LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
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Probability Theory, 6 hp
732A63

Exam in Probability Theory, 6 credits

Exam time:	8-12
Allowed:	Pocket calculator. Table with common formulas and moment generating functions (distributed with the exam). Table of integrals (distributed with the exam). Table with distributions from Appendix B in the course book (distributed with the exam).
Examinator:	Mattias Villani.
Assisting teacher:	Per Sidén, phone *0704-977175*
Grades:	Grades: Maximum is 20 points. A: 19 points B: 17 points C: 14 points D: 12 points E: 10 points F: <10 points

- Write clear and concise answers to the questions.
- Make sure to specify the definition region for all density functions.

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1. Assume $X \sim N(0,1)$ and $Y \sim N(0,4)$ and that X and Y are independent.

   (a) Compute $E[(X+1)(Y-1)]$. 1p.

Solution: Since X and Y are independent

$$E[(X+1)(Y-1)] = E[X+1]E[Y-1] = 1 \cdot (-1) = -1.$$

   (b) Compute $E[Y|X+Y=6]$. 1.5p.

Solution: Let $W = X+Y$ and use that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\mu, \Sigma) = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right).$$

Let

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} Y \\ W \end{pmatrix} \sim N(B\mu, B\Sigma B')$$

that is

$$\begin{pmatrix} Y \\ W \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}\right).$$

From the formula of the conditional distribution for bivariate normal we have

$$E\left[Y|W=6\right]=\mu_Y+\rho\frac{\sigma_Y}{\sigma_W}\left(6-\mu_W\right)=0+\frac{4}{\sqrt{4\cdot 5}}\frac{\sqrt{4}}{\sqrt{5}}\left(6-0\right)=\frac{24}{5}.$$

(c) Derive the distribution of $U=\frac{X}{Y}$. Does it belong to a known distribution? 2.5p.

Solution: Use the transformation

$$\begin{cases} U=\frac{X}{Y} \\ V=Y \end{cases}$$

which has the inverse

$$\begin{cases} X=UV \\ Y=V \end{cases}$$

and Jacobian

$$J=\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix}=\begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}=v.$$

Since

$$f_{X,Y}(x,y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot\frac{1}{\sqrt{2\pi\cdot 4}}e^{-\frac{1}{2}\frac{y^2}{4}}=\frac{1}{4\pi}e^{-\frac{1}{2}\left(x^2+\frac{y^2}{4}\right)}$$

the transformation theorem gives us $f_{U,V}(u,v)=f_{X,Y}(h_1(u,v),h_2(u,v))\cdot|J|=\frac{1}{4\pi}e^{-\frac{1}{2}\left(u^2v^2+\frac{v^2}{4}\right)}\cdot|v|$ .Now

$$f_U(u)=\int_{-\infty}^{\infty}f_{U,V}(u,v)dv=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}v^2\left(u^2+\frac{1}{4}\right)}|v|\,dv=\frac{2}{4\pi}\int_{0}^{\infty}e^{-\frac{1}{2}v^2\left(u^2+\frac{1}{4}\right)}vdv$$

where the last step holds since the integrand is symmetric around $v=0$. Using IT61 with $x=v$ and $a=\frac{1}{2}\left(\frac{1}{4}+u^2\right)$

$$f_U(u)=\frac{1}{2\pi}\int_{0}^{\infty}e^{-\frac{1}{2}v^2\left(u^2+\frac{1}{4}\right)}vdv=\frac{1}{2\pi}\left[-\frac{1}{2\left(\frac{1}{2}\left(\frac{1}{4}+u^2\right)\right)}e^{-\frac{1}{2}v^2\left(u^2+\frac{1}{4}\right)}\right]_0^{\infty}=$$

$$=\frac{1}{2\pi}\cdot\frac{1}{\frac{1}{4}+u^2}=\frac{1}{\pi}\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2+u^2},\qquad -\infty<u<\infty.$$

So, $U\sim C(0,\frac{1}{2})$.

