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A Numerical Solution of the Convective Diffusion Equation with Constant Coefficient

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Abstract: A numerical method of constant coefficient convection diffusion equation is studied. An implicit difference scheme is established for convective dominance and diffusion dominance respectively. The truncation error, convergence and stability of this scheme are discussed. Theoretical analysis and experimental results show that when the grid ratio is properly selected, the difference scheme is quite stable. It can be extended directly to the two-dimensional problem, and all the advantages of the difference scheme can be maintained. On this basis, an implicit split center difference algorithm is presented.

Keywords: convection dominant; convection diffusion equations; implicit difference scheme

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0 Introduction

Convection dominated convection diffusion equation is often used in mechanics, physics and other fields of application, which can describe many physical phenomena as mass, heat transport process and reflect the expansion process of pollution[1-4]. There are a lot of papers have discussed the numerical solution of convection-diffusion equation with high accuracy, good stability and suitable for small diffusion coefficient in recent years. For the purpose of eliminating the numerical oscillation caused by convection dominance, Dougals-Russell[5] proposed the characteristic finite element method and the characteristic finite difference method, which has attracted the attention of many scholars because of its effectiveness in numerical calculation[6-9]. These methods have been widely used in many fields in recent decades, see references [10-13].

Moreover, some new methods are proposed, such as upwind difference method, generalized upwind difference method, modified finite element method, feature line method, streamline diffusion method, mixed finite element method, finite volume method (generalized difference method), double mesh algorithm and so on. The above method improves the traditional spoiler method, but it also has many insurmountable effects. The streamline diffusion method reduces the numerical diffusion, but artificially imposes the streamline direction. The modified finite element method can be flexible with large time step without reducing the precision of approximation and has high stability at the front of flow front, which eliminates the numerical diffusion phenomenon, but too dissipate. The mixed finite element method for the convection dominant diffusion equation, for the convection part of the equation is discretion by the characteristic difference method, the diffusion part of the equation is discretion by the mixed element method. The algorithm can improve the efficiency without reducing the approximation accuracy and avoid the numerical diffusion phenomenon. However, for the larger time

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step, there is still a non-physical oscillation.

It is difficult to getting numerical solutions for strong convection-dominant problems. There are lot of scholars proposed corresponding improvement measures, as combined with finite difference method and finite element method[14-18]. Dougals and Russell[5] proposed the feature finite element method and characteristic finite difference method.

Besides, there's the main method is upwind discrete technique. Axelsson and Gustafsson[19] gave the modified upwind format exhibits second-order accuracy. But for the strong convective dominance problem, the format is still only one order accuracy, the usual upwind format to produce serious numerical dissipation phenomenon.

The above method improves the traditional method, but it also has many insurmountable defects: the streamline dissipation method reduces the numerical diffusion, the disadvantage is: artificially impose the direction of the streamline;

The modified finite element method can adopt a large time step without reducing the rescission of the approximation, has high stability, eliminates the numerical diffusion phenomenon, but also reflects the dissipation phenomenon. Douglas[16] proposed a feature line correction digitization method for the flow diffusion equation. This method can overcome the numerical oscillation effectively and guarantee the stability of the numerical solution by using the physical and mechanical properties of the convection diffusion problem.

At present, the feature difference method is still a hot topic at home and abroad, and new processing techniques are emerging. There's a lot of theory and research on this topics. Many scholars have carried out a lot of research in this field. However, methods above usually used to linear interpolation on convection terms, it does not oscillate, but the accuracy is low, in the spatial direction only quadratic interpolation although the accuracy is 2, but it is possible to oscillation on the boundary layer problem and gradient problem.

Application is glorious of numerical methods in many fields. Soil physics scholars, environmental scientists and agricultural sub-family have also carried out a large number of indoor and outdoor water-salt experimental research and numerical simulation.

In a word, for the convection dominant diffusion problem, the low-order scheme has serious numerical dissipation, and the high-order scheme is prone to numerical dispersion and non-physical oscillation. Therefore, it is necessary to construct a numerical method with high precision, good stability, suitable for small diffusion coefficient and can reflect the characteristic properties of hyperbolic equation. Many scholars have studied the numerical solution in the difference way, we use the finite difference method to solving the convective diffusion equation.

1 One dimensional convection dominant problem

We consider one dimensional convective-diffusion equation with constant coefficients[20]

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},x\in\mathbf{R},t>0,\tag{1}$$

in which a,ν is constant and $\nu>0$ (when $a\leqslant 0,\nu\leqslant 0$ then the initial-value problem is called not well-posed). We define $Pe=\frac{a}{\nu}$. If the Pe is small, the convection effect is relatively weak. In such problems, diffusion dominates, and the equations are elliptic or parabolic. If the number of Pe is large, the diffusion of solute molecules is slow relative to the fluid velocity. In such problems, the convection is dominant and the equations have the characteristics of hyperbolic equations. Since when $a>\nu$, the problem performs convection-dominant cases, if the problem becomes strong convection dominance problem.

