Problem Set - 3 (Continuous Random Variables and Joint Distributions)

ECO 104 (Section 11)
Instructor: Shaikh Tanvir Hossain

Due Date: Final Exam Day

§

Please Note: Please submit individually in class.

Continuous Random Variables

1. Suppose following is a density function for a random variable X that takes any value between 0 and 1

$$f(x) = 3x^2, \text{ where } 0 \leq x \leq 1$$

   (a) Is this is a valid density function?
   (b) Calculate $\mathbb{P}(X < 0.5)$
   (c) Calculate $\mathbb{P}(X > 0.6)$
   (d) Calculate $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$

Solution:

(a) To check if this is a valid density function, we need to check two things,

1. $f(x) \geq 0$ for all x in the support of X, which is $[0, 1]$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

For the first condition, we can see that $f(x) = 3x^2 \geq 0$ for all $x \in [0, 1]$. So this condition is satisfied.

For the second condition, we can calculate the integral,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_0^1 3x^2 dx \\ &= 3\left[\frac{x^3}{3}\right]_0^1 \\ &= [x^3]_0^1 = 1^3 - 0^3 = 1\end{aligned}$$

2. From Anderson et al. (2020) Chapter 6, #1

The random variable X is known to be uniformly distributed between 1.0 and 1.5 (**in notation we write** $X \sim$ **Unif**$(1, 1.5)$, they both mean same thing!)

   (a) Show the graph of the probability density function.
   (b) Compute $\mathbb{P}(X = 1.25)$.
   (c) Compute $\mathbb{P}(1.0 \leq X \leq 1.25)$.
   (d) Compute $\mathbb{P}(1.20 < X < 1.5)$.

Solution:

(a) The density function of X is

$$f(x) = \frac{1}{b-a} = \frac{1}{1.5 - 1.0} = \frac{1}{0.5} = 2 \quad \text{for } 1.0 \leq x \leq 1.5$$

This is a simple plot, we did the plot in the class

(b) X is a continuous random variable, so this means

$$\mathbb{P}(X = 1.25) = 0$$

this holds for any value, not just 1.25

(c) There are two ways to do this, the hard way is,

$$\begin{aligned}
\mathbb{P}(1.0 \leq X \leq 1.25) &= \int_{1.0}^{1.25} f(x)dx \\
&= \int_{1.0}^{1.25} 2dx \\
&= 2[x]_{1.0}^{1.25} = 2(1.25 - 1.0) = 2(0.25) = 0.5
\end{aligned}$$

the easy way is drawing the picture of the density function, and then applying the formula for the area of a rectangle,

$$\mathbb{P}(1.0 \leq X \leq 1.25) = (1.25 - 1.0) \times f(x) = (1.25 - 1.0) \times 2 = 0.5$$

3. From Anderson et al. (2020) Chapter 6, #2

The random variable X is known to be uniformly distributed between 10 and 20 (in notation we write $X \sim$ Unif$(10, 20)$)

(a) Show the graph of the probability density function.

(b) Compute $\mathbb{P}(X < 15)$ or Calculate $F(15)$.

(c) Compute $\mathbb{P}(12 \leq X \leq 18)$.

(d) Compute $\mathbb{E}(X)$.

(e) $\text{Var}(X)$.

Solution:

- a) This is a simple plot, we did a similar plot in the class
- b) Here $F(15) = \mathbb{P}(X < 15)$ means a cumulative probabilities upto 15. Let's derive the general formula to calculate the cumulative probability for a Uniform distribution with parameter a and b, The density function of the uniform is given by

$$f(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b$$

Now we can get cumulative probability at c, with

$$F(c) = \mathbb{P}(X \leq c) = \int_a^c \frac{1}{b-a}dx = \frac{1}{b-a}\int_a^c dx = \frac{1}{b-a}[x]_a^c = \frac{c-a}{b-a}$$

Now we know that $a = 10$, $b = 20$, and $c = 15$, so we can calculate the cumulative probability at 15, finish it...Now to calculate probabilities for the uniform distribution we have this nice formula, but you cannot use this formula to calculate cumulative probabilities for any other distribution, this is only for the uniform distribution. For example, for the normal distribution we used the table in Anderson et al. (2020).

