1.5 Stationary Processes

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Stationarity refers to the time invariance of some statistical properties of a stochastic process. We have already seen some forms of time invariance in the previous sections:

- The homogeneity of Markov chain, X_n, which implies the time invariance of the conditional probabilities: $P(X_n|X_m) = P(X_{n+k}|X_{m+k})$, for any $k \geq 0$, $n \geq m \geq 0$.
- The time invariance property of a SRW, which implies that $P(X_{n+h} - X_h) = P(X_n)$.

In this section we introduce two forms of stationarity which are particularly interesting in statistical signal processing applications. In its more restrictive form, stationarity implies the invariance of all joint distributions in the process.

1.5.1 Strict-Sense Stationarity

Definition 9 (Strict-Sense Stationarity) A two-sided stationary process is said to be Strict-Sense Stationary (or simply SSS) if all joint distributions are invariant to a time shift, that is, for any $N > 0$, any sampling instants $n_0, \ldots, x_{n_{N-1}}$ and any time shift $m \in \mathbb{Z}$

$$P(X_{n_0}, X_{n_1}, \ldots, X_{n_{N-1}}) = P(X_{n_0+m}, X_{n_1+m}, \ldots, X_{n_{N-1}+m}) \tag{1.35}$$

Thus, if a stochastic process X_n is SSS, the process $Y_n = X_{n+m}$ has the same statistical properties as X_n.

Example 10 (Bernoulli process)

A Bernoulli process, $\mathscr{B}(p)$ given by (1.7) is independent and, thus, its joint probability mass function will have the form

$$P_{X_{n_0},\ldots,X_{n_{N-1}}}(x_0, \ldots, x_{N-1}) = \prod_{i=0}^{N-1} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=0}^{N-1} x_i}(1-p)^{N-\sum_{i=0}^{N-1} x_i} \tag{1.36}$$

which does not depend on $n_0, \ldots, n_{N-1}$. Thus, the Bernoulli process is SSS.

In general, all IID processes are SSS. In general, independent but not identically distributed processes are not IID. This is the case of non-constant deterministic process, as the following example shows

Example 11 (Deterministic process)

Consider the deterministic process given by $X_n = 2n$ (with probability 1), that is, for each time n, X_n is a discrete random variable with PMF

$$P_{X_n}(k) = \delta[k - 2n] \tag{1.37}$$

which depends on n. Thus, it is not SSS.

Unless for these and other simple examples, strict stationarity is difficult to check and overly restrictive, and weaker forms of stationarity are more interesting. In the following, we will focus in the invariance of first and second order statistics[1]

1.5.2 First and Second-Order Statistics

In order to define wide-sense stationarity, we will first define some first and second-order statistics of stochastic processes. We will assume, in general, stochastic processes which can take complex values:

Definition 10 (First and second order statistics) Given stochastic process X_n we define the following statistics:

- **Mean**:
$$\mu_X[n] = \mathbb{E}\{X_n\} \tag{1.38}$$
- **Autocorrelation**:
$$R_X[n_1, n_2] = \mathbb{E}\{X_{n_1} X^*_{n_2}\} \tag{1.39}$$
- **Cross correlation**
$$R_{XY}[n_1, n_2] = \mathbb{E}\{X_{n_1} Y^*_{n_2}\} \tag{1.40}$$

Given stochastic processes X_n and Y_n with means $\mu_X[n]$ and $\mu_Y[n]$, we define the following joint statistics:

- **Cross covariance** between X_n and Y_n
$$C_{XY}[n_1, n_2] = \mathbb{E}\{(X_{n_1} - \mu_X[n_1])(Y_{n_2} - \mu_Y[n_2])^*\} \tag{1.41}$$
- **Autocovariance**
$$C_X[n_1, n_2] = \mathbb{E}\{(X_{n_1} - \mu_X[n_1])(X_{n_2} - \mu_X[n_2])^*\} \tag{1.42}$$

Note that $C_X[n, n]$ is the variance of X_n. It is easy to verify that the autocovariance and the autocorrelation are related by

$$C_X[n_1, n_2] = R_X[n_1, n_2] - \mu_X[n_1]\mu^*_X[n_2] \tag{1.43}$$

Example 12 (Bernoulli process)

