About determinants

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Determinants

Determinants are defined for any square matrix (number of rows and columns are equal) but not for others. The determinant for a square matrix $\mathbf{A}$ is a real number and is denoted by $\det(\mathbf{A})$ or by $|\mathbf{A}|$ (note that even if the latter notation is used, it can be negative).

Determinants for 2×2-matrices

The determinant for the 2×2-matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is the product of diagonal minus product of off-diagonal,

$$\det(\mathbf{A}) = |\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

Determinants for 3×3- and larger matrices

There exists also a formula for the determinant of a 3×3-matrix which some might know. But this way of computing cannot be used for 4×4- or larger matrices. Therefore, we explain instead the general way of computing determinants. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

You can choose any row or column to "develop" the determinant. We choose here the first row with its elements a_{11}, a_{12}, and a_{13}. For each of these three elements, we compute the determinant of the 2×2-matrix which we get when we delete the row and the column of this element. Then, we multiply this determinant with the element. Finally, we sum the three results where we every second term with -1. More exactly, we multiply a term belonging to a_{ij} with $(-1)^{i+j}$.
Example: Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

Then

$$\det(\mathbf{A}) = 0 \cdot \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 3 & 4 \\ 6 & 7 \end{pmatrix}.$$

The first term is 0. Calculating the determinants of the other two 2×2-matrices, we obtain

$$\det(\mathbf{A}) = 0 - 1 \cdot (24 - 30) + 2 \cdot (21 - 24) = 6 - 6 = 0.$$

Some useful rules

The determinant can be calculated in `R` simply by `det(A)`. If determinants of large matrices have to be computed repeatedly, it is usful to know some rules which might then speed up computation drastically. Two important examples are below.

Determinant of inverse matrix

We have $\det(A^{-1}) = 1/\det(A)$. This simple rule means: For computing $s = \det(A^{-1})$, you should rather write `s <- 1/det(A)` and not `s <- det(solve(A))` since the latter is slower for large matrices. This saves usually 70-80% computation time. Below an example for a repeated calculation in the case of a 5×5-matrix. Time was reduced from 5 s to 1.52 s.

```
> set.seed(2021)
> size <- 5
> rep  <- 100000
> A <- matrix(rnorm(size^2), ncol=size)
> det(solve(A))
[1] -0.5090438
> 1/det(A)
[1] -0.5090438
>
> stime <- proc.time()
> for (i in 1:rep){
+   s <- det(solve(A))
+ }
> proc.time() - stime
   user  system elapsed
   5.00    0.06    5.06
>
> stime <- proc.time()
> for (i in 1:rep){
+   s <- 1/det(A)
+ }
> proc.time() - stime
   user  system elapsed
   1.52    0.03    1.55
```

Determinant of sparse matrices

Sometimes in statistics and machine learning, we have large matrices with many 0's (so-called sparse matrices). For example, if $\mathbf{A}$ is a so-called block-diagonal matrix with blocks of non-zero elements in the diagonal and 0 otherwise, then the determinant of the whole matrix is the product of the determinants of the blocks. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix},$$

then

$$\det(\mathbf{A}) = (1 \cdot 4 - 2 \cdot 3) \cdot (5 \cdot 8 - 6 \cdot 7) = 4.$$

If two rows of a matrix are switched, the determinant is multiplied with -1. The same holds for the switch of two columns. Therefore:

$$\det \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 5 & 0 & 6 \\ 3 & 0 & 4 & 0 \\ 0 & 7 & 0 & 8 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 7 & 8 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix} = 4.$$