Risk Parity Model Based on MVT

DTFF Final Assignment

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Overview

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Risk Measures

Since investors pay more attention to the downside risk of financial assets, two indicators to measure the downside risk are raised.

Definition 1: Mean Downside Semi Variance

$$\mathrm{SV}_m = \frac{1}{T}\sum_{t=1}^{T} \max\left[0, E(R_t) - R_t\right]^2$$

where $E(R_t)$ is the average rate of return.

Definition 2: Target Downside Semi Variance

$$\mathrm{SV}_t = \frac{1}{T}\sum_{t=1}^{T} \max\left[0, R^* - R_t\right]^2$$

where R^* is the target rate of return.

Risk Measures

Besides the risk metrics, there are three indicators to measure value at risk.

Definition 3: Value at Risk

$$\mathrm{VaR}_\alpha(X) = -q_\alpha(X) = -\inf\{x : P[X \leq x] \geq \alpha\}$$

where X is a given random variable, and α is the confidence level.

Definition 4: Conditional Value at Risk

$$\mathrm{CVaR}_\alpha(X) = -E[X \leq q_\alpha(X)]$$

Definition 5: Expected Shortfall

$$ES(\alpha) = \frac{\int_0^\alpha \mathrm{VaR}(u)du}{\alpha}$$

Risk Parity Model

The core idea of All Weather strategy is to equates the risk of each asset by determining their weights in the asset portfolio.

Definition of Total Risk $\mathcal{R}_{\mathbf{p}}(\vec{w})$ of a Portfolio

$$\mathcal{R}_{\mathbf{p}} = \sqrt{\vec{w}^T \Sigma \vec{w}}$$

Definition of the Marginal Risk and the Risk Contribution

- **Marginal Risk $\mathcal{MR}(X_i)$ of each asset i of a Portfolio**

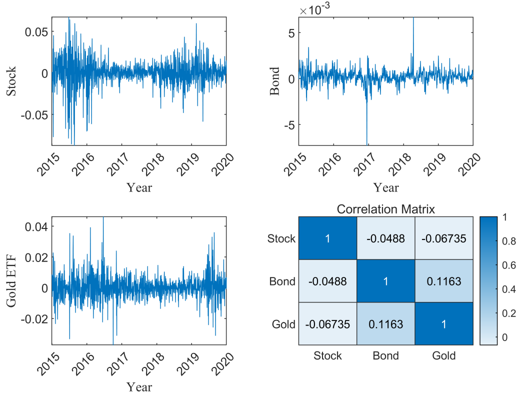
$$\mathcal{MR}\left(X_i\right) = \frac{\partial \mathcal{R}_{\mathbf{p}}}{\partial w_i}$$

- **Risk Contribution $\mathcal{RC}(X_i)$ of each asset i of a Portfolio**

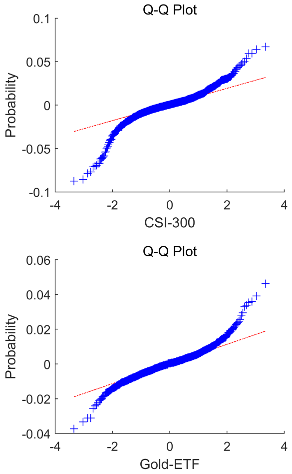
$$\mathcal{RC}\left(X_i\right) = w_i \frac{\partial \boldsymbol{R}_{\mathbf{p}}}{\partial w_i} = w_i \frac{(\Sigma \vec{w})_i}{\sqrt{\vec{w}^T \sum \vec{w}}}$$

Overview

We use CSI 300, CSI Aggregate Bond Index and Gold Exchange Traded Fund to represent the performance of stock, marketable securities, and commodity markets, respectively. The following figures represent the Daily Return of CSI-300, CSI-ABI, and Gold- ETF (lower left) and the Correlation Matrix.



Normality Test



Characteristics of the Data

Three Characteristics

- **Volatility:** The volatility of the stock market is the highest while that of the bond market is the lowest, and the volatility of the gold market is in the middle.
- **Leptokurtic:** All of the three types of assets have kurtosis greater than 3, and have bias. It is known as "peak and fat tail".
- **Normality Test:** From the Q-Q plot, it is clear that the logarithmic return of the three assets does not follow the normal distribution.

Parameter Estimation of MVT Distribution

Since the normal distribution is not effective to describe the return on financial assets, so we introduce the multivariate t distribution (MVT).

