Math Competition

Eighth Grade

Sunday, 18 October 2009

1. In square meters, what is the area of a rhombus with both diagonals measuring $10\sqrt{2}$ cm?
2. Sherry and Jenny are trying to find each other but they both have a terrible sense of direction. Sherry starts out exactly ninety meters west of Jenny. She walks thirty meters north, turns and walks eighty meters west, then turns again and walks ten meters south. Jenny walks fifty meters south, then turns and walks seventy meters east. How far apart in meters are they now?
3. How many solutions does the equation $x = \sqrt{16}$ have?
4. The average of Steve's five tests is 99. The average of Bill's three tests is 17. What average must Eric get on his two tests such that the average of the ten tests is at least 73?
5. Point B lies in the same plane as line segment $\overline{AC}$ such that B is nine units away from C, eleven units away from A, and not on the line $\overline{AC}$. If the length of $\overline{AC}$ is an integer, how many possible values are there for the length of $\overline{AC}$?
6. How many integers from 1 to 100 are relatively prime to 140?
7. Kelvin Wang sneaks into an intergalatic convention. Two species of aliens are attending: Dinasauri (singular: Dinosaurus) and Yahaos. Given that each Dinosaurus has seven heads, each Yahao has fifteen heads, and Kelvin correctly counted a total of eighty-two heads, find the total number of aliens taking part in this convention (not including Kelvin).
8. Suppose a right triangle has a hypotenuse of length 3. Given that its area is 1, find its perimeter.
9. Evaluate $20_3 + 2_3 + 0.2_3 + 0.02_3$. Express your answer as a fraction in base 10.
10. Mike Sun has a floating chair that moves at 3 mph in water with no current. He gets onto his floating chair, and facing north, rides a current going south at 1 mph. After x minutes, he turns his chair around 180^o and rides the same current going south. He notices that he arrives back to his starting point exactly one hour after he first left. Find x.
11. At what time between 4:00 and 5:00 do the minute hand and the hour hand overlap on an analog clock? Round your answer to the nearest minute.
12. A triangle has three positive integral sides, $x-10$, x, and $x+10$. How many such triangles are obtuse?
13. Kevin solves ten math problems on the first day. Each day he increases the number of math problems solved per day by three problems (so on the second day he solves thirteen problems, sixteen problems on the third day, etc). The first day is Monday. By the end of Sunday, how many problems will he have solved?
14. Let x be the answer to this problem. What is $x^2 - 21x + 121$?
15. Compute the sum of all distinct values for x such that $\dfrac{x^3 - 3x + 2}{x^3 - 7x + 6} = 0$.
16. Alex's bus number is 534. He once noticed that it had three consecutive positive digits (3, 4, and 5, though not necessarily in that order.) Bored, he correctly computed the total number of 3-digit numbers that contain three consecutive positive digits. What was the result of his computation?
17. How many integers from 1 to 1000 are multiples of 2 and 3 but not 5?
18. Evaluate $\left(-\dfrac{1}{2}\right)^{-1^{100}}$.

19. Let $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers. Given that $f(1) = 15$, $f(2) = 24$, and $f(3) = 35$, compute $f(6)$.

20. Two sides of a right triangle are 3 and 4. Find all possible areas of the triangle.

21. If x and y are real numbers satisfying $x^2 + y^2 = 1$, find the greatest possible value of $x + y$.

22. Let $i = \sqrt{-1}$. Evaluate $i^{1^2} + i^{2^2} - i^{3^2} - i^{4^2} + i^{5^2} + i^{6^2} - i^{7^2} - i^{8^2} + \cdots + i^{2009^2} + i^{2010^2}$.

23. What is the minimum distance from the point $(0, 0)$ to a point on the line $5x + 12y = 60$?

24. If a number has more than one digit, then it decreases when we take the sum of its digits. Therefore, if we constantly repeat the process of adding the digits of the number, we ultimately end up with a single digit. For any integer x, denote this single digit by $f(x)$. Find $f(6^{3^2})$.

25. If a circle centered at (1, 2) is tangent to the line $y = -x - 3$, what is the radius of the circle?

26. Peter runs two laps around a track. He runs the first lap in 30 yards per minute. He wants to set his goal for his average pace of the two laps to x yards per minute. What is the smallest $x > 30$ such that no matter how fast he decides to run, he can never achieve his goal?

27. Two real numbers a and b are such that $a = 4b$ and $\log_{10} a + \log_{10} b = \log_{10}(a + b)$. Find $a \cdot b$.

28. Suppose that $A \cdot B \cdot C + D \cdot E - F = 28$, where A, B, C, D, E, and F are the digits 1 through 6 with no digit repeated. If $A > B > C$ and $D > E$, what is the six-digit number $ABCDEF$?