The initial value is given by

$$u(x,0)=g(x),\ x\in \mathbf{R}, \tag{2}$$

which describes the diffusion of a substance in a medium that is moving with speed v, the unknown function $u(x,t)$ is the concentration of the diffusing substance. Generally it is difficult to write down a formula for a classical solution. We consider a numerical solution which depends upon its compatibility, stability and convergence. The initial-boundary problem

$$\begin{cases} \dfrac{\partial u}{\partial t}+a\dfrac{\partial u}{\partial x}=\nu\dfrac{\partial^2 u}{\partial x^2},\ x\in\Omega,\ t>0, \\ u(x,0)=g(x),\ 0\leqslant x\leqslant l, \\ u(0,t)=\phi_1(t),\ u(1,t)=\phi_2(t),\ t>0. \end{cases}$$

where $\Omega=\{x|\ 0<x<l\}$. When $x\in\Omega$, $u=u(x,t)$ is the sufficient smooth solution of equation (1), $\partial\Omega(x,t)$ is boundaries.

2 Numerical solution

Here we establish the implicit difference scheme

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{a}{2}\left(\frac{u_{j+1}^n-u_{j-1}^n}{2h}+\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h}\right)=\frac{\nu}{2}\left(\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}\right), \tag{3}$$

the initial condition discretion $u_j^0=g_j$. It is a two-layers implicit difference scheme which involves six points $u_{j-1}^{n+1}, u_j^{n+1}, u_{j+1}^{n+1}, u_{j-1}^n, u_j^n, u_{j+1}^n$.

Rewrite the initial-value problem (1) as

$$L(u,t)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}=0,\ u(x,0)=g(x), \tag{4}$$

and the difference scheme (3) as

$$\begin{cases} D(u_j,t_n)=\dfrac{u_j^{n+1}-u_j^n}{\tau}+\dfrac{a}{2}\displaystyle\sum_{k=n,n+1}\dfrac{u_{j+1}^k-u_{j-1}^k}{2h}-\dfrac{\nu}{2}\displaystyle\sum_{k=n,n+1}\dfrac{u_{j+1}^k-2u_j^k+u_{j-1}^k}{h^2}=0, \\ u_j^0=g_j. \end{cases} \tag{5}$$

Now we consider the compatibility of (3) with initial-value problem (1).

2.1 Truncated error

Suppose $u(x,t)$ be a sufficient smooth solution of initial-valued problem (1). Each item of (5) gives Taylor expanding at the point of (x_j,t_n), we have

$$\frac{u_j^{n+1}-u_j^n}{\tau}=\left[\frac{\partial u}{\partial t}\right]_j^n+\frac{1}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n\tau+\frac{1}{6}\left[\frac{\partial^3 u}{\partial t^3}\right]_j^n\tau^2+o(\tau^3). \tag{6}$$

$$\begin{aligned}\frac{u_{j+1}^n-u_{j-1}^n}{2h}=&\left[\left[\frac{\partial u}{\partial x}\right]_j^n+\frac{1}{6}\left[\frac{\partial^3 u}{\partial x^3}\right]_j^n h^2+\left[\frac{\partial u}{\partial t}\right]_j^n\frac{\tau}{h}\right]+\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n\tau^2 \\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left[\frac{\partial u}{\partial t}\right]_j^n\tau h+\frac{1}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n\tau^2 h\right]_j^n,\end{aligned} \tag{7}$$

$$\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}=\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n+\frac{1}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^n h^2+\frac{\partial}{\partial t}\left[\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n+\frac{1}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^n\right]\tau h^2$$

$$+\frac{1}{2}\frac{\partial^2}{\partial t^2}[[\frac{\partial^2 u}{\partial x^2}]_j^n+\frac{1}{12}[\frac{\partial^4 u}{\partial x^4}]_j^n]\tau^2h^2, \tag{8}$$

substitute (6),(7),(8) and the Talyor expandings into (5), then

$$D(u_j,t_n)=[\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}]_j^n+\frac{1}{2}[\frac{\partial^2 u}{\partial t^2}]_j^n\tau+\frac{a}{3}[\frac{\partial^3 u}{\partial x^3}]_j^n h^2+\frac{\nu}{12}[\frac{\partial^4 u}{\partial x^4}]_j^n h^2$$
$$+\frac{\partial}{\partial t}[a[\frac{\partial u}{\partial x}]_j^n+\frac{a}{3}[\frac{\partial^3 u}{\partial x^3}]_j^n+\nu[\frac{\partial^2 u}{\partial x^2}]_j^n+\frac{\nu}{12}[\frac{\partial^4 u}{\partial x^4}]_j^n]\tau h^2+O(\tau^2+\tau h^2+h^2), \tag{9}$$

then we have

$$\lim_{h\to 0,\tau\to 0}T(x,\tau)=\lim_{h\to 0}(D(u_j,t_n)-L(u,t))\to 0.$$

The difference equation tends to initial–valued problem, the compatibility is hold on. The differ–ence scheme (3) improves precision on time space $O(\tau^2+h^2+\tau h^2)$.

2.2 Stability

Generally use the Fourier transformation method for stability of the constant coefficient problem. Rearrange the difference scheme (3), we have

$$u_j^{n+1}+\frac{1}{4}\lambda(u_{j+1}^{n+1}-u_{j-1}^{n+1})-\frac{1}{2}\mu(u_{j+1}^{n+1}-2u_{j1}^{n+1}+u_{j-1}^{n+1})=u_j^n-\frac{1}{4}\lambda(u_{j+1}^n-u_{j+1}^n)+\frac{1}{2}\mu(u_{j+1}^n-2u_j^n+u_{j-1}^n), \tag{10}$$

where $\lambda=\frac{\tau}{h}>0,\mu=\frac{\tau}{h^2}>0.$

Theorem 1 The difference scheme (3) is stable .

