8th Grade Competition Solutions

19 October 2008

1. Before taking the USAMO, a student notices that he has two bags of Doritos, two bags of Fritos, three cans of soda, a bottle of water, four bags of skittles, half a box of Cheez-its, two bags of Cheetos, and five mini Hersheys bars on his desk. Each bag of Doritos has 130 calories, each bag of Fritos has 120 calories, each can of soda has 140 calories, each bag of Skittles has 60 calories, each box of Cheez-its has 1200 calories, each mini Hersheys bar has 60 calories, each bag of Cheetos has 110 calories, and each bottle of water has no calories. If, during the test, the student consumes everything on his desk, how many calories will he have consumed?
Answer: 2280

$$2\cdot 130+2\cdot 120+3\cdot 140+1\cdot 0+4\cdot 60+\tfrac{1200}{2}+2\cdot 110+5\cdot 60 = 260+240+420+0+240+600+220+300 = 500+660+1120 = 2280.$$

2. $8+88+888+8888+88888 = ?$
Answer: 98760
$8+88+888+8888+88888 = 8(1+11+111+1111+11111) = 8\cdot 12345 = 98760.$

3. On Monday, Matt hits a golf ball 100 yards. Disappointed, he goes to the gym and lifts weights for two hours. The next day, he can hit the golf ball 25 yards further than the day before. Still disappointed, he goes back to the gym that night and lifts weights again. Wednesday, he could hit the ball 25 yards further than on Tuesday. If this process continues through Thursday night, then how many yards will Matt be able to hit a golf ball on Friday of that week?
Answer: 200
On Friday, Matt will be able to hit the ball $100+4\cdot 25 = 200$ yards.

4. Josh and Kun-Soo have 2008 Frosted Flakes each. Josh gives Kun-Soo half of his flakes. After this, Kun-Soo gives a quarter of his new number of flakes back to Josh. How many flakes does Josh have now?
Answer: 1757
After the first exchange, Josh has $\dfrac{2008}{2} = 1004$ flakes and Kun-Soo has $2008+1004 = 3012$ flakes. After the second exchange, Josh will have $1004+\dfrac{3012}{4} = 1004+753 = 1757$ flakes.

5. If someone buys 3 cokes at \$1.77 each, and tax is 35 cents total, how much change would that person get from a 10 dollar bill?
Answer: \$4.34
The person will end up paying $3\cdot \$1.77+\$.35 = \$5.66$, and will thus receive $\$10-\$5.66 = \$4.34$ in change.

6. Compute $123456789+876543211$. Write your answer in the form 10^a, where a is a whole number.
Answer: 10^9
$123456789+876543211 = 1{,}000{,}000{,}000 = 10^9$.

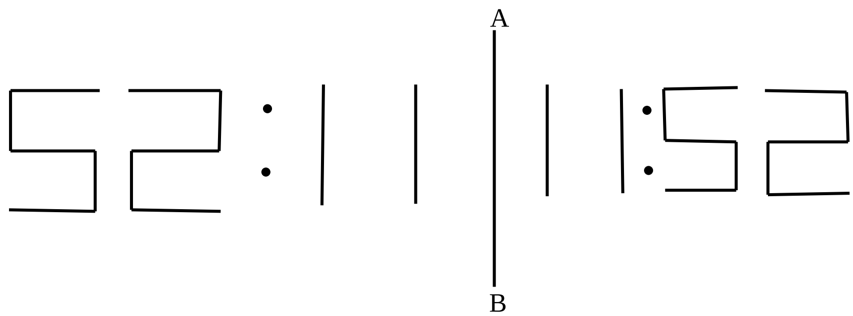
7. David Rush is in a hurry to get to class. His dorm at Philips Exeter is 840 meters away from his first class. He runs the first quarter of the distance at a brisk 7 meters per second. After this, he (instantly) slows down to 3 meters per second and finishes running at this rate. How many seconds did it take him to get to class?
Answer: 240
The first quarter of the distance (210 meters) took David $\dfrac{210}{7} = 30$ seconds. The rest of the run took him $\dfrac{630}{3} = 210$ seconds. Thus, the whole trip took him $30+210 = 240$ seconds.

