Stochastic Processes: Problems

31 de enero de 2025

1. Markov Processes

MP1

Let $X_k, k \geq 0$ be a Markov chain with state space $\mathcal{Z} = \{0, 1\}$ and transition probabilities $P\{X_k = 1 | X_{k-1} = 0\} = 0.8$ and $P\{X_k = 0 | X_{k-1} = 1\} = 0.4$.

(a) Draw the corresponding transition graph

(b) Assume that the initial state is $X_0 = 1$. Compute $P\{X_2 = 1\}$

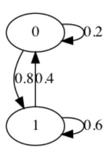
(c) Compute the stationary distribution.

Solution:

(a) The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

and, thus, the transition graph is



(b)

$$P\{X_2 = 1\} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{P}^\intercal \mathbf{P}^\intercal \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.68$$

(c) The stationary distribution is the solution of

$$\mathbf{P}^\intercal \boldsymbol{\pi} = \boldsymbol{\pi}$$

with $(1,1)\boldsymbol{\pi} = 1$, that is:

$$0.2\pi_0 + 0.4\pi_1 = \pi_0$$

and taking $\pi_1 = 1 - \pi_0$, we get

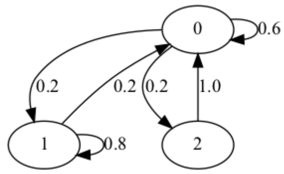
$$0.4(1 - \pi_0) = 0.8\pi_0$$

so that $\pi_0 = \frac{1}{3}$ and

$$(\pi_0, \pi_1) = \left(\frac{1}{3}, \frac{2}{3}\right)$$

MP2

Let $X_k, k \geq 0$ be a Markov chain with state space $\mathcal{Z} = \{0, 1, 2\}$ and the transition graph shown in the figure.



(a) Show the transition matrix

(b) Compute $P\{X_{22} = 1 | X_{20} = 2\}$

(c) Compute the stationary distribution.

Solution:

(a)

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.8 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b)

$$P\{X_{22} = 1 | X_{20} = 2\} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathbf{P}^\intercal \mathbf{P}^\intercal \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.2$$

(c) This is the solution of

$$\begin{pmatrix} \mathbf{P}^\intercal - \mathbf{I} \\ \mathbb{1}^\intercal \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

that is

$$\begin{pmatrix} -0.4 & 0.2 & 0.2 \\ 0.2 & -0.2 & 0 \\ 0.2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which, removing the first row (which is redundant), reduces to

$$\begin{pmatrix} 0.2 & -0.2 & 0 \\ 0.2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

whose solution is

$$\boldsymbol{\pi} = \frac{1}{11} \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix}$$

MP3

Let $X_k, k \geq 0$ be a Markov chain with state space $\mathcal{Z} = \{0, 1, 2, 3\}$. The initial state is 0, that is, $P\{X_0 = 0\} = 1$. If, at time n, the process is in state $i < 3$, at time $n+1$ it will remain in the same state with probability $1-p$ or jump to state $i+1$ with probability p.

$$
\begin{aligned}
P\{X_{n+1} = i+1 \mid X_n = i\} &= p \\
P\{X_{n+1} = i \mid X_n = i\} &= 1-p
\end{aligned}
$$

If the process is in state 3, it will remain in the same state with probability 1.

(a) Find the transition matrix

(b) Show the transition graph

(c) Compute $P\{X_2 = 1\}$

(d) Compute $P\{X_n = 0\}$, for any $n > 0$

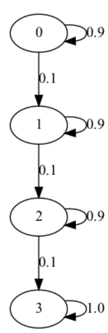
(e) Find a stationary distribution for this process

Solution:

(a)

$$
\mathbf{P} = \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 1-p & p & 0 \\ 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

(b) .



(c)

$$
P\{X_2 = 1\} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \mathbf{q}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \mathbf{P}^\intercal \mathbf{P}^\intercal \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} p & 1-p & 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p \\ p \\ 0 \\ 0 \end{pmatrix} = 2p(1-p)
$$

(d) $X_n = 0$ if and only if there are no transitions from 0 to 1, that is $X_0 = X_1 = \ldots = X_n = 1$. This will happen with probability

$$
P\{X_n = 0\} = (1-p)^n
$$

(e) Eventually, the process will reach state 3 and remain there. Thus $\boldsymbol{\pi} = (0, 0, 0, 1)$ must be a stationary distribution. Indeed,

$$\mathbf{P}^\intercal \boldsymbol{\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{\pi}$$

MP4

A video game consists of N consecutive levels, $0, 1, ..., N-1$. The player starts at level 0. If a player passes level i, she enters level $i+1$, if not, she returns back to level 0. It is known that all phases have the same difficulty, so, if a player is at level i, she reaches level $i+1$ with probability q, and returns back to 0 with probability $1-q$.

When the player reaches stage $N-1$, she gets a medal, returns to level 0 and the game continues.

Let X_k be the stochastic process that represents the sequence of levels during a game, such that $X_k = i$ means that the player was at level i at time k. The game begins at $X_0 = 0$.

(a) Formulate the problem as a stationary Markov process, and draw the transition graph for $N = 6$.

(b) Assuming $N \geq 2$, compute $P\{X_2 = 1\}$.

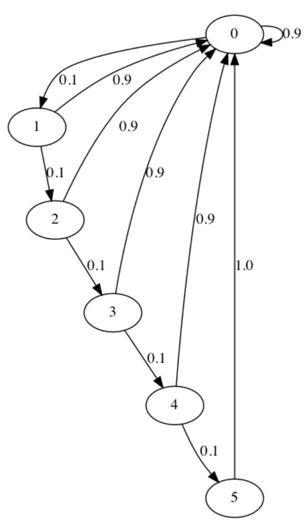
(c) Assuming $N \geq 2$, determine the probability of obtaining a medal exactly at time k, that is $P\{X_k = N-1\}$, for $k = 0, 1, \ldots, N$.

