LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time: 8-12

Allowed: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).

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Grades: Grades: Maximum is 20 points.
A=19-20 points
B=17-18 points
C=12-16 points
D=10-11 points
E=8-9 points
F=0-7 points

- Write clear and concise answers to the questions.
- Remember that you may get points from explaining how you would solve the problem, even if you don't fully solve it.

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1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} \exp\left[-\left(x + \frac{y}{x}\right)\right] & \text{if } 0 \leq x \leq \infty \text{ and } 0 \leq y \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal density function for X. Does it belong to any of the known distributions? 1.5p.

Solution: The marginal density of X is

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = \exp(-x) \int_0^\infty \exp\left(-\frac{y}{x}\right) dy$$

where

$$\int_0^\infty \exp\left(-\frac{y}{x}\right) dy = \left[\exp\left(-\frac{y}{x}\right)(-x)\right]_0^\infty = [0 - 1 \cdot (-x)] = x.$$

So

$$f_X(x) = x \cdot \exp(-x)$$

for $0 \leq x \leq \infty$. This can be recognized as a $X \sim \Gamma(2,1)$ density.

(b) Compute the conditional density of $Y|X = x$. Does it belong to any of the known distributions? 1.5p.

Solution: The conditional density of $Y|X = x$ is

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\exp\left[-\left(x + \frac{y}{x}\right)\right]}{x \cdot \exp(-x)} = \frac{1}{x}\exp\left(-\frac{y}{x}\right),$$

which means that $Y|X = x \sim Exp(x)$.

(c) Compute $Cov(X, Y)$ 2p.

Solution: We know that

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$$

where $E(X) = 2 \cdot 1 = 2$ (since $X \sim \Gamma(2, 1)$). To compute $E(Y)$ we use the law of iterated expectation

$$E(Y) = E\left[E(Y|X)\right] = E\left(X\right) = 2$$

where we have used that $Y|X = x \sim Exp(x)$. Now

$$E(XY) = E\left[E(XY|X)\right] = E\left[X \cdot E(Y|X)\right] = E\left[X \cdot X\right] = E(X^2) = V(X) + \left[E(X)\right]^2 = 2 \cdot 1^2 + 2^2 = 6.$$

Hence

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y) = 6 - 2 \cdot 2 = 2.$$

2. Let $Y|(\Lambda = \lambda) \sim Po(\lambda)$ and $\Lambda \sim Exp(a)$.

(a) Calculate the expected value and variance of Y. 2p.

Solution: By the law of iterated expectation we have

$$E\left[E(Y|\Lambda)\right] = E\left(\Lambda\right) = a$$

and by the law of total variance

$$V(Y) = E\left[V(Y|\Lambda)\right] + V\left[E(Y|\Lambda)\right] = E\left(\Lambda\right) + V\left(\Lambda\right) = a + a^2 = a(1 + a).$$

(b) Calculate the moment generating function Y. 2p.

Solution: The moment generating function of Y is most easily computed by the law of iterated expectation as follows

$$\psi_Y(t) = E\left(e^{tY}\right) = E\left[E\left(e^{tY}|\Lambda = \lambda\right)\right] = E\left[e^{\lambda(e^t - 1)}\right] = \psi_\Lambda(e^t - 1)$$

where

$$\psi_\Lambda(t) = \frac{1}{1 - at}$$

is the moment generating function for the $Exp(a)$ distribution, which exists if $t < 1/a$. Hence

$$\psi_Y(t) = \psi_\Lambda(e^t - 1) = \frac{1}{1 - a(e^t - 1)},$$

which exists if $e^t - 1 < 1/a$, that is if $e^t < 1 + \frac{1}{a}$ or $t < \ln(1 + \frac{1}{a})$.

(c) Calculate the moment generating function of $Z = nY$, where n is a positive integer. 1p.

Solution: The moment generating function of $Z = nY$ is by definition

$$E(e^{tZ}) = E\left(e^{(tn)Y}\right) = \psi_Y(tn) = \frac{1}{1 - a(e^{tn} - 1)}.$$

3. Let $X_k \sim Exp(a),\ k = 1, 2, \ldots$ be independent random variables.

   (a) Derive the density of $Z_n = \min(X_1, X_2, \ldots, X_n)$. [for full points it is not enough to just write down the formula, you have to derive it.] 2p.

