Stochastic Processes: Problems

7 de marzo de 2022

1. Markov Processes

Exercise MP1 (Markov Chain)

Let $X_k, k \geq 0$ be a Markov chain with state space $\mathcal{Z} = \{0, 1\}$ and transition probabilities $P\{X_k = 1 | X_{k-1} = 0\} = 0.8$ and $P\{X_k = 0 | X_{k-1} = 1\} = 0.4$.

(a) Draw the corresponding transition graph

(b) Assume that the initial state is $X_0 = 1$. Compute $P\{X_2 = 1\}$

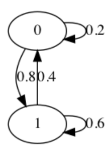
(c) Compute the stationary distribution.

Solution:

(a) The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

and, thus, the transition graph is



(b)

$$P\{X_2 = 1\} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{P}^\intercal \mathbf{P}^\intercal \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.68$$

(c) The stationary distribution is the solution of

$$\mathbf{P}^\intercal \boldsymbol{\pi} = \boldsymbol{\pi}$$

with $(1, 1)\boldsymbol{\pi} = 1$, that is:

$$0.2\pi_0 + 0.4\pi_1 = \pi_0$$

and taking $\pi_1 = 1 - \pi_0$, we get

$$0.4(1 - \pi_0) = 0.8\pi_0$$

so that $\pi_0 = \frac{1}{3}$ and

$$(\pi_0, \pi_1) = \left(\frac{1}{3}, \frac{2}{3}\right)$$

Exercise MP2 (Markov Process)

A video game consists of N consecutive levels, $0, 1, ..., N-1$. The player starts at level 0. If a player passes level i, she enters level $i+1$, if not, she returns back to level 0. It is known that all phases have the same difficulty, so, if a player is at level i, she reaches level $i+1$ with probability q, and returns back to 0 with probability $1-q$.

When the player reaches stage $N-1$, she gets a medal, returns to level 0 and the game continues.

Let X_k be the stochastic process that represents the sequence of levels during a game, such that $X_k = i$ means that the player was at level i at time k. The game begins at $X_0 = 0$.

(a) Formulate the problem as a stationary Markov process, and draw the transition graph for $N = 6$.

(b) Assuming $N \geq 2$, compute $P\{X_2 = 1\}$.

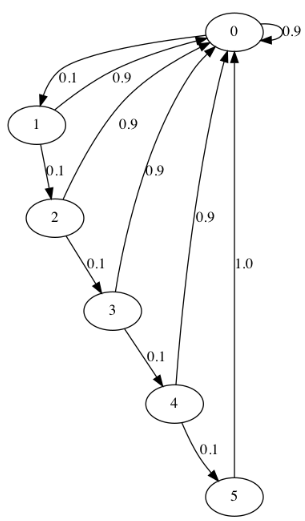
(c) Assuming $N \geq 2$, determine the probability of obtaining a medal exactly at time k, that is $P\{X_k = N-1\}$, for $k = 0, 1, \ldots, N$.

(d) For $N = 2$, determine the stationary distribution.

(e) For $N = \infty$, determine the stationary distribution

Solution:

(a) The transition graph is shown in the figure for $q = 0.1$.



(b)

$$P\{X_2 = 1\} = qP\{X_1 = 0\} = q(1-q)P\{X_0 = 0\} = q(1-q)$$

(c) The probability of obtaining a medal at time k is $Q_k = P\{X_k = N-1\}$. Since at least $N-1$ steps are required to reach level $N-1$, we have

$$Q_k = 0, \qquad \text{for } k = 0, \ldots, N-2$$

Reaching level $N-1$ at time $k = N-1$ is possible only if the player does not fail at any time, so that

$$Q_{N-1} = q^{N-1}$$

Reaching level $N-1$ at time $k=N$ is possible only if the player fails at time 0 only, so that.

$$Q_N = (1-q)q^{N-1}$$

(d) Since

$$\pi_1 = q\pi_0$$

and $\pi_0 + \pi_1 = 1$, we have

$$\pi_1 = \frac{q}{1+q}$$

(e) For $i > 0$, we have

$$\pi_i = q\pi_{i-1} = q^2\pi_{i-2} = \ldots = q^i\pi_0$$

and, for $i = 0$,

$$\pi_0 = \sum_{i=0}^{\infty}(1-q)\pi_i = (1-q)\sum_{i=0}^{\infty}\pi_i = 1-q$$

so that

$$\pi_i = (1-q)q^i$$

2. Stationary Processes

Exercise SP1 (Autocorrelation, Power Spectrum)

Let X_n be i.i.d. stochastic process with probability density function

$$p_X(x) = x\exp(-x), \qquad x \geq 0$$

Assume that X_n is the input to a linear system with impulse response

$$h_n = \delta[n] + 0.5\delta[n-1]$$

the system output Y_n, is corrupted by a Gaussian i.i.d noise E_n (independent of X_n) with mean zero and unit variance, to produce the final process

$$Z_n = Y_n + E_n$$

(a) Compute the autocorrelation functions $r_X[n]$ and $r_E[n]$ of X_n and E_n, respectively.

