LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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2013-11-25
Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time:	9-13
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani, phone. $0709-373799$
Grades:	Grades: Maximum is 20 points. A=19-20 points B=17-18 points C=12-16 points D=10-11 points E=8-9 points F=0-7 points

- Write clear and concise answers to the questions.

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1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} (1+x)y^x \exp(-x) & \text{if } 0 \leq x \leq \infty \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are X and Y independent? 1p.
   (b) Compute the marginal density of X. Does it belong to any of the known distributions? 1p.
   (c) Compute the conditional density of $Y|X=x$. Does it belong to any of the known distributions? 1.5p.
   (d) Compute $E[(X+2)Y]$. 1.5p.

2. Let $Y|\theta \sim Bin(n,\theta)$, where n is a known positive integer. Let the density of θ be

$$p(\theta) = 3 \cdot (1-\theta)^2$$

   for $\theta \in [0,1]$ and $p(\theta) = 0$ otherwise.

   (a) Calculate the expected value and variance of Y. 2p.
   (b) Calculate the moment generating function for Y. 1.5p.
   (c) Compute the density of Y. 1.5p.

3. Let $X_k \sim Tri(0,1),\ k = 1, 2, ...$ be independent random variables. ($Tri(a,b)$ is the triangular density over the interval $[a,b]$)

   (a) Derive the density of $Z_n = \max(X_1, X_2, ..., X_n)$. 2p.

   (b) Let $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. Show that $Y_n \xrightarrow{p} \frac{1}{2}$ as $n \to \infty$. 1p.

   (c) Let W_n be a sequence of random variables with finite mean μ and variance σ^2. Let $\bar{W}_n = \frac{1}{n}\sum_{i=1}^{n} W_i$. Show that $\bar{W}_n \cdot Y_n$ converges in distribution as $n \to \infty$, and find the limiting distribution. 2p.

4. Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$ be independent Gamma variables.

   (a) Show that $X + Y$ and $X/(X+Y)$ are independent 1.5p.

   (b) Find the marginal density of $X/(X+Y)$. 1.5p.

   (c) What is the moment generating function of $Z = \frac{c\cdot(X+Y)^2 + d\cdot X}{X+Y}$, where c and d are positive constants? 2p.

Good luck!

Mattias