AVDASI: Potential Flow
Example Sheet

1. The non-lifting flow over an oval can be modelled as a combination of which three flows?

2. The lifting flow over a cylinder can be modelled as a combination of which three flows?

3. A flow can be modelled as a horizontal onset flow of 4 ms^{-1} and a source of strength 5 m^2s^{-1}. Find the horizontal and vertical velocity at the point O, 10 m above the source.

   [Ans: 4, 0.0796]

4. An air flow over a body with a semi-circular cross-section can be modelled as a horizontal onset flow of 4 ms^{-1} and a doublet of strength 128π m^2s^{-1}. Find the height, h, of the body and the velocity, u, at the highest point of the body.

   [Ans: 4, 8, 0]

5. An air flow over a long spinning circular cylinder, with a radius of $0.5m$, can be modelled by a horizontal onset flow of 10 ms^{-1} a doublet of strength 5π m^2s^{-1} and a vortex of strength 5π m^2s^{-1}. Find the streamwise velocity, u, at the top of the cylinder assuming potential flow.

   [Ans: 25]

6. Using Bernoulli's equation for potential flow, derive an expression for the pressure coefficient in terms of the velocity.

7. A wind of speed $10m/s$ is blowing over a rounded cliff; the situation is modelled as a source of strength $\Lambda = 130\pi m^2/s$ at the origin combined with a uniform stream $U_\infty = 10m/s$ in the x – direction
   a. Draw a sketch of the streamlines of the flow (internal & external to the cliff).
   b. Find the location of the stagnation point.
   c. The height of the cliff as $x \to \infty$
   d. Directly above the source, the cliff is half its ultimate downstream size, what is the percentage difference relative to the free stream velocity of the velocity measured by a sensor located 2m above the surface of the cliff?

   [Ans: -6.5, 20.42, 13.29%]

8. A steady wind of velocity 10m/s is blowing straight in off the sea to a rounded cliff face which extends laterally for several kilometres. The flow over the cliff face can be assumed to be two-dimensional and to be represented by the flow due to a point source at sea level. The ultimate height of the cliff face is 100m. A glider is flying straight and level along the length of the cliff face, directly above the foot of the cliff. Relative to the local air speed the glider is falling vertically at 1.5m/s (this is known as the sinking speed). At what altitude is it flying? Remember that relative to the ground the glider height is fixed. If the glider moves inland, how much higher could it fly at the same sinking speed and still maintain altitude.

   [Ans: 207m, 5m higher]

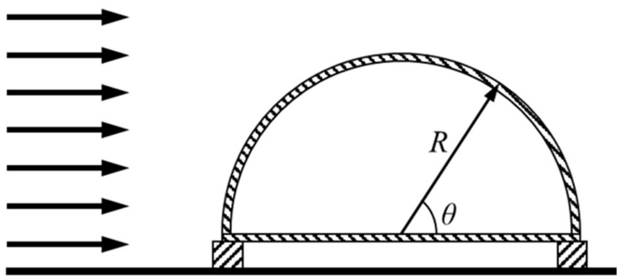
9. An airflow velocity sensor is to be mounted either in front, or above, the nose of the wing of a light aircraft, to give an indication of the speed of the aircraft. To obtain some feel for the situation, assume that the local flow is two-dimensional (i.e. the wing is straight and long relative to its chord length) and that the nose of the wing is symmetrical and at zero incidence. In this case, the wing leading edge can be represented by a point source in a uniform flow, and the ultimate thickness of the body so obtained is taken to be the maximum thickness of the wing section (which is 0.3m in this case)
   a. If the sensor is mounted on the centreline of the section, how far forward of the nose of the section must it be for the velocity error to be (i) 1% and (ii) 10%
   b. If the sensor is mounted directly above the position of the source, where the thickness is half the ultimate thickness, how far above the surface must it be for the error to be 1%
   c. A pressure transducer measures the difference in pressure between two static tapings on the surface; one directly above the source and the other at the nose, What factor would multiply this differential pressure value to give the freestream dynamic pressure.