8. James wants to paint his 12×9 ft^2 bedroom wall. However, he does not need to paint his bedroom door, which has dimensions of 3×6 ft^2. If each square foot of wall requires 2 milliliters of paint, then how many milliliters of paint does James need to complete his job?

Answer: 180
The area of wall that must be painted is $12 \cdot 9 - 3 \cdot 6 = 108 - 18 = 90$ ft^2. This will require $90 \cdot 2 = 180$ milliliters of paint.

9. A square has perimeter 104. What is the area of the square?
Answer: 676
The side length of the square must be $\frac{104}{4} = 26$, and so the area of the square is $26^2 = 676$.

10. Angel sees the time 52:11 on his digital clock. He then realizes this does not make sense because he is looking into a mirror. What time is it, assuming his clock is correct?
Answer: 11:52
The image on the left is reflected over line segment $\overline{AB}$.



11. n minutes before the time in problem 10, the time in the mirror matched the time in real life. What is the smallest possible value of n?
Answer: 41
The last time the time in the mirror matched the time in real life was at 11:11, which was $52 - 11 = 41$ minutes ago.

12. Find $76648 \div 13$.
Answer: 5896
By straight calculation, $76648 \div 13 = 5896$.

13. If $a@b = \dfrac{a^2 - 2b^2}{a-b}$, compute $3@4 - 4@7$.
Answer: $\frac{-13}{3}$
By definition, $3@4 - 4@7 = \dfrac{3^2 - 2 \cdot 4^2}{3-4} - \dfrac{4^2 - 2 \cdot 7^2}{4-7} = \dfrac{9-32}{-1} - \dfrac{16-98}{-3} = 23 - \dfrac{82}{3} = \dfrac{69-82}{3} = \dfrac{-13}{3}$.

14. If $T(x,y) = \dfrac{x^2}{y^x}$, what is $T(T(1,2), T(2,1))$?
Answer: $\frac{1}{8}$
By definition, $T(T(1,2), T(2,1)) = T\left(\dfrac{1^2}{2^1}, \dfrac{2^2}{1^2}\right) = T\left(\dfrac{1}{2}, 4\right) = \dfrac{(\frac{1}{2})^2}{2} = \dfrac{1}{4 \cdot 2} = \dfrac{1}{8}$.

15. Simplify $\dfrac{7+\imath}{1-2\imath}$ given that $\imath = \sqrt{-1}$.
Answer: 1 + 3$\imath$
$\dfrac{7+\imath}{1-2\imath} \cdot 1 = \dfrac{7+\imath}{1-2\imath} \cdot \dfrac{1+2\imath}{1+2\imath} = \dfrac{5+15\imath}{5} = 1 + 3\imath$

16. What is the probability of rolling an 11 with a pair of fair 6-sided dice?
Answer: $\frac{1}{18}$

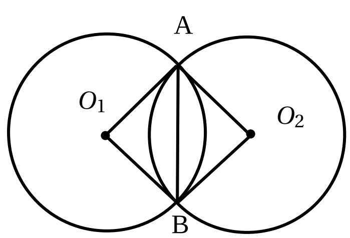
We must roll a 5 and a 6 in any order. That has probability $2 \cdot \dfrac{1}{6} \cdot \dfrac{1}{6} = \dfrac{1}{18}$ of happening.

17. If $\gamma = \frac{1}{2}$, then evaluate $\gamma + \gamma^2 + \gamma^3 + \gamma^4$.
Answer: $\frac{15}{16}$
$\gamma + \gamma^2 + \gamma^3 + \gamma^4 = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} = \dfrac{8+4+2+1}{16} = \dfrac{15}{16}$.

18. Jenny and Hyungie are playing a variant of baseball. Jenny has a $\frac{4}{7}$ chance of winning an inning while Hyungie has a $\frac{3}{7}$ of winning the inning. (There are no ties.) What is the probability that Hyungie will be winning after two innings?
Answer: $\frac{9}{49}$
In order to be winning after two innings, Hyungie must win both innings. The chances of this happening are $\dfrac{3}{7} \cdot \dfrac{3}{7} = \dfrac{9}{49}$.

