关于二次约束二次规划问题强对偶性的几个结果*

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SEVERAL RESULTS ON THE STRONG DUALITY OF QCQP

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Abstract In this paper, we revisit the strong duality of the quadratically constrained quadratic programming (QCQP) problem. We first generalize a known result for the rank-one decomposition of matrices and then apply it to consider the strong duality for more general QCQP scenarios, including the cases with one constraint, two constraints while at least one being inactive on the optimal solution point, multiple constraints, and an interval constraint. A sufficient condition ensuring the strong duality of more general QCQP problems is studied as well. We also extend our results to the QCQP problems with complex variables.

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1 Introduction

Duality theory plays a fundamental role in studying constrained optimization problems. For convex optimization problems, it is known that the strong duality always holds under the Slater condition, see, e.g. [4]. For nonconvex optimization problems, the strong duality generally does not hold even for quadratic programming problems. But for some special cases such as a quadratically constrained quadratic programming (QCQP) problem with one constraint, the strong duality holds despite the nonconvexity of the problem.

Let us consider the canonical QCQP model:

$$\begin{aligned} (P) \qquad & \min \quad x^T A_0 x + 2b_0^T x + c_0 \\ & \text{s. t.} \quad x^T A_1 x + 2b_1^T x + c_1 \le 0, \end{aligned}$$

where $A_i \in S^n, b_i \in R^n, c_i \in R$, for $i = 0, 1$. Then, its dual problem (D) is

$$\begin{aligned} (D) \qquad & \max \quad \gamma \\ & \text{s. t.} \quad \begin{bmatrix} A_0 + yA_1 & b_0 + yb_1 \\ b_0^T + yb_1^T & c_0 + yc_1 - \gamma \end{bmatrix} \succeq 0, \quad y \ge 0. \end{aligned}$$

If the Slater condition holds for (P), i.e., there exists $x_0 \in R^n$ such that $x_0^T A_1 x_0 + 2b_1^T x_0 + c_1 < 0$, then the optimal values of (P) and (D) coincide. To discuss the strong duality of QCQP problems, the Yakubovich's S-Procedure (or called S-Lemma) in [14]is important. We refer to [7][11] for extended study in the context of cones related to nonnegative quadratic functions, and [5] for the extension to polynomial systems. More references can be found in, e.g., [3][4][9][12]. In particular, a "modern approach" based on the rank-one decomposition of matrices has been developed in [6][11][17]and it has been widely used in the literature. For more literatures, we refer to, e.g., [6][7][15][16].

In this paper, we follow the "modern approach" based on the rank-one decomposition of matrices and further discuss the strong duality of QCQP problems for various general scenarios. We shall generalize the rank-one decomposition result in [6][11][17] and then apply the generalized result to prove the strong duality of general QCQP problems where there is one constraint, and where there are two constraints with at least one being inactive at the optimal solution point. A byproduct is the strong duality of QCQP with an interval constraint. A sufficient condition ensuring the strong duality of general QCQP cases is also given. Our analysis is then extended to the QCQP case with complex variables.

The rest of the paper is organized as follows. In Section 2, we generalize the known rank-one decomposition of a symmetric matrix in [6][11][17]. In Section 3, using the generalized result, we provide a new proof for the strong duality of the canonical QCQP problem (P) and the QCQP case where there are two constraints under certain assumptions. A sufficient condition ensuring the strong duality of some more general QCQP cases is given as well. Moreover, the strong duality of the QCQP case with an interval constraint is established. In Section 4, we extend the generalized rank-one decomposition result of a symmetric matrix to the case of Hermitian matrix, and then prove the strong duality of complex-variable QCQP cases with multiple constraints. The cases with an inequality constraint and with an interval constraint are discussed as well. Finally, we make some conclusions in Section 5.

Throughout, we use the following notation. Let S^n denote the space of $n\times n$ symmetric matrices, S^n_+ the set of positive semidefinite matrices in S^n, H^n the space of $n\times n$ Hermitian matrices, H^n_+ the set of positive semidefinite matrices in H^n. Let $A\circ B$ denote the inner product of two matrixes, namely $A\circ B = tr(A^TB)$. Finally, let "$conv$" denote the convex hull of a set, and "$A\succeq 0$" means A is positive semidefinite (PSD).

2 A General Rank-one Decomposition Result

In this section, we generalize the result of rank-one decomposition of a symmetric matrix in [6][11][17]. This more general result makes it possible to discuss more general QCQP cases where there are more than one constraint. Let us first recall the result in [6][11][17].

Lemma 2.1 Assume $G\in S^n$, and X is an $n\times n$ positive semidefinite matrix with rank r. Then there are linearly independent vectors $\{p_i\}_{i=1}^r$ such that $X=\sum\limits_{i=1}^r p_ip_i^T$, and

$$p_i^TGp_i = \frac{G\circ X}{r}, \quad i=1,\cdots,r.$$

In particular, if $G\circ X<0$, then $p_i^TGp_i<0, i=1,\cdots,r$; if $G\circ X=0$, then $p_i^TGp_i=0, i=1,\cdots,r$.

To study QCQP problems, it is usual to homogenize the involved quadratic functions and consider their semidefinite programming (SDP) relaxations. Thus, for $X\in S^n_+$, it is usual to augment it with $x\in R^n$ and obtain the $(n+1)\times(n+1)$ PSD matrix:

$$Z=\begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix}.$$

Now, we generalize Lemma 2.1 and propose a more general version which will be the basis of our later analysis.

Proposition 2.2 Assume $x_0 \in R^n$ such that $x_0^T A x_0 + 2b^T x_0 + c < 0, A \in S^n, b \in R^n, c \in R$. If $Z \in S_+^{n+1}, r(Z) = k \geq 1, z_{n+1,n+1} = 1$, and it satisfies

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \circ Z < 0,$$

then there exists a rank-one decomposition of $Z = \sum_{i=1}^{k} \eta^i {\eta^i}^T$ such that

$${\eta^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \eta^i < 0, \quad \eta_{n+1}^i \neq 0, \qquad i = 1, \cdots, k,$$

where η_{n+1}^i denotes the $(n+1)$-th component of η^i.