   [Ans: a. 4.73m, 0.43m b. 0.262m c. 0.712]

10. (Extension Question) The non-lifting flow over a cylinder can be modelled using a free stream and a doublet. By finding the strength of the doublet in terms of the cylinder radius it can be shown that the pressure distribution on the cylinder is given by

$$p(\theta) = p_\infty + \frac{1}{2}\rho U_\infty^2(1 - 4\,sin^2\,\theta)$$

Where U_∞ and p_∞ are the values of the speed of air and pressure far upstream respectively, ρ is the density, and θ is the anticlockwise angle made with the freestream direction.



(a) A hut can be represented as a closed semi cylinder whose radius is R. It is mounted on blocks as shown in in the figure. Ignore viscous effects and assume that the flow field over the top of the hut is identical to the flow over a non-lifting cylinder for $0 \leq \theta \leq \pi$. The lower surface can be ignored when calculating the flow over the upper surface of the hut. The air under the hut is at rest, with pressure equal to the total pressure, p_0. Hence, by considering the forces acting on a small element of the upper surface of the hut, show that the overall lift per unit length on the hut is

$$l = 2p_0R - \int_0^\pi p(\theta)R\,sin\,\theta\,d\theta$$

(b) Hence, using the given pressure distribution show that the net lift force acting on the hut is equal to

$$l = \frac{8}{3}\rho_\infty U_\infty^2 R$$

You will need to use the relationship:

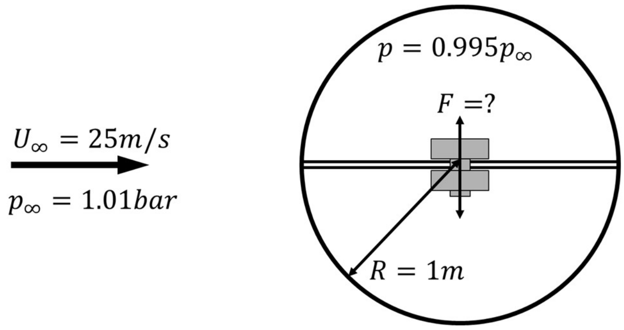
$$\int \sin^3\theta\,d\theta = -\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta + C$$

11. (Extension Question) A "yawmeter" consists of a circular cylinder with its axis normal to the flow, in which there are two static pressure tappings, 60° apart in the same plane normal to the axis. The angle of the incident flow is found by turning the cylinder until the pressure at the two tappings is equal (at which point they are 30° either side of the front stagnation point). Assuming inviscid flow, calculate the sensitivity of the yawmeter (i.e. the rate of change of pressure difference with flow angle) scaled by the free stream dynamic pressure, in the vicinity of the null position (where the pressures at the two tappings are equal). Further, find the angle between the pressure tappings that would give the maximum sensitivity and what is the value of that maximum sensitivity.

[Ans: 0.121 Pa per degree, 90°, 0.140 Pa per degree]

12. (Extension Question) A pressurised cylinder of 1m radius is formed by bolting two semi-cylindrical channels together on the inside, as shown in the figure to the right. The internal pressure is 99.5% of the atmospheric pressure of 1.01bar. If one bolt is used per metre, what is the tensile force on each bolt if the cylinder is placed in a 25 m/s wind as shown (air density =1.2kg/m³). You will need to use the relationship:

$$\int \sin^3\theta\,d\theta = -\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta + C$$



[Ans:480N]

13. (Extension Question) The pressure distribution on a cylinder in separated flow can by approximated by potential flow for the front semi-cylinder and a constant pressure for the rear semi-cylinder. Assuming the pressure at the rear of the cylinder is equal to the potential flow pressure at the junction of the front and rear semi-cylinders, find the drag coefficient. You will need to use the relationship:

$$\int \cos^3\theta\,d\theta = \frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\sin\theta + C$$

[Ans 8/3]

14. Consider a spinning cylinder whose radius is R. The lift per unit length of such a cylinder is

$$l = \rho U_{\infty} \Gamma$$

(a) Find the lift per unit length on a spinning cylinder of diameter $1m$ in a $4ms^{-1}$ headwind if only a single stagnation point is on the surface of the cylinder. Take density as $1.2kg/m^3$

(b) The rate of rotation of same cylinder is reduced so that the lift per unit length becomes $19.2\pi\ N$. Find the angle from the horizontal that a line from the origin through the stagnation points makes for this lift case.