Proof Use the Fourier transforming method to get growth factor of the difference scheme, let $u_j^n=\nu^n\cdot e^{ikjh}$ on (10), we have

$$\nu^{n+1}\cdot e^{ikjh}+\frac{1}{4}\lambda(\nu^{n+1}\cdot e^{ik(j+1)h}-\nu^{n+1}\cdot e^{ik(j-1)h})-\frac{1}{2}\mu(\nu^{n+1}\cdot e^{ik(j+1)h}-2\nu^{n+1}\cdot e^{ikjh}+\nu^{n+1}\cdot e^{ik(j-1)h})$$
$$=\nu^n\cdot e^{ikjh}-\frac{1}{4}\lambda(\nu^n\cdot e^{ik(j+1)h}-\nu^n\cdot e^{ik(j-1)h})+\frac{1}{2}\mu(\nu^n\cdot e^{ik(j+1)h}-2\nu^n\cdot e^{ikjh}+\nu^n\cdot e^{ik(j-1)h}).$$

Multiple e^{-ikjh} to both side of equation, we have

$$[1+\frac{1}{4}\lambda(e^{ikh}-e^{-ikh})-\frac{1}{2}\mu(e^{ikh}+e^{-ikh})+\mu]\nu^{n+1}=[1-\frac{1}{4}\lambda(e^{ikh}-e^{-ikh})+\frac{1}{2}\mu(e^{ikh}+e^{-ikh})-\mu]\nu^n,$$

then

$$\nu^{n+1}=\frac{1-\frac{1}{4}\lambda(e^{ikh}-e^{-ikh})+\frac{1}{2}\mu(e^{ikh}+e^{-ikh})-\mu}{1+\frac{1}{4}\lambda(e^{ikh}-e^{-ikh})-\frac{1}{2}\mu(e^{ikh}+e^{-ikh})+\mu}\nu^n.$$

So we have that the growth factor

$$G(\tau,k)=\frac{1-\frac{1}{4}\lambda(2i\sin kh)+\frac{1}{2}\mu(2\cos kh)-\mu}{1+\frac{1}{4}\lambda(2i\sin kh)-\frac{1}{2}\mu(2\cos kh)+\mu}=\frac{1-\mu+\mu\cos kh-i\frac{\lambda}{2}\sin kh}{1+\mu-\mu\cos kh+i\frac{\lambda}{2}\sin kh},$$

and check it's second power of mode, we have

$$|G|^2=\frac{(1-\mu+\mu\cos kh)^2-(\frac{\lambda}{2}\sin kh)^2}{(1+\mu-\mu\cos kh)^2+(\frac{\lambda}{2}\sin kh)^2},$$

i e,

$$|G|^2-1=\frac{-4\mu(1-\cos kh)}{(1+\mu-\mu\cos kh)^2+(\frac{\lambda}{2}\sin kh)^2},$$

Because $1-\cos kh \geqslant 0$, since the denominator of the above is positive, then $|G|^2-1\leqslant 0$. the difference scheme (3) is stable.

2.3 Convergence

Lax and Richmyer[21] gives the Lax equivalence theorem which helps to determine the convergence when we don't know the exact solution.

Theorem 2[20] (Lax equivalence theorem)Given a well-posed linear initial boundary value problem and a compatible difference scheme, the stability of the difference scheme is a necessary and sufficient condition of it's convergences.

The Lax equivalence theorem can be used when the initial value problem is linear ,well-posed and has the periodic initial and boundary condition. The problem (1) is a well-posed first-Dirichlet initial-boundary problem with periodic initial condition.

When $\tau\to 0, h\to 0$, we have $e_j^n = u(x_j,t_n)-u_j^n\to 0$, the difference scheme (3) is converges to problem (1).

2.4 Greedy algorithm

Theorem 3 The problem (1) is uniquely solved by (3) directly.

Proof According to the time layers, rewrite (3),

$$(-\frac{a\lambda}{2}-\nu\mu)u_{j-1}^{n+1}+(2+2\nu\mu)u_j^{n+1}+(\frac{a\lambda}{2}-\nu\mu)u_{j+1}^{n+1}=(\frac{a\lambda}{2}+\nu\mu)u_{j-1}^{n}+(2-2\nu\mu)u_j^{n}+(-\frac{a\lambda}{2}+\nu\mu)u_{j+1}^{n}, \quad (11)$$

where $\lambda=\frac{\tau}{h}, \mu=\frac{\tau}{h^2}$. Let $j=1,2,\cdots,J-1$, then (11) represents a linear system, the coefficient matrices are

$$A=\begin{bmatrix} 2+2\nu\mu & \frac{a\lambda}{2}-\nu\mu & 0 & 0 & 0 \\ -\frac{a\lambda}{2}-\nu\mu & 2+2\nu\mu & \frac{a\lambda}{2}-\nu\mu & 0 & 0 \\ 0 & -\frac{a\lambda}{2}-\nu\mu & 2+2\nu\mu & \frac{a\lambda}{2}-\nu\mu & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2+2\nu\mu & \frac{a\lambda}{2}-\nu\mu \\ 0 & 0 & 0 & -\frac{a\lambda}{2}-\nu\mu & 2+2\nu\mu \end{bmatrix}, \quad B=\begin{bmatrix} 2-2\nu\mu & -\frac{a\lambda}{2}+\nu\mu & 0 & 0 & 0 \\ \frac{a\lambda}{2}+\nu\mu & 2-2\nu\mu & -\frac{a\lambda}{2}+\nu\mu & 0 & 0 \\ 0 & \frac{a\lambda}{2}+\nu\mu & 2-2\nu\mu & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2-2\nu\mu & -\frac{a\lambda}{2}+\nu\mu \\ 0 & 0 & 0 & \frac{a\lambda}{2}+\nu\mu & 2-2\nu\mu \end{bmatrix},$$

$$f=((-\frac{a\lambda}{2}-\nu\mu)u_0^{n+1}+(\frac{a\lambda}{2}+\nu\mu)u_0^n, 0\cdots, 0,(\frac{a\lambda}{2}-\nu\mu)u_M^{n+1}+(-\frac{a\lambda}{2}+\nu\mu)u_M^n)^T.$$