(b) Compute the autocorrelation function of Y_n, $r_Y[n]$

(c) Compute the autocorrelation function of Z_n, $r_Z[n]$

(d) Compute the power spectrum of Z_n, $S_Z(\omega)$..

Solution:

(a) Since X_n is zero-mean i.i.d, we have

$$
\begin{aligned}
r_X[n] = \mathbb{E}\{X_k X_{k+n}\} &= \left[\begin{array}{ll} \mathbb{E}\{X_k^2\}, & n = 0 \\ \mathbb{E}\{X_k\}\mathbb{E}\{X_{k+n}\}, & n \neq 0 \end{array} \right. \\
&= \mathbb{E}\{X_k^2\}\delta[n] + \mathbb{E}\{X_k\}^2(1 - \delta[n]) \\
&= \left(\mathbb{E}\{X_k^2\} - \mathbb{E}\{X_k\}^2\right)\delta[n] + \mathbb{E}\{X_k\}^2
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\mathbb{E}\{X_k\} &= \int_0^\infty x^2 \exp(-x)dx = 2 \\
\mathbb{E}\{X_k^2\} &= \int_0^\infty x^3 \exp(-x)dx = 6
\end{aligned}
$$

we get

$$
r_X[n] = 2\delta[n] + 4
$$

Since E_n is zero-mean i.i.d, we have

$$
r_E[n] = \sigma_E^2\delta[n] = \delta[n]
$$

(b) Since

$$
Y_n = X_n * h_n
$$

we have

$$
\begin{aligned}
r_Y[n] &= r_X[n] * h_n * h_{-n} \\
&= (2\delta[n] + 4) * (\delta[n] + 0.5\delta[n-1]) * (\delta[n] + 0.5\delta[n+1]) \\
&= (2\delta[n] + 4) * (1.25\delta[n] + 0.5\delta[n-1] + 0.5\delta[n+1]) \\
&= 2.5\delta[n] + \delta[n-1] + \delta[n+1] + 9
\end{aligned}
$$

(c) Since Y_n and E_n are independent and E_n is zero-mean

$$
r_Z[n] = r_Y[n] + r_E[n] = 3.5\delta[n] + \delta[n-1] + \delta[n+1] + 9
$$

(d) Computing the Fourier transform of the autocorrelation function, we get

$$
S_Z(\omega) = 3.5 + 2\cos(\omega) + 18\pi\delta(\omega), \qquad \omega \in [-\pi, \pi]
$$

Exercise SP2 (Autocorrelation, Power Spectrum)

The stochastic process X_n is given by the pair of equations

$$
\begin{aligned}
X_n &= S_n \cdot R_n \\
S_n &= W_n - \frac{1}{2}W_{n-1}
\end{aligned}
$$

where W_n is a Gaussian i.i.d. process with mean 0 and variance v, and R_n is stationary processes with autocorrelation function

$$
r_R[n] = 2^{-|n|}
$$

Processes W_n and R_n are mutually independent.

(a) Compute the autocorrelation function of W_n, $r_W[n]$

(b) Compute and draw the autocorrelation function of S_n, $r_S[n]$

(c) Compute and draw the autocorrelation function of X_n, $r_X[n]$

(d) Compute the power spectrum of the process $Z_n = \sum_{k=0}^{\infty} 2^{-k} X_{n-k}$

Solution:

(a) Since W_n is zero-mean i.i.d, $r_W[n] = v\delta[n]$

(b)

$$\begin{aligned} r_S[n] &= \mathbb{E}\{S_k \cdot S_{k+n}\} \\ &= \mathbb{E}\{(W_k - \frac{1}{2}W_{k-1})(W_{k+n} - \frac{1}{2}W_{k+n-1})\} \\ &= \frac{5}{4}v\delta[n] - \frac{1}{2}v\delta[n-1] - v\frac{1}{2}\delta[n+1] \end{aligned}$$

