732A96/TDDE15 Advanced Machine Learning
Hidden Markov Models

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Lecture 6: Hidden Markov Models

Contents

Literature

- Main source
  - Bishop, C. M. *Pattern Recognition and Machine Learning*. Springer, 2006. Chapter 13.1-13.2.
- Additional source
  - Ghahramani, Z. An Introduction to Hidden Markov Models and Bayesian Networks. *International Journal of Pattern Recognition and Artificial Intelligence* 15, 9-42, 2001.

Dynamic Bayesian Networks

- To model **sequential data**, e.g. time series data.
- **Simplification**: Time is discretized in equal width intervals, i.e. $t = 0, 1, \ldots$
- Consider a finite set of discrete random variables $X^t = \{X_1^t, \ldots, X_n^t\}$ representing the state at time t of a system described by $X = \{X_1, \ldots, X_n\}$.
- A **dynamic Bayesian network** (DBN) is a BN over $X^{0:T} = \{X^0, \ldots, X^T\}$. Thus, it defines $p(x^{0:T})$.

$$
\begin{tikzcd}
X_1^0 \arrow[r] \arrow[d] \arrow[rrrr, bend left] & X_1^1 \arrow[dr] \arrow[rrr, bend left] & X_1^2 \arrow[d] \arrow[rr, bend left] & X_1^3 \arrow[r] & X_1^4 \arrow[d] \arrow[dd, bend left] \\
X_2^0 \arrow[r] \arrow[d] & X_2^1 \arrow[u] \arrow[dr] & X_2^2 \arrow[r] & X_2^3 & X_2^4 \arrow[d] \\
X_3^0 \arrow[r] & X_3^1 \arrow[u] & X_3^2 \arrow[r] \arrow[u] & X_3^3 & X_3^4
\end{tikzcd}
$$

- **Assumption**: The system is Markovian, i.e. $X^{t+1} \perp_p X^{0:t-1} | X^t$.

$$
\begin{tikzcd}
X_1^0 \arrow[r] \arrow[d] & X_1^1 \arrow[dr] & X_1^2 \arrow[d] & X_1^3 \arrow[r] & X_1^4 \arrow[d] \arrow[dd, bend left] \\
X_2^0 \arrow[r] \arrow[d] & X_2^1 \arrow[u] \arrow[dr] & X_2^2 \arrow[r] & X_2^3 & X_2^4 \arrow[d] \\
X_3^0 \arrow[r] & X_3^1 \arrow[u] & X_3^2 \arrow[r] \arrow[u] & X_3^3 & X_3^4
\end{tikzcd}
$$

- **Assumption**: The system is stationary, i.e. $p(x^{t+1}|x^t) = p(x'|x)$.

$$
\begin{tikzcd}
X_1^0 \arrow[r] \arrow[d] & X_1^1 \arrow[r] & X_1^2 \arrow[r] & X_1^3 \arrow[r] & X_1^4 \\
X_2^0 \arrow[d] & X_2^1 \arrow[u] & X_2^2 \arrow[u] & X_2^3 \arrow[u] & X_2^4 \arrow[u] \\
X_3^0 \arrow[r] & X_3^1 \arrow[r] \arrow[u] & X_3^2 \arrow[r] \arrow[u] & X_3^3 \arrow[r] \arrow[u] & X_3^4 \arrow[u]
\end{tikzcd}
$$

Dynamic Bayesian Networks

- Then, a DBN over $X^{0:T}$ can be defined as
  - a BN over X^0, and
  - a BN over $X^t \cup X^{t+1}$ where the nodes in X^t are parentless.

Initial model	Transition model
$\begin{array}{c} X_1^0 \\ \downarrow \\ X_2^0 \\ \downarrow \\ X_3^0 \end{array}$	$\begin{array}{ccc} X_1^t & \rightarrow & X_1^{t+1} \\ & & \uparrow \\ X_2^t & & X_2^{t+1} \\ & & \uparrow \\ X_3^t & \rightarrow & X_3^{t+1} \end{array}$

- DBN unrolled for $T = 4$.

