Chapter 1

Recap of Essential Math Concepts

Shaikh Tanvir Hossain

October 14, 2022

East West University

Outline

1. **Sets**

2. **Functions**

3. **Counting Methods**
   - Multiplication rule
   - Permutation
   - Combination

- A set is any collection of items or objects thought of as a whole, where the members of the sets are called *elements*. We usually use uppercase letters to denote a set, for example, S, and lower case letters to denote the element of a set. If S is a set and a is an element, we can write $a \in S$. The notation $\in$ means "belongs to".
- For example, think about the set of even numbers between 1 and 11, if we enumerate then we can write this set as $S = \{2, 4, 6, 8, 10\}$. Note in this case, $2, 4, 6, 8, 10$ are the elements, so we can write $2 \in S$, $4 \in S$, and so on...
- Note that the same set can also be written as $S = \{x : x$ is an even number between 1 and $11\}$. It means "S is a set of x, such that x is an even number between 1 and 11" (note the symbol ":" means "such that"). This is another way of writing sets. It is called Set builder notation.
- Sometimes you will also see a slightly different notation "|". This is also is used for "such that". So we can also write $S = \{x \mid x$ is an even number between 1 and $11\}$. This means the same thing.

Some remarks regarding notations

- Reading Mathematics means you need to read lot's of notations.
- Notations are just symbols to represent different things.
- For example we have already used the symbol ":" for "such that". So this is a notation.
- There will be many notations that we will use time to time, do please don't get worried when you see notations.
- In the end notations mean somethings, and your job is to understand what does a particular notation or symbol mean.

- Let's see some more definitions related to Set.
- **Empty set:** When a set has no element, then we call this an *empty set*. This set is denoted by $\emptyset$ or $\{\}$.
- **Equal Sets:** Two sets X and Y are equal if they contain exactly the same elements. and we write $X = Y$
- **Subset:** If we have two sets X and Y, and all the elements of a set X are also elements of the set Y, then X is a called a *subset* of Y, and we use the notation $X \subset Y$. Note that in this case Y is also called a *superset* of X.
- **Proper Subset:** Note when we write $X \subset Y$, then the set X may have exactly same elements as Y or may have less than Y. If all the elements in set X are in a set Y, but not all the elements of Y are in X, then X is called a *proper subset* of Y, and we write $X \subset Y$. In this case X must have less number of elements. The notation is $X \subsetneq Y$.

Let's see an example,

Example (Empty set, Equal Set and Subset) 1.1

Suppose we have following sets,

$$A = \{a, b, c\}$$
$$B = \{a, b, c\}$$
$$C = \{b, c\}$$
$$D = \{c\}$$

Then we can see that $A = B$, but $A \neq C$ and also $B \neq C$ and also $C \neq D$. Also note $A \subset B$, and also $B \subset A$, $C \subset B$ and also $D \subset C$, and also $C \subsetneq B$ (Question: Is it correct to write $D \in C$, Ans: No, why?)

- **Union of two sets:** The union of two sets A and B is the set of elements that belong *either* set A *or* set B or *both* of the sets. The notation we will use is $A \cup B$. So this means

$$A \cup B = \{x : x \in A \text{ OR } x \in B\}$$

- **Intersection of two sets:** The intersection of two sets A and B is the set of elements that belong to *both* A and B. We will use the notation $A \cap B$. So

$$A \cap B = \{x : x \in A \text{ AND } x \in B\}$$

- **Difference between two sets:** The difference (sometimes also called relative difference) of A and B is the set of elements that *belong to* A but *not belong to* B. The notation is $A \setminus B$. We can write,

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

- **Product of two sets:** If A and B are sets, then the *Cartesian product* of A and B, is the *set* of all *ordered pairs* (a, b) such that $a \in A$ and $b \in B$. The notation is $A \times B$, and we can write.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- Note in the Cartesian Product ordering is important. For two sets A and B, $A \times B$ is not same as $B \times A$ (look at the next example).