- c) This is similar to a problem we solved in the class (and we have in the slides), we can solve it using a shortcut formula for the uniform, which is

$$\mathbb{P}(c \leq X \leq d) = (c - d) \times \frac{1}{b - a} = 1$$

In this case,

$$\mathbb{P}(12 \leq X \leq 18) = (18 - 12) \times \frac{1}{20 - 10} = 6 \times 0.1 = 0.6$$

Question is why does this formula work don't we need integral? The answer is if we do integral we will get the same answer,

$$\mathbb{P}(c \leq X \leq d) = \int_c^d \frac{1}{b - a} dx = \frac{1}{b - a} \int_c^d dx = \frac{1}{b - a} [x]_c^d = (d - c) \times \frac{1}{b - a}$$

if you draw it, you should see that this will be a rectangular area, and the area of a rectangle is the product of the length and width, which is $(d - c) \times \frac{1}{b-a}$.

Optional: We can also use cumulative probability to calculate this,

$$\mathbb{P}(12 \leq X \leq 18) = \mathbb{P}(X \leq 18) - \mathbb{P}(X \leq 12) = \frac{18 - 10}{20 - 10} - \frac{12 - 10}{20 - 10} = \frac{8}{10} - \frac{2}{10} = \frac{6}{10} = 0.6$$

- d and e. To solve d) and e) you can directly use the general formula for the Expectation and Variance of the Uniform distribution, so if $X \sim \mathcal{U}_{[a,b]}$, then

$$\mathbb{E}(X) = \frac{a + b}{2}$$
$$\mathbb{V}\text{ar}(X) = \frac{(b - a)^2}{12}$$

Now since we know $a = 10$ and $b = 20$, we can calculate $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$, finish it.

Completely Optional (but good for understanding): You can also calculate the expectation and variance using the definition. In the sides we have a proof of expectation formula for the Uniform distribution. Let's see how can we derive the variance,

$$
\begin{aligned}
\mathbb{V}\text{ar}(X) &= \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] \\
&= \mathbb{E}\left[\left(X - \frac{a+b}{2}\right)^2\right] \\
&= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\
&= \frac{1}{b-a} \int_a^b \left(x^2 - 2x\frac{a+b}{2} + \left(\frac{a+b}{2}\right)^2\right) dx \\
&= \frac{1}{b-a} \int_a^b x^2 dx - (a+b) \int_a^b x dx + \left(\frac{a+b}{2}\right)^2 \int_a^b dx \\
&= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b - (a+b) \left[\frac{x^2}{2}\right]_a^b + \left(\frac{a+b}{2}\right)^2 [x]_a^b \\
&= \ldots\ldots \\
&= \ldots\ldots \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

The calculation is very messy, don't worry it's not a math exam, so in the exam you won't have these calculations.... There is another way the calculation becomes a bit easier, note that now we know there is another way we can write variance, that is from question 5

$$\mathbb{V}\text{ar}(X) = \mathbb{E}\left[\left(X - \mathbb{E}\left(X\right)\right)^2\right] = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

So we already know $\mathbb{E}(X) = \frac{a+b}{2}$, now we need to calculate $\mathbb{E}(X^2)$, this is a bit easier (apply LOTUS)

$$
\begin{aligned}
\mathbb{E}(X^2) &= \int_a^b x^2 \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x^2 dx \\
&= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b \\
&= \frac{1}{3(b-a)} \left(b^3 - a^3\right) \\
&= \frac{1}{3(b-a)} (b-a) \left(b^2 + ab + a^2\right) \\
&= \frac{1}{3} \left(b^2 + ab + a^2\right)
\end{aligned}
$$

So now

$$\mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{3}\left(b^2 + ab + a^2\right) - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

So this means $\mathbb{V}\text{ar}(X) = \frac{(b-a)^2}{12}$, when we are calculating variance of the Uniform distribution, with parameters a and b

