PROBABILITY THEORY

LECTURE 6

Mattias Villani

Division of Statistics
Dept. of Computer and Information Science
Linköping University

OVERVIEW LECTURE 6

- Modes of convergence
  - almost surely
  - in probability
  - in r-mean
  - in distribution
- Law of large numbers
- Central limit theorem
- Convergence of sums, differences and products.

INTRODUCTION

- We are often interested in the **large sample**, or **asymptotic,** behavior of random variables.
- We are considering a **sequence** of random variables $X_1, X_2, ...$, also denoted by $\{X_n\}_{n=1}^{\infty}$.
- Example: what can we say about the sample mean $X_n = n^{-1}\sum_{i=1}^{n} Y_i$ in large samples?
  - Does it converge to a single number? (**law of large numbers**)
  - How fast? (central limit theorem)
  - What is the distribution of the sample mean in large samples? (**central limit theorem**)
- The usual limit theorems from mathematics will not do. Need to consider that X_n is a **random** variable.

MARKOV AND CHEBYSHEV'S INEQUALITIES

- **Markov's inequality**. For a positive random variable X and constant $a > 0$

$$Pr(X \geq a) \leq \frac{E(X)}{a}$$

Markov and Chebyshev's inequalities

- **Markov's inequality**. For a positive random variable X and constant $a > 0$

$$Pr(X \geq a) \leq \frac{E(X)}{a}$$

- Proof: $a \cdot I_{X \geq a} \leq X$. Then $E(a \cdot I_{X \geq a}) = a \cdot Pr(X \geq a) \leq E(X)$.

Markov and Chebyshev's inequalities

- **Markov's inequality**. For a positive random variable X and constant $a > 0$

$$Pr(X \geq a) \leq \frac{E(X)}{a}$$

- Proof: $a \cdot I_{X \geq a} \leq X$. Then $E(a \cdot I_{X \geq a}) = a \cdot Pr(X \geq a) \leq E(X)$.
- **Chebyshev's inequality**. Let Y be a random variable with finite mean m and variance σ^2. Then

$$Pr(|Y - m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

Markov and Chebyshev's inequalities

- **Markov's inequality**. For a positive random variable X and constant $a > 0$

$$Pr(X \geq a) \leq \frac{E(X)}{a}$$

- Proof: $a \cdot I_{X \geq a} \leq X$. Then $E(a \cdot I_{X \geq a}) = a \cdot Pr(X \geq a) \leq E(X)$.
- **Chebyshev's inequality**. Let Y be a random variable with finite mean m and variance σ^2. Then

$$Pr(|Y - m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

- Proof: Use Markov's inequality with $X = (Y - m)^2$and $a = \varepsilon^2$, and that $E(X) = E(Y - m)^2 = \sigma^2$. We then have

$$Pr\left((Y - m)^2 \geq \epsilon^2\right) \leq \frac{\sigma^2}{\varepsilon^2}$$

and therefore

$$Pr\left(|Y - m| \geq \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2}.$$

Almost sure convergence

- $X_1, \ldots X_n$ and X are random variables on the same probability space.

Def X_n converges **almost surely** (a.s.) to X as $n \to \infty$ iff

$$P\left(\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\right) = 1.$$

- Denoted by $X_n \stackrel{a.s.}{\to} X$.

Almost sure convergence

- $X_1, \ldots X_n$ and X are random variables on the same probability space.

DEF X_n converges **almost surely** (a.s.) to X as $n \to \infty$ iff

$$P\left(\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\right) = 1.$$

- Denoted by $X_n \stackrel{a.s.}{\to} X$.
- For a given $\omega \in \Omega$, $X_n(\omega)$ $(n = 1, 2, \ldots)$ and $X(\omega)$ are real numbers (not random variables).
- Almost sure convergence: check if the sequence of real numbers $X_n(\omega)$ converges to the real number $X(\omega)$ for all ω, except those ω that have probability zero.
- Example: roll two dice $(\Omega = \{(1,1), (1,2), \ldots, (6,6)\})$. Let $Y_n(\omega)$ be the sum of the two dice in the nth roll. Let $X_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} Y_i$. [Show simulation in R.]