Let $U^{n+1}=(u_1^{n+1},u_2^{n+1},\cdots,u_{M-1}^{n+1})^T, U^n=(u_1^n,u_2^n,\cdots,u_{M-1}^n)^T$, the linear system (11) represented as

$$U^{n+1}=A^{-1}BU^n+A^{-1}f. \quad (12)$$

Then we have the Greedy algorithm here.

Greedy Algorithm:

Step 1. Input $\tau,\lambda,\nu,u^0=u_j^0,u_0^n,u_M^n$;

Step 2. Calculating f,A,B and $A^{-1}B, A^{-1}f$;

Step 3. Iterating $U^{n+1}=A^{-1}BU^n+A^{-1}f$;

Step 4. If $e_j^n=u_j^n-u(x_j,t_n)\to 0$, then stop, or go back step 3;

Step 5: Output u_j^{n+1}, and give the Matlab picture.

Theorem 4 The sufficient conditions to find a unique solution of the linear system (12) using the catch-up method is these inequalities are hold on at a time:

1) $|\beta_1| > |\gamma_1|$; 2) $|\beta_1| > |\alpha_j| + |\gamma_1|$, $\alpha_j \cdot \gamma_j \neq 0, j = 2, \cdots, J-2$; 3) $|\beta_{J-1}| > |\alpha_{J-1}|$,

where $\alpha_j = -\dfrac{a\lambda}{2} - \nu\mu, \beta_j = 2 + 2\nu\mu, \gamma_j = \dfrac{a\lambda}{2} - \nu\mu, j = 2, \cdots, J-1$.

Linear system (12) proper with the convection dominant cases, which is second ordered accuracy on convection terms, time step and diffusion terms.

Theorem 5 If $0 \leqslant \dfrac{a\lambda}{2} - \nu\mu \leqslant 2$, then the coefficient matrix A in (12) is diagonally dominant.

Proof Because $\lambda \geqslant 0, \mu \geqslant 0, \nu \geqslant 0, a \geqslant 0$, and $0 \leqslant \dfrac{a\lambda}{2} - \nu\mu \leqslant 2$ hold on, then

$$|2 + 2\nu\mu| - \left(\left|-\frac{a\lambda}{2} - \nu\mu\right| + \left|\frac{a\lambda}{2} - \nu\mu\right|\right) \geqslant 0. \tag{13}$$

So A is the diagonally-dominant matrix. If $\dfrac{a\lambda}{2} - \nu\mu \leqslant 0$, then inequality (13) is hold.

Corollary 1 If theorem 5 holds on, then theorem 4 holds on and (12) has a unique solution.

Theorem 6 If $A \in \mathrm{R}^{n\times n}$ and $\delta_A = \min\limits_{1\leqslant j\leqslant n}\left(|a_{jj}| - \sum\limits_{i=1, i\neq j}|a_{ij}|\right) \geqslant 0$, then A^{-1} exists, and $\|A^{-1}\| \leqslant \dfrac{1}{\delta_A}$.

Proof

$$\begin{aligned} U^{n+1} &= A^{-1}BU^n + A^{-1}f = (A^{-1}B(A^{-1}BU^{n-1} + A^{-1}f) + A^{-1}f) \\ &= (A^{-1}B)^{n+1}U^0 + (A^{-1}B)^n A^{-1}f + \cdots + (A^{-1}B)^1 A^{-1}f + A^{-1}f \\ &= (A^{-1}B)^{n+1}U^0 + \frac{1-(A^{-1}B)^n}{1-A^{-1}B}A^{-1}f. \end{aligned} \tag{14}$$

If $\rho(A^{-1}B) < 1$, then the difference iteration method is convergent.

Because of (14), we have

$$\begin{aligned} \|U^{n+1}\| &\leqslant \|(A^{-1}B)^{n+1}\| \cdot \|U^0\| + \left\|\frac{1-(A^{-1}B)^n}{1-A^{-1}B}\right\| \cdot \|A^{-1}\| \cdot \|f\| \\ &\leqslant \|(A^{-1}B)^{n+1}\| \cdot \|U^0\| + \frac{\|f\|}{\delta_A^{n+1}} \cdot \left\|\frac{A^n - B^n}{A-B}\right\|, \end{aligned} \tag{15}$$

where $A-B$ is a diagonal matrix, and $\|A-B\|, \|A^n - B^n\|$ is a positive real number, then there is a real number K, suppose $\left\|\dfrac{A^n - B^n}{A-B}\right\| < K$, and let $G = A^{-1}B, F = A^{-1}f$, then (15) becomes

$$\|U^{n+1}\| \leqslant \|G^{n+1}\| \cdot \|U^0\| + \frac{K \cdot \|F\|}{\delta_A^n}. \tag{16}$$

Because $\rho(A^{-1}B) < 1$, then the difference scheme (3) is convergent to initial-boundary problem (1), and linear system (12) solves (1) directly, (1) has a unique numerical solution. Although difference scheme (3) is unconditionally stable in theories, in practice it's still receives certain restrictions.