4. From Anderson et al. (2020) Chapter 6, #9

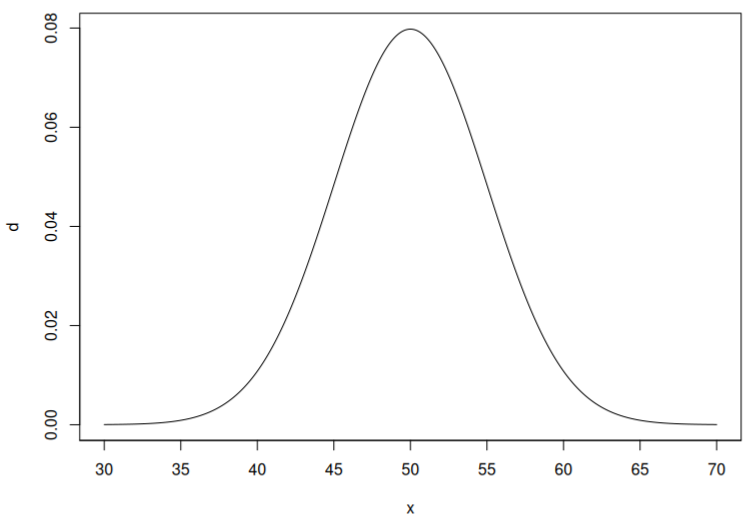
A random variable X is normally distributed with a mean of $\mu = 50$ and a standard deviation of $\sigma = 5$ (in notation we write $x \sim \mathcal{N}(50, 25)$)

(a) Sketch a normal curve for the probability density function (Using Figure 6.6 in Anderson et al. (2020) as a guide). Label the horizontal axis with values of $35, 40, 45, 50, 55, 60,$ and 65.

(b) What is the probability the random variable will assume a value between 45 and 55? This means $\mathbb{P}(45 < X < 55) =?$

(c) What is the probability the random variable will assume a value between 40 and 60. This means $\mathbb{P}(40 < X < 60) =?$

Solution: a). We did it in the class. This is a simple plot.



b) The solution is very easy,

$$\begin{aligned}
\mathbb{P}(45 < X < 55) &= \mathbb{P}\left(\frac{45-50}{5} < \frac{X-\mu}{\sigma} < \frac{55-50}{5}\right) \\
&= \mathbb{P}\left(\frac{45-50}{5} < Z < \frac{55-50}{5}\right) \\
&= \mathbb{P}\left(-1 < Z < 1\right) \\
&= \mathbb{P}\left(Z < 1\right) - \mathbb{P}\left(Z < -1\right) \\
&= 0.8413 - 0.1587 = 0.6826
\end{aligned}$$

In the second line we used Z transformation to bring everything into standard normal. The probabilities will be same, so we can do this. At the end we used the Z table in Anderson et al. (2020) and found $\mathbb{P}\left(Z < 1\right) = 0.8413$ and $\mathbb{P}\left(Z < -1\right) = 0.1587$.

Interpretation: *So finally we calculated $\mathbb{P}(45 < X < 55) = 0.6826$ Question - What does 0.6826 mean? Or what is the interpretation of this number?*

Ans: We can comment on two ways,

- We can say that the probability of X taking values between 45 and 55 is 0.6826. Or there are 68.26% values of X that are within this range .
- We can also say that if we randomly pick 100 numbers from all possible values of X (which ranges from $-\infty$ to ∞), then almost 69 times we will get values between 45 and 65, the rest 31 times we won't.

Note that, the two comments are coming from classical definition and frequency definition of probability.

c) This is similar to b), you should try this! We will talk about this later.

5. From Anderson et al. (2020) Chapter 6, #10

   Suppose we have a random variable which is distributed as a standard normal distribution (this means $Z \sim \mathcal{N}(0, 1)$. Label the horizontal axis at values of $-3, -2, -1, 0, 1, 2,$ and 3. Then use the table in Anderson et al. (2020) to compute the following probabilities.

   (a) $\mathbb{P}(Z \leq 1.5)$

   (b) $\mathbb{P}(Z \leq 1)$

   (c) $\mathbb{P}(1 \leq Z \leq 1.5)$

   (d) $\mathbb{P}(0 < Z < 2.5)$

   Solution: This is easy!

6. From Anderson et al. (2020) Chapter 6, #12. Given that $Z \sim \mathcal{N}(0, 1)$, compute the following probabilities.

   (a) $\mathbb{P}(0 \leq Z \leq .83)$

   (b) $\mathbb{P}(-1.57 \leq Z \leq 0)$

   (c) $\mathbb{P}(Z > .44)$

   (d) $\mathbb{P}(Z \geq -.23)$

   (e) $\mathbb{P}(Z < 1.20)$

   (f) $\mathbb{P}(Z \leq -.71)$

7. From Anderson et al. (2020) Chapter 6, #15.Given that $Z \sim \mathcal{N}(0, 1)$, find z for each situation.

   (a) The area to the left of z is $.2119$.

