1 Grade 8 Solutions

1. The answer is $\boxed{2}$.

2. $\dfrac{0.4+0.04+0.004+0.0004}{4} = \dfrac{0.4444}{4} = \boxed{0.1111}$.

3. It is easy to see that the largest prime less than 93 is 89. Therefore, the answer is $93-89 = \boxed{4}$ dollars.

4. Our sorted list is 1, 2, 2, 3, 5, 6, 8, 9. The median of these numbers is the average of the two middle ones, so it is $\dfrac{3+5}{2} = \boxed{4}$.

5. $(-3)^{(-3)} - 3 = \dfrac{1}{(-3)^3} - 3 = -\dfrac{1}{27} - 3 = \boxed{-\dfrac{82}{27}}$.

6. Since x^2y is constant for all values of x and y, we have that x^2y is always equal to $6^2 \cdot 4 = 144$. Therefore, we need y such that $4^2 \cdot y = 144$, so $y = \boxed{9}$.

7. $\dfrac{12}{5}$ is equal to 2.4. $2\dfrac{1}{3}$ is equal to 2.33...., which is clearly less than 2.4. Also, $\dfrac{1}{2.2} = 1 < 2.4$, and clearly $2.2 < 2.4$. Thus, $\boxed{\dfrac{12}{5}}$ is the largest of these numbers.

8. The number halfway between $\dfrac{1}{2}$ and $\dfrac{1}{3}$ is $\dfrac{\frac{1}{2}+\frac{1}{3}}{2} = \dfrac{\frac{3}{6}+\frac{2}{6}}{2} = \dfrac{5}{12}$. The number halfway between $\dfrac{1}{3}$ and $\dfrac{5}{12}$ is $\dfrac{\frac{1}{3}+\frac{5}{12}}{2} = \dfrac{\frac{4}{12}+\frac{5}{12}}{2} = \dfrac{9}{24} = \boxed{\dfrac{3}{8}}$.

9. $g(1) = 2$, $f(g(1)) = f(2) = 3$, $g(f(g(1)) = g(3) = 6$, $f(g(f(g(1)))) = 7$, $g(f(g(f(g(1))))) = 14$, so $f(g(f(g(f(g(1)))) = \boxed{15}$.

10. A triangle with side lengths of 6, 8, and 10 is a right triangle with legs of lengths 6 and 8 and a hypotenuse of length 10 because the triangle satisfies the Pythagorean Theorem ($6^2 + 8^2 = 10^2$.) The area is therefore half the product of the legs: $\dfrac{1}{2} \times 6 \times 8 = \boxed{24}$.

11. There are 4 choices for the first person, 3 for the second, 2 for the third, and 1 for the fourth. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$ ways to line up the people.

12. They would have made $93500 \cdot \dfrac{12}{11} = \boxed{102000}$ dollars.

13. Since a Galleon is worth 17 Sickles, 2 Galleons are worth 34 Sickles. Thus, 2 Galleons and 2 Sickles are worth 36 Sickles. Since there are 29 Knuts in a Sickle, the answer is $36 \cdot 29 = \boxed{1044}$ Knuts.

14. After every 20 minutes, the number of parasprites doubles: each of the original parasprites remains, and each of them spawns a new parasprite. Since six sets of 20 minutes pass in two hours, there will be a total of $2^6 = \boxed{64}$ parasprites after 2 hours.

15. Note that $49^{27x} = (7^2)^{27x} = 7^{54x}$. As a result, $\dfrac{49^{27x}}{7^{9x}} = 7^{45x}$. Since this is equal to $49 = 7^2$, we have that $x = \boxed{\dfrac{2}{45}}$.

16. The resulting figure has one circular face, of surface area πr^2, one lateral surface of a cylinder, with surface area $2\pi rh$, and one-half of a sphere, which has surface area $\frac{1}{2}(4\pi r^2) = 2\pi r^2$. Thus, the total surface area is $3\pi r^2 + 2\pi rh = 3\pi(5^2) + 2\pi(5)(10) = \boxed{175\pi}$.

17. If the product of three numbers is odd, each individual number must be odd. The probability that any single roll of a die results in an odd number is $\frac{1}{2}$, so our answer is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{8}}$.

18. Prime factorize n as $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$. The number of divisors of n is just $(e_1 + 1)(e_2 + 1) \cdots (e_m + 1)$. Thus, if n has 9 divisors, n is either the product of 8 identical prime numbers or the product of two squares of primes The smallest prime taken to the 8th power will be equal to $2^8 = 256$, while the product of the two smallest primes whose square roots are prime is equal to $4 \times 9 = 36$. Thus, the smallest possible value of n is $\boxed{36}$.

