732A96/TDDE15 Advanced Machine Learning

Reinforcement Learning

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Lectures 8: Q-Learning Algorithm

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- Learning through Interaction
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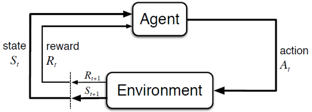
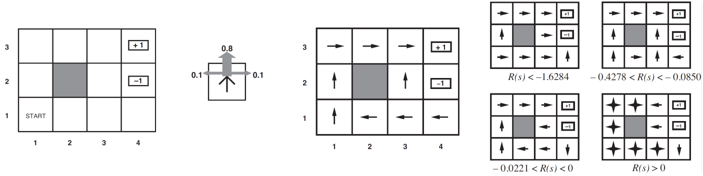
Literature

- Main source
  - Sutton, R. S. and Barto, A. G. *Reinforcement Learning: An Introduction*. The MIT Press, 2018. Chapters 1-7.
- Additional source
  - Russel, S. and Norvig, P. *Artifical Intelligence: A Modern Approach*. Pearson, 2010. Chapters 16, 17 and 21.

Learning through Interaction

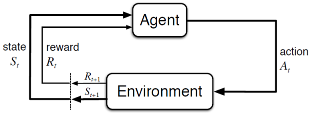


- **Agent**: The learner and decision maker.
- **Environment**: The agent interacts with it.
  - **State**: State of the agent and the environment.
  - **Action**: The agent decides the next action on the basis of the current state.
  - **Reward**: Numerical response to the action chosen by the agent. The agent aims to learn how to act so as to maximize the cumulative reward.
  - **Trajectory**: $S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, S_3, \ldots$
- **Policy**: Probability of doing an action in a state. The agent acts according to it. The agent aims to learn an optimal one.
- Example: A robot moves with probability 0.8 in the intended direction, and at the right angles of it otherwise. The reward for non-terminal states is $R(s) = -0.04$. All this is unknown to the robot. Optimal policies shown.



- RL is neither supervised nor unsupervised learning.

Markov Decision Processes



- We assume that the agent-environment interaction follows a finite **Markov decision process**:
  - Finite sets of states, actions and rewards. Fully observable state.
  - Markovian and stationary transition model:

$$p(S_t, R_t | S_{0:t-1}, A_{0:t-1}, R_{1:t-1}) = p(S_t, R_t | S_{t-1}, A_{t-1}) = p(S', R | S, A).$$

- The **transition model** is typically unknown to the agent. Note the randomness of the next state and reward. Note also the effect of the action on the next state.
- The objective of the agent is to learn a policy $\pi(a|s)$ that maximizes the **expected discounted return**:

$$E_\pi[G_t] = E_\pi\Big[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\Big] = E_\pi[R_{t+1} + \gamma G_{t+1}]$$

where $0 < \gamma \leq 1$ describes our preference between present and future rewards. Note the infinite horizon. However, the expectation is finite if $\gamma < 1$. For episodic tasks, $\gamma = 1$ and $R_{t+k+1} = 0$ for all $t + k + 1 > T$.

Markov Decision Processes

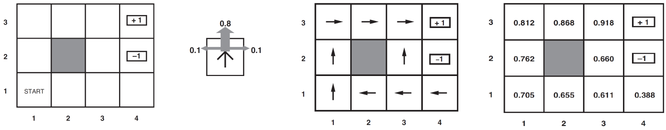
- The **state value** function $v_\pi(s)$ is the expected return of following policy π starting from state s:

$$v_\pi(s) = E_\pi[G_t|S_t = s] = \sum_a \pi(a|s) E_\pi[G_t|S_t = s, A_t = a] = \sum_a \pi(a|s) q_\pi(s,a).$$

- The **action value** function $q_\pi(s,a)$ is the expected return of doing action a in state s and then following policy π:

$$q_\pi(s,a) = E_\pi[G_t|S_t = s, A_t = a] = E_\pi[R_{t+1} + \gamma G_{t+1}|s,a] = \sum_{s',r} p(s',r|s,a)(r + \gamma v_\pi(s')).$$

- Example: Environment, policy and state values.



- We can define the objective of the agent as learning a policy π_* such that

$$v_*(s) \geq v_\pi(s) \text{ for all } \pi, s.$$

For MDPs, there is always **at least one** such optimal policy.

Bellman Equations

- The state value function satisfies a recursive relationship known as **Bellman equation**:

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)(r + \gamma v_\pi(s')).$$

- Moreover, v_π is the **unique solution** to the equations. Note that there are as many equations as unknowns. Since the equations are linear, they can be solved by linear algebra methods in $O(n^3)$. But this requires knowing the transition model.
- Likewise for the action value function:

$$q_\pi(s,a) = \sum_{s',r} p(s',r|s,a)(r + \gamma \sum_{a'} \pi(a'|s') q_\pi(s',a')).$$

- The Bellman equations of an **optimal policy** are

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)(r + \gamma v_*(s'))$$

and

$$q_*(s,a) = \sum_{s',r} p(s',r|s,a)(r + \gamma \max_{a'} q_*(s',a')).$$

- As before, v_* and q_* are the **unique solutions** to these equations. Note that the equations are now non-linear due to the max operator and, thus, harder to solve. Again, this requires knowing the transition model.

Bellman Equations

- Once we have v_* or q_*, it is easy to determine an **optimal policy**:

$$\pi_*(a|s) = \arg\max_a q_*(s, a)$$

  or

$$\pi_*(a|s) = \arg\max_a \sum_{s',r} p(s', r|s, a)(r + \gamma v_*(s')).$$

- Note that the optimal policy is **deterministic**. So, we can consider only deterministic policies without loss of generality, i.e. $\pi(s)$ instead of $\pi(a|s)$.

