LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time:	8-12
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani.
Assisting teacher:	Per Sidén, phone *0704-977175*
Grades:	Grades: Maximum is 20 points. A=19-20 points B=17-18 points C=12-16 points D=10-11 points E=8-9 points F=0-7 points

- Write clear and concise answers to the questions.
- Make sure to specify the definition region for all density functions.

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1. The random variable X has the distribution function

$$F_X(x) = \begin{cases} a\left(1 - \frac{1}{x}\right), & 1 < x < \infty \\ 0, & otherwise \end{cases}$$

and the conditional probability density of Y given X is

$$f_{Y|X=x}(y) = \begin{cases} by, & 0 < y < x \\ 0, & otherwise \end{cases}.$$

(a) Determine the constant a and the probability density function of X. 1p.

Solution:

$$1 = \lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} a\left(1 - \frac{1}{x}\right) = a \Rightarrow a = 1.$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{1}{x^2},\ 1 < x < \infty.$$

(b) Determine the constant b as a function of x and compute $E[Y|X=4]$. 1p.

Solution:

$$1 = \int_0^x by\,dy = \left[\frac{by^2}{2}\right]_0^x = \frac{bx^2}{2} \Rightarrow b = \frac{2}{x^2}.$$

$$E[Y|X=4] = \int y \cdot f_{Y|X=4}(y)\,dy = \int_0^4 \frac{2y^2}{16}dy = \left[\frac{2}{16}\frac{y^3}{3}\right]_0^4 = \frac{8}{3}.$$

(c) Compute the joint density function of X and Y. Are X and Y independent? 1p.

Solution:

$$f_{X,Y}(x,y) = f_{Y|X=x}(y)\,f_X(x) = \frac{2y}{x^2}\cdot\frac{1}{x^2} = \frac{2y}{x^4}, \quad \begin{matrix} \max(1,y) < x < \infty \\ 0 < y < x \end{matrix}.$$

X and Y are not independent because of the non-rectangular definition region.

(d) Compute the marginal density of Y and the probability $P(Y<2)$. 2p.

Solution:

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_{\max(1,y)}^{\infty} \frac{2y}{x^4}dx = \left[-\frac{2y}{3x^3}\right]_{\max(1,y)}^{\infty} = \begin{cases} \frac{2}{3}y & , 0<y<1 \\ \frac{2}{3y^2} & , 1<y<\infty \\ 0 & , otherwise \end{cases}.$$

$$P(Y<2) = \int_0^2 f_Y(y)\,dy = \int_0^1 \frac{2}{3}ydy + \int_1^2 \frac{2}{3y^2}dy = \left[\frac{y^2}{3}\right]_0^1 + \left[-\frac{2}{3y}\right]_1^2 = \frac{1}{3} - \frac{1}{3} + \frac{2}{3} = \frac{2}{3}.$$

2. Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}(2x+y) & , \begin{cases} 0<x<1 \\ 0<y<1 \end{cases} \\ 0 & , otherwise \end{cases}.$$

(a) Compute $E[2X+Y]$. 2p.

Solution:

$$\begin{aligned} E[2X+Y] &= \iint (2x+y)\,f_{X,Y}(x,y)\,dxdy = \int_{y=0}^{1}\int_{x=0}^{1} \frac{2}{3}(2x+y)^2\,dxdy \\ &= \int_{y=0}^{1}\int_{x=0}^{1} \frac{2}{3}\left(4x^2+4xy+y^2\right)dxdy = \int_{y=0}^{1}\left[\frac{2}{3}\left(\frac{4}{3}x^3+2x^2y+y^2x\right)\right]_0^1 dy \\ &= \int_{y=0}^{1}\frac{2}{3}\left(\frac{4}{3}+2y+y^2\right)dy = \left[\frac{2}{3}\left(\frac{4}{3}y+y^2+\frac{y^3}{3}\right)\right]_0^1 = \frac{2}{3}\left(\frac{4}{3}+1+\frac{1}{3}\right) = \frac{16}{9}. \end{aligned}$$

(b) Determine the distribution of $2X+Y$. 3p.

Solution: Define

$$\begin{cases} U &= 2X+Y \\ V &= X \end{cases} \Leftrightarrow \begin{cases} X &= V \\ Y &= U-2V \end{cases} \Rightarrow |J| = \left|\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array}\right| = \left|\begin{array}{cc} 0 & 1 \\ 1 & -2 \end{array}\right| = |-1| = 1.$$

The transformation theorem gives

$$f_{U,V}(u,v) = f_{X,Y}(v,u-2v)\cdot|J| = \frac{2}{3}(2v+u-2v)\cdot 1 = \frac{2}{3}u, \quad \begin{cases} 2v<u<2v+1 \\ \max\left(0,\frac{u-1}{2}\right) < v < \min\left(1,\frac{u}{2}\right) \end{cases}.$$