   (b) The area between $-z$ and z is $.9030$.

   (c) The area between $-z$ and z is $.2052$.

   (d) The area to the left of z is $.9948$.

   (e) The area to the right of z is $.6915$.

   Solution:

   - a) This is asking for the z value such that

     $$\mathbb{P}(Z < z) = 0.2119$$

     Using the table we can directly calculate this, and this will be $z = -0.8$. This is the z value for which we have 21.19% cumulative probabilities. We have also learned that here -0.8 is called the 0.2119^{th} quantile or 21.19^{th} percentile.

   - b) The question tells us that there is a z such that the area between $-z$ and z is $.9030$. So we have

     $$\mathbb{P}(-z < Z < z) = 0.9030$$

     This is the probability in center, the probabilities in two tails would be $1 - .9030 = 0.097$ This means the tail probabilities together is 0.097. Now since the distribution is symmetric around 0. To get the probability in the left tail, we can simply divide this number by 2, so we have

     $$\mathbb{P}(Z < z^*) = 0.097/2 = 0.0485$$

     here z^* is the z value for which we have 0.0485 cumulative probabilities (this means probabilities on the left). If you see the table then this value is -1.66. So now we know that

$$\mathbb{P}(-1.66 < Z < 1.66) = 0.9030$$

So here

- $-z = -1.66$ and $z = 1.66$

c) Same as b)

d) We need to find z such that

$$\mathbb{P}(Z < z) = .9948$$

Finish this....

e) We need to find z such that

$$\mathbb{P}(Z > z) = .6915$$

but since we know

$$\mathbb{P}(Z > z) = 1 - \mathbb{P}(Z < z)$$

so we can write

$$\begin{aligned} 1 - \mathbb{P}(Z < z) &= .6915 \\ \mathbb{P}(Z < z) &= 1 - .6915 = 0.3085 \end{aligned}$$

Now you should be able to figure out z, and from the table we get $z = -.5$.

8. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #33.

   Consider a random variable following exponential probability density function.

$$f(x) = \frac{1}{3}e^{-x/3} \quad \text{for } x \geq 0$$

   (a) Write the formula for $P(x \leq x_0)$.
   (b) Find $\mathbb{P}(X \leq 2)$.
   (c) Find $\mathbb{P}(X \geq 3)$.
   (d) Find $\mathbb{P}(X \leq 5)$.
   (e) Find $\mathbb{P}(2 \leq X \leq 5)$.

§. **Applied Problems** Here are some applied problems from Anderson et al. (2020)

9. From Anderson et al. (2020) Chapter 6, #3.

   Cincinnati to Tampa Flight Time. Delta Airlines quotes a flight time of 2 hours, 5 minutes for its flights from Cincinnati to Tampa. Suppose we believe that actual flight times are uniformly distributed between 2 hours and 2 hours, 20 minutes.

   (a) Show the graph of the probability density function for flight time.
   (b) What is the probability that the flight will be no more than 5 minutes late?
   (c) What is the probability that the flight will be more than 10 minutes late?
   (d) What is the expected flight time?

Solution:

- a. First of all note that here $a = 120$ and $b = 140$. Now we can use the formula for the Uniform distribution to calculate the density function,

$$f(x) = \frac{1}{b-a} = \frac{1}{140-120} = 0.05$$

- No more than 5 minutes late means it must be less than $120 + 5 = 125$ mins, so this means we need to calculate

$$\mathbb{P}(X \leq 125) = 0.05 \times (125 - 120) = 0.05 \times 5 = 0.25$$

- More than 10 min late means $\mathbb{P}(X \geq 135)$, so in this case

$$\mathbb{P}(X \geq 135) = 0.05 \times (140 - 135) = 0.05 \times 5 = 0.25$$

- We can use the formula for the expectation of the Uniform distribution,

$$\mathbb{E}(X) = \frac{a+b}{2} = \frac{120+140}{2} = 130$$

10. From Anderson et al. (2020) Chapter 6, #7.

Bidding on Land. Suppose we are interested in bidding on a piece of land and we know one other bidder is interested. The seller announced that the highest bid in excess of $\$10,000$ will be accepted. Assume that the competitor's bid X is a random variable that is uniformly distributed between $\$10,000$ and $\$15,000$.