For a Bernoulli process, $\mathscr{B}(p)$, using (1.7) we get

$$\mu_X[n] = \mathbb{E}\{X_n\} = p, \tag{1.44}$$

[1] In this document, first and second order statistics refer to the expected value of polynomial functions of a random variable up to degree 2, that, it, the mean, the correlation and other related statistics, like the covariance. This should not be confused with the so called "order statistics".

also

$$R_X[n_1,n_2] = \mathbb{E}\{X_{n_1}X_{n_2}\} = \begin{bmatrix} \mathbb{E}\{X_{n_1}^2\} & \text{if } n_1 = n_2 \\ \mathbb{E}\{X_{n_1}\}\mathbb{E}\{X_{n_2}\} & \text{if } n_1 \neq n_2 \end{bmatrix} = \begin{bmatrix} p & \text{if } n_1 = n_2 \\ p^2 & \text{if } n_1 \neq n_2 \end{bmatrix}$$
$$= p^2 + p(1-p)\delta[n_1 - n_2] \tag{1.45}$$

and

$$C_X[n_1,n_2] = p(1-p)\delta[n_1 - n_2] \tag{1.46}$$

respectively.

Example 13 (Deterministic process)
Consider the deterministic process in Example 11, we have

$$\mu_X[n] = \mathbb{E}\{X_n\} = \mathbb{E}\{2n\} = 2n \tag{1.47}$$

and,

$$R_X[n_1,n_2] = \mathbb{E}\{X_{n_1}X_{n_2}\} = \mathbb{E}\{4n_1n_2\} = 4n_1n_2 \tag{1.48}$$

and, using (1.43)

$$C_X[n_1,n_2] = 0 \tag{1.49}$$

Example 14 Consider the stochastic process X_n given by

$$X_n = (1+n^2)Y \tag{1.50}$$

where Y is a real Gaussian random variable with mean 0 and unit variance.
The mean and the autocorrelation of the process can be easily calculated as

$$\mu_X[n] = \mathbb{E}\{Y\} \cdot (1+n^2) = 0 \tag{1.51}$$

$$R_X(n,m) = \mathbb{E}\{X_nX_m\} = (1+n^2)\cdot(1+m^2)\cdot\mathbb{E}\left\{Y^2\right\} = (1+n^2)\cdot(1+m^2) \tag{1.52}$$

Since X_m has zero mean, the autocovariance is equal to the autocorrelation, and the variance is

$$\sigma_X^2[n] = R_n[n,n] = \left(1+n^2\right)^2 \tag{1.53}$$

Example 15 (Simple Random Walk)
For the SRW X_n given by (1.8), the value of the process at any time n depends on the value of a set of n random variables $(Y_1, \ldots Y_n)$. We have seen that the process is zero-mean with variance n. The autocorrelation (equal to the autocovariance) can be computed as

$$R_X[n_1,n_2] = \mathbb{E}\{X_{n_1}X_{n_2}\} = \sum_{k=1}^{n_1}\sum_{\ell=0}^{n_2}\mathbb{E}\{Y_kY_\ell\} = \sum_{k=1}^{n_1}\sum_{\ell=0}^{n_2}\delta[k-\ell]$$
$$= \min(n_1,n_2) \tag{1.54}$$

1.5.3 Wide-Sense Stationarity

We are ready to define a weaker form of stationarity:

Definition 11 (Wide-Sense Stationarity) A two-sided stationary process is Wide-Sense Stationary (or simply WSS) if its mean and autocorrelation functions are invariant to a time shift, that is, for any integers n, n_1, n_2 and any time shift $m \in \mathbb{Z}$,

$$\mu_X[n] = \mu_X[n+m] \tag{1.55}$$

and

$$R_X[n_1, n_2] = R_X[n_1+m, n_2+m] \tag{1.56}$$

Thus, if a process is WSS, its mean is constant and its autocorrelation only depends on the difference between n_1 and n_2. For this reason, we will typically use the notation

$$\mu_X = \mu_X[n] \tag{1.57}$$

$$R_X[n] = R_X[m+n, m] \tag{1.58}$$

Also, combining (1.57) and (1.58) with (1.43), it is straighforward to see that the covariance function is also invariant to a time shift, and we can write

$$C_X[n] = C_X[m+n, m] \tag{1.59}$$

Example 16 Let's go back to the examples from the previous section.

