Notes

PROBABILITY THEORY
LECTURE 2

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Notes

OVERVIEW LECTURE 2

- ▶ Conditional distributions
- ▶ Conditional expectation, conditional variance
- ▶ Distributions with random parameters and the Bayesian approach
- ▶ Regression and Prediction

Notes

CONDITIONAL DISTRIBUTIONS

- ▶ For events [if $P(B) > 0$]

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- ▶ A and B are **independent** if and only if $P(A|B) = P(A)$.
- ▶ For **discrete** random variables

$$p_{Y|X=x}(y) = p(Y = y|X = x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

$$p_{Y|X=x}(y) = \frac{p_{X,Y}(x,y)}{\sum_y p_{X,Y}(x,y)}.$$

- ▶ For **continuous** random variables

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,z)dz}$$

Notes

CONDITIONAL EXPECTATION

- ▶ **Conditional expectation of Y given $X = x$ is**

$$E(Y|X = x) = \begin{cases} \sum_y y \cdot p_{Y|X=x}(y) & \text{if } Y \text{is discrete} \\ \int_{-\infty}^{\infty} y \cdot f_{Y|X=x}(y)dy & \text{if } Y \text{is continuous} \end{cases}$$

- ▶ If $h(x) = E(Y|X = x)$, note that $h(X) = E(Y|X)$ is a random variable that only depends on X.
- ▶ Ex. 2.1 page 33. $X \sim U(0,1)$, $Y|X = x \sim U(0,x)$.

LAW OF ITERATED EXPECTATION

- Theorem 2.1. **Law of iterated expectation**.

$$E\left[E(Y|X)\right] = E(Y)$$

- Note that the *inner expectation* $(E(Y|X))$ is with respect to $f_{Y|X}(y)$, while the *outer expectation* is with respect to $f_X(x)$.
- The law of iterated expectation is an "expectation version" of the law of total probability.
- $E(Y|X) = E(Y)$ if X and Y are independent.

Notes

CONDITIONAL VARIANCE

- **Conditional variance of Y given $X = x$** is

$$Var(Y|X = x) = E\left[(Y - E(Y|X = x))^2 \,|X = x\right]$$

- Note that $v(X) = Var(Y|X)$ is a random variable that only depends on X.
- Corollary 2.3.1

$$Var(Y) = E\left[Var(Y|X)\right] + Var\left[E(Y|X)\right]$$

- Note the naive version $Var(Y) = E\left[Var(Y|X)\right]$ misses the uncertainty in Y that comes from not knowing X in $E(Y|X)$.

Notes

DISTRIBUTIONS WITH RANDOM PARAMETERS

- $X|\theta \sim f_X(x; \theta)$ and θ is a random variable.
- Example 1:
  - $X|N = n \sim Bin(n, p)$ and $N \sim Po(\lambda)$.
  - If the number of potential bidders in an auction is $N = n$ and each of them bids with probability p, then $X \sim Bin(n, p)$ bids will be placed.
  - The number of potential bidders is uncertain, $N \sim Po(\lambda)$.
  - The marginal distribution for X is $Po(\lambda \cdot p)$

- Example 2:
  - $X|(\sigma^2 = 1/\lambda) \sim N(0, 1/\lambda)$ and $\lambda \sim \Gamma\left(\frac{n}{2}, \frac{2}{n}\right)$, then $X \sim t(n)$.
  - X is daily stock market returns. $X|\lambda \sim N(0, 1/\lambda)$, where $1/\lambda$ is the daily variance.
  - The daily variance varies from day to day according to $\lambda \sim \Gamma\left(\frac{n}{2}, \frac{2}{n}\right)$. Turbulent day: realization of λ is very small.

Notes

BAYESIAN COIN TOSSING

- X_n=number of heads after n tosses.

$$X_n|P = p \sim Bin(n, p)$$

- **Prior distribution**: $P \sim U(0, 1)$.
- **Posterior distribution**: $P|(X_n = k) \sim Beta(k + 1, n + 1 - k)$.
- Marginal of X_n

$$X_n \sim U(\{1, 2, ..., n\})$$

Notes

Bayesian coin tossing

- Conditional of X_{n+1} given X_n **and** p

$$P(X_{n+1} = n+1 | X_n = n, P = p) = p$$

- Conditional of X_{n+1} given X_n

$$P(X_{n+1} = n+1 | X_n = n) = \frac{n+1}{n+2} \to 1 \text{ as } n \to \infty$$

- Coin flips are no longer independent when p is uncertain and we learn about p from data.

Regression and Prediction

- The **regression function**

$$h(\mathbf{x}) = h(x_1, ..., x_n) = E(Y | X_1 = x_1, ..., X_n = x_n) = E(Y | \mathbf{X} = \mathbf{x})$$

- **Predictor**: $\hat{Y} = d(\mathbf{X})$.
- **Linear predictor** $d(\mathbf{X}) = a_0 + a_1 X_1 + ... a_n X_n$.
- Expected **quadratic prediction error**: $E[Y - d(\mathbf{X})]^2$
- The **best predictor** of Y [minimizes expected quadratic prediction error] is the regression function $E(Y | \mathbf{X} = \mathbf{x})$.
- Best **linear** predictor - **least squares**:

$$\hat{Y} = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(X - \mu_x)$$

- When (X, Y) is jointly normal, $E(Y | X = x)$ is linear. For other distributions, this is not true in general.

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