常系数扩散方程的三层九点差分格式的稳定性*

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STABILITY OF THREE-LEVEL NINE-POINT DIFFERENCE SCHEME FOR CONSTANT COEFFICIENT DIFFUSION EQUATIONS

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Abstract In this paper, we study a numerical solution of diffusion equation. We propose a three level-nine-point implicit difference scheme and prove the difference scheme is compatible with diffusion equation, second order convergent, unconditionally stable. A numerical experiments show, the difference scheme works well inside domain, but not near the discontinuous initial-boundary points, there are still has a vibration even though it was proved unconditionally stable theoretically. We take an action to solve the disturbance, give an Algorithm, Algorithm says, we must do some primal work at the discontinuous-initial-boundary points, then starting numerical solution according the three level-nine-point implicit difference scheme we proposed in this paper. The numerical example is done once again, and there is no disturbance or vibration, our Algorithm performed well all in domain and on the boundary points with small error and good accuracy, so the Algorithm we recommended is feasible and effective.

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1 Introduction

Diffusion equation is very important in study of fluid flow and heat conducting. It has an important physical background. Numerical solution of the diffusion equation is a hot topic in numerical mathematics. There are three methods of numerical solution, as finite difference method, finite volume method, and finite element method etc. [1], [2], [5], [6], [12] have done systematic researches about finite difference methods of the diffusion equation. [6], [9], [10], [11], [12] gave some implicit difference schemes, still not solve the small vibration beyond the discontinuous initial boundary condition. This paper is organized as follows. In section 2, we construct the three-level nine-point implicit difference scheme for diffusion equation with constant coefficient. In Section 3,4, the truncated error and stability of the difference scheme are analyzed. In Section 5, the numerical example is given. Experimentation shows, if we chose the ratio of time step to the square of space step is bigger, the difference scheme which was unconditionally stable have vibrations unexpectedly near the discontinuous primary boundary points. Then we give an Algorithm in Section 5, eliminate the vibrations. The results of calculation show that the difference scheme is unconditionally stable, the error is small, the accuracy is high, and has no disturbance near the initial boundary value points.

2 Question Raised

Constant coefficient diffusion equation[1]

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \quad x \in R, \quad t > 0, \tag{1}$$

where a is a positive constant. If given initial conditions

$$u(x,0) = g(x), \quad x \in R, \tag{2}$$

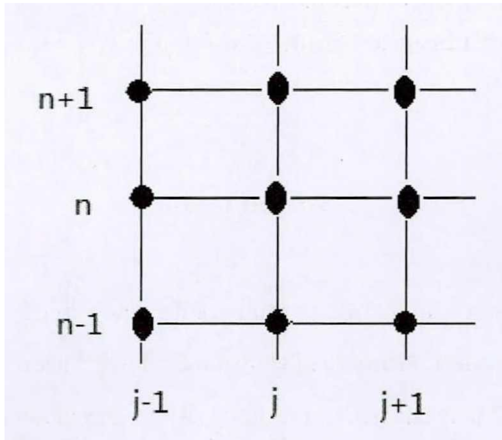

Fig. 1 three-level nine-point implicit difference scheme

then the initial value problem is formed. First, we give the three- level nine-point difference scheme approximation of differential equation (1),

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}-a\frac{1}{3h^2}(\delta_x^2u_j^{n+1}+\delta_x^2u_j^n+\delta_x^2u_j^{n-1})=0, \tag{3}$$

in which τ denotes time step and h denotes space step

$$\delta_x^2u_j=u_{j+1}-2u_j+u_{j-1}. \tag{4}$$

The specific three - level nine-point implicit format is composed when the (4)of each layer is substituted into (3),then we have

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}-\frac{a}{3h^2}$$
$$\left[\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)+\left(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}\right)+\left(u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}\right)\right]=0. \tag{5}$$

These points used in scheme (5) are shown in Figure 1.