$$\begin{tikzcd}
X_1^0 \arrow[r] \arrow[d] & X_1^1 \arrow[r] & X_1^2 \arrow[r] & X_1^3 \arrow[r] & X_1^4 \\
X_2^0 \arrow[d] & X_2^1 \arrow[u] & X_2^2 \arrow[u] & X_2^3 \arrow[u] & X_2^4 \arrow[u] \\
X_3^0 \arrow[r] & X_3^1 \arrow[r] \arrow[u] & X_3^2 \arrow[r] \arrow[u] & X_3^3 \arrow[r] \arrow[u] & X_3^4 \arrow[u]
\end{tikzcd}$$

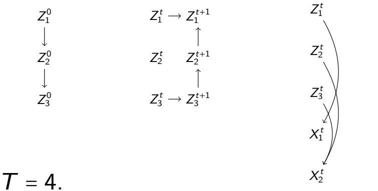
- The DBN defines

$$p(x^{0:T}) = p(x^0) \prod_{t=0}^{T-1} p(x^{t+1}|x^t) = [\prod_{i=1}^{n} p(x_i^0|pa_i^0)][\prod_{t=0}^{T-1} \prod_{i=1}^{n} p(x_i^{t+1}|pa_i^{t+1})]$$

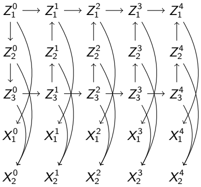
Hidden Markov Models

- To overcome the **Markovian limitation** of DBNs, while keeping sparsity.
- A **hidden Markov model** (HMM) over $\{Z^{0:T}, X^{0:T}\}$ where $X^{0:T}$ are **observed** and $Z^{0:T}$ are **unobserved** consists of
  - a DBN over $Z^{0:T}$, and
  - a BN over $Z^t \cup X^t$ where the nodes in Z^t are parentless.

Initial model	Transition model	Emission model



- HMM unrolled for $T = 4$.



- A HMM is a DBN that defines

$$p(z^{0:T}, x^{0:T}) = p(z^0) \prod_{t=1}^{T-1} p(z^{t+1}|z^t) \prod_{t=0}^{T} p(x^t|z^t)$$

Hidden Markov Models: Lab

- You are asked to model the behavior of a robot that walks around a ring. The ring is divided into 10 sectors. At any given time point, the robot is in one of the sectors and decides with equal probability to stay in that sector or move to the next sector. You do not have direct observation of the robot. However, the robot is equipped with a tracking device that you can access. The device is not very accurate though: If the robot is in the sector i, then the device will report that the robot is in the sectors $[i-2, i+2]$ with equal probability.

$$
\begin{array}{ccccccccc}
Z^0 & \longrightarrow & Z^1 & \longrightarrow & Z^2 & \longrightarrow & Z^3 & \longrightarrow & Z^4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^0 & & X^1 & & X^2 & & X^3 & & X^4
\end{array}
$$

- Where is the robot at time t ? I.e., compute the **filtered** distribution $p(z^t|x^{0:t})$ via the **forward-backward algorithm**.
- Where was the robot at time t ? I.e., compute the **smoothed** distribution $p(z^t|x^{0:T})$ via the **forward-backward algorithm**.
- Which is the most likely path that the robot followed ? I.e., run the **Viterbi algorithm**.

Hidden Markov Models: Forward-Backward Algorithm

- Smoothing:

$$
\begin{aligned}
p(z^t|x^{0:T}) &= \frac{p(x^{0:T}|z^t)p(z^t)}{p(x^{0:T})} \\
&= \frac{p(x^{0:t}|z^t)p(z^t)p(x^{t+1:T}|z^t)}{p(x^{0:T})} \text{ by } X^{0:t} \perp_p X^{t+1:T} | Z^t \\
&= \frac{p(x^{0:t}, z^t)p(x^{t+1:T}|z^t)}{p(x^{0:T})} = \frac{\alpha(z^t)\beta(z^t)}{\sum_{z^t}\alpha(z^t)\beta(z^t)}
\end{aligned}
$$