Proof We prove it by induction. For the case where $k = 1$, it is trivial. When $k = 2$, by Lemma 2.1, there exist η^1 and η^2 such that $Z = \eta^1 {\eta^1}^T + \eta^2 {\eta^2}^T$ and

$${\eta^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \eta^i < 0, \quad i = 1, 2.$$

Clearly, we have $(\eta_{n+1}^1)^2 + (\eta_{n+1}^2)^2 = 1$. If $\eta_{n+1}^1 \eta_{n+1}^2 \neq 0$, then the conclusion immediately follows. Otherwise, we can assume $\eta_{n+1}^1 = 0$ and thus have $(\eta_{n+1}^2)^2 = 1$. Let us set $y^1 = \eta^2 + u\eta^1$ and $y^2 = \eta^1 - u\eta^2$. By the continuity, when $|u| > 0$ is sufficiently small, we have the following assertions:

$${y^1}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y^1 < 0, \quad {y^2}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} y^2 < 0,$$

$$\begin{aligned} & y^1 {y^1}^T + y^2 {y^2}^T \\ = & \eta^2 {\eta^2}^T + u(\eta^2 {\eta^1}^T + \eta^1 {\eta^2}^T) + u^2 \eta^1 {\eta^1}^T \\ & + \eta^1 {\eta^1}^T - u(\eta^1 {\eta^2}^T + \eta^2 {\eta^1}^T) + u^2 \eta^2 {\eta^2}^T \\ = & (1 + u^2)(\eta^1 {\eta^1}^T + \eta^2 {\eta^2}^T) \\ = & (1 + u^2) Z. \end{aligned}$$

Thus, it holds that $Z = (y^1 {y^1}^T + y^2 {y^2}^T)/(1 + u^2)$. Denote $z^i = y^i / \sqrt{1 + u^2}, i = 1, 2$. Then, we have $Z = z^1 {z^1}^T + z^2 {z^2}^T$ and

$${z^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} z^i < 0, i = 1, 2.$$

From the definitions of z^i, we see $z^i_{n+1} \neq 0, i = 1, 2$.

Furthermore, the proposition holds if $r(Z) = m < k$. That is, there exists $Z = \sum_{i=1}^{m} \eta^i {\eta^i}^T$ such that

$$ {\eta^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \eta^i < 0, \quad \eta^i_{n+1} \neq 0, \quad i = 1, \cdots, m. $$

We shall show that for $r(Z) = m + 1 \leq k$, the assertion holds. Indeed, by Lemma 2.1, there are $m + 1$ linearly independent vectors η^i such that $Z = \sum_{i=1}^{m+1} \eta^i {\eta^i}^T$ and the following inequalities hold:

$$ {\eta^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \eta^i < 0, \quad i = 1, \cdots, m+1. $$

Obviously, if $\eta^i_{n+1} \neq 0, i = 1, \cdots, m+1$, the assertion holds. Otherwise, there exists at least one i such that $\eta^i_{n+1} = 0$. Without loss of generality, we assume $\eta^1_{n+1} = 0$ and $\eta^2_{n+1} \neq 0$. With the same procedure for proving the case of $r(Z) = 2$, we have z^1 and z^2 such that $z^1 {z^1}^T + z^2 {z^2}^T = \eta^1 {\eta^1}^T + \eta^2 {\eta^2}^T$, and

$$ {z^i}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} z^i < 0, \quad z^i_{n+1} \neq 0, \quad i = 1, 2. $$

Therefore, we show that $Z = z^1 {z^1}^T + z^2 {z^2}^T + \sum_{i=3}^{m+1} \eta^1 {\eta^1}^T$. Repeating this process at most m times enables us to obtain the assertion. The proof is complete.

3 Strong Duality of Various QCQP Problems

In this section, we establish the strong duality for various QCQP problems, including the canonical case (P), the case with two constraints but at least one constraint being inactive at the optimal solution point, the case with an interval constraint. Last, a sufficient condition enduring the strong duality for these general QCQP cases is provided.

Proposition 2.2 is the foundation of our analysis. We first prove two more propositions that will be used in later analysis. The first one is obtained by applying Proposition 2.2 to a quadratic function.

Proposition 3.1 If the Slater condition holds, that is, there exists a $x_0 \in R^n$ such that

$$ \begin{bmatrix} x_0 \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} < 0, $$

then it holds that

$$\operatorname{conv}\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^T \middle| \begin{bmatrix}x\\1\end{bmatrix}^T\begin{bmatrix}A & b\\b^T & c\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}<0, x\in R^n\right\}$$

$$=\left\{Z\in S_+^{n+1} \middle| \begin{bmatrix}A & b\\b^T & c\end{bmatrix}\circ Z<0, Z_{n+1,n+1}=1\right\}\triangleq C^o.$$

Proof Clearly, the left-hand-side set is contained in the right-hand-side one, and both are nonempty. We just need to prove that the right-hand-side set is contained by the left-hand-side one as well. For $Z\in C^o$, assume $r(Z)=k>1$. According to Proposition 2.2, we know that there exist k linearly independent vectors η^i such that $Z=\sum_{i=1}^{k}\eta^i{\eta^i}^T$ and the following inequalities hold:

$${\eta^i}^T\begin{bmatrix}A & b\\b^T & c\end{bmatrix}\eta^i<0, \eta^i_{n+1}\neq 0,\quad i=1,\cdots,k.$$

Setting $z^i=\eta^i/\eta^i_{n+1}$, then we have $z^i_{n+1}=1, i=1,\cdots,k$; and for each i, it holds that

$${z^i}^T\begin{bmatrix}A & b\\b^T & c\end{bmatrix}z^i<0.$$

Since $Z=\sum_{i=1}^{k}(\eta^i_{n+1})^2z^i{z^i}^T$, and $z_{n+1,n+1}=1$, one has that $\sum_{i+1}^{k}(\eta^i_{n+1})^2=1$. So, we have

$$Z\in\operatorname{conv}\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^T \middle| \begin{bmatrix}x\\1\end{bmatrix}^T\begin{bmatrix}A & b\\b^T & c\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}<0, x\in R^n\right\}.$$

The proof is complete.

For QCQP problems, the inequality constraints are generally not strictly. So we obtain the following proposition.

Proposition 3.2 Assume the Slater condition holds. Then, it holds that

$$\operatorname{conv}\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^T \middle| \begin{bmatrix}x\\1\end{bmatrix}^T\begin{bmatrix}A & b\\b^T & c\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}\leq 0, x\in R^n\right\}$$

$$= \left\{ Z \in S_+^{n+1} \,\middle|\, \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \circ Z \le 0, Z_{n+1,n+1} = 1 \right\} \triangleq C.$$

Proof It suffices to prove that the right-hand-side set is contained in the left–hand-side one. Assume $\bar{Z} \in C$. Then we have a series of $\{Z^j\}$ in C^o such that $Z^j \to \bar{Z}$ $(j \to \infty)$. Note that

$$Z^j \in \operatorname{conv}\left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, x \in R^n \right\}.$$

Letting $j \to \infty$, we get

$$\bar{Z} \in \operatorname{conv}\left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \,\middle|\, \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, x \in R^n \right\}.$$

This completes the proof.