- **Bernoulli process**: The Bernoulli process X_n from Example 10 is SSS, since their joint probability function, (1.36), does not depend on the time variables. Consequently, it is also WSS, as can be seen by observing that the average in (1.44) is constant and the autocorrelation in (1.45) depends on the time difference only.
- **Deterministic process**: For the deterministic process $X_n = 2n$ in Example 11, we have $\mu_X[n] = 2n$, which depends on n. Thus, it is not WSS.
- The process X_n in Example 14 has zero mean (thus, the mean is invariant to a time shift), but the autocorrelation $R_X[n_1, n_2]$ (1.52) cannot be expressed as function of the time difference $n_1 - n_2$ only. Thus, the process is not WSS.
- **Simple Random Walk**: The SRW from Example 15 is not stationary since its autocorrelation (1.54) is not invariant to a time shift: for any $m > 0$

$$\begin{aligned} R_X[n_1+m, n_2+m] &= \min(n_1+m, n_2+m) = \min(n_1, n_2) + m = R_X(n_1, n_2) + m \\ &\neq R_X[n_1, n_2] \end{aligned} \tag{1.60}$$

1.5.3.1 Properties of stationary processes

Theorem 2 *The autocorrelation WSS process X_n satisfies the following properties:*

1. ***Hermiticity**: $R_X[n] = R_X^*[-n]$.*
2. *$R_X[0] = \mathbb{E}\{|X_n|^2\}$.*
3. *It is **positive-definite**: for any integer $N > 0$ and any coefficients $c_n \in \mathbb{C}, n = 0, \ldots, N-1$,*

$$\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} c_m c_n^* R_X[n-m] \geq 0 \tag{1.61}$$

4. ***Maximality*** *at the origin:* $|R_X[n]| \leq R_X[0]$.

Proof Hermiticity is a direct consequence of the definition, as well as property 2. To prove the positive definiteness, note that, for any coefficients c_n, we have

$$0 \leq \mathbb{E}\left\{\left|\sum_{n=0}^{N-1} c_n\right|^2\right\} = \mathbb{E}\left\{\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} c_m c_n^* R_X[n-m]\right\} \tag{1.62}$$

The maximality at the origin is a consequence of the positive definiteness: since (1.61) is true for any complex coefficients, we can take $c_0 = 1$, $c_1 = c_2 = \ldots = c_{n-1} = 0$ to get

$$(1+|a_n X_n|^2)R_X[0] + c_n R_X[n] + c_n^* R_x^*[n] \geq 0 \tag{1.63}$$

and, taking $c_n = -\frac{R_X^*[n]}{|R_X[n]|}$ (which implies $|c_n|^2 = 1$), we get

$$2R_X[0] - 2\,|R_X[n]|| \geq 0 \tag{1.64}$$

which completes the proof. □

1.5.3.2 WSS and SSS processes

Note that stationarity in the strict sense implies the wide sense stationarity, but the opposite is not generally true: there are SSS processes that are not WSS, as the following example shows:

Example 17 Let X_n be a zero mean stationary stochastic process and $Y_n = (-1)^n X_n$. It is easy to check that

$$\mathbb{E}\{Y_n\} = (-1)^n \mathbb{E}\{X_n\} = 0 \tag{1.65}$$

and

$$R_Y[n_1, n_2] = (-1)^{n_1+n_2} R_X[n_2 - n_1] \tag{1.66}$$

Since $(-1)^{n_1+n_2} = (-1)^{n_2-n_1}$, it follows

$$R_Y[n_1, n_2] = (-1)^{n_2-n_1} R_X[n_2 - n_1] \tag{1.67}$$

which only depends on $n_2 - n_1$. Therefore, Y_n is WSS. However, we can check that, for example,

$$\mathbb{E}\{Y_n^3\} = (-1)^n \mathbb{E}\{X_n^3\} \tag{1.68}$$

which, in general, depends on n. For example, if, for all n, X_n follows the discrete distribution

$$p_X[k] = \frac{3}{4}\delta[k-1] + \frac{1}{4}\delta[k+3] \tag{1.69}$$

we have $\mathbb{E}\{X_n\} = 0$ and $\mathbb{E}\{X_n^3\} = -3/2$, then

$$\mathbb{E}\{Y_n^3\} = -\frac{3}{2}(-1)^n \tag{1.70}$$

which depends on n. Therefore, some statistics from X_n are not invariant to a time shift and, thus, X_n is not SSS.

