LINKÖPINGS UNIVERSITET
Institutionen för datavetenskap
Avdelningen för statistik
Mattias Villani

2013-10-24
Probability Theory, 6 hp
732A40

Exam in Probability Theory, 6 credits

Exam time: 8-12

Allowed: Pocket calculator.
Table with common formulas and moment generating functions (distributed with the exam).
Table of integrals (distributed with the exam).
Table with distributions from Appendix B in the course book (distributed with the exam).

Examinator: Mattias Villani, phone. $0709 - 373799$

Grades: Grades: Maximum is 20 points.
A=19-20 points
B=17-18 points
C=12-16 points
D=10-11 points
E=8-9 points
F=0-7 points

- Write clear and concise answers to the questions.
- Remember that you may get points from explaining how you would solve the problem, even if you don't fully solve it.

---

1. The random variables X and Y have a joint probability density of the form

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}\exp\left[-\left(x+\frac{y}{x}\right)\right] & \text{if } 0 \le x \le \infty \text{ and } 0 \le y \le \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute the marginal density function for X. Does it belong to any of the known distributions? 1.5p.

   (b) Compute the conditional density of $Y|X = x$. Does it belong to any of the known distributions? 1.5p.

   (c) Compute $Cov(X,Y)$ 2p.

2. Let $Y|(\Lambda = \lambda) \sim Po(\lambda)$ and $\Lambda \sim Exp(a)$.

   (a) Calculate the expected value and variance of Y. 2p.

   (b) Calculate the moment generating function Y. 2p.

   (c) Calculate the moment generating function of $Z = nY$, where n is a positive integer. 1p.

3. Let $X_k \sim Exp(a),\ k = 1, 2, ...$ be independent random variables.

   (a) Derive the density of $Z_n = \min(X_1, X_2, ..., X_n)$. [for full points it is not enough to just write down the formula, you have to derive it.] 2p.

   (b) Derive the distribution of $Y_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, ...$ 1p.

   (c) Let again $Y_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, ...$ Find a such that $Y_n \xrightarrow{p} 1$ as $n \to \infty$. 2p.

4. Let X_1, X_2 and X_3 follow a multivariate normal distribution with mean vector $\mu = (0, 1, 2)'$ and covariance matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

   (a) What is the bivariate distribution of X_1 and X_3? 1p.

   (b) Compute the correlation between X_1 and X_3. 1p.

   (c) Compute $Var(c_1 \cdot X_1 + c_2 \cdot X_2)$ where c_1 and c_2 are constants. 1p.

   (d) What is the conditional distribution of $X_1 + X_2 + X_3$ given $X_1 - X_3 = c$ for some constant c? 2p.

Good luck!

Mattias