3 Truncation error

Give the Taylor expansion for each item of (5),we have

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}=\left[\frac{\partial u}{\partial t}\right]_j^n+\frac{1}{3!}\left[\frac{\partial^3u}{\partial t^3}\right]_j^n\tau^2+\cdots \tag{6}$$

$$\begin{aligned}
u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}= &\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n+1}h^2+\frac{1}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n+1}h^4+\cdots\\
= &\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}h^2+\frac{1}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}h^4\\
&+2\frac{\partial}{\partial t}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau h^2\\
&+\frac{1}{2!}\frac{\partial^2}{\partial t^2}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau^2 h^2+\cdots
\end{aligned}\tag{7}$$

$$u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}=2\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}h^2+\frac{2}{4!}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}h^4+\cdots\tag{8}$$

$$\begin{aligned}
u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}= &\left(u_{j+1}^{n-1}-u_j^{n-1}\right)-\left(u_j^{n-1}-u_{j-1}^{n-1}\right)\\
= &\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}h^2+\frac{1}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}h^4-2\frac{\partial}{\partial t}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau h^2\\
&+\frac{1}{2!}\frac{\partial^2}{\partial t^2}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau^2 h^2+\cdots
\end{aligned}\tag{9}$$

Substitute (6), (7),(8) and (9) into the the difference format (5), and we get

$$\begin{aligned}
&\frac{u_j^{n+1}-u_j^{n}}{2\tau}-\frac{a}{3h^2}\\
&\left[\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)+\left(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}\right)+\left(u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}\right)\right]\\
&=\left[\left[\frac{\partial u}{\partial t}\right]_j^{n}-a\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}\right]+\frac{1}{3!}\left[\frac{\partial^3 u}{\partial t^3}\right]_j^{n}\tau^2-\frac{a}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}h^2\\
&-\frac{a}{3}\frac{\partial^2}{\partial t^2}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau^2+\cdots
\end{aligned}\tag{10}$$

Because (1) is smooth, then we have

$$T(x,\tau)=\frac{1}{3!}\left[\frac{\partial^3 u}{\partial t^3}\right]_j^{n}\tau^2-\frac{a}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}h^2-\frac{a}{3}\frac{\partial^2}{\partial t^2}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n}+\frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^{n}\right]\tau^2+\cdots$$

$T(x,\tau)$ is called truncated error. Therefore, the truncation error of the difference scheme (3) is second order accuracy as $O(\tau^2+h^2)$. Because of $\tau\to 0, h\to 0 \ \Rightarrow T(x,\tau)\ \to 0$. The definition of compatibility shows that the difference scheme (3) is compatible with the differential equation (1).

4 Stability

Reforming the difference scheme(3) into the convenient form of calculation

$$\frac{1}{2}\left(u_j^{n+1}-u_j^{n-1}\right)=\frac{1}{3}a\lambda\left(\delta_x^2 u_j^{n+1}+\delta_x^2 u_j^n+\delta_x^2 u_j^{n-1}\right)$$

in which $\lambda=\frac{\tau}{h^2}$, then we have

$$(1-\frac{2}{3}a\lambda\delta_x^2)u_j^{n+1}=(1+\frac{2}{3}a\lambda\delta_x^2)u_j^{n-1}+\frac{2}{3}a\lambda\delta_x^2 u_j^n. \tag{11}$$

For stability , we give the two - layer equations which is equivalent to (11)

$$\begin{cases}(1-\frac{2}{3}a\lambda\delta_x^2)u_j^{n+1}=(1+\frac{2}{3}a\lambda\delta_x^2)v_j^n+\frac{2}{3}a\lambda\delta_x^2 u_j^n\\ v_j^{n+1}=u_j^n\end{cases} \tag{12}$$

Substitute $\delta_x^2 u_j=u_{j+1}-2u_j+u_{j-1}$ into the two-layer difference equations above,

$$\begin{cases}u_j^{n+1}-\frac{2}{3}a\lambda\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)=v_j^n+\frac{2}{3}a\lambda\left(v_{j+1}^n-2v_j^n+v_{j-1}^n\right)\\ \qquad +\frac{2}{3}a\lambda\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\\ v_j^{n+1}=u_j^n\end{cases} \tag{13}$$