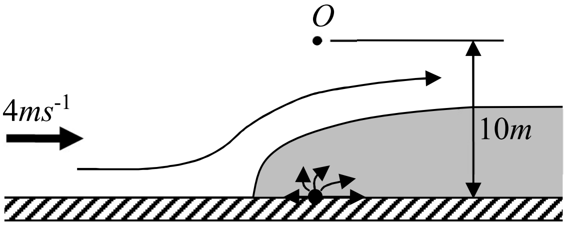
[Ans: $120.64N, 30^o$]

15. The circulation on a lifting cylinder is such that the rear stagnation point is displaced by 30° downwards. Calculate the magnitudes of the velocities at the top and bottom of the cylinder in terms of the freestream velocity and the corresponding surface pressure coefficients. Using Kutta-Joukowski, evaluate the lift coefficient per unit span.

[Ans: $3U_{\infty}$, U_{∞} , $-8, 0, 2\pi$]

Solutions

1. Uniform flow, source and sink of equal strength
2. Uniform flow, doublet and vortex
3.



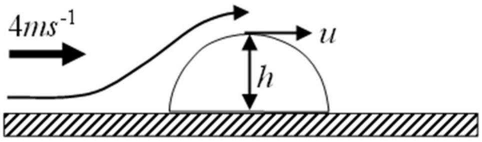
From given equations, source provides horizontal & vertical velocities:

$u = \frac{+\Lambda}{2\pi}\frac{x}{(x^2+y^2)}$ $\quad v = \frac{+\Lambda}{2\pi}\frac{y}{(x^2+y^2)}$

Velocity at $x = 0, y = 10$

$$u = 4 + 0 = 4ms^{-1} \qquad v = \frac{5}{2\pi}\frac{10}{(0^2+10^2)} = 0.0796ms^{-1}$$

4. From given equations, doublet provides horizontal & vertical velocities:



$$u = \frac{-\kappa}{2\pi}\frac{(x^2-y^2)}{(x^2+y^2)^2}, \qquad v = \frac{-\kappa}{2\pi}\frac{2xy}{(x^2+y^2)^2}$$

Calculate the position of the stagnation point at the front of the hemisphere.

Stagnation point at leading edge, placing origin at the centre of the doublet then

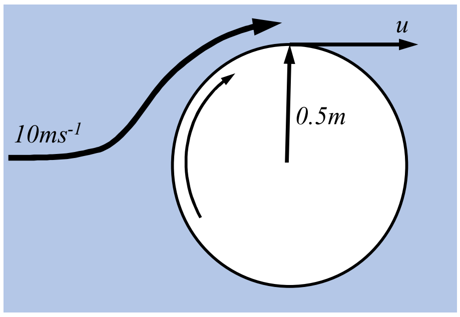
$0 = U_\infty - \frac{\kappa}{2\pi}\frac{1}{x^2}, \qquad x^2 = \frac{\kappa}{2\pi}\frac{1}{U_\infty} = \frac{128\pi}{2\pi}\frac{1}{4} = 16 \qquad x = \pm 4$

Flow is that over a hemisphere so height = 4m

Velocities at top therefore

$$u = 4 + \frac{-128\pi}{2\pi}\frac{(-4^2)}{(4^2)^2} = 8ms^{-1}, \qquad v = 0$$

5. No vertical component of velocity so just sum the horizontal velocity components at the top of the cylinder



$$u = U + \frac{+\Gamma}{2\pi}\frac{y}{(x^2+y^2)} + \frac{-\kappa}{2\pi}\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$u = U + \frac{+\Gamma}{2\pi}\frac{1}{0.5} + \frac{\kappa}{2\pi}\frac{1}{(0.5)^2} = 10 + \frac{5\pi}{2\pi}\frac{1}{0.5} + \frac{5\pi}{2\pi}\frac{1}{(0.5)^2}$$
$$= 10 + 5 + 10 = 25m/s$$

