Probability Theory

Lecture 2

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Overview Lecture 2

- Conditional distributions
- Conditional expectation, conditional variance
- Distributions with random parameters and the Bayesian approach
- Regression and Prediction

Conditional distributions

- For events [if $P(B) > 0$]

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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- For **continuous** random variables

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

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Conditional expectation

- **Conditional expectation of Y given $X = x$ is**

$$E(Y|X=x) = \begin{cases} \sum_y y \cdot p_{Y|X=x}(y) & \text{if } Y \text{is discrete} \\ \int_{-\infty}^{\infty} y \cdot f_{Y|X=x}(y)dy & \text{if } Y \text{is continuous} \end{cases}$$

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- Ex. 2.1 page 33. $X \sim U(0,1)$, $Y|X = x \sim U(0,x)$.

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- The law of iterated expectation is an "expectation version" of the law of total probability.
- $E(Y|X) = E(Y)$ if X and Y are independent.

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- Corollary 2.3.1

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- Note the naive version $Var(Y) = E\left[Var(Y|X)\right]$ misses the uncertainty in Y that comes from not knowing X in $E(Y|X)$.

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  - X is daily stock market returns. $X|\lambda \sim N(0, 1/\lambda)$, where $1/\lambda$ is the daily variance.
  - The daily variance varies from day to day according to $\lambda \sim \Gamma\left(\frac{n}{2}, \frac{2}{n}\right)$. Turbulent day: realization of λ is very small.

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- Coin flips are no longer independent when p is uncertain and we learn about p from data.

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- When (X, Y) is jointly normal, $E(Y|X = x)$ is linear. For other distributions, this is not true in general.