Example 1.2 Suppose we have following sets,

$$A = \{a, b, c\}$$
$$B = \{a, b, c\}$$
$$C = \{b, c\}$$
$$D = \{c\}$$
$$E = \{1, 2\}$$

Now $A \cup B = \{a, b, c\}, C \cup D = \{b, c\}, C \cap D = \{c\}, B \setminus C = \{a\}$

For Cartesian Products,
$A \times E = \{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$, but
$E \times A = \{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Note that, *ordering matters* for the product of sets, so $(a, 1) \neq (1, a)$. So $A \times E \neq E \times A$.

- Often (depending upon the problem) we have a *universal set*, and usually we denote this set with U.
- Once we have a universal set, then we can find a complement of any set (which is a subset of the speicified universal set).
- If $A \subset U$, then $A^c = U \setminus A$ (for complement sometimes there is another notation $\bar{A}$)

Example 1.3 Suppose we have following sets,

$$\begin{aligned} U &= \{a, b, c, 1, 2\} & A &= \{a, b, c\} \\ B &= \{a, b, c\} & C &= \{b, c\} \\ D &= \{c\} & E &= \{1, 2\} \end{aligned}$$

Now note that $A^c = \{1, 2\}$. Calculate B^c and C^c.

We can have a visual understanding of the set operations (union, intersection, complement, difference) using a diagram called *Venn diagram*.

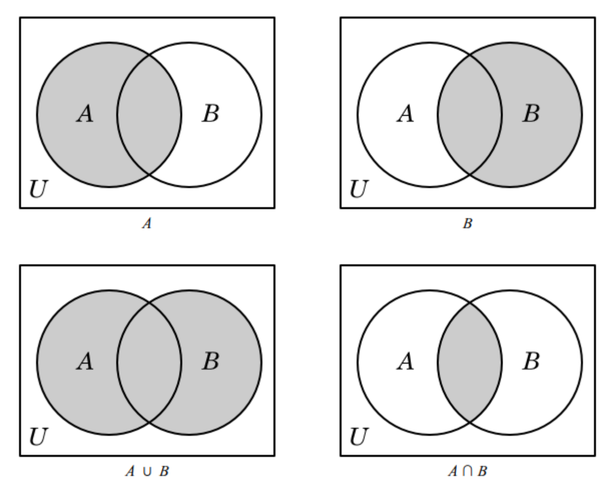


Figure 1: Venn diagram for $A \cup B$, $A \cap B$

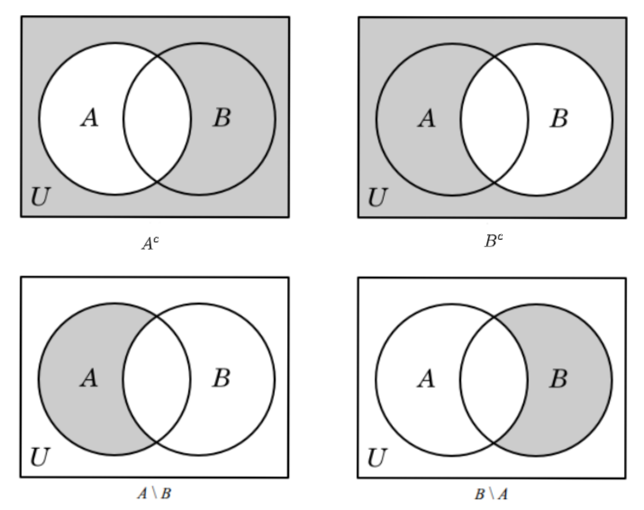


Figure 2: Venn diagram for A^c, B^c, $A \setminus B$, and $B \setminus A$

- Unions, Intersections, Cartesian Product, Difference are called *set operations*.
- Now here *union, intersection and products* can be extended to more than two sets.
- For example if we have a *sequence of sets* $A_1, A_2, A_3, \ldots$, then we can write, $A_1 \cup A_2 \cup A_3 \cup \ldots$ or maybe $A_1 \cap A_2 \cap A_3 \cap \ldots$
- Two important laws of sets are
  - ✓ Associative Law -
    - $A \cup (B \cup C) = (A \cup B) \cup C$
    - $A \cap (B \cap C) = (A \cap B) \cap C$
  - ✓ Distributive Law -
    - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
    - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- There is a nice rule which helps to find complements of unions and intersections. This is known as *Demorgan's Law*. This law says
  - ✓ $(A \cap B)^c = A^c \cup B^c$
  - ✓ $(A \cup B)^c = A^c \cap B^c$?
- We will do some math problems using these laws.