19. If 96346791 beads are split among 12 people so that each person has the same whole number of beads, how many are left over?
Answer: 3
The sum of the digits of 96346791 is 45, so it is divisible by 3. If there were three less beads, the total number of beads would be divisible by 3 and also 4 (since it would end in a two- digit number that is divisible by 4), and there would be no beads left over. So, when 96346791 beads are split among 12 people, there will be 3 beads left over.

20. The circles centered at O_1 and O_2 both have radii of length 4. They intersect at A and B, and $\angle AO_1B = \angle AO_2B = 90^o$. The area of the intersection of both circles can be written in the form $p\pi + q$. Find $p + q$.
Answer: –8
Draw line $\overline{AB}$. The area of half the intersection is the difference between the area of a 90^o sector of a circle and the area of $\triangle AO_2B$. That difference is $\frac{1}{4} \cdot 16\pi - \frac{1}{2} \cdot 4 \cdot 4 = 4\pi - 8$, and the area of the entire intersection is $2(4\pi - 8) = 8\pi - 16$. Then our answer is $8 - 16 = -8$.



21. How many 1s does Nikhil need to write if he writes all the page numbers for a book that has 416 pages and starts on page 1?
Answer: 189
The digit 1 appears in the one's place $\lfloor \frac{416}{10} \rfloor + 1 = 42$ times. It appears in the ten's place $4 \cdot 10 + 7 = 47$ times. It appears in the hundred's place 100 times. Thus, Nikhil must write $42 + 47 + 100 = 189$ 1's.

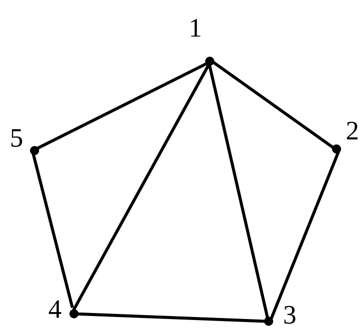
22. Find the largest integer value of a for which the statement "6144 is divisible by 2^a" is true.
Answer: 11
$6144 = 3 \cdot 2048 = 3 \cdot 2^{11}$, so the largest value of a is 11.

23. Compute all real x such that $x^6 - x^3 = 0$.
Answer: 0 and 1
$x^6 - x^3 = 0 \Rightarrow x^3(x^3 - 1) = 0 \Rightarrow x^3(x - 1)(x^2 + x + 1) = 0$. The first two factors contribute the solutions 0 and 1, respectively. The third factor contributes no real solutions because the discriminant in the quadratic formula, $b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3$, is negative.

24. Express $\dfrac{b^3-1}{b-1}$ in base b. Write your answer without the subscript b.
Answer: 111
$\dfrac{b^3-1}{b-1} = b^2+b+1$ which, when written in base b, is 111.

25. Nikhil is writing the page numbers in a book that has 186 pages. After a certain page number, Nikhil notices that he has written the same number of digits as he still has to write in order to finish numbering all 186 pages. What page number is this?
Answer: 111
The one-digit page numbers contribute 9 digits, the two-digit page numbers contribute $2\cdot 90 = 180$ digits, and the three-digit page numbers contribute $3\cdot 87 = 261$ digits. When Nikhil is halfway done, he has written $\dfrac{9+180+261}{2} = \dfrac{450}{2} = 225$ page numbers. This means he has $\dfrac{225}{3} = 75$ three-digit page numbers left to write, which means that he has written $186-75 = 111$ page numbers.

26. A construction company must pave every road in Aganon, a region shown below. There are five cities in Aganon. If every road must be traversed exactly once, then the construction company may start in only two cities, cities a and b. Find $10a+b$, given that $a<b$.
Answer: 34
Define the $degree$ of a city to be the number of roads that have an endpoint at that city. For example, the degree of city 1 is 4. Now, if the construction company does not start or end in a certain city, then the company paves an even number of roads that go into that city because each time the company gets to that city, it comes in on one road and leaves on another road, thus visiting two roads each time it goes into that city. Thus, if all roads are to be paved, the cities with even degree cannot be the start- or end-points. The only cities with odd degree are cities 3 and 4, and our answer is $10\cdot 3+4 = 34$.