Convergence in probability

DEF X_n **converges in probability** to X as $n \to \infty$ iff

$$P\left(|X_n - X| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \stackrel{p}{\to} X$.

Convergence in probability

DEF X_n **converges in probability** to X as $n \to \infty$ iff

$$P\left(|X_n - X| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \stackrel{p}{\to} X$.

✏ Let $X_n \sim Beta(n, n)$ show that $X_n \stackrel{p}{\to} \frac{1}{2}$ as $n \to \infty$.

Convergence in probability

DEF X_n **converges in probability** to X as $n \to \infty$ iff

$$P\left(|X_n - X| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \stackrel{p}{\to} X$.

✏ Let $X_n \sim Beta(n, n)$ show that $X_n \stackrel{p}{\to} \frac{1}{2}$ as $n \to \infty$.

Solution: $E(X_n) = \frac{n}{n+n} = \frac{1}{2}$ and

$$Var(X_n) = \frac{n \cdot n}{(n+n)^2(n+n+1)} = \frac{1}{4(2n+1)}.$$

By Chebyshev's inequality, for all $\varepsilon > 0$

$$Pr\left(|X_n - 1/2| \geq \varepsilon\right) \leq \frac{1}{4(2n+1)\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

Convergence in r-mean

Def X_n **converges in** r**-mean** to X as $n \to \infty$ iff

$$E|X_n - X|^r \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \xrightarrow{r} X$.

Convergence in r-mean

Def X_n **converges in** r**-mean** to X as $n \to \infty$ iff

$$E|X_n - X|^r \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \xrightarrow{r} X$.

Let X_n be a random variable with

$$P(X_n = 0) = 1 - \frac{1}{n}\ ,\ P(X_n = 1) = \frac{1}{2n} \text{ and } P(X_n = -1) = \frac{1}{2n}.$$

Show that $X_n \xrightarrow{r} 0$ as $n \to \infty$.

Convergence in r-mean

Def X_n **converges in** r**-mean** to X as $n \to \infty$ iff

$$E|X_n - X|^r \to 0 \quad \text{as } n \to \infty.$$

- Denoted by $X_n \xrightarrow{r} X$.

Let X_n be a random variable with

$$P(X_n = 0) = 1 - \frac{1}{n}\ ,\ P(X_n = 1) = \frac{1}{2n} \text{ and } P(X_n = -1) = \frac{1}{2n}.$$

Show that $X_n \xrightarrow{r} 0$ as $n \to \infty$.

Solution: we have

$$E|X_n - X|^r = |0-0|^r \cdot \left(1 - \frac{1}{n}\right) + |1-0| \cdot \frac{1}{2n} + |-1-0|^r \cdot \frac{1}{2n}$$
$$= \frac{1}{n} \to 0.$$

as $n \to \infty$ for all $r > 0$.

Convergence in distribution

Def X_n **converges in distribution** to X as $n \to \infty$ iff

$$F_{X_n}(x) \to F(x) \quad \text{as } n \to \infty$$

at all continuity points of X.

- Denoted by $X_n \xrightarrow{d} X$.

Suppose $X_n \sim Bin(n, \lambda/n)$. Show that $X_n \to Po(\lambda)$ as $n \to \infty$.

Convergence in distribution

Def X_n **converges in distribution** to X as $n \to \infty$ iff

$$F_{X_n}(x) \to F(x) \quad \text{as } n \to \infty$$

at all continuity points of X.

- Denoted by $X_n \xrightarrow{d} X$.
- ✏ Suppose $X_n \sim Bin(n, \lambda/n)$. Show that $X_n \to Po(\lambda)$ as $n \to \infty$.