- Now let's talk about the *Power Set*. The power set of a set A, is a *set of sets* where we have the *set of all subsets of A*. The notation for power set of A is $\mathcal{P}(A)$.

Example 1.4 (Power Set)

- ✓ If we have a set $A = \{1, 2\}$, then $\mathcal{P}(A) = \{.\{\}, \{1\}, \{2\}, \{1, 2\}\}.$.
- ✓ If we have a set $A = \{1, 2, 4\}$, then

$$\mathcal{P}(A) = \{.\{\}, \{1\}, \{2\}, \{1, 2\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}.$$

$$= \{.\emptyset, \{1\}, \{2\}, \{1, 2\}, \{4\}, \{1, 4\}, \{2, 4\}, A\}.$$

- ✓ If we have a set $B = \{a, b, c, d\}$, calculate $\mathcal{P}(B)$
- ✓ If we have a set $\Omega = \{H, T\}$, calculate $\mathcal{P}(\Omega)$
- ✓ If we have a set $\Omega = \{1, 2, 3, 4, 5, 6\}$, calculate $\mathcal{P}(\Omega)$

- So in simple words, a *power set is a set of sets* which has all the subsets that we can construct with the elements of A.
- Often a *set of sets* is called a *family of sets*, so power set is a *family of sets*.

- There is a nice trick which we can use to count the number of elements in the power set if the given set is countable and finite. For example if A is a *countable and finite set** then $\mathcal{P}(A)$ will have

$$2^{(\text{number of elements in } A)}$$

- For example, if $A = \{1, 2\}$, then the power set $\mathcal{P}(A)$ will have $2^2 = 4$ elements. Note that this matches with our earlier answer.
- Can you think about power set of $\mathbb{R}$, so this means can you think about $\mathcal{P}(\mathbb{R})$? (This is huge right but maybe an idea?)

*Question is what does countable mean in Mathematics? It means - we can *enumerate* (পরপর উল্লেখ করা) the elements. A set can be countably finite, countably infinite and uncountable. Uncountable sets are always infinite, it is not possible to have uncountable but finite sets. Examples.....(on board)

◇ There are different *sets of numbers* in mathematics.

✓ *Set of Real numbers*, we use the notation $\mathbb{R}$ to denote this set. This set include all numbers that you can possibly think about†. This is a huge set which is *uncountable* and of course *infinite*.

✓ *Set of Natural numbers*, we use the notation $\mathbb{N}$. This set include all positive integer numbers $1, 2, 3, 4, \ldots,$. This is a *countable set* but an infinite set, so this is a *countably infinite* set‡.

✓ *Set of Integer numbers*, we use the notation $\mathbb{Z}$. This set include all the positive and negative integer numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. This is also a *countably infinite set*. Note that $\mathbb{N} \subsetneq \mathbb{Z}$

✓ *Set of Rational numbers*, we use the notation $\mathbb{Q}$. This set include the numbers which can be written as fractions p/q, where p and q are both integers. This set has numbers like $2/3, 10/3$ and also all positive and negative integers are also part of this set (why?). This is also a *countably infinite set*. $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$

✓ *Set of Irrational numbers*. Everything that is NOT Rational but in $\mathbb{R}$ is part of this set, for example $\sqrt{2}$, We can write this set with $\mathbb{R} \setminus \mathbb{Q}$.

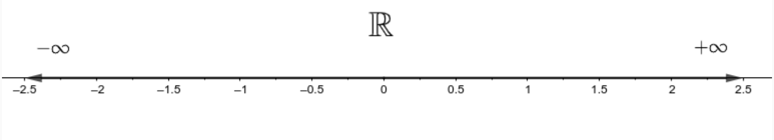
✓ This means we can write $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$

---

† except complex numbers, which we don't need now!

‡ Are you wondering what does countable mean? ...

- The set of real numbers $\mathbb{R}$ can be also visualized in the numberline, here is the numberline that you are probably familiar with



- At the center, we have the number 0 (this is often called the origin or center), at left the number goes to $-\infty$ and right it goes to ∞.
- We can point any number that belongs to $\mathbb{R}$ in the numberline. Here we showed only a few.