(d) For $N = 2$, determine the stationary distribution.

(e) For $N = \infty$, determine the stationary distribution

Solution:

(a) The transition graph is shown in the figure for $q = 0.1$.



(b)

$$P\{X_2 = 1\} = qP\{X_1 = 0\} = q(1-q)P\{X_0 = 0\} = q(1-q)$$

(c) The probability of obtaining a medal at time k is $Q_k = P\{X_k = N-1\}$. Since at least $N-1$ steps are required to reach level $N-1$, we have

$$Q_k = 0, \qquad \text{for } k = 0, \ldots, N-2$$

Reaching level $N-1$ at time $k = N-1$ is possible only if the player does not fail at any time, so that

$$Q_{N-1} = q^{N-1}$$

Reaching level $N-1$ at time $k = N$ is possible only if the player fails at time 0 only, so that.

$$Q_N = (1-q)q^{N-1}$$

(d) Since

$$\pi_1 = q\pi_0$$

and $\pi_0 + \pi_1 = 1$, we have

$$\pi_1 = \frac{q}{1+q}$$

(e) For $i > 0$, we have

$$\pi_i = q\pi_{i-1} = q^2\pi_{i-2} = \ldots = q^i\pi_0$$

and, for $i = 0$,

$$\pi_0 = \sum_{i=0}^{\infty}(1-q)\pi_i = (1-q)\sum_{i=0}^{\infty}\pi_i = 1-q$$

so that

$$\pi_i = (1-q)q^i$$

MP5

A game has three players, named G_0, G_1 and G_2 and consists of a sequence of rounds. At each round, only one of the players enters the game.

The result of each round can be win or lose. If the active player wins a round, she can play the next one. If she loses, she must pass the turn to one of the other players, which is chosen at random with equal probabilities.

Based on the game skills of the players, it is known that the winning probabilities are 0.4 (for G_0), 0.6 (G_1) and 0.8 (G_2).

Let X_k be the one-sided stochastic process that represents the sequence of active players during a game, such that $X_k = i$ means that the active player at time k is G_i.

The game always starts with player 0, that is, $X_0 = 0$.

(a) Formulate the problem as a stationary Markov chain: compute the transition matrix and draw the transition graph.

(b) Compute the probability that the first players in the sequence are $0, 1, 2, 1$ (i.e., $P\{X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 1\}$)).

(c) Compute $P\{X_2 = 1\}$.

(d) Compute the stationary distribution

(e) Players G_0 and G_1 are unsatisfied because G_2, with better skills, plays most rounds. They decide to make the game fairer, in the sense that, in the long term, everyone plays with the same frequency. To do so, they proceed as follows:

1. If G_0 is the active player and loses, she passes the turn to player G_1 with probability q and to G_2 with probability $1-q$.
2. If G_1 is the active player and loses, she passes the turn to player G_0 with probability r and to G_2 with probability $1-r$.

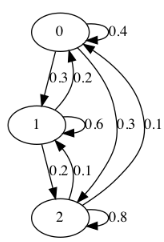
Determine if G_0 and G_1 will succeed in making a fair game, that is, if there exist values q and r so that the stationary distribution is uniform, i.e., $\boldsymbol{\pi} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ If so, compute them.

Solution:

(a) The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

The transition graph is shown in the figure.



(b) Since X_k is a Markov chain

$$\begin{aligned} &P\{X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 1\} \\ &\qquad = P\{X_0 = 0\}P\{X_1 = 1|X_0 = 0\}P\{X_2 = 2|X_1 = 1\}P\{X_3 = 1|X_2 = 2\} \\ &\qquad = 1 \cdot 0.3 \cdot 0.2 \cdot 0.1 = 0.006 \end{aligned}$$

(c) At time $k = 2$, we have

$$P\{X_2 = 1\} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathbf{P}^\intercal \mathbf{P}^\intercal \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.6 & 0.1 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = 0.33$$

(d) The stationary distribution is the solution of

$$\begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.2 & 0.8 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ 1 \end{pmatrix}$$

Since the first 3 equations are linearly dependent, we can remove the first one,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.2 & -0.2 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\end{pmatrix} = \frac{1}{11}\begin{pmatrix}2\\ 3\\ 6\end{pmatrix}$$

(e) If the stationary distribution is uniform, we have

$$\begin{pmatrix}0.4 & 0.4r & 0.1\\ 0.6q & 0.6 & 0.1\\ 0.6(1-q) & 0.4(1-r) & 0.8\end{pmatrix}\begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

Using the first to equations, we have

$$0.4 + 0.4r + 0.1 = 1$$
$$0.6q + 0.6 + 0.1 = 1$$

The unique solution is $q = 0.5$, $r = 1.25$. Since r is not a probability value, there is no way to make the game fair.

MP6

Consider the markov chain X_k given by state space $\mathcal{S} = \{0, 1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix}1-p & p & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1-p & p\\ 0 & p & 0 & 1-p\end{pmatrix}$$

It is known that the initial state is $X_0 = 0$ with probability 1.

(5 %) (a) Draw the transition graph.

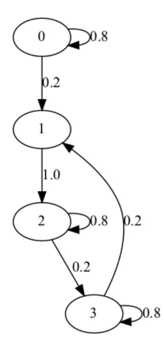
(7 %) (b) Compute the state probability distribution at time $k = 2$.