(a) Suppose you bid $\$12,000$. What is the probability that your bid will be accepted?

(b) Suppose you bid $\$14,000$. What is the probability that your bid will be accepted?

(c) What amount should you bid to maximize the probability that you get the property?

Solution:

Here your competitor's bid X is uniformly distributed between $\$10,000$ and $\$15,000$, so we can write

$$X \sim \textbf{Unif}(10000, 15000)$$

The density function in this case is,

$$f(x) = \frac{1}{15000 - 10000} = \frac{1}{5000} \quad \text{for } 10000 \leq x \leq 15000$$

(a) Recall here X is the competitor's bid. It's a bidding problem, in this case you will win if your competitor's bid is less than your bid. So if you bid $\$12,000$, then you will win if $X < 12000$. So we need to calculate

$$\mathbb{P}(X < 12000) = \frac{1}{5000} \times (12000 - 10000) = \frac{1}{5000} \times 2000 = 0.4$$

So the probability of your bid being accepted is 0.4 or 40%.

(b) Similar to the last one, in this case your competitor needs to bid less than $14,000$, so

$$\mathbb{P}(X < 14000) = \frac{1}{5000} \times (14000 - 10000) = \frac{1}{5000} \times 4000 = 0.8$$

So the probability of your bid being accepted is 0.8 or 80%.

(c) Obviously if your competitor's bidding range is between $10,000$ and $15,000$, then you should bid more than $15,000$ to maximize the probability of winning. So the maximum probability of winning is 1 or 100% if you bid more than $15,000$, since

$$\mathbb{P}(X < 15000) = \frac{1}{5000} \times (15000 - 10000) = \frac{1}{5000} \times 5000 = 1$$

11. From Anderson et al. (2020) Chapter 6, #17.

Height of Dutch Men. Males in the Netherlands are the tallest, on average, in the world with an average height of 183 centimeters (cm) (BBC News website). Assume that the height of men in the Netherlands is normally distributed with a mean of 183 cm and standard deviation of 10.5 cm.

(a) What is the probability that a Dutch male is shorter than 175 cm ?

(b) What is the probability that a Dutch male is taller than 195 cm ?

(c) What is the probability that a Dutch male is between 173 and 193 cm ?

(d) Out of a random sample of 1000 Dutch men, how many would we expect to be taller than 190 cm ?

Solution:

In this case we assume the height of the Dutch men X is normally distributed with mean $\mu = 183$ cm and standard deviation $\sigma = 10.5$ cm, so this means

$$X \sim \mathcal{N}(183, 10.5^2)$$

(a) Here we need to calculate

$$\begin{aligned}\mathbb{P}(X < 175) &= \mathbb{P}\left(\frac{X - 183}{10.5} < \frac{175 - 183}{10.5}\right) \\ &= \mathbb{P}\left(Z < -0.76\right) = 0.2236 \text{ (We get this from the table)}\end{aligned}$$

(b) Here we need to calculate

$$\begin{aligned}\mathbb{P}(X > 195) &= \mathbb{P}\left(\frac{X - 183}{10.5} > \frac{195 - 183}{10.5}\right) \\ &= \mathbb{P}\left(Z > 1.14\right) = 1 - \mathbb{P}\left(Z < 1.14\right) = 1 - 0.8729 = 0.1271\end{aligned}$$

(c) Here we need to calculate

$$\begin{aligned}\mathbb{P}(173 < X < 193) &= \mathbb{P}(\frac{173 - 183}{10.5} < \frac{X - \mu}{\sigma} < \frac{193 - 183}{10.5}) \\ &= \mathbb{P}(-0.95 < Z < 0.95) \\ &= \mathbb{P}(Z < 0.95) - \mathbb{P}(Z < -0.95) \\ &= 0.8289 - 0.1711 = 0.6578\end{aligned}$$

(d) For this problem first find the probability,

$$\mathbb{P}(X > 190) = \mathbb{P}\left(\frac{X - 183}{10.5} > \frac{190 - 183}{10.5}\right)$$
$$= \mathbb{P}(Z > 0.67) = 1 - \mathbb{P}(Z < 0.67) = 1 - 0.7486 = 0.2514 = 25.14\%$$

So this means out of 100 there are around 25 people who are taller than 190 cm. So out of 1000 people we expect around 250 people who are taller than 190 cm.