Density Function of d-dimensional Student t Distribution $T_\nu(\mu, \Sigma)$

$$p(x \mid v, \mu, \Sigma) = \frac{\Gamma\left(\frac{d+v}{2}\right)}{\Gamma\left(\frac{v}{2}\right) v^{\frac{d}{2}} \pi^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \frac{1}{\left(1 + \frac{1}{v}(x-\mu)^{\mathrm{T}} \Sigma^{-1}(x-\mu)\right)^{\frac{d+v}{2}}}$$

with $v > 0$ degrees of freedom, location paramter $\mu \in \mathbb{R}^d$ and symmetric, positive definite scatter matrix $\Sigma \in \mathrm{SPD}(\mathrm{d})$

Parameter Estimation of MVT Distribution

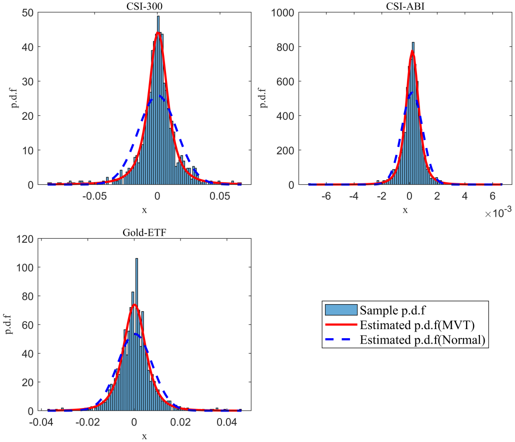
Through the MMF method, we obtain the estimated parameters

$$\hat{\nu} = 3.4273,\ \hat{\mu} = [8.519, 1.783, 2.064] \times 10^{-4}$$

$$\hat{\Sigma} = \begin{bmatrix} 998 & -2.55 & -7.24 \\ -2.55 & 2.59 & 3.25 \\ -7.24 & 3.25 & 279 \end{bmatrix} \times 10^{-7}$$

Parameter Estimation of MVT Distribution

Using the estimated parameters, we draw the probability density function.



VaR and ES based on MVT

Theoretical Formula

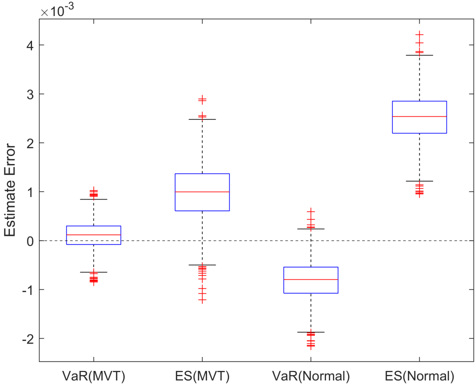
Assume the return $r = (r_1, r_2, r_3)^T \sim t_\nu(\mu, \Sigma)$ is a 3-variate t-distribution random variable, then the return of a portfolio with weight $w = (w_1, w_2, w_3)^T$ is $r_p(w) = w^T r \sim t_\nu(w^T\mu, w^T\Sigma w)$.

$$
\begin{aligned}
\mathrm{VaR}_\alpha\left(r_p(w)\right) &= \mu_p(w) + \Sigma_p(w) t_\nu^{-1}(\alpha) \\
\mathrm{ES}_\alpha\left(r_p(w)\right) &= \mu_p(w) + \Sigma_p(w) \mathrm{ES}_\alpha\left(t\right) \\
&= \mu_p(w) - \Sigma_p(w) \frac{f_{t_\nu}\left(t_\alpha^{-1}(\nu)\right)}{F_{t_\nu}\left(t_\alpha^{-1}(\nu)\right)} \cdot \frac{\left(\nu + (t_\alpha^{-1}(\nu))^2\right)}{\nu - 1}
\end{aligned}
\tag{1}
$$

where $\mu_p(w) = w^T\mu$ and $\sigma_p^2(w) = w^T\Sigma w$

VaR and ES based on MVT

To compare the estimation effect of risk measurement based on MVT and normal distribution, we construct the Monte Carlo Simulation.



Overview

Data Selection

In order to test whether the risk parity model based on MVT can perform well in the actual market.

- We construct the risk parity model for CSI 300, CSI ABI and Gold ETF and carry out the dynamic backtesting.
- In terms of the time duration, we still applied the logarithmic rate of return data from 2015 to 2019.
- However, due to the need to estimate parameters and calculate risk metrics during the backtesting process, the actual backtesting interval is from April 2015 to December 2019.