(8 %) (c) Find the stationary distribution.

(5 %) (d) Find the expected value of the time to reach state 1 for the first time.

Solution:

(a) The figure shows the transition graph for $p = 0.2$:



(b) Since $X_0 = 0$, the initial state vector is $q_0 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^\top$. Thus, the probability distribution at $k = 2$ is

$$\begin{aligned}
\mathbf{q}_2 &= \mathbf{P}^\top \mathbf{P}^\top \mathbf{q}_0 \\
&= \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}^\top \cdot \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}^\top \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}^\top \cdot \begin{pmatrix} 1-p \\ p \\ 0 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} (1-p)^2 \\ p(1-p) \\ p \\ 0 \end{pmatrix}}
\end{aligned}$$

(c) The stationary distribution $\boldsymbol{\pi}$ is the solution of

$$\begin{pmatrix} -p & 0 & 0 & 0 \\ p & -1 & 0 & p \\ 0 & 1 & -p & 0 \\ 0 & 0 & p & -p \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The first line shows that $\pi_0 = 0$. Taking it into account and removing the 4th equation, which is redundant, we get the simplified equation

$$\begin{pmatrix} -1 & 0 & p \\ 1 & -p & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

whose solution is $\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \frac{1}{2+p} \begin{pmatrix} p \\ 1 \\ 1 \end{pmatrix}$. Therefore

$$\boxed{\boldsymbol{\pi} = \frac{1}{2+p} \begin{pmatrix} 0 \\ p \\ 1 \\ 1 \end{pmatrix}}$$

(d) Let T be the time to reach state 1 for the first time. Since, at time $k = 1$, X_k can be 0 or 1 only, we can write

$$\begin{aligned}
\mathbb{E}\{T\} &= \mathbb{E}\{T \mid X_1 = 1\} P\{X_1 = 1\} + \mathbb{E}\{T \mid X_1 = 0\} P\{X_1 = 0\} \\
&= 1 \cdot p + (1 + \mathbb{E}\{T\})(1-p)
\end{aligned}$$

therefore

$$\boxed{\mathbb{E}\{T\} = \frac{1}{p}}$$

[An alternative path to the solution is

$$\begin{aligned}
\mathbb{E}\{T\} &= \sum_{k=0}^{\infty} k P\{T = k\} = p \sum_{k=0}^{\infty} k(1-p)^{k-1} \\
&= \frac{p}{1-p} \lim_{n\to\infty} \frac{1 - p - (n+1)(1-p)^{n+1} + n(1-p)^{n+2}}{p^2} = \frac{1}{p}
\end{aligned}$$

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2. Stationary Processes

SP1

Let X_n be i.i.d. stochastic process with probability density function

$$p_X(x) = x\exp(-x), \qquad x \geq 0$$

Assume that X_n is the input to a linear system with impulse response

$$h_n = \delta[n] + 0.5\delta[n-1]$$

the system output Y_n, is corrupted by a Gaussian i.i.d noise E_n (independent of X_n) with mean zero and unit variance, to produce the final process

$$Z_n = Y_n + E_n$$

(a) Compute the autocorrelation functions $r_X[n]$ and $r_E[n]$ of X_n and E_n, respectively.

(b) Compute the autocorrelation function of Y_n, $r_Y[n]$

(c) Compute the autocorrelation function of Z_n, $r_Z[n]$

(d) Compute the power spectrum of Z_n, $S_Z(\omega)$..

Solution:

(a) Since X_n is zero-mean i.i.d, we have

$$\begin{aligned} r_X[n] = \mathbb{E}\{X_k X_{k+n}\} &= \left[\begin{array}{ll} \mathbb{E}\{X_k^2\}, & n = 0 \\ \mathbb{E}\{X_k\}\mathbb{E}\{X_{k+n}\}, & n \neq 0 \end{array} \right. \\ &= \mathbb{E}\{X_k^2\}\delta[n] + \mathbb{E}\{X_k\}^2(1-\delta[n]) \\ &= \left(\mathbb{E}\{X_k^2\} - \mathbb{E}\{X_k\}^2\right)\delta[n] + \mathbb{E}\{X_k\}^2 \end{aligned}$$

Noting that

$$\begin{aligned} \mathbb{E}\{X_k\} &= \int_0^\infty x^2\exp(-x)dx = 2 \\ \mathbb{E}\{X_k^2\} &= \int_0^\infty x^3\exp(-x)dx = 6 \end{aligned}$$

we get

$$r_X[n] = 2\delta[n] + 4$$

Since E_n is zero-mean i.i.d, we have

$$r_E[n] = \sigma_E^2\delta[n] = \delta[n]$$

(b) Since

$$Y_n = X_n * h_n$$

we have

$$\begin{aligned} r_Y[n] &= r_X[n] * h_n * h_{-n} \\ &= (2\delta[n] + 4) * (\delta[n] + 0.5\delta[n-1]) * (\delta[n] + 0.5\delta[n+1]) \\ &= (2\delta[n] + 4) * (1.25\delta[n] + 0.5\delta[n-1] + 0.5\delta[n+1]) \\ &= 2.5\delta[n] + \delta[n-1] + \delta[n+1] + 9 \end{aligned}$$

(c) Since Y_n and E_n are independent and E_n is zero-mean

$$r_Z[n] = r_Y[n] + r_E[n] = 3.5\delta[n] + \delta[n-1] + \delta[n+1] + 9$$

(d) Computing the Fourier transform of the autocorrelation function, we get

$$S_Z(\omega) = 3.5 + 2\cos(\omega) + 18\pi\delta(\omega), \qquad \omega \in [-\pi, \pi]$$