3.1 QCQP with One Constraint

We first provide a new proof for the strong duality of the QCQP with one constraint, i.e., the model (P), under the Slater condition. Its proof can be found in, e.g., Appendix of [5].

Theorem 3.1 For the QCQP problem (P), assume that its objective function has a finite lower bound and its dual problem has a strictly feasible solution point. Then, we have $v(P) = v(D)$, where $v(P)$ and $v(D)$ denote the optimal values of (P) and (D), respectively.

Proof First, recall that the dual problem of (P) can be written as (see, e.g. Appendix B of [4])

$$\begin{aligned} (D) \qquad & \max \quad \gamma \\ & \text{s. t.} \quad \begin{bmatrix} A_0 + yA_1 & b_0 + yb_1 \\ b_0^T + yb_1^T & c_0 + yc_1 - \gamma \end{bmatrix} \succeq 0, \quad y \ge 0, \end{aligned}$$

and furthermore, the dual problem of (D) is the semidefinite programming(SDP) relaxation of (P):

$$\begin{aligned} (RP) \qquad & \min \quad A_0 \circ X + 2b_0^T x + c_0 \\ & \text{s. t.} \quad A_1 \circ X + 2b_1^T x + c_1 \le 0, \\ & \qquad\quad X \succeq xx^T. \end{aligned}$$

As assumed, the Slater condition holds for (P), and it also holds for (RP). Because the problem (D) is assumed to have a strictly feasible solution point, by the strong duality of SDP, we know that $v(RP) = v(D)$, and both (D) and (RP) are attainable since (RP) is also strictly feasible. Let (X^*, x^*) be an optimal solution of (RP), $Z^* = \begin{bmatrix} X^* & x^* \\ x^{*T} & 1 \end{bmatrix}$. Then, it holds that

$$\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} \circ Z^* \le 0.$$

Assume $r(Z^*) = k$, by Proposition 3.2 we know that there exist an integer k, $z^i, z^i_{n+1} = 1 (i = 1, \cdots, k)$ and $q_i (i = 1, \cdots, k)$ such that $Z^* = \sum_{i=1}^k q_i z^i {z^i}^T, q_i > 0, \sum_{i=1}^k q_i = 1$ and

$${z^i}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} z^i \le 0.$$

Let us denote

$$z^i = \begin{bmatrix} x^i \\ 1 \end{bmatrix}, x^i \in R^n.$$

Then each x^i is a feasible solution point of (P) and thus we have

$$\begin{aligned} v(RP) = \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \circ Z^* &= \sum_{i=1}^k q_i \begin{bmatrix} x^i \\ 1 \end{bmatrix}^T \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \begin{bmatrix} x^i \\ 1 \end{bmatrix} \\ &\ge {x^l}^T A_0 x^l + 2b_0^T x^l + c^0 \ge v(P), \end{aligned}$$

where $x^l = \arg\min\limits_{1 \le i \le k} \{{x^i}^T A_0 x^i + 2b_0^T x^i + c_0\}$. So we obtain $v(D) = v(RP) = v(D)$ and it becomes trivial to see that all x^i, $i = 1, \cdots, k$, are solution points of the QCQP model (P).

3.2 QCQP with Two Constraints

In this subsection, we study the strong duality of the QCQP problem with two constrains:

$$\begin{aligned} (P_2) \qquad & \min \quad x^T A_0 x + 2b_0^T x + c_0 \\ & \text{s. t.} \quad x^T A_i x + 2b_i^T x + c_i \le 0, \ i = 1, 2, \end{aligned}$$

where $A_i \in S^n, b_i \in R^n, c_i \in R$ for $i = 0, 1, 2$. It is well-known that the dual problem of

(P_2) can be written as

$$
\begin{aligned}
(D_2) \qquad & \max \quad \gamma \\
& \text{s. t.} \quad \begin{bmatrix} A_0 + y_1 A_1 + y_2 A_2 & b_0 + y_1 b_1 + y_2 b_2 \\ b_0^T + y_1 b_1^T + y_2 b_2^T & c_0 + y_1 c_1 + y_2 c_2 - \gamma \end{bmatrix} \succeq 0 \\
& \qquad\ \ y \geq 0.
\end{aligned}
$$

We first prove the strong duality for (P_2) for the case where there is at least one inactive constraint at a solution point of its semidefinite relaxation. Proposition 3.2 will be used.

Theorem 3.4 Suppose Slater condition holds for (P_2) and its dual problem (D_2) has a strictly feasible solution. Let (X^*, x^*) be a solution point of the SDP relaxation of (P_2) and at least one of the constraints of the SDP relaxation of (P_2) is inactive at (X^*, x^*). Then the strong duality holds for (P_2).

Proof The dual problem of (D_2) is the SDP relaxation of (P_2):

$$
\begin{aligned}
(RP_2) \qquad & \min \quad A_0 \circ X + 2b_0^T x + c_0 \\
& \text{s. t.} \quad A_i \circ X + 2b_i^T x + c_i \leq 0, i = 1, 2, \\
& \qquad\ \ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0.
\end{aligned}
$$

The assumptions guarantee that the problems (D_2) and (RP_2) have the same (attainable) optimal values. Because there is at least one inactive constraint at (X^*, x^*), without loss of generality, let us assume it is the second one. That is, $A_2 \circ X^* + 2b_2^T x^* + c_2 < 0$. This implies that (X^*, x^*) is also a solution point of the following problem:

$$
\begin{aligned}
(RP_3) \qquad & \min \quad A_0 \circ X + 2b_0^T x + c_0 \\
& \text{s. t.} \quad A_1 \circ X + 2b_1^T x + c_1 \leq 0, \\
& \qquad\ \ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0.
\end{aligned}
$$

Denote $Z^* = \begin{bmatrix} X^* & x^* \\ x^{*T} & 1 \end{bmatrix}$. Then, it follows from Proposition 3.2 that there exist $z^i, z_{n+1}^i =$

$1(i=1,\cdots,k)$ and $q_i(i=1,\cdots,k)$ such that $Z^*=\sum_{i=1}^{k}q_i z^i {z^i}^T, q_i>0, \sum_{i=1}^{k}q_i=1$ and

$$ {z^i}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} z^i \le 0. $$

Note that Z^* satisfies

$$ \begin{bmatrix} A_2 & b_2 \\ {b_2}^T & c_2 \end{bmatrix} \circ Z^* < 0. $$

Thus, we know that at least there is one i, say $i=1$, such that $(z^1)^T \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} z^1 < 0$.

