}$ must coincide with one of the endpoints of the interval $[0, 1]$. Since the derivative is positive on that interval, $p_{S|X}(s|x)$ is increasing in the domain, and hence

$$\hat{s}_{\text{MAP}} = 1$$

ET8

Random variables S and X are defined by the joint distribution

$$p_{S,X}(s, x) = 2e^{-3x}, \qquad x \ge 0, \qquad 0 \le s \le e^x$$

(a) Compute $p_{S|X}(s|x)$

(b) Compute the minimum mean absolute deviation estimate of S given X, $\hat{S}_{\text{MAD}}$.

(c) Compute the minimum mean square error (MSE) estimate of S given X, $\hat{S}_{\text{MSE}}$.

(d) Consider the random variable $T = S^3$. Compute the minimum MSE estimate of T given X, $\hat{T}_{\text{MSE}}$.

Solution:

(a) $p_X(x) = \int_0^{e^x} 2e^{-3x} ds = 2e^{-3x}e^x = 2e^{-2x}$, for $x \geq 0$, therefore,

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = e^{-x}, \qquad x \geq 0, \quad 0 \leq s \leq e^x$$

(b) Since $p_{S|X}(s|x)$ is uniform in $[0, e^x]$, the minimum MAD estimate is the midpoint

$$\hat{s}_{\text{MAD}} = \frac{1}{2}e^x$$

(c)

$$\begin{aligned}\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} &= \int_{-\infty}^{\infty} s p_{S|X}(s|x) ds \\ &= \int_0^{e^x} e^{-x} s ds = e^{-x} \left. \frac{s^2}{2} \right]_0^{\exp(x)} = \frac{1}{2} e^{-x} e^{2x} = \frac{1}{2} e^x\end{aligned}$$

(d)

$$\begin{aligned}\hat{t}_{\text{MSE}} = \mathbb{E}\{T|x\} = \mathbb{E}\{S^3|x\} &= \int_{-\infty}^{\infty} s^3 p_{S|X}(s|x) ds \\ &= \int_0^{e^x} e^{-x} s^3 ds = e^{-x} \left. \frac{s^4}{4} \right]_0^{\exp(x)} = \frac{1}{4} e^{-x} e^{4x} = \frac{1}{4} e^{3x}\end{aligned}$$

ET9

Consider the estimation of a random variable S from another random variable X, given the joint probability density function (pdf)

$$p_{S,X}(s,x) = \frac{6}{7}(x+s)^2, \qquad 0 \leq x \leq 1, \quad 0 \leq s \leq 1$$

(a) Find $p_X(x)$.

(b) Find $p_{S|X}(s|x)$.

(c) Compute the minimum MSE estimator of S given X, $\hat{S}_{\text{MSE}}$.

(d) Compute the MAP estimator of S given X, $\hat{S}_{\text{MAP}}$.

(e) Compute the bias and the variance of the MAP estimator.

Solution:

(a)

$$\begin{aligned}p_X(x) &= \int_0^1 p_{S,X}(s,x) ds = \int_0^1 \frac{6}{7}(x+s)^2 ds \\ &= \frac{2}{7}\left(3x^2 + 3x + 1\right), \qquad 0 \leq x \leq 1\end{aligned}$$

(b)

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{(x+s)^2}{x^2+x+\frac{1}{3}}, \qquad 0 \le x \le 1, \quad 0 \le s \le 1$$

(c)

$$\begin{aligned}\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} &= \int_0^1 s p_{S|X}(s|x) ds = \frac{1}{x^2+x+\frac{1}{3}} \int_0^1 s\,(x+s)^2\, ds \\ &= \frac{\frac{x^2}{2}+\frac{2x}{3}+\frac{1}{4}}{x^2+x+\frac{1}{3}}.\end{aligned}$$

(d) Given that $p_{S|X}(s|x)$ increases with s for $0 \le x \le 1$ and $0 \le s \le 1$, $\hat{S}_{\text{MAP}} = 1$.

(e) Since

$$\begin{aligned}p_S(s) &= \int_0^1 p_{S,X}(s,x) ds = \int_0^1 \frac{6}{7}\,(x+s)^2\, dx \\ &= \frac{2}{7}\left(3s^2+3s+1\right), \ \ 0 < s < 1\end{aligned}$$

we have

$$\mathbb{E}\left\{S\right\} = \int_0^1 s p_S(s) ds = \frac{2}{7}\int_0^1 s\left(3s^2+3s+1\right) ds = \frac{9}{14},$$

and, thus, the expected bias is

$$\mathbb{E}\{\hat{S}_{\text{MAP}}\} - \mathbb{E}\{S\} = 1 - \frac{9}{14} = \frac{5}{14}$$

Since $\hat{S}_{\text{MAP}} = 1$ (constant and independent of X), its variance is zero.

ET10

The joint distribution of S and X is given by:

$$p_{S,X}(s,x) = \frac{4}{(x+s)^3}, \qquad s \ge 1, \quad x \ge 1.$$

(a) Find $p_{S|X}(s|x)$.

(b) Find the MAP estimator of S given X.

(c) Find the minimum absolute deviation (MAD) estimator of S given X.

Solution:

(a) $p_{S|X}(s|x) = \dfrac{2(x+1)^2}{(x+s)^3}, \qquad , x \ge 1.$

(b) $\hat{S}_{\text{MAP}} = 1$.

(c) $\hat{S}_{\text{MAD}} = (\sqrt{2}-1)X + \sqrt{2}$.

ET11

Consider the estimation of a random variable S based on the observation of X, where the two variables satisfy the following joint pdf:

$$p_{S,X}(s,x) = \begin{cases} 6s, & 0 < s < x-1, \;\; 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Obtain:

(a) The minimum MSE estimator of S given X, $\hat{S}_{\text{MSE}}$.
(b) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\text{MAP}}$.
(c) The minimum mean absolute error estimator of S given X, $\hat{S}_{\text{MAD}}$.
(d) The unconditional bias of all previous estimators.

Solution:

(a) $\hat{S}_{\text{MSE}} = \frac{2}{3}(X-1)$

(b) $\hat{S}_{\text{MAP}} = X - 1$

(c) $\hat{S}_{\text{MAD}} = \frac{1}{\sqrt{2}}(X-1)$

(d) $\mathbb{E}\left\{\hat{S}_{\text{MSE}} - S\right\} = 0 \quad \mathbb{E}\left\{\hat{S}_{\text{MAP}} - S\right\} = 0.25 \quad \mathbb{E}\left\{\hat{S}_{\text{MAD}} - S\right\} = 0.03$

ET12

The random variables S, T and X are related through the following probability density function

$$p_{S|T,X}(s|t,x) = t \cdot e^{-st}, \qquad s \geq 0, \quad t \geq 0, \quad x \geq 0$$

$$p_{T,X}(t,x) = xe^{-x-tx} \qquad t \geq 0, \quad x \geq 0,$$

(8%) (a) Compute the minimum MSE estimate of S given (T,X), $\hat{S}_{\text{MSE}}$.
(8%) (b) Compute the MAP estimate of S given X, $\hat{S}_{\text{MAP}}$.
(7%) (c) Compute the minimum absolute deviation estimate of S given X, $\hat{S}_{\text{MAD}}$

Solution:

(a)

$$\hat{S}_{\text{MSE}} = \mathbb{E}\{S|t,x\} = \int_0^\infty st \cdot e^{-st} ds = \frac{1}{t}$$

(b) Since

$$\begin{aligned} p_{S,T,X}(s,t,x) &= p_{S|T,X}(s|t,x) \cdot p_{T,X}(t,x) \\ &= xt \cdot e^{-x-tx-st}, \qquad s \geq 0, \quad t \geq 0, \quad x \geq 0 \end{aligned}$$

we have

$$\begin{aligned} p_{S,X}(s,x) &= \int_0^\infty p_{S,T,X}(s,t,x) dt \\ &= xe^{-x} \int_0^\infty t \cdot e^{-t(x+s)} dt = \frac{x}{(x+s)^2} e^{-x} \end{aligned}$$

Therefore

$$\hat{S}_{\text{MAP}} = \underset{s}{\operatorname{argmax}}\, p_{S|X}(s|x) = \underset{s}{\operatorname{argmax}}\, p_{S,X}(s,x) = 0$$

(c)

$$\begin{aligned} p_X(x) &= \int_0^\infty p_{S,X}(s,x)ds = e^{-x}\int_0^\infty \frac{x}{(x+s)^2}ds \\ &= -xe^{-x}\left[\frac{1}{x+s}\right]_0^\infty = e^{-x} \end{aligned}$$

Therefore

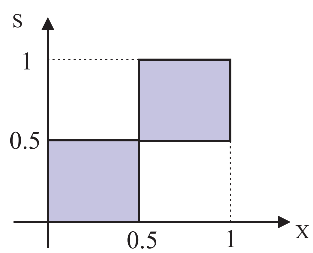
$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{x}{(x+s)^2}$$

and the MAD estimate is the solution of

$$\begin{aligned} \int_0^{\hat{s}_{\text{MAD}}} p_{S|X}(s|x)ds = \frac{1}{2} \qquad &\Rightarrow \qquad x\int_0^{\hat{s}_{\text{MAD}}} \frac{1}{(s+x)^2}ds = \frac{1}{2} \\ &\Rightarrow \qquad x\left[-\frac{1}{s+x}\right]_0^{\hat{s}_{\text{MAD}}} = \frac{1}{2} \\ &\Rightarrow \qquad 1 - \frac{x}{\hat{s}_{\text{MAD}} + x} = \frac{1}{2} \\ &\Rightarrow \qquad \hat{s}_{\text{MAD}} = x \end{aligned}$$

ET13

In the plot below, the shaded region shows the domain of a joint distribution of S and X, i.e., the set of points for which $p_{X,S}(x,s) \neq 0$.



Please, provide justified answers to the following questions:

(a) If it is known that $p_{X,S}(x,s)$ is constant in its domain, which is the MSE estimator of S given X? Provide a graphical representation of this estimator.

(b) Is there any $p_{X,S}(x,s)$ with the previous domain for which the MSE estimator of S given X is $\hat{S}_{\text{MSE}} = X/2$?

(c) Justify if there exists any $p_{X,S}(x,s)$ with the previous domain, so that $\hat{S} = 0.5$ is:

- The minimum mean square error estimator of S given X.
- The minimum mean absolute deviation estimator of S given X.
- The maximum *a posteriori* estimator of S given X.

Solution:

(a) $\hat{S}_{\mathrm{MSE}} = 0.25$ for $0 < x < 0.5$ and $\hat{S}_{\mathrm{MSE}} = 0.75$ for $0.5 < x < 1$

(b) When $0.5 < x < 1$, $p_{S|X}(s|x)$ is non-zero for $0.5 < s < 1$, thus $X/2$ can never be the mean of $p_{S|X}(s|x)$ for that range of X.

(c) $\hat{S} = 0.5$ cannot be the mean or the median of $p_{S|X}(s|x)$, but it can be its maximum. Therefore, $\hat{S} = 0.5$ can just be $\hat{S}_{\mathrm{MAP}}$ (but not $\hat{S}_{\mathrm{MSE}}$ or $\hat{S}_{\mathrm{MAD}}$).

1.3 Other Bayesian estimators

ET14

The joint distribution of random variables S and X is given by:

$$p_{S,X}(s,x) = 12s^2, \qquad 0 < s < x, \qquad 0 < x < 1$$

(a) Determine the estimate of S, $\hat{S}_{\mathrm{c}}$, which minimizes the risk for the cost function

$$c(s,\hat{s}) = \frac{(s-\hat{s})^2}{s}$$

(b) Determine the conditional risk given observation x, $\mathbb{E}\{c(S,\hat{S}_{\mathrm{c}})|X=x\}$, for the estimator obtained in part a).

Solution:

(a) $\hat{S}_{\mathrm{c}} = \frac{2X}{3}$

(b) $\mathbb{E}\{c(S,\hat{S}_{\mathrm{c}})|X=x\} = \frac{x}{12}$

1.4 Bayesian Estimation: MSE, MAD, MAP, and others

ET15

The posterior distribution of random variable S given X is

$$p_{S|X}(s|x) = \frac{1}{s\ln(x)}, \qquad 1 \le s \le x.$$

We want to estimate S based on the observation of X.

(a) Compute the minimun mean square error estimate, $\hat{s}_{\mathrm{MMSE}}$.

(b) Compute the maximum *a posteriori* (MAP) estimate.

(c) Compute the estimate with minimum absolute deviation (MAD).

(d) Compute the Bayesian estimator, $\hat{s}_{\mathrm{B}}$, for a cost $c(s,\hat{s}) = s(s-\hat{s})^2$.

Solution:

(a) $\hat{s}_{\mathrm{MSE}} = \dfrac{x-1}{\ln(x)}$

(b) $\hat{s}_{\mathrm{MAP}} = 1$

(c) $\hat{s}_{\text{MAD}} = \sqrt{x}$

(d) $\hat{s}_{\text{B}} = \frac{1}{2}(1+x)$

ET16

Random variables S and X follow the joint distribution

$$p_{S,X}(s,x) = 6e^{2x}(e^{-x} - s)s, \qquad 0 \le s \le e^{-x}, \qquad x \ge 0$$

(7%) (a) Compute the maximum a posteriori estimate of S given X, $\hat{S}_{\text{MAP}}$.

(8%) (b) Compute the minimum MSE estimate of S given X, $\hat{S}_{\text{MSE}}$.

(8%) (c) Compute the Bayesian estimate $\hat{S}^*$ minimizing the risk given by the cost

$$c(S,\hat{S}) = \frac{(S-\hat{S})^2}{S}$$

Solution:

(a) The MAP estimate is

$$\begin{aligned}\hat{S}_{\text{MAP}} &= \operatorname*{argmax}_s p_{S|X}(s|x) = \operatorname*{argmax}_s \frac{p_{S,X}(s,x)}{p_X(x)} = \operatorname*{argmax}_s p_{S,X}(s,x) \\ &= \operatorname*{argmax}_s \{(e^{-x} - s)s\} = \frac{1}{2}e^{-x}, \qquad x \ge 0\end{aligned}$$

(b) The marginal distribution is

$$\begin{aligned}p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds \\ &= 6e^{2x}\int_0^{e^{-x}} (e^{-x} - s)sds = 6e^{2x}\left[\frac{e^{-x}}{2}s^2 - \frac{1}{3}s^3\right]_0^{e^{-x}} \\ &= e^{-x}, \qquad x \ge 0\end{aligned}$$

Therefore, the posterior distribution is

$$p_{S|X}(s|x) = 6e^{3x}(e^{-x} - s)s, \qquad 0 \le s \le e^{-x}, \qquad x \ge 0$$

and the minimum MSE estimate is

$$\begin{aligned}\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} &= \int_{-\infty}^{\infty} s p_{S|X}(s|x)ds \\ &= 6e^{2x}\int_0^{e^{-x}} (e^{-x} - s)s^2ds = 6e^{2x}\left[\frac{e^{-x}}{3}s^3 - \frac{1}{4}s^4\right]_0^{e^{-x}} \\ &= \frac{1}{2}e^{-x}, \qquad x \ge 0\end{aligned}$$

(c) The conditional risk is given by

$$\begin{aligned}\mathbb{E}\{c(S,\hat{s}) \mid x\} &= \mathbb{E}\left\{\frac{(S-\hat{s})^2}{S} \mid x\right\} \\ &= \mathbb{E}\left\{S - 2\hat{s} + \frac{\hat{s}^2}{S} \mid x\right\} \\ &= \mathbb{E}\left\{S \mid x\right\} - 2\hat{s} + \hat{s}^2\mathbb{E}\left\{\frac{1}{S} \mid x\right\}\end{aligned}$$

which is minimized, with respect to $\hat{s}$, for

$$\hat{s}^* = \frac{1}{\mathbb{E}\left\{\frac{1}{S}|x\right\}}$$

For the given distribution

$$\begin{aligned}\mathbb{E}\left\{\frac{1}{S} \mid x\right\} &= \int_{-\infty}^{\infty} \frac{1}{s} p_{S|X}(s|x) ds \\ &= 6e^{3x} \int_0^{e^{-x}} (e^{-x} - s) ds = 6e^{3x} \left[e^{-x} s - \frac{1}{2} s^2\right]_0^{e^{-x}} \\ &= 3e^x, \qquad x \geq 0\end{aligned}$$

Therefore,

$$\hat{s}^* = \frac{1}{3} e^{-x}, \qquad x \geq 0$$

2 Bayesian and non-Bayesian estimators

ET17

Random variables S and X are jointly distributed according to

$$p_{S,X}(s,x) = \alpha s x^2, \qquad 0 \leq s \leq 1 - x, \quad 0 \leq x \leq 1$$

α being a parameter that needs to be determined.

(a) Find the expressions for the marginal probability density functions $p_X(x)$ and $p_S(s)$.
(b) Obtain the MAP estimator of S given X, $\hat{S}_{\mathrm{MAP}}(X)$.
(c) Obtain the ML estimator of S given X, $\hat{S}_{\mathrm{ML}}(X)$.
(d) Obtain the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}(X)$.
(e) Compare the previous estimators according to the mean square errors given X in which they incur.

