LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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2015-10-14
Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time:	8-12
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani.
Assisting teacher:	Per Sidén, phone *0704-977175*
Grades:	Grades: Maximum is 20 points. A=19-20 points B=17-18 points C=12-16 points D=10-11 points E=8-9 points F=0-7 points

- Write clear and concise answers to the questions.

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1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} ax^2y & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are X and Y independent? 1p.

   (b) Determine the constant a. 1p.

   (c) Compute the marginal density of Y. 1p.

   (d) Compute the conditional density of $X|Y = y$. 1p.

   (e) Compute $E\left[\exp\left\{X^3\right\}|Y = y\right]$. 1p.

2. Let $N, X_1, X_2, X_3, \ldots$ be independent random variables such that $N \sim Fs\,(p)$ and $X_k \sim L\,(a)$ for all k. Define

$$S_N = \sum_{k=1}^{N} X_k.$$

   (a) Calculate the expected value and variance of S_N. 1.5p.

   (b) Calculate the characteristic function for S_N. 2.5p.

   (c) Show that $\sqrt{p}S_N \sim L\,(a)$. 1p.

3. Let X_1, X_2 and X_3 follow a multivariate normal distribution with mean vector $\mu = (0, 0, 2)'$ and covariance matrix

$$\Lambda = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 1 \\ -2 & 1 & 6 \end{pmatrix}.$$

(a) What is the bivariate distribution of X_2 and X_3? 1p.

(b) What is the conditional distribution of X_1 given $X_1 + X_2 - X_3 = c$ for some constant c? 2p.

(c) Define $Y_1 = \frac{1}{X_1}$. Derive the conditional probability density function of Y_1 given $X_1 + X_2 - X_3 = c$ for some constant c. 2p.

4. Assume that a car insurance company has received n incoming claims (requests for payment) in a given year. Assume that the size of each claim can be modeled as independent with a Gamma distribution with mean 3θ and variance $3\theta^2$.

(a) Derive the probability density function of the size of the largest claim this year. 2p.

(b) What is the distribution of the sum of all claims this year? 1.5p.

(c) Now, let $n \to \infty$ and Y_n be the sum of all claims this year. Show that

$$\frac{Y_n - 3n\theta}{\sqrt{n}}$$

converges in distribution and find the limiting distribution. 1.5p.

Good luck!

Per