SP2

The stochastic process X_n is given by,

$$X_n = \exp(-S_n)$$

where S_n is an i.i.d. process with probability density function

$$p_S(s) = \lambda \exp(-\lambda s), \qquad s \geq 0, \qquad \lambda > 0$$

Assume that X_n is the input to a linear and time-invariant system with impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

with output Y_n

(a) Compute the mean of the process, $\mu_X = \mathbb{E}\{X_n\}$.
(b) Compute the autocorrelation function, $r_X[n]$.
(c) Compute the power spectrum of the process Y_n for $\lambda = 1$

Solution:

(a) The mean is given by

$$\mu_X = \mathbb{E}\{\exp(-S_n)\} = \int_{-\infty}^{\infty} \exp(-s) \cdot p_S(s) ds = \int_0^{\infty} \exp(-s) \cdot \lambda \exp(-\lambda s) ds$$
$$= \lambda \int_0^{\infty} \exp(-(\lambda+1)s) ds = \frac{\lambda}{\lambda+1}$$

(b) Since X_k is i.i.d., the autocorrelation is

$$r_X[n] = \mathbb{E}\{X_k X_{k+n}\} = \left[\begin{array}{ll} \mathbb{E}\{X_k\}\mathbb{E}\{X_{k+n}\}, & n \neq 0 \\ \mathbb{E}\{X_k^2\}, & n = 0 \end{array} \right.$$

Noting that

$$\mathbb{E}\{X_k^2\} = \mathbb{E}\{\exp(-2S_n)\} = \lambda \int_0^{\infty} \exp(-(\lambda+2)s) ds = \frac{\lambda}{\lambda+2}$$

we get

$$r_X[n] = \mathbb{E}\{X_k X_{k+n}\} = \left[\begin{array}{ll} \frac{\lambda^2}{(\lambda+1)^2}, & n \neq 0 \\ \frac{\lambda}{\lambda+2}, & n = 0 \end{array} \right.$$

(c) For $\lambda = 1$, we get

$$r_X[n] = \left[\begin{array}{ll} \frac{1}{4}, & n \neq 0 \\ \frac{1}{3}, & n = 0 \end{array} \right] = \frac{1}{4} + \frac{1}{12}\delta[n]$$

therefore

$$S_X(\omega) = \frac{\pi}{2}\delta(\omega) + \frac{1}{12}, \qquad -\pi \le \omega \le \pi$$

and

$$\begin{aligned} S_Y(\omega) = S_X(\omega)|H(\omega)|^2 &= \left(\frac{\pi}{2}\delta(\omega) + \frac{1}{12}\right)|1-\exp(-j\omega)|^2 \\ &= \frac{1}{12}|1-\exp(-j\omega)|^2 = \frac{1}{6}(1-\cos(\omega)) \end{aligned}$$

(Alternatively, we can also compute $S_Y(\omega)$ from $r_X[n]$ through $r_Y[n]$:

$$\begin{aligned} r_Y[n] &= r_X[n] * h[n] * h[-n] \\ &= r_X[n] * (\delta[n] - \delta[n-1]) * (\delta[-n] - \delta[-n-1]) \\ &= r_X[n] * (\delta[n] - \delta[n-1]) * (\delta[n] - \delta[n+1]) \\ &= r_X[n] * (2\delta[n] - \delta[n-1] - \delta[n+1]) \\ &= (2r_X[n] - r_X[n-1] - r_X[n+1]) \\ &= \frac{1}{12}(2\delta[n] - \delta[n-1] - \delta[n+1]) \end{aligned}$$

and, applying the Fourier transform,

$$S_Y(\omega) = \frac{1}{12}\left(2 - e^{-j\omega} - e^{j\omega}\right) = \frac{1}{6}(1-\cos(\omega))$$

SP3

Let S_n be a two-sided IID process with

$$p_{S_n}(s) = 6s(1-s), \qquad 0 \le s \le 1.$$

Also, let X_n be the process given by

$$X_n = S_n(1-S_n),$$

and let Y_n be the stochastic process given by

$$Y_n = X_n - \frac{1}{2}X_{n-1} - \frac{1}{2}X_{n+1}$$

(a) Compute the autocorrelation function of X_n, $r_X[n]$.

(b) Compute the power spectrum of Y_n, $S_Y(e^{j\omega})$.

Solution:

(a) The autocorrelation is

$$r_X[n] = \left[\begin{array}{ll} \mathbb{E}\{X_m^2\} & n = 0 \\ \mathbb{E}\{X_m\}\mathbb{E}\{X_{m+n}\} & n \neq 0 \end{array}\right.$$

Noting that

$$\mathbb{E}\{X_m\} = \mathbb{E}\{S_n(1-S_n)\} = 6\int_0^1 s^2(1-s)^2 ds = \frac{1}{5}$$
$$\mathbb{E}\{X_m^2\} = \mathbb{E}\{S_n^2(1-S_n)^2\} = 6\int_0^1 s^3(1-s)^3 ds = \frac{3}{70}$$

we get

$$r_X[n] = \left[\begin{array}{ll} \frac{3}{70} & n = 0 \\ \frac{1}{25} & n \neq 0 \end{array}\right.$$

(b) Noting that

$$r_X[n] = \frac{1}{25} + \frac{1}{350}\delta[n]$$

we have

$$S_X(e^{j\omega}) = \frac{2\pi}{25}\delta(\omega) + \frac{1}{350}$$

Process Y_n is the output of a linear time invariant filter with impulse response

$$h[n] = \delta[n] - \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+1]$$

and, thus

$$S_Y(e^{j\omega}) = S_X(e^{j\omega})|H(e^{j\omega})|^2 = \left(\frac{2\pi}{25}\delta(\omega) + \frac{1}{350}\right)\left|1 - \frac{1}{2}\exp(-j\omega) - \frac{1}{2}\exp(j\omega)\right|^2$$
$$= \frac{1}{350}(1-\cos(\omega))^2$$

SP4

The stochastic process X_n is given by the pair of equations

$$X_n = S_n \cdot R_n$$
$$S_n = W_n - \frac{1}{2}W_{n-1}$$

where W_n is a Gaussian i.i.d. process with mean 0 and variance v, and R_n is stationary processes with autocorrelation function

$$r_R[n] = 2^{-|n|}$$

Processes W_n and R_n are mutually independent.