Solution:

(a) Parameter α must take the value that makes the integral of the distribution a unity. Since

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{S,X}(s,x) ds dx &= \int_0^1 \int_0^{1-x} \alpha s x^2 ds dx = \alpha \int_0^1 x^2 \int_0^{1-x} s ds dx \\ &= \alpha \int_0^1 x^2 \left[\frac{1}{2} s^2\right]_0^{1-x} dx = \frac{\alpha}{2} \int_0^1 x^2 (1-x)^2 dx \\ &= \frac{\alpha}{60}\end{aligned}$$

we have $\alpha = 60$ and, thus

$$\begin{aligned}p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x) ds = \int_0^{1-x} 60 s x^2 ds = 60 x^2 \int_0^{1-x} s ds \\ &= 30 x^2 (1-x)^2, \qquad 0 \leq x \leq 1\end{aligned}$$

$$p_S(s) = \int_{-\infty}^{\infty} p_{S,X}(s,x)dx = \int_0^{1-s} 60sx^2 ds = 60s\int_0^{1-s} x^2 ds$$
$$= 20s(1-s)^3, \qquad 0 \le s \le 1$$

(b)

$$\begin{aligned}\hat{s}_{\text{MAP}} &= \operatorname*{argmax}_s p_{S|X}(s|x) = \operatorname*{argmax}_s \frac{p_{S,X}(s,x)}{p_X(x)} = \operatorname*{argmax}_s p_{S,X}(s,x)\\ &= \operatorname*{argmax}_{s\in[0,1-x]} 60sx^2 = \operatorname*{argmax}_{s\in[0,1-x]} s\\ &= 1-x\end{aligned}$$

(c) Since the likelihood function is

$$p_{X|S}(x|s) = \frac{p_{S,X}(s,x)}{p_S(s)} = \frac{60sx^2}{20s(1-s)^3} = \frac{3x^2}{(1-s)^3}, \qquad 0 \le s \le 1-x, \quad 0 \le x \le 1$$

the ML estimator is

$$\begin{aligned}\hat{s}_{\text{ML}} &= \operatorname*{argmax}_s p_{X|S}(x|s) = \operatorname*{argmax}_{s\in[0,1-x]} \frac{3x^2}{(1-s)^3} = \operatorname*{argmax}_{s\in[0,1-x]} \frac{1}{(1-s)^3} = \operatorname*{argmin}_{s\in[0,1-x]} (1-s)^3\\ &= 1-x\end{aligned}$$

(d) Since the posterior distribution is

$$p_{S|x}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{60sx^2}{30x^2(1-x)^2} = \frac{2s}{(1-x)^2}, \qquad 0 \le s \le 1-x, \quad 0 \le x \le 1$$

the minimum MSE estimator will be

$$\begin{aligned}\hat{s}_{\text{MSE}} &= \mathbb{E}\{S|x\} = \int_{-\infty}^{\infty} s p_{S|X}(s|x)ds = \frac{2}{(1-x)^2}\int_0^{1-x} s^2 ds\\ &= \frac{2}{3}(1-x)\end{aligned}$$

(e)

$$\begin{aligned}\mathbb{E}\left\{\left(S-\hat{S}_{\text{MAP}}\right)^2 \mid x\right\} &= \mathbb{E}\left\{(S-(1-x))^2 \mid x\right\} = \int_{-\infty}^{\infty} (s-(1-x))^2 p_{S|X}(s|x)ds\\ &= \frac{2}{(1-x)^2}\int_0^{1-x} s\left(s-(1-x)\right)^2 ds\\ &= \frac{1}{6}(1-x)^2\end{aligned}$$

Since $\hat{S}_{\text{ML}} = \hat{S}_{\text{MAP}}$, its MSE will be identycal,

$$\mathbb{E}\left\{\left(S-\hat{S}_{\text{ML}}\right)^2 \mid x\right\} = \frac{1}{6}(1-x)^2$$

Finally,

$$\begin{aligned}\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}(X)\right)^2 \mid x\right\} &= \mathbb{E}\left\{\left(S-\frac{2}{3}(1-x)\right)^2 \mid x\right\} = \int_0^{1-x} \frac{2s\left(s-\frac{2}{3}(1-x)\right)^2}{(1-x)^2} ds\\ &= \frac{2}{(1-x)^2}\int_0^{1-x} s\left(s-\frac{2}{3}(1-x)\right)^2 ds\\ &= \frac{1}{18}(1-x)^2\end{aligned}$$

ET18

Consider the estimation of a r.v. S from another random variable X. The joint p.d.f. of the two variables is given by:

$$p_{X,S}(x,s) = \begin{cases} 6x, & 0 \le x \le s, \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(b) Obtain the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

(c) Find the probability density function of the previous estimators, $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s})$ and $p_{\hat{S}_{\mathrm{ML}}}(\hat{s})$, and provide a plot of them.

(d) Find the mean and the variance of the error of both estimators.

Solution:

(a) $\hat{S}_{\mathrm{MSE}}(X) = \frac{1}{2}(1+X)$

(b) $\hat{S}_{\mathrm{ML}}(X) = X$

(c) $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s}) = 24(2\hat{s}-1)(1-\hat{s}), \qquad \frac{1}{2} \le \hat{s} \le 1$

$p_{\hat{S}_{\mathrm{ML}}}(\hat{s}) = 6\hat{s}(1-\hat{s}), \qquad 0 \le \hat{s} \le 1$

(d) $\mathbb{E}\{S - \hat{S}_{\mathrm{ML}}\} = \dfrac{1}{4}, \qquad \mathbb{E}\{S - \hat{S}_{\mathrm{MSE}}\} = 0$

$\mathrm{Var}\{S - \hat{S}_{\mathrm{ML}}\} = \dfrac{13}{80}, \qquad Var\{S - \hat{S}_{\mathrm{MSE}}\} = \dfrac{1}{40}$

3 Transformations of random variables

ET19

We want to estimate the value of a positive random variable S using a random observation X, which is related with S via

$$X = R/S$$

R being a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Obtain the likelihood of S, $p_{X|S}(x|s)$.

(b) Find the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The joint p.d.f. of S and X, $p_{S,X}(s,x)$, and the *a posteriori* distribution of S, $p_{S|X}(s|x)$.

(d) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(e) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(f) The bias of estimators $\hat{S}_{\mathrm{MAP}}$ and $\hat{S}_{\mathrm{MSE}}$.

Solution:

(a) $p_{X|S}(x|s) = s\ \exp(-xs), \quad x > 0.$

(b) $\hat{S}_{\mathrm{ML}} = \dfrac{1}{X}.$

(c) $p_{X,S}(x,s) = s\ \exp\left(-s(x+1)\right), \quad x, s > 0;$

$p_{S|X} = (x+1)^2 s\ \exp\left(-s(x+1)\right), \quad s > 0.$

(d) $\hat{S}_{\mathrm{MAP}} = \dfrac{1}{X+1}.$

(e) $\hat{S}_{\mathrm{MSE}} = \dfrac{2}{X+1}.$

(f) $\mathbb{E}\{\hat{S}_{\mathrm{MAP}} - S\} = -\dfrac{1}{2}; \quad \mathbb{E}\{\hat{S}_{\mathrm{MSE}} - S\} = 0.$

ET20

Random variables S and X are related through the stochastic equation:

$$X = S + N$$

where the prior pdf of S is

$$p_S(s) = s\exp(-s) \qquad s \geq 0$$

and where N is an additive noise, independent of S, with distribution

$$p_N(n) = \exp(-n) \qquad n \geq 0$$

Find:

(a) The maximum likelihood estimator of S, $\hat{S}_{\mathrm{ML}}$.

(b) The joint pdf of X and S, $p_{X,S}(x,s)$, and the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) The maximum a posteriori estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(e) The bias of all previous estimators, $\hat{S}_{\mathrm{ML}}$, $\hat{S}_{\mathrm{MAP}}$ and $\hat{S}_{\mathrm{MSE}}$.

(f) Which of the previous estimators has a minimum variance? Justify your answer without calculating the variances of the estimators.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = X$

(b) $p_{X,S}(x,s) = s\exp(-x), \qquad 0 \leq s \leq x$

$p_{S|X}(s|x) = \dfrac{2s}{x^2}, \qquad 0 \leq s \leq x$

(c) $\hat{S}_{\mathrm{MAP}} = X$

(d) $\hat{S}_{\mathrm{MSE}} = \dfrac{2}{3}X$

(e) $\mathbb{E}\left\{\hat{S}_{\mathrm{ML}} - S\right\} = \mathbb{E}\left\{\hat{S}_{\mathrm{MAP}} - S\right\} = 1$

$\mathbb{E}\left\{\hat{S}_{\mathrm{MSE}} - S\right\} = 0$

$\mathrm{var}\left\{\hat{S}_{\mathrm{MSE}}\right\} < \mathrm{var}\left\{\hat{S}_{\mathrm{MAP}}\right\} = \mathrm{var}\left\{\hat{S}_{\mathrm{ML}}\right\}$

ET21

We want to estimate the value of a random variable S using a random observation X, which is related to S via

$$X = S - R$$

where R is a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Find the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

Solution:

(a) $p_{X|S}(x|s) = \exp(x - s), \quad x < s.$
$\hat{S}_{\mathrm{ML}} = X.$

(b) $p_{X,S}(x, s) = \exp\left(-x - 2s\right), \quad s > 0; \ x < s;$
$$\hat{s}_{\mathrm{MAP}} = \begin{cases} 0 & \text{if } \ x < 0 \\ x & \text{if } \ x > 0 \end{cases}$$

(c) $$p_{S|X} = \begin{cases} 2\exp\left(-2s\right), \quad s > 0 & \text{si } \ x < 0 \\ 2\exp\left(2x\right)\exp\left(-2s\right), \quad s > x & \text{if } \ x > 0 \end{cases}$$
$$\hat{s}_{\mathrm{MSE}} = \begin{cases} \frac{1}{2} & \text{if } \ x < 0 \\ \frac{1}{2} + x & \text{if } \ x > 0 \end{cases}$$

4 Bayesian estimation with constraints

4.1 Minimum MSE estimation

ET22

We wish to design a linear minimum mean square error estimator for the estimation of random variable S based on the observation of random variables X_1 and X_2. It is known that:

$$\begin{array}{lll} \mathbb{E}\left\{S\right\} = \dfrac{1}{2} & \mathbb{E}\left\{X_1\right\} = 1 & \mathbb{E}\left\{X_2\right\} = 0 \\ \mathbb{E}\left\{SX_1\right\} = 1 & \mathbb{E}\left\{SX_2\right\} = 2 & \mathbb{E}\left\{X_1X_2\right\} = \dfrac{1}{2} \\ \mathbb{E}\left\{S^2\right\} = 4 & \mathbb{E}\left\{X_1^2\right\} = \dfrac{3}{2} & \mathbb{E}\left\{X_2^2\right\} = 2 \end{array}$$

Obtain the weights of estimator $\hat{S}_{\mathrm{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$, and calculate its mean square error $\mathbb{E}\left\{(S - \hat{S}_{\mathrm{LMSE}})^2\right\}$.

Solution: A video resolution of this problem (in Spanish) can be found in
`http://decisionyestimacion.blogspot.com/2013/05/p1-estimacion.html`

$$w_0 = \frac{1}{2} \quad w_1 = 0 \quad w_2 = 1$$
$$\mathbb{E}\left\{(S - \hat{S}_{\text{LMSE}})^2\right\} = \frac{7}{4}$$

ET23

The joint pdf of two random variables S and X is given by:

$$p_{S,X}(s,x) = \frac{1}{3}(x+s) \quad 0 < x < 2, \;\; 0 < s < 1.$$

(a) Obtain the Linear Minimum Mean Square Error estimator, $\widehat{S}_{\text{LMSE}} = w_0 + w_1 X$.

(b) Calculate the mean square error of the estimator.

Solution:

(a) $\hat{s}_{\text{LMSE}} = \dfrac{14}{23} - \dfrac{1}{23}x.$

(b) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}\right)^2\right\} = \dfrac{11}{138}$

ET24

The joint p.d.f. of random variables X and S is given by

$$p_{X,S}(x,s) = \begin{cases} x+s & 0 \le x \le 1 \text{ and } 0 \le s \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}} = w_0 + w_1 X$.

Solution: $\hat{S}_{\text{LMSE}} = \dfrac{7}{11} - \dfrac{X}{11}$

ET25

The joint distribution of random variables S and X is given by

$$p_{S,X}(s,x) = 15s, \qquad 0 \le x \le 1, \quad x^2 \le s \le x$$

We want to estimate S after observing X.

(a) Compute the minimun mean square error estimate.

(b) Compute the maximum a posteriori estimate.

(c) Compute the minimun mean square error estimate in the form $\hat{S} = wX$

Solution:

(a) $\hat{s}_{\text{MSE}} = \dfrac{2}{3}\dfrac{x - x^4}{1 - x^2}$

(b) $\hat{s}_{\text{MAP}} = x$

(c) $w = \dfrac{7}{8}$

ET26

We wish to estimate a random variable S based on the observation of X. Their joint pdf is given by:

$$p_{S,X}(s,x) = 24xs, \quad 0 \leq s \leq 1-x,\ 0 < x < 1$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(b) The MAP estimator of S given X, $\hat{S}_{\text{MAP}}$.

(c) The MAD estimator of S given X, $\hat{S}_{\text{MAD}}$.

(d) The estimator in the form $\hat{S}_{\text{q}} = wX^2$ that minimizes the mean square error.

Solution:

(a) $\hat{S}_{\text{MSE}} = \dfrac{2}{3}(1-X)$.

(b) Since $p_{S|X}(s|x)$ is strictly increasing with respect to s, $\hat{S}_{\text{MAP}} = 1 - X$.

(c) $\hat{S}_{\text{MAD}} = \dfrac{1-X}{\sqrt{2}}$.

(d) $w = 0.8$

ET27

We want to estimate the value of a random variable S using a random observation X, from which the joint probability distribution is known

$$p_{S,X}(s,x) = 4x, \qquad 0 \leq s \leq x^2, \qquad 0 \leq x \leq 1$$

Obtain:

(a) The maximum likelihood estimator of S given X, $\hat{S}_{\text{ML}}$.

(b) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\text{MAP}}$.

(c) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(d) The linear estimator of S, with minimum mean square error, given X, $\hat{S}_{\text{LMSE}} = w_0 + w_1 x$.

Solution:

(a) The prior distribution is

$$p_S(s) = \int_{-\infty}^{\infty} p_{S,X}(s,x)dx = \int_{\sqrt{s}}^{1} 4x dx = 2(1-s)$$

Therefore, the likelihood function is

$$p_{X|S}(x|s) = \frac{p_{X,S}(x,s)}{p_S(s)} = \frac{x}{1-s} \qquad 0 \leq s \leq x^2, \qquad 0 \leq x \leq 1$$

which is an increasing function of s. Thus, the ML estimate is the maximum value of s in the domain of the likelihood function, that is

$$\hat{S}_{\text{ML}} = X^2$$

(b)

$$\hat{s}_{\text{MAP}} = \operatorname*{argmax}_s p_{S|X}(s|x) = \operatorname*{argmax}_s p_{S,X}(s,x) = \operatorname*{argmax}_s 4x$$

Thus, $\hat{S}_{\text{MAP}}$ is not unique: any value of $s \in [0, X^2]$ is a MAP estimate.

(c) Since the marginal distribution is

$$p_X(x) = \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^{x^2} 4x ds = 4x^3$$

the posterior distribution is

$$p_{S|X}(s|x) = \frac{p_{X,S}(x,s)}{p_X(x)} = \frac{1}{x^2} \qquad 0 \le s \le x^2 \le 1$$

Therefore

$$\hat{s}_{\text{MSE}} = \mathbb{E}\{S|x\} = \int_{-\infty}^{\infty} s p_{S|X}(s|x)ds = \int_0^{x^2} \frac{s}{x^2} ds = \frac{x^2}{2}$$

(d) Since $\hat{S}_{\text{LMSE}} = \mathbf{w}^\intercal \mathbf{Z}$, where $\mathbf{Z} = (1, X)^\intercal$, we have

$$\mathbf{w} = \mathbf{R}_{\mathbf{Z}}^{-1} \mathbf{r}_{S\mathbf{Z}}$$
$$\mathbf{R}_{\mathbf{Z}} = \mathbb{E}\{\mathbf{Z}\mathbf{Z}^\intercal\} = \begin{pmatrix} 1 & \mathbb{E}\{X\} \\ \mathbb{E}\{X\} & \mathbb{E}\{X^2\} \end{pmatrix}$$
$$\mathbf{r}_{S\mathbf{Z}} = \mathbb{E}\{S\mathbf{Z}\} = \begin{pmatrix} \mathbb{E}\{S\} \\ \mathbb{E}\{SX\} \end{pmatrix}$$

Noting that

$$\mathbb{E}\{X\} = \int_0^1 4x^4 dx = \frac{4}{5}$$
$$\mathbb{E}\{X^2\} = \int_0^1 4x^5 dx = \frac{2}{3}$$
$$\mathbb{E}\{S\} = \int_0^1 2s(1-s)ds = \frac{1}{3}$$
$$\mathbb{E}\{SX\} = \int_0^1 \mathbb{E}\{Sx|x\} p_X(x) dx = \int_0^1 \frac{x^3}{2} 4x^3 dx = \frac{2}{7}$$

we get

$$\mathbf{w} = \begin{pmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & \frac{2}{3} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{7} \end{pmatrix} = \begin{pmatrix} -\frac{5}{21} \\ \frac{5}{7} \end{pmatrix}$$

Thus $\hat{S}_{\text{LMSE}} = \frac{5}{7}X - \frac{5}{21}$

ET28

We wish to estimate random variable S from random variable X. They are related as

$$X = S \cdot T$$

where S and T are independent random variables, both uniformly distributed between 0 and 1.

(a) Obtain the mean and the variance of S.
(b) Obtain the maximum likelihood estimator of S as a function of X, $\hat{S}_{\mathrm{ML}}$.
(c) Plot the support of the joint distribution of S and X. Calculate also the joint probability density function of S and X, $p_{S,X}(s,x)$.
(d) Obtain the mean of X and its mean quadratic value. (Hint: It may be convenient not to compute $p_X(x)$ as an intermediate result).
(e) Design the linear minimum mean square error estimator, $\hat{S}_{\mathrm{LMSE}} = w_0^* + w^* X$.
(f) Plot the estimators that have been designed in this problem on top of the same coordinate axis X-S, and discuss which of them incurs in the smallest mean square error.

