LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time:	8-12
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani.
Assisting teacher:	Per Sidén, phone *0704-977175*
Grades:	Grades: Maximum is 20 points. A=19-20 points B=17-18 points C=12-16 points D=10-11 points E=8-9 points F=0-7 points

- Write clear and concise answers to the questions.

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1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} ax^2y & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? 1p.

Solution: No, since the definition space is not rectangular.

(b) Determine the constant a. 1p.

Solution:

$$\begin{aligned} 1 &= \int_{x=0}^{1}\int_{y=x}^{1} ax^2 y dy dx = \int_{x=0}^{1} ax^2 \left[y^2/2\right]_x^1 dx = \\ &= \frac{a}{2}\int_{x=0}^{1} x^2\left(1-x^2\right)dx = \frac{a}{2}\left[x^3/3 - x^5/5\right]_0^1 = \frac{a}{15} \\ \Rightarrow a &= 15 \end{aligned}$$

(c) Compute the marginal density of Y. 1p.

Solution:

$$\begin{aligned} f_Y(y) &= \int_{x=0}^{y} 15x^2 y dx = 15y\left[x^3/3\right]_0^y = \\ &= 5y^4,\, 0 < y < 1 \end{aligned}$$

(d) Compute the conditional density of $X|Y = y$. 1p.

Solution:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{15x^2y}{5y^4} = \frac{3x^2}{y^3},\, 0 < x < y$$

(e) Compute $E\left[\exp\left\{X^3\right\}|Y = y\right]$. 1p.

Solution:

$$E\left[\exp\left\{X^3\right\}|Y=y\right] = \int_0^y \exp\left\{x^3\right\}\frac{3x^2}{y^3}dx = \left[\frac{\exp\left\{x^3\right\}}{y^3}\right]_0^y = \frac{\exp\left\{y^3\right\}-1}{y^3}$$

2. Let $N, X_1, X_2, X_3, \ldots$ be independent random variables such that $N \sim Fs(p)$ and $X_k \sim L(a)$ for all k. Define

$$S_N = \sum_{k=1}^{N} X_k.$$

(a) Calculate the expected value and variance of S_N. 1.5p.

Solution:

$$E\left[S_N|N=n\right] = E\left[\sum_{k=1}^{n} X_k\right] = n\cdot 0 = 0$$

$$E\left[S_N\right] = E\left[E\left(S_N|N\right)\right] = E\left[0\right] = 0$$

$$Var\left[S_N|N=n\right] = Var\left[\sum_{k=1}^{n} X_k\right] = n\cdot 2a^2$$

$$\begin{aligned} Var\left[S_N\right] &= E\left[Var\left(S_N|N\right)\right] + Var\left[E\left(S_N|N\right)\right] = \\ &= E\left[N\cdot 2a^2\right] = \frac{2a^2}{p} \end{aligned}$$

(b) Calculate the characteristic function for S_N. 2.5p.

Solution: First note that

$$\varphi_{S_N|N=n}(t) = \left(\varphi_{X_k}(t)\right)^n = \frac{1}{\left(1+a^2t^2\right)^n}$$

and that

$$\begin{aligned} g_N(t) &= E\left[t^N\right] = \sum_{k=1}^{\infty} t^k p q^{k-1} = pt\sum_{j=0}^{\infty}(tq)^j = \\ &= \frac{pt}{1-qt},\, with\, q = 1-p. \end{aligned}$$

Now

$$\begin{aligned} \varphi_{S_N}(t) &= E\left[e^{itS_N}\right] = E\left[E\left(e^{itS_N}|N\right)\right] = E\left[\varphi_{S_N|N=n}(t)\right] \\ &= E\left[\left(\frac{1}{1+a^2t^2}\right)^N\right] = g_N\left(\frac{1}{1+a^2t^2}\right) = \frac{p\frac{1}{1+a^2t^2}}{1-q\frac{1}{1+a^2t^2}} \\ &= \frac{p}{1+a^2t^2-q} = \frac{p}{p+a^2t^2} \end{aligned}$$

(c) Show that $\sqrt{p}S_N \sim L(a)$. 1p.

Solution:

$$\varphi_{\sqrt{p}S_N}(t) = E\left[e^{it\sqrt{p}S_N}\right] = \varphi_{S_N}(t\sqrt{p}) = \frac{p}{p + a^2\left(t\sqrt{p}\right)^2} = \frac{1}{1+a^2t^2}.$$

Thus $\sqrt{p}S_N \sim L(a)$ by theorem 3.4.2.

3. Let X_1, X_2 and X_3 follow a multivariate normal distribution with mean vector $\mu = (0,0,2)'$ and covariance matrix

$$\Lambda = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 1 \\ -2 & 1 & 6 \end{pmatrix}.$$

(a) What is the bivariate distribution of X_2 and X_3? 1p.

Solution:

$$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}\right)$$

(b) What is the conditional distribution of X_1 given $X_1 + X_2 - X_3 = c$ for some constant c? 2p.

