Stochastic Processes: Problems

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1. Markov Processes

Exercise MP1 (Markov Chain)

Let $X_k, k \geq 0$ be a Markov chain with state space $\mathcal{Z} = \{0, 1\}$ and transition probabilities $P\{X_k = 1|X_{k-1} = 0\} = 0.8$ and $P\{X_k = 0|X_{k-1} = 1\} = 0.4$.

(a) Draw the corresponding transition graph

(b) Assume that the initial state is $X_0 = 1$. Compute $P\{X_2 = 1\}$

(c) Compute the stationary distribution.

Exercise MP2 (Markov Process)

A video game consists of N consecutive levels, $0, 1, ..., N - 1$. The player starts at level 0. If a player passes level i, she enters level $i + 1$, if not, she returns back to level 0. It is known that all phases have the same difficulty, so, if a player is at level i, she reaches level $i + 1$ with probability q, and returns back to 0 with probability $1 - q$.

When the player reaches stage $N - 1$, she gets a medal, returns to level 0 and the game continues.

Let X_k be the stochastic process that represents the sequence of levels during a game, such that $X_k = i$ means that the player was at level i at time k. The game begins at $X_0 = 0$.

(a) Formulate the problem as a stationary Markov process, and draw the transition graph for $N = 6$.

(b) Assuming $N \geq 2$, compute $P\{X_2 = 1\}$.

(c) Assuming $N \geq 2$, determine the probability of obtaining a medal exactly at time k, that is $P\{X_k = N - 1\}$, for $k = 0, 1, \ldots, N$.

(d) For $N = 2$, determine the stationary distribution.

(e) For $N = \infty$, determine the stationary distribution

Exercise MP3 (Markov Processes)

A game has three players, named G_0, G_1 and G_2 and consists of a sequence of rounds. At each round, only one of the players enters the game.

The result of each round can be win or lose. If the active player wins a round, she can play the next one. If she loses, she must pass the turn to one of the other players, which is chosen at random with equal probabilities.

Based on the game skills of the players, it is known that the winning probabilities are 0.4 (for G_0), 0.6 (G_1) and 0.8 (G_2).

Let X_k be the one-sided stochastic process that represents the sequence of active players during a game, such that $X_k = i$ means that the active player at time k is G_i.

The game always starts with player 0, that is, $X_0 = 0$.

(4 %) (a) Formulate the problem as a stationary Markov chain: compute the transition matrix and draw the transition graph.

(4 %) (b) Compute the probability that the first players in the sequence are $0, 1, 2, 1$ (i.e., $P\{X_0 = 0, X_1 = 1, X_2 = 2, X_3 = 1)$).

(5 %) (c) Compute $P\{X_2 = 1\}$.

(5 %) (d) Compute the stationary distribution

(5 %) (e) Players G_0 and G_1 are unsatisfied because G_2, with better skills, plays most rounds. They decide to make the game fairer, in the sense that, in the long term, everyone plays with the same frequency. To do so, they proceed as follows:

1. If G_0 is the active player and loses, she passes the turn to player G_1 with probability q and to G_2 with probability $1 - q$.
2. If G_1 is the active player and loses, she passes the turn to player G_0 with probability r and to G_2 with probability $1 - r$.

Determine if G_0 and G_1 will succed in making a fair game, that is, if there exist values q and r so that the stationary distribution is uniform, i.e., $\boldsymbol{\pi} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ If so, compute them.

2. Stationary Processes

Exercise SP1 (Autocorrelation, Power Spectrum)

Let X_n be i.i.d. stochastic process with probability density function

$$p_X(x) = x \exp(-x), \qquad x \geq 0$$

Assume that X_n is the input to a linear system with impulse response

$$h_n = \delta[n] + 0.5\delta[n-1]$$

the system output Y_n, is corrupted by a Gaussian i.i.d noise E_n (independent of X_n) with mean zero and unit variance, to produce the final process

$$Z_n = Y_n + E_n$$

(a) Compute the autocorrelation functions $r_X[n]$ and $r_E[n]$ of X_n and E_n, respectively.

(b) Compute the autocorrelation function of Y_n, $r_Y[n]$

(c) Compute the autocorrelation function of Z_n, $r_Z[n]$

(d) Compute the power spectrum of Z_n, $S_Z(\omega)$..

