Probability Theory

Lecture 5

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Overview Lecture 5

- Linear algebra recap
- Multivariate normal distribution

Linear algebra recap

- **Eigen-decomposition** of an $n \times n$ symmetric matrix $\mathbf{A}$

$$\mathbf{C}'\mathbf{A}\mathbf{C} = \mathbf{D}$$

where $\mathbf{D} = Diag(\lambda_1, ..., \lambda_n)$ and $\mathbf{C}$ is an orthogonal matrix.

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  - $\det \mathbf{C} = \pm 1$

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  - $\det \mathbf{C} = \pm 1$
- The columns of $\mathbf{C} = (\mathbf{c}_1, ..., \mathbf{c}_n)$ are the eigenvectors, and λ_i is the ith largest eigenvalue.
- $\det \mathbf{A} = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.

Quadratic forms and positive-definiteness

- **Quadratic form**
$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$$
- $Q(\mathbf{x})$ is **positive-definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- $Q(\mathbf{x})$ is **positive-semidefinite** if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

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- $Q(\mathbf{x})$ is **positive-definite** iff all eigenvalues of $\mathbf{A}$ are positive.
- $Q(\mathbf{x})$ is **positive-semidefinite** iff all eigenvalues of $\mathbf{A}$ are non-negative.

Matrix square root

- If $D = diag(\lambda_1, ..., \lambda_n)$ is diagonal, then $\tilde{D} = diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$ is the square root of D:
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- The **square root of a positive definite matrix** A
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- Check:
$$A^{1/2}A^{1/2} = C\tilde{D}C'C\tilde{D}C' = C\tilde{D}\tilde{D}C' = CDC' = A$$
- We also have
$$\left(A^{-1}\right)^{1/2} = \left(A^{1/2}\right)^{-1}$$
which is denoted by $A^{-1/2}$.

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Th Every covariance matrix is positive semidefinite.

- $\det \Lambda \geq 0$.

Linear transformations

- Recall that if $Y = aX + b$, where $E(X) = \mu$ and $Var(X) = \sigma^2$ then

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Th **Multivariate linear transformation**

Let $\mathbf{Y} = \mathbf{B}\mathbf{X} + \mathbf{b}$, where $\mathbf{X}$ is $n \times 1$ and $\mathbf{B}$ is $m \times n$.
Assume $E\mathbf{X} = \mu$ and $Cov(\mathbf{X}) = \Lambda$. Then,

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Th Let $\mathbf{X} = (X_1, ..., X_n)'$ where $X_1, ..., X_n \overset{iid}{\sim} N(0, 1)$. Then

$$\mathbf{Y} = \mu + \Lambda^{1/2}\mathbf{X} \sim N(\mu, \Lambda)$$

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- **Bivariate normal** $(n = 2)$

$$\Lambda = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $-1 \leq \rho \leq 1$ is the correlation coefficient.

Properties of the normal distribution

- Let $\mathbf{X} \sim N(\mu, \Lambda)$.

TH Linear combinations: $\mathbf{Y} = \mathbf{B}\mathbf{X} + \mathbf{b}$, where $\mathbf{X}$ is $n \times 1$ and $\mathbf{B}$ is $m \times n$. Then

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$$\mathbf{X}_1 \sim N(\mu_1, \Lambda_1)$$

where μ_1 are the n_1 first elements of μ and Λ_1 is the $n_1 \times n_1$ submatrix of Λ.

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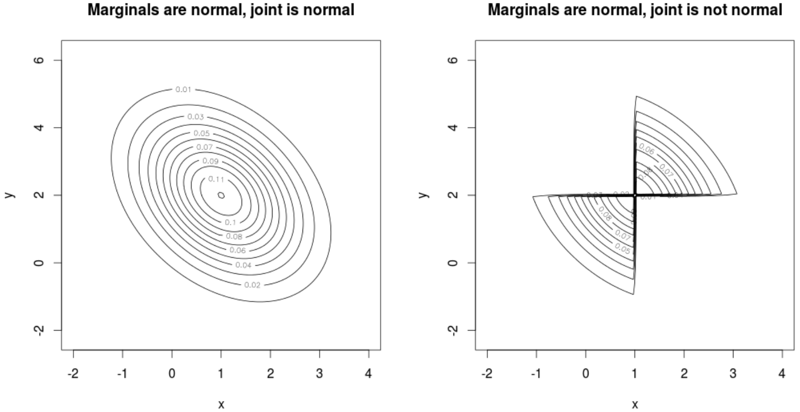
- We know that $\mathbf{X} \sim N(\mu, \Lambda)$ implies that all marginals are normal.
- The converse does not hold. Normal marginals does not imply that the joint distribution is normal.



Conditional distributions from $N(\mu, \Lambda)$

- Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \Lambda)$, where

$$\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}$$

- Then

$$Y|X = x \sim N\left[\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x),\ \sigma_y^2(1 - \rho^2)\right]$$

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Th Let $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$ and partition μ and Λ accordingly as

$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$. Then

$$\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2 \sim N\left[\mu_1 + \Lambda_{12}\Lambda_{22}^{-1}(x_2 - \mu_2),\ \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}\right]$$

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- Correlation measures **linear** association (dependence).
- In general: Uncorrelated $\nrightarrow$ Independence.
- In the normal distribution: Uncorrelated $\leftrightarrow$ Independence.
- Remember that: X and Y are jointly normal $\rightarrow$ the regression function is linear $\rightarrow$the linear predictor is optimal.
- $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ are independent.

Principal components

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Th Let $\mathbf{X} \sim N(\mu, \Lambda)$ and set $\mathbf{Y} = \mathbf{C}'\mathbf{X}$, then

$$\mathbf{Y} \sim N(\mathbf{C}'\mu, \mathbf{D})$$

so that the components of $\mathbf{Y}$ are independent and $Var(Y_i) = \lambda_i$.