Solution: For fixed k we have

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \to e^{-\lambda}\frac{\lambda^k}{k!}$$

as $n \to \infty$

More on convergence

- Theorem 6.1.2 tells us that if $X_n \to X$ and $X_n \to Y$, then $X = Y$ almost surely ($X \overset{d}{=} Y$ for convergence in distribution).
- The different notions of convergence are related as follows:

$$\begin{array}{ccccc} X_n \xrightarrow{a.s.} X & \Rightarrow & X_n \xrightarrow{p} X & \Rightarrow & X_n \xrightarrow{d} X \\ & & \Uparrow & & \\ & & X_n \xrightarrow{r} X & & \end{array}$$

- So $\xrightarrow{a.s.}$ is stronger than $\xrightarrow{p}$ which is stronger than $\xrightarrow{d}$.

Convergence via transforms

- Let $X, X_1, X_2, ...$ be random variables. What if the moment generating function of X_n converges to the moment generation function of X? Does that mean that X_n converges to X?

Th Let $X, X_1, X_2, ...$ be random variables, and suppose that

$$\varphi_{X_n}(t) \to \varphi_X(t) \text{ as } n \to \infty$$

then

$$X_n \xrightarrow{d} X \text{ as } n \to \infty.$$

Th The converse also holds. If $X_n \xrightarrow{d} X$, then $\varphi_{X_n}(t) \to \varphi_X(t)$.

- Similar theorems hold for the generating function and moment generating function (Th 6.4.1-6.4.3).

Law of large numbers - some preliminaries

- Let $X_1, .., X_n$ be independent variables with mean μ and variance σ^2.
- Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the sample mean of n observations.
- We then have

$$E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{1}{n}n\mu = \mu$$

and

$$Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i) = \frac{\sigma^2}{n}.$$

Law of large numbers

- **(Weak) law of large numbers**. Let $X_1, .., X_n$ be independent variables with mean μ and finite variance σ^2. Then
$$\bar{X}_n \xrightarrow{p} \mu.$$
- Proof: By Chebychev's inequality
$$Pr\left(|\bar{X}_n - \mu| > \varepsilon\right) \leq \frac{\sigma^2/n}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$
- This version of the law of large numbers requires a population variance which is finite. Theorem 6.5.1 gives a version where only the mean needs to be finite.
- The strong law of large numbers proves that $\bar{X}_n \xrightarrow{a.s.} \mu$ if the mean is finite.
- The assumption of a finite mean is important. Example: if $X_1, X_2, \ldots$ are independent $C(0,1)$, then $\bar{X}_n \overset{d}{=} X_1$ for all n. The law of large numbers does not hold since the Cauchy does not exist.

Central limit theorem

Th Let $X_1, X_2, \ldots$ be iid random variables with finite expectation μ and variance σ^2. Then
$$\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right) \xrightarrow{d} N(0,1) \text{ as } n \to \infty.$$

- Proof by showing that
$$\varphi_{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}(t) \to \varphi_{N(0,1)}(t) = e^{-t^2/2}.$$
- Application: empirical distribution function
$$F_n(x) = \frac{\#\text{observations} \leq x}{n}$$
then as $n \to \infty$
$$F_n(x) \xrightarrow{p} F(x)$$
$$\sqrt{n}\left(F_n(x) - F(x)\right) \xrightarrow{d} N\left(0, \sigma^2(x)\right)$$
where $\sigma^2(x) = F(x)\left[1 - F(x)\right]$.

Convergence of sums of sequences of RVs

Th If $X_n \to X$ and $Y_n \to Y$, then $X_n + Y_n \to X + Y$.

- Holds for $a.s.$, p and r-convergence without assuming independence.
- The theorem also holds for d-convergence if we assume independence.

Convergence of sums of sequences of RVs

Th If $X_n \to X$ and $Y_n \to Y$, then $X_n + Y_n \to X + Y$.

- Holds for $a.s.$, p and r-convergence without assuming independence.
- The theorem also holds for d-convergence if we assume independence.