If we take $U=(u,v)^T$, then write the above in vector form, we have

$$\begin{aligned}&\begin{bmatrix}-\frac{2}{3}a\lambda & 0\\ 0 & 0\end{bmatrix}U_{j+1}^{n+1}+\begin{bmatrix}1+\frac{4}{3}a\lambda & 0\\ 0 & 1\end{bmatrix}U_j^{n+1}+\begin{bmatrix}-\frac{2}{3}a\lambda & 0\\ 0 & 0\end{bmatrix}U_{j-1}^{n+1}\\ =&\begin{bmatrix}\frac{2}{3}a\lambda & \frac{2}{3}a\lambda\\ 0 & 0\end{bmatrix}U_{j+1}^n+\begin{bmatrix}-\frac{4}{3}a\lambda & 1-\frac{4}{3}a\lambda\\ 1 & 0\end{bmatrix}U_j^n+\begin{bmatrix}\frac{2}{3}a\lambda & \frac{2}{3}a\lambda\\ 0 & 0\end{bmatrix}U_{j-1}^n\end{aligned} \tag{14}$$

If $U_j^n = V^n e^{ikjh}$

$$\begin{bmatrix} 1+\frac{4}{3}a\lambda & 0 \\ 0 & 1 \end{bmatrix} V^{n+1}e^{ikjh} + \begin{bmatrix} -\frac{2}{3}a\lambda & 0 \\ 0 & 0 \end{bmatrix} V^{n+1}e^{ik(j+1)h} + \begin{bmatrix} -\frac{2}{3}a\lambda & 0 \\ 0 & 0 \end{bmatrix} V^{n+1}e^{ik(j-1)h} = \begin{bmatrix} \frac{2}{3}a\lambda & \frac{2}{3}a\lambda \\ 0 & 0 \end{bmatrix} V^n e^{ik(j+1)h} + \begin{bmatrix} -\frac{4}{3}a\lambda & 1-\frac{4}{3}a\lambda \\ 1 & 0 \end{bmatrix} V^n e^{ikjh} + \begin{bmatrix} \frac{2}{3}a\lambda & \frac{2}{3}a\lambda \\ 0 & 0 \end{bmatrix} V^n e^{ik(j-1)h} \tag{15}$$

Eliminate the common factor e^{ikjh}, we have

$$\begin{bmatrix} 1+\frac{4}{3}a\lambda-\frac{2}{3}a\lambda(e^{ikh}+e^{-ikh}) & 0 \\ 0 & 1 \end{bmatrix} V^{n+1} = \begin{bmatrix} \frac{2}{3}a\lambda(e^{ikh}+e^{-ikh})-\frac{4}{3}a\lambda & 1-\frac{4}{3}a\lambda+\frac{2}{3}a\lambda(e^{ikh}+e^{-ikh}) \\ 1 & 0 \end{bmatrix} V^n, \tag{16}$$

and Since $e^{ikh} = \cos kh + i\sin kh$, $e^{-ikh} = \cos kh - i\sin kh$, we have

$$\begin{bmatrix} 1+\frac{4}{3}a\lambda(1-\cos kh) & 0 \\ 0 & 1 \end{bmatrix} V^{n+1} = \begin{bmatrix} -\frac{4}{3}a\lambda(1-\cos kh) & 1-\frac{4}{3}a\lambda(1-\cos kh) \\ 1 & 0 \end{bmatrix} V^n$$

And use $1-\cos kh = 2\sin^2\dfrac{kh}{2}$, then

$$\begin{bmatrix} 1+\frac{8}{3}a\lambda\sin^2\frac{kh}{2} & 0 \\ 0 & 1 \end{bmatrix} V^{n+1} = \begin{bmatrix} -\frac{8}{3}a\lambda\sin^2\frac{kh}{2} & 1-\frac{8}{3}a\lambda\sin^2\frac{kh}{2} \\ 1 & 0 \end{bmatrix} V^n$$

So the growth factor is available.

$$G(\tau,k)=\begin{bmatrix}1+\frac{8}{3}a\lambda\sin^2\frac{kh}{2} & 0\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}-\frac{8}{3}a\lambda\sin^2\frac{kh}{2} & 1-\frac{8}{3}a\lambda\sin^2\frac{kh}{2}\\ 1 & 0\end{bmatrix}$$

If take $\alpha=\dfrac{8}{3}a\lambda\sin^2\dfrac{kh}{2}$, we have

$$G(\tau,k)=\begin{bmatrix}1+\alpha & 0\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}-\alpha & 1-\alpha\\ 1 & 0\end{bmatrix}=\begin{bmatrix}\frac{-\alpha}{1+\alpha} & \frac{1-\alpha}{1+\alpha}\\ 1 & 0\end{bmatrix}$$