Exercise SP2 (Autocorrelation, Power Spectrum)

The stochastic process X_n is given by,

$$X_n = \exp(-S_n)$$

where S_n is an i.i.d. process with probability density function

$$p_S(s) = \lambda \exp(-\lambda s), \qquad s \geq 0, \qquad \lambda > 0$$

Assume that X_n is the input to a linear and time-invariant system with impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

with output Y_n

(a) Compute the mean of the process, $\mu_X = \mathbb{E}\{X_n\}$.

(b) Compute the autocorrelation function, $r_X[n]$.

(c) Compute the power spectrum of the process Y_n for $\lambda = 1$

Exercise SP3 (Autocorrelation, Power Spectrum)

The stochastic process X_n is given by the pair of equations

$$X_n = S_n \cdot R_n$$
$$S_n = W_n - \frac{1}{2}W_{n-1}$$

where W_n is a Gaussian i.i.d. process with mean 0 and variance v, and R_n is stationary processes with autocorrelation function

$$r_R[n] = 2^{-|n|}$$

Processes W_n and R_n are mutually independent.

(a) Compute the autocorrelation function of W_n, $r_W[n]$

(b) Compute and draw the autocorrelation function of S_n, $r_S[n]$

(c) Compute and draw the autocorrelation function of X_n, $r_X[n]$

(d) Compute the power spectrum of the process $Z_n = \sum_{k=0}^{\infty} 2^{-k} X_{n-k}$

Exercise SP4 (Autocorrelation, Power Spectrum)

The stochastic process X_n is the sum of two i.i.d. stochastic processes S_n and R_n,

$$X_n = S_n + R_n$$

with probability density functions

$$p_S(s) = s\exp(-s), \qquad s \geq 0$$

and

$$p_R(r) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right)$$

respectively. The processes S_n and R_n are mutually independent. Assume that X_n is the input to a linear and time-invariant system with impulse response

$$h_n = \frac{1}{2^n}u[n]$$

with output Y_n

(a) Compute the autocorrelation function, $r_X[n]$, and the power spectrum, $S_X(\omega)$, of X_n

(b) Compute the autocorrelation function, $r_Y[n]$, and the power spectrum, $S_Y(\omega)$, of Y_n

Exercise SP5 (Autocorrelation, Power Spectrum)

Suppose that X_n is a two-sided binary Bernoulli(p) process, that is, an IID process given by

$$P_{X_n}(k) = \begin{bmatrix} p, & k=1 \\ 1-p, & k=0 \end{bmatrix}, \qquad\qquad n \in \mathbb{Z}$$

Suppose that W_n is another binary Bernoulli(α) process, statistically independent of process X_n (that is, any collection of samples from X_n is independent from any collection of samples from W_n).

Using X_n and W_n, we define the following random processes

$$Y_n = X_n \oplus W_n,$$
$$Z_n = X_n \oplus X_{n-1}$$

where operator $\oplus$ denotes mod 2 addition

(a) Compute the probability mass function of Y_n, that is, $P_{Y_n}(k) = P\{Y_n = k\}$, $k \in \{0, 1\}$.

(b) Compute the autocorrelation function, $r_Y[n]$, of Y_n.

(c) Compute the power spectrum of Y_n, $S_Y(\omega)$.

(d) Compute the probability mass function of Z_n, $P_{Z_n}(k)$, $k \in \{0, 1\}$. To simplify some expressions, you can express your results as a function of variable $h = p(1 - p)$.

(e) Compute the autocorrelation function, $r_Z[n]$, of Z_n.

(f) Compute the power spectrum of Z_n, $S_Z(\omega)$.

Exercise SP6 (Autocorrelation, Power Spectrum)

Suppose that X_n is a two-sided binary Bernoulli(p) process, that is, an IID process given by

$$P_{X_n}(k) = \begin{bmatrix} p, & k = 1 \\ 1 - p, & k = 0 \end{bmatrix}, \qquad n \in \mathbb{Z}$$

Using X_n, we define the following random processes

$$\begin{aligned} T_n &= X_n \cdot X_{n-1} \\ U_n &= X_n \cdot X_{n-1}, \cdots X_{n-\ell}, \qquad m \geq 1 \end{aligned}$$

where operator $\oplus$ denotes mod 2 addition

(a) Compute the probability mass function of T_n, $P_{T_n}(k)$, $k \in \{0, 1\}$.

(b) Compute the autocorrelation function, $r_T[n]$, of T_n.

(c) Compute the power spectrum of T_n, $S_T(\omega)$.

(d) Compute the probability mass function of U_n, $P_{U_n}(k)$, $k \in \{0, 1\}$.

(e) Compute the autocorrelation function, $r_U[n]$, of U_n.