Probability Theory

Lecture 3

Per Sidén

Division of Statistics
Dept. of Computer and Information Science
Linköping University

Overview Lecture 3

- Transforms
- Probability generating function
- Moment generating function
- Characteristic function
- Transforms and distributions with random parameters

TRANSFORMS

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- $X \stackrel{d}{=} Y \iff g_X(t) = g_Y(t)$
- **Summation** of independent variables corresponds to **multiplication of transforms**. Nice!

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Probability generating function, cont.

Cor Let $X_1, X_2, ..., X_n$ be independent and identically distributed. Then

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where $g_X^{(n)}(t)$ is the nth derivative of $g_X(t)$ wrt to t.

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- Moments can be computed

$$\mathrm{E}X = g_X'(1)$$

$$\mathrm{Var}X = g_X''(1) + g_X'(1) - \left(g_X'(1)\right)^2$$

Probability generating function - examples

- ✓ binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
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$$g_X(t) = \prod_{i=1}^{n} g_{X_i(t)} = \prod_{i=1}^{n} (q+pt) = (q+pt)^n$$

so $X \sim Bin(n,p)$.

PROBABILITY GENERATING FUNCTION - EXAMPLES

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$$g_X(t) = \sum_{k=0}^{\infty} t^k \frac{e^{-m} m^k}{k!} = e^{-m} \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} = e^{m(t-1)}$$

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$$g_{X_1+X_2}(t) = e^{m_1(t-1)}e^{m_2(t-1)} = e^{(m_1+m_2)(t-1)}$$

so $X_1 + X_2 \sim Po(m_1 + m_2)$.

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- Taylor expansion around $t = 0$ [note $\frac{\partial^k e^{tX}}{\partial t^k} = X^k e^{tX}$]

$$e^{tX} = 1 + \sum_{n=1}^{\infty} \frac{t^n X^n}{n!}$$

so

$$Ee^{tX} = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} EX^n$$

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$$\psi_X(t) = Ee^{tX} = qe^{t \cdot 0} + pe^{t \cdot 1} = q + pe^t$$

  - $\psi_X'(t) = pe^t$ so $E(X) = \psi_X'(0) = p$.
  - $\psi_X''(t) = pe^t$ so $E(X^2) = \psi_X''(0) = p$.
  - $Var(X) = E(X^2) - [E(X)]^2 = p - p^2 = pq$

- $X \sim \Gamma(p, a)$

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- $X \sim \Gamma(p, a)$

$$\psi_X(t) = \frac{1}{(1-at)^p}$$

  - $\psi'_X(t) = \frac{ap}{(1-at)^{p+1}}$ so $E(X) = \psi'_X(0) = ap$.
  - $\psi''_X(t) = \frac{a^2 p(p+1)}{(1-at)^{p+2}}$ so $E(X^2) = \psi''_X(0) = a^2 p(p+1)$.
  - $Var(X) = E(X^2) - [E(X)]^2 = a^2 p(p+1) - a^2 p^2 = a^2 p$.

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$$\psi_X(t) = \frac{1}{(1-dt)^p}$$

$$\psi_Y(t) = \frac{1}{(1-d\sigma t)^p},$$

which is the mgf of $\Gamma(d\sigma, p)$. Gamma family is closed under scaling.

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$$\varphi_X(t) = Ee^{itX} = E\left(\cos tX + i \sin tX\right)$$

where i is the imaginary number ($i^2 = -1$).

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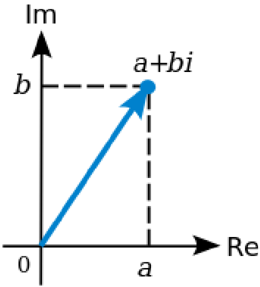
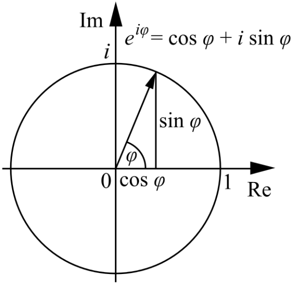
$$\varphi_X\left(t\right) = \frac{e^{itb} - e^{ita}}{it\left(b - a\right)}$$

Complex numbers

- Complex number $z = a + b \cdot i$
- $Re(z) = a$ is the real part of z
- $Im(z) = b$ is the imaginary part of z
- Complex conjugate $\bar{z} = a - b \cdot i$
- Addition: $z_1 + z_2 = a_1 + a_2 + (b_1 + b_2) \cdot i$
- Multiplication: $z_1 z_2 = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$
- Modulus: $|z| = \sqrt{a^2 + b^2}$. Length of vector.
- Complex exponentials: $e^{ix} = \cos x + i \cdot \sin x$





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$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt$$

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TH Linear combinations

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Transforms - distributions with random parameters

- Transforms are expected values (or t^X, e^{tX} or e^{itX}), so the law of iterated expectation is useful.
- Let $X|(N = n) \sim Bin(n, p)$ and $N \sim Po(\lambda)$. What is the marginal distribution of X?

TRANSFORMS - DISTRIBUTIONS WITH RANDOM PARAMETERS

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- Let $X|(N = n) \sim Bin(n, p)$ and $N \sim Po(\lambda)$. What is the marginal distribution of X? X is non-negative and integer-valued, so $g_X(t)$ is defined.

$$g_X(t) = E\left(E(t^X|N)\right) = Eh(N)$$

where

$$h(n) = E(t^X|N = n) = (q + pt)^n.$$

We then have

$$g_X(t) = E\left((q + pt)^N\right) = g_N(q + pt) = e^{\lambda[(q+pt)-1]} = e^{\lambda p(t-1)}.$$

- $X|y \sim N(0, y)$ and $y \sim \mathrm{Exp}(1)$, then $X \sim L(1/\sqrt{2})$. Prove using characteristic functions.

Transforms - sums of random number of random variables

Th Let $S_n = X_1 + X_2 + ... + X_n$ be a sum of i.i.d variables and N be a non-negative integer valued random variable. Then

$$\begin{aligned} g_{S_N}(t) &= g_N(g_X(t)) \\ \psi_{S_N}(t) &= g_N(\psi_X(t)). \\ \varphi_{S_N}(t) &= g_N(\varphi_X(t)) \end{aligned}$$

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$$\begin{aligned} \psi_{S_N}(t) &= g_N(\psi_X(t)) = \frac{1}{1-\frac{t}{p}} \\ &\Rightarrow S_N \sim Exp(1/p) \end{aligned}$$