Value Iteration

- We can avoid solving the Bellman equations for the state values of an optimal policy by turning them into update rules.

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $\quad v \leftarrow V(s)$
| $\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
| $\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$

- VI converges asymptotically for $\gamma < 1$. Since the Bellman optimality equations have a unique solution, VI converges to v_*.
- VI still requires knowing the transition model.

Policy Iteration

- Policy evaluation: Turn the ordinary Bellman equations into update rules.

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\quad \Delta \leftarrow 0$
   $\quad$ Loop for each $s \in \mathcal{S}$:
   $\quad\quad v \leftarrow V(s)$
   $\quad\quad V(s) \leftarrow \sum_{s',r} p(s', r \,|\, s, \pi(s)) \big[r + \gamma V(s') \big]$
   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad$ *old-action* $\leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \,|\, s, a) \big[r + \gamma V(s') \big]$
   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Policy Iteration

- Theorem: If $q_\pi(s, \pi'(s)) \geq q_\pi(s, \pi(s))$ for all s, then $v_{\pi'}(s) \geq v_\pi(s)$ for all s. Thus, we can modify π into a better policy π' by doing

$$\pi'(s) = \arg\max_a q_\pi(s, a) \text{ for all } s.$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

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   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Policy Iteration

- PI terminates since each iteration improves the policy and there is a finite number of policies. When PI halts, the Bellman optimality equations hold and, thus, π is optimal. Again, PI requires knowing the transition model.

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

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   policy-stable $\leftarrow$ *true*
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   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Monte Carlo Method

On-policy first-visit MC control (for ε-soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter: small $\varepsilon > 0$

Initialize:

$\pi \leftarrow$ an arbitrary ε-soft policy

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)

For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

Monte Carlo Method

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$\quad Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

$\quad$ Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$\quad G \leftarrow 0$

$\quad$ Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$\quad\quad G \leftarrow \gamma G + R_{t+1}$

$\quad\quad$ Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

$\quad\quad\quad$ Append G to $Returns(S_t, A_t)$

$\quad\quad\quad Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$\quad\quad\quad A^* \leftarrow \arg\max_a Q(S_t, a)$ $\quad$ (with ties broken arbitrarily)

$\quad\quad\quad$ For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

- To ensure that each state-action pair is selected infinitely often so as to produce reliable estimates, an **ϵ-greedy** policy is used: Choose the action with maximal estimated value with probability $1 - \epsilon$, and a random one with probability ϵ.
- The algorithm converges asymptotically to the optimal ϵ-greedy policy.
- The algorithm updates the action value estimates and policy at the end of an episode. So, it is not suitable for step-by-step incremental (i.e., online) computation.
- The algorithm does **not** require knowing the transition model.

Q-Learning

- If the transition model were so that s and a are always followed by s' and r, then $q_*(s,a) = r + \gamma \max_{a'} q_*(s',a')$ by the Bellman optimality equation and, thus, $0 = r + \gamma \max_{a'} q_*(s',a') - q_*(s,a)$. We can try to enforce this constraint by executing π one step from s and a and, then, updating the estimate of $q_*(s,a)$ as

$$q_*(s,a) \leftarrow q_*(s,a) + \alpha(r + \gamma \max_{a'} q_*(s',a') - q_*(s,a)).$$

where $\alpha > 0$ is the learning rate.

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters: step size $\alpha \in (0,1]$, small $\varepsilon > 0$
Initialize $Q(s,a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma \max_a Q(S',a) - Q(S,A)\big]$
    $S \leftarrow S'$
  until S is terminal

Q-Learning

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
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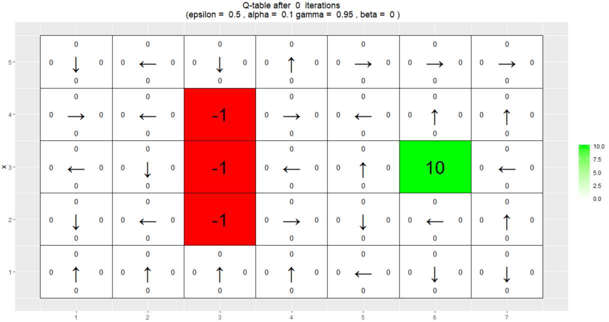
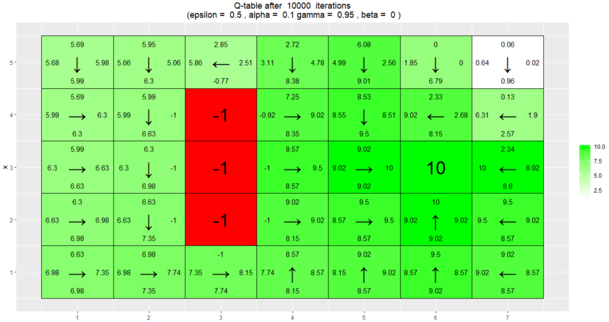
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
    $S \leftarrow S'$
  until S is terminal

- Q-learning converges asymptotically if e.g. $\alpha(t) = O(1/N(s,t))$.
- Q-learning converges asymptotically to q_* if e.g. an ϵ-greedy policy is used to keep updating all the state-action pairs.
- Q-learning also works for stochastic transition models, since the number of times that s and a are followed by s' and r in the sampled episodes is proportional to the transition probability.
- Q-learning does **not** require knowing the transition model.

Example: Grid Worlds





Summary

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- Interested in more ? Check out AlphaGo - The Movie.

Thank you