19. The area which Mike can roam is $\frac{3}{4}$ of a circle with radius 4. Hence, the answer is $\frac{3}{4}(4^2\pi) = \boxed{12\pi}$.

20. First of all, $n = 0$ is a solution. Now, suppose that $n \neq 0$, and divide both sides of the equation by n. Then, we obtain $\frac{2 + 2^n}{n+1} = 2 \implies 2 + 2^n = 2n + 2$. Therefore, $2^n = 2n$, and the only possible solutions for this equation are $n = 1, 2$. Therefore, the answers are $\boxed{0, 1, 2}$.

21. There are 7! ways to arrange 7 distinct letters. However there are three A's in the word. If we treat these as distinct letters we have the above number. However because we cannot distinguish between the A's, we need to divide by the number of times we counted the A's as if they were distinct, which is the same as the number of ways we can arrange the A's, which is 3!. There are therefore $\frac{7!}{3!} = \boxed{840}$ arrangements of letters in the word ANAGRAM.

22. It is easy to see that the number n appears between positions $(1 + 2 + \cdots + (n-1)) + 1$ and $(1 + 2 + \cdots + n)$ inclusive, that is, $\frac{n(n-1)}{2} + 1$ and $\frac{n(n+1)}{2}$ inclusive. Since 200 lies between $190 = \frac{20 \cdot 19}{2} + 1$ and $210 = \frac{20 \cdot 21}{2}$ inclusive, our answer is $\boxed{20}$.

23. Subtract the second equation from the first to get $3x^2 - 8x = 35 \implies 3x^2 - 8x - 35 = 0$. This factors as $(3x + 7)(x - 5) = 0 \implies x = 5$ or $x = -\frac{7}{3}$. However, only one of these is a solution to the first equation, namely $x = \boxed{5}$.

24. $1.9\overline{8} = \frac{19 + 0.\overline{8}}{10} = \frac{19 + \frac{8}{9}}{10} = \frac{171 + 8}{90} = \boxed{\frac{179}{90}}$.

25. First of all, we must consider that 2012 is a leap year so we must consider the extra day (hence 2012 will have 366 days instead of 365). We must also consider that October 16, 2011, the day of the contest, is a Sunday. One year from now, which is 366 days later, October 16, 2012 will be in 52 weeks and 2 days, so October 16 will be a Tuesday. Therefore, October 17 will be a $\boxed{\text{Wednesday}}$.

26. The number 9^n ends with 1 if n is even, and ends with 9 if n is odd. Since $8^{7^{6^{5^{4^{3^{2^{1}}}}}}}$ is even, the last digit of $9^{8^{7^{6^{5^{4^{3^{2^{1}}}}}}}}$ is $\boxed{1}$.

27. If the first roll is a 1, we have six choices for the second roll. If it is a 2, that gives us three choices for the second roll (2, 4, 6). If it is a 3, then there are only two choices (3, 6), and for 4, 5, and 6, there is only one choice (itself). That gives us a probability of $\frac{6+3+2+1+1+1}{36} = \boxed{\frac{7}{18}}$.

28. Let y be the length of the side parallel to the river, and let x be the lengths of the other two sides. We must have $2x + y = 200$. We seek to maximize the area, xy, that is enclosed by the fence and the river. Rearranging, we get $y = 200 - 2x$, so the area is $xy = x \cdot (200 - 2x) = 200x - 2x^2$, which is a quadratic in x. The vertex, or maximum (because the coefficient of x^2 is negative), occurs at $x = \frac{-200}{(2 \cdot (-2))} = 50$. When $x = 50, y = 200 - 2 \cdot 50 = 100$. The area is then $xy = 50 \cdot 100 = \boxed{5000}$.

29. $((3!)!)! = (6!)! = 720!$; the number of zeroes at the end of 720 is $\lfloor \frac{720}{5} \rfloor + \lfloor \frac{720}{25} \rfloor + \lfloor \frac{720}{125} \rfloor + \lfloor \frac{720}{625} \rfloor = 144 + 28 + 5 + 1 = \boxed{178}$.