2. Let $Y|\theta\sim Bin(n,\theta)$, where n is a known positive integer. Let the density of θ be

$$f_\theta(\theta)=a\cdot\theta^3$$

for $\theta\in[0,1]$ and $f_\theta(\theta)=0$ otherwise.

(a) Determine the constant a, so that $f_\theta\left(\theta\right)$ is a proper density. 1p.

Solution: A density should integrate to 1 so

$$1=\int_0^1 a\cdot\theta^3=\left[\frac{a\theta^4}{4}\right]_0^1=\frac{a}{4}\Rightarrow a=4.$$

(b) Calculate the variance of Y. 2p.

Solution: Note first that $\theta \sim Beta(4,1)$ which has $E(\theta) = 4/5$ and $V(\theta) = 4/(5^2 \cdot 6) = 2/75$. By the law of iterated expectation

$$E(Y) = E\left(E(Y|\theta)\right) = E\left[n\theta\right] = \frac{4n}{5}$$

$$\begin{aligned} V(Y) &= E\left[V(Y|\theta)\right] + V\left[E(Y|\theta)\right] \\ &= E\left[n\theta(1-\theta)\right] + V\left[n\theta\right] \\ &= nE(\theta) - nE(\theta^2) + \frac{2n^2}{75} \\ &= \frac{4n}{5} - n\left(2/75 + \left(\frac{4}{5}\right)^2\right) + \frac{2n^2}{75} \\ &= \frac{n\left(2n+10\right)}{75}. \end{aligned}$$

(c) Compute the density of Y. 2p.

Solution:

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|\theta=\theta}(y) f_\theta(\theta) d\theta = \int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} 4 \cdot \theta^3 d\theta \\ &= 4\binom{n}{y} \int_0^1 \theta^{y+3}(1-\theta)^{n-y} d\theta \end{aligned}$$

this integral is easy to solve by realizing that $\theta^{y+3}(1-\theta)^{n-y}$ is proportional to the $Beta(y+4, n-y+1)$ density, so

$$\int_0^1 \theta^{y+3}(1-\theta)^{n-y} d\theta = \frac{\Gamma(y+4)\Gamma(n-y+1)}{\Gamma(n+5)}.$$

This is enough for full points, but the distribution can be simplified further as

$$f_Y(y) = 4\binom{n}{y}\frac{\Gamma(y+4)\Gamma(n-y+1)}{\Gamma(n+5)} = 4\frac{n!}{y!(n-y)!}\frac{(y+3)!\,(n-y)!}{(n+4)!} = \frac{4\,(y+3)\,(y+2)\,(y+1)}{(n+4)\,(n+3)\,(n+2)\,(n+1)}$$

for $y = 0, 1, ..., n$ and zero otherwise. We have used that $\Gamma(x+1) = x!$ if x is a positive integer.

3. Consider a fair die with probability $1/6$ of rolling a six. Consider a game where the die is rolled until a six comes up and let the random variable X denote the number of die rolls required until this event happens.

(a) Determine the probability function of X and state if it belongs to a known distribution. 1p.

Solution: This is the First success distribution, $X \sim Fs\left(\frac{1}{6}\right)$, with probabilty function

$$p_X\left(x\right) = \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}, x = 1, 2, \ldots.$$

(b) Assume that the same game is played n times and let X_{max} be the maximum value of X across all games. Derive the distribution function of X_{max}. 2p.