27. Austin runs an animal hospital that takes care of cats, dogs, and birds. The cats here only have three legs and one tail each, the birds have one leg and two tails each, and the dogs have four legs each but no tail. Victoria, the hospital inspector, walks in one day and counts 15 heads, 42 legs, and 13 tails. How many cats were there?
Answer: 3
Let c be the number of cats, b be the number of birds, and d be the number of dogs. Then $3c+b+4d = 42$, $c+2b = 13$, and $b+c+d = 15$. Subtracting four times the last equation from the sum of the first two, we find $(4c+3b+4d)-(4b+4c+4d) = 55-60 \Rightarrow -b = -5 \Rightarrow b = 5$. Plugging this into the second equation, we find $c = 13-2\cdot 5 = 3$, and we are done.

28. Scoring in a frisbee game requires a long pass, two medium passes, or four short passes. A team cannot combine different types of passes to score. If the probabilities of successfully completing these passes are 0.3, 0.6, and 0.8 respectively, which is the best strategy to score? Your answer should be exactly one of the words 'long,' 'short,' and 'medium.'
Answer: short

The probability of a team scoring using a long pass is .3. The probability of a team scoring using medium passes is $.6^2 = .36$. The probability of a team scoring using short passes is $.8^4 = .64^2 = .4096$. The best passes to use are short ones.

29. What are the last two digits of 7^{2008}?

Answer: 01

Since 7^4 ends in the two digits 01, we have $7^{2008} = (7^4)^{502} = (100a+1)^{502}$ for some positive integer a. (In fact, a is equal to 24.) It is not hard to see that this number will end in the digits 01.

30. Two positive integers are relatively prime if they share no factors other than 1. $\phi(n)$ is the defined as the number of numbers less than or equal to n that are relatively prime to n. For example, $\phi(6) = 1$. Find $\phi(120)$.

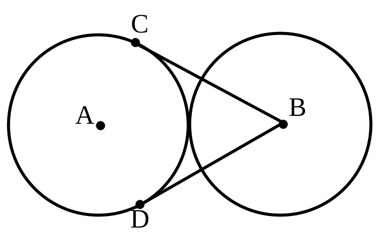
Answer: 32

If $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$, where each of the p_i are prime numbers and each of the e_i are positive integers, then $\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_k}\right)$. (Why?) Since $120 = 2^3 \cdot 3 \cdot 5$, we have $\phi(120) = 120\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = 120 \cdot \frac{4}{15} = 8 \cdot 4 = 32$. Alternatively, since any number not divisible by 2, 3, or 5 is relatively prime to 120, one could arrive at the answer by (very carefully) counting these non-multiples of 2, 3, and 5.

31. If you place a nickel on the table and surround the nickel with other nickels so that the inner nickel is touching every other nickel and the outer nickels are each touching the inner nickel and two other nickels, how many nickels are on the table?

Answer: 7

Consider two tangent circles. We will find $\angle CBD$, and then we will know there are at most $\frac{360}{\angle CBD}$ circles tangent to the center one. Consider $\triangle ABC$. $\angle ACB = 90^o$, and $\overline{AB} = 2\overline{AC}$ since the circles are congruent. Thus, $\triangle ABC$ is a 30-60-90 triangle with $\angle ABC = 30^o$. Then $\angle CBD = 60^o$, there are $\frac{360}{60} = 6$ circles tangent to the center one, and there are 7 coins on the table.



32. Let A be the set of 3-digit positive palindromes and let B be the set of 4-digit numbers that are divisible by 3. If $|X|$ is the number of elements in set X, then compute $|B| - |A|$.

Answer: 2910

A 3-digit palindrome is determined by its first two digits. Since there are 9 choices for the first digit and 10 for the second, there are $9 \cdot 10 = 90$ 3-digit palindromes. Next, there are $\frac{9999}{3} - \frac{1002}{3} + 1 = 3333 - 334 + 1 = 3000$ 4-digit numbers that are divisible by 3. Finally, $|B| - |A| = 3000 - 90 = 2910$.