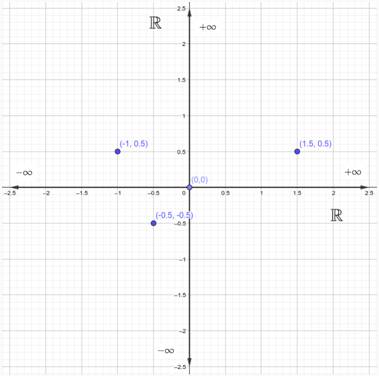
- We can also have different kinds of intervals in $\mathbb{R}$, which are also subsets of $\mathbb{R}$. For example we can construct following intervals (here a and b can be any number in $\mathbb{R}$)

TABLE A.1 Types of intervals

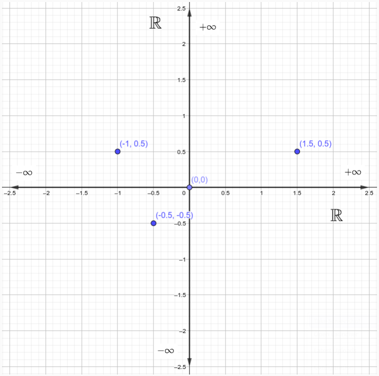
Notation	Set description	Type	Picture
(a, b)	$\{x \mid a < x < b\}$	Open	
$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
(a, ∞)	$\{x \mid x > a\}$	Open	
$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
$(-\infty, b)$	$\{x \mid x < b\}$	Open	
$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

- Intervals like (a, b) is called *open intervals*, intervals like $[a, b]$ *closed intervals*, intervals like $(a, b]$ or $[a, b)$ are called *half-open* intervals.

- Recall the idea of Cartesian product? Can you think about the Cartesian product $\mathbb{R} \times \mathbb{R}$? Although it is impossible to write the set $\mathbb{R} \times \mathbb{R}$, but we can visualize it.
- Just put another numberline vertically on top of the horizontal one, then we will have something which is known as *Cartesian Coordinate* or $x - y$ *Coordinate* or $x - y$ *plane*
- Now here we have a horizontal axis, known as x-axis, and the vertical axis, known as y-axis.



- Here we can show any pair of numbers (x, y), where the first number is on the x-axis and the second is on the y-axis, and togther we can locate the point (x, y) on this $x - y$ plane.
- We have showed the center, which is at $(0, 0)$, and also other three points, $(-1, 0.5)$, $(1.5, 0.5)$ and $(-0.5, -0.5)$



1. Sets

2. Functions

3. Counting Methods
   - Multiplication rule
   - Permutation
   - Combination

- If you have taken any math courses, whether in college or courses like MAT100 or MAT110, you have definitely seen functions.
- For example following are all examples of functions,
  - ✓ $f(x) = x^2$
  - ✓ $f(x) = 2x + 1$
  - ✓ $f(x) = 3x^3 + 2x^2 + 1$
- It is important to note, these are all examples of functions, but what is the *definition* of a function? or *what is a function*?

Definition 1.1 (Function)

Given any two sets A and B, a function $f : A \to B$ is a *mapping* between the elements of A and B such that the following condition is satisfied

- ✓ For every element of A there is a unique element in B.

In this case, the set A is called the *domain* of the function f and B is called the *codomian* of the function.

- It is important to mention that although in the definition we wrote one condition, that is *"For every element of A there is a unique element in B."*, this actually means two points,
  - ✓ First, Since we are saying "For *every* element of A...", the word "every" here automatically means all elements of A needs to be used for mapping .
  - ✓ Second, when we say for each element of A, there must be a unique element in B. This means it will never happen that a single point from A is mapped to two different points in B.

◇ Let's see some examples, suppose we have two sets $A = \{p, q, r, s\}$ and $B = \{1, 2, 3, 4, 5\}$. Here the domain is A and the codomain is B. Question - Is the following mapping a function? Answer - Yes it is, how?