Solution: Let $Z = (Z_1, Z_2)' = BX$ with

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N(B\mu, B\Lambda B')$$

that is

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 4 & 14 \end{pmatrix}\right).$$

Now,

$$X_1|(X_1+X_2+X_3=c) = Z_1|(Z_2=c) \sim N\left[\mu_1 + \rho_{12}\frac{\sigma_1}{\sigma_2}(c-\mu_2), \sigma_1^2(1-\rho_{12}^2)\right]$$

where the indicies are with respect to the random vector Z. Filling in the right values, we obtain

$$Z_1|(Z_2=c) \sim N\left[0 + \frac{2}{7}(c+2), 2(1-\frac{4}{7})\right] = N\left[\frac{2c+4}{7}, \frac{6}{7}\right]$$

since $\rho_{12} = \frac{Cov(Z_1,Z_2)}{\sigma_1\sigma_2} = \frac{4}{\sqrt{2}\sqrt{14}} = \frac{2}{\sqrt{7}}$, so $\rho_{12}^2 = \frac{4}{7}$ and $\rho_{12}\frac{\sigma_1}{\sigma_2} = \frac{4}{\sqrt{2}\sqrt{14}}\frac{\sqrt{2}}{\sqrt{14}} = \frac{2}{7}$.

(c) Define $Y_1 = \frac{1}{X_1}$. Derive the conditional probability density function of Y_1 given $X_1 + X_2 - X_3 = c$ for some constant c. 2p.

Solution: From b) we have

$$f_{X_1|X_1+X_2+X_3=c}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$$

with $\mu = \frac{2c+4}{7}$ and $\sigma^2 = \frac{6}{7}$ here. Using the transformation theorem and that

$$|J| = \left|\frac{dx}{dy}\right| = \left|-\frac{1}{y^2}\right| = \frac{1}{y^2}$$

we have

$$
\begin{aligned}
f_{Y_1|X_1+X_2+X_3=c}(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{\left(\frac{1}{y}-\mu\right)^2}{\sigma^2}\right\}\frac{1}{y^2} \\
&= \frac{1}{\sqrt{2\pi\frac{6}{7}}} \exp\left\{-\frac{1}{2}\frac{\left(\frac{1}{y}-\frac{2c+4}{7}\right)^2}{\frac{6}{7}}\right\}\frac{1}{y^2}.
\end{aligned}
$$

4. Assume that a car insurance company has received n incoming claims (requests for payment) in a given year. Assume that the size of each claim can be modeled as independent with a Gamma distribution with mean 3θ and variance $3\theta^2$.

(a) Derive the probability density function of the size of the largest claim this year. 2p.

Solution: Let X_k be the size of claim k. Then

$$
\begin{aligned}
f_{X_k}(x) &= \frac{1}{\Gamma(3)}x^2\frac{1}{\theta^3}e^{-x/\theta} = \frac{1}{2\theta^3}x^2e^{-x/\theta},\ x>0 \\
F_{X_k}(x) &= \int_0^x \frac{1}{2\theta^3}z^2e^{-z/\theta}dz = \begin{bmatrix} IT55 \\ a=-1/\theta \end{bmatrix} \\
&= \frac{1}{2\theta^3}\left[\left(-\theta z^2-2\theta^2 z-2\theta^3\right)e^{-z/\theta}\right]_0^x = \frac{1}{2\theta^3}\left(\left(-\theta x^2-2\theta^2 x-2\theta^3\right)e^{-x/\theta}+2\theta^3\right) \\
&= 1-\frac{x^2+2\theta x+2\theta^2}{2\theta^2}e^{-x/\theta},\ x>0
\end{aligned}
$$

Now let $X_{(n)}=\max\{X_1,\dots,X_n\}$ and we have

$$
\begin{aligned}
f_{X_{(n)}}(x) &= n\left(F_{X_k}(x)\right)^{n-1}f_{X_k}(x) \\
&= n\left(1-\frac{x^2+2\theta x+2\theta^2}{2\theta^2}e^{-x/\theta}\right)^{n-1}\frac{1}{2\theta^3}x^2e^{-x/\theta},\ x>0.
\end{aligned}
$$

(b) What is the distribution of the sum of all claims this year? 1.5p.

Solution: The characteristic function for X_k is

$$\varphi_{X_k}(t) = \frac{1}{(1-\theta it)^3}.$$

Let $Y_n=\sum_{k=1}^n X_k$, then

$$\varphi_{Y_n}(t) = \left(\frac{1}{(1-\theta it)^3}\right)^n = \frac{1}{(1-\theta it)^{3n}}.$$

So, $Y_n \sim \Gamma(3n,\theta)$.

(c) Now, let $n\to\infty$ and Y_n be the sum of all claims this year. Show that

$$\frac{Y_n-3n\theta}{\sqrt{n}}$$

converges in distribution and find the limiting distribution. 1.5p.

Solution: The central limit theorem states that

$$\frac{Y_n-n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1),$$

with $\mu=3\theta$ and $\sigma=\sqrt{3}\theta$. Thus

$$\frac{Y_n-3n\theta}{\sqrt{n}} = \frac{Y_n-3n\theta}{\sqrt{3}\theta\sqrt{n}}\sqrt{3}\theta \xrightarrow{d} N(0,3\theta),\ n\to\infty.$$