LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time:	9-13
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani, phone. $0709 - 373799$
Grades:	Grades: Maximum is 20 points. A=19-20 points B=17-18 points C=12-16 points D=10-11 points E=8-9 points F=0-7 points

- Write clear and concise answers to the questions.

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1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} (1+x)y^x \exp(-x) & \text{if } 0 \leq x \leq \infty \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? 1p.

Solution: The joint density cannot be factorized as $f_{X,Y}(x,y) = g(x)g(y)$ so X and Y are not independent.

(b) Compute the marginal density of X. Does it belong to any of the known distributions? 1p.

Solution:

$$\begin{aligned} f_X(x) &= \int_0^1 (1+x)y^x \exp(-x)dy = (1+x)\exp(-x)\int_0^1 y^x dy \\ &= (1+x)\exp(-x)\left[\frac{1}{1+x}y^{x+1}\right]_0^1 = (1+x)\exp(-x)\frac{1}{1+x} = \exp(-x) \end{aligned}$$

so $X \sim Exp(1)$.

(c) Compute the conditional density of $Y|X = x$. Does it belong to any of the known distributions? 1.5p.

Solution:

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(1+x)y^x \exp(-x)}{\exp(-x)} = (1+x)y^x = (1+x)y^{x+1-1}(1-y)^{b-1}$$

where $b = 1$. Note that this looks like a $Beta(x+1,1)$ density, if we can show that the normalizing constant of this density is $(1+x)$. The normalizing constant of the $Beta(x+1,1)$ density is

$$\frac{\Gamma(x+1+1)}{\Gamma(x+1)\Gamma(1)} = \frac{\Gamma(x+2)}{\Gamma(x+1)} = \frac{(x+1)\Gamma(x+1)}{\Gamma(x+1)} = 1+x$$

since $\Gamma(a+1) = a\Gamma(a)$ for any $a > 0$. Thus, $Y|X = x \sim Beta(x+1,1)$.

(d) Compute $E\left[(X+2)Y\right]$. 1.5p.

Solution: By the law of iterated expectation

$$\begin{aligned} E\left[(X+2)Y\right] &= EE\left[(X+2)Y|X\right] = E\left((X+2)E\left[Y|X\right]\right) \\ &= E\left((X+2)\frac{X+1}{X+2}\right) = E\left(X+1\right) = 1+1 = 2 \end{aligned}$$

since $Y|X = x \sim Beta(x+1,1)$ and $X \sim Exp(1)$.

2. Let $Y|\theta \sim Bin(n,\theta)$, where n is a known positive integer. Let the density of θ be

$$p(\theta) = 3 \cdot (1-\theta)^2$$

for $\theta \in [0,1]$ and $p(\theta) = 0$ otherwise.

(a) Calculate the expected value and variance of Y. 2p.

Solution: Note first that $\theta \sim Beta(1,3)$ which has $E(\theta) = 1/4$ and $V(\theta) = 3/(4^2 \cdot 5) = 3/80$. By the law of iterated expectation

$$E(Y) = EE(Y|\theta) = E\left[n\theta\right] = \frac{n}{4}$$

$$\begin{aligned} V(Y) &= E\left[V(Y|\theta)\right] + V\left[E(Y|\theta)\right] \\ &= E\left[n\theta(1-\theta)\right] + V\left[n\theta\right] \\ &= nE(\theta) - nE(\theta^2) + \frac{3n}{80} \\ &= \frac{n}{4} - n\left(3/80 + \left(\frac{n}{4}\right)^2\right) + \frac{3n}{80} \\ &= \frac{n}{4} - \frac{n^3}{16} = \frac{n}{4}\left(1 - \frac{n^2}{4}\right) \end{aligned}$$

(b) Calculate the moment generating function for Y when $n = 1$. 1.5p.

Solution:

$$E^{tY} = E\left(e^{tY}|\theta\right) = E\left(1-\theta+\theta e^t\right) = \frac{3}{4} + \frac{1}{4}e^t$$

(c) Compute the density of Y. 1.5p.