2.5 Numerical example 1

Considering the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = v\dfrac{\partial^2 u}{\partial x^2}, \ 0 < x < 1, t > 0, \\ u(x,0) = \sin \pi x, \ 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \ t > 0. \end{cases} \tag{17}$$

The analytical solution is $u(t,x) = \mathrm{e}^{-\pi^2 t}\sin \pi x$. We seek the numerical solution of $u(t,x)$ in domain. Initial value problem has a first-order derivative $\dfrac{\partial u}{\partial t}$, and second order in x from the derivative with two boundary condition. This comes from the typical application of specification of conditions at the boundaries of a physical system.

Let $J=10, h=\frac{1}{10}, x_j=jh(j=0,1,\cdots,J), \lambda=\frac{\tau}{h}, \mu=\frac{\tau}{h^2}$ is grid ratio. Discreet the initial and boundary condition

$$u(x,0)=g(x)=\sin\pi x, x\in\mathbf{R},\ U^0=u_j^0=g(x_j)=\sin\pi x_j, x_j\in\mathbf{R}, j=0,1,\cdots,J-1,$$
$$u(0,t)=u(1,t)=0,\ u_0^n=u_J^n=0,\ u_0^{n+1}=u_J^{n+1}=0.$$

From (12), we can see that $A^{-1}B$ is a constant matrix, we solve the initial–boundary problem (17) by solving linear system below

$$U^{n+1}=A^{-1}BU^n=\cdots=(A^{-1}B)^n U^0. \tag{18}$$

Because $\rho(A^{-1}B)<1$ holds on, we use Matlab to solve the problem directly. When $a=\nu$, the numerical solution is very smooth, and converges the analytic solution quickly, see Fig.1. On the convection term, time step and diffusion term in (17), which proper with the convection dominant cases. It's second or–dered accuracy on convection terms.

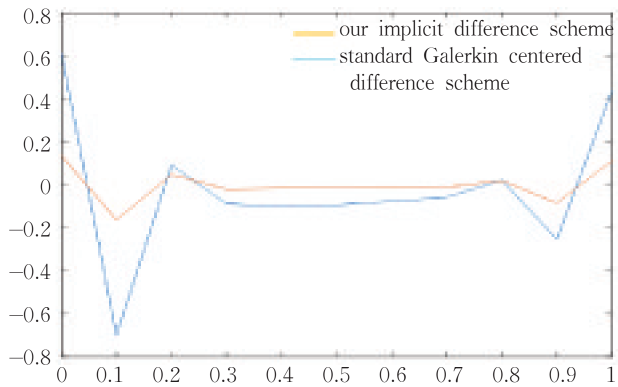

Fig.1 The implicit difference method(3) and the standard Galerkin center difference scheme

For the same mesh ratio, when $a>\nu>1>0$, the numerical solution begins to dispersion. If $a>1>\nu>0$, then the numerical solution appears severe dispersion. The difference scheme proposed in this paper really embodies its advantages at this time, see Fig.2(a). When the mesh ratio is too large or too small, the numerical solution is far above or below the analytic solution, and convergences need is too long time, see Fig.2(b).

If the mesh ratio is quite appropriate, and $a\geqslant\nu>1$, the convergences of (3) is ideal. When se–lect the mesh ratio casually, ν approximate to a, the difference format (3) is still ideal.

For $a>1>\nu>0$ cases, we divided into a in the difference scheme, since we can effectively eliminate the dissipation, see Fig.3.

Therefore, for the severe convection dominant cases, the numerical solution affected by mesh–grid ratio, and the coefficients a, ν. The necessary condition of convergences of (3) is $0\leqslant a\lambda/2-\nu\mu\leqslant 2$.

Corollary 2 There are statements about the relationship between a, ν and grid ratio:

1) For fixed ν, if $Pe\approx 1$, the convergence of numerical solutions is optimal.

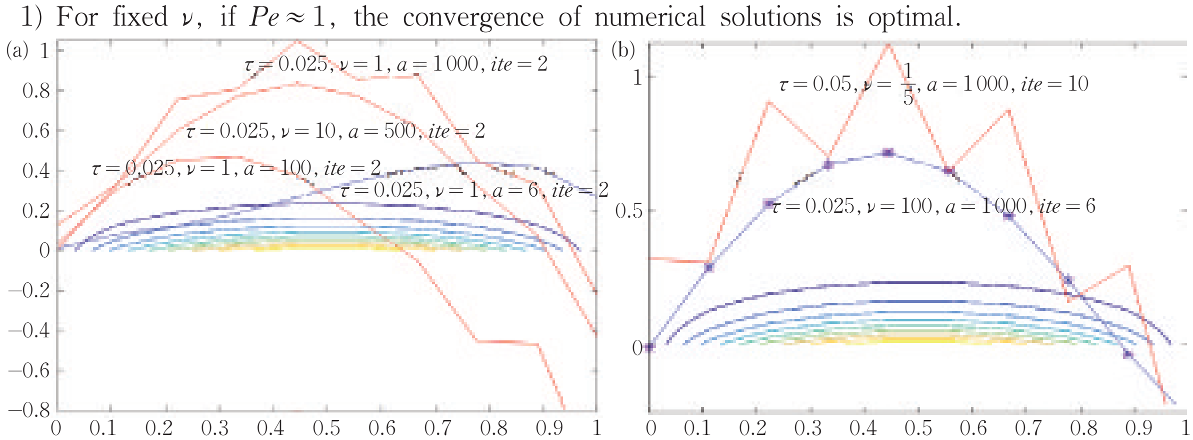

Fig.2 The dispersion is serious when (a) $a\geqslant\nu$ for the same τ; (b) τ is bigger for $a\geqslant\nu$