Solution:

(a) $\mathbb{E}\{S\} = \frac{1}{2}$ y $v_x = \frac{1}{12}$

(b) Since $\hat{s}_{\mathrm{ML}} = \operatorname{argmax}_s p_{X|S}$, we will compute first the likelihood function,

$$
\begin{aligned}
p_{X|S}(x|s) &= \frac{d}{dx} F_{X|S}(x|s) = \frac{d}{dx} P\{X \le x | S = s\} = \frac{d}{dx} P\{ST \le x | S = s\} \\
&= \frac{d}{dx} P\left\{T \le \frac{x}{s}\right\} = \frac{d}{dx} F_T\left(\frac{x}{s}\right) = \frac{1}{s} p_T\left(\frac{x}{s}\right) \\
&= \frac{1}{s}, \qquad 0 \le x \le s \le 1
\end{aligned}
$$

Thus,

$$
\hat{s}_{\mathrm{ML}} = \operatorname*{argmax}_s p_{X|S}(x|s) = \operatorname*{argmax}_{s \in [x,1]} \frac{1}{s} = x \tag{1}
$$

(c)

$$
p_{S,X}(s,x) = p_{X|S}(x|s) p_X(x) = \frac{1}{s}, \qquad 0 \le x \le s \le 1
$$

(d)

$$
\begin{aligned}
\mathbb{E}\{X\} &= \mathbb{E}\{ST\} = \mathbb{E}\{S\}\mathbb{E}\{T\} = \frac{1}{4} \\
\mathbb{E}\{X^2\} &= \mathbb{E}\{S^2 T^2\} = \mathbb{E}\{S^2\}\mathbb{E}\{T^2\} = \frac{1}{9}
\end{aligned}
$$

(e) The LMSE estimate is $\hat{S}_{\mathrm{LMSE}} = w_0^* + w^* X = \mathbf{w}^{*\intercal} \mathbf{Z}$ donde $\mathbf{Z} = (1, X)^\intercal$. Thus,

$$
\mathbf{w}^* = \mathbf{R}_{\mathbf{Z}}^{-1} \mathbf{r}_{s\mathbf{z}} = \begin{pmatrix} 1 & \mathbb{E}\{X\} \\ \mathbb{E}\{X\} & \mathbb{E}\{X^2\} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{E}\{S\} \\ \mathbb{E}\{SX\} \end{pmatrix}
$$

Knowing that $\mathbb{E}\{SX\} = \mathbb{E}\{S^2\}\mathbb{E}\{T\} = \frac{1}{6}$, we get

$$
\mathbf{w}^* = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{9} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 2 \\ 6 \end{pmatrix} \tag{2}
$$

that is $\hat{S}_{\mathrm{LMSE}} = \frac{2}{7} + \frac{6}{7} X$

(f) Since both estimators are linear, necessarily $\hat{S}_{\mathrm{LMSE}}$ has a smaller MSE.

ET29

An electric company owns two wind farms. The total generated power can be modeled as

$$S = 10(2X_1 + X_2),$$

where S is the generated power, and $X_i,\ i = 1, 2$, represents wind speed at each farm. It is further known that the joint distribution of X_1 and X_2 is given by

$$p_{X_1,X_2}(x_1, x_2) = \frac{1}{a\ b} \exp\left(-\frac{x_1}{a}\right), \quad \text{for } 0 < x_1 < \infty \quad \text{and} \quad x_1 < x_2 < x_1 + b \tag{3}$$

with $b = a \ln 2$.

We wish to design a linear minimum mean square error estimator of S based just on observation X_1. In order to do so:

(a) Sketch the support of the joint distribution of X_1 and X_2.

(b) Obtain the marginal distribution of X_1. Calculate the mean and mean square value of such variable.

(c) Find the mean of X_2 and the correlation between X_1 and X_2: $\mathbb{E}\{X_1X_2\}$.

(d) Find the optimum weights of the desired LMSE estimator. $\hat{S}_{\text{LMSE}} = w_0^* + w_1^* X_1$

Solution:

(b) $p_{X_1}(x_1) = \frac{1}{a} \exp\left(-\frac{x_1}{a}\right)$, for $x_1 > 0$

$\mathbb{E}\{X_1\} = a$ y $\mathbb{E}\{X_1^2\} = 2a^2$

(c) $\mathbb{E}\{X_2\} = a + \frac{b}{2}$

$\mathbb{E}\{X_1X_2\} = 2a^2 + \frac{ba}{2}$

ET30

Let X and S be two random variables with joint pdf

$$p_{X,S}(x, s) = \begin{cases} \alpha & ; \quad 0 < x < 1,\ 0 < s < 2(1-x) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

with α a constant.

(a) Plot the support of the pdf, and use it to determine the value of α.

(b) Obtain the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) Find the minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$

(d) Find the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}}$.

Solution:

(a) $\alpha = 1$

(b) $p_{S|X}(s|x) = \frac{1}{2(1-x)}$

(c) $\hat{S}_{\text{MSE}} = 1 - X$

(d) $\hat{S}_{\text{LMSE}} = 1 - X$

ET31

Two random iid variables X and Y follow a uniform pdf between 0 and 1. Two new random variables are generated as $U = \max(X, Y)$ and $V = \min(X, Y)$, i.e., they are defined as the maximum and minimum, respectively, of the original uniform iid variables.

(a) Compute $p_{U|X}(u|x)$ and $p_{V|X}(v|x)$.

(b) Compute $p_U(u)$ and $p_V(v)$.

(c) Compute $\mathbb{E}\{U\}$, $\mathbb{E}\{U^2\}$ and $\mathbb{E}\{V\}$.

(d) Compute the linear estimator of V with minimum MSE given U, $\hat{v}_{\text{LMSE}}(u)$.

Hint: In case you didn't notice, in part (a) you are performing a change of variable. You might find it helpful to draw $\min(x, y)$ and $\max(x, y)$ as functions of y for a fixed value of x. The flat parts of these two functions will produce Dirac deltas in the corresponding results.

Solution:

(a) $p_{U|X}(u|x) = x\delta(u - x) + 1$ for $x \leq u \leq 1$, and 0 otherwise

$p_{V|X}(v|x) = (1 - x)\delta(v - x) + 1$ for $0 \leq v \leq x$, and 0 otherwise

(b) $p_U(u) = 2u$ for $0 \leq u \leq 1$, and 0 otherwise

$p_V(v) = 2(1 - v)$ for $0 \leq v \leq 1$, and 0 otherwise

(c) $\mathbb{E}\{U\} = 2/3$, $\mathbb{E}\{U^2\} = 1/2$ and $\mathbb{E}\{V\} = 1/3$

(d) $\hat{v}_{\text{LMSE}}(u) = u/2$.

ET32

Random variables S and X are characterized by the following joint distribution:

$$p_{S,X}(s, x) = c, \quad 0 < s < 1, \quad s < x < 2s$$

with c a constant.

(a) Plot the support of the p.d.f., and use it to calculate the value of c.

(b) Give the expressions for the marginal p.d.f. of the random variables: $p_S(s)$ and $p_X(x)$.

(c) Find the minimum mean square error estimator of S based on the observation of X, $\hat{S}_{\text{MSE}}(X)$. Plot the estimator on the same plot as the support of $p_{S,X}(s, x)$, and discuss whether it would had been possible to obtain the estimator without analytical derivations.

(d) Calculate the mean square error $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$ incurred by the previous estimator.

(e) Now, find the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}}(X)$. Again, plot the estimator together with the support of $p_{S,X}(s, x)$. Discuss your result.

(f) Obtain the mean square error $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}(X)\right)^2\right\}$ of the linear estimator, and compare it with $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$.

(g) It is perceived (e.g., visualizing several samples of (X, S)) that there exist different statistical behaviors for $0 < X < 1$ and $1 < X < 2$. What would occur if, based on this, different optimal linear estimators where designed for each of the intervals ($\hat{S}_{A,\text{LMSE}}(X)$ y $\hat{S}_{B,\text{LMSE}}(X)$, respectively)? Verify analytically the proposed solution.

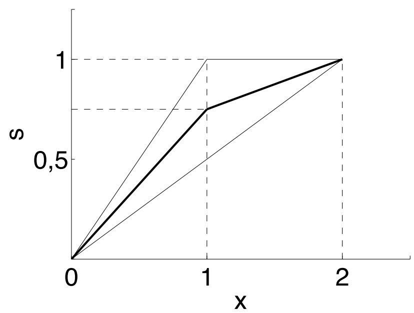
Solution:

(a) Since the area of the support of $p_{S,X}(s,x)$ is $1/2$, $c=2$.

(b) $p_S(s) = 2s,\ 0 < s < 1; \qquad p_X(x) = \begin{cases} x & ,0 < x < 1 \\ 2-x & ,1 < x < 2 \end{cases}$

(c)

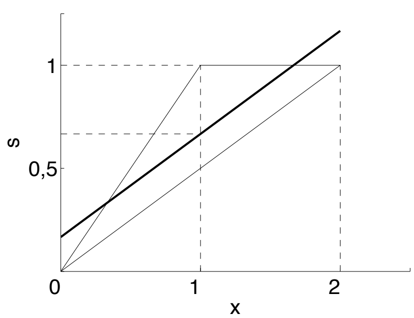
$$\hat{S}_{\text{MSE}}(X) = \begin{cases} \dfrac{3X}{4}, & 0 < X < 1 \\ \\ \dfrac{1}{2}\left(\dfrac{X}{2}+1\right), & 1 < X < 2 \end{cases}$$



Since for every value X we have a uniform *a posteriori distribution* $p_{S|X}(s|x)$, the MSE estimator is given as the average between the minimum and maximum values of S (for each X).

(d) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\} = \dfrac{1}{96}$

(e) $\hat{S}_{\text{LMSE}}(X) = \dfrac{X}{2} + \dfrac{1}{6}$



(f) $\mathbb{E}\left\{\left(S - \hat{S}_{\text{LMSE}}(X)\right)^2\right\} = \dfrac{11}{24}$, which is larger than $\mathbb{E}\left\{\left(S - \hat{S}_{\text{MSE}}(X)\right)^2\right\}$

(g) $\hat{S}_{A,\text{LMSE}}(X) = \dfrac{3X}{4}$ and $\hat{S}_{B,\text{LMSE}}(X) = \dfrac{1}{2}\left(\dfrac{X}{2}+1\right)$. When jointly considered, these estimators compose $\hat{S}_{\text{MSE}}(X)$.
$p_A(s,x)$ and $p_B(s,x)$ are uniform, and now the linear estimators will also be optimal.

The random variables S, T and X are related through the equation

$$X = S \cdot T$$

We know that S and T are statistically independent and given by the probability density functions

$$p_S(s) = \frac{1}{2}s^2 \exp(-s), \qquad s \geq 0$$
$$p_T(t) = 2t, \qquad 0 \leq t \leq 1$$

respectively.

(a) Find the linear minimum mean square error estimate of S given X with the form $\hat{S} = wX$.

(b) Find $p_{X|S}(x|s)$.

(c) Find $p_{S|X}(s|x)$

(d) Find the minimum MSE estimate of S given X, $\hat{S}_{\text{MSE}}$.

Solution:

(a) The coefficient of the linear minimum MSE estimate is given by

$$\begin{aligned} w &= \frac{\mathbb{E}\{SX\}}{\mathbb{E}\{X^2\}} = \frac{\mathbb{E}\{S^2T\}}{\mathbb{E}\{S^2T^2\}} = \frac{\mathbb{E}\{S^2\}\mathbb{E}\{T\}}{\mathbb{E}\{S^2\}\mathbb{E}\{T^2\}} = \frac{\mathbb{E}\{T\}}{\mathbb{E}\{T^2\}} \\ &= \frac{\int_0^1 2t^2 dt}{\int_0^1 2t^3 dt} = \frac{4}{3} \end{aligned}$$

therefore, the estimate is

$$\hat{S} = \frac{4}{3}X$$

(b) Given $S = s$, random variable X is a function of random variable T only and, thus, we can compute the likelihood by the method of transformation of variables, as follows:

$$\begin{aligned} p_{X|S}(x|s) &= \frac{\partial}{\partial x}F_{X|S}(x|s) = \frac{\partial}{\partial x}P\{X \leq x|S = s\} = \frac{\partial}{\partial x}P\left\{T \leq \frac{x}{s} \mid S = s\right\} \\ &= \frac{\partial}{\partial x}F_T\left(\frac{x}{s}\right) = \frac{1}{s}p_T\left(\frac{x}{s}\right) \\ &= \frac{2x}{s^2} \qquad 0 \leq x \leq s \end{aligned}$$

(c) From the Bayes rule,

$$p_{S|X}(s|x) = \frac{p_{X|S}(x|s)p_S(s)}{p_X(x)}$$

The marginal likelihood is

$$\begin{aligned} p_X(x) &= \int_0^\infty p_{X|S}(x|s)p_S(s)ds = x\int_x^\infty \exp(-s)ds \\ &= x\exp(-x) \qquad x \geq 0 \end{aligned}$$

Therefore

$$p_{S|X}(s|x) = \frac{x\exp(-s)}{x\exp(-x)} = \exp(x-s) \qquad 0 \leq x \leq s$$

(d)

$$\hat{S}_{\text{MSE}} = \mathbb{E}\{S|x\} = \int_x^\infty s \exp(x-s) ds = \int_0^\infty (s'+x)\exp(-s')ds'$$
$$= 1 + x$$

ET34

We wish to estimate a random variable S based on the observation of X. They jointly follow a Pareto distribution with a pdf given by:

$$p_{S,X}(s,x) = \frac{2x^2}{s^3}, \qquad 1 \leq s \leq x, \quad 1 \leq x \leq 2$$

(a) Find the MAP estimate of S given X, $\hat{S}_{\text{MAP}}$.

(b) Find the minimum MSE estimate of S given X, $\hat{S}_{\text{MSE}}$.

(c) Consider the estimator is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to MSE cost, i.e., $\hat{S}_{\text{LMSE}} = wX^2$.

(d) Obtain the mean square error of the linear estimator, $\mathbb{E}\{(S-\hat{S}_{\text{LMSE}})^2\}$. Should this error be higher or lower than the associated to the MMSE estimator, $\mathbb{E}\{(S-\hat{S}_{\text{MSE}})^2\}$?

Solution:

(a) We can solve this in different ways. The easiest would be to use the following

$$\underset{s}{\operatorname{argmax}}\, p_{S|X}(s|x) = \underset{s}{\operatorname{argmax}} \frac{p_{S,X}(s,x)}{p_X(x)} = \underset{s}{\operatorname{argmax}}\, p_{S,X}(s,x)$$

So, the joint distribution is monotonically decreasing with respect to S, therefore,

$$\boxed{\hat{S}_{\text{MAP}} = 1}$$

(b) The MSE estimate corresponds to the expected value of S given X

$$\hat{S}_{\text{MSE}} = \mathbb{E}\{S|X\} = \int s p_{S|X}(s|x) ds = \int_1^x s \frac{2}{s^3} \frac{x^2}{x^2-1} ds = 2\frac{x^2}{x^2-1}\int_1^x \frac{1}{s^2} ds$$
$$= 2\frac{x^2}{x^2-1}\left[-s^{-1}\right]_1^x = 2\frac{x^2}{x^2-1}\left(1-\frac{1}{x}\right) = \boxed{\frac{2x}{x+1}}$$

(c) The coefficient of the linear minimum MSE estimate is given by

$$w = \mathbb{E}\{X^2X^2\}^{-1}\mathbb{E}\{SX^2\},$$

where the first term corresponds to

$$\mathbb{E}\{X^2X^2\} = \int_1^2 x^4 p_X(x) dx = \int_1^2 x^4(x^2-1)dx = \left[\frac{x^7}{7} - \frac{x^5}{5}\right]_1^2$$
$$= \frac{2^7-1}{7} - \frac{2^5-1}{5} = \frac{5\cdot 2^7 - 5 - 7\cdot 2^5 + 7}{35} = \frac{418}{35} \approx 11.94$$

and the second term corresponds to

$$\mathbb{E}\{SX^2\} = \int_1^2 \mathbb{E}\{SX^2|x\}p_X(x)dx = \int_1^2 x^2\mathbb{E}\{S|x\}\left(x^2-1\right)dx = \int_1^2 x^2\hat{s}_{\text{MSE}}\left(x^2-1\right)dx$$
$$= 2\int_1^2 x^2\frac{x^2-x}{x^2-1}\left(x^2-1\right)dx = 2\int_1^2\left(x^4-x^3\right)dx = 2\left[\frac{x^5}{5}-\frac{x^4}{4}\right]_1^2$$
$$= 2\frac{2^5-1}{5} - 2\frac{2^4-1}{4} = \frac{4\cdot 2^5 - 4 - 5\cdot 2^4 + 5}{10} = \frac{49}{10} = 4.9.$$

Therefore, if we substitute both values, we get

$$\boxed{\hat{S}_{\text{LMSE}} = wX^2 = \frac{49\cdot 35}{10\cdot 418}X^2 \approx 0.41X^2}$$

(d) Let's start by substituting the estimator we previously obtained:

$$\mathbb{E}\left\{\left(S-\hat{S}_{\text{LMSE}}\right)^2\right\} = \mathbb{E}\left\{\left(S-0.41X^2\right)^2\right\} = \mathbb{E}\{S^2 - 0.82SX^2 + 0.17X^4\}$$
$$= \mathbb{E}\{S^2\} - 0.82\mathbb{E}\{SX^2\} + 0.17\mathbb{E}\{X^4\}$$

Where we have all the expectations with the exception of $\mathbb{E}\{S^2\}$. Now, to calculate this expectation, we need the prior:

$$p_S(s) = \int_s^2 p_{S,X}(s,x)dx = \int_s^2 \frac{2x^2}{s^3}dx = \frac{2}{s^3}\left[\frac{x^3}{3}\right]_s^2 = \frac{2}{s^3}\left[\frac{x^3}{3}\right]_s^2 = \frac{2}{s^3}\left(\frac{8-s^3}{3}\right)$$

This way, we have

$$\mathbb{E}\{S^2\} = \int_1^2 s^2 p_S(s)ds = \int_1^2 \frac{2}{s}\left(\frac{8-s^3}{3}\right)ds = \frac{2}{3}\int_1^2\left(\frac{8}{s}-s^2\right)ds$$
$$= \frac{16\ln(2) - 2\frac{\left(2^3-1^3\right)}{3}}{3} = \frac{16\ln(2)}{3} - \frac{14}{9},$$

Finally we have that

$$\mathbb{E}\left\{\left(S-\hat{S}_{\text{LMSE}}\right)^2\right\} = \mathbb{E}\{S^2\} - 0.82\mathbb{E}\{SX^2\} + 0.17\mathbb{E}\{X^4\}$$
$$= \frac{16\ln(2)}{3} - \frac{14}{9} - 2\frac{\frac{49}{10}}{\frac{418}{35}}\frac{418}{35} + \left(\frac{49\cdot 35}{10\cdot 418}\right)^2\frac{49}{10}$$
$$= \frac{16\ln(2)}{3} - \frac{14}{9} - 9.8 + \frac{49^3\cdot 35^2}{10^3\cdot 418^2} = \frac{16\ln(2)}{3} - 1.56 - 4.9 + 0.82$$
$$= \frac{16\ln(2)}{3} - 5.64 = \boxed{-1.94}$$

The error associated with the MMSE estimator will always be lower than or equal to the LMMSE estimator. In this case, $\mathbb{E}\left\{\left(S-\hat{S}_{\text{MSE}}\right)^2\right\} < \mathbb{E}\left\{\left(S-\hat{S}_{\text{LMSE}}\right)^2\right\}$, since both estimators are not the same.