33. How many distinct rearrangements of the letters in 'ippississim' are there?

Answer: 34650

Although there are 11 letters, 4 of them are i's, 4 are s's and 2 are p's. Thus, there are $\frac{11!}{4!4!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 11 \cdot 5 \cdot 3 \cdot 7 \cdot 6 \cdot 5 = 55 \cdot 630 = 34650$.

34. Ben is making a five hour lecture to a class of 20 students, $\frac{9}{10}$ of which are always paying attention. If the probability of a student who is not always paying attention being asleep at any time is $\frac{1}{5}$, then what is the expected number of minutes that the entire class will be awake?

Answer: 192

At any point in time, the chances the whole class is awake are the same as the chances of the two students who might fall asleep being awake; these chances are $\frac{4}{5}\cdot\frac{4}{5}=\frac{16}{25}$. Since the lecture lasts $5\cdot 60=300$ minutes, the expected amount of time that the entire class is awake is $300\cdot\frac{16}{25}=12\cdot 16=192$ minutes.

35. Evaluate $2+\dfrac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}}$.

Answer: $1+\sqrt{2}$

Let $x=2+\dfrac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}}$. Then $\dfrac{1}{x-2}=x\Rightarrow 1=x^2-2x\Rightarrow x^2-2x-1=0$. By the quadratic formula, the solutions to this equation are $\dfrac{2\pm\sqrt{4-4\cdot-1\cdot 1}}{2}=\dfrac{2\pm 2\sqrt{2}}{2}=1\pm\sqrt{2}$. Clearly, $x>0$, and our answer is $1+\sqrt{2}$.

36. By definition, $|a+b\imath|=\sqrt{a^2+b^2}$, and $\imath=\sqrt{-1}$. Find $\left|\dfrac{5+12\imath}{3-4\imath}\right|$.

Answer: $\frac{13}{5}$

$\left|\dfrac{5+12\imath}{3-4\imath}\right|=\dfrac{|5+12\imath|}{|3-4\imath|}=\dfrac{\sqrt{5^2+12^2}}{\sqrt{3^2+(-4)^2}}=\dfrac{\sqrt{169}}{\sqrt{25}}=\dfrac{13}{5}$.

37. The area of an isosceles triangle with base length 4 is 30. Find the perimeter of the triangle.

Answer: $4+2\sqrt{229}$

The height h to the vertex of the triangle satisfies $\frac{1}{2}\cdot 4\cdot h=30\Rightarrow 2h=30\Rightarrow h=15$. Then the length of a leg is $\sqrt{15^2+2^2}=\sqrt{225+4}=\sqrt{229}$. Since 229 is prime, this radical is fully simplified, and the perimeter of the triangle is $4+2\sqrt{229}$.

38. A sphere inside a cylinder is tangent to both of the cylinder's bases and touches the rounded part. If the area of the cylinder is 96, compute the area of the sphere.

Answer: 64

Let the radius of the sphere be r. Then the volume of the sphere is $\frac{4}{3}\pi r^3$, and the volume of the cylinder is $(\pi r^2)\cdot 2r=2\pi r^3$. The ratio of these volumes is $\dfrac{\frac{4}{3}\pi r^3}{2\pi r^3}=\dfrac{4}{3\cdot 2}=\dfrac{2}{3}$. Thus, the volume of the sphere is $\frac{2}{3}\cdot 96=64$.

39. Evaluate $\log_2(\log_4 16)$.

Answer: 1

$\log_2(\log_4 16)=\log_2(2)=1$.

40. Your mom is mixing batter for cookies one day. She puts in 100 milliliters of flour and .9 liters of water and starts stirring. However, .2 liters of the evenly-mixed mixture spills out. She accidentally replaces the lost batter with .1 liters of hydrochloric acid and keeps stirring. After the hydrochloric acid is evenly distributed, she pours 300 milliliters of the final mixture into a cup. How many milliliters of water are in the cup?

Answer: 240

The original mixture has 900 milliliters (henceforth abbreviated as mL) of water and 100 mL of flour. After some of the batter spills, there are $900\cdot .8=720$ mL of water and 80 milliliters of flour left. Then 100 mL of acid are added, making the total volume of the mixture 900 mL. Finally, a third of the mixture is poured into a cup. This portion of the mixture includes $720\div 3=240$ mL of water.