2) If $Pe=\frac{|a|h}{\nu}>1$, or Pe is very large, then the numerical solution appears severe non-physical oscillation. A large magnitude and unacceptable numerical oscillations appear at the local maximum and the grid-block interface. Under this circumstance, a continuous solution across the interface of global blocks is unachievable.

3) When Pe is very large, if we choose the smaller mesh grid, dispersion can be eliminated as much as possible.

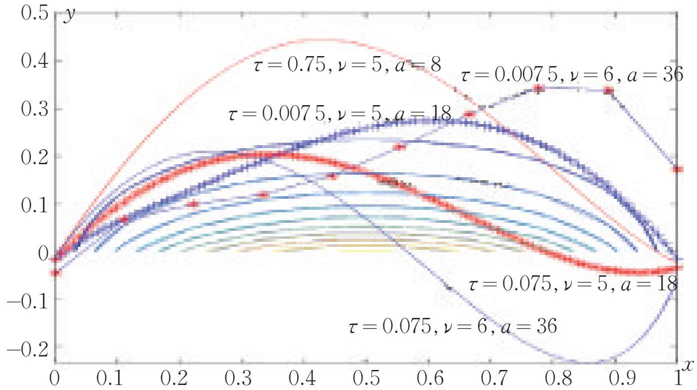

Fig.3 Fig.3 The dispersion is controlled by (19) when τ is proper for the $a>1>\nu>0$ cases

2.6 Adjustment of difference format

When Pe is very large, we amendments at the time layers, the difference scheme

$$\left(\frac{u_{j+1}^{n}-u_{j-1}^{n}}{2h}+\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2h}\right)=\frac{v}{a}\left(\frac{u_{j+1}^{n}-2u_{j}^{n}+u_{j-1}^{n}}{h^{2}}+\frac{u_{j+1}^{n+1}-2u_{j}^{n+1}+u_{j-1}^{n+1}}{h^{2}}\right)$$
$$-\frac{2}{a}\left(\frac{1}{6}\frac{u_{j-1}^{n+1}-u_{j-1}^{n}}{\tau}+\frac{2}{3}\frac{u_{j}^{n+1}-u_{j}^{n}}{\tau}+\frac{1}{6}\frac{u_{j+1}^{n+1}-u_{j+1}^{n}}{\tau}\right). \tag{19}$$

Rewrite (19), we have

$$\left(\frac{1}{3}-\frac{a\lambda}{2}-\nu\mu\right)u_{j-1}^{n+1}+\left(\frac{4}{3}+2\nu\mu\right)u_{j}^{n+1}+\left(\frac{1}{3}+\frac{a\lambda}{2}-\nu\mu\right)u_{j+1}^{n+1}$$
$$=\left(\frac{1}{3}+\frac{a\lambda}{2}+\nu\mu\right)u_{j-1}^{n}+\left(\frac{4}{3}-2\nu\mu\right)u_{j}^{n}+\left(\frac{1}{3}-\frac{a\lambda}{2}+\nu\mu\right)u_{j+1}^{n}.$$

It's growth factor is

$$G(\tau,k)=\frac{\frac{1}{3}(4+\cos kh)-2\nu\mu(1-\cos kh)-a\lambda\sin ikh}{\frac{1}{3}(4+\cos kh)+2\nu\mu(1-\cos kh)+a\lambda\sin ikh},$$

$$|G(\tau,k)|^{2}=\frac{\left(\frac{1}{3}(4+\cos kh)-2\nu\mu(1-\cos kh)\right)^{2}+a^{2}\lambda^{2}\sin^{2}kh}{\left(\frac{1}{3}(4+\cos kh)+2\nu\mu(1-\cos kh)\right)^{2}+a^{2}\lambda^{2}\sin^{2}kh}\leqslant 1.$$

This is a third ordered convergent on time layers, second ordered on convection terms, unconditionally stable. When $1\pm\frac{3a\lambda}{2}-3\nu\mu>0\rightarrow|a\lambda|<\frac{2}{3}-2\nu\mu$, linear system of (19) has a unique numerical solution.

2.7 Numerical example 2

Do the numerical example 1 by using (19). Let $\tau=0.00315, \nu=3, a=6, ite=8$, and $\tau=0.005$, $\nu=4, a=8, ite=10$, the numerical solution converges to analytic solution quickly. If mesh grid ratio too big or too small, numerical solution stays upper or under the analytic solution, convergences is too hard. If the mesh grid ratio chooses properly, six or less iterations can obtain convergent numerical solutions.

We choose $\tau=0.005, \nu=10, a=1, ite=0.18+ite(4)$, the numerical solution down to the analytic solution, we add artificially 0.18+iteration(4), then the numerical solution converges. If we choose v bigger, for example $v=10, 100, 1000$ or more bigger, and let the grid ratio bigger or smaller, numerical solution begin to shrinkage or deformation.