30. Call the roots of $x^2 + 4x - 2 = 0$ a and b. From Vieta's formulas, we know that $ab = -2$ and $a + b = -4$. Write the desired quadratic as $x^2 - qx + r$, for some q and r. Again by Vieta's forumlas, we have that $r = (ab)^2$ and $q = a^2+b^2$. r is simply $(-2)^2 = 4$. To find q, we note that $(a+b)^2 = a^2+2ab+b^2$. Rearranging, we have that $a^2+b^2 = (a+b)^2-2ab = (-4)^2-2(-2) = 20$. The quadratic equation we seek is therefore $\boxed{x^2 - 20x + 4 = 0}$.

31. Note that the line segments A_1B_1, B_1C_1, C_1A_1 divides triangle ABC into four identical smaller triangles, and similarly A_2B_2, B_2C_2, C_2A_2 divides triangle $A_1B_1C_1$ into four even smaller identical triangles. Therefore, the area of triangle ABC is four times that of triangle $A_1B_1C_1$, which in turn has area four times that of $A_2B_2C_2$. Hence, the ratio of the area of triangle $A_2B_2C_2$ to the area of triangle ABC is $\boxed{\frac{1}{16}}$.

32. $1^2 - 2^2 + 3^2 - 4^2 + \cdots - 50^2 + 51^2 = 1 + (3+2)(3-2) + (5+4)(5-4) + \cdots + (51+50)(51-50)$. (Note: $a^2 - b^2 = (a + b)(a - b)$.) This is equal to $1 + 5 + 9 + \cdots + 101 = 13 \cdot 102 = \boxed{1326}$.

33. The expression factors to $(x + 4)(x - 2)$, which can only be a prime number if one of the two expressions is equal to 1 or -1 and the other one is a prime number or a negative number whose absolute value is prime. The only values of x that work are $\boxed{-5}$ and $\boxed{3}$.

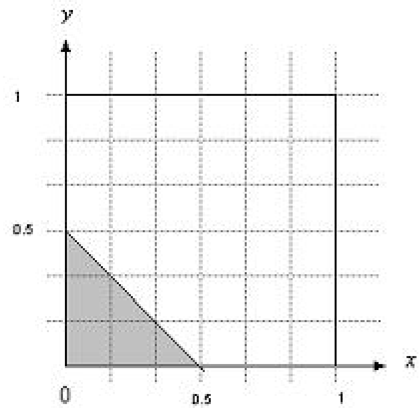
34. Let $x = BD$ and $y = DC$. Clearly, $x+y = 5$. Also, by the angle bisector theorem, $\frac{x}{y} = \frac{4}{6} = \frac{2}{3}$, so $3x = 2y = 2(5 - x)$, so $5x = 10$, so $x = BD = \boxed{2}$.

35. If $x - \frac{1}{x} = 3$, then $\left(x - \frac{1}{x}\right)^2 = 9$, or $x^2 + \frac{1}{x^2} - 2 = 9$, so $x^2 + \frac{1}{x^2} = \boxed{11}$.

36. By the factor theorem, $x^3 - 6x^2 + 5x + 1$ factors as $(x - r)(x - s)(x - t)$. Thus, $(2 - r)(2 - s)(2 - t) = 2^3 - 6 \cdot 2^2 + 5 \cdot 2 + 1 = 8 - 24 + 10 + 1 = \boxed{-5}$.

37. There are 50 cards from which Steven can draw. The cards he can draw to keep from going above 21 are each of the four aces, twos, threes, fours, and fives, the three remaining sixes, the three remaining sevens, and the four eights. There are 30 such cards in total, so the probability is $\frac{30}{50} = \boxed{\frac{3}{5}}$.

38. Consider the attached diagram. If we choose x, y randomly in this 1×1 square, the probability that their sum is less than $\frac{1}{2}$ is the probability that (x, y) lies in the shaded region. The area of the shaded region is $\frac{1}{8}$, while the area of the whole region is 1, so our answer is $\boxed{\frac{1}{8}}$.



39. Consider the 3 balls Bob picks. The only way one of your balls will not be among Bob's balls is if you pick the two he did not pick. This only happens with probability $\frac{1}{\binom{5}{2}} = \frac{1}{10}$, so the probability one of your balls will be among Bob's chosen ones is $1 - \frac{1}{10} = \boxed{\frac{9}{10}}$.