Random vector $\mathbf{X} = [X_1, X_2, X_3]^T$ follows a p.d.f. with mean $\mathbf{m} = \mathbf{0}$ and covariance matrix

$$\mathbf{V_{XX}} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) Obtain the coefficients (w_0, w_1 and w_2) of the linear minimum mean square error estimator of X_3 given X_1 and X_2,

$$\hat{X}_{3,\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Calculate the mean square error of the estimator $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\text{LMSE}}\right)^2\right\}$.

Solution:

(a) Defining

$$\mathbf{Z} = \begin{bmatrix} 1 \\ X_1 \\ X_2 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

the LMSE estimator will be given by coefficients

$$\begin{aligned}
\mathbf{w} = \mathbf{R}_{\mathbf{Z}}^{-1}\mathbf{r}_{X_3\mathbf{Z}} &= \mathbb{E}\{\mathbf{Z}\mathbf{Z}^\intercal\}^{-1}\mathbb{E}\{X_3\mathbf{Z}\} \\
&= \begin{bmatrix} 1 & \mathbb{E}\{X_1\} & \mathbb{E}\{X_2\} \\ \mathbb{E}\{X_1\} & \mathbb{E}\{X_1^2\} & \mathbb{E}\{X_1X_2\} \\ \mathbb{E}\{X_2\} & \mathbb{E}\{X_1X_2\} & \mathbb{E}\{X_2^2\} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}\{X_3\} \\ \mathbb{E}\{X_1X_3\} \\ \mathbb{E}\{X_2X_3\} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{4}{5} \end{bmatrix}
\end{aligned}$$

therefore

$$\hat{X}_{3,\text{LMSE}} = -\frac{1}{5}X_1 + \frac{4}{5}X_2$$

(b)

$$\begin{aligned}
\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\text{LMSE}}\right)^2\right\} &= \mathbb{E}\left\{\left(X_3 + \frac{1}{5}X_1 - \frac{4}{5}X_2\right)^2\right\} \\
&= \mathbb{E}\left\{X_3^2\right\} + \frac{1}{25}\mathbb{E}\left\{X_1^2\right\} + \frac{16}{25}\mathbb{E}\left\{X_2^2\right\} + \frac{2}{5}\mathbb{E}\left\{X_3X_1\right\} - \frac{8}{5}\mathbb{E}\left\{X_3X_2\right\} - \frac{8}{25}\mathbb{E}\left\{X_1X_2\right\} \\
&= 3 + \frac{3}{25} + \frac{16}{25}\cdot 3 + \frac{2}{5}\cdot 1 - \frac{8}{5}\cdot 2 - \frac{8}{25}\cdot 2 \\
&= \frac{8}{5}
\end{aligned}$$

ET36

The joint p.d.f. of two random variables S and X is given by:

$$p_{S,X}(s,x) = \alpha, \quad -1 < x < 1, \quad 0 \le s \le |x|$$

(a) Obtain the marginal p.d.f. of X, $p_X(x)$, specifying the value of α.

(b) Find the estimators of S based on variable X that minimize the mean square error (MSE), ($\overline{C}_{\mathrm{MSE}} = \mathbb{E}\left\{(S-\hat{S})^2\right\}$) and mean absolute deviation (MAD) ($\overline{C}_{\mathrm{MAD}} = \mathbb{E}\left\{|S-\hat{S}|\right\}$), $\hat{S}_{\mathrm{MSE}}$ and $\hat{S}_{\mathrm{MAD}}$, respectively.

(c) If the estimator is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to cost MSE, i.e., $\hat{S}_{q,\mathrm{MSE}} = w_1 X^2$.

(d) (*Very hard*) If the estimator is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to MAD, i.e., $\hat{S}_{q,\mathrm{MAD}} = w_2 X^2$.

Solution:

(a)

$$p_X(x) = \int_0^{|x|} \alpha dx = \alpha|x|, \quad -1 < x < 1$$

Since the area of the pdf must be unity,

$$\int_{-1}^{1} p_X(x)dx = \int_{-1}^{1} \alpha|x|dx = \alpha = 1$$

therefore $\alpha = 1$ and

$$p_X(x) = |x|, \quad -1 < x < 1$$

(b) The posterior distribution is

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{1}{|x|}, \quad 0 \le s \le |x|$$

which is a uniform distribution. Therefore, both the mean and the median are in the middle point:

$$\hat{S}_{\mathrm{MMSE}} = \hat{S}_{\mathrm{MAD}} = |X|/2$$

(c)

$$\begin{aligned} w_1 &= \frac{\mathbb{E}\{SX^2\}}{\mathbb{E}\{X^4\}} = \frac{\int_{-1}^{1} \mathbb{E}\{SX^2|x\}|x|dx}{\int_{-1}^{1} x^4|x|dx} = \frac{\int_{-1}^{1} x^2\mathbb{E}\{S|x\}|x|dx}{\int_{-1}^{1} x^4|x|dx} \\ &= \frac{2\int_0^1 \frac{1}{2}x^4 dx}{2\int_0^1 x^5 dx} = \frac{3}{5} \end{aligned}$$

Therefore

$$\hat{S}_{q,\mathrm{MSE}}(X) = 3X^2/5$$

(d) The MAD for any estimator in the form w_2X^2 is given by

$$\begin{aligned} \overline{C}_{\mathrm{MAD}} &= \mathbb{E}\left\{|S-\hat{S}|\right\} = \int_{-1}^{1}\int_0^{|x|} |s - w_2x^2||x|dsdx \\ &= 2\int_0^1\int_0^x |s - w_2x^2|dsxdx \end{aligned}$$

For $w_2 \le 0$ we have

$$\begin{aligned} \overline{C}_{\mathrm{MAD}} &= 2\int_0^1\int_0^x (s - w_2x^2)dsxdx = \int_0^1 \left[(x - w_2x^2)^2 - w_2^2x^4\right] xdx \\ &= \int_0^1 \left[x^3 - 2w_2x^4\right] dx = \frac{1}{4} - \frac{2}{5}w_2 \end{aligned}$$

and, for $w_2 > 0$,

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= 2\left(\int_0^1 \int_0^{\min(x,w_2x^2)} (w_2x^2 - s)dsxdx + \int_0^1 \int_{\min(x,w_2x^2)}^{x} (s - w_2x^2)dsxdx\right) \\
&= \int_0^1 \left[-(w_2x^2 - s)^2\right]_0^{\min(x,w_2x^2)} xdx + \int_0^1 \left[(s - w_2x^2)^2\right]_{\min(x,w_2x^2)}^{x} xdx \\
&= \int_0^1 \left[w_2^2x^4 - (w_2x^2 - \min(x, w_2x^2))^2\right] xdx + \int_0^1 \left[(x - w_2x^2)^2 - (\min(x, w_2x^2) - w_2x^2)^2\right] xdx
\end{aligned}
$$

Now, for $0 \le w_2 \le 1$, since $0 \le x \le 1$ we have $\min(x, w_2x^2) = w_2x^2$, so that

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= \int_0^1 w_2^2x^5dx + \int_0^1 (x - w_2x^2)^2xdx \\
&= \frac{1}{6}w_2^2 + \frac{1}{4} - \frac{2}{5}w_2 + \frac{1}{6}w_2^2 = \frac{1}{3}w_2^2 - \frac{2}{5}w_2 + \frac{1}{4}
\end{aligned}
$$

Finally, for $w_2 > 1$, we get

$$
\begin{aligned}
\overline{C}_{\text{MAD}} &= \int_0^{\frac{1}{w_2}} w_2^2x^5dx + \int_{\frac{1}{w_2}}^1 \left[w_2^2x^4 - (w_2x^2 - x)^2\right] xdx + \int_0^{\frac{1}{w_2}} \left[(x - w_2x^2)^2\right] xdx \\
&= \int_0^{\frac{1}{w_2}} w_2^2x^5dx + \int_{\frac{1}{w_2}}^1 \left[2w_2x^3 - x^2\right] dx + \int_0^{\frac{1}{w_2}} \left[x^3 - 2w_2x^4 + w_2^2x^5\right] dx \\
&= \frac{1}{6}\frac{w_2^2}{w_2^6} + \left[\frac{w_2}{2} - \frac{1}{3} - \frac{w_2}{2w_2^4} + \frac{1}{3w_2^3}\right] + \left[\frac{1}{4w_2^4} - \frac{2w_2}{5w_2^5} + \frac{w_2^2}{6w_2^6}\right] \\
&= \frac{1}{6w_2^4} + \frac{w_2}{2} - \frac{1}{3} - \frac{1}{2w_2^3} + \frac{1}{3w_2^3} + \frac{1}{4w_2^4} - \frac{2}{5w_2^4} + \frac{1}{6w_2^4} \\
&= \frac{w_2}{2} - \frac{1}{3} - \frac{1}{6w_2^3} + \frac{11}{60w_2^4}
\end{aligned}
$$

Since, for $w_2 > 0$,

$$
\frac{d\overline{C}_{\text{MAD}}}{dw_2} = \frac{1}{2} + \frac{1}{2w_2^4} - \frac{11}{15w_2^5} = \frac{1}{30w_2^5}\left(15w_2^5 + 15w_2 - 22\right) > 0
$$

the risk grows for $w_2 > 1$ and, thus, the minimum is in $[0, 1]$. Therefore,

$$
w_2^* = \underset{w_2\in[0,1]}{\operatorname{argmin}} \left\{\frac{1}{3}w_2^2 - \frac{2}{5}w_2 + \frac{1}{4}\right\} = \frac{3}{5}
$$

4.2 Other cost functions

ET37

Random variables S and X have a joint probability density function given by

$$
p_{S,X}(s, x) = \begin{cases} 10s, & 0 < s < x^2 \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
$$

Consider the estimation of S based on the observation of X, with the objective to minimize the following cost function:

$$c(S,\hat{S}) = S^2\left(S-\hat{S}\right)^2$$

Find:

(a) The Bayesian estimator, $\hat{S}_C$, for the given cost.

(b) The linear estimator $\hat{S}_L = wX$ which minimizes the risk $\mathbb{E}\left\{c(S,\hat{S})\right\}$.

(c) The risk of both estimators: $\mathbb{E}\left\{c(S,\hat{S}_C)\right\}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\}$.

(d) The unconditional bias of both estimators.

(e) The variance of the error both estimators: $\operatorname{var}\left\{S-\hat{S}_C\right\}$ and $\operatorname{var}\left\{S-\hat{S}_L\right\}$.

Solution:

(a)

$$\begin{aligned}
\hat{s}_c &= \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{c(S,\hat{s})|x\} = \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{S^2\left(S-\hat{s}\right)^2|x\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{S^4 - 2S^3\hat{s} + S^2\hat{s}^2|x\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\left\{\mathbb{E}\{S^4|x\} - 2\mathbb{E}\{S^3|x\}\hat{s} + \mathbb{E}\{S^2|x\}\hat{s}^2\right\} \\
&= \frac{\mathbb{E}\{S^3|x\}}{\mathbb{E}\{S^2|x\}}
\end{aligned}$$

Noting that

$$p_X(x) = \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = 10\int_0^{x^2} s ds = \frac{5x^4}{2}$$

and, thus,

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{4s}{x^4}, \qquad 0 \le s \le x^2, \quad 0 \le x \le 1$$

therefore

$$\hat{s}_c = \frac{\mathbb{E}\{S^3|x\}}{\mathbb{E}\{S^2|x\}} = \frac{\int_{-\infty}^{\infty} s^3 p_{S|X}(s|x)ds}{\int_{-\infty}^{\infty} s^2 p_{S|X}(s|x)ds} = \frac{\frac{4}{x^4}\int_0^{x^2} s^4 ds}{\frac{4}{x^4}\int_0^{x^2} s^3 ds} = \frac{\frac{x^{10}}{5}}{\frac{x^8}{4}} = \frac{4}{5}x^2$$

(b)

$$\begin{aligned}
w &= \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{c(S,wX)\} = \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{S^2\left(S-wX\right)^2\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\,\mathbb{E}\{S^4 - 2S^3Xw + S^2X^2w\} \\
&= \underset{\hat{s}}{\operatorname{argmin}}\left\{\mathbb{E}\{S^4\} - 2\mathbb{E}\{S^3X\}w + \mathbb{E}\{S^2X^2\}w^2\right\} \\
&= \frac{\mathbb{E}\{S^3X\}}{\mathbb{E}\{S^2X^2\}}
\end{aligned}$$

Noting that, for any $m \leq 0$, $n \leq 0$

$$\begin{aligned}
\mathbb{E}\{S^m X^n\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^m x^n p_{S,X}(s,x) ds dx \\
&= 10\int_0^1 x^n \int_0^{x^2} s^{m+1} ds dx = \frac{10}{m+2}\int_0^1 x^{2m+n+4} dx \\
&= \frac{10}{(m+2)(2m+n+5)}
\end{aligned}$$

we can write

$$w = \frac{\mathbb{E}\{S^3 X\}}{\mathbb{E}\{S^2 X^2\}} = \frac{\frac{1}{6}}{\frac{5}{22}} = \frac{11}{15}$$

Therefore, the linear estimation minimizing the overall risk is

$$\hat{S}_L = \frac{11}{15} X$$

(c) For any estimator $\hat{S}$, the overall risk is

$$\mathbb{E}\left\{c(S,\hat{S})\right\} = \mathbb{E}\{S^2\left(S-\hat{S}\right)^2\} = \mathbb{E}\{S^4\} - 2\mathbb{E}\{S^3\hat{S}\} + \mathbb{E}\{S^2\hat{S}^2\} = \frac{5}{39} - 2\mathbb{E}\{S^3\hat{S}\} + \mathbb{E}\{S^2\hat{S}^2\}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left\{c(S,\hat{S}_C)\right\} &= \frac{5}{39} - \frac{8}{5}\mathbb{E}\{S^3X^2\} + \frac{16}{25}\mathbb{E}\{S^2X^4\} \\
&= \frac{5}{39} - \frac{8}{5}\cdot\frac{2}{13} + \frac{16}{25}\cdot\frac{5}{26} = \frac{1}{195} \\
\mathbb{E}\left\{c(S,\hat{S}_L)\right\} &= \frac{5}{39} - \frac{22}{15}\mathbb{E}\{S^3X\} + \frac{11^2}{15^2}\mathbb{E}\{S^2X^2\} \\
&= \frac{5}{39} - \frac{22}{15}\cdot\frac{1}{6} + \frac{11^2}{15^2}\cdot\frac{5}{22} = \frac{7}{1170}
\end{aligned}$$

(d) The bias is

$$\begin{aligned}
B_C &= \mathbb{E}\left\{\hat{S}_C - S\right\} = \frac{4}{5}\mathbb{E}\left\{X^2\right\} - \mathbb{E}\{S\} = \frac{4}{7} - \frac{10}{21} = \frac{2}{21} \\
B_L &= \mathbb{E}\left\{\hat{S}_L - S\right\} = \frac{11}{15}\mathbb{E}\left\{X\right\} - \mathbb{E}\{S\} = \frac{11}{18} - \frac{10}{21} = \frac{17}{126}
\end{aligned}$$

(e) Using the bias-variance decomposition,

$$\begin{aligned}
\mathrm{Var}\left\{S-\hat{S}_C\right\} &= \mathbb{E}\left\{\left(S-\hat{S}_C\right)^2\right\} - B_C^2 \\
&= \mathbb{E}\left\{S^2\right\} + \mathbb{E}\left\{\hat{S}_C^2\right\} - 2\mathbb{E}\left\{S\hat{S}_C\right\} - B_C^2 \\
&= \mathbb{E}\left\{S^2\right\} + \frac{16}{25}\mathbb{E}\left\{X^4\right\} - \frac{8}{5}\mathbb{E}\left\{SX^2\right\} - B_C^2 \\
&= \frac{5}{18} + \frac{16}{25}\cdot\frac{5}{9} - \frac{8}{5}\frac{10}{27} - \frac{4}{441} = \frac{419}{13230} \approx 0.03167
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\mathrm{Var}\left\{S-\hat{S}_L\right\} &= \mathbb{E}\left\{S^2\right\} + \mathbb{E}\left\{\hat{S}_L^2\right\} - 2\mathbb{E}\left\{S\hat{S}_L\right\} - B_L^2 \\
&= \mathbb{E}\left\{S^2\right\} + \frac{121}{225}\mathbb{E}\left\{X^2\right\} - \frac{22}{15}\mathbb{E}\left\{SX\right\} - B_L^2 \\
&= \frac{5}{18} + \frac{121}{225}\cdot\frac{5}{7} - \frac{22}{15}\frac{5}{12} - \frac{17^2}{126^2} = \frac{2587}{79380} \approx 0.03259
\end{aligned}$$

ET38

Consider the family of cost functions given by

$$c(S,\hat{S}) = \frac{1}{N+1}\hat{S}^{N+1} + \frac{1}{N(N+1)}S^{N+1} - \frac{1}{N}S\hat{S}^{N}$$

where N is a non-negative and odd integer, and assume that

$$p_{S,X}(s,x) = \frac{1}{\lambda x}\exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right) \qquad s \geq 0, \qquad x \geq 0, \qquad \lambda > 0$$

(a) Find the Bayesian estimator of S given X for the given costs.

(b) Obtain the minimum risk.

(c) Determine the coefficient w that minimizes the risk of an estimator with analytical shape

$$\hat{S}_L = wX^m$$

m being a positive integer.