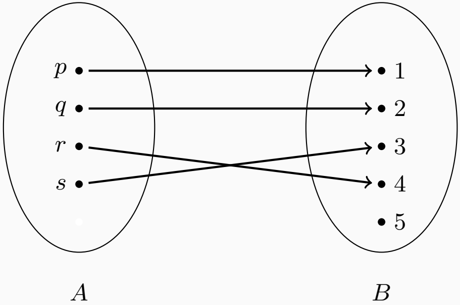


Figure 3: This is a mapping between two sets A and B. In general mapping can be any relation between the two sets. However this mapping is a function (check the condition) and we can call this function f, also we can write $f : A \to B$.

We can also define function using the Cartesian Product but I am not using this definition here.

◇ It is important to understand that here there is a viloation of the condition. In particular it violates the second point in page 23. So it is NOT a function.

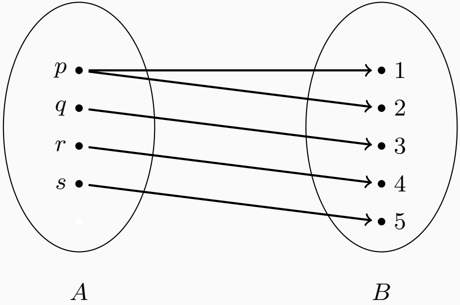


Figure 4: Note that this violates the condition, because for the element p we don't have a unique element in B, rather we have two elements 1 and 2.

- Following is a function, and there is no problem with the condition.

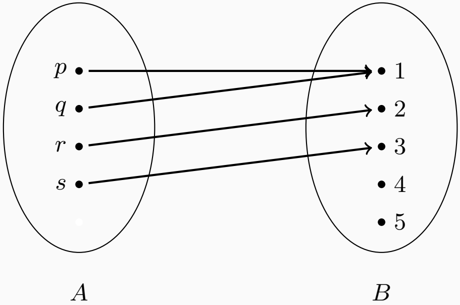


Figure 5: Here although both p and q are mapped to the same elements but still it does not violate the condition.

- For the notation of functions, usually we use the letters f, g, h, etc. Sometimes when we write many functions we also use index $1, 2, 3, \ldots,$. For example $f_1, f_2, f_3, \ldots,$ and so on.
- Always remember when we write $f : X \to Y$, this means f is a function, X is the domain and Y is the codomain.
- The notation that you are used to is $y = f(x)$, here we can also use in Figure 3, we can write $f(p) = 1$, $f(q) = 2$, $f(r) = 4$ and $f(s) = 3$, and we can write $f : A \to B$.
- There is another set in function, which is called *range of a function*. Range is simply the subset of the co-domain which is used the in the mapping. So for the function in Figure 3, the set $B = \{1, 2, 3, 4, 5\}$ is the *codomain* and $\{1, 2, 3, 4\}$ is the *range*.

- Question - Note we did not use all the elements in B in Figure 3 to represent a function. Is this a problem with the definition of a function? (Answer is NO, why?)
- Question - For these two sets can you draw some mappings which are not functions? (Try this now, Hint: Just intentionally violate the two points mentioned in page 23.)

- The functions that you already know or have seen, for example,
  - ✓ 1. $f(x) = x^2$
  - ✓ 2. $f(x) = 2x + 1$
  - ✓ 3. $f(x) = 3x^3 + 2x^2 + 1$

  are all examples functions where we used *algebraic expressions*. We write functions in this way when the domain and codomain are infinite or uncountable sets.
- Note that for above three functions,
  - ✓ 1. $f(x) : \mathbb{R} \to \mathbb{R}$, domain and codomain - $\mathbb{R}$ and range $\mathbb{R}_{\geq 0}$
  - ✓ 2. $f(x) : \mathbb{R} \to \mathbb{R}$, domain, codomain and range $\mathbb{R}$
  - ✓ 3. $f(x) : \mathbb{R} \to \mathbb{R}$, domain, codomain and range $\mathbb{R}$

◇ Here the functions are mapping between two huge sets, so we cannot draw pictures like Figure 3. But definitely when we have a *graph of the functions* then we can see also see the connections between the domian $\mathbb{R}$ and the co-domain $\mathbb{R}$.