40. Let us consider $n \mod 3$. When $n \equiv 0 \mod 3$, $n^2 + 2n + 3 \equiv 0 + 0 + 3 \equiv 0 \mod 3$. When $n \equiv 1 \mod 3$, $n^2 + 2n + 3 \equiv 1 + 2 + 3 \equiv 0 \mod 3$. Finally, when $n \equiv 2 \mod 3$, $n^2 + 2n + 3 \equiv 4 + 4 + 3 \equiv 2 \mod 3$. Thus, $3 | n^2 + 2n + 1$ exactly when n is congruent to 0 or 1 mod 3. There are 670 numbers that are $0 \mod 3$, namely $3 \cdot 1, 3 \cdot 2, \ldots, 3 \cdot 670 = 2010$. There are 671 numbers that are $1 \mod 3$, namely $3 \cdot 0 + 1, 3 \cdot 1 + 1, \ldots, 3 \cdot 670 + 1 = 2011$. Thus there are $670 + 671 = \boxed{1341}$ numbers that satisfy the restrictions.

41. Consider the top left hand corner square of an $n \times n$ square. Notice that when $n = 1$, there are 6^2 choices for the top left hand corner square, when $n = 2$, there are 5^2 choices, and, in general, for an $k \times k$ square there are $(7-k)^2$ choices for the top left hand corner square. Thus, since k ranges from 1 to 6, the total number of squares is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \boxed{91}$.

42. $11_2 + 11_3 + 11_4 + \cdots + 11_100 = (1 \times 2 + 1) + (1 \times 3 + 1) + (1 \times 4 + 1) + \cdots + (1 \times 100 + 1) = 3 + 4 + 5 + \cdots + 101$. This is the same as the sum of positive integers up to 101 without 1 and 2. The sum of positive integers from 1 to n can be expressed as $\frac{n(n+1)}{2}$. The sum is therefore $\frac{101(102)}{2} - (1 + 2) = \boxed{5148}$.

43. The area of a polygon with 20,000 sides can be approximated by the area of a circle. The polygon has a diagonal of length 20, which corresponds to a diameter of 20 in the approximating circle. A circle with a diameter of 20 has a radius of 10 for an area of $\pi \cdot 10^2 = 100\pi$, which is approximately $\boxed{314}$.

44. Let a, t, r be the percentage of the apples that Applejack, Twilight Sparkle, and Rainbow Dash can pick in one day, respectively. We are given that $3(a+t) = 4(a+r) = 6(t+r) = 1$, so $a+t = \frac{1}{3}$, $a+r = \frac{1}{4}$, and $t+r = \frac{1}{6}$. Adding these up and dividing by 2 gives $a+r+t = \frac{3}{8}$, so it will take $\frac{1}{a+r+t} = \boxed{\frac{8}{3}}$ days for them to pick all the apples.

45. If $a^2 = b^2 + p$, where p is prime, then $p = a^2 - b^2 = (a-b)(a+b)$. If $a - b > 1$, then p is the product of two integers larger than 1, contradicting its primality. Thus, $b = a-1$, so $p = 2a-1$. a is maximized when p is maximized, when happens when $p = 97$, so $a = \frac{97+1}{2} = \boxed{49}$.

46. The region in question is a right triangular pyramid with legs of length 12, 15, and 20 (since its vertices are $(0,0,0), (0,0,12), (0,15,0)$, and $(20,0,0)$. The volume of this pyramid is $\frac{12 \cdot 15 \cdot 20}{6} = \boxed{600}$.

47. If $x - \frac{1}{x} = 3$, then $\left(x - \frac{1}{x}\right)^2 = 9$, or $x^2 + \frac{1}{x^2} - 2 = 9$, so $x^2 + \frac{1}{x^2} = \boxed{11}$.

48. First, notice that $\frac{x^5-1}{x-1} = x^4+x^3+x^2+x+1$. Letting $x = 10^{20}$ gives us that the subtrahend is $(10^{80}+10^{60}+10^{40}+10^{20}+1)$. Next, we realize that $10^{100}-1 = 999\ldots999$, or a string of 100 9's, which has a sum of 900. Since we are subtracting the number $(10^{80}+10^{60}+10^{40}+10^{20}+1)$, which is composed of a total of 5 1's and many 0's, and since no borrowing is done, the answer is just $900 - 5 = \boxed{895}$.

49. In general, $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$. Thus, $2 \cdot 929 = (23-20)^2 + (23+20)^2 = 46^2 + 3^2$, so our answer must be $43 + 3 = \boxed{46}$. (It can be verified that this decomposition is unique.)

50. Divide the equation $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$ by $\frac{1}{2^2}$ to get

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{24}.$$

Subtract this from the original equation to get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$