41. Find $\cos 30^o \tan 30^o$.

Answer: $\frac{1}{2}$

$\cos 30^o\tan 30^o=\cos 30^o\dfrac{\sin 30^o}{\cos 30^o}=\sin 30^o=\dfrac{1}{2}$. This last step can be seen using right triangle trigonometry on a 30-60-90 triangle.

42. Compute $\dfrac{1}{\sqrt{2}+\sqrt{1}}+\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{\sqrt{4}+\sqrt{3}}+\ldots+\dfrac{1}{\sqrt{99}+\sqrt{98}}+\dfrac{1}{\sqrt{100}+\sqrt{99}}$.

Answer: 9

$\dfrac{1}{\sqrt{m}+\sqrt{n}}\cdot 1=\dfrac{1}{\sqrt{m}+\sqrt{n}}\cdot\dfrac{\sqrt{m}-\sqrt{n}}{\sqrt{m}-\sqrt{n}}=\dfrac{\sqrt{m}-\sqrt{n}}{m-n}$. Therefore, the given expression is equal to $(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\ldots+(\sqrt{100}-\sqrt{99})=\sqrt{100}-\sqrt{1}=10-1=9$.

43. A and B are digits. Find the sum of all two-digit numbers AB (not the product AB but rather the two-digit number 10A + B) such that 836A7B is divisible by 44.

Answer: 68

In order for 836A7B to be divisible by 4, B must be either 2 or 6. If B is 2, then the sum of the odd-placed digits is 3 + A + 2 = 5 + A, and the sum of the even-placed digits is 8 + 6 + 7 = 21. We must have 21 – (5 + A) = 16 – A divisible by 11, so A must be 5. Similarly, if B is 6, then we must have 21 – (9 + A) = 12 – A divisible by 11, so A must be 1. Finally, the sum asked for is 52 + 16 = 68.

44. Express 31 in base 2.

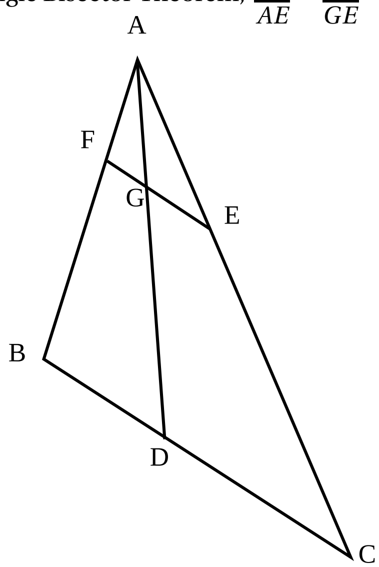
Answer: 11111

$31=16+8+4+2+1=1\cdot 2^4+1\cdot 2^3+1\cdot 2^2+1\cdot 2^1+1\cdot 2^0=11111_2$.

45. In the diagram, $\angle BAD=\angle DAC$ and $\angle AFE=\angle ABC$. Given that $\overline{AF}=5$, $\overline{AE}=10$, $\overline{FG}=3$, and $\overline{EC}=20$, compute $\overline{BC}$.

Answer: 27

First, $\overline{FE}//\overline{BC}$. Thus, $\triangle AFE$ is similar to $\triangle ABC$. By the given lengths, we know the ratio must be 10:30 or 1:3. By the Angle Bisector Theorem, $\dfrac{\overline{AF}}{\overline{AE}}=\dfrac{\overline{FG}}{\overline{GE}}\Rightarrow\dfrac{5}{10}=\dfrac{3}{\overline{GE}}\Rightarrow\overline{GE}=6$. Then $\overline{FE}=6+3=9$ and $\overline{BC}=3\cdot\overline{FE}=3\cdot 9=27$.