The numerical results shows, the adjusted difference format (19) eliminate the oscillations effec-tively which caused by convection-dominant. When the initial-boundary condition are discontinuous, and $a > v$, the difference scheme (3) of larger or smaller mesh ratio is always disperse, and the finite difference method can not get a good numerical solution. The difference scheme (19) reduces the dis-persion to a certain extent, obtains a better numerical solution, and also used for larger time steps.

For strong convection dominant cases, if the initial boundary condition is continuous and mesh grid ratio is proper, then (19) works well(see Fig.4(a)). If initial boundary condition is discontinuous, and mesh grid ratio is so big or too smallthen there are strong oscillation occurs(see Fig.4(b)).

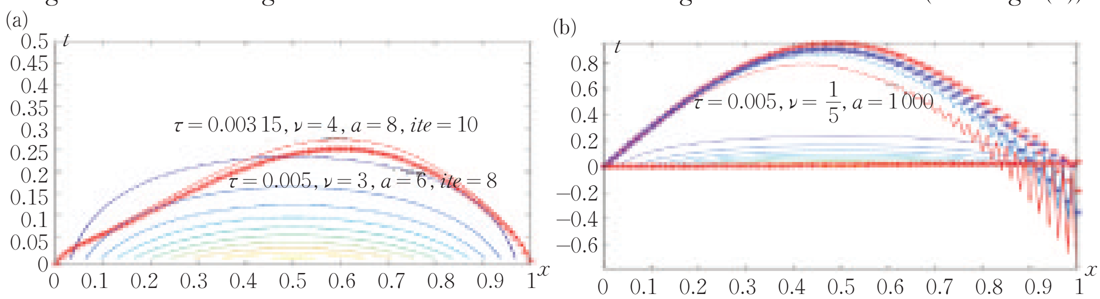

Fig.4 The numerical solutions for the numercal example 2

3 One dimensional diffusion-dominant problem

If $\nu > a > 0$, then (1) performs diffusion dominant cases. Diffusion makes the physical quantity move from high value to low value in the flow field. Diffusion dominated convection-diffusion prob-lems are well studied in the literature since the 1970s.

3.1 The basic analysis

Our methods concentrate in the diffusion coefficient v is constant real numbers. Indeed, there is a good reason for such a choice, because the variable coefficient cases can be handled directly. When $v > a > 0$ will appears severe numerical dissipation. So we need to adjust the difference scheme. Note that $Pe = \frac{|a|}{v}h$ in (1), where h is the maximum size of the cell, $|a|$ is the maximum velocity component in the cell, then (1) can be write as

$$\frac{\partial^2 u}{\partial x^2} = Pe\frac{\partial u}{\partial x} + \rho'\frac{\partial u}{\partial t},\ \rho' = \frac{1}{v}. \tag{20}$$

We use a difference scheme

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} = \frac{a}{v}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h}\right) + \frac{2}{v}\left(\frac{1}{12}\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\tau} + \frac{5}{6}\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{12}\frac{u_{j+1}^{n+1} - u_{j+1}^n}{\tau}\right). \tag{21}$$

Theorem 7 The difference scheme (21) is unconditionally stable.

Proof The process of proving stability is same as theorem 1, we omit the proof here.

Rewrite (21), we have

$$\left(\frac{1}{6} - \frac{a\lambda}{2} - \nu\mu\right)u_{j-1}^{n+1} + \left(\frac{5}{3} + 2\nu\mu\right)u_j^{n+1} + \left(\frac{1}{6} + \frac{a\lambda}{2} - \nu\mu\right)u_{j+1}^{n+1} = \left(\frac{1}{6} + \frac{a\lambda}{2} + \nu\mu\right)u_{j-1}^n + \left(\frac{5}{3} - 2\nu\mu\right)u_j^n + \left(\frac{1}{6} - \frac{a\lambda}{2} + \nu\mu\right)u_{j+1}^n. \tag{22}$$

Difference scheme (21) uses the six points which same as (3). It's growth factor is

$$G(\tau,k)=\frac{\left(\frac{5}{3}-2\nu\mu\right)+\left(\frac{1}{3}+2\nu\mu\right)\cos kh-a\lambda\sin ikh}{\left(\frac{5}{3}+2\nu\mu\right)+\left(\frac{1}{3}-2\nu\mu\right)\cos kh+a\lambda\sin ikh},\ \left|G(\tau,k)\right|^{2}-1<0,$$

since (22) is unconditionally stable, and second ordered convergent on convection terms, in time layers is a third ordered, as $O(\tau^3+h^2+\tau^2h^2)$.

3.2 Numerical example 3

Do the numerical example 1 by using (21).

Let $\tau=0.025, v=1, a=1, ite=6$, the numerical solution converges to analytic solution needs iterate six times only. When $\tau=0.025, v=10, a=1, ite=0.18+ite(4)$, the numerical solution down to the analytic solution, we add 0.18+iteration(4) artificially, then the numerical solution converges. If we choose v bigger, as $v=10, 100, 1\,000$ or more bigger, and let the grid ratio bigger or smaller, numerical solution begin to shrinkage or deformation. (22) works well in discontinuous cases.

Corollary 3 There are statements about the relationship between ν, a and grid ratio:

1) For a fixed a, the greater ν, the greater dissipation;

2) The smaller mesh ratio, the greater match ν, the better convergence;

3) For the smaller mesh ratio, the greater match ν, the better convergence;

4) If the ν is larger, choose the larger or much smaller grid ratio, the numerical solution is above or below the analytic solution, it is difficult to convergent(see Fig.5).