Hint: $\int_0^\infty x^N \exp(-x)\, dx = N!$

Solution:

(a) The conditional risk is given by

$$\begin{aligned}
R_{\hat{s}} &= \mathbb{E}\{c(S,\hat{s})|\mathbf{x}\} \\
&= \mathbb{E}\left\{\frac{1}{N+1}\hat{s}^{N+1} + \frac{1}{N(N+1)}S^{N+1} - \frac{1}{N}S\hat{s}^{N} \mid x\right\} \\
&= \frac{1}{N+1}\hat{s}^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}\mathbb{E}\left\{S \mid x\right\}\hat{s}^{N}
\end{aligned}$$

Since this risk is a differentiable function of $\hat{s}$, the minimum must be at a stationary point

$$\begin{aligned}
\frac{\partial R_{\hat{s}}}{\partial \hat{s}} = 0 \quad &\Leftrightarrow \quad \hat{s}^N - \mathbb{E}\left\{S \mid x\right\}\hat{s}^{N-1} = 0 \\
&\Leftrightarrow \quad \hat{s}^{N-1}\left(\hat{s} - \mathbb{E}\left\{S \mid x\right\}\right) = 0
\end{aligned}$$

Thus the minimizer of the conditional risk is

$$\hat{s}^* = \mathbb{E}\{S \mid x\}.$$

To compute the conditional mean, we need the posterior distribution of S. Noting that

$$\begin{aligned}
p_X(x) &= \int_{-\infty}^{\infty} p_{S,X}(s,x)ds = \int_0^\infty \frac{1}{\lambda x}\exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right)ds \\
&= \frac{1}{\lambda x}\exp\left(-\frac{x}{\lambda}\right)\int_0^\infty \exp\left(-\frac{s}{x}\right)ds = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right)
\end{aligned}$$

we have

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)} = \frac{1}{x}\exp\left(-\frac{s}{x}\right)$$

so that

$$\hat{s}^* = \mathbb{E}\{S \mid x\} = \int_{-\infty}^{\infty} s p_{S|X}(s|x)ds = \int_0^\infty \frac{s}{x}\exp\left(-\frac{s}{x}\right)ds = x$$

(b) Since the minimum conditional risk is

$$
\begin{aligned}
R_{\hat{s}} &= \frac{1}{N+1}\left(\hat{s}^*\right)^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}\mathbb{E}\left\{S \mid x\right\}\left(\hat{s}^*\right)^N \\
&= \frac{1}{N+1}x^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1} \mid x\right\} - \frac{1}{N}x^{N+1} \\
&= \frac{1}{N(N+1)}\left(\int_0^\infty \frac{s^{N+1}}{x}\exp\left(-\frac{s}{x}\right)ds - x^{N+1}\right) \\
&= \frac{(N+1)!-1}{N(N+1)}x^{N+1}
\end{aligned}
$$

the minimum risk can be computed as

$$
\begin{aligned}
\mathbb{E}\{c(S,\hat{S}\} &= \int_{-\infty}^{\infty}\mathbb{E}\{c(S,\hat{s})|\mathbf{x}\}p_X(x)dx \\
&= \frac{(N+1)!-1}{\lambda N(N+1)}\int_0^\infty x^{N+1}\exp\left(-\frac{x}{\lambda}\right)dx \\
&= \frac{(N+1)!-1}{N(N+1)}(N+1)!\lambda^{N+1} \\
&= (N+1)!-1)(N-1)!\lambda^{N+1}
\end{aligned}
$$

(c) If $\hat{S} = wX^m$, the risk is given by

$$
\begin{aligned}
R &= \mathbb{E}\{c(S,\hat{s})\} \\
&= \frac{1}{N+1}\mathbb{E}\left\{\hat{S}^{N+1}\right\} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1}\right\} - \frac{1}{N}\mathbb{E}\left\{S\hat{S}^N\right\} \\
&= \frac{1}{N+1}\mathbb{E}\left\{X^{m(N+1)}\right\}w^{N+1} + \frac{1}{N(N+1)}\mathbb{E}\left\{S^{N+1}\right\} - \frac{1}{N}\mathbb{E}\left\{SX^{mN}\right\}w^N
\end{aligned}
$$

By differentiation, this is minimum when

$$
\mathbb{E}\left\{X^{m(N+1)}\right\}w^N - \mathbb{E}\left\{SX^{mN}\right\}w^{N-1} = 0
$$

that is

$$
w = \frac{\mathbb{E}\left\{SX^{mN}\right\}}{\mathbb{E}\left\{X^{m(N+1)}\right\}}
$$

The numerator can be computed as

$$
\begin{aligned}
\mathbb{E}\left\{SX^{mN}\right\} &= \int_0^\infty \mathbb{E}\left\{SX^{mN} \mid x\right\}p_X(x)dx \\
&= \int_0^\infty x^{mN}\mathbb{E}\left\{S \mid x\right\}p_X(x)dx \\
&= \frac{1}{\lambda}\int_0^\infty x^{mN+1}\exp\left(-\frac{x}{\lambda}\right)dx = \lambda^{mN+1}(mN+1)!
\end{aligned}
$$

and the denominator is

$$
\begin{aligned}
\mathbb{E}\left\{X^{m(N+1)}\right\} &= \int_0^\infty x^{m(N+1)}p_X(x)dx \\
&= \frac{1}{\lambda}\int_0^\infty x^{m(N+1)}\exp\left(-\frac{x}{\lambda}\right)dx = \lambda^{m(N+1)}(m(N+1))!
\end{aligned}
$$

Therefore

$$
w = \frac{(Nm+1)!}{(Nm+m)!\lambda^{m-1}}
$$

5 Gaussian distributions

ET39

We have taken two measurements X_1 and X_2 about the value of some unknown variable S. We know that X_1 and X_2 are related to S by means of

$$X_1 = S + T_1$$
$$X_2 = S + T_2$$

where T_1 and T_2 are independent random variables (and independent from S), with zero mean and variances 0.1 y 0.3, respectively

Also, we know that S is a Gaussian random variable with mean 4 y and variance 0.9.

(a) Compute the minimum MSE estimator of S given X_1. Denote it as $\hat{S}_1$.

(b) Compute the MSE of estimator $\hat{S}_1$.

(c) Compute the minimum MSE estimator of S given $Z = \frac{1}{2}(X_1 + X_2)$. Denote it as $\hat{S}_z$.

(d) Compute the probability of $\hat{S}_1$ being higher than $\hat{S}_z$, i.e., $P\{\hat{S}_1 > \hat{S}_z\}$. In case you cannot compute the solution analytically, express it by means of the function

$$F(x) = \int_{-\infty}^{x} \frac{1}{2\pi} \exp\left(-\frac{z^2}{2}\right) dz$$

Solution:

(a) $\hat{S}_1 = 0.9X_1 + 0.4$.

(b) $\text{MSE} = 0.09$

(c) $\hat{S}_z = 0.9Z + 0.4$.

(d) $P\{\hat{S}_1 > \hat{S}_z\} = \frac{1}{2}$.

ET40

Random variables S and X are defined by the equation

$$X = \mathbf{v}^\intercal \mathbf{R}$$
$$S = \mathbf{w}^\intercal \mathbf{R}$$

where $\mathbf{v} = (v_0, v_1)^\intercal$ and $\mathbf{w} = (w_0, w_1)^\intercal$ are known parameter vectors and $\mathbf{R} = (R_0, R_1)^\intercal$ is a random vector. Random variables R_0 and R_1 are independent Gaussian random variables with zero mean and variance σ_R^2.

(a) Compute $\mathbb{E}\{X\}$, $\mathbb{E}\{S\}$, $\mathbb{E}\{X^2\}$, $\mathbb{E}\{S^2\}$ and $\mathbb{E}\{SX\}$.

(b) Compute the minimum MSE estimate of S given X, $\hat{S}_{\text{MSE}}$.

(c) Compute the MSE of estimate $\hat{S}_{\text{MSE}}$.

(d) The minimum MSE may depend on parameter vectors $\mathbf{v}$ and $\mathbf{w}$. Compute the value of $\mathbf{v}$ minimizing the minimum MSE.

(e) Compute the minimum MSE estimate of S given R_0, $\hat{S}'_{\text{MSE}}$.

Solution:

(a)

$$\begin{aligned}
\mathbb{E}\{X\} &= \mathbf{v}^\intercal \mathbb{E}\{\mathbf{R}\} = 0 \\
\mathbb{E}\{S\} &= \mathbf{w}^\intercal \mathbb{E}\{\mathbf{R}\} = 0 \\
\mathbb{E}\{X^2\} &= \mathbf{v}^\intercal \mathbb{E}\{\mathbf{R}\mathbf{R}^\intercal\}\mathbf{v} = \sigma_R^2 \|\mathbf{v}\|^2 \\
\mathbb{E}\{S^2\} &= \mathbf{w}^\intercal \mathbb{E}\{\mathbf{R}\mathbf{R}^\intercal\}\mathbf{w} = \sigma_R^2 \|\mathbf{w}\|^2 \\
\mathbb{E}\{SX\} &= \mathbf{v}^\intercal \mathbb{E}\{\mathbf{R}\mathbf{R}^\intercal\}\mathbf{w} = \sigma_R^2 \mathbf{v}^\intercal \mathbf{w}
\end{aligned}$$

(b) Since

$$\begin{pmatrix} X \\ S \end{pmatrix} = \begin{pmatrix} \mathbf{v}^\intercal \\ \mathbf{w}^\intercal \end{pmatrix} \mathbf{R} \tag{4}$$

we can see that (X, S) is a linear function of two Gaussian random variables (R_0 and R_1). Thus, X and S are jointly Gaussian, and the minimum MSE estimate is given by

$$\hat{S}_{\text{MSE}} = m_S + \frac{V_{SX}}{V_X}(X - m_X) = \frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{v}\|^2} X$$

(c)

$$\begin{aligned}
MSE &= \mathbb{E}\{(S - \hat{S}_{\text{MSE}})^2\} \\
&= \mathbb{E}\left\{\left(S - \frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{w}\|^2} X\right)^2\right\} \\
&= \mathbb{E}\left\{S^2\right\} - 2\frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{v}\|^2} \mathbb{E}\left\{SX\right\} + \left(\frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{v}\|^2}\right)^2 \mathbb{E}\left\{X^2\right\} \\
&= \sigma_R^2 \|\mathbf{w}\|^2 - 2\frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{v}\|^2} \sigma_R^2 \mathbf{v}^\intercal \mathbf{w} + \left(\frac{\mathbf{v}^\intercal \mathbf{w}}{\|\mathbf{v}\|^2}\right)^2 \sigma_R^2 \|\mathbf{v}\|^2 \\
&= \sigma_R^2 \|\mathbf{w}\|^2 \left(1 - \frac{(\mathbf{v}^\intercal \mathbf{w})^2}{\|\mathbf{v}\|^2 \|\mathbf{w}\|^2}\right)
\end{aligned}$$

(d) From the relations $X = \mathbf{v}^\intercal \mathbf{R}$ and $S = \mathbf{w}^\intercal \mathbf{R}$, we can see that, if $\mathbf{v} = \mathbf{w}$, we have $S = X$. In that case the minimum MSE estimate is $\hat{S}_{\text{MSE}} = X$ with $MSE = 0$. Therefore, the minimum value of the minimum MSE is 0.

(e)

$$\begin{aligned}
\hat{S}'_{\text{MSE}} &= \mathbb{E}\{S|R_0\} = \\
&= \mathbb{E}\{w_0 R_0 + w_1 R_1 | R_0\} \\
&= w_0 \mathbb{E}\{R_0|R_0\} + w_1 \mathbb{E}\{R_1|R_0\} \\
&= w_0 R_0
\end{aligned}$$

ET41

We have access to the two following observations for estimating a random variable S:

$$\begin{aligned}
X_1 &= S + N_1 \\
X_2 &= \alpha S + N_2
\end{aligned}$$

where α is a known constant, and S, N_1, and N_2 are independent Gaussian random variables, with zero mean and variances v_s, v_n, and v_n, respectively.

(a) Obtain the minimum mean square error estimator of S given X_1 and X_2, $\hat{S}_1$ and $\hat{S}_2$, respectively.

(b) Calculate the mean square error of each of the estimators from the previous section. Which of the two provides a smaller MSE? Justify your answer for the different values of parameter α.

(c) Obtain the minimum mean square error estimator of S based on the joint observation of variables X_1 and X_2, i.e., as a function of the observation vector $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\hat{S}_{\text{MSE}}$.

Solution:

(a) S y X_2 are jointly Gaussian, with means

$$m_S = 0$$
$$m_{X_2} = \alpha m_S + \mathbb{E}\{N_2\} = 0,$$

variances v_s and

$$\begin{aligned} v_{X_2} &= \mathbb{E}\{(X_2 - m_{X_2})^2\} = \mathbb{E}\{X_2^2\} = \mathbb{E}\left\{(\alpha S + N_2)^2\right\} \\ &= \alpha^2 \mathbb{E}\{S^2\} + 2\alpha\mathbb{E}\{SN_2\} + \mathbb{E}\{N_2^2\} \\ &= \alpha^2 v_s + v_n \end{aligned}$$

respectively, and covariance

$$\begin{aligned} v_{SX_2} &= \mathbb{E}\{(S - m_S)(X_2 - m_{X_2})\} = \mathbb{E}\{SX_2\} = \mathbb{E}\{S(\alpha S + N_2)\} \\ &= \alpha v_s \end{aligned}$$

Thus, the MMSE estimate of S given X_2 is

$$\begin{aligned} \hat{s}_2 &= m_{S|X_2} = m_S + \frac{v_{SX_2}}{v_{X_2}}(x_2 - m_{X_2}) = \frac{v_{SX_2}}{v_{X_2}} x_2 \\ &= \frac{\alpha v_s}{\alpha^2 v_s + v_n} x_2 \end{aligned}$$

On the other hand, given that the relation between X_1 and S is formally equivalent to that of X_2 and S for $\alpha = 1$, it is straighforward to see that the MMSE estimate of S given X_1 es equivalent to take $\alpha = 1$ in the expression above, that is

$$\hat{s}_1 = \frac{v_s}{v_s + v_n} x_2$$

(b) The mean square error $\hat{S}_2$ can be computed as

$$\begin{aligned} \mathbb{E}\left\{\left(S - \hat{S}_2\right)^2\right\} &= \mathbb{E}\left\{\left(S - \frac{\alpha v_S}{\alpha^2 v_s + v_n} X_2\right)^2\right\} \\ &= \mathbb{E}\left\{S^2\right\} - 2\frac{\alpha v_s}{\alpha^2 v_s + v_n}\mathbb{E}\{SX_2\} + \left(\frac{\alpha v_s}{\alpha^2 v_s + v_n}\right)^2 \mathbb{E}\{X_2^2\} \\ &= v_s - 2\frac{\alpha v_s}{\alpha^2 v_s + v_n} v_{SX_2} + \left(\frac{\alpha v_s}{\alpha^2 v_s + v_n}\right)^2 v_{X_2} \\ &= v_s - \frac{\alpha^2 v_s^2}{\alpha^2 v_s + v_n} \\ &= \frac{v_s v_n}{\alpha^2 v_s + v_n} \end{aligned}$$

(alternativelly, it can be computed in a more straightforward manner taking into account that the minimum MSE must be equal to $v_{S|X_2}$).

In a simliar way, the MSE of estimate $\hat{S}_1$ is equivalent to take $\alpha = 1$ in the previous expression,

$$\mathbb{E}\left\{\left(S - \hat{S}_1\right)^2\right\} = \frac{v_s v_n}{v_s + v_n}$$

For $|\alpha| > 1$ we can see that the MSE of $\hat{S}_2$ is smaller thant that of $\hat{S}_1$.

(c) Defining vectors $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ y $\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, we can express the model equation as

$$\mathbf{X} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} S + \mathbf{N}$$

S a $\mathbf{X}$ are jointly Gaussian, with means

$$m_S = 0$$
$$\mathbf{m_X} = \begin{bmatrix} m_{X_1} \\ m_{X_2} \end{bmatrix} = 0$$

variances v_s y

$$\begin{aligned}
\mathbf{V_X} &= \mathbb{E}\{(\mathbf{X} - \mathbf{m_X})(\mathbf{X} - \mathbf{m_X})^\intercal\} = \mathbb{E}\{\mathbf{X}\mathbf{X}^\intercal\} \\
&= \mathbb{E}\left\{\left(\begin{bmatrix} 1 \\ \alpha \end{bmatrix} S + \mathbf{N}\right)\left(\begin{bmatrix} 1 \\ \alpha \end{bmatrix} S + \mathbf{N}\right)^\intercal\right\} \\
&= \begin{bmatrix} 1 \\ \alpha \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \end{bmatrix}^\intercal \mathbb{E}\{S^2\} + \begin{bmatrix} 1 \\ \alpha \end{bmatrix}\mathbb{E}\{S\mathbf{N}^\intercal\} + \mathbb{E}\{S\mathbf{N}\}\begin{bmatrix} 1 \\ \alpha \end{bmatrix}^\intercal + \mathbb{E}\{\mathbf{N}\mathbf{N}^\intercal\} \\
&= v_s \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{bmatrix} + v_n \mathbf{I} \\
&= \begin{bmatrix} v_s + v_n & v_s\alpha \\ v_s\alpha & v_s\alpha^2 + v_n \end{bmatrix},
\end{aligned}$$

respectively, and covariances

$$\mathbf{V}_{S\mathbf{X}} = \begin{bmatrix} v_{SX_1} \\ v_{SX_2} \end{bmatrix}^\intercal = \begin{bmatrix} v_s \\ \alpha v_s \end{bmatrix}^\intercal$$

Thes, the MMSE estimate of S given $\mathbf{X}$ is

$$\begin{aligned}
\mathbf{m}_{S|\mathbf{X}} &= m_S + \mathbf{V}_{S\mathbf{X}}\mathbf{V_X}^{-1}(\mathbf{x} - \mathbf{m_X}) = \mathbf{V}_{S\mathbf{X}}\mathbf{V_X}^{-1}\mathbf{x} \\
&= v_s \begin{bmatrix} 1 \\ \alpha \end{bmatrix}^\intercal \begin{bmatrix} v_s + v_n & v_s\alpha \\ v_s\alpha & v_s\alpha^2 + v_n \end{bmatrix}^{-1} \mathbf{x} \\
&= \frac{v_s}{(1 + \alpha^2)v_s + v_n}(x_1 + \alpha x_2)
\end{aligned}$$

ET42

Let X be a measurement of the instantaneous voltage at a circuit node. In such node exists a signal component with value S, contaminated by Gaussian aditive noise with mean zero and variance v, which is independent of S. A priori, S follows a Gaussian pdf with both the mean and variance equal to 1.

(a) Assuming v is known, obtain the maximum likelihood estimator of S, $\hat{s}_{\text{ML}}(x)$.
(b) Calculate mean square error incurred by estimator $\hat{S}_{\text{ML}}(x)$.
(c) Obtain the likelihood of v given x, i.e. $p_{X|v}(x|v)$.
(d) Calculate the maximum likelihood estimator of v, $\hat{v}_{\text{ML}}(x)$.