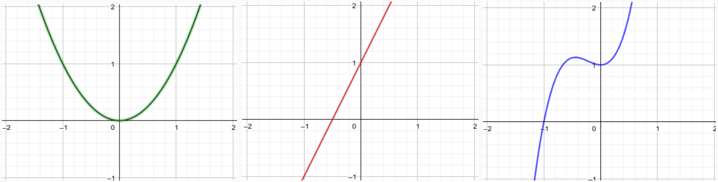
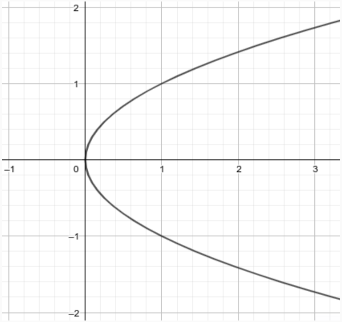


Figure 6: From left - $f(x) = x^2$, $f(x) = 2x + 1$ and $f(x) = 3x^3 + 2x^2 + 1$

The plotting of these functions using a software called *Geogebra* is really easy, just go to `https://www.geogebra.org/calculator` and plot the functions. I will show this on the class.

- Question - If we draw any line on the $x - y$ coordinate, is it always going to be a function?
- For example, is the following a function? We can write this equation as $x = y^2$



- NO! why? Do a vertical line test.
- There is one more concept we need to review that is *limit of a function*, but we will do this recap when we need it.

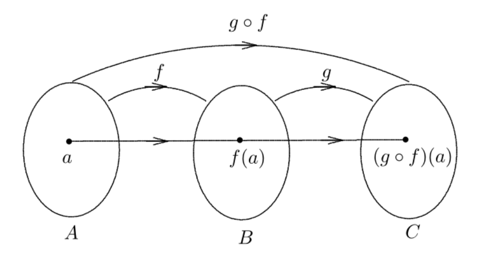
There is a way we can combine two (or more) functions, it's called *composition* of functions. Here is the definition,

Definition 1.2 (Composition of Functions)

Let A, B and C be sets and suppose that we have two functions, $f : A \to B$ and $g : B \to C$. The composition $g \circ f$ is the function from A to C defined by

$$(g \circ f)(a) = g(f(a)) \text{ for } a \in A.$$

So $(g \circ f)$ is new a function (that we got from composition). The domain of this new function is the domain of f and the codomain is the codomain of g.



1. Sets

2. Functions

3. **Counting Methods**
   - Multiplication rule
   - Permutation
   - Combination

Counting Methods

Multiplication rule

- The *multiplication rule* helps to solve some counting problems when we have a process with more than one parts or steps. With this we can count how many ways the entire process can be performed. First let us see the rule and then it will be more clear once we see some examples.

Multiplication Principle

If a procedure or a process consists of k *independent parts* (where $k \geq 2$) and the i^{th} part can be performed in n_i possible ways (where $i = 1, 2, \ldots, k$), then the entire process can be performed in $n_1 \times n_2 \times \ldots, \times n_k$ possible ways.

- Here i^{th} part can be performed in n_i possible ways means,
  - ✓ when $i = 1$, 1^{st} process can be performed with n_1 ways
  - ✓ when $i = 2$, 2^{nd} process can be performed with n_2 ways
  - ✓ $\vdots$
  - ✓ when $i = k$, k^{th} process can be performed with n_k ways
- *independent parts* means one part is not dependent on the other.

Example 1.1 (Multiplication Rule)

- Suppose we have three cities, A, B and C. We can go from city A to B in 2 ways and and from city B to C in 3 ways.
- Question is - *how many ways we can go from city A to C via city B?*
- Since the process has two parts, this means here we have $k = 2$. Now for the first part we have $n_1 = 2$ and for the second part we have $n_2 = 3$.
- So the whole process can be performed in $n_1 \times n_2 = 2 \times 3 = 6$ possible ways.
- For the multiplication problems, the tree diagram (figure on the right) might be useful to visualize.