6. Using Bernoulli's equation for incompressible flow gives

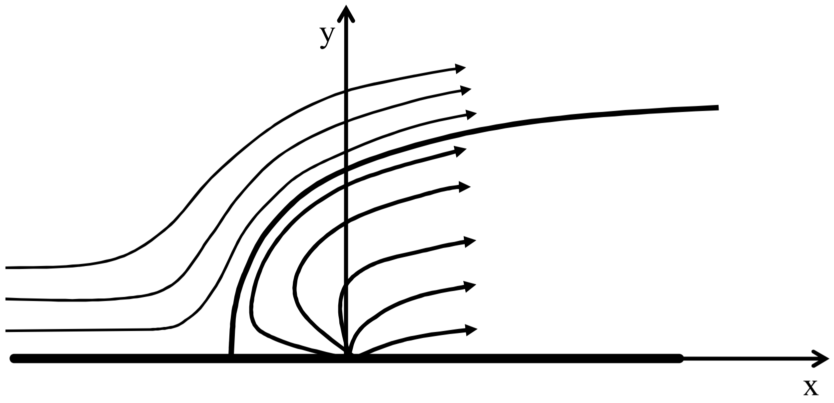
$$p_\infty + \frac{1}{2}\rho_\infty U_\infty^2 = p + \frac{1}{2}\rho_\infty V^2$$

By definition

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$$

$$C_p = \frac{\frac{1}{2}\rho_\infty U_\infty^2 - \frac{1}{2}\rho_\infty V^2}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{U_\infty^2 - V^2}{U_\infty^2} = 1 - \frac{V^2}{U_\infty^2}$$

7. a)



b) To find the location of the stagnation point at the nose the first step is to determine the velocity components. Thus, total horizontal velocity is given by

$$u = 10 + 65\frac{x}{x^2 + y^2}$$

and the vertical velocity is

$$v = 65\frac{y}{x^2 + y^2}$$

Then find the stagnation point when *u=v=0*

$v = 65\frac{y}{x^2+y^2} = 0$ when *y=0* i.e., along the x-axis.

$u = 10 + 65\frac{x}{x^2+y^2} = 10 + 65\frac{1}{x}$ when *y=0*

Therefore u=0 when $x = -6.5$. The stagnation point is on the centre line a distance $6.5m$ forward of the source.

(c) To determine the cliff height (*h*) far downstream as $x \to \infty$, use the fact that the source strength is equal to the volume flow rate per unit depth and the source has no impact on the velocity far downstream. Therefore, the flow rate equals the width between the dividing streamlines multiplied by the uniform onset flow speed:

$\Lambda = 130\pi = U_\infty\, 2h = 20h$

The ultimate cliff height is thus 6.5πm = 20.42 m

(d) from (b) and (c) the correct *x* location is *x=0*, so next work out the velocities at this location

$$u = 10 + 65\frac{x}{x^2 + y^2} = 10m/s$$

$$v = 65\frac{y}{x^2 + y^2} = \frac{65}{y}m/s$$

Then the velocity at this point is

$$V = \sqrt{u^2 + v^2}$$

The height of the cliff where the velocity is being measured is $y = 3.25\pi + 2$. Then substituting into the above equation gives

$$V = \sqrt{100 + \frac{65^2}{(3.25\pi + 2)^2}} = 11.329$$

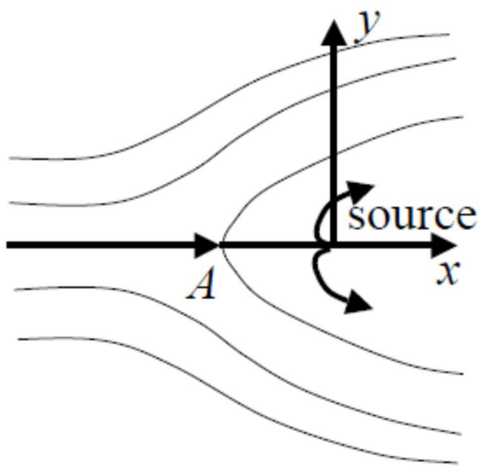
This is larger than the freestream velocity, hence the percentage difference relative to the free stream is

$$100\left(\frac{V - U_\infty}{U_\infty}\right) = 100\left(\frac{1.329}{10}\right) = 13.29\%$$