Then the eigenvalue function of $G(\tau,k)$ can be writeten as

$$\mu^2+\frac{\alpha}{1+\alpha}\mu-\frac{1-\alpha}{1+\alpha}=0 \tag{17}$$

Lemma 3.1 The module of G is less than or equal to 1 determined by $|b|\leq 1-c$, $c\leq 1$, as the necessary and sufficient condition of the quadratic equation with real coefficients like

$$\mu^2-b\mu-c=0.$$

From (17), we have $b=\dfrac{\alpha}{1+\alpha}$, $c=-\dfrac{1-\alpha}{1+\alpha}$ and $|b|<1-c=1+\dfrac{1-\alpha}{1+\alpha}=\dfrac{2}{1+\alpha}$ and $c=-\dfrac{1-\alpha}{1+\alpha}<1$ as $\alpha\geq 0$. Use the Lemma 3.1, we can get $|G|\leq 1$, so $|\mu_i|\leq 1\ (i=1,2)$, then the **Von Neumann** condition is satisfied, thus the difference scheme (3) is unconditionally stable.

5 Numerical Example

Given the diffusion equation

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}, & 0<x<1,t>0\\ u(x,0)=\sin\pi x, & 0\leq x\leq 1\\ u(0,t)=u(1,t)=0, & t\geq 0\end{cases} \tag{18}$$

The analytical solution can be obtained by the method of variable separation as

$$u(x,t)=e^{-\pi^2 t}\sin\pi x\ ,\ 0\leq x\leq 1,\ t\geq 0$$

discrete the domain of definition of (18)

$$J=10,\ h=\frac{1}{10},\ x_j=jh(j=0,1,2,\cdots,J),\qquad t_n=n\tau\ (\ n=1,2,\cdots,N)$$

in which τ is time step, $\lambda=\frac{\tau}{h^2}$ is grid ratio. The computational results of different enhanced implicit schemes are compared with analytical solutions. Rearrange the (5), we have

$$\begin{aligned}&-\frac{2}{3}a\lambda u_{j-1}^{n+1}+(1+\frac{4}{3}a\lambda)u_j^{n+1}-\frac{2}{3}a\lambda u_{j+1}^{n+1}\\&=\frac{2}{3}a\lambda u_{j-1}^{n-1}+(1+\frac{4}{3}a\lambda)u_j^{n-1}+\frac{2}{3}a\lambda u_{j+1}^{n-1}\\&+\frac{2}{3}a\lambda u_{j-1}^{n}+\frac{4}{3}a\lambda u_j^{n}+\frac{2}{3}a\lambda u_{j+1}^{n}\end{aligned}\tag{19}$$

Let $j=1:J-1$, we have

$$\begin{aligned}&\begin{pmatrix}1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & 0 & 0 & \cdots & 0\\ -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & 0 & \cdots & 0\\ 0 & -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda\end{pmatrix}\begin{pmatrix}u_1^{n+1}\\u_2^{n+1}\\u_3^{n+1}\\\vdots\\u_{J-1}^{n+1}\end{pmatrix}=\\&\begin{pmatrix}-\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & 0 & 0 & \cdots & 0\\ \frac{2}{3}a\lambda & -\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & 0 & \cdots & 0\\ 0 & \frac{2}{3}a\lambda & -\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \frac{2}{3}a\lambda & -\frac{4}{3}a\lambda\end{pmatrix}\begin{pmatrix}u_1^{n}\\u_2^{n}\\u_3^{n}\\\vdots\\u_{J-1}^{n}\end{pmatrix}+\end{aligned}\tag{20}$$

$$
\begin{pmatrix}
1-\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & 0 & 0 & \cdots & 0 \\
\frac{2}{3}a\lambda & 1-\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & 0 & \cdots & 0 \\
0 & \frac{2}{3}a\lambda & 1-\frac{4}{3}a\lambda & \frac{2}{3}a\lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \frac{2}{3}a\lambda & 1-\frac{4}{3}a\lambda
\end{pmatrix}
\begin{pmatrix}
u_1^{n-1} \\ u_2^{n-1} \\ u_3^{n-1} \\ \vdots \\ u_{J-1}^{n-1}
\end{pmatrix}
$$

$$
+\begin{bmatrix}
-\frac{2}{3}a\lambda u_{j-1}^{n+1}+\frac{2}{3}a\lambda u_{j-1}^{n}+\frac{2}{3}a\lambda u_{j-1}^{n-1} \\
0 \\
0 \\
\vdots \\
-\frac{2}{3}a\lambda u_{J}^{n+1}+\frac{2}{3}a\lambda u_{J}^{n}+\frac{2}{3}a\lambda u_{J}^{n-1}
\end{bmatrix}
$$