(c)

$$\begin{aligned} r_X[n] &= \mathbb{E}\{X_k \cdot X_{k+n}\} \\ &= \mathbb{E}\{S_k \cdot S_{k+n}\}\mathbb{E}\{R_k \cdot R_{k+n}\} \\ &= r_S[n]r_R[n] \\ &= 2^{-|n|} \cdot \left(\frac{5}{4}v\delta[n] - \frac{1}{2}v\delta[n-1] - v\frac{1}{2}\delta[n+1]\right) \\ &= \frac{5}{4}v\delta[n] - \frac{1}{4}v\delta[n-1] - \frac{1}{4}v\delta[n+1]) \end{aligned}$$

(d)

$$Z_n = X_n * h[n]$$

where $h_n = 2^{-n}u[n]$. Therefore

$$\begin{aligned} S_Z(\omega) &= S_X(\omega)|H(\omega)|^2 \\ &= \frac{v}{2} \cdot \left(\frac{5}{2} - \cos(\omega)\right) \left|\frac{1}{1 - \frac{1}{2}\exp(-j\omega)}\right|^2 \\ &= \frac{v}{2} \cdot \frac{\frac{5}{2} - \cos(\omega)}{\left|1 - \frac{1}{2}\exp(-j\omega)\right|^2} \\ &= v \cdot \frac{5 - 2\cos(\omega)}{5 - 4\cos(\omega)} \end{aligned}$$

Exercise SP3 (Autocorrelation, Power Spectrum)

The stochastic process X_n is the sum of two i.i.d. stochastic processes S_n and R_n,

$$X_n = S_n + R_n$$

with probability density functions

$$p_S(s) = s\exp(-s), \qquad s \geq 0$$

and

$$p_R(r) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right)$$

respectively. The processes S_n and R_n are mutually independent. Assume that X_n is the input to a linear and time-invariant system with impulse response

$$h_n = \frac{1}{2^n} u[n]$$

with output Y_n

(a) Compute the autocorrelation function, $r_X[n]$, and the power spectrum, $S_X(\omega)$, of X_n

(b) Compute the autocorrelation function, $r_Y[n]$, and the power spectrum, $S_Y(\omega)$, of Y_n

Solution:

(a)

$$\begin{aligned}
r_X[n] &= \mathbb{E}\{X_k X_{k+n}\} \\
&= \mathbb{E}\{(S_k + R_k)(S_{k+n} + R_{k+n})\} \\
&= r_S[n] + r_R[n] + \mathbb{E}\{S_k\}\mathbb{E}\{R_{k+n}\} + \mathbb{E}\{R_k\}\mathbb{E}\{S_{k+n}\} \\
&= r_S[n] + r_R[n]
\end{aligned}$$

Since S_n and R_n are i.i.d.

$$\begin{aligned}
r_S[n] &= \mathbb{E}\{S_k S_{k+n}\} \\
&= \mathbb{E}\{S_k^2\}\delta[n] + \mathbb{E}\{S_k\}\mathbb{E}\{S_{k+n}\}(1 - \delta[n]) \\
&= \int_0^\infty s^3 \exp(-s) ds \cdot \delta[n] + \left(\int_0^\infty s^2 \exp(-s) ds\right)^2 (1 - \delta[n]) \\
&= 3!\delta[n] + 4(1 - \delta[n]) \\
&= 4 + 2\delta[n]
\end{aligned}$$

and

$$\begin{aligned}
r_R[n] &= \mathbb{E}\{R_k R_{k+n}\} \\
&= \mathbb{E}\{R_k^2\}\delta[n] + \mathbb{E}\{R_k\}\mathbb{E}\{R_{k+n}\}(1 - \delta[n]) = \delta[n]
\end{aligned}$$

Therefore

$$r_X[n] = 4 + 3\delta[n]$$

and the power spectrum is

$$S_X(w) = 3 + 8\pi\delta(\omega), \qquad -\pi \le \omega \le \pi$$

(b)

$$\begin{aligned}
r_Y[n] &= r_X[n] * h[n] * h[-n] \\
&= \frac{4}{3}\left(\frac{1}{2}\right)^{-|n|} * (4 + 3\delta[n]) = 16 + 4\left(\frac{1}{2}\right)^{-|n|} \\
S_Y(w) &= S_X(w)|H(\omega)|^2 \\
&= \frac{3 + 8\pi\delta(\omega)}{\left|1 - \frac{1}{2}e^{-j\omega}\right|^2}
\end{aligned}$$

(This expression can be further simplified to

$$S_Y(w) = \frac{3 + 8\pi\delta(\omega)}{\frac{5}{4} - \cos(\omega)} = \frac{3}{\frac{5}{4} - \cos(\omega)} + 32\pi\delta(\omega)$$

).