Denote $z^1=\begin{bmatrix} x^1 \\ 1 \end{bmatrix}$. So one has that x^1 is a feasible solution point of (P_2). Recall the proof of Theorem 2.1. We know that x^1 is a solution point of the following QCQP problem with one constraint:

$$ \begin{aligned} \min \quad & x^TA_0x+2b_0^Tx+c_0 \\ \text{s. t.} \quad & x^TA_1x+2b_1^Tx+c_1 \le 0. \end{aligned} $$

Thus we obtain that

$$ v(P_2) \le {x^1}^T A_0 x^1 + 2b_0^T x^1 + c_0 = \begin{bmatrix} A_0 & b_0 \\ {b_0}^T & c_0 \end{bmatrix} \circ Z^* = v(RP_2) = v(D_2) \le v(P_2). $$

This implies the strong duality of (P_2) and the proof is complete.

Remark 3.5 The proof of Theorem 3.1 in [15] is based on the technique of rank-one decomposition of matrices. Our new proof seems to be simpler and more general to include the QCQP case with two inactive constraints.

Now, we consider the The following QCQP model with an interval constraint:

$$ \begin{aligned} (P_4) \qquad \min \quad & x^TA_0x+2b_0^Tx+c_0 \\ \text{s. t.} \quad & l \le x^TA_1x+2b_1^Tx+c_1 \le u, \end{aligned} $$

where $l<u$ are finite real numbers. Obviously, it is a special case of (P_2). Indeed, either for (P_4) or for its SDP relaxation, the involved two constraints cannot be active simultaneously because of $l<u$. The strong duality of the model (P_4) can be summarized in the following corollary.

Corollary 3.6 Assume Slater condition holds for (P_4), and the dual problem (D_4) of (P_4) has a strictly feasible solution point. Then, we have $v(P_4) = v(D_4)$, and these two problems have the same attainable optimal values.

In recent papers [10][12], for QCQP problems with interval constraints, the strong duality was proven by different methods. For example, the proof in [12] is based on the new S-lemma presented in [13]. Our proof seems to be simpler than existing ones.

Also, we call the following conditions (in our notation) of Assumption 1 in [10] to ensure the strong duality of (P_4):

Item 2 (P_4) is feasible.

Item 3 There is some (X, x) such that $X - xx^T$ is positive definite and $l < A_1 \circ X + 2b_1^T x + c_1 < u$.

The following theorem was proved in [12]. Here we provide an alternative proof based on Proposition 2.2.

Theorem 3.7 Item 3 holds if and only if the Slater condition holds for (P_4).

Proof The sufficient direction is trivial. Now we only prove the necessary direction. Since $X - xx^T \in S^n_{++}$, by the Schur complement we know that $Z = \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix}$ is positive. Note that

$$l < A_1 \circ X + 2b_1^T x + c_1 = \begin{bmatrix} A_1 & b_1 \\ {b_1}^T & c_1 \end{bmatrix} \circ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} < u$$

and the last inequality is equivalent to

$$\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 - u \end{bmatrix} \circ Z < 0.$$

It follows from Proposition 2.2 that $Z = \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} = \sum_{i=1}^{n+1} q_i z^i {z^i}^T, q_i > 0, \sum_{i=1}^{n+1} q_i = 1$ such that

$${z^i}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 - u \end{bmatrix} z^i < 0.$$

We thus have

$$\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 - l \end{bmatrix} \circ Z = \sum_{i=1}^{n+1} q_i {z^i}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 - l \end{bmatrix} z^i > 0,$$

which implies that there exist at least an i, say $i=1$, such that ${z^1}^T \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 - l \end{bmatrix} z^1 > 0$.

Denote $z^1 = \begin{bmatrix} x^1 \\ 1 \end{bmatrix}$, then x^1 is a strictly feasible solution point of (P_4). This completes the proof.

3.3 QCQP with multiple constraints

In general, the strong duality does not hold for QCQP problems with two or more constraints; more assumptions are needed to ensure the strong duality. In this subsection, we give a sufficient condition to guarantee the strong duality of QCQP problems with multiple constraints:

$$
\begin{aligned}
(P_5) \qquad & \min \quad x^T A_0 x + 2b_0^T x + c_0 \\
& \text{s. t.} \quad x^T A_i x + 2b_i^T x + c_i \le 0, \quad i = 1, \cdots, m,
\end{aligned}
$$

where $A_i \in S^n, b_i \in R^n, c_i \in R$, for $i = 0, 1, \cdots, m$.

Theorem 3.8 Assume Slater condition holds for (P_5). If the dual problem (D_5) of (P_5) admits a strictly feasible solutions point, then we have $v(D_5) = v(RP_5)$ and these two problems can attain their optimal values, where (D_5) and (RP_5) denote the dual problem of (P_5) and its SDP relaxation, respectively. If Z^* is a solution point of (RP_5) and it can be expressed as

$$
Z^* = \sum_{k=1}^{l} \alpha_k \begin{bmatrix} x_k \\ 1 \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix}^T, \alpha_k \ge 0, \sum_{k=1}^{l} \alpha_k = 1,
$$

and

$$
\begin{bmatrix} x_k \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} \le 0, \quad \forall i = 1, \cdots, m, \quad k = 1, \cdots, l.
$$

Then the strong duality holds for (P_5).

Proof The assumptions ensure that (D_5) and (RP_5) can attain their optimal values, respectively, and $v(D_5) = v(RP_5)$. Since

$$
Z^* = \sum_{k=1}^{l} \alpha_k \begin{bmatrix} x_k \\ 1 \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix}^T,
$$

we have

$$v(RD_5) = \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \circ Z^* = \sum_{k=1}^{l} \alpha_k \begin{bmatrix} x^k \\ 1 \end{bmatrix}^T \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \begin{bmatrix} x^k \\ 1 \end{bmatrix}$$
$$\geq {x^l}^T A_0 x^l + 2b_0^T x^l + c^0 \geq v(P_5).$$

So one has $v(P_5) = v(D_5)$,and it is easy to see that all $x^i, i = 1, \cdots, k$ are optimal solution points of (P_5).

The proposed sufficient condition can be satisfied by some examples. Below we list such an example.