(a) Compute the autocorrelation function of W_n, $r_W[n]$

(b) Compute and draw the autocorrelation function of S_n, $r_S[n]$

(c) Compute and draw the autocorrelation function of X_n, $r_X[n]$

(d) Compute the power spectrum of the process $Z_n = \sum_{k=0}^{\infty} 2^{-k} X_{n-k}$

Solution:

(a) Since W_n is zero-mean i.i.d, $r_W[n] = v\delta[n]$

(b)

$$\begin{aligned} r_S[n] &= \mathbb{E}\{S_k \cdot S_{k+n}\} \\ &= \mathbb{E}\{(W_k - \frac{1}{2}W_{k-1})(W_{k+n} - \frac{1}{2}W_{k+n-1})\} \\ &= \frac{5}{4}v\delta[n] - \frac{1}{2}v\delta[n-1] - v\frac{1}{2}\delta[n+1] \end{aligned}$$

(c)

$$\begin{aligned} r_X[n] &= \mathbb{E}\{X_k \cdot X_{k+n}\} \\ &= \mathbb{E}\{S_k \cdot S_{k+n}\}\mathbb{E}\{R_k \cdot R_{k+n}\} \\ &= r_S[n]r_R[n] \\ &= 2^{-|n|} \cdot \left(\frac{5}{4}v\delta[n] - \frac{1}{2}v\delta[n-1] - v\frac{1}{2}\delta[n+1]\right) \\ &= \frac{5}{4}v\delta[n] - \frac{1}{4}v\delta[n-1] - \frac{1}{4}v\delta[n+1]) \end{aligned}$$

(d)

$$Z_n = X_n * h[n]$$

where $h_n = 2^{-n}u[n]$. Therefore

$$\begin{aligned} S_Z(\omega) &= S_X(\omega)|H(\omega)|^2 \\ &= \frac{v}{2}\left(\frac{5}{2} - \cos(\omega)\right)\left|\frac{1}{1 - \frac{1}{2}\exp(-j\omega)}\right|^2 \\ &= \frac{v}{2} \cdot \frac{\frac{5}{2} - \cos(\omega)}{\left|1 - \frac{1}{2}\exp(-j\omega)\right|^2} \\ &= v \cdot \frac{5 - 2\cos(\omega)}{5 - 4\cos(\omega)} \end{aligned}$$

SP5

The stochastic process X_n is the sum of two i.i.d. stochastic processes S_n and R_n,

$$X_n = S_n + R_n$$

with probability density functions

$$p_S(s) = s\exp(-s), \qquad s \geq 0$$

and

$$p_R(r) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{r^2}{2}\right)$$

respectively. The processes S_n and R_n are mutually independent. Assume that X_n is the input to a linear and time-invariant system with impulse response

$$h_n = \frac{1}{2^n}u[n]$$

with output Y_n

(a) Compute the autocorrelation function, $r_X[n]$, and the power spectrum, $S_X(\omega)$, of X_n

(b) Compute the autocorrelation function, $r_Y[n]$, and the power spectrum, $S_Y(\omega)$, of Y_n

Solution:

(a)

$$\begin{aligned} r_X[n] &= \mathbb{E}\{X_k X_{k+n}\} \\ &= \mathbb{E}\{(S_k + R_k)(S_{k+n} + R_{k+n})\} \\ &= r_S[n] + r_R[n] + \mathbb{E}\{S_k\}\mathbb{E}\{R_{k+n}\} + \mathbb{E}\{R_k\}\mathbb{E}\{S_{k+n}\} \\ &= r_S[n] + r_R[n] \end{aligned}$$

Since S_n and R_n are i.i.d.

$$\begin{aligned} r_S[n] &= \mathbb{E}\{S_k S_{k+n}\} \\ &= \mathbb{E}\{S_k^2\}\delta[n] + \mathbb{E}\{S_k\}\mathbb{E}\{S_{k+n}\}(1-\delta[n]) \\ &= \int_0^\infty s^3 \exp(-s)ds \cdot \delta[n] + \left(\int_0^\infty s^2 \exp(-s)ds\right)^2 (1-\delta[n]) \\ &= 3!\delta[n] + 4(1-\delta[n]) \\ &= 4 + 2\delta[n] \end{aligned}$$

and

$$\begin{aligned} r_R[n] &= \mathbb{E}\{R_k R_{k+n}\} \\ &= \mathbb{E}\{R_k^2\}\delta[n] + \mathbb{E}\{R_k\}\mathbb{E}\{R_{k+n}\}(1-\delta[n]) = \delta[n] \end{aligned}$$

Therefore

$$r_X[n] = 4 + 3\delta[n]$$

and the power spectrum is

$$S_X(w) = 3 + 8\pi\delta(\omega), \qquad -\pi \le \omega \le \pi$$

(b)

$$\begin{aligned} r_Y[n] &= r_X[n] * h[n] * h[-n] \\ &= \frac{4}{3}\left(\frac{1}{2}\right)^{|n|} * (4 + 3\delta[n]) = 16 + 4\left(\frac{1}{2}\right)^{|n|} \\ S_Y(w) &= S_X(w)|H(\omega)|^2 \\ &= \frac{3 + 8\pi\delta(\omega)}{\left|1 - \frac{1}{2}e^{-j\omega}\right|^2} \end{aligned}$$

(This expression can be further simplified to

$$S_Y(w) = \frac{3 + 8\pi\delta(\omega)}{\frac{5}{4} - \cos(\omega)} = \frac{3}{\frac{5}{4} - \cos(\omega)} + 32\pi\delta(\omega)$$

).