12. From Anderson et al. (2020) Chapter 6, #20.

Gasoline Prices. Suppose that the average price for a gallon of gasoline in the United States is $\$3.73$ and in Russia is $\$3.40$. Assume these averages are the population means in the two countries and that the probability distributions are normally distributed with a standard deviation of $\$.25$ in the United States and a standard deviation of $\$.20$ in Russia.

(a) What is the probability that a randomly selected gas station in the United States charges less than $\$3.50$ per gallon?

(b) What percentage of the gas stations in Russia charge less than $\$3.50$ per gallon?

(c) What is the probability that a randomly selected gas station in Russia charged more than the mean price in the United States?

Solution:

So here we have two random variables, X_1 is the gasoline price of US and X_2 is the gasoline price of Russia. We know that both of these random variables are normally distributed,

In the case of US we know the mean of US gasoline price is $\mu_1 = 3.73$ and standard deviation is $\sigma_1 = 0.25$. So we can write

$$X_1 \sim \mathcal{N}(3.73, 0.25^2)$$

and for Russia we know the mean is $\mu_2 = 3.40$ and standard deviation is $\sigma_2 = 0.20$. So we can write

$$X_2 \sim \mathcal{N}(3.40, 0.20^2)$$

(a) This is for US

We need to calculate

$$\mathbb{P}(X_1 < 3.50) = \mathbb{P}\left(\frac{X_1 - 3.73}{0.25} < \frac{3.50 - 3.73}{0.25}\right)$$
$$= \mathbb{P}(Z < -0.92) = 0.1788$$

So the probability that a randomly selected gas station in the United States charges less than $\$3.50$ per gallon is 0.1788 or 17.88%.

(b) This is for Russia

We need to calculate

$$\mathbb{P}(X_2 < 3.50) = \mathbb{P}\left(\frac{X_2 - 3.40}{0.20} < \frac{3.50 - 3.40}{0.20}\right)$$
$$= \mathbb{P}(Z < 0.50) = 0.6915$$

So the percentage of the gas stations in Russia charge less than $\$3.50$ per gallon is 0.6915 or 69.15%.

(c) Now the last question asks for Russia, when it charges more than US mean or more than $\mu_1 = \$3.73$. So we need to calculate

$$\begin{aligned}\mathbb{P}(X_2 > \mu_1) = \mathbb{P}(X_2 > 3.73) = \mathbb{P}\left(\frac{X_2 - 3.40}{0.20} > \frac{3.73 - 3.40}{0.20}\right) = \\ = \mathbb{P}\left(Z > 1.65\right) = 1 - \mathbb{P}\left(Z < 1.65\right) = 1 - 0.9505 = 0.0495\end{aligned}$$

13. From Anderson et al. (2020) Chapter 6, #19.

Automobile Repair Costs. Automobile repair costs continue to rise with an average 2015 cost of \$367 per repair (U.S. News & World Report website). Assume that the cost for an automobile repair is normally distributed with a standard deviation of \$88. Answer the following questions about the cost of automobile repairs.

(a) What is the probability that the cost will be more than \$450 ?

(b) What is the probability that the cost will be less than \$250 ?

(c) What is the probability that the cost will be between \$250 and \$450 ?

(d) If the cost for your car repair is in the lower 5% of automobile repair charges, what is your cost?

Solution:

Here automobile repair costs are normally distributed with a mean of $\mu = 367$ and standard deviation of $\sigma = 88$. We can write this as $X \sim \mathcal{N}(367, 88^2)$, where X is a random variable which represents automobile costs.