Solution:

(a) According to the statement,

$$X = S + R$$

(where R is the noise). R zero-mean Gaussian with variance v, a S is unity-mean Gaussian with variance 1. Therefore, $X|S$ is Gaussian, with mean

$$m_{X|s} = \mathbb{E}\{X|s\} = \mathbb{E}\{S + R|s\} = \mathbb{E}\{S|s\} + \mathbb{E}\{R|s\} = s + \mathbb{E}\{R\} = s.$$

and variance

$$\mathbb{E}\{(X - m_{X|s})^2|s\} = \mathbb{E}\{(X - s)^2|s\} = \mathbb{E}\{R^2|s\} = v$$

Thus

$$p_{X|S}(x|s) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-s)^2}{2v}\right)$$

and, maximizing with respect to s,

$$\hat{s}_{\text{ML}}(x) = x$$

(b) The MSE is

$$\mathbb{E}\{(\hat{S}_{\text{ML}} - S)^2\} = \mathbb{E}\{(X - S)^2\} = \mathbb{E}\{(S + R - S)2\} = \mathbb{E}\{R^2\} = v$$

(c) X is Gaussian with mean

$$\mathbb{E}\{X\} = \mathbb{E}\{S\} + \mathbb{E}\{R\} = 1$$

and variance

$$\mathbb{E}\{(X-1)^2\} = \mathbb{E}\{(S-1)^2\} + \mathbb{E}\{R^2\} = v + 1$$

Therefore,

$$p_X(x) = \frac{1}{\sqrt{2\pi(v+1)}} \exp\left(-\frac{(x-1)^2}{2(v+1)}\right)$$

(v is a deterministic parameter of the distribution, so that $p_X(x) \equiv p_{X|v}(x|v)$

(d)

$$\begin{aligned}
\hat{v}_{\text{ML}}(x) &= \underset{v\geq 0}{\operatorname{argmax}}\, p_{X|v}(x|v) = \underset{v\geq 0}{\operatorname{argmax}} \log(p_{X|v}(x|v)) \\
&= \underset{v\geq 0}{\operatorname{argmax}} \left\{-\frac{1}{2}\log(2\pi(v+1)) - \frac{(x-1)^2}{2(v+1)}\right\} \\
&= \underset{v\geq 0}{\operatorname{argmin}} \left\{\log(v+1) + \frac{(x-1)^2}{v+1}\right\}
\end{aligned}$$

The value of v with zero-derivative of the log-likelihood is

$$v = (x-1)^2 - 1$$

but it could be negative. In that case, the ML estimator is the closest non-negative value, that is, 0. Thus,

$$\hat{v}_{\mathrm{ML}}(x) = \text{máx}[(x-1)^2 - 1, 0]$$

ET43

We wish to estimate a random variable S from the observation of another random variable X given by:

$$X = S + N_1 + N_2$$

where S is Gaussian-distributed, with mean and variance m_s and v_s, respectively, and where N_1 and N_2 are two noise random variables, independent of S, and with joint p.d.f.

$$p_{N_1,N_2}(n_1,n_2) \sim G\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}v_1 & v_{12}\\ v_{12} & v_2\end{bmatrix}\right)$$

Obtain:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(b) The maximum a posteriori estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(c) The bias and variance of both estimators.

Solution:

(a)

$$\hat{S}_{\mathrm{MSE}} = m_S + \frac{v_{SX}}{v_X}(X - m_X)$$

where

$$\begin{aligned}
m_X &= \mathbb{E}\{X\} = \mathbb{E}\{S + N_1 + N_2\} = \mathbb{E}\{S\}\\
v_{SX} &= \mathbb{E}\{(S-m_S)(X-m_X)\} = \mathbb{E}\left\{(S-m_S)\left(S - m_S + N_1 + N_2\right)\right\}\\
&= \mathbb{E}\left\{(S-m_S)^2\right\} + \mathbb{E}\left\{(S-m_S)(N_1+N_2)\right\}\\
&= v_S + \mathbb{E}\left\{(S-m_S)\right\}\mathbb{E}\left\{(N_1+N_2)\right\}\\
&= v_S\\
v_X &= \mathbb{E}\{(X-m_X)^2\} = \mathbb{E}\left\{(S-m_S+N_1+N_2)^2\right\}\\
&= \mathbb{E}\left\{(S-m_S)^2\right\} + 2\mathbb{E}\left\{(S-m_S)(N_1+N_2)\right\} + \mathbb{E}\left\{(N_1+N_2)^2\right\}\\
&= v_S + \mathbb{E}\left\{(S-m_S)\right\}\mathbb{E}\left\{(N_1+N_2)\right\}\\
&= v_S + v_1 + v_2 + 2v_{12}
\end{aligned}$$

Therefore

$$\hat{S}_{\mathrm{MSE}} = m_S + \frac{v_S}{v_S + v_1 + v_2 + 2v_{12}}(X - m_S)$$

(b) $\hat{S}_{\mathrm{MAP}} = \hat{S}_{\mathrm{MSE}}$

(c) $\mathbb{E}\left\{\hat{S}_{\text{MSE}} - S\right\} = \mathbb{E}\left\{\hat{S}_{\text{MAP}} - S\right\} = 0$

$$
\begin{aligned}
\text{Var}\left\{\hat{S}_{\text{MAP}}\right\} = \text{Var}\left\{\hat{S}_{\text{MSE}}\right\} &= \mathbb{E}\left\{\left(\hat{S}_{\text{MSE}} - m_S\right)^2\right\} \\
&= \mathbb{E}\left\{\left(\frac{v_S}{v_S + v_1 + v_2 + 2v_{12}}(X - m_S)\right)^2\right\} \\
&= \frac{v_S^2}{(v_S + v_1 + v_2 + 2v_{12})^2}(v_S + v_1 + v_2 + 2v_{12}) \\
&= \frac{v_S^2}{v_S + v_1 + v_2 + 2v_{12}}
\end{aligned}
$$

ET44

Consider the estimation of a random vector $\mathbf{S}$ from a statistically related observation vector $\mathbf{X}$:

$$\mathbf{X} = \mathbf{H}\mathbf{S} + \mathbf{R}$$

where $\mathbf{H}$ is a known matrix, $\mathbf{R}$ a noise vector with distribution $\mathcal{N}(\mathbf{0}, v_r\mathbf{I})$, and $\mathbf{S}$ the random vector to be estimated, whose distribution is $\mathcal{N}(\mathbf{m_S}, \mathbf{V_S})$. It is also known that $\mathbf{S}$ and $\mathbf{R}$ are independent random vectors:

(a) Find the ML estimator of $\mathbf{S}$, $\hat{\mathbf{S}}_{\text{ML}}$.

(b) Is the ML estimator unbiased? Justify your answer.

(c) As it is known, the MSE estimator of $\mathbf{S}$ is given by:

$$\hat{\mathbf{S}}_{\text{MSE}} = \left(\mathbf{H}^\top\mathbf{H} + v_r\mathbf{V_S}^{-1}\right)^{-1}\mathbf{H}^\top\mathbf{X}$$

Obtain the bias of $\hat{\mathbf{S}}_{\text{MSE}}$ and indicate under which conditions such bias vanishes.

Solution:

(a) $\hat{\mathbf{S}}_{\text{ML}} = \left(\mathbf{H}^\top\mathbf{H}\right)^{-1}\mathbf{H}^\top\mathbf{X}$

(b) The estimator is unbiased.

(c) $\mathbb{E}\left\{\hat{\mathbf{S}}_{\text{MSE}} - \mathbf{S}\right\} = \left(\mathbf{H}^\top\mathbf{H} + v_r\mathbf{V_S}^{-1}\right)^{-1}\mathbf{H}^\top\mathbf{H}\mathbf{m_S} - \mathbf{m_S}$. The bias goes to zero as the noise power decreases towards 0.

ET45

The random variables S, X_1 and X_2 are jointly Gaussian. The parameters of its joint distribution are unknown, but it is known that:

- The marginal distribution of X_1 and X_2 is

$$(X_1, X_2) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\right)$$

- The minimum mean square error (MSE) estimator of S based on X_1 only is:

$$\widehat{S}_{\text{MSE},1} = \frac{1}{2}X_1,$$

- The minimum MSE estimator of S based on X_2 only is:

$$\widehat{S}_{\text{MSE},2} = X_2.$$

We want to estimate S given a new random variable X_3 given by the following linear combination of X_1 and X_2

$$X_3 = X_1 + X_2.$$

(a) Compute the mean value of X_3, $\mathbb{E}\{X_3\}$, its variance, v_{X_3}, and its covariance with S, v_{S,X_3}.

(b) Compute the minimum MSE estimator of S based on X_3 only, $\hat{S}_{\text{MSE},3}$.

(c) Given that $\mathbb{E}\{S^2\} = 1$, compute the MSE of $\hat{S}_{\text{MSE},3}$.

Solution:

(a) $\mathbb{E}\{X_3\} = 0$,$v_{X_3} = 3$, $v_{S,X_3} = 3/2$.

(b) $\hat{S}_{\text{MSE},3} = 1/2 X_3$.

(c) $\mathbb{E}\{(S - \hat{S}_{\text{MSE},3})^2\} = 1/4$.

ET46

An unknown deterministic parameter s, $s > 0$ is measured using two different systems, which provide observations

$$X_i = A_i s + N_i, \quad i = 1, 2$$

where $\{A_i\}, \{N_i\}$, are independent Gaussian random vectors, with means $\mathbb{E}\{A_i\} = 1$, $\mathbb{E}\{N_i\} = 0$, and variances $\{v_{Ai}\}$, $\{v_{Ni}\}$, respectively ($i = 1, 2$).

(a) State the expression that defines the ML estimator of s, $\hat{S}_{\text{ML}}$.

(b) Obtain $\hat{S}_{\text{ML}}$ for the particular case $v_{Ai} = 0, i = 1, 2$.

(c) Obtain $\hat{S}_{\text{ML}}$ for the particular case $v_{Ni} = 0, i = 1, 2$.

Solution:

(a) $\hat{S}_{\text{ML}} = \text{argmin}_s \left\{ \ln\left[(s^2 v_{A1} + v_{N1})(s^2 v_{A2} + v_{N2})\right] + \dfrac{(s - X_1)^2}{s^2 v_{A1} + v_{N1}} + \dfrac{(s - X_2)^2}{s^2 v_{A2} + v_{N2}} \right\}$

(b) $\hat{S}_{\text{ML}} = \dfrac{v_{N2} X_1 + v_{N1} X_2}{v_{N1} + v_{N2}}$

(c) $\hat{S}_{\text{ML}} = \dfrac{1}{4}\sqrt{\left(\dfrac{X_1}{v_{A1}} + \dfrac{X_2}{v_{A2}}\right)^2 + 8\left(\dfrac{X_1^2}{v_{A1}} + \dfrac{X_2^2}{v_{A2}}\right)} - \left(\dfrac{X_1}{v_{A1}} + \dfrac{X_2}{v_{A2}}\right)$

ET47

Two independent Gaussian variables Z_1 and Z_2 have means 2 and 1, respectively. Both variables have unit variance. We wish to estimate the difference $S = Z_1 - Z_2$.

(a) Find $p_S(s)$, the minimum MSE estimator of S, and the mean square error of such estimator if no other information is available.

(b) Consider now we can observe $X = Z_1 + Z_2 = 3$. Find $p_{S|X}(s|x)$, the minimum MSE estimator of S given X, and the mean square error of such estimator. Discuss your result in relation to your answers to the previous subsection.

Solution:

(a) $\hat{S}_{\mathrm{MSE}} = 1$; $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MSE}}\right)^2\right\} = 2$.

(b) X is independent of S, thus the answers are the same as in subsection (a).

6 Estimation from IID samples

ET48

Random variable X is characterized by the following density:

$$p_X(x) = a^2 x \exp\left(-ax\right), \qquad x \geq 0$$

where a is an unknown parameter.

(a) Obtain the expression of the ML estimator of a for a given set $\mathcal{C} = \{x_k, k = 0, \ldots, K-1\}$ of K independent realizations of X.

(b) Given the set of observations $\mathcal{C} = \{0.2, 0.5, 0.8, 1\}$, find the value of $\hat{A}_{\mathrm{ML}}$.

Solution:

(a) $\hat{A}_{\mathrm{ML}} = \dfrac{2K}{\sum_{k=0}^{K-1} x_k}$

(b) $\hat{A}_{\mathrm{ML}} = 3.2$

ET49

We have access to a collection of observations $\{x_k, k = 0, \ldots, K-1\}$ independently drawn from a Rayleigh distribution with parameter b, i.e.,

$$p_{X|b}(x|b) = \frac{2}{b} x \exp\left(-\frac{x^2}{b}\right), \qquad x \geq 0$$

with $b > 0$.

(a) Determine the ML estimator of b, $\hat{b}_{\mathrm{ML}}$.

(b) If the observations are $\{2, 0, 1, 1\}$, what would be the most likelihood value of b?

(c) Find the bias of the estimator.

Solution:

(a) $\hat{b}_{\mathrm{ML}} = \dfrac{1}{K} \sum_{k=0}^{K-1} x_k^2$

(b) $\hat{b}_{\mathrm{ML}} = 1.5$

(c) The estimator is unbiased.

ET50

The random variables S and X are related through the likelihood function

$$p_{X|S}(x|s) = \frac{1}{6} x^2 s^3 \exp(-sx), \qquad s \geq 0, \quad x \geq 0$$

The prior distribution of S is given by

$$p_S(s) = \exp(-s), \qquad s \geq 0$$

(a) Compute the MAP estimate of S given X, $\hat{S}_{\text{MAP}}$

(b) Compute the ML estimate of S given a collection of IID observations $\{x_k, k = 0, \dots K-1\}$ from X.

Solution:

(a)

$$\begin{aligned}\hat{S}_{\text{MAP}} &= \underset{s}{\operatorname{argmax}}\, p_{S|X}(s|x) = \underset{s}{\operatorname{argmax}}\, p_{X|S}(x|s)p_S(s) \\ &= \underset{s\geq 0}{\operatorname{argmax}} \left(x^2 s^3 \exp(-(x+1)s)\right) = \underset{s\geq 0}{\operatorname{argmax}} \left(3\log(s) - (x+1)s\right) \\ &= \frac{3}{x+1}\end{aligned}$$

(b)

$$\begin{aligned}\hat{S}_{\text{ML}} &= \underset{s}{\operatorname{argmax}} \sum_{k=0}^{K-1} \log\left(p_{X|S}(x_k|s)\right) = \underset{s}{\operatorname{argmax}} \left(3K\log(s) - s\sum_{k=0}^{K-1} x_k\right) \\ &= \frac{3K}{\sum_{k=0}^{K-1} x_k}\end{aligned}$$

ET51

An order-N Erlang probability density is characterized by the following expression:

$$p_X(x) = \frac{a^N x^{N-1} \exp(-ax)}{(N-1)!} \qquad x > 0, \quad a > 0$$

Assume that N is known. Considering that the mean of the distribution is given by $m = N/a$, obtain:

(a) The ML estimator of the mean using K independent observations of the variable, $\hat{M}_{\text{ML}}$.

(b) The conditional bias of $\hat{M}_{\text{ML}}$.

(c) Is $\hat{M}_{\text{ML}}$ MSE-consistent?

Solution:

(a) $\hat{M}_{\text{ML}} = \dfrac{1}{K}\sum_{k=1}^{K} X_k$

(b) The estimator is unbiased.

(c) $\text{var}\left\{\hat{M}_{\text{ML}}\right\} = \dfrac{v_x}{K}$; therefore, the estimator is MSE-consistent.

ET52

Random variable X is driven by the likelihood function

$$p_{X|S}(x|s) = \ln\left(\frac{1}{s}\right) s^x, \qquad x \geq 0, \qquad 0 \leq s \leq 1$$

where s is an unknown parameter.

(a) Compute the ML estimate of s given x.

(b) Compute the value of the maximum likelihood for $x = 1$.

(c) Compute the ML estimate of s given K a set $\mathcal{C} = \{x_k, k = 0, \ldots, K-1\}$ of K independent realizations of X.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = \exp\left(-\frac{1}{X}\right)$

(b) $p_{X|S}(1|s_{\mathrm{ML}}) = \frac{1}{e}$

(c) $\hat{S}_{\mathrm{ML}} = \exp\left(-\frac{K}{\sum_{k=0}^{K-1} x_k}\right)$

ET53

The random variable X follows an Inverse Gamma distribution with parameters α and β

$$p_X(x) = \frac{\beta^\alpha}{(\alpha-1)!} x^{-\alpha-1} e^{-\frac{\beta}{x}}, \qquad 0 \leq x,$$

where α is a positive integer.

The following expression gives the posterior distribution that relates S and X

$$p_{S|X}(s|x) = \frac{x^\alpha}{(\alpha-1)!} s^{\alpha-1} e^{-xs}, \qquad 0 \leq s$$

(a) Calculate the MSE estimate of S given the observation, X.

(b) Assuming that α is known, compute the ML estimate of β given a set of i.i.d. observations $\{x_0, x_1, \ldots, x_{K-1}\}$.

Solution:

(a)

$$\begin{aligned} \hat{s}_{\mathrm{MSE}} = \mathbb{E}\{s|x\} &= \int_0^\infty s p_{S|X}(s|x) ds = \int_0^\infty s x^\alpha \frac{1}{(\alpha-1)!} s^{\alpha-1} e^{-xs} ds \\ &= \frac{x^\alpha}{(\alpha-1)!} \int_0^\infty s^\alpha e^{-xs} ds = \frac{x^\alpha}{(\alpha-1)!} \int_0^\infty \frac{u^\alpha}{x^\alpha} e^{-u} \frac{du}{x} = \frac{\alpha!}{(\alpha-1)!} \frac{1}{x} = \frac{\alpha}{x} \end{aligned}$$

(b)

$$\begin{aligned} \beta_{\mathrm{ML}} &= \operatorname*{argmax}_\beta \sum_{k=0}^{K-1} \log(p_{X|\beta}(x_k|\beta)) = \operatorname*{argmax}_\beta \sum_{k=0}^{K-1} \log\left(\frac{\beta^\alpha}{(\alpha-1)!} x_k^{-\alpha-1} e^{-\frac{\beta}{x_k}}\right) \\ &= \operatorname*{argmax}_\beta \sum_{k=0}^{K-1} \left(\alpha \log(\beta) - \log((\alpha-1)!) - (\alpha+1)\log(x_k) - \frac{\beta}{x_k}\right) \\ &= \operatorname*{argmax}_\beta \left(K\alpha \log(\beta) - \sum_{k=0}^{K-1} \frac{\beta}{x_k}\right) \end{aligned}$$

Therefore

$$K\frac{\alpha}{\hat{\beta}_{\mathrm{ML}}} - \sum_{k=0}^{K-1}\frac{1}{x_k} = 0$$

and, thus,

$$\hat{\beta}_{\mathrm{ML}} = \frac{\alpha K}{\sum_{k=0}^{K-1}\frac{1}{x_k}}$$

ET54

Random variables S and X are related by the likelihood function

$$p_{X|S}(x|s) = xs^2\exp\left(-sx\right), \qquad x \geq 0,\ s \geq 0$$

and prior distribution

$$p_S(s) = \exp(-s), \qquad s \geq 0$$

(a) Compute the ML estimate of S given X, $\hat{S}_{\mathrm{ML}}$.
(b) Compute the MAP estimate of S given X, $\hat{S}_{\mathrm{MAP}}$.
(c) Compute the minimum MSE estimate, $\hat{S}_{\mathrm{MSE}}$.
(d) Compute the ML estimate based on the set of i.i.d. observations $\{x^{(0)}, x^{(1)}, \ldots, x^{(K-1)}\}$.

