

Exam in Probability Theory, 6 credits

Exam time: 9-13

Allowed: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).

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Grades: Grades: Maximum is 20 points.
A=19-20 points
B=17-18 points
C=12-16 points
D=10-11 points
E=8-9 points
F=0-7 points

- Write clear and concise answers to the questions.

1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} (1+x)y^x \exp(-x) & \text{if } 0 \leq x \leq \infty \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Are X and Y independent? 1p.
 - (b) Compute the marginal density of X . Does it belong to any of the known distributions? 1p.
 - (c) Compute the conditional density of $Y|X = x$. Does it belong to any of the known distributions? 1.5p.
 - (d) Compute $E[(X+2)Y]$. 1.5p.
2. Let $Y|\theta \sim \text{Bin}(n, \theta)$, where n is a known positive integer. Let the density of θ be

$$p(\theta) = 3 \cdot (1 - \theta)^2$$

for $\theta \in [0, 1]$ and $p(\theta) = 0$ otherwise.

- (a) Calculate the expected value and variance of Y . 2p.
- (b) Calculate the moment generating function for Y . 1.5p.
- (c) Compute the density of Y . 1.5p.

3. Let $X_k \sim Tri(0, 1)$, $k = 1, 2, \dots$ be independent random variables. ($Tri(a, b)$ is the triangular density over the interval $[a, b]$)
- (a) Derive the density of $Z_n = \max(X_1, X_2, \dots, X_n)$. 2p.
 - (b) Let $Y_n = \frac{1}{n} \sum_{k=1}^n X_k$. Show that $Y_n \xrightarrow{P} \frac{1}{2}$ as $n \rightarrow \infty$. 1p.
 - (c) Let W_n be a sequence of random variables with finite mean μ and variance σ^2 . Let $\bar{W}_n = \frac{1}{n} \sum_{i=1}^n W_i$. Show that $\bar{W}_n \cdot Y_n$ converges in distribution as $n \rightarrow \infty$, and find the limiting distribution. 2p.
4. Let $X \sim \Gamma(\alpha, 1)$ and $Y \sim \Gamma(\beta, 1)$ be independent Gamma variables.
- (a) Show that $X + Y$ and $X/(X + Y)$ are independent 1.5p.
 - (b) Find the marginal density of $X/(X + Y)$. 1.5p.
 - (c) What is the moment generating function of $Z = \frac{c \cdot (X+Y)^2 + d \cdot X}{X+Y}$, where c and d are positive constants? 2p.

GOOD LUCK!

MATTIAS