SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

PER SIDÉN

3. Transforms

Exercise 3.1

Suppose X has probability function $p_X(k) = (1-\theta)\,\theta^{k-1}$, $k = 1, 2, \ldots$, for $0 \leq \theta \leq 1$ and that Y has density function $f_Y(y) = 2y$, $0 \leq y \leq 1$.

(a) Derive the probability generating function of X.

(b) Derive the moment generating function of Y.

(c) Let $S_X = Y_1 + Y_2 + \ldots + Y_X$ be the sum of X i.i.d. random variables with the same distribution as Y in (b), and assume that $Y_1, Y_2, \ldots, Y_X$ are all independent of X, where X is distributed as in (a). Compute the characteristic function of S_X.

Exercise 3.2 (3.1 in Gut's book)

The non-negative, integer-valued, random variable X has generating function $g_X(t) = \log(1/(1-qt))$. Determine $P(X = k)$ for $k = 0, 1, 2, \ldots$, $E(X)$, and $Var(X)$.

Exercise 3.3 (3.6 in Gut's book)

Show, by using moment generating functions, that if $X \sim L(1)$, then $X \stackrel{d}{=} Y_1 - Y_2$, where Y_1 and Y_2 are independent, exponentially distributed random variables.

Exercise 3.4 (3.34 in Gut's book)

Suppose that X is a nonnegative, integer-valued random variable, and let n and m be non-negative integers. Show that

$$g_{nX+m}(t) = t^m \cdot g_X(t^n).$$

Exercise 3.5* (3.5 in Gut's book)

Let $Y \sim \beta(n, m)$ $(n, m$ integers).

(a) Compute the moment generating function of $-\log Y$.

(b) Show that $-\log Y$ has the same distribution as $\sum_{k=1}^{m} X_k$, where $X_1, X_2, \ldots$ are independent, exponentially distributed random variables.

Remark. The formula $\Gamma(r+s)/\Gamma(r) = (r+s-1)\cdots(r+1)\,r$, which holds when s is an integer, might be useful.

Exercise 3.6* (3.26 in Gut's book)
The number of cars passing a road crossing during an hour is $Po\,(b)$-distributed. The number of passengers in each car is $Po\,(p)$-distributed. Find the generating function of the total number of passengers, Y, passing the road crossing during one hour, and compute $E\,(Y)$ and $Var\,(Y)$.