Let X_n be a discrete two-sided IID process with probability density function

$$p_X(x) = \frac{1}{2}, \qquad -1 \le x \le 1$$

Let Y_n be the process defined by

$$Y_n = X_n^3$$

Let Z_n be the output of a linear time-invariant filter with impulse response:

$$h[n] = \frac{u[n]}{3^n}$$

when the input is Y_n.

(a) Is X_n wide-sense-stationary (WSS)? Is it strict-sense stationary (SSS)?

(b) Compute the mean $\mu_Y[n]$ and the autocorrelation function, $r_Y[n]$, of Y_n.

(c) Compute the power spectrum of Z_n, $S_Z\left(e^{j\omega}\right)$

(d) Find the impulse response $g[n]$ of a linear-time invariant system such that, if V_n is the output of the system for input Z_n, the autocorrelation function of V_n is

$$r_V[n] = \delta[n]$$

Solution:

(a) Since X_n is IID, it is WSS and SSS.

(b)

$$\mu_Y[n] = \mathbb{E}\{Y_n\} = \mathbb{E}\{X_n^3\} = \frac{1}{2}\int_{-1}^{1} x^3 dx = 0$$

$$\begin{aligned} r_Y[n] = \mathbb{E}\{Y_m Y_{m+n}\} &= \left[\begin{array}{ll} \mathbb{E}\{X_m^6\}, & n = 0, \\ \mathbb{E}\{X_m^3\}\mathbb{E}\{X_{m+n}^3\}, & n \neq 0, \end{array} \right. \\ &= \mathbb{E}\{X_m^6\}\delta[n] = \frac{1}{2}\int_{-1}^{1} x^6 dx \delta[n] = \frac{1}{7}\delta[n] \end{aligned}$$

(c)

$$\begin{aligned} S_Z\left(e^{j\omega}\right) =& S_Y\left(e^{j\omega}\right)\left|H\left(e^{-j\omega}\right)\right|^2 \\ =& \frac{1}{7\left|1 - \frac{1}{3}e^{-j\omega}\right|^2} \end{aligned}$$

(d) If the autocorrelation is $r_V[n] = \delta[n]$, the power spectrum is $S_V\left(e^{j\omega}\right) = 1$. Since

$$S_V\left(e^{j\omega}\right) = S_Z\left(e^{j\omega}\right)\left|G\left(e^{j\omega}\right)\right|^2$$

we have

$$\frac{1}{7\left|1 - \frac{1}{3}e^{-j\omega}\right|^2}\left|G\left(e^{j\omega}\right)\right|^2 = 1$$

therefore

$$\left|G\left(e^{j\omega}\right)\right|^2 = 7\left|1 - \frac{1}{3}e^{-j\omega}\right|^2$$

that is

$$G\left(e^{j\omega}\right)G^*\left(e^{j\omega}\right) = 7\left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{j\omega}\right)$$

Therefore, we can take, for instance

$$G\left(e^{j\omega}\right) = \sqrt{7}\left(1 - \frac{1}{3}e^{-j\omega}\right)$$

so that

$$g[n] = \sqrt{7}\left(\delta[n] - \frac{1}{3}\delta[n-1]\right)$$

SP7

Suppose that X_n is a two-sided binary Bernoulli(p) process, that is, an IID process given by

$$P_{X_n}(k) = \begin{bmatrix} p, & k=1 \\ 1-p, & k=0 \end{bmatrix}, \qquad n \in \mathbb{Z}$$

Using X_n, we define the following random processes

$$\begin{aligned} T_n &= X_n \cdot X_{n-1} \\ U_n &= X_n \cdot X_{n-1}, \cdots X_{n-\ell}, \qquad \ell \geq 1 \end{aligned}$$

where operator $\oplus$ denotes mod 2 addition

(a) Compute the probability mass function of T_n, $P_{T_n}(k)$, $k \in \{0, 1\}$.

(b) Compute the autocorrelation function, $r_T[n]$, of T_n.

(c) Compute the power spectrum of T_n, $S_T(\omega)$.

(d) Compute the probability mass function of U_n, $P_{U_n}(k)$, $k \in \{0, 1\}$.

(e) Compute the autocorrelation function, $r_U[n]$, of U_n.