- a) we need find $\mathbb{P}(X > 450)$. Use Z transformation and standard normal table
- b) we need find $\mathbb{P}(X < 250)$. Again use Z transformation and standard normal table.
- c) we need find $\mathbb{P}(250 < X < 450)$. Again use Z transformation and standard normal table.
- d) we need find x such that $\mathbb{P}(X < x) = 0.05$. But we cannot directly find this x, because we don't have the table for $\mathcal{N}(367, 88^2)$, we only have table for $\mathcal{N}(0, 1)$. First, we need to find z such that $\mathbb{P}(Z < z) = 0.05$, then we can use Z transformation to find x.
  - step 1: Let's find z such that $\mathbb{P}(Z < z) = 0.05$. From the table we see that $z = -1.645$.
  - step 2: Now we can use Z transformation to find x,

  $$x = \mu + \sigma z = 367 + 88 \times (-1.645) = 225.82$$

  Why we can do this? We can do this because of the relation between standard normal and any other normal distribution. The probabilities (or area) are same, this means

  $$\underbrace{\mathbb{P}(X < 225.82)}_{\text{this is probability under } \mathcal{N}(367, 88^2)} = \underbrace{\mathbb{P}(Z < -1.645)}_{\text{this is probability under } \mathcal{N}(0,1)}$$

So if I know the cost for my car repair is in the lower 5% of automobile repair charges, then I know my cost would be less than \$225.82.

14. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #34.

Phone Battery Life. Battery life between charges for a certain mobile phone is 20 hours when the primary use is talk time, and drops to 7 hours when the phone is primarily used for Internet applications over a cellular network. Assume that the battery life in both cases follows an exponential distribution.

(a) Show the probability density function for battery life for this phone when its primary use is talk time.

(b) What is the probability that the battery charge for a randomly selected phone will last no more than 15 hours when its primary use is talk time?

(c) What is the probability that the battery charge for a randomly selected phone will last more than 20 hours when its primary use is talk time?

(d) What is the probability that the battery charge for a randomly selected phone will last no more than 5 hours when its primary use is Internet applications?

15. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #38.

Boston 911 Calls. The Boston Fire Department receives 911 calls at a mean rate of 1.6 calls per hour (Mass.gov website). Suppose the number of calls per hour follows a Poisson probability distribution.

(a) What is the mean time between 911 calls to the Boston Fire Department in minutes?

(b) Using the mean in part (a), show the probability density function for the time between 911 calls in minutes.

(c) What is the probability that there will be less than one hour between 911 calls?

(d) What is the probability that there will be 30 minutes or more between 911 calls?

(e) What is the probability that there will be more than 5 minutes, but less than 20 minutes between 911 calls?

Joint, Marginal and Conditional Distributions

16. Suppose we have two random variables X and Y,

Let: - X is a discrete random variable takes three values, $\{1, 2, 3\}$, and Y is another discrete random variable takes three values $\{10, 20, 30\}$, following is the joint PMF,

	$Y = 10$	$Y = 20$	$Y = 30$
$X = 1$	0.1	0.1	0.2
$X = 2$	0.1	0.1	0.1
$X = 3$	0.1	0.05	0.15

(a) Calculate the Marginal PMF of X, which is $f(x)$ and marginal PMF of Y, which is $f(y)$.

(b) Following (a), calculate expected value of X, $\mathbb{E}(X)$ and expected value of Y, $\mathbb{E}(Y)$.

(c) Following (a) and (b), calculate variances $\mathbb{V}ar(X)$ and $\mathbb{V}ar(Y)$.

(d) How many conditional PMF of X given Y can we calculate? and similarly how many conditional PMF of Y given X can we calculate?

(e) Calculate conditional PMF of X given $Y = 10$, which is $f(x|y = 10)$, with this calculate $\mathbb{E}(X|Y = 10)$ and $\mathbb{V}ar(X|Y = 10)$.

(f) Calculate conditional PMF of Y given $X = 1$, which is $f(y|x = 1)$, with this calculate $\mathbb{E}(Y|X = 1)$ and $\mathbb{V}ar(Y|X = 1)$.

(g) Calculate the Covariance between X, Y $\mathbb{C}ov(X, Y)$ and Correlation between X and Y $\rho(X, Y)$ (Note: these are population quantities). What can you say about the relationship between X and Y?

(h) Are the two variables X and Y independent? Justify your answer.

(i) If you start from the Marginal PMF of X and Y can you now construct and new joint PMF of X and Y such that they are independent.

Remarks: All problems are taken from Anderson et al. (2020). If possible you should do more problems from there.

References:

Anderson, D. R., Sweeney, D. J., Williams, T. A., Camm, J. D., Cochran, J. J., Fry, M. J. and Ohlmann, J. W. (2020), *Statistics for Business & Economics*, 14th edn, Cengage, Boston, MA.