Let A, B and C be the coefficient matrix, then we have

$$AU^{n+1}=BU^{n}+CU^{n-1}+f$$

in which

$$
f=\begin{bmatrix}
-\frac{2}{3}a\lambda u_{j-1}^{n+1}+\frac{2}{3}a\lambda u_{j-1}^{n}+\frac{2}{3}a\lambda u_{j-1}^{n-1} \\
0 \\
0 \\
\vdots \\
-\frac{2}{3}a\lambda u_{J}^{n+1}+\frac{2}{3}a\lambda u_{J}^{n}+\frac{2}{3}a\lambda u_{J}^{n-1}
\end{bmatrix}
$$

Because we have the initial-boundary condition

$$u(0,t)=u(1,t)=0, \quad t\geq 0$$

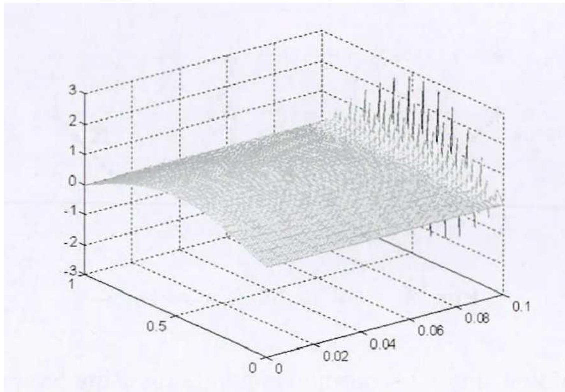

Fig. 2 Crank-Nicolson implicit difference scheme is unconditionally stable, however perturbed unexpectedly near the discontinuous initial-boundary points

so $f = 0$, then the problem becomes

$$U^{n+1} = A_1^{-1} A_2 U^n + A_1^{-1} A_3 U^{n-1}. \tag{21}$$

When $n = 1, U^0$ is known initial condition, U^1 is unknown, and we need to find U^2. How do we obtain U^1? Using (21)directly is impossible, using some explicit method not effective, because there is a discontinuous initial-boundary points laying out there, the discontinuity of these points will affect the stability of the difference scheme[4] see figure 2.

Measures to Eliminate the Vibration

We proposed to use the Saul'ev difference scheme[1] which used initial and boundary condition left to right and right to left

$$\check{u}_j^n = \frac{1 - a\lambda}{1 + a\lambda} u_j^{n-1} + \frac{a\lambda}{1 + a\lambda}(u_{j-1}^n + u_{j+1}^{n-1}) \tag{22}$$

and

$$\hat{u}_j^n = \frac{1 - a\lambda}{1 + a\lambda} u_j^{n-1} + \frac{a\lambda}{1 + a\lambda}(u_{j-1}^n + u_{j+1}^{n-1}) \tag{23}$$

Let $U^n = \frac{1}{2}(\hat{u}_j^n + \check{u}_j^n)$, then we obtain the first level value legally. In addition, Saul'ev scheme is compatible with given partial difference function, whose truncated error is $O(\frac{\tau}{h} + \tau + h^2)$, and it is unconditionally stable. It is worth to showing that Saul'ev Scheme dexterous not to use u_0^0 and u_J^0 , which are the discontinuous initial-boundary points.See the Figure 3.