Also, note that all IID processes are SSS, with autocovariance

$$C_X[n] = \mathbb{E}\{(X_n - \mu_X)(X_0 - \mu_X)^*\} = \sigma_X^2 \delta[n] \tag{1.71}$$

If the process is zero mean, we have

$$R_X[n] = \sigma_X^2 \delta[n] \tag{1.72}$$

In general, a SP whose autocovariance is a delta function is said to be white. All zero-mean IID processes are white, but the opposite is not true in general: some white processes may be non IID.

1.5.3.3 Jointly stationary processes.

The two-side stationary processes X_n and Y_n are said to be jointly stationary *in the strict sense* if their joint statistical properties do not vary with a displacement in time, that is, if, for any values of N and N', any value of m and any instants $n_0, \ldots, n_{N-1}$ and $n'_0, \ldots, n'_{N'-1}$,

$$P(X_{n_0}, \ldots, X_{n_{N-1}}, Y_{n'_0}, \ldots, Y_{n'_{N'-1}}) = P(X_{n_0+m}, \ldots, X_{n_{N-1}+m}, Y_{n'_0+m}, \ldots, Y_{n'_{N'-1}+m}) \tag{1.73}$$

Likewise, we will say that X_n and Y_n are jointly wide-sense stationary if both are WSS and for any integer values of n_1, n_2 and m we have

$$R_{XY}[n_1, n_2] = R_{XY}[n_1 + m, n_2 + m] \tag{1.74}$$

which allows us to describe the cross-correlation function using a single variable

$$R_{XY}[n] = R_{XY}[m + n, m] \tag{1.75}$$

1.5.4 Ergodicity

If the expected value of a random variable X is unknown, it can be estimated as the sample average of K independent realizations $x_0, \ldots, x_{K-1}$

$$\mathbb{E}\{X\} \approx \frac{1}{K} \sum_{k=0}^{K-1} x_k \tag{1.76}$$

This estimation is supported by the (weak and strong) laws of large numbers, that guarantee, under quite general conditions, the convergence of the sample average to the mean as the number of samples goes to infinity.

We can estimate the mean of a stochastic process in the same way, by averaging multiple realizations. However, in many practical applications, this is not possible because only a single sample of the process is available.

If a stochastic process X_n is WSS, the mean is constant, and we can try to estimate it as the average of all samples from a single realization. Thus, we may wonder if time averages of the SP converge to the mean. A process satisfying this property is called mean-ergodic.

Definition 12 (Mean-ergodicity)

A WSS process X_n with mean μ_X is mean-ergodic if the time average

$$S_N = \frac{1}{2N+1} \sum_{n=-N}^{N} X_n \tag{1.77}$$

converges in squared mean to the mean, μ_X, that is

$$\lim_{N\to\infty} \mathbb{E}\left\{ |S_N - \mu_X|^2 \right\} = 0 \tag{1.78}$$

Note that S_N is itself a random variable with the same mean as the process

$$\mathbb{E}\{S_N\} = \frac{1}{2N+1} \sum_{n=-N}^{N} \mathbb{E}\{X_n\} = \mu_X \tag{1.79}$$

and the expectation in (1.78) is the variance. Expanding the expression of this variance, it is straightforward to obtain the following condition for ergodicity:

Theorem 3 *A WSS process X_n with mean μ is mean-ergodic iff*

$$\lim_{N\to\infty} \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} C_X[n-m] = 0 \tag{1.80}$$

Proof Noting that

$$\begin{aligned} \mathbb{E}\{|S_N - \mu_X|^2\} =& E\left\{ \left| \frac{1}{2N+1} \sum_{n=-N}^{N} (X_n - \mu_X) \right|^2 \right\} = \\ =& \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} \mathbb{E}\{(X_n - \mu_X)(X_m^* - \mu_X^*)\} \\ =& \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} C_X[n-m] \end{aligned} \tag{1.81}$$

the proof is completed. □

We can use the above theorem to state a sufficient condition for ergodicity, that is satisfied by many WSS processes of practical interest:

Theorem 4 *If the covariance function of a WSS process X_n is absolutely summable, that is*

$$\sum_{n=-\infty}^{\infty} |C_X[n]| < \infty \tag{1.82}$$

the process is mean-ergodic.