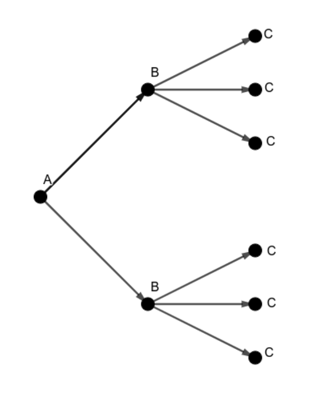


Figure 7: Tree diagram of the problem. *How many ways we can go from city A to C via city B?* The answer is $2 \times 3 = 6$

Example 1.2 (Multiplication Rule)

- Suppose a retail store sells windbreaker jackets in small (S), medium (M), large (L), and extra large (XL). All are available in color "blue" or "red". If a customer wants to buy how many options/choices does he have?
- Applying multiplication rule, we get in total there are $4 \times 2 = 8$ possible choices. We can actually list them

$$\{(S, \text{Blue}), (S, \text{Red}), (M, \text{Blue}), (M, \text{Red}), (L, \text{Blue}), (L, \text{Red}), (XL, \text{Blue}), (XL, \text{Red})\}$$

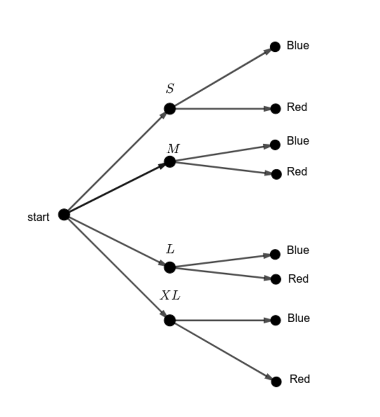


Figure 8: Tree diagram of the problem. *How many combined choices are there?* The answer is $4 \times 2 = 8$

The above 2 examples that we have discussed have only 2 parts in the process, but it is possible to have more than 2 parts.

Example 1.3 (Multiplication Rule)

- Suppose a coin is tossed 6 times, how many possible outcomes are there.
- Actually there will be $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 36$ possible outcomes. Can you list all possible outcomes. For example one outcomes is $HTTHHH$ (can you think about the tree diagram here?)
- Can you draw the tree diagram (yes but this is cumbersome)?

Example 1.4 (Multiplication Rule)

- Suppose we have a 3 digit combination lock where each digit can be from 0 to 9. How many possible combination locks we can set?
- There will be $10 \times 10 \times 10 = 10^3 = 1000$ possible combination locks.
- Can you draw the tree diagram (yes but this is cumbersome)?

Counting Methods

Permutation

- We saw the multiplication rule when different parts are independent, what if they are dependent?
- Consider the last combination lock problem, let's think about the problem in a different way. Think about three empty boxes and then we want to know how many possible ways we can fill the boxes?
- For the first box we have 10 possible options, for the second we also have 10, and for the third we also have 10. This gives the following picture

$$\boxed{10} \times \boxed{10} \times \boxed{10}$$

- This means we have $10 \times 10 \times 10 = 1000$ possible options for locks. The result is same.
- Now suppose we want the number of combinations where *there should be no repetition of digits*. This means the same digit cannot appear more than once. Or in other words all three digits should be different. For example we don't want to count combinations like - 0, 0, 1 or 1, 1, 1.

- For this problem the first place for the lock has 10 digits, the second place for the lock has 9 digits, and the third place for the lock has 8 digits.

$$\boxed{10} \times \boxed{9} \times \boxed{8}$$

- So we have $10 \times 9 \times 8 = 720$ possible combinations.
- Although we are applying multiplication rule but here each part is dependent on other parts. This problem is known as *ordering problem* where order matters.
- This means when we are counting we are treating $1, 2, 3$ and $2, 1, 3$ as a separate count.
- For ordering problem, we don't have to think about the combination lock, the idea is if we have 10 digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we ask how many ways we can *order* any 3 digits. The answer is

$$^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8$$

- This is the famous permutation formula that you already know[§]
- So the ordering problem is actually a multiplication problem. If someone asks you "if we have n objects then how many ways we can order k of them if we pick one at a time?" the answer is

$$^{n}P_{k} = \frac{n!}{(n-k)!}$$

- In this case you can think n objects and k empty boxes and we are trying to fill them one by one.
- What if we have n objects and n boxes, then $^{n}P_{n} = n!$ (note that $0! = 1$) This means $n!$ gives the total number of orderings when we want to order n objects and we have n empty boxes.

[§]The word "permutation" in English just means "rearrangement". For example if we have three letters a,b,c then a another permutation (or ordering) is b,a,c. So when we ask total number of permutations, this is same asking total number of arrangements or total number of orderings.