Example 1

$$\begin{array}{lll} (P_6) & \min & x_1^2 + x_2^2 \\ & \text{s. t.} & x_1^2 \geq 1, \\ & & x_2^2 \geq 4. \end{array}$$

It is easy to verify that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 2 \end{bmatrix}$$

are solution points of (P_6). The dual problem of (P_6) is

$$\begin{array}{lll} (D_6) & \max & \gamma \\ & \text{s. t.} & \begin{bmatrix} 1-\lambda_1 & & \\ & 1-\lambda_2 & \\ & & \lambda_1 + 4\lambda_2 - \gamma \end{bmatrix} \succeq 0 \\ & & \lambda_1 \geq 0, \lambda_2 \geq 0, \end{array}$$

which amounts to

$$\begin{array}{ll} \max & \gamma \\ \text{s. t.} & 0 \leq \lambda_1 \leq 1 \\ & 0 \leq \lambda_2 \leq 1 \\ & \lambda_1 + 4\lambda_2 - \gamma \geq 0. \end{array}$$

It is easy to check that

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 5\end{bmatrix}$$

is a solution point of (D_6). Notice that

$$\begin{aligned}
&\mathrm{conv}\left\{\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T \middle| \begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T\begin{bmatrix}-1&&\\ &0&\\ &&1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\le 0, \begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T\begin{bmatrix}0&&\\ &-1&\\ &&4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\le 0\right\}\\
=&\mathrm{conv}\left\{\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T \middle| x_1^2\ge 1, x_2^2\ge 4\right\}\\
=&\left\{\begin{bmatrix}z_{11}&z_{12}&z_{13}\\ z_{21}&z_{22}&z_{23}\\ z_{31}&z_{32}&1\end{bmatrix}\succeq 0 \middle| \begin{bmatrix}-1&&\\ &0&\\ &&1\end{bmatrix}\circ Z\le 0, \begin{bmatrix}0&&\\ &-1&\\ &&4\end{bmatrix}\circ Z\le 0\right\}\\
=&\left\{\begin{bmatrix}z_{11}&z_{12}&z_{13}\\ z_{21}&z_{22}&z_{23}\\ z_{31}&z_{32}&1\end{bmatrix}\succeq 0 \middle| z_{11}\ge 1, z_{22}\ge 4\right\}.
\end{aligned}$$

The (P_6) and its SDP relaxation, namely the dual problem of (D_6), can be written as following, respectively:

$$\begin{aligned}
(P_6)\quad &\min \begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\begin{bmatrix}1&\\ &1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\
&\text{s. t.} \begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T\begin{bmatrix}-1&&\\ &0&\\ &&1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\le 0,\\
&\qquad \begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}^T\begin{bmatrix}0&&\\ &-1&\\ &&4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix}\le 0,
\end{aligned}$$

$$(RP_6)\qquad \min \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \circ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix}$$

$$\text{s. t.} \begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix} \circ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} 0 & & \\ & -1 & \\ & & 4 \end{bmatrix} \circ \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0.$$

It is easy to see that

$$Z^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

is a solution point of (RP_6) and it can be expressed as

$$Z^* = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}^T + \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Here,$[1, 2]^T$ and $[-1, 2]^T$ are both feasible solution points of (P_6). So the proposed sufficient condition is satisfied and the strong duality holds for this problem.

In the following we give another example showing that the strong duality does not hold.

Example 2

$$(P_7) \qquad \min \quad -2x_1^2+2x_2^2+4x_1$$
$$\text{s. t.} \quad x_1^2+x_2^2-4\le 0,$$
$$x_1^2+x_2^2-4x_1+3\le 0.$$

The SDP relaxation of (P_7) is

$$(RP_7) \qquad \min \quad \begin{bmatrix} -2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \circ Z$$
$$\text{s. t.} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \circ Z \le 0,$$
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix} \circ Z \le 0$$
$$Z \succeq 0, Z_{3,3}=1.$$

It is easy to see that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is an optimal solution point of (P_7) and the corresponding objective function value is 0. But the optimal function value of (RP_7) is -1. Hence, the strong duality does not hold for this example. In fact, it is easy to find that

$$Z^* = \begin{bmatrix} 4 & 0 & 7/4 \\ 0 & 0 & 0 \\ 7/4 & 0 & 1 \end{bmatrix} \tag{3.1}$$

is a solution point of (RP_7). Next we illustrate that there does not exist a decomposition of Z^* to guarantee the conditions posed in Theorem 2.3. Without loss of generality, we assume that there Z^* can be decomposed as following:

$$Z^* = \sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_i \\ 0 \\ 1 \end{bmatrix}^T = \sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i^2 & 0 & x_i \\ 0 & 0 & 0 \\ x_i & 0 & 1, \end{bmatrix}, \tag{3.2}$$

where $0 \le \lambda_i \le 1, i = 1, 2$, and $\sum_{i=1}^{2} \lambda_i = 1$. For $i = 1, 2$, if

$$\begin{bmatrix} x_i \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_i \\ 0 \\ 1 \end{bmatrix}^T$$

satisfies the conditions in Theorem 2.2, then we have

$$\begin{cases} x_i^2 - 4 \le 0 \\ x_i^2 - 4x_i + 3 \le 0. \end{cases}$$

So $1 \le x_i \le 2,\ i = 1, 2$. By (1)(2), and $Z^*_{1,1} = 4$, it follows that $x_i = 2,\ i = 1, 2$. This contradicts the fact $Z^*_{1,3} = 7/4$.

From Theorem 3.8 we see that the strong duality holds if and only if there exists a decomposition of Z^* satisfying the conditions in Theorem 3.8. Furthermore, if the strong duality holds, there must be a rank-one solution for the SDP relaxation of (P). So if we get an optimal solution point Z^* of (RP), then the practical procedure in [2][8]may be used to find another optimal solution of the (SDP) problem with a lower rank whenever it exists. If the procedure stops at a solution point while its rank is great than 1, then this shows that the strong duality does not hold. So the procedure in [2][8]provides a performable procedure to check whether or not the strong duality for a QCQP problem holds.

4 QCQP Problems with Complex Variables

In [1][6][17], the rank-one decomposition of symmetric matrix is systematically extended to Hermitian matrices and the strong duality of some QCQP problems with complex variables are addressed therein. In this section, we extend our result in Section 2 to the Hermitian matrix case, and then analyze the strong duality for some QCQP problems with complex variables. First we recall some known results established in [17][7].