Solution: The distribution function of Z_n is

$$F_{Z_n}(z) = P(Z_n \le z) = 1 - P(Z_n > z) = 1 - P(X_1 > z, X_2 > z, \ldots, X_n > z)$$

since Z_n is the minimum and $Z_n > z$ is therefore the same as all $X_1, \ldots, X_n$ being larger than z. Now, because of independence and that all X_k have same distribution, we have

$$F_{Z_n}(z) = 1 - (1 - F_X(z))^n = 1 - \left(1 - \left[1 - \exp(-\frac{z}{a})\right]\right)^n = 1 - \exp\left(-\frac{nz}{a}\right)$$

and the density is therefore

$$f_Z(z) = \frac{1}{(a/n)} \exp\exp\left(-\frac{z}{a/n}\right)$$

which shows that $Z \sim Exp(\frac{a}{n})$.

   (b) Derive the distribution of $Y_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, \ldots$ 1p.

Solution: Since the X_k are independent and identically distributed we know that

$$\psi_{Y_n}(t) = \left(\frac{1}{1 - at}\right)^n = \frac{1}{(1 - at)^n} \sim \Gamma(n, a)$$

for $t < 1$.

   (c) Let again $Y_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, \ldots$. Find a such that $Y_n \xrightarrow{p} 1$ as $n \to \infty$. 2p.

Solution: The expected value of $Y_n \sim \Gamma(n, a)$ is na and the variance is na^2. For $a = 1/n$ we therefore have $E(Y_n) = 1$ and $V(Y_n) = 1/n$. We need to prove that for any $\varepsilon > 0$

$$P(|Y_n - 1| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

We can use Chebyshev's inequality

$$P(|Y_n - 1| > \varepsilon) \le \frac{\sigma^2}{\varepsilon^2} = \frac{1/n}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \to 0$$

as $n \to \infty$.

4. Let X_1, X_2 and X_3 follow a multivariate normal distribution with mean vector $\mu = (0, 1, 2)'$ and covariance matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

   (a) What is the bivariate distribution of X_1 and X_3? 1p.

Solution: Since the distribution is normal, the distribution is obtained by just picking out the relevant elements of μ and Λ:

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\right)$$

   (b) Compute the correlation between X_1 and X_3. 1p.

Solution: From Λ we see that $Cov(X_1, X_3) = 1$. We know that $\rho_{X_1,X_3} = \frac{Cov(X_1,X_3)}{\sigma_{X_1}\sigma_{X_3}} = \frac{1}{\sqrt{1}\sqrt{4}} = \frac{1}{2}$.

   (c) Compute $Var(c_1 \cdot X_1 + c_2 \cdot X_2)$ where c_1 and c_2 are constants. 1p.

Solution: From Λ we see that $Cov(X_1, X_2) = 0$ so $\rho_{X_1,X_2=0}$ and, because the variables are normal, this means that X_1 and X_2 are independent. We therefore have

$$Var(c_1 \cdot X_1 + c_2 \cdot X_2) = c_1^2 Var(X_1) + c_2^2 Var(X_2) = c_1^2 + 2c_2^2.$$

(d) What is the conditional distribution of $X_1 + X_2 + X_3$ given $X_1 - X_3 = c$ for some constant c? 2p.

Solution: Let us first derive joint distribution of $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 - X_2$. Since these are linear transformations and X_1, X_2 and X_3 are jointly normal, we have

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = BX$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

We then know that

$$Y \sim N\left(B\mu, B\Lambda B'\right)$$

that is

$$Y \sim N\left(\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 11 & -4 \\ -4 & 3 \end{pmatrix}\right)$$

Now,

$$Y_1|(Y_2 = c) \sim N\left[\mu_1 + \rho_{12}\frac{\sigma_1}{\sigma_2}(c - \mu_2), \sigma_1^2(1 - \rho_{12}^2)\right]$$

where the indicies are with respect to the random vector Y. Filling in the right values, we obtain

$$Y_1|(Y_2 = c) \sim N\left[3 - \frac{4}{3}(c + 2), 11(1 - \frac{16}{11 \cdot 3})\right] = N\left[\frac{1 - 4c}{3}, \frac{17}{3}\right]$$

since $\rho_{12} = \frac{Cov(Y_1,Y_2)}{\sigma_1\sigma_2} = \frac{-4}{\sqrt{11}\sqrt{3}}$, so $\rho_{12}^2 = \frac{16}{11 \cdot 3}$ and $\rho_{12}\frac{\sigma_1}{\sigma_2} = \frac{-4}{\sqrt{11}\sqrt{3}}\frac{\sqrt{11}}{\sqrt{3}} = -\frac{4}{3}$.

Good luck!

Mattias