Proof Since $C_X[n]$ is absolutely summable, $B = \sum_{n=-\infty}^{\infty} |C_X[n]|$ is finite, and we can use B to upper bound the variance in (1.83) as

$$\left|\mathbb{E}\{|S_N - \mu_X|^2\}\right| \leq \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} |C_X[n-m]| = \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} B = \frac{B}{2N+1} \tag{1.83}$$

Since the upper bound converges to zero for large N, the process in mean-ergodic. □

Note that all white processes (and, in particular, IID processes) satisfy (1.82). Thus, they are mean-ergodic.

Example 18 Consider the stationary process given by $X_n = Y_n + 0.8Y_{n-1}$, where Y_n is a Gaussian IID process of unit variance. It is easy to see that X_n is a stationary process with zero mean and autocovariance

$$C_X[n] = 1.64\delta[n] + 0.8\delta[n-1] + 0.8\delta[n+1] \tag{1.84}$$

Since the autocovariance vanishes for $|k| > 1$,the process is mean-ergodic. Figure 1.4 represents the first samples (from $n = 0$ to $n = 50$) of 4 realizations. The average of 50 realizations is represented in the lower part. The 50 sample average of each realization is shown on the right margin. As a consequence of stationarity, the average signal in the bottom approaches a constant signal around the mean, which is zero. As a consequence of the ergodicity, the averages of each signal also approaches the mean.

1.5.4.1 Ergodicity in the autocorrelation

In a similar way, ergodicity is defined in autocorrelation. A WSS process X_n is autocorrelation-ergodic if

$$R_N = \frac{1}{2N+1} \sum_{m=-N}^{N} X_{m+n} X_m^* \tag{1.85}$$

converges to $R_X[n]$ in squared mean. In general, X_n is autocorrelation-ergodic if, for every value of m, the process $Z_n[m] = X_{m+n} X_m^*$ is mean-ergodic.

1.5.5 Linear systems with stochastic inputs

Let X_n be a stochastic process that is the input to a linear time-invariant system with impulse response $h[n]$. The output process Y_n is given by

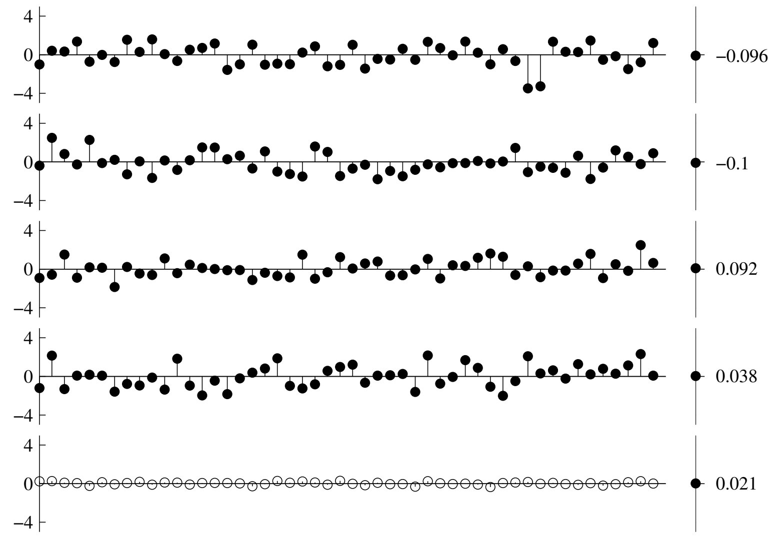

Fig. 1.4 4 realizations of the stochastic process of the example 18 (top), the average of 50 realizations (bottom) and the averages of the represented samples (right). Because of the stationarity, the average signal approaches a constant of value equal to the mean of the process. Because of the ergodicity, time averages also approximate the mean.

$$Y_n = X_n * h[n] \tag{1.86}$$

In general, the statistical characterization of Y_n is not easy, except in particular cases (such as the Gaussian), but we can find some useful expressions for some first and second order moments.

For example, the mean of the output process is

$$\begin{aligned}\mu_Y[n] =& \mathbb{E}\{Y_n\} = \mathbb{E}\{h[n] * X_n\} = h[n] * \mathbb{E}\{X_n\}\\ =& h[n] * \mu_X[n],\end{aligned} \tag{1.87}$$

the cross-correlation of the input and output is

$$\begin{aligned}R_{YX}[n_1,n_2] =& \mathbb{E}\{Y_{n_1} X^*_{n_2}\} = E\left\{X^*_{n_2} \sum_{k=-\infty}^{\infty} X_k h[n_1-k]\right\}\\ =& \sum_{k=-\infty}^{\infty} h[n_1-k] R_X[k,n_2]\\ =& h[n_1] * R_X[n_1,n_2]\end{aligned} \tag{1.88}$$