Lemma 4.1 Assume $A \in H^n, B \in H^n, Z \in H_+^n, r(Z) = k$. Then there exist $Z = \sum_{i=1}^{k} p^i {p^i}^H, p^i \in C^n$ such that

$$\frac{A \circ Z}{k} = {p^i}^H A p^i, \frac{B \circ Z}{k} = {p^i}^H B p^i, i = 1, \cdots, k.$$

Particularly, if $A \circ Z \leq 0, B \circ Z \leq 0$, then ${p^i}^H A p^i \leq 0, {p^i}^H A p^i \leq 0, i = 1, \cdots, k$.

The following result can be regarded as an extension of Proposition 2.2 to Hermitian matrices.

Proposition 4.2 Assume $A, B \in H^n, a, b \in C^n, c_1, c_2 \in R$. If $Z \in H_+^{n+1}, r(Z) = k, Z_{n+1.n+1} = 1$ such that

$$\begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \circ Z < 0, \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \circ Z < 0.$$

Then there is an expression $Z = \sum_{i=1}^{k} \eta^i {\eta^i}^H, \eta_{n+1}^i \neq 0, i = 1, \cdots, k$, with

$${\eta^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \eta^i < 0, {\eta^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \eta^i < 0, \quad i = 1, \cdots, k.$$

Proof Similar as the proof of Proposition 2.2, we prove it by mathematical induction. The case of $k = 1$ is trivial. If $k = 2$, by Lemma 4.1, one has that there exist η^1 and η^2 such that $Z = \eta^1 {\eta^1}^H + \eta^2 {\eta^2}^H$ and

$${\eta^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \eta^i < 0, \quad {\eta^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \eta^i < 0, \quad i = 1, 2,$$

and $|\eta_{n+1}^1|^2 + |\eta_{n+1}^2|^2 = 1$. If $|\eta_{n+1}^1||\eta_{n+1}^2| \neq 0$, then the conclusion follows immediately. Otherwise, we may assume $\eta_{n+1}^1 = 0$, then $|\eta_{n+1}^2|^2 = 1$.

Set $y^1 = \eta^2 + u\eta^1$, $y^2 = \eta^1 - u\eta^2$, where u is real. Then when $|u| > 0$ is small enough, one has

$${y^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} y^i < 0, \quad {y^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} y^i < 0, i = 1, 2.$$

Note that

$$\begin{aligned}
&y^1{y^1}^H + y^2{y^2}^H \\
=&\eta^2{\eta^2}^H + u(\eta^2{\eta^1}^H + \eta^1{\eta^2}^H) + u^2\eta^1{\eta^1}^H \\
&+\eta^1{\eta^1}^H - u(\eta^1{\eta^2}^H + \eta^2{\eta^1}^H) + u^2\eta^2{\eta^2}^H \\
=&(1+u^2)(\eta^1{\eta^1}^H + \eta^2{\eta^2}^H) \\
=&(1+u^2)Z.
\end{aligned}$$

So $Z = (y^1{y^1}^H + y^2{y^2}^H)/(1+u^2)$. Denote $z^i = y^i/\sqrt{1+u^2}, i=1,2$, then $Z = z^1{z^1}^H + z^2{z^2}^H$ and

$$ {z^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} z^i < 0, {z^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} z^i < 0, \quad i=1,2. $$

According to the definition of z^i, we have $z^i_{n+1} \neq 0, i=1,2$. Assume the assertion holds when $r(Z) = m < k$, that is, $Z = \sum_{i=1}^{m} \eta^i{\eta^i}^H$, and

$$ {\eta^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \eta^i < 0, {\eta^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \eta^i < 0 \quad \eta^i_{n+1} \neq 0, \quad i=1,\cdots,m. $$

We shall prove that the assertion for $r(Z) = m+1 \le k$. By Lemma 4.1, there exists linearly independent vectors $\{\eta^i\}$ such that $Z = \sum_{i=1}^{m+1} \eta^i{\eta^i}^H$ and

$$ {\eta^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \eta^i < 0, {\eta^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \eta^i < 0, \quad i=1,\cdots,m+1. $$

Clearly, if $\eta^i_{n+1} \neq 0, i=1,\cdots,m+1$, the proposition holds. Otherwise, there is at least an integer i such that $\eta^i_{n+1} = 0$. We may assume $\eta^1_{n+1} = 0, \eta^2_{n+1} \neq 0$. Similar to the argument for the case of $r=2$, one has z^1, z^2 such that $z^1{z^1}^H + z^2{z^2}^H = \eta^1{\eta^1}^H + \eta^2{\eta^2}^H$ and

$$ {z^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} z^i < 0, {z^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} z^i < 0, \quad z^i_{n+1} \neq 0, \quad i=1,2. $$

Therefore we get $Z = z^1{z^1}^H + z^2{z^2}^H + \sum_{i=3}^{m+1} \eta^1{\eta^1}^H$. Repeating this procedure for at most m times, we get the assertion.

Applying this extended rank-one decomposition result for complex quadratic functions gives us the following propositions.

Proposition 4.3 Assume $x_0 \in C^n$ satisfies

$$\begin{bmatrix} x_0 \\ 1 \end{bmatrix}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} < 0, \begin{bmatrix} x_0 \\ 1 \end{bmatrix}^H \begin{bmatrix} B & b \\ B^H & c_2 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} < 0.$$

Then, we have

$$\operatorname{conv}\left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^H \middle| \begin{bmatrix} x \\ 1 \end{bmatrix}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, \begin{bmatrix} x \\ 1 \end{bmatrix}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, x \in C^n \right\}$$

$$= \left\{ Z \in H_+^{n+1} \middle| \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \circ Z < 0, \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \circ Z < 0, Z_{n+1,n+1} = 1 \right\} \triangleq C_1^o.$$

Proof Clearly, the sets in both sides are both nonempty and the left-hand one is contained in the right-hand one. Next we show the the other direction. For a given $Z \in C_1^o$, let $r(Z) = k(> 1)$. By Proposition 4.2 we know that there exists η^i such that $Z = \sum_{i=1}^k \eta^i {\eta^i}^H$ and

$$ {\eta^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \eta^i < 0, {\eta^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \eta^i < 0, \quad \eta_{n+1}^i \neq 0, \quad i = 1, \cdots, k.$$

Set $z^i = \eta^i / \eta_{n+1}^i$. Then, we have $z_{n+1}^i = 1, i = 1, \cdots, k$, and

$$ {z^i}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} z^i < 0, {z^i}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} z^i < 0.$$

Since $Z = \sum_{i=1}^k |\eta_{n+1}^i|^2 z^i {z^i}^H$ and $z_{n+1,n+1} = 1$, we get $\sum_{i+1}^k |\eta_{n+1}^i|^2 = 1$. So, it holds that

$$Z \in \operatorname{conv}\left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^H \middle| \begin{bmatrix} x \\ 1 \end{bmatrix}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, \begin{bmatrix} x \\ 1 \end{bmatrix}^H \begin{bmatrix} B & b \\ b^H & c_2 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, x \in C^n \right\}.$$

This proof is complete.