Solution:

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|X=x}(y) f_X(x) dx = \int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} 3 \cdot (1-\theta)^2 d\theta \\ &= 3\binom{n}{y} \int_0^1 \theta^y (1-\theta)^{n-y+2} d\theta \end{aligned}$$

this integral is easy to solve by realizing that $\theta^y(1-\theta)^{n-y+2}$ is proportional to the $Beta(y+1, n-y+3)$ density, so

$$\int_0^1 \theta^y(1-\theta)^{n-y+2}d\theta = \frac{\Gamma(y+1)\Gamma(n-y+3)}{\Gamma(n+4)}.$$

This is enough for full points, but the distribution can be simplified further as

$$f_Y(y) = 3\binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+3)}{\Gamma(n+4)} = 3\frac{n!}{y!(n-y)!}\frac{y!(n-y+2)!}{(n+3)!} = \frac{3(n-y+2)(n-y+1)}{(n+3)(n+2)(n+1)}$$

for $y = 0, 1, ..., n$ and zero otherwise. I have used that $\Gamma(x+1) = x!$ if x is a positive integer.

3. Let $X_k \sim Tri(0,1),\ k = 1, 2, ...$ be independent random variables. ($Tri(a, b)$ is the triangular density over the interval $[a, b]$)

   (a) Derive the density of $Z_n = \max(X_1, X_2, ..., X_n)$. 2p.

Solution: The distribution function is by definition

$$\begin{aligned}F_{Z_n}(z) &= P(Z_n \le z) = P(X_1 \le z, X_2 \le z, ..., X_n \le z)\\ &= \prod_{i=1}^{n} F_{X_i}(z) = [F_X(z)]^n\end{aligned}$$

(the maximum is smaller than z means that all the X_i are smaller than z). We need to compute $F_X(z)$. From Appendix B we know that the density of $Tri(0, 1)$ is

$$f_X(x) = 2\left(1 - 2\left|x - \frac{1}{2}\right|\right)$$

so

$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x 2\left(1 - 2\left|x - \frac{1}{2}\right|\right) dt.$$

Let look first at the case $x \le 1/2$. Then

$$\int_0^x 2\left(1 - 2\left|t - \frac{1}{2}\right|\right) dt = 2\int_0^x \left(1 + 2\left(t - \frac{1}{2}\right)\right) dt = 2\int_0^x 2t dt = 4\left[\frac{1}{2}t^2\right]_0^x = 2x^2$$

If $x \ge 1/2$ we have

$$\begin{aligned}F_X(x) &= \frac{1}{2} + \int_{1/2}^x f_X(t)dt = \frac{1}{2} + \int_{1/2}^x 2\left(1 - 2\left(t - \frac{1}{2}\right)\right) dt = \frac{1}{2} + 2\int_{1/2}^x (2(1-t))\, dt\\ &= \frac{1}{2} + 4\left[t\left(1 - \frac{t}{2}\right)\right]_{1/2}^x = \frac{1}{2} + 4\left[x\left(1 - \frac{x}{2}\right) - \frac{1}{2}\left(1 - \frac{1}{4}\right)\right] = 4x\left(1 - \frac{x}{2}\right) - 1\end{aligned}$$

So

$$F_{Z_n}(z) = [F_X(z)]^n = \begin{cases} 2^n z^{2n} & \text{for } z \le 1/2 \\ \left[4z\left(1 - \frac{z}{2}\right) - 1\right]^n & \text{for } z > 1/2 \end{cases}$$

and

$$f_{Z_n}(z) = \begin{cases} 2^{n+1} n z^{2n-1} & \text{for } z \le 1/2 \\ n\left[4z\left(1 - \frac{z}{2}\right) - 1\right]^{n-1} 4(1-z) & \text{for } z > 1/2 \end{cases}$$

   (b) Let $Y_n = \frac{1}{n}\sum_{k=1}^n X_k$. Show that $Y_n \xrightarrow{p} \frac{1}{2}$ as $n \to \infty$. 1p.

Solution: We will use Chebyshev inequality. Note that $E(Y_n) = E(X_k) = 1/2$ and $\sigma^2 = V(Y_n) = \frac{1}{n^2}\sum_{i=1}^n V(X_n) = \frac{1}{n}Var(X)$ where $X \sim Tri(0, 1)$. From Appendix B, we have $Var(X) = \frac{1}{24}$, so $\sigma^2 = V(Y_n) = \frac{1}{24n}$. Now, from Chebyshev's inequality

$$Pr\left(\left|Y_n - \frac{1}{2}\right| \le \varepsilon\right) \le \frac{\sigma^2}{\varepsilon^2} = \frac{1}{24n\varepsilon^2} \to 0$$

for any $\varepsilon > 0$ as $n \to \infty$. By the definition of convergence in probability, this shows that $Y_n \xrightarrow{p} \frac{1}{2}$ as $n \to \infty$.