and the autocorrelation of the output is

$$
\begin{aligned}
R_Y[n_1,n_2] =& \mathbb{E}\{Y_{n_1}Y^*_{n_2}\} = E\left\{Y_{n_1}\sum_{k=-\infty}^{\infty} h^*[k]X^*_{n_2-k}\right\} \\
=& \sum_{k=-\infty}^{\infty} h^*[k]R_{YX}[n_1,n_2-k] \\
=& R_{YX}[n_1,n_2] * h^*[n_2]
\end{aligned}
\tag{1.89}
$$

Combining (1.88) and (1.89), we get

$$
R_Y[n_1,n_2] = h[n_1] * R_X[n_1,n_2] * h^*[n_2] \tag{1.90}
$$

In the above equation, there are two convolution operations. We must perform the first on the variable n_1 and the second on the variable n_2. Equations (1.87) and (1.90) show that both the mean and the autocorrelation of the output depend exclusively on the mean and the autocorrelation of the input, respectively, as well as on the impulse response of the system.

1.5.5.1 Stationary processes.

We can use the general expressions for the output mean and the output autocorrelation to derive specific expression for WSS processes. If X_n is stationary, $\mu_X[n] = \mu_X$, and

$$
\mu_Y = \mu_X \sum_{n=-\infty}^{\infty} h[n] \tag{1.91}
$$

Also, by the properties of the convolution operator, for any $m \in \mathbb{Z}$ we can write

$$
\begin{aligned}
R_Y[n_1+m,n_2+m] &= h[n_1] * R_X[n_1+m,n_2+m] * h^*[n_2] = h[n_1] * R_X[n_1,n_2] * h^*[n_2] \\
&= R_Y[n_1,n_2]
\end{aligned}
\tag{1.92}
$$

and, therefore Y_n is also stationary. In such case, we can express the autocorrelation as a function of a single variable. By calling $n = n_1 - n_2$, the cross-correlation of the input and output is

$$
\begin{aligned}
R_{YX}[n_1,n_2] =& h[n_1] * R_X[n_1-n_2] = \sum_{k=-\infty}^{\infty} h[n_1-k]R_X[k-n_2] \\
=& \sum_{k=-\infty}^{\infty} h[n_2+n-k]R_X[k-n_2] = \sum_{k=-\infty}^{\infty} h[n-k]R_X[k] \\
=& h[n] * R_X[n]
\end{aligned}
\tag{1.93}
$$

Finally, the autocorrelation of the output is

$$
\begin{aligned}
R_Y[n] =& R_Y[n_1,n_2] = R_{YX}[n_1,n_2] * h^*[n_2] \\
=& \sum_{k=-\infty}^{\infty} h^*[k] R_{YX}[n_1-n_2+k] = \sum_{k=-\infty}^{\infty} h^*[k] \sum_{\ell=-\infty}^{\infty} h[n+k-\ell] R_X[\ell] \\
=& \sum_{\ell=-\infty}^{\infty} R_X[\ell] \sum_{k=-\infty}^{\infty} h^*[k] h[n+k-\ell] = R_X[n] * h[n] * h^*[-n] \\
=& R_X[n] * r_h[n]
\end{aligned}
\tag{1.94}
$$

where $r_h[n] = h[n] * h^*[-n]$.

Example 19 Let X_n be a unit variance white process. The stationary process Y_n resulting from passing X_n through a linear and invariant and causal filter, given by the difference equation

$$Y_n = 0.6 \cdot Y_{n-1} + X_n \tag{1.95}$$

is stationary. According to (1.87), since the impulse response of this system is $h[n] = 0.6^n u[n]$, the process mean will be,

$$\mu_Y[n] = h[n] * \mu_X[n] = 0 \tag{1.96}$$

and its autocorrelation

$$R_Y[n] = R_X[n] * h[n] * h[-n] = \frac{0.6^{|n|}}{0.64} \tag{1.97}$$

1.5.6 Gaussian processes

Definition 13 (Gaussian Process) A real SP X_n is a Gaussian Process (GP) if, for any value of N and any arbitrary set of N time instants $\{n_0,\ldots,n_{N-1}\}$, the pdf of order N,

$$p_{X_{n_0},\ldots,X_{n_{N-1}}}(x_0,\ldots,x_{N-1}) \tag{1.98}$$

is Gaussian.