8. Define the velocities

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}\ , \qquad v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2}$$

Find the stagnation point, from symmetry $y_s = 0$, subbing into u equation



$$u = 0 = U_\infty + \frac{\Lambda}{2\pi}\frac{1}{x_s} \qquad \rightarrow \qquad x_s = -\frac{\Lambda}{2\pi U_\infty}$$

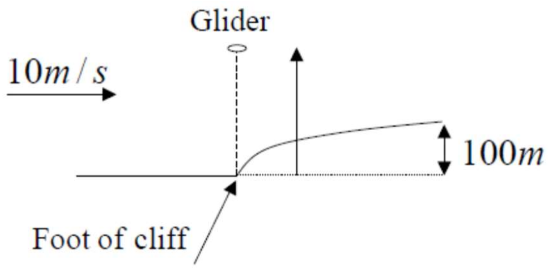
Now need to find the ultimate height of the cliff. Using the same logic as used in the notes

- Far downstream the velocity due to the source is zero
- So, all flow must have returned to freestream conditions
- All mass (or volume as incompressible) from the source must remain inside the dividing streamline
- So, inside the dividing streamline flow speed is U_∞ and volume flow rate is Λ .
- We take unit length into the page so area between dividing streamlines is the distance between them
- So ultimate width of dividing streamlines $= \frac{\Lambda}{U_\infty}$

Ultimate height of headland $=100 = \frac{\Lambda}{2U_\infty} \qquad \rightarrow \qquad \Lambda = 200U_\infty = 2{,}000$

Subbing back into stagnation location $\quad x_s = -\frac{2000}{20\pi} = -\frac{100}{\pi}$

In finding the height of the glider we are effectively finding the location above the stagnation point where the vertical velocity is 1.5m/s. Substituting our stagnation point location and source strength into the vertical velocity equation :



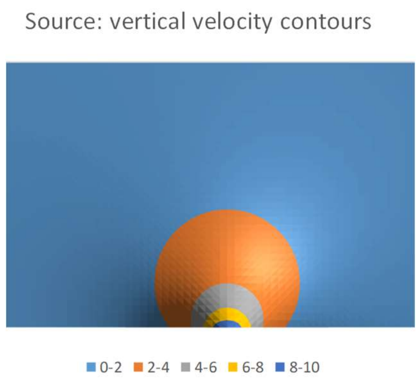
$$v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} = \frac{1000\pi y}{100^2 + \pi^2 y^2} = 1.5$$

$$3y^2 - \frac{2000}{\pi}y + \frac{30{,}000}{\pi^2} = 0$$

$$y = \frac{2000 \pm \sqrt{2000^2 - 360{,}000}}{6\pi} = \frac{2000 \pm 100\sqrt{364}}{6\pi} = 207.319, 4.887$$

Only one of these is physically sensible, so we choose **207m**. (4.9m above the base is less than 0.5m away from the land surface).

The second part of the question is asking for the position where we can find a vertical velocity of 1.5m/s at the highest location. Remember that the vertical velocity comes only from the source placed at the origin. To help visualise what this means for the vertical velocity, see the figure to the right. We can see that the position of maximum height to attain any specified velocity will be directly above the origin. (N.B. looking at the figure, we can also see why there were two solutions for v=1.5m/s above the base of the headland).



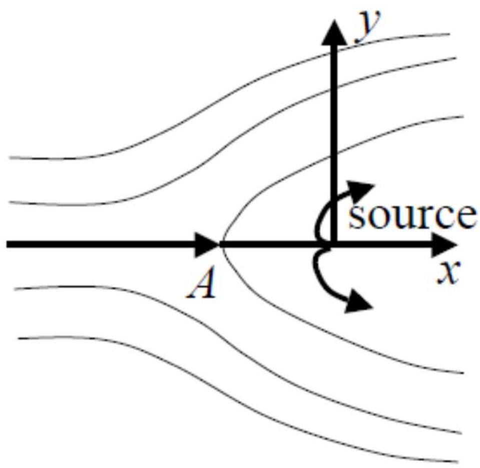
Substituting our source strength and $x = 0$ into the vertical velocity equation :

$$v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} = \frac{1000}{\pi y} = 1.5 \qquad \rightarrow \qquad y = 212.207$$