Solution:

(a)

$$\begin{aligned} P_T(1) &= P\{X_n = 1, X_{n-1} = 1\} = p^2 \\ P_T(0) &= 1 - p^2 \end{aligned}$$

(b) For T_n we have

$$\begin{aligned} r_T[n] &= \mathbb{E}\{T_m T_{m+n}\} \\ &= \left[\begin{array}{ll} \mathbb{E}\{X_m^2 X_{m-1}^2\}, & n = 0 \\ \mathbb{E}\{X_{m-1} \cdot X_m^2 \cdot X_{m+1}\}, & n \in \{-1, 1\} \\ \mathbb{E}\{X_{m-1} \cdot X_m \cdot X_{m+n-1} \cdot X_{m+n}\}, & |n| > 1 \end{array}\right. \end{aligned}$$

and, noting that, since X_m is a binary process, $X_m^2 = X_m$,

$$\begin{aligned}
r_T[n] &= \mathbb{E}\{T_m T_{m+n}\} \\
&= \left[\begin{array}{ll} \mathbb{E}\{X_m\}\mathbb{E}\{X_{m-1}\}, & n=0 \\ \mathbb{E}\{X_{m-1}\}\cdot\mathbb{E}\{X_m\}\cdot\mathbb{E}\{X_{m+1}\}, & n\in\{-1,1\} \\ \mathbb{E}\{X_{m-1}\}\cdot\mathbb{E}\{X_m\}\cdot\mathbb{E}\{X_{m+n-1}\}\mathbb{E}\{X_{m+n}\}, & |n|>1 \end{array}\right. \\
&= \left[\begin{array}{ll} p^2, & n=0 \\ p^3, & n\in\{-1,1\} \\ p^4, & |n|>1 \end{array}\right. \\
&= (p^2-p^4)\delta[n] + (p^3-p^4)\left(\delta[n-1]+\delta[n+1]\right) + p^4
\end{aligned}$$

(c) The power spectrum is

$$S_T(\omega) = (p^2-p^4) + 2(p^3-p^4)\cos(\omega) + 2\pi p^4\delta(\omega)$$

(d)

$$\begin{aligned}
&P_U(1) = P\{X_n=1, X_{n-1}=1,\ldots,X_{n-m}=1\} = p^m \\
&P_U(0) = 1-p^m
\end{aligned}$$

(e) Since U_{m+n} and U_m have common factors if and only if $|n|\le\ell$, we can write:

$$\begin{aligned}
r_U(n) &= \left[\begin{array}{ll} \mathbb{E}\{X_{m-\ell}\cdot\ldots\cdot X_m\cdot X_{m+n-\ell}\cdot\ldots\cdot X_{m+n}\}, & |n|>\ell \\ \mathbb{E}\{X_{m-\ell}\cdot\ldots\cdot X_{m+n}\}, & 0\le n\le\ell \\ \mathbb{E}\{X_{m+n-\ell}\cdot\ldots\cdot X_m\}, & -\ell\le n\le 0 \end{array}\right. \\
&= \left[\begin{array}{ll} p^{2\ell+2}, & |n|>\ell \\ p^{|n|+\ell+1}, & |n|\le\ell \end{array}\right.
\end{aligned}$$

SP8

Suppose that X_n is a two-sided binary Bernoulli(p) process, that is, an IID process given by

$$P_{X_n}(k) = \left[\begin{array}{ll} p, & k=1 \\ 1-p, & k=0 \end{array}\right], \qquad n\in\mathbb{Z}$$

Suppose that W_n is another binary Bernoulli(α) process, statistically independent of process X_n (that is, any collection of samples from X_n is independent from any collection of samples from W_n).

Using X_n and W_n, we define the following random processes

$$\begin{aligned}
Y_n &= X_n \oplus W_n, \\
Z_n &= X_n \oplus X_{n-1}
\end{aligned}$$

where operator $\oplus$ denotes mod 2 addition

(a) Compute the probability mass function of Y_n, that is, $P_{Y_n}(k) = P\{Y_n=k\}$, $k\in\{0,1\}$.

(b) Compute the autocorrelation function, $r_Y[n]$, of Y_n.

(c) Compute the power spectrum of Y_n, $S_Y(\omega)$.

(d) Compute the probability mass function of Z_n, $P_{Z_n}(k)$, $k\in\{0,1\}$. To simplify some expressions, you can express your results as a function of variable $h = p(1-p)$.

(e) Compute the autocorrelation function, $r_Z[n]$, of Z_n.

(f) Compute the power spectrum of Z_n, $S_Z(\omega)$.

Solution:

(a)

$$\begin{aligned} P_Y(1) &= P\{Y_n = 1\} = P\{X_n = 0, W_n = 1\} + P\{X_n = 1, W_n = 0\} \\ &= P\{X_n = 0\} \cdot P\{W_n = 1\} + P\{X_n = 1\} \cdot P\{W_n = 0\} \\ &= (1-\alpha)p + \alpha(1-p) \\ P_Y(0) &= 1 - P_Y(1) \\ &= 1 - (1-\alpha)p - \alpha(1-p) \end{aligned}$$

(b) Since X_n and W_n are IID processes, so it is Y_n, therefore,

$$\begin{aligned} r_Y(n) &= \mathbb{E}\{Y_m Y_{n+m}\} \\ &= \mathbb{E}\{Y_m^2\}\delta[n] + \mathbb{E}\{Y_m\}\mathbb{E}\{Y_{m+n}\}(1-\delta[n]) \\ &= \mathbb{E}\{Y_m\}\delta[n] + \mathbb{E}\{Y_m\}^2(1-\delta[n]) \\ &= v^2 + v(1-v)\delta[n] \end{aligned}$$

where

$$v = \mathbb{E}\{Y_m\} = (1-\alpha)p + \alpha(1-p)$$

(c) The power spectrum is

$$S_Y(\omega) = 2\pi v^2 \delta(\omega) + v(1-v)$$

(d) Since X_n and X_{n-1} are independent,

$$\begin{aligned} P_Z(1) &= P\{Z_n = 1\} \\ &= P\{X_n = 1, X_{n-1} = 0\} + P\{X_n = 0, X_{n-1} = 1\} \\ &= P\{X_n = 1\} \cdot P\{X_{n-1} = 0\} + P\{X_n = 0\} \cdot P\{X_{n-1} = 1\} \\ &= 2p(1-p) = 2h \\ P_Z(0) &= 1 - 2p(1-p) = 1 - 2h \end{aligned}$$