Gaussian processes have several interesting properties:

1. Since the parameters of the (multidimensional) Gaussian distribution are the mean and the covariance matrix, any GP is completely characterized by its mean and its autocorrelation function.
2. Thus, if a GP is WSS, it is SSS.
3. Since any linear combination of Gaussian random variables is Gaussian, if a GP is the input to a linear time-invariant system, the output is also GP characterized by the output mean and the output autocorrelation driven by Eqs. (1.87) and (1.108).

1.5.7 Power Spectral Density

The Power Spectral Density is the basic tool to extend the Fourier Analysis of signals to stochastic processes.

Since the realizations of a stochastic process are signals, it is tempting to define the Fourier Transform of a process stochastic X_n as a new process whose realizations are the Fourier transforms of the realizations of X_n, x_n. However, there is an insurmountable difficulty in this definition: the realizations of the stationary stochastic processes are signals of finite and non-zero average power and infinite energy and, therefore, they do not have a Fourier Transform.

However, in general, the Fourier Transform of a truncated process (limited in time) can be computed. Let X_n be a stochastic process of mean $\mu_X[n]$ and autocorrelation $R_X[n_1, n_2]$, and consider the truncated process $X_{N,n}$ given by

$$X_{N,n} = \begin{cases} X_n, & \text{if } |n| \leq N \\ 0, & \text{if } |n| > N \end{cases} \tag{1.99}$$

The Fourier Transform of $X_{N,n}$, given by

$$X_N\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} X_{N,n} e^{-j\omega n} = \sum_{n=-N}^{N} X_n e^{-j\omega n} \tag{1.100}$$

is, in turn, a stochastic process of mean

$$\mathbb{E}\{X_N\left(e^{j\omega}\right)\} = \sum_{n=-N}^{N} \mu_X[n] e^{-j\omega n} \tag{1.101}$$

and squared mean

$$\mathbb{E}\left\{\left|X_N\left(e^{j\omega}\right)\right|^2\right\} = E\left\{\left|\sum_{n=-N}^{N} X_n e^{-j\omega n}\right|^2\right\} \tag{1.102}$$

If the process is stationary, the mean quadratic value grows indefinitely with N until it becomes infinity, but we can avoid this effect by normalizing (1.102) with respect to the length of the integration interval, $2N$. We define the *power spectral density* or PSD of a stochastic process to the limit of the expression above.

Definition 14 (Power Spectral Density) The power spectal density (PSD) of a stochastic process X_n is defined as

$$S_X(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{2N+1} \mathbb{E}\{|X_N(e^{j\omega})|^2\} \tag{1.103}$$

Note that, by definition, the PSD is always real and not negative: for all ω,

$$S_X\left(e^{j\omega}\right) \geq 0 \tag{1.104}$$

Also, by the properties of the Fourier Transform, the PSD of a real stochastic process is an even function.

The expression in (1.103) is not practical, but we can find an alternative expression for stationary processes, that allows to compute the PSD using the autocorrelation, using a classical result from Wiener and Khinchine that we state without proof.

Theorem 5 (Wiener-Khinchine theorem)
If X_n is WSS and $R_X[n]$ is absolutelly summable,

$$S_X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} R_X[n]e^{-j\omega n} \tag{1.105}$$

Therefore, the power spectral density of a stationary process X_n is equal to the Fourier Transform of its autocorrelation function and, conversely

$$R_X[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(e^{j\omega})e^{j\omega n}d\omega \tag{1.106}$$

Example 20 White processes
If X_n is a white WSS process with autocorrelation $R_X[k] = \sigma_X^2\delta[k]$, its power spectral density is constant at all frequencies.

$$S_X(e^{j\omega}) = \sigma_X^2 \tag{1.107}$$

1.5.7.1 Power spectral density at the output of linear systems

It is interesting to analyze the power spectral density of stochastic processes that pass through linear filters. Starting from the expression for the output autocorrelation obtained in (1.90), we can write

$$R_Y[n] = R_X[n] * h[n] * h^*[-n] \tag{1.108}$$

and applying the convolution property to (1.108), we get

$$S_Y\left(e^{j\omega}\right) = S_X\left(e^{j\omega}\right)\left|H\left(e^{j\omega}\right)\right|^2 \tag{1.109}$$

References

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