29. Chan has twenty-seven unit cubes, which he makes into a 3 by 3 by 3 cube. He paints the exterior red. He throws the cube into the air and it splits into twenty-seven cubes that are all resting on the ground. What is the expected number of red faces that are visible?

30. What is the remainder when 2^{32} is divided by 25?

31. Pavel slices a perfectly spherical orange with a 2-inch radius into two parts. He notices that when each part is placed on its freshly-cut surface, one piece is one inch tall and the other is three inches tall. What is the radius (in inches) of the freshly-cut surfaces?

32. Find the sum of all positive integers n for which $\dfrac{10n + 77}{n + 1}$ is also an integer.

33. In hexagon $ABCDEF$, all angles between adjacent sides are equal. If $\overline{AB} = 8$, $\overline{CD} = 15$, and $\overline{DE} = 17$, compute $\overline{AF}$.

34. What is the largest n such that the difference $100! - 99!$ is divisible by 10^n? ($n!$ is $n \cdot (n - 1) \cdots 1$)

35. Mike was looking at the numbers one day and he stumbled upon the number 236. He decided to call this number a "growing" number, because the integers increased in value from left to right. How may four-digit growing numbers are there?

36. Find all values of x for which $3x^2 - 39x + 126 < 0$.

37. Victoria and Weili are competing in a hot dog eating contest, which consists of five matches. Because he fasted for a week, Weili has a $\frac{4}{5}$ chance of winning a match and Victoria has a $\frac{1}{5}$ chance of winning a match (there are no ties). If the probability that Victoria will win at least two matches and at most four matches is $\frac{p}{q}$, where p and q are relatively prime, what is $p + q$?

38. Find the smallest positive integer n such that 7 divides $\underbrace{111 \cdots 11}_{\text{n}}$.

39. Jordan and Ian arrange a meeting to talk about Star Wars. Jordan plans to arrive at a random time between 1 o'clock PM and 4 o'clock PM, and will wait for an hour for Ian. Ian, on the other hand, plans to arrive at a random time between 12 o'clock PM and 5 o'clock PM, but will wait only 30 minutes for Jordan. What are the chances that the two meet to talk about Star Wars?

40. Find the minimum positive integer N such that $x^4 + N$ can be factored into two quadratic polynomials with integer coefficients.

41. Triangle $\triangle ABC$ has sides $\overline{AB} = 6$, $\overline{BC} = 8$, and $\overline{CA} = 10$. There is a point P such that it is distance R from A, B, and C. What is πR^2?

42. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If x is a randomly chosen number between 0 and 42, compute the probability that $\lfloor 17x \rfloor = 17\lfloor x \rfloor$.

43. If the equation $x^2 + 4x + y^2 + 6y = m$ has exactly one real solution (x, y), find m.

44. Yumi was playing a computer game when she noticed her score was a three-digit palindrome. She also noticed she was 382 points away from another palindromic score. What is the highest score she could have had at that moment?

45. Suppose a dartboard with radius 1 has an equilateral triangle inscribed in it. Find the probability that when a dart is thrown, it lands inside the triangle. All darts land on the dartboard and are equally as likely to land anywhere on it.

46. The polynomial $x^5 + x + 1$ can be expressed in the form $(x^2 + ax + b)(x^3 + cx^2 + dx + e)$, for some integers a, b, c, d, and e. Compute $a + b + c + d + e$.

47. Suppose a triangle $\triangle ABC$ has sides $\overline{AB} = \sqrt{73}$, $\overline{BC} = 10$, and $\overline{CA} = 9$. Points D and E lie on $\overline{AC}$ such that $\overline{AD} = \overline{DE} = \overline{EC}$. Point F is the midpoint of $\overline{BC}$. Let P be the intersection of lines $\overline{BE}$ and $\overline{DF}$. If $\overline{BD}$ is perpendicular to $\overline{AC}$, what is the sum of the areas of $\triangle BPF$ and $\triangle DPE$?

48. Suppose $f(x) = x^2 - 10x + 28$. Find the two integer solutions to the equation $f(f(x)) = x$.

49. A scientist develops a procedure to test for a terrible disease, which occurs in 1% of the population. If the subject is indeed diseased, the test will be report this 95% of the time. However, even if a subject is healthy, the test will give a false positive x% of the time. If a subject tests positive on the test, there is a 50% chance that this subject has the disease. What is x, rounded to two decimal places?

50. Suppose the sequence of numbers $a_0, a_1, \ldots, a_n$ are defined as

$$2^{a_k} = 2^{2^k} + 1$$

What is the integer nearest to $a_0 + a_1 + \ldots + a_6$?