   (c) Let W_n be a sequence of random variables with finite mean μ and variance σ^2. Let $\bar{W}_n = \frac{1}{n}\sum_{i=1}^n W_i$. Show that $\bar{W}_n \cdot Y_n$ converges in distribution as $n \to \infty$, and find the limiting distribution. 2p.

Solution: By the central limit theorem we know that $\bar{W}_n \xrightarrow{d} N\left(\mu, \frac{\sigma^2}{n}\right)$. Since $Y_n \xrightarrow{p} \frac{1}{2}$, it follows from Slutsky's theorem that $\bar{W}_n \cdot Y_n \xrightarrow{d} \frac{1}{2} \cdot N\left(\mu, \frac{\sigma^2}{n}\right)$ (with some abuse of notation). That is, $\bar{W}_n \cdot Y_n \xrightarrow{d} N\left(\frac{\mu}{2}, \frac{\sigma^2}{4n}\right)$.

4. Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$ be independent Gamma variables.

   (a) Show that $X + Y$ and $X/(X + Y)$ are independent 1.5p.

Solution: First note that the density of $X \sim \Gamma(\alpha, 1)$ is of the form

$$f_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$$

Let $U = X + Y$ and $V = X/(X + Y)$. The inverse transformation is $X = UV$ and $Y = U - UV = U(1 - V)$. The Jacobian is

$$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -vu - u(1 - v) = -u$$

so $|J| = u$. Then

$$\begin{aligned} f_{U,V}(u, v) = f_X(uv) f_Y(u(1 - v)) \cdot |J| &= \frac{1}{\Gamma(\alpha)} (uv)^{\alpha-1} e^{-uv} \frac{1}{\Gamma(\beta)} (u(1 - v))^{\beta-1} e^{-u(1-v)} \cdot u \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha} u^{\beta-1} e^{-u} v^{\alpha-1} (1 - v)^{\beta-1} \end{aligned}$$

since $f_{U,V}(u, v)$ can be factorized as $f_{U,V}(u, v) = g(u)h(v)$, u and v are independent.

   (b) Find the marginal density of $X/(X + Y)$. 1.5p.

      i. From 4a) we know that the joint density is

$$f_{U,V}(u, v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha+\beta-1} e^{-u} v^{\alpha-1} (1 - v)^{\beta-1}$$

which directly shows (because of the independence) that the marginal density of V is proportional to

$$v^{\alpha-1}(1 - v)^{\beta-1}$$

which implies that $V = X/(X + Y) \sim Beta(\alpha, \beta)$.

   (c) What is the moment generating function of $Z = \frac{c \cdot (X+Y)^2 + d \cdot X}{X+Y}$, where c and d are positive constants? 2p.

      i. Note that

$$Z = c \cdot (X + Y) + d \frac{X}{X + Y} = cU + dV$$

and the we know from 4a that U and V are independent. The moment generating function of Z is

$$E(e^{tZ}) = E\left[e^{t(c \cdot U + dV)}\right] = E\left[e^{tcU}\right] E\left[e^{tdV}\right] = \varphi_U(tc) \cdot \varphi_V(td)$$

We know that $V \sim Beta(\alpha, \beta)$ so $\varphi_V(\cdot)$ is the moment generating function of the $Beta(\alpha, \beta)$. We can compute $\varphi_U(tc)$ directly from its definition, but we rather try to find the distribution of U and then read off the moment generating function from the Tables. From above, the marginal density of U is proportional to

$$u^{\alpha+\beta-1} e^{-u}$$

which means that $U \sim \Gamma(\alpha+\beta, 1)$ and so

$$\varphi_U(t) = \frac{1}{(1-t)^{\alpha+\beta}}$$

for $t < 1$. Thus

$$E(e^{tZ}) = \frac{1}{(1-tc)^{\alpha+\beta}} \cdot \varphi_V(td),$$

where $\varphi_V(t)$ is the moment generating function of the $Beta(\alpha, \beta)$.

Good luck!

Mattias