Model Derivation

The risk measure $\boldsymbol{\mathcal{R}}(.)$ is a mapping from $\mathcal{F}$ to space $\mathbb{R}$. The risk measure is consistent if it satisfies the following properties:

Consistency in Risk Metrics

- Monotonicity: $\forall X_1 \geq X_2 \in \mathcal{F}, \mathcal{R}\left(\mathrm{X}_1\right) \geq \mathcal{R}\left(\mathrm{X}_2\right)$
- Homogeneity: $\forall X \in \mathcal{F}, c \geq 0, \mathcal{R}(\mathrm{cX}) = c\mathcal{R}(\mathrm{X})$
- Translation Invariance: $\forall X \in \mathcal{F}, c \in \mathbb{R}, \mathcal{R}(\mathrm{X} + \mathrm{c}) = \mathcal{R}(\mathrm{X}) - c$
- Subadditivity: $\forall X_1, X_2, \ldots, X_N \in \mathcal{F}, \mathcal{R}\left(\sum_{i=1}^{N} X_i\right) \leq \sum_{i=1}^{N} \mathcal{R}\left(X_i\right)$

Model Derivation

Model Extension

- **Model based on Standard Deviation**

$$\mathcal{RC}(X_i) = w_i \frac{\partial \mathcal{R}_p}{\partial w_i} = w_i \frac{(\Sigma \vec{w})_i}{\sqrt{\vec{w}^T \sum \vec{w}}}$$

- **Model based on $\mathbf{VaR}_\alpha$**

$$\mathcal{RC}(X_i) = -w_i \mu_i - w_i \frac{(\Sigma \vec{w})_i}{\sqrt{\vec{w}^T \sum \vec{w}}} t_\nu^{-1}(\alpha)$$

- **Model based on $\mathbf{ES}_\alpha$**

$$\mathcal{RC}(X_i) = -w_i \mu_i + w_i \frac{(\Sigma \vec{w})_i}{\sqrt{\vec{w}^T \sum \vec{w}}} \frac{f_{t_\nu}\left(t_\alpha^{-1}(\nu)\right)}{F_{t_\nu}\left(t_\alpha^{-1}(\nu)\right)} \cdot \frac{\left(\nu + (t_\alpha^{-1}(\nu))^2\right)}{\nu - 1}$$

Model Backtesting

We use the method of quarterly rebalancing for backtesting. There are 1, 2, ..., 19 quarters in total. When repositioning for time t, the following steps are performed:

1. Calculate the parameters of MVT and the risk measures each asset in the t^{th} quarter.
2. Use the risk parity model to calculate the weight of each asset $w_t = (w_1, w_2, w_3)^T$ in period t.
3. Calculate the indicators used to evaluate the performance of the models.

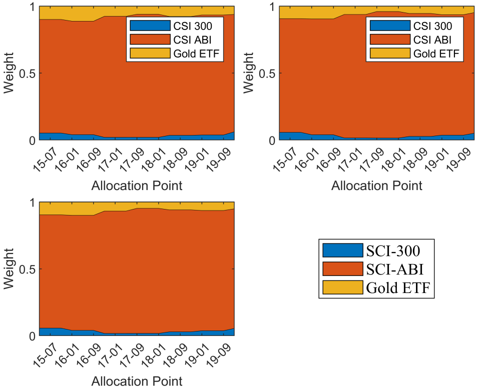
Result Analysis

The following table shows the evaluation indexes of models and assets from Apr,2015 to Dec,2019 under the condition of $R_f = 3\%$.

Index	Annual Return	Std	MDD	Sharp Ratio
std Model	5.39%	1.83%	4.00%	1.3080
VaR Model	5.25%	1.71%	3.94%	1.3157
ES Model	5.28%	1.73%	3.94%	1.3161
CSI 300	0.23%	23.86%	61.71%	-0.1159
CSI ABI	4.81%	1.15%	4.47%	1.5745
GOLD ETF	7.79%	11.50%	15.64%	0.4167

Result Analysis

The following figure shows that the weight of assets based on Std Model, VaR Model, and ES Model respectively.



Result Analysis

1. **Allocation Models All Outperform Individual Assets**
   - The annualized rate of return is higher than that of CSI 300 and CSI ABI, and lower than gold ETF.
   - The annualized volatility is much lower than that of CSI 300 and gold ETF, while it is slightly higher than that of CSI ABI.
   - Sharpe ratio is significant higher than that of a single asset.
2. **Different Models Have Their Own Characteristics**
   - Compared with the standard deviation models, the performance of the VaR model and the ES model is basically the same.
   - With respect to the Sharpe ratio, the standard deviation model is the lowest and the expected shortfall model is the highest.
3. **Weight Allocation with More Debt and Less Shares**
   - The average allocation ratio of the VaR model is the lowest in the CSI 300 Index and the gold ETF, and the highest in the CSI ASI.
   - The expected shortfall model is more sensitive to the changes of the upward and downward trend.
   - The asset allocation ratio has more bonds than stocks.

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