(e) For Z_n we have

$$\begin{aligned} r_Z[n] &= \mathbb{E}\{Z_m Z_{m+n}\} \\ &= \left[\begin{array}{ll} \mathbb{E}\{Z_m^2\}, & n = 0 \\ \mathbb{E}\{Z_m Z_{m+1}\}, & n = -1, n = 1 \\ \mathbb{E}\{Z_n\}\mathbb{E}\{Z_{n+m}\}, & |n| > 1 \end{array} \right. \\ &= \left[\begin{array}{ll} \mathbb{E}\{Z_m\}, & n = 0 \\ \mathbb{E}\{(X_n \oplus X_{n-1}) \cdot (X_{n+1} \oplus X_n)\}, & n \in \{-1, 1\} \\ \mathbb{E}\{Z_n\}^2, & |n| > 1 \end{array} \right. \\ &= \left[\begin{array}{ll} 2h, & n = 0 \\ \mathbb{E}\{(X_n \oplus X_{n-1}) \cdot (X_{n+1} \oplus X_n)\}, & n \in \{-1, 1\} \\ 4h^2, & |n| > 1 \end{array} \right. \end{aligned}$$

Noting that $(X_n \oplus X_{n-1}) \cdot (X_{n+1} \oplus X_n) = 1$ if and only if $(X_{n+1} = X_{n-1} = 1, X_n = 0)$ or $(X_{n+1} = X_{n-1} = 0, X_n = 1)$, we have

$$\mathbb{E}\{(X_n \oplus X_{n-1}) \cdot (X_{n+1} \oplus X_n)\} = p^2(1-p) + p(1-p)^2 = p(1-p) = h$$

Therefore

$$\begin{aligned} r_Z[n] &= \begin{bmatrix} 2h, & n = 0 \\ h, & n = -1, n = 1 \\ 4h^2, & |n| > 1 \end{bmatrix} \\ &= (2h - 4h^2)\delta[n] + (h - 4h^2)\left(\delta[n+1] + \delta[n-1]\right) + 4h^2 \end{aligned}$$

(f) The power spectrum is

$$S_Z(\omega) = (2h - 4h^2) + 2(h - 4h^2)\cos(\omega) + 8\pi h^2 \delta(\omega)$$

SP9

The IID stochastic process X_n is given by a probability density function

$$p_X(x) = \frac{1}{2}, \qquad -1 \le x \le 1$$

The stochastic process given by $Y_n = X_n^p$, where p is an arbitrary positive and odd integer, is the input to a linear and time invariant system with impulse response

$$h[n] = \delta[n] - \delta[n-2]$$

Let Z_n be the output of this filter.

(a) Compute the autocorrelation function and the power spectrum of X_n.

(b) Compute the autocorrelation function and the power spectrum of Y_n.

(c) Compute the autocorrelation function and the power spectrum of Z_n.

Solution:

(a)

$$r_X[n] = \mathbb{E}\{X_m X_{m+n}\} = \begin{cases} \mathbb{E}\{X_m^2\}, & n = 0 \\ \mathbb{E}\{X_m\}\mathbb{E}\{X_{m+n}\}, & n \neq 0 \end{cases}$$

Knowing that,

$$\mathbb{E}\{X_n\} = 0$$

(by the symmetry of the density function), and

$$\mathbb{E}\{X_n^2\} = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{3}$$

we get

$$\boxed{r_X[n] = \frac{1}{3}\delta[n]}$$

and

$$\boxed{S_X\left(e^{j\omega}\right) = \frac{1}{3}}$$

(b)

$$r_Y[n] = \mathbb{E}\{Y_m Y_{m+n}\} = \left[\begin{array}{ll} \mathbb{E}\{X_m^{2p}\}, & n = 0 \\ \mathbb{E}\{X_m^p\}\mathbb{E}\{X_{m+n}^p\}, & n \neq 0 \end{array} \right]$$

Knowing that,

$$\mathbb{E}\{X_n^{2p}\} = \int_{-1}^{1} \frac{1}{2} x^{2p} dx = \frac{1}{2p+1}\left(1^{2p+1} - (-1)^{2p+1}\right) = \frac{1}{2p+1}$$

$$\mathbb{E}\{X_n^{p}\} = \int_{-1}^{1} \frac{1}{2} x^{p} dx = \frac{1}{p+1}\left(1^{p+1} - (-1)^{p+1}\right) = 0$$

we get

$$r_Y[n] = \mathbb{E}\{Y_m Y_{m+n}\} = \left[\begin{array}{ll} \frac{1}{2p+1}, & n = 0 \\ 0, & n \neq 0 \end{array} \right] = \boxed{\frac{1}{2p+1}\delta[n]}$$

and, thus,

$$\boxed{S_Y\left(e^{j\omega}\right) = \frac{1}{2p+1}}$$

(c) Since $Z_n = Y_n * h[n]$, we have

$$S_Z\left(e^{j\omega}\right) = S_Y\left(e^{j\omega}\right)\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{2p+1}\left|1 - e^{-2j\omega}\right|^2$$

$$= \boxed{\frac{2}{2p+1}\left(1 - \cos(2\omega)\right)}$$

Therefore

$$\boxed{r_Z[n] = \frac{2}{2p+1}\left(2\delta[n] - \delta[n-2] - \delta[n+2]\right)}$$