Th If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$, where a is a constant, then as $n \to \infty$
$$X_n + Y_n \xrightarrow{d} X + a$$
$$X_n - Y_n \xrightarrow{d} X - a$$
$$X_n \cdot Y_n \xrightarrow{d} X \cdot a$$
$$\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{a} \quad \text{for } a \neq 0$$

Convergence of sums of sequences of RVs

TH If $X_n \to X$ and $Y_n \to Y$, then $X_n + Y_n \to X + Y$.

- Holds for *a.s.*, *p* and *r*-convergence without assuming independence.
- The theorem also holds for *d*-convergence if we assume independence.

TH If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} a$, where a is a constant, then as $n \to \infty$

$$X_n + Y_n \xrightarrow{d} X + a$$
$$X_n - Y_n \xrightarrow{d} X - a$$
$$X_n \cdot Y_n \xrightarrow{d} X \cdot a$$
$$\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{a} \quad \text{for } a \neq 0$$

Let $X_1, X_2, \ldots$ be independent $U(0,1)$. Show that

$$\frac{X_1 + X_2 + \ldots + X_n}{X_1^2 + X_2^2 + \ldots + X_n^2} \xrightarrow{p} \frac{3}{2} \text{ as } n \to \infty.$$

Convergence of functions of convergent RVs

TH Let $X_1, X_2, \ldots$ be random variables such that $X_n \xrightarrow{p} a$ for some constant a. Let $g()$ be a function which is continuous at a. Then

$$g(X_n) \xrightarrow{p} g(a).$$

Convergence of functions of convergent RVs

TH Let $X_1, X_2, \ldots$ be random variables such that $X_n \xrightarrow{p} a$ for some constant a. Let $g()$ be a function which is continuous at a. Then

$$g(X_n) \xrightarrow{p} g(a).$$

Let $X_1, X_2, \ldots$ be iid random variables with finite mean $\mu \geq 0$. Show that $\sqrt{\bar{X}_n} \xrightarrow{p} \sqrt{\mu}$ as $n \to \infty$.

Convergence of functions of convergent RVs

TH Let $X_1, X_2, \ldots$ be random variables such that $X_n \xrightarrow{p} a$ for some constant a. Let $g()$ be a function which is continuous at a. Then

$$g(X_n) \xrightarrow{p} g(a).$$

Let $X_1, X_2, \ldots$ be iid random variables with finite mean $\mu \geq 0$. Show that $\sqrt{\bar{X}_n} \xrightarrow{p} \sqrt{\mu}$ as $n \to \infty$.

Solution: from the law of large numbers we have $\bar{X}_n \xrightarrow{p} \mu$. Since $g(x) = \sqrt{x}$ is continuous at $x = \mu$ the above theorem proves that $\sqrt{\bar{X}_n} \xrightarrow{p} \sqrt{\mu}$ as $n \to \infty$.

Convergence of functions of convergent RVs

TH Let $X_1, X_2, \ldots$ be random variables such that $X_n \xrightarrow{p} a$ for some constant a. Let $g()$ be a function which is continuous at a. Then

$$g(X_n) \xrightarrow{p} g(a).$$

✏ Let $X_1, X_2, \ldots$ be iid random variables with finite mean $\mu \geq 0$. Show that $\sqrt{\bar{X}_n} \xrightarrow{p} \sqrt{\mu}$ as $n \to \infty$.

Solution: from the law of large numbers we have $\bar{X}_n \xrightarrow{p} \mu$. Since $g(x) = \sqrt{x}$ is continuous at $x = \mu$ the above theorem proves that $\sqrt{\bar{X}_n} \xrightarrow{p} \sqrt{\mu}$ as $n \to \infty$.

✏ Let $Z_n \sim N(0,1)$ and $V_n \sim \chi^2(n)$ be independent RVs. Show that

$$T_n = \frac{Z_n}{\sqrt{\frac{V_n}{n}}} \overset{d}{\sim} N(0,1) \quad \text{as } n \to \infty.$$