Proposition 4.4 Assume $x_0 \in C^n$ such that

$$\begin{bmatrix} x_0 \\ 1 \end{bmatrix}^H \begin{bmatrix} A & a \\ a^H & c_1 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} \leq 0, \begin{bmatrix} x_0 \\ 1 \end{bmatrix}^H \begin{bmatrix} B & b \\ B^H & c_2 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} \leq 0.$$

Then

$$conv\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^H \middle| \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}A & a\\a^H & c_1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}\le 0, \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}B & b\\b^H & c_2\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}\le 0, x\in C^n\right\}$$

$$=\left\{Z\in H_+^{n+1}\middle| \begin{bmatrix}A & a\\a^H & c_1\end{bmatrix}\circ Z\le 0, \begin{bmatrix}B & b\\b^H & c_2\end{bmatrix}\circ Z\le 0, Z_{n+1,n+1}=1\right\}\triangleq C_1.$$

Proof It suffices to prove that the right-hand set is contained in the left-hand one. Assume $\bar{Z}\in C_1$. Then there exist a series of $\{Z^j\}$, $Z^j\in C_1^o$ such that $Z^j\to\bar{Z}(j\to\infty)$. Note that

$$Z^j\in \mathrm{conv}\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^H \middle| \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}A & a\\a^H & c_1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}< 0, \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}B & b\\b^H & c_2\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}< 0, x\in C^n\right\}.$$

Let $j\to\infty$. We have that

$$\bar{Z}\in \mathrm{conv}\left\{\begin{bmatrix}x\\1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}^H \middle| \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}A & a\\a^H & c_1\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}\le 0, \begin{bmatrix}x\\1\end{bmatrix}^H\begin{bmatrix}B & b\\b^H & c_2\end{bmatrix}\begin{bmatrix}x\\1\end{bmatrix}\le 0, x\in C^n\right\}.$$

This completes the proof.

Based on the previous propositions, we shall show the strong duality between the following complex QCQP:

$$\begin{aligned}(CP)\qquad &\min\ \ x^HA_0x+2Re(b_0^Hx)+c_0\\ &\text{s. t.}\ \ x^HA_1x+2Re(b_1^Hx)+c_1\le 0\\ &\qquad\ \ x^HA_2x+2Re(b_2^Hx)+c_2\le 0,\end{aligned}$$

and its dual problem:

$$\begin{aligned}(DCP)\qquad &\max\ \ \gamma\\ &\text{s. t.}\ \ \begin{bmatrix}A_0+yA_1+zA_2 & b_0+yb_1+zb_2\\ b_0^H+yb_1^H+zb_2^H & c_0+yc_1+zc_2-\gamma\end{bmatrix}\succeq 0,\\ &\qquad\ \ y\ge 0, z\ge 0,\end{aligned}$$

where $A_i\in H^n, b_i\in C^n, c_i\in R, i=0,1,2$, and Re denotes the real part of a complex number. As usual, we assume the Slater condition holds for (CP). That is, there exist $x_0\in C^n$ such that $x_0^HA_1x_0+2Re(b_1^Hx_0)+c_1<0$ and $x_0^HA_2x_0+2Re(b_2^Hx_0)+c_2<0$.

Theorem 4.5 Assume (CP) and its dual problem (DCP) both have strictly feasible solution points. Then, the strong duality holds. That is, $v(CP) = v(DCP)$ where $v(CP)$ and $v(DCP)$ denote the optimal function values of (CP) and (DCP), respectively. Furthermore, both $v(CP)$ and (DCP) are attainable.

Proof The dual problem of (DCP) is the SDP relaxation of (CP), namely

$$
\begin{aligned}
(RCP) \qquad & \min \quad A_0 \circ X + 2Re(b_0^H x) + c_0 \\
& \text{s. t.} \quad A_1 \circ X + 2Re(b_1^H x) + c_1 \le 0, \\
& \qquad\quad A_2 \circ X + 2Re(b_2^H x) + c_2 \le 0, \\
& \qquad\quad \begin{bmatrix} X & x \\ x^H & 1 \end{bmatrix} \succeq 0.
\end{aligned}
$$

Since the Slater condition holds for both (DCP) and (RCP), the strong duality of (SDP) guarantees that $v(DCP) = v(RCP)$ and they are both attainable. Assume (X^*, x^*) is an optimal solution point of (RCP). Denote $Z^* = \begin{bmatrix} X^* & x^* \\ x^{*H} & 1 \end{bmatrix}$, then we have

$$
\begin{bmatrix} A_1 & b_1 \\ b_1^H & c_1 \end{bmatrix} \circ Z^* \le 0, \begin{bmatrix} A_2 & b_2 \\ b_2^H & c_2 \end{bmatrix} \circ Z^* \le 0.
$$

Assume that the rank of Z^* is k. By Proposition 4.4 there exist $z^i, z^i_{n+1} = 1 (i = 1, \cdots, k)$ such that $Z^* = \sum\limits_{i=1}^{k} q_i z^i {z^i}^H$, and

$$
{z^i}^H \begin{bmatrix} A_1 & b_1 \\ b_1^H & c_1 \end{bmatrix} z^i \le 0, {z^i}^H \begin{bmatrix} A_2 & b_2 \\ b_2^H & c_2 \end{bmatrix} z^i \le 0,
$$

and $q_i \ge 0$ with $\sum\limits_{i=1}^{k} q_i = 1$. Denote

$$
z^i = \begin{bmatrix} x^i \\ 1 \end{bmatrix}, x^i \in C^n.
$$

Then, every x^i is a feasible solution point of (CP). Note that

$$
\begin{aligned}
v(RCD) = \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 \end{bmatrix} \circ Z^* &= \sum_{i=1}^{k} q_i \begin{bmatrix} x^i \\ 1 \end{bmatrix}^H \begin{bmatrix} A_0 & b_0 \\ b_0^H & c_0 \end{bmatrix} \begin{bmatrix} x^i \\ 1 \end{bmatrix} \\
&\ge {x^l}^H A_0 x^l + 2\mathrm{Re}(b_0^H x^l) + c^0 \ge v(CP).
\end{aligned}
$$

This implies that $v(DCP) = v(RCD) = v(CP)$ and every x^i is optimal solution point of (CP).