46. Compute $0\binom{7}{0}+1\binom{7}{1}+2\binom{7}{2}+3\binom{7}{3}+\ldots+6\binom{7}{6}+7\binom{7}{7}$.

Answer: 448

By straight calculation, it is equal to $0\cdot1+1\cdot7+2\cdot21+3\cdot35+4\cdot35+5\cdot21+6\cdot7+7\cdot1=0+7+42+105+140+105+42+7=49+350+49=448$. In general, $\sum_{k=0}^{n}k\binom{n}{k}=\sum_{k=0}^{n}k\dfrac{n!}{k!(n-k)!}=\sum_{k=0}^{n}\dfrac{n!}{(k-1)!(n-k)!}=n\sum_{k=1}^{n}\binom{n-1}{k-1}=n\sum_{k=0}^{n-1}\binom{n-1}{k}=n\cdot 2^{n-1}$.

47. Find all ordered pairs (p,q) of prime numbers with $p>q$ such that p^2-q^2 is a prime number.

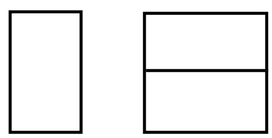
Answer: (3,2)

$p^2 - q^2 = (p+q)(p-q)$. Therefore, $p^2 - q^2$ is prime if and only if $p - q = 1$ and $p + q$ is a prime number. The only consecutive prime numbers that differ by one are 2 and 3. Since $2+3=5$ is prime, $(3,2)$ is the only solution.

48. Tolga is playing Young Duck in a game involving coin flips. Tolga wins if the coin comes up heads two times in a row while Young Duck wins after the coin comes up tails just once. If P(Y) is the probability that Young Duck wins, and P(T) is the probability of Tolga winning, then find P(Y) - P(T).
Answer: $\frac{1}{2}$
The probability of Tolga winning is $(\frac{1}{2})^2 = \frac{1}{4}$ because the coin must come up heads each of the first two flips – otherwise, Young Duck wins. Thus, the probability that Young Duck winning must be $1 - \frac{1}{4} = \frac{3}{4}$. Then P(Y) - P(T) = $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

49. Express $BCA_{13} + 15_6 + 1337_8$ in base 10.
Answer: 2776
$BCA_{13} + 15_6 + 1337_8 = (11 \cdot 13^2 + 12 \cdot 13^1 + 10 \cdot 13^0) + (1 \cdot 6^1 + 5 \cdot 6^0) + (1 \cdot 8^3 + 3 \cdot 8^2 + 3 \cdot 8^1 + 7 \cdot 8^0) = (1859 + 156 + 10) + (6+5) + (512 + 192 + 24 + 7) = 2025 + 11 + 735 = 2776$.

50. In how many ways can a 2×9 board be tiled with nine 2×1 tiles? Overlap of tiles is not allowed.
Answer: 55
In this solution, the board is viewed horizontally. Observe that if a tile is placed horizontally, there must be a tile that lies entirely above it or entirely below it – horizontally-placed tiles may not be staggered. (Otherwise, the area on either side of this arrangement would be odd and could not be tiled by tiles of area 2). Define a *unit* to be either a vertically-placed tile or two horizontally-placed tiles with one on top of the other.



If the tiling is made by 9 units, then all 9 must be vertical tiles. This tiling may be done in $\binom{9}{0}$ ways. If the tiling is done with 8 units, then 7 of them are vertical tiles and the last is a unit made of two horizontal tiles. This may be done in $\binom{8}{1}$ ways. Similarly, there are $\binom{7}{2}$ ways to do the tiling with 7 units. By this calculation, we can see the answer is $\binom{9}{0} + \binom{8}{1} + \binom{7}{2} + \binom{6}{3} + \binom{5}{4} = 1 + 8 + 21 + 20 + 5 = 55$.
OR
We will show by induction that the number of ways to tile a $2 \times k$ board is F_k, where the Fibonacci numbers F_i are defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-1}$ for $n > 1$. Clearly, there is one way to tile both a 0×2 board and a 1×2 board. Now assume that the number of ways to tile a $2 \times m$ board is F_m for $0 \leq m < k$. We define a *unit* as we did above, and consider the rightmost unit of a tiling of a $2 \times k$ board. If this rightmost unit is a vertical domino, then there are F_{k-1} ways to tile the rest of the board. If the rightmost unit is two horizontal tiles, then there are F_{k-2} ways to tile the rest of the board. These are the only two cases. Thus, the number of ways to tile a $2 \times k$ board is $F_{k-1} + F_{k-2} = F_k$, and we are done.