Solution:

(a)

$$\begin{aligned}\hat{s}_{\mathrm{ML}} &= \operatorname*{argmax}_s p_{X|S}(x|s) = \operatorname*{argmax}_s xs^2\exp\left(-sx\right)\\ &= \operatorname*{argmax}_s \left\{2\ln s - sx\right\} = \frac{2}{x}\end{aligned}$$

(b)

$$\begin{aligned}\hat{s}_{\mathrm{MAP}} &= \operatorname*{argmax}_s p_{X|S}(x|s)p_S(s) = \operatorname*{argmax}_s xs^2\exp\left(-s(1+x)\right)\\ &= \operatorname*{argmax}_s \left\{2\ln s - s(1+x)\right\} = \frac{2}{1+x}\end{aligned}$$

(c) Noting that

$$p_X(x) = \int_0^\infty p_{X|S}(x|s)p_S(s)ds = \int_0^\infty xs^2\exp\left(-s(1+x)\right)ds$$

and applying the variable change $u = (1+x)s$, we get

$$p_X(x) = \frac{x}{(1+x)^3}\int_0^\infty u^2\exp\left(-u\right)du = \frac{2x}{(1+x)^3}$$

Therefore

$$p_{S|X}(s|x) = \frac{p_{X|S}(x|s)p_S(s)}{p_X(x)} = \frac{xs^2 \exp\left(-s(1+x)\right)}{\frac{2x}{(1+x)^3}}$$
$$= \frac{1}{2}(1+x)^3 s^2 \exp\left(-s(1+x)\right)$$

and, thus

$$\hat{s}_{\mathrm{MSE}} = \mathbb{E}\{S|x\} = \int_0^\infty \frac{1}{2}(1+x)^3 s^3 \exp\left(-s(1+x)\right) ds$$
$$= \frac{1}{2(1+x)} \int_0^\infty s^3 \exp\left(-u\right) du = \frac{3!}{2(1+x)} = \frac{3}{1+x}$$

(d)

$$\hat{s}_{\mathrm{ML}} = \underset{s}{\operatorname{argmax}} \prod_{k=0}^{K-1} p_{X|S}(x_k|s)$$
$$= \underset{s}{\operatorname{argmax}} \left(\prod_{k=0}^{K-1} x_k\right) s^{2K} \exp\left(-s\sum_{k=0}^{K-1} x_k\right)$$
$$= \underset{s}{\operatorname{argmax}} \left\{2\ln s - s\frac{1}{K}\sum_{k=0}^{K-1} x_k\right\} = \frac{2K}{\sum_{k=0}^{K-1} x_k}$$

ET55

The joint distribution of random variables S and X is known to be:

$$p_{X|s}(x|s) = s\exp(-s\exp(-x)+x), \qquad , \quad x \in \mathbb{R}.$$

(a) Compute the ML estimate of S given x.

(b) Compute the ML estimate of S given K independent and identically distributed observations, $\{X_k, \quad k = 0, \ldots, K-1\}$.

(c) Let $K = 2$ and assume that $x_1 = 0$, $x_2 = -\ln 2$. Compute the value of the maximum likelihood for these observations.

Solution:

(a) $\hat{S}_{\mathrm{ML}} = \exp(X)$.

$$\hat{s}_{\mathrm{ML}} = \underset{s}{\operatorname{argmax}} \log p_{X|S}(x|s)$$
$$= \underset{s}{\operatorname{argmax}} \log\left(s\exp\left(-s\exp(-x)+x\right)\right)$$
$$= \underset{s}{\operatorname{argmax}} \left(\log s - s\exp(-x) + x\right)$$
$$= \exp(x)$$

(b)

$$\begin{aligned}
\hat{s}_{\mathrm{ML}} &= \operatorname*{argmax}_{s} \sum_{k=0}^{K-1} \log p_{X|S}(x_k|s) \\
&= \operatorname*{argmax}_{s} \sum_{k=0}^{K-1} \log \left(s \exp\left(-s \exp\left(-x_k\right) + x^{(k)} \right) \right) \\
&= \operatorname*{argmax}_{s} \left(K \log s - s \sum_{k=0}^{K-1} \exp\left(-x_k\right) + \sum_{k=0}^{K-1} x_k \right) \\
&= \frac{K}{\sum_{k=0}^{K-1} \exp(-x_k)}
\end{aligned}$$

(c) $p_{X_1,X_2|s}(0, -\ln(2)|\hat{s}_{\mathrm{ML}}) = \dfrac{2\exp(-2)}{9}$.

ET56

We have access to a set of observations $\{x_k, k = 0, \ldots, K-1\}$, independently drawn from a Pareto distribution with deterministic parameters α and β, i.e.,

$$p_{X|\alpha,\beta}(x|\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \qquad x \geq \beta$$

with $\alpha > 1$ and $\beta > 0$.

(a) Assuming that β is known, find the ML estimator of α, $\hat{\alpha}_{\mathrm{ML}}$.

(b) For $K = 1$ (i.e., just one observation), find the ML estimator of β, $\hat{\beta}_{\mathrm{ML}}$.

(c) For $K = 1$, compute the bias of $\hat{\beta}_{\mathrm{ML}}$.

Solution:

(a) $\hat{\alpha}_{\mathrm{ML}} = \frac{K}{\sum_{k=0}^{K-1} \ln \frac{x_k}{\beta}}$

(b) $\hat{\beta}_{\mathrm{ML}} = x_0$

(c) $\mathrm{bias}\left\{\hat{\beta}_{\mathrm{ML}}|\beta\right\} = \frac{1}{\alpha-1}\beta$

ET57

Consider a random variable X with p.d.f.

$$p_X(x) = a \exp\left[-a(x-d)\right], \qquad x \geq d$$

where $a > 0$ and d are two parameters.

Find the maximum likelihood estimators of both parameters, $\hat{a}_{\mathrm{ML}}$ and $\hat{d}_{\mathrm{ML}}$, as a function of K samples of X independently drawn, $\{x_k\}_{k=0}^{K-1}$.

Solution: The ML estimates a and d are given by

$$(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) = \operatorname*{argmax}_{a,d} \prod_{k=1}^{K} \left(a \exp\left(-a(x_k - d)\right) u(x_k - d) \right)$$

Note that if $d > x_k$ for some sample x_k, we have $u(x_k - d) = 0$ and, thus, the total likelihood is 0. Therefore, $\hat{d}_{\mathrm{ML}} \leq x_k$, for all k, or, equivalently, $\hat{d}_{\mathrm{ML}} \leq \min_k \{x_k\}$, and we can write

$$(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) = \underset{a,d|d\geq x_{\min}}{\operatorname{argmax}} \prod_{k=1}^{K} \left(a \exp\left(-a(x_k - d)\right)\right)$$

where $x_{\min} = \max_k \{x_k\}$.

Minimizing the logarithm, we can write

$$\begin{aligned}
(\hat{a}_{\mathrm{ML}}, \hat{d}_{\mathrm{ML}}) &= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \sum_{k=1}^{K} \left(\log(a) - a(x_k - d)\right) \\
&= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \left(K \log(a) - a\left(\sum_{k=1}^{K} x_k - Kd\right)\right) \\
&= \underset{a,d|d\leq x_{\min}}{\operatorname{argmax}} \left(K \log(a) + Kad - a\sum_{k=1}^{K} x_k\right)
\end{aligned}$$

Given that the function to maximize increases with d, $\hat{d}_{\mathrm{ML}}$ will be the highest values of d in the feasible interval, that is,

$$\hat{d}_{\mathrm{ML}} = x_{\min} = \min_k \{x_k\}$$

and, thus,

$$\begin{aligned}
\hat{a}_{\mathrm{ML}} &= \underset{a}{\operatorname{argmax}} \left(K \log(a) + Ka \cdot \hat{d}_{\mathrm{ML}} - a\sum_{k=1}^{K} x_k\right) \\
&= \frac{K}{\sum_{k=1}^{K} \left(x_k - \min_k\{x_k\}\right)}
\end{aligned}$$

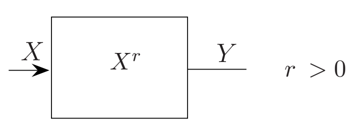
(where the maximum has been computed by differentiation)

ET58

A random variable X with p.d.f.

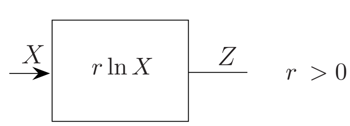
$$p_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

is transformed as indicated in the figure, producing a random observation Y.



(a) Obtain the maximum likelihood estimator of r, $\hat{R}_{\mathrm{ML}}$, based on K independently drawn observations of Y.

(b) Now, consider the following situation



and obtain $\hat{R}_{\text{ML}}$ using K independent observations of random variable Z. Discuss your result.

Solution:

(a) $\hat{R}_{\text{ML}} = -\frac{1}{K}\sum_{k=0}^{K-1} \ln Y_k$. The unknown parameter of the transformation is being identified.

(b) $\hat{R}_{\text{ML}} = -\frac{1}{K}\sum_{k=0}^{K-1} Z_k$. It is coherent with the previous estimator since $Z = \ln Y$, which is a deterministic (and invertible) transformation of Y.

ET59

The random variable S is related to random variable X following

$$p_{S|X}(s|x) = \frac{1}{5}\left(\frac{1}{2x} + \frac{s}{x^2}\right)\exp\left(-\frac{s}{2x}\right), \qquad x > 0, \quad s > 0$$

(a) Obtain the minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(b) Compute the MSE given X associated with the estimator, $\mathbb{E}\left(S - \hat{S}_{\text{MSE}}\right)^2|X$.

Simultaneously, X follows a distribution given bellow with a distribution parameter a:

$$p_{X|a}(x|a) = \frac{1}{16}\frac{x^2}{a^3}\cdot\exp\left(-\frac{x}{2a^2}\right), \qquad x > 0, \quad a > 0$$

(c) Compute the Maximum Likelihood (ML) estimate of parameter a given a set of i.i.d. observations $\mathcal{D} = \{x_0, x_1, \ldots, x_{K-1}\}$.

(d) Consider we have the following observations $\mathcal{D} = \{1, 2, 6\}$. Compute the estimate of a and the associated value of the maximum likelihood for these observations.

Solution:

(a) We can use the posterior distribution to calculate the MMSE estimator

$$\begin{aligned}\hat{s}_{\text{MSE}} = \mathbb{E}\{S|X = x\} &= \int_0^\infty s p_{S|X}(s|x)ds = \int_0^\infty s\frac{1}{5}\left(\frac{1}{2x} + \frac{s}{x^2}\right)\exp\left(-\frac{s}{2x}\right)ds \\ &= \frac{1}{5}\int_0^\infty \frac{s}{2x}\exp\left(-\frac{s}{2x}\right)ds + \frac{1}{5}\int_0^\infty \frac{s^2}{x^2}\exp\left(-\frac{s}{2x}\right)ds\end{aligned}$$

Applying the variable change $u = \frac{s}{2x}$ in both integrals we get

$$\hat{s}_{\text{MSE}} = \frac{1}{5}\int_0^\infty u\exp(-u)\,2xdu + \frac{1}{5}\int_0^\infty 4u^2\exp(-u)\,2xdu = \frac{2x}{5} + \frac{16x}{5} = \frac{18x}{5}$$

(b) The conditional MSE is

$$\begin{aligned}\mathbb{E}\left\{(S - \hat{S}_{\text{MSE}})^2|X = x\right\} &= \mathbb{E}\left\{\left(S - \frac{18X}{5}\right)^2|x\right\} = \mathbb{E}\{S^2|x\} - \frac{36}{5}\mathbb{E}\{SX|x\} + \frac{18^2}{5^2}\mathbb{E}\{X^2|x\} \\ &= \mathbb{E}\{S^2|x\} - \frac{36}{5}x\mathbb{E}\{S|x\} + \frac{18^2}{5^2}x^2\end{aligned}$$

where $\mathbb{E}\{S|x\} = \hat{s}_{\text{MSE}}$ and we can calculate $\mathbb{E}\{S^2|x\}$ equivalently

$$\mathbb{E}\{S^2|x\} = \int_0^\infty s^2 p_{S|X}(s|x)ds = \int_0^\infty s^2 \frac{1}{5}\left(\frac{1}{2x} + \frac{s}{x^2}\right)\exp\left(-\frac{s}{2x}\right)ds$$
$$= \frac{1}{5}\int_0^\infty \frac{s^2}{2x}\exp\left(-\frac{s}{2x}\right)ds + \frac{1}{5}\int_0^\infty \frac{s^3}{x^2}\exp\left(-\frac{s}{2x}\right)ds$$

having that we can apply the same variable change, $u = \frac{s}{2x}$, for both integrals

$$\mathbb{E}\{S^2|x\} = \frac{1}{5}\int_0^\infty 2xu^2\exp(-u)\,2xdu + \frac{1}{5}\int_0^\infty 8xu^3\exp(-u)\,2xdu = \frac{8x^2}{5} + \frac{96x^2}{5} = \frac{104x^2}{5}$$

Therefore, the conditional error is given by

$$\mathbb{E}\left\{(S - \hat{S}_{\text{MSE}})^2|x\right\} = \frac{104}{5}x^2 - \frac{18^2}{25}x^2 = 7.84x^2$$

(c) We can calculate the ML estimate of a by maximizing $p_{X|a}(x|a)$

$$\hat{a}_{ML} = \underset{a}{\operatorname{argmax}} \sum_{k=1}^{K}\log(p_{x|a}(x_k|a)) = \underset{a}{\operatorname{argmax}}\left(\sum_{k=1}^{K}\log\left(\frac{1}{16}\frac{x_k^2}{a^3}\cdot\exp\left(-\frac{x_k}{2a^2}\right)\right)\right)$$
$$= \underset{a}{\operatorname{argmax}} \sum_{k=1}^{K}\left(\log\left(\frac{1}{16}\right) + 2\log(x_k) - 3\log(a) - \frac{x_k}{2a^2}\right)$$
$$= \underset{a}{\operatorname{argmax}}\left(-K\log(16) + 2\sum_{k=1}^{K}\log(x_k) - 3K\log(a) - \sum_{k=1}^{K}\frac{x_k}{2a^2}\right)$$

$$\frac{\partial\sum_{k=1}^{K}\log(p_{x|a}(x_k|a))}{\partial a} = -\frac{3K}{a} + \sum_{k=1}^{K}\frac{x_k}{a^3}$$
$$-\frac{3K}{\hat{a}_{\text{ML}}} + \sum_{k=1}^{K}\frac{x_k}{\hat{a}_{\text{ML}}^3} = 0$$
$$\hat{a}_{\text{ML}} = \sqrt{\frac{1}{3N}\sum_{k=1}^{K}x_k}$$

where the solution $\hat{a}_{\text{ML}} = 0$ is not possible because $a > 0$.

(d) Giving the set of observations, $\mathcal{D} = \{1, 2, 6\}$, we can calculate the estimate of a using the result from previous question:

$$\hat{a}_{\text{ML}} = \sqrt{\frac{1}{3K}\sum_{k=1}^{K}x_k} = \sqrt{\frac{1}{3\cdot 3}(1+2+6)} = 1$$

Substituting both the set of observations and the estimate of the parameter on the likelihood we have:

$$p_{\mathcal{D}|a}(\mathcal{D}|\hat{a}_{\text{ML}}) = \prod_{k=0}^{K-1}\left[\frac{1}{16}\frac{x_k^2}{\hat{a}_{\text{ML}}^3}\cdot\exp\left(-\frac{x_k}{2\hat{a}_{\text{ML}}^2}\right)\right] = \frac{1}{16^3}\left[\prod_{k=0}^{K-1}x_k^2\right]\cdot\exp\left(-\sum_{k=0}^{K-1}\frac{x_k}{2}\right)$$
$$= \frac{1}{16^3}1^2\cdot 2^2\cdot 6^2\cdot\exp\left(-\frac{9}{2}\right) = \frac{9}{256}\cdot\exp\left(-\frac{9}{2}\right)$$

ET60

We have access to a set of K samples, $\{X_k\}_{k=0}^{K-1}$, independently drawn from a random variable X with p.d.f.

$$p_X(x) = \frac{1}{bx^2} \exp\left(-\frac{1}{bx}\right), \qquad x \geq 0$$

with $b > 0$ a constant.

(a) Find the ML estimator of b as a function of the available samples, $\hat{B}_{\mathrm{ML}}$.

(b) Verify that random variable $Y = 1/X$ is characterized by a unilateral exponential p.d.f. $p_Y(y)$, and obtain the value of the mean of such distribution.

(c) Considering your answers to the previous sections, is $\hat{B}_{\mathrm{ML}}$ an unbiased estimator?

Solution:

(a) Maximizing the log-likelihood, we can write (assuming that, according to the probability model, all samples are non-negative)

$$\begin{aligned}
\hat{b}_{\mathrm{ML}} &= \underset{b}{\operatorname{argmax}} \sum_{k=0}^{K-1} \log\left(p_X(x_k)\right) \\
&= \underset{b}{\operatorname{argmax}} \sum_{k=0}^{K-1} \log\left(\frac{1}{bx_k^2} \exp\left(-\frac{1}{bx_k}\right)\right) \\
&= \underset{b}{\operatorname{argmax}} \left(-K\log(b) - 2\sum_{k=0}^{K-1} \log\left(x_k\right) - \frac{1}{b}\sum_{k=0}^{K-1} \frac{1}{x_k}\right) \\
&= \underset{b}{\operatorname{argmax}} \left(-K\log(b) - \frac{1}{b}\sum_{k=0}^{K-1} \frac{1}{x_k}\right) \\
&= \frac{1}{K}\sum_{k=0}^{K-1} \frac{1}{x_k}
\end{aligned}$$

where the last step has been solved for derivation.