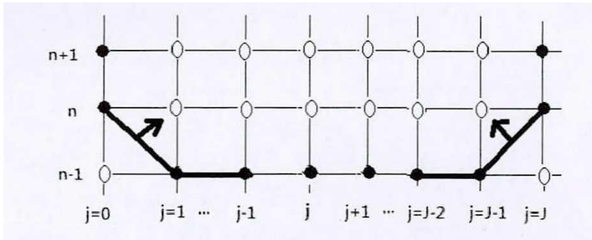

Fig. 3 Saul'ev difference scheme

Thus we have avoiding these discontinuous points by using Saul'ev scheme for the first layers . For example,when $n=0, j=1$ we use(22)

$$\breve{u}_1^1=\frac{1-a\lambda}{1+a\lambda}u_1^0+\frac{a\lambda}{1+a\lambda}(u_0^1+u_2^0)$$

in which u_0^1 is the left boundary condition ,not to use u_0^0. Then still use (22) for $j=2,\ldots,J-1$ we can find U^1. When $n=0, j=J-1$ we use (23)

$$\hat{u}_{J-1}^1=\frac{1-a\lambda}{1+a\lambda}u_{J-1}^0+\frac{a\lambda}{1+a\lambda}(u_J^1+u_{J-2}^0)$$

in which u_J^1 is the right boundary condition , u_J^0 is not used,after then still use (23) for $j=J-2, J-3,\ldots,1$ we can find also U^1.

Let $U^1=\frac{1}{2}((22)+(23))$, thus we obtain the reliable value of U^1.Then back to (21), and use iterating method for $n=1,2,\ldots,N$, three-level-nine-point implicit difference scheme becomes explicit iteration scheme. Even though each step of iteration have to solve the matrix linear system whose coefficient matrix is constant, in the parallel computation system, solving the problem (21) is not as complex as it's representing.

Now we give some information about coefficient matrices. In (21) A_1 represents

$$\begin{pmatrix} 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & 0 & 0 & \cdots & 0 \\ -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & 0 & \cdots & 0 \\ 0 & -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda \end{pmatrix}$$

Let

$$X=\begin{pmatrix} 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda & 0 \\ -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda & -\frac{2}{3}a\lambda \\ 0 & -\frac{2}{3}a\lambda & 1+\frac{4}{3}a\lambda \end{pmatrix}, \quad Y=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A_1=\begin{pmatrix} X & -\frac{2}{3}a\lambda Y & 0 & \cdots & 0 \\ -\frac{2}{3}a\lambda Y & X & -\frac{2}{3}a\lambda Y & \cdots & 0 \\ 0 & -\frac{2}{3}a\lambda Y & X & -\frac{2}{3}a\lambda Y & 0 \\ \vdots & \vdots & \ddots & \ddots & -\frac{2}{3}a\lambda Y \\ 0 & 0 & 0 & -\frac{2}{3}a\lambda Y & X \end{pmatrix}$$

it is clear that A_1 is tri-diagonal block matrix, so are the A_2 and A_3 . According the following theorem, A_1^{-1} exists.

Theorem 1[3] If $A \in \mathbb{R}^{n\times n}$ and

$$\delta=\min_{1\le j\le n}\left(|a_{jj}|-\sum_{\substack{i=1\\ i\ne j}}|a_{ij}|\right)\ge 0 \tag{24}$$

then

$$\|A^{-1}\|_1\le\frac{1}{\delta}.$$

We can see that $A_1^{-1}A_2$ and $A_1^{-1}A_3$ are decomposed tri-diagonal block matrices that makes it easy to solve the linear system (21).

Theorem 2 The finite difference scheme (5) is uniquely solvable by (21).

We will give an Algorithm which no-vibration.

Step 1 Input initial and boundary condition of Heat Equation;

Step 2 Input coefficient matrices A_1, A_2, A_3, calculating A_1^{-1} and $A_1^{-1}A_2, A_1^{-1}A_3$;

Step 3 initial condition $\rightarrow U^0$, calculating U^1 by (22);

Step 4 iteration step:n=1:N ,iterating (21).

$$U^{n+1} = A_1^{-1} A_2 U^n + A_1^{-1} A_3 U^{n-1}. \tag{25}$$

Step 5 Comparing two successive iterations by checking the following inequality

$$\|U^{n+1} - U^n\| \le \epsilon, \forall \epsilon \ge 0$$

if the inequality holds, stop iteration, then go to step 6, else go back step 4;

Step 6 Numerical Solution of diffusion equation is U^{n+1}. Solving the the numerical example (18) according to the computational phases above , we have the good numerical solution we need, see the table 1 and figure 4.