Thus

$$
\begin{aligned}
f_U(u) &= \int f_{U,V}(u,v)\,dv = \int\limits_{\max\left(0,\frac{u-1}{2}\right)}^{\min\left(1,\frac{u}{2}\right)} \frac{2}{3}u\,dv = \left[\frac{2}{3}uv\right]_{\max\left(0,\frac{u-1}{2}\right)}^{\min\left(1,\frac{u}{2}\right)} \\
&= \begin{cases} \frac{2}{3}u\left(\frac{u}{2}-0\right) & ,0<u<1 \\ \frac{2}{3}u\left(\frac{u}{2}-\frac{u-1}{2}\right) & ,1<u<2 \\ \frac{2}{3}u\left(1-\frac{u-1}{2}\right) & ,2<u<3 \end{cases} = \begin{cases} \frac{u^2}{3} & ,0<u<1 \\ \frac{1}{3}u & ,1<u<2 \\ u-\frac{u^2}{3} & ,2<u<3 \end{cases}.
\end{aligned}
$$

3. Let X_1 and X_2 follow a multivariate normal distribution with mean vector $\mu = (1,0)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Define Y_1, Y_2 and Y_3 through

$$\begin{cases} Y_1 &= X_1 + X_2 \\ Y_2 &= -X_1 + 2X_2 \\ Y_3 &= X_2 - 1 \end{cases}.$$

(a) What is the joint distribution of Y_1, Y_2 and Y_3 ? 1.5p.

Solution: Let $Y = (Y_1, Y_2, Y_3)' = BX + b$ with

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Then

$$Y \sim N\left(B\mu + b, B\Sigma B'\right)$$

that is

$$Y \sim N\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 & -3 & 0 \\ -3 & 12 & 3 \\ 0 & 3 & 1 \end{pmatrix}\right).$$

(b) Are any of Y_1, Y_2 and Y_3 independent? 1p.

Solution: Yes, Y_1 and Y_3 are independent since element $(3,1)$ in the covariance matrix is zero. Y_2 is dependent with both others.

(c) Suppose $X_n \sim Bin(n, \lambda/n)$. Show that $X_n \xrightarrow{d} Po(\lambda)$ as $n \to \infty$. 2.5p.

Solution:

$$
\begin{aligned}
g_{Bin(n,p)}(t) &= (q+pt)^n \\
\Rightarrow g_{X_n}(t) &= \left(1 - \frac{\lambda}{n} + \frac{\lambda t}{n}\right)^n \\
&= \left(1 + \frac{\lambda(t-1)}{n}\right)^n \to e^{\lambda(t-1)}, \; n \to \infty.
\end{aligned}
$$

Since $g_{Po(\lambda)}(t) = e^{\lambda(t-1)}$ we have

$$X_n \xrightarrow{d} Po(\lambda)$$

through Theorem 6.4.1.

4. Let X_k, $k = 1, 2, \ldots$ be independent random variables, with common density $f_X(x)$ and distribution $F_X(x)$. Also, let N be a positive integer-valued random variable with probability generating function $g_N(t)$. Assume that N and $X_1, X_2, \ldots$ are independent. Define

$$Z_N = \max(X_1, X_2, \ldots, X_N).$$

(a) Derive the density of $Z_N|N = n$. 2p.

Solution: Denote $Z_N|N = n$ as Z_n. The distribution function of Z_n is

$$F_{Z_n}(z) = P(Z_n \leq z) = P(X_1 \leq z, X_2 \leq z, ..., X_n \leq z)$$

since Z_n is the maximum and $Z_n \leq z$ is therefore the same as all $X_1, ..., X_n$ being smaller than z. Now, because of independence and that all X_k have same distribution, we have

$$F_{Z_n}(z) = (F_X(z))^n .$$

Taking the derivative of both sides yields

$$f_{Z_n}(z) = n\left(F_X(z)\right)^{n-1} f_X(z) .$$

(b) Show that $F_{Z_N}(z) = g_N(F_X(z))$. 1.5p.

Solution:

$$\begin{aligned} F_{Z_N}(z) &= \sum_n F_{Z_N|N=n}(z)\, p_N(n) = \sum_n (F_X(z))^n\, p_N(n) = E\left[(F_X(z))^N\right] \\ &= g_N(F_X(z)) . \end{aligned}$$

(c) Now, assume $X_1, X_2, \ldots$ are all $U(0,1)$-distributed and $N \sim Ge\left(\frac{1}{2}\right)$. Compute $F_{Z_n}(z)$. 1.5p.

Solution:

$$\begin{aligned} F_X(x) &= x,\ 0 < x < 1 \\ g_N(t) &= \frac{\frac{1}{2}}{1 - \frac{1}{2}t} = \frac{1}{2 - t}. \end{aligned}$$

Thus

$$F_{Z_N}(z) = \frac{1}{2 - z},\ 0 < z < 1.$$