Next we provide a simple proof for the strong duality of QCQP problems with complex variables. The following case with three constraints is considered (corresponding to the real-variable QCQP problem with two constraints):

$$\begin{array}{lll}(CP_1) & \min & x^H A_0 x + 2Re(b_0^H x) + c_0 \\ & \text{s. t.} & x^H A_i x + 2Re(b_i^H x) + c_i \le 0, \quad i = 1, 2, 3.\end{array}$$

Its dual problem is:

$$\begin{array}{lll}(DCP_1) & \max & \gamma \\ & \text{s. t.} & \begin{bmatrix} A_0 + y_1 A_1 + y_2 A_2 + y_3 A_3 & b_0 + y_1 b_1 + y_2 b_2 + y_3 b_3 \\ b_0^H + y_1 b_1^H + y_2 b_2^H + y_3 b_3^H & c_0 + y_1 c_1 + y_2 c_2 - \gamma \end{bmatrix} \succeq 0, \\ & & y \ge 0.\end{array}$$

Theorem 4.6 Suppose Slater condition holds for (CP_1) and the dual problem (DCP_1) has a strictly feasible solution point. If at least one constraint is inactive at the solution of (RCP_1), then $v(CP_1) = v(DCP_1)$, and this optimal value is attainable for (CP_1) as well as (DCP_1).

Proof The dual problem of (DCP_1) is the SDP relaxation of (CP_1):

$$\begin{array}{lll}(RCP_1) & \min & A_0 \circ X + 2Re(b_0^H)x + c_0 \\ & \text{s. t.} & A_i \circ X + 2Re(b_i^H x) + c_i \le 0, i = 1, 2, 3, \\ & & \begin{bmatrix} X & x \\ x^H & 1 \end{bmatrix} \succeq 0.\end{array}$$

The assumptions imply that (DCP_1) and (RCP_1) have the same attainable optimal values. Assume $Z^* = \begin{bmatrix} X^* & x^* \\ x^{*H} & 1 \end{bmatrix}$ is a solution point of (RCP_1). Because there exists at least one inactive constraint at Z^*, we assume that it is the third one at Z^*. That is, $A_3 \circ X^* +$

$2Re(b_3^H x^*) + c_3 < 0$. This impliesthat Z^* is also a solution point of the following problem:

$$
\begin{aligned}
(RCP_2) \qquad & \min \quad A_0 \circ X + 2Re(b_0^H x) + c_0 \\
& \text{s. t.} \quad A_1 \circ X + 2Re(b_1^H x) + c_1 \le 0, \\
& \qquad\quad A_2 \circ X + 2Re(b_2^H x) + c_2 \le 0, \\
& \qquad\quad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0.
\end{aligned}
$$

By Lemma 4.4 we have that there exist a group of $z^i, z^i_{n+1} = 1 (i = 1, \cdots, k)$ and $q_i (i = 1, \cdots, k)$ such that $Z^* = \sum_{i=1}^k q_i z^i {z^i}^H, q_i > 0, \sum_{i=1}^k q_i = 1$ and

$$
{z^i}^H \begin{bmatrix} A_i & b_i \\ b_i^H & c_i \end{bmatrix} z^i \le 0, i = 1, 2,
$$

with $q_i > 0$ with $\sum_{i=1}^k q_i = 1$. Note that Z^* satisfies

$$
\begin{bmatrix} A_3 & b_3 \\ {b_3}^H & c_3 \end{bmatrix} \circ \begin{bmatrix} X^* & x^* \\ x^{*H} & 1 \end{bmatrix} < 0,
$$

from which we know that there exists i, say $i = 1$, such that $(z^1)^H \begin{bmatrix} A_3 & b_3 \\ b_3^H & c_3 \end{bmatrix} z^1 < 0$. Denote $z^1 = \begin{bmatrix} x^1 \\ 1 \end{bmatrix}$. Then, x^1 is a feasible solution point of CP_1. It follows from Theorem 4.1 that x^1 is a solution point of the following QCQP with two constraints:

$$
\begin{aligned}
& \min \quad x^H A_0 x + 2Re(b_0^H x) + c_0 \\
& \text{s. t.} \quad x^H A_i x + 2Re(b_i^H x + c_i \le 0, i = 1, 2.
\end{aligned}
$$

Thus we obtain that

$$
v(CP_1) \le {x^1}^H A_0 x^1 + 2Re(b_0^H x^1) + c_0 = \begin{bmatrix} A_0 & b_0 \\ {b_0}^H & c_0 \end{bmatrix} \circ Z^* = v(RCP_1) = v(DCP_1) \le v(CP_1).
$$

This implies the strong duality of (CP_1).

In [17][16], the proof of Theorem 4.6 is given based on the matrix rank-one decomposition of Hermitian matrices but only for the case of one constraint being inactive. Here, we generate the results and our discussion is valid for other cases where two or three constraints are inactive. With the help of Proposition 4.2, our proof seems to be simpler.

For the following QCQP with an interval constraint and an inequality constraint

$$
\begin{aligned}
(CP_2) \qquad \min \quad & x^H A_0 x + 2Re(b_0^H x) + c_0 \\
\text{s. t.} \quad & x^H A_1 x + 2Re(b_1^H x) + c_1 \le 0, \\
& l \le x^H A_2 x + 2Re(b_2^H x) + c_2 \le u,
\end{aligned}
$$

it is easy to know that it is a special situation of (CP_1) and thus the strong duality holds. We summarize the result in the following corollary. Let us denote by (RCP_2) the SDP relaxation of (CP_2). Since there are at most two active constraints at the optimal solution point of (RCP_2), this corollary follows immediately from Theorem 4.6.

Corollary 4.7 If both (CP_2) and its dual problem (DCP_2) have strictly feasible solution points, then $v(CP_2) = v(DCP_2)$, and this optimal value is attainable for both (CP_2) and (DCP_2).

Last, we would like to mention that a sufficient condition similar as Theorem 3.8 for real-variable cases can be proposed for complex-variable cases. For succinctness, we omit it.

5 Conclusions

In this paper we revisit the strong duality of several scenarios of the quadratically constrained quadratic programming (QCQP) problem. Some QCQP cases with one or multiple constraints are discussed, and both the real- and complex-variable cases are discussed. Some new and simpler proofs are given for the strong duality. Our analysis is based on an extended result of the rank-one matrix decomposition for positive semidefinite and Hermitian matrices. As well studied in the literature, the S-lemma has wide applications in other areas such as control theory, we may take into a closer look at the strong duality of the models arising in such problems (e.g., [2]) and try to find some simpler proofs. We leave it as a topic for future research.

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