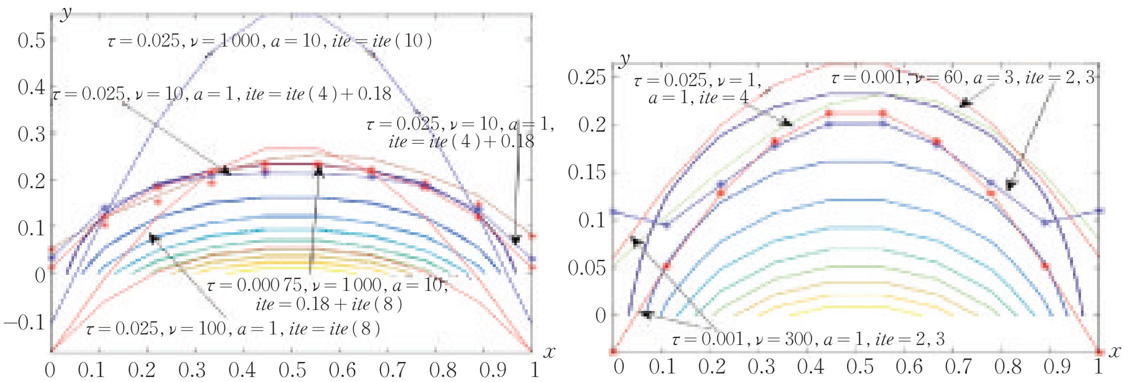

Fig.5 The numerical solutions for numerical example 3

The computational complexity of these method is not ideal, need each iteration contains more or less vibration. If we solve the continuous diffusion dominant cases by using the difference scheme, the numerical solution of (1) works quite ideal.

Given values of a, ν, it is only possible to satisfy the condition of $Pe<2$ if the velocity is small, or the grid spacing is small. This gives stability problems and unbounded solutions under certain flow conditions. Similarly the west coefficient can become negative when the flow is in the negative direction. The quick scheme is therefore conditionally stable.

For the diffusion dominant case, the different scheme are given for presents dispersion cases when ν is too big or too small, or the time step chosen too large or too small.

Here we give some modified implicit difference scheme based on the (3), which proper to diffusion dominant cases. The difference scheme (21) overcomes the disadvantages of the characteristic line direction method, which is a low-order difference in the time layer. Difference scheme (21) is third ordered accuracy in the time layer and avoids non-physical oscillations.

3.3 Guaranty of stability

The difference scheme is established to ensure that its solution satisfies the properties of mass, momentum and total energy conservation on the whole solution region and even on each grid. In a word, the study on the conservation of difference scheme should include two contents, one is how to construct the appropriate conservation scheme, the other is that the results of numerical solution should be tested when the conservation scheme can not or is difficult to design. It is obvious, the conservation of this practical calculation result is closely related to the grid scale effect, differential remainder effect and numerical boundary effect and conservation of format mentioned above. Therefore, conservation should be used as a test of the calculation process, according to which the grid scale should be checked and adjusted dissipation dispersion and numerical boundaries to ensure the stability of the calculated results.

The difference scheme (21), which proper to strong diffusion dominant problem, and which can be extended to the problem of time dominance. For the diffusion dominant case, the scheme (21) presents dispersion cases when the v is too big or too small, or the time step chosen too large or too small. Though the difference scheme is unconditionally stable, however, in the process of practicing, the stability of the difference scheme always be effected by grid scale effect, residual effect of difference scheme, numerical boundary effect, conservation effect of scheme, etc.

If the inequality system $0<\frac{a\lambda}{2}+\mu\nu\leqslant\frac{1}{6}$, $\frac{a\lambda}{2\nu}-\mu>0$ hold on, then (21) has a unique numerical solution. When $\mu\to 0$, equation (20) produces moving discontinuities that provide a very stringent test of any numerical procedure. Convection terms in (20) a special case of the Navies–Stokes equations of fluid mechanics, and it can be approximated more ideally.

Difference equation (3) and (20) can easily be extended to more than one spatial dimension, and the analytical solutions for several dimensions are also available. Thus, they are can be used to test numerical methods in one, two, three and more dimensional convection–diffusion equations and still standing unconditionally stable and higher accuracy on time step and convection terms.

4 Conclusion

If the numerical example has discontinuous initial–boundary condition, the others numerical methods makes perturbation near them. Since our implicit difference scheme is second ordered convergent on the space layers.

For those cases where the processing is not good enough, such as convection dominance and diffusion dominance, the number of meshes should be increased and the length of mesh grid step is reduced. However, no more in–depth study was undertaken in this area. This method to this practical background is a meaningful application and this method can be tested and corrected and directly extending to two dimensional convention dffusion equation.

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常系数对流扩散方程的一种数值解法

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摘要: 讨论了一维常系数对流扩散方程的一种数值解法.对于对流占优和扩散占优的情况,分别构造了相应的隐式差分格式,分析了截断误差、收敛和稳定性.理论分析和数值实验结果表明,该隐式差分格式有效地消除了对流主导所引起的数值振荡,控制了扩散主导引起的耗散问题,并且可以在保持差分格式的所有优点的基础上直接推广到二维问题.最后在此基础上给出了一种隐式分裂中心差分算法.

关键词: 对流占优; 扩散占优; 二维对流–扩散方程; 隐式差分格式