Solution: The distribution function of X is

$$F_X\left(x\right) = P\left(X \leq x\right) = \sum_{i=1}^{x} p_X\left(i\right) = \frac{1}{6}\sum_{i=1}^{x}\left(\frac{5}{6}\right)^{i-1} = \frac{1}{6}\frac{\left(\frac{5}{6}\right)^x - 1}{\frac{5}{6} - 1} = 1 - \left(\frac{5}{6}\right)^x$$

The distribution of X_{max} is

$$\begin{aligned} F_{X_{max}}(x) &= P(X_{max} \le x) = P(X_1 \le x, \ldots, X_n \le x) = \prod_{i=1}^{n} P(X_i \le x) \\ &= (P(X \le x))^n = (F_X(x))^n = \left(1 - \left(\frac{5}{6}\right)^x\right)^n. \end{aligned}$$

(c) How many times does one have to play the game for having at least 50% chance of obtaining a value of X_{max} that is greater than or equal to 30? 2p.

Solution: We have

$$\begin{aligned} 0.5 &< P(X_{max} \ge 30) = 1 - \left(1 - \left(\frac{5}{6}\right)^{29}\right)^n \\ \Rightarrow \left(1 - \left(\frac{5}{6}\right)^{29}\right)^n &> 0.5 \\ \Rightarrow n &> \frac{\log(0.5)}{\log\left(1 - \left(\frac{5}{6}\right)^{29}\right)} = 136.8. \end{aligned}$$

So one needs to play the game at least 137 times.

4. Let X and Y be random variables such that

$$Y|X = x \sim N(0, x)$$

with $X \sim Po(\lambda)$.

(a) Find the characteristic function of Y. 1p.

Solution:

$$E(e^{itY}) = E\left[E(e^{itY}|X)\right] = E\left(\exp(-\frac{1}{2}t^2 X)\right) = m_X(-t^2/2)$$

where

$$m_X(t) = \exp\left[\lambda(e^t - 1)\right]$$

is the moment generating function of the $Po(\lambda)$ distribution. So

$$\varphi_Y(t) = E(e^{itY}) = \exp\left[\lambda(e^{-t^2/2} - 1)\right]$$

is the characteristic function of Y.

(b) Show that

$$\frac{Y}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1)$$

as $\lambda \to \infty$. 2p.

Solution: The characteristic function of $Y/\sqrt{\lambda}$ is (Theorem 3.4.8)

$$\varphi_{Y/\sqrt{\lambda}}(t) = \varphi_Y(t/\sqrt{\lambda}) = \exp\left[\lambda(e^{-t^2/2\lambda} - 1)\right]$$

Now,

$$\lim_{\lambda\to\infty} \varphi_{Y/\sqrt{\lambda}}(t) = \lim_{\lambda\to\infty} \exp\left[\lambda(e^{-t^2/2\lambda} - 1)\right] = \lim_{\lambda\to\infty} \exp\left[\lambda\left\{\left(e^{-t^2/2}\right)^{1/\lambda} - 1\right\}\right].$$

To calculate this limit, consider the variable substitution

$$\begin{bmatrix} 1/\lambda = h \\ e^{-t^2/2} = a \\ \lambda \to \infty \Leftrightarrow h \to 0^+ \end{bmatrix}.$$

Now using a Taylor expansion we find that

$$\lim_{\lambda\to\infty} \exp\left[\lambda\left\{\left(e^{-t^2/2}\right)^{1/\lambda} - 1\right\}\right] = \lim_{h\to 0^+} \exp\left[\frac{a^h - 1}{h}\right] = \lim_{h\to 0^+} \exp\left[\frac{0 + \frac{\ln a \cdot a^h}{1!}h + \frac{(\ln a)^2 a^h}{2!}h^2 + o(h^2)}{h}\right] =$$

$$= \lim_{h\to 0^+} \exp\left[\ln a \cdot a^h + \frac{(\ln a)^2 a^h}{2}h + o(h)\right] = \exp[\ln a] = \exp\left[\ln e^{-t^2/2}\right] = e^{-t^2/2}.$$

which is the characteristic function of the $N(0,1)$ distribution. By Theorem 6.4.3 we then have that $Y/\sqrt{\lambda} \xrightarrow{d} N(0,1)$.

(c) Formulate and prove the Central Limit Theorm. 2p.

Solution: See page 162-163 in the course book.