Table 1 The approximate value of the solution is calculated by using the three-level nine-point implicit scheme.Error is the difference between thee-level nine-point and exact solution

λ	$\theta = \frac{1}{2}$ Crank-Nicolson	three − levels nine − point differencescheme($\alpha = 1$)	Exact solution	Error
0.25	0.0189519	0.0189516	0.0183519	0.3×10^{-6}
0.5	0.0189407	0.0189517	0.0183519	0.2×10^{-6}
1.0	0.0188963	0.0183519	0.0183519	0
2.0	0.0187186	0.0183514	0.0183519	0.5×10^{-6}
4.0	0.0180093	0.0183517	0.0183519	0.2×10^{-6}
8.0	0.0152002	0.0183519	0.0183519	0

6 Summary

The three-level nine-point difference scheme (3) which is discussed in this paper is unconditionally stable. However when the λ is bigger near the discontinuous initial-boundary points, the implicit difference scheme gived in this paper is perturbed unexpectedly . This

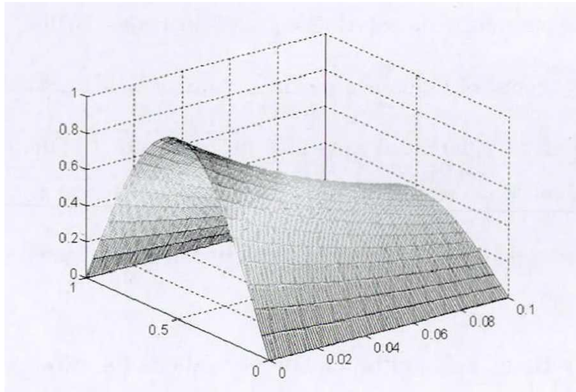

Fig. 4 three-level nine point implicit difference scheme combining Saul'ev scheme

phenomenon alerts us to avoiding these discontinuous initial-boundary points as far as possible , thus we using Saul'ev scheme at the first level before using the the three-level nine-point implicit difference scheme.

The result of numerical experiment shows that Saul'ev scheme combining with the three-level nine-point difference scheme performs well with small error and good accuracy via big λ . All the processing of algorithm shows the three-level nine-point difference scheme (3) which discussed in this paper is unconditionally stable, compatible, feasible and effective.

参考文献

[1] Lu J P and Guan Z. Numerical Solution Of partial differential equation. Tsinghua University Press (Second edition), 2015, (33): 86,122.

[2] Hu J W and Tang M. Numerical Solution of partial differential equation. Nan Kai University press, 1998.

[3] Gene H, Golub, Charles F. Van Loan. Matrix Computation. The Johns Hopkins University Press, Baltimore, Maryland, 2013, 196–199.

[4] Amina Sabir, Abdurexit Abduwali.Study the variation near the boundary of the approximate solution of the heat transfer equation Jornal of capital Normal University(Second Edition)Apr, 2013.

[5] Steven Wray. Alternating Direction Implicit Finite Difference Methods for the Heat Equation on General Domains in Two and Three Dimensions.Science in Applied Mathematics and

Statistics-Computational and Applied Mathematics Specialty, 2019.

[6] Samarskii A A. The theory of difference schemes. Marcel Dekker, New York, 2001.

[7] Selvadurai A P S. Partial differential equations in mechanics 1: Fundamentals, Laplace equation, diffusion equation, wave equation. Springier-Vera, Berlin, 2000.

[8] Thomas J W. Numerical partial differential equations: Finite difference methods. Springer-Verlag, New York, 1995.

[9] Mehdi Dehghan. Fully implicit finite differences methods for two-dimensional diffusion with a non-local boundary condition.Journal of Computational and Applied Mathematics, 1999, (106): 255–269.

[10] Jiraporn Janwised, Ben Wongsaijai, Thanasak Mouktonglang, and Kanyuta Poochinapan.A Modified Three-Level Average Linear-Implicit Finite Difference Method for the Rosenau-Burgers Equation.Advances in Mathematical Physics,Volume, 2014, 11-55.

[11] Yang W Y, Cao W W and ,Tae-Sang Chung. John Morris. Applied Numerical Methods Using MATLAB. A John Wileysons INC Publication, 2005: 416–415.

[12] David Kincaid,Ward Cheney Wang G R. Numerical Analysis Mathematics of Scientific Computing. 2005.