(b)

$$\begin{aligned}
p_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{d}{dy}P\{Y \leq y\} = \frac{d}{dy}P\left\{\frac{1}{X} \leq y\right\} \\
&= \frac{d}{dy}P\left\{X \geq \frac{1}{y}\right\} = \frac{d}{dy}\left(1 - F_X\left(\frac{1}{y}\right)\right) = \frac{1}{y^2}p_X\left(\frac{1}{y}\right) \\
&= \frac{1}{b}\exp\left(-\frac{y}{b}\right), \qquad y \geq 0
\end{aligned}$$

(c) Given that

$$\hat{B}_{\mathrm{ML}} = \frac{1}{K}\sum_{k=0}^{K-1} \frac{1}{X_k}$$

the mean of the estimator is

$$\begin{aligned}
\mathbb{E}\{\hat{B}_{\mathrm{ML}}\} &= \frac{1}{K}\sum_{k=0}^{K-1} \mathbb{E}\left\{\frac{1}{X_k}\right\} = \mathbb{E}\left\{\frac{1}{X}\right\} \\
&= \mathbb{E}\left\{Y\right\} = \int_0^{\infty} y\frac{1}{b}\exp\left(-\frac{y}{b}\right) = b
\end{aligned}$$

Thus $\hat{B}_{\mathrm{ML}}$ is unbiased

ET61

A random variable S follows an exponential pdf

$$p_S(s) = \lambda e^{-\lambda s}, \qquad s > 0$$

with $\lambda > 0$. Consider now a discrete random variable X related to S via a Poisson distribution, i.e.,

$$P_{X|S}(x|s) = \frac{s^x e^{-s}}{x!}, \qquad x = 0, 1, 2, \ldots$$

(a) Determine the ML estimator of S given x.

(b) Assume now that we have access to K independent realizations $\{(x_k, s_k), k = 0, \ldots, K-1\}$ of (X, S). Find the ML estimator of λ based on these observations.

(c) Find the MAP estimation of S for $x = 1$.

Solution:

(a) $\hat{S}_{\text{ML}} = X$

(b) $\hat{\lambda}_{\text{ML}} = K \left(\sum_{k=0}^{K-1} s_k \right)^{-1}$

(c) $\hat{S}_{\text{MAP}} = \dfrac{X}{1+\lambda}$

ET62

The manager of an IT company intends to analyze the productivity of his employees by estimating the time S they need to implement a certain computer program. With this goal, at 12:00 a.m. the manager requests the implementation of the program; instead of directly starting the coding task, the employees need to finish first whatever task they are currently carrying out, what requires an additional time N. As a consequence, the total elapsed time between the request of the program implementation and each employee's notification indicating the conclusion of the task is $X = S + N$.

It is known that the time N that the employees need to finish the tasks and start the program implementation can be modeled as the following exponential distribution

$$p_N(n) = a \exp(-an) \quad n > 0,$$

whereas the time for coding the program, S, follows also an exponential distribution, in this case characterized by the expression

$$p_S(s) = b \exp(-b(s-c)) \quad s > c.$$

(a) Before the described process, a simulation has been carried out using a control group, and measuring directly the times N and S required by the members of this group. As a result of the test, four independent observations were obtained for each variable. Concretely, the four observations for N were 6, 10, 12, and 20 minutes, whereas the observations for S were 6, 12, 18, and 36 minutes. Based on these observations, estimate using maximum likelihood the values of constants a, b, and c.

Consider in the following $a = 10$ minutes, $b = 10$ minutes, and $c = 5$ minutes.

(b) For the actual productivity test, the manager receives notifications from three different employees indicating that they have finished the implementation of the program at 12:25, 12:30, and 12:40 a.m. Estimate using maximum likelihood the time that each employee needed for the implementation of the program.

(c) Repeat the estimation of the previous subsection if a minimum mean square error estimator were used.

Solution:

(a) $\hat{a}_{\mathrm{ML}} = \frac{K}{\sum_{k=0}^{K-1} n_k} = \dfrac{1}{12}$ minutes^{-1};
$\hat{c}_{\mathrm{ML}} = \min_k \{s_k\} = 6$ minutes;
$\hat{b}_{\mathrm{ML}} = \frac{K}{\sum_{k=0}^{K-1}(s_k - \hat{c}_{\mathrm{ML}})} = \dfrac{1}{12}$ minutes^{-1}.

(b) $\hat{s}_{\mathrm{ML}} = x$. $\hat{s}_{\mathrm{ML}}(x = 25) = 25$, $\hat{s}_{\mathrm{ML}}(x = 30) = 30$,$\hat{s}_{\mathrm{ML}}(x = 40) = 40$.

(c) $\hat{s}_{\mathrm{MSE}} = \frac{x+5}{2}$, $\hat{s}_{\mathrm{MSE}}(x = 25) = 15$, $\hat{s}_{\mathrm{MSE}}(x = 30) = 17.5$,$\hat{s}_{\mathrm{MSE}}(x = 40) = 22.5$.

ET63

An energetic student gets up early every morning and reaches the bus stop exactly at 8.00 am, the scheduled arrival time of the only bus that can take take him to the university. The bus is usually late and never arrives before its scheduled arrival time. The pdf of the delay of the bus

$$p_{T_{\mathrm{B}}|\Lambda_{\mathrm{B}}}(t_{\mathrm{B}}|\lambda_{\mathrm{B}}) = \lambda_{\mathrm{B}} \exp(-\lambda_{\mathrm{B}} t_{\mathrm{B}}), \quad 0 < t_{\mathrm{B}} \text{ min}$$

where t_{B} are the minutes of delay. The delays are iid for each day.

A second, lazy student makes use of the same bus, but isn't as punctual as the energetic one. The delay in the arrival of the lazy student to the bus stop follows the pdf

$$p_{T_{\mathrm{L}}|\Lambda_{\mathrm{L}}}(t_{\mathrm{L}}|\lambda_{\mathrm{L}}) = \lambda_{\mathrm{L}} \exp(-\lambda_{\mathrm{L}} t_{\mathrm{L}}), \quad 0 < t_{\mathrm{L}} \text{ min}$$

where T_{L} is the delay wrt to the energetic student in reaching the bus stop. These delays are iid for each day. Finally, T_{B} and T_{E} are independent.

(a) Modeling: The first five days of the course the bus arrived to the bus stop 0, 6, 15, 20 and 24 minutes late, whereas the lazy student arrived to the bus stop 15, 10, 12, 5 and 3 minutes late. Estimate λ_{B} and λ_{L} using ML. Specify units.

Consider this ML estimates as the true values for λ_{B} and λ_{L} for the remainder of the exercise.

(b) Compute the expected waiting time for the energetic student at the bus stop.

(c) The sixth day, the lazy student arrives to the bus stop at 8.05 am. He meets there the energetic student and asks him how much longer are both expected to wait (the expected time) until the bus comes. Compute this quantity and contrast it with your answer to b).

Hint: You are being asked to compute $\mathbb{E}[t_{\mathrm{B}} - 5 \text{ min}|t_{\mathrm{B}} > 5 \text{ min}]$.

(d) If the lazy student misses the bus, he won't attend university that day. Assuming this arrival process for the bus and the lazy student is repeated for the rest of the course, compute the *expected* percentage of days that each student attends university.

Solution:

(a) $\lambda_{\mathrm{B}} = \frac{1}{13}$ min^{-1}, $\lambda_{\mathrm{V}} = \frac{1}{9}$ min^{-1} y $\mathbb{E}[t_{\mathrm{B}}] = 13$ min.

(b) $\mathbb{E}[t_{\mathrm{B}} - 5 \text{ min}|t_{\mathrm{B}} > 5 \text{ min}] = 13$ min. Arriving 5 min late doesn't save waiting time. On average, he has to wait as much as the willful student the remaining days. This apparent paradox arises from the fact that there is an additional information in the sixth day: the bus will be more than 5 minutes late.

(c) $p(t_{\mathrm{V}} < t_{\mathrm{B}}) = \frac{\lambda_{\mathrm{v}}}{\lambda_{\mathrm{v}} + \lambda_{\mathrm{B}}}$, in percentage $\frac{100\lambda_{\mathrm{v}}}{\lambda_{\mathrm{v}} + \lambda_{\mathrm{B}}}$%. The willful student always goes.

ET64

A research company is working on a new communication prototype able to modify the noise distribution before being mixed with the signal. In this way, the receiver observes the following signal:

$$X = S + N_{mod},$$

where S is a Gaussian r.v. with zero mean and variance v_s, and N_{mod} is the new noise whose value is given by the following expression:

$$N_{\text{mod}} = \lambda N_1 + (1-\lambda)N_2$$

N_1 and N_2 being two independent Gaussian random variables, independent from S, with zero mean and variance v; whereas λ is a control parameter which takes values from 0 to 1.

(a) Find the distribution of N_{mod} as a function of λ.

(b) Obtain the minimum MSE estimator of S given X for the new system and compute the mean square error of this estimator as a function of λ.

(c) Compute the value of λ, λ_{opt}, which provides the minimum mean square error.

(d) Obtain, in terms of reduction of the mean square error, the advantage provided by this model when λ is set to λ_{opt} with respect to a model using $\lambda = 0$ or $\lambda = 1$. Compute this error reduction when $v_s = v = 1$.

The technician in charge of designing the system generating N_{mod} has gone on vacation, leaving the system online but without specifying what value of λ is being used. The new intern is tasked with obtaining a set of independent observations of the noise at the system's output $\{N_{\text{mod}}^{(k)}\}_{k=0}^{K-1}$, and computing a maximum likelihood estimation of the value of λ, $\hat{\lambda}_{ML}$.

(e) Considering $v = 3$ and the observation set $\{-1, 0, 2, 1\}$, obtain the value of $\hat{\lambda}_{ML}$.

Solution:

(a) N_{mod} is Gaussian with zero mean and variance $v_{MOD} = \lambda^2 v + (1-\lambda)^2 v$.

(b) $\hat{s}_{\text{MSE}} = \frac{v_s}{v_s + v_{MOD}} x \qquad MSE_{MOD} = \frac{(\lambda^2 + (1-\lambda)^2) v v_s}{v_s + (\lambda^2 + (1-\lambda)^2) v}$

(c) $\lambda_{\text{opt}} = 0.5$

(d) $\Delta MSE = \frac{1}{6}$

(e) $\hat{\lambda}_{ML} = 0.5$

ET65

Consider a communication system in which the transmitted symbol S is sent through two different channels, generating observations X_1 and X_2:

$$X_1 = \alpha(S + N_1)$$

$$X_2 = 2\alpha(S + N_2)$$

with α a constant associated to the attenuation of the channels.

It is known that the noise values N_1 and N_2, which are independent of each other, can be modeled as Gaussian random variables with mean zero and variance 0.5. Furthermore, the transmitted symbol, S, is also a Gaussian random variable with mean zero and variance one, and is independent of N_1 and N_2. At the receiver, just the sum of both outputs can be observed, i.e.,

$$X = X_1 + X_2$$

(a) Obtain the minimum mean square error estimate of S given X, $\hat{S}_{\text{MSE}}$.

(b) Compute the mean square error of estimator $\hat{S}_{\text{MSE}}$.

(c) Obtain the maximum likelihood estimate of S given X, $\hat{S}_{\text{ML}}$.

(d) Find the marginal distribution, $p_X(x)$.

(e) The marginal distribution depends on the unknown attenuation parameter, α. Find the ML estimate of α, $\hat{\alpha}_{\text{ML}}$, based on K values, $\{x_k\}_{k=0}^{K-1}$, independently drawn from the marginal distribution.

Solution:

(a) $\hat{S}_{\text{ML}} = \dfrac{x}{3\alpha}$

(b) $\hat{S}_{\text{MSE}} = \dfrac{3\alpha}{11.5\alpha^2}x$

(c) $\hat{\alpha}_{\text{ML}} = \sqrt{\frac{2}{23}\sum_k x_k^2}$

(d) $\hat{\alpha}_{\text{ML}} = 2$

ET66

The current intensity through the branch of a circuit can be characterized by equation

$$i(t) = A\cos(\omega_o t)\exp^{-\alpha t} + B\sin(\omega_o t)\exp^{-\alpha t} + C$$

ω_o and α being two known constants. For determining the other model parameters, A, B, and C, we have access to a set of meaures of $i(t)$ for K different time instants, i.e., a set of pairs $\{t_k, i(t_k)\}_{k=1}^{K}$.

Provide expressions that can be used to calculate the values of parameters A, B, and C, which minimize the quadratic error of the model averaged over the set of available samples.

Solution: Defining $x_{1,k} = \cos(\omega_o t_k)\exp^{-\alpha t_k}$ and $x_{2,k} = \sin(\omega_o t_k)\exp^{-\alpha t_k}$, and denoting by $\mathbf{X}_e$ the extended observations matrix $\{x_{1,k}, x_{2,k}\}_{k=0}^{K-1}$ and by $\mathbf{i}$ a vector with components $\{i(t_k)\}_{k=1}^{K}$, we have that:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \left(\mathbf{X}_e^T \mathbf{X}_e\right)^{-1} \mathbf{X}_e^T \mathbf{i}$$

7 Bias and Variance

ET67

The variance v of a zero-mean r.v. X is estimated from K independent observations of the variable, $\{X_k\}_{k=0}^{K-1}$, using the following estimator:

$$\hat{V} = \frac{1}{K}\left[\sum_{k=0}^{K-1} X_k\right]^2$$

(a) Find the bias of such an estimator.

(b) For $K = 2$, and assuming that $\mathbb{E}\left\{X^4\right\} = \alpha$ is known, find the variance of the estimator.

Solution:

(a) The estimator is unbiased.

(b) $\operatorname{var}\left\{\hat{V}\right\} = \frac{1}{2}\left[\alpha + v^2\right]$

ET68

Consider the estimation of a r.v. X from a set of K independent observations $\{X_k\}_{k=0}^{K-1}$, and also consider the following estimator:

$$\hat{M} = \frac{a}{K}\sum_{k=0}^{K-1} X_k,$$

with a a constant to determine.

(a) Obtain the bias and the variance of the estimator as a function of a.

(b) Which value of a minimizes the variance? Is there any value of a that produces an unbiased estimator?

(c) Obtain the mean square error of the estimator, and find the value of a that minimizes it.

Solution:

(a) $\mathbb{E}\left\{\hat{M} - m\right\} = (a-1)m$;

$\operatorname{var}\left\{\hat{M}\right\} = \frac{a^2 v}{K}$.

(b) The variance is minimized by $a = 0$, whereas the bias is null for $a = 1$.

(c) $\mathbb{E}\left\{(\hat{M} - m)^2\right\} = (a-1)^2 m^2 + \frac{a^2 v}{K}$. Thus, the value of a minimizing it is $a^* = \frac{m^2}{m^2 + v/K}$.

ET69

A random variable X follows a unilateral exponential distribution with parameter $a > 0$:

$$p_X(x) = \frac{1}{a}\exp\left(-\frac{x}{a}\right) \quad x > 0$$

As it is known, the mean and variance of X are given by a and a^2, respectively.

(a) Obtain the maximum likelihood estimator of a, $\hat{A}_{\text{ML}}$, based on a set of K independent observations of random variable X, $\{X_k\}_{k=0}^{K-1}$.

(b) Consider now a new estimator based on the previous one, and characterized by expression:

$$\hat{A} = c \cdot \hat{A}_{\text{ML}},$$

where $0 \le c \le 1$ is shrinkage constant that allows re-scaling the ML estimator. Find the bias squared, the variance, and the Mean Square Error (MSE) o the new estimator, and represent them all together in the same plot as a function of c.

(c) Find the value of c which minimizes the MSE, c^*, and discuss its evolution as the number of available observations increases. Calculate the MSE of the estimator associated to c^*.

(d) Determine the range of values of c for which the MSE of $\hat{A}$ is smaller than the MSE of the ML estimator, and explain how such range changes as $K \longrightarrow \infty$. Discuss your result.

Solution:

(a) $\hat{A}_{\mathrm{ML}} = \dfrac{1}{K}\sum_{k=1}^{K} X_k$

(b) $\hat{A} = \dfrac{c}{K}\sum_{k=1}^{K} X_k$

$$\mathbb{E}\left\{\hat{A} - a\right\}^2 = (c-1)^2 a^2,$$

$$\mathrm{var}\left\{\hat{A}\right\} = \frac{c^2 a^2}{K},$$

$$\mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = (c-1)^2 a^2 + \frac{c^2 a^2}{K}$$

(c) $c^* = \dfrac{K}{K+1}$, $c^* \to 1\ (K \to \infty)$,

$$\mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = \frac{a^2}{K+1}\ (c = c^*)$$

(d) The range of values is: $c \in \left[\dfrac{K-1}{K+1}, 1\right]$, which narrows as K increases.

ET70

We want to estimate the mean m of a random variable X with variance v, using a set of $K+1$ independent observations of such random variable, $\{X_k\}_{k=0}^{K}$. Consider the following estimators:

$$\hat{M}_1 = \frac{a}{K}\sum_{k=0}^{K-1} X_k \qquad \hat{M}_2 = X_K \qquad \hat{M}_3 = \lambda \hat{M}_1 + (1-\lambda)\hat{M}_2$$

a being a positive constant, strictly less than one, and λ another constant to be set.

(a) Compare the bias and variance of estimators $\hat{M}_1$ and $\hat{M}_2$.

(b) Find the bias, the variance, and mean square error (MSE) of estimator $\hat{M}_3$, simplifying your result for $K \longrightarrow \infty$.

Solution:

(a)
$$\mathbb{E}\left\{\hat{M}_1 - m\right\} = (a-1)\,m \qquad \mathbb{E}\left\{\hat{M}_2 - m\right\} = 0$$
$$\mathrm{Var}\left\{\hat{M}_1\right\} = \frac{a^2 v}{K} \qquad \mathrm{Var}\left\{\hat{M}_2\right\} = v$$

(b) $\mathbb{E}\left\{\hat{M}_3 - m\right\} = \lambda(a-1)\,m \qquad \mathrm{Var}\left\{\hat{M}_3\right\} = \dfrac{\lambda^2 a^2 v}{K} + v(1-\lambda)^2$

$$\mathbb{E}\left\{\left(\hat{M}_3 - m\right)^2\right\} = \frac{\lambda^2 a^2 v}{K} + v(1-\lambda)^2 + \lambda^2 (a-1)^2 m^2$$

If $K \to \infty$, $\mathrm{Var}\left\{\hat{M}_3\right\} = v(1-\lambda)^2$ and

$$\mathbb{E}\left\{\left(\hat{M}_3 - m\right)^2\right\} = v(1-\lambda)^2 + \lambda^2 (a-1)^2 m^2.$$