Example 1.5 (Permutation)

If we have 5 letters a, b, c, d, e, then

- How many ways we can order them?
- How many ways we can order 3 of them (taking one at a time)?

The answer to the first question is $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. To answer the second question, we can follow one of the two approaches

- **Empty box approach (perhaps this is more intuitive):** We have 3 empty boxes, so for the first box we have 5 options, for the second box we have 4 and for the third we have 3. So in total we have $5 \times 4 \times 3 = 60$. So there are 60 possible ways we can order them
- **Directly applying the formula:** Since this is a direct ordering problem we can apply the permutation formula, ${}^5P_3 = \frac{5!}{(5-3)!} = 60$.

There is a nice website which can show all possible permutations, check this out - `https://www.dcode.fr/partial-k-permutations`

- Here are all 60 permutations if we pick 3 letters out of 5.

$$\begin{array}{cccccc}
a,b,c & b,a,c & c,a,b & a,c,b & b,c,a & c,b,a \\
a,b,d & b,a,d & d,a,b & a,d,b & b,d,a & d,b,a \\
a,b,e & b,a,e & e,a,b & a,e,b & b,e,a & e,b,a \\
a,c,d & c,a,d & d,a,c & a,d,c & c,d,a & d,c,a \\
a,c,e & c,a,e & e,a,c & a,e,c & c,e,a & e,c,a \\
a,d,e & d,a,e & e,a,d & a,e,d & d,e,a & e,d,a \\
b,c,d & c,b,d & d,b,c & b,d,c & c,d,b & d,c,b \\
b,c,e & c,b,e & e,b,c & b,e,c & c,e,b & e,c,b \\
b,d,e & d,b,e & e,b,d & b,e,d & d,e,b & e,d,b \\
c,d,e & d,c,e & e,c,d & c,e,d & d,e,c & e,d,c
\end{array}$$

- Note that ordering matters. Now it should be clear to you what does it mean - Take the first row where we have different permutations of the letter a, b, c. In total there are $3 \times 2 = 6$ permutations (look at row 1), ordering matters means when we count, we count all 6 of them.
- Similarly in every row we have 6 permutations of three letters and when we count we count all of them.
- What if we treat them as a single count?

Counting Methods

Combination

- The answer to the question in the last line of the last slide is - we get combination.
- For the last problem in the permutation section if we ask - “how many ways we can *select* 3 letters out of 5”, the answer is 10.
- Notice the word “select”, the idea is when we select we don’t care about orders we just pick.
- Now how did we get 10? Just count one for each row where we show all possible permutations. Since there are 10 rows we have 10 possible combinations.
- Let’s see the definition now and hopefully things will be clear.

Definition 1.6 (Combinations)

If we have a set of n elements. Each *subset* of size k chosen from this set is called a *combination of n elements taken k at a time*. We denote the number of *distinct* such combinations by the symbol ${}^nC_k$. And we can count this number by

$$ {}^nC_k = \frac{n!}{k!\,(n-k)!} $$

- Notice the last formula can be written as
$$ {}^nC_k = \frac{{}^nP_k}{k!} $$
- How does this formula come? We can explain this via permutation. The idea is let's think permutations as being constructed in two steps or two parts.
- *Step 1* A combination of k elements is chosen out of n, this is ${}^nC_k$
- *Step 2* those k elements are arranged in a specific order within themselves. This is $k!$
- Now we can use multiplication rule and we see that
$$ {}^nP_k = {}^nC_k \times k! $$

- From here we get our formula

$$^{n}C_{k} = \frac{^{n}P_{k}}{k!}$$

- So we can say that the number of distinct subsets of size k that can be chosen from a set of size n is $^{n}C_{k}$.
- Or if someone asks you *"how many ways you can select k objects from n?"*, then the answer is $^{n}C_{k} = \frac{^{n}P_{k}}{k!} = \frac{n!}{k!(n-k)!}$

There is a very useful application of ${}^{n}C_k$, we this call *Binomial Theorem*, here is the theorem.

Theorem 1.7 (Binomial Theorem.)

For all numbers x and y and each positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

where $\binom{n}{k}$ is same as ${}^{n}C_k$, so this is possible number of combinations of k objects out of n. In this case this is also known as *Binomial co-efficient*