- Filtering:

$$
p(z^t|x^{0:t}) = \frac{\alpha(z^t)}{\sum_{z^t}\alpha(z^t)}
$$

Hidden Markov Models: Forward-Backward Algorithm

$$
\begin{aligned}
\boldsymbol{\alpha(z^t)} &= p(x^{0:t}, z^t) = p(x^{0:t}|z^t)p(z^t) = p(x^t|z^t)p(z^t)p(x^{0:t-1}|z^t) \text{ by } X^{0:t-1} \perp_p X^t | Z^t \\
&= p(x^t|z^t)p(x^{0:t-1}, z^t) = p(x^t|z^t) \sum_{z^{t-1}} p(x^{0:t-1}, z^t|z^{t-1})p(z^{t-1}) \\
&= p(x^t|z^t) \sum_{z^{t-1}} p(x^{0:t-1}|z^{t-1})p(z^t|z^{t-1})p(z^{t-1}) \text{ by } X^{0:t-1} \perp_p Z^t | Z^{t-1} \\
&= p(x^t|z^t) \sum_{z^{t-1}} p(x^{0:t-1}, z^{t-1})p(z^t|z^{t-1}) = p(x^t|z^t) \sum_{z^{t-1}} \boldsymbol{\alpha(z^{t-1})} p(z^t|z^{t-1}) \\
\alpha(z^0) &= p(x^0, z^0) = p(x^0|z^0)p(z^0)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\beta(z^t)} &= p(x^{t+1:T}|z^t) = \sum_{z^{t+1}} p(x^{t+1:T}, z^{t+1}|z^t) = \sum_{z^{t+1}} p(x^{t+1:T}|z^{t+1}, z^t)p(z^{t+1}|z^t) \\
&= \sum_{z^{t+1}} p(x^{t+1:T}|z^{t+1})p(z^{t+1}|z^t) \text{ by } X^{t+1:T} \perp_p Z^t | Z^{t+1} \\
&= \sum_{z^{t+1}} p(x^{t+2:T}|z^{t+1})p(x^{t+1}|z^{t+1})p(z^{t+1}|z^t) \text{ by } X^{t+2:T} \perp_p X^{t+1} | Z^{t+1} \\
&= \sum_{z^{t+1}} \boldsymbol{\beta(z^{t+1})} p(x^{t+1}|z^{t+1})p(z^{t+1}|z^t) \\
\beta(z^T) &= 1 \text{ by } p(z^T|x^{0:T}) = \frac{\alpha(z^T)\beta(z^T)}{p(x^{0:T})} = p(z^T|x^{0:T})\beta(z^T)
\end{aligned}
$$

Hidden Markov Models: Forward-Backward Algorithm

FB algorithm

$\alpha(z^0) := p(x^0|z^0)p(z^0)$
For $t = 1, \ldots, T$ do
$\quad \alpha(z^t) := p(x^t|z^t) \sum_{z^{t-1}} \alpha(z^{t-1})p(z^t|z^{t-1})$
$\beta(z^T) := 1$
For $t = T-1, \ldots, 0$ do
$\quad \beta(z^t) := \sum_{z^{t+1}} \beta(z^{t+1})p(x^{t+1}|z^{t+1})p(z^{t+1}|z^t)$
Return $\alpha(z^0), \ldots, \alpha(z^T), \beta(z^0), \ldots, \beta(z^T)$

- Filtering: $p(z^t|x^{0:t}) = \frac{\alpha(z^t)}{\sum_{z^t} \alpha(z^t)}$.
- Smoothing: $p(z^t|x^{0:T}) = \frac{\alpha(z^t)\beta(z^t)}{\sum_{z^t} \alpha(z^t)\beta(z^t)}$.
- Note that the FB algorithm consists of two independent (parallelizable) steps.

Hidden Markov Models: Viterbi Algorithm

- To compute the most probable configuration for HMMs, i.e.

$$z_{\max}^{0:T} = \arg\max_{z^{0:T}} p(z^{0:T}|x^{0:T})$$

Viterbi algorithm

$\omega(z^0) := \log p(z^0) + \log p(x^0|z^0)$
For $t = 0, \dots, T-1$ do
  $\omega(z^{t+1}) := \log p(x^{t+1}|z^{t+1}) + \max_{z^t}[\log p(z^{t+1}|z^t) + \omega(z^t)]$
  $\psi(z^{t+1}) := \arg\max_{z^t}[\log p(z^{t+1}|z^t) + \omega(z^t)]$
$z_{\max}^T = \arg\max_{z^T} \omega(z^T)$
For $t = T-1, \dots, 0$ do
  $z_{\max}^t := \psi(z_{\max}^{t+1})$
Return $z_{\max}^{0:T}$

- **Exercise**. Prove that the Viterbi algorithm is correct.

Contents

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Thank you