LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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2015-11-24
Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time: 8-12

Allowed: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).

Examinator: Mattias Villani.
Assisting teacher: Per Sidén, phone *0704-977175*

Grades: Grades: Maximum is 20 points.
A=19-20 points
B=17-18 points
C=12-16 points
D=10-11 points
E=8-9 points
F=0-7 points

- Write clear and concise answers to the questions.
- Make sure to specify the definition region for all density functions.

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1. The random variable X has the distribution function

$$F_X(x) = \begin{cases} a\left(1 - \frac{1}{x}\right), & 1 < x < \infty \\ 0, & otherwise \end{cases}$$

and the conditional probability density of Y given X is

$$f_{Y|X=x}(y) = \begin{cases} by, & 0 < y < x \\ 0, & otherwise \end{cases}.$$

(a) Determine the constant a and the probability density function of X. 1p.
(b) Determine the constant b as a function of x and compute $E[Y|X=4]$. 1p.
(c) Compute the joint density function of X and Y. Are X and Y independent? 1p.
(d) Compute the marginal density of Y and the probability $P(Y < 2)$. 2p.

2. Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}(2x+y) & , \begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \\ 0 & , otherwise \end{cases}.$$

(a) Compute $E[2X+Y]$. 2p.
(b) Determine the distribution of $2X+Y$. 3p.

3. Let X_1 and X_2 follow a multivariate normal distribution with mean vector $\mu = (1,0)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Define Y_1, Y_2 and Y_3 through

$$\begin{cases} Y_1 & = X_1 + X_2 \\ Y_2 & = -X_1 + 2X_2 \\ Y_3 & = X_2 - 1 \end{cases}.$$

(a) What is the joint distribution of Y_1, Y_2 and Y_3 ? 1.5p.

(b) Are any of Y_1, Y_2 and Y_3 independent? 1p.

(c) Suppose $X_n \sim Bin\,(n, \lambda/n)$. Show that $X_n \xrightarrow{d} Po\,(\lambda)$ as $n \to \infty$. 2.5p.

4. Let X_k, $k = 1, 2, \ldots$ be independent random variables, with common density $f_X(x)$ and distribution $F_X(x)$. Also, let N be a positive integer-valued random variable with probability generating function $g_N(t)$. Assume that N and $X_1, X_2, \ldots$ are independent. Define

$$Z_N = \max(X_1, X_2, \ldots, X_N).$$

(a) Derive the density of $Z_N | N = n$. 2p.

(b) Show that $F_{Z_N}(z) = g_N(F_X(z))$. 1.5p.

(c) Now, assume $X_1, X_2, \ldots$ are all $U(0,1)$-distributed and $N \sim Ge\left(\frac{1}{2}\right)$. Compute $F_{Z_n}(z)$. 1.5p.

Good luck!

Per