

## Exam in Probability Theory, 6 credits

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Exam time:	8-12
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani.
Assisting teacher:	Per Sidén, phone 0704-977175
Grades:	Grades: Maximum is 20 points. A: 19 points B: 17 points C: 14 points D: 12 points E: 10 points F: <10 points

- Write clear and concise answers to the questions.
  - Make sure to specify the definition region for all density functions.
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1. Assume  $X \sim N(0, 1)$  and  $Y \sim N(0, 4)$  and that  $X$  and  $Y$  are independent.

- (a) Compute  $E[(X + 1)(Y - 1)]$ . 1p.
- (b) Compute  $E[Y|X + Y = 6]$ . 1.5p.
- (c) Derive the distribution of  $U = \frac{X}{Y}$ . Does it belong to a known distribution? 2.5p.

2. Let  $Y|\theta \sim \text{Bin}(n, \theta)$ , where  $n$  is a known positive integer. Let the density of  $\theta$  be

$$f_{\theta}(\theta) = a \cdot \theta^3$$

for  $\theta \in [0, 1]$  and  $f_{\theta}(\theta) = 0$  otherwise.

- (a) Determine the constant  $a$ , so that  $f_{\theta}(\theta)$  is a proper density. 1p.
- (b) Calculate the variance of  $Y$ . 2p.
- (c) Compute the density of  $Y$ . 2p.

3. Consider a fair die with probability  $1/6$  of rolling a six. Consider a game where the die is rolled until a six comes up and let the random variable  $X$  denote the number of die rolls required until this event happens.

- (a) Determine the probability function of  $X$  and state if it belongs to a known distribution. 1p.
- (b) Assume that the same game is played  $n$  times and let  $X_{max}$  be the maximum value of  $X$  across all games. Derive the distribution function of  $X_{max}$ . 2p.
- (c) How many times does one have to play the game for having at least 50% chance of obtaining a value of  $X_{max}$  that is greater than or equal to 30? 2p.

4. Let  $X$  and  $Y$  be random variables such that

$$Y|X = x \sim N(0, x)$$

with  $X \sim Po(\lambda)$ .

- (a) Find the characteristic function of  $Y$ . 1p.
- (b) Show that

$$\frac{Y}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1)$$

as  $\lambda \rightarrow \infty$ . 2p.

- (c) Formulate and prove the Central Limit Theorem. 2p.

GOOD LUCK!

PER