Therefore, extra height gained $= 212.207 - 207.319 = \boldsymbol{4.887m}$

9. Just as the start of the previous question. Define the velocities

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}\ , \qquad v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2}$$



Find the stagnation point, from symmetry $y_s = 0$, subbing into u equation

$$u = 0 = U_\infty + \frac{\Lambda}{2\pi}\frac{1}{x_s} \qquad \rightarrow \qquad x_s = -\frac{\Lambda}{2\pi U_\infty}$$

Now need to find the ultimate width of wing. Using the same logic as used in the notes (reproduce in a written exam)

– The ultimate width of dividing streamlines $= \frac{\Lambda}{U_\infty}$

Ultimate width of wing $=0.3 = \frac{\Lambda}{U_\infty} \qquad \rightarrow \qquad \Lambda = 0.3U_\infty$

Subbing back into stagnation location $x_s = -\frac{0.3U_\infty}{2\pi U_\infty} = -\frac{0.3}{2\pi} = -0.0477$

Location of wing "nose" is $x_n = -\frac{0.3}{2\pi} = -0.048m$

a) The velocity sensor is mounted on the centreline and so $y = 0$ and $v = 0$, the velocity is therefore

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2} = U_\infty + \frac{\Lambda}{2\pi x} = U_\infty + \frac{0.3U_\infty}{2\pi x}$$

Now the actual velocity is U_∞ and the percentage relative error is

$$e = \frac{U_\infty - u}{U_\infty} \times 100 = \frac{0.3}{2\pi x} \times 100$$

For a 1% error $\quad 1 = \frac{30}{2\pi x} \qquad \rightarrow \qquad x = \mathbf{4.775m}$

For a 10% error $\quad 10 = \frac{30}{2\pi} \qquad \rightarrow \qquad x = \mathbf{0.477m}$

b) The question tells us the position of the surface directly above the source (half the ultimate height) $y = 0.075m$. However, we could have worked out the height at this point using the same logic we used for finding the ultimate width. Along the y-axis the only velocity in the x-direction is U_∞. Therefore, all the mass/volume coming from the front half of the source must pass between the dividing stream lines at a speed of U_∞ in the x-direction. So, half the mass/volume at the same speed means half the width.
The velocity sensor is mounted on the y-axis so $x = 0$, the velocity is therefore

$$u = U_\infty \qquad v = \frac{\Lambda}{2\pi y}$$

The speed is therefore $V = \sqrt{u^2+v^2} = \sqrt{U_\infty^2 + \left(\frac{0.3U_\infty}{2\pi y}\right)^2} = U_\infty\sqrt{1+\left(\frac{0.3}{2\pi y}\right)^2}$

and the percentage relative error is

$$e = \frac{U_\infty - V}{U_\infty} \times 100 = \left(1 - \sqrt{1+\left(\frac{0.3}{2\pi}\right)^2\frac{1}{y^2}}\right) \times 100$$

For a 1% error $\quad \pm 1 = \left(1 - \sqrt{1+\left(\frac{0.3}{2\pi}\right)^2\frac{1}{y^2}}\right) \times 100 \qquad \rightarrow \qquad \sqrt{1+\left(\frac{0.3}{2\pi}\right)^2\frac{1}{y^2}} = 1 \pm 0.01$

Looking at terms inside the root

$$\sqrt{1+\left(\frac{0.3}{2\pi}\right)^2\frac{1}{y^2}} = 1.01 \qquad \rightarrow \qquad \left(\frac{0.3}{2\pi}\right)^2\frac{1}{y^2} = 0.0201 \qquad \rightarrow \qquad y = \frac{0.3}{2\pi}\sqrt{\frac{1}{0.0201}} = 0.337m$$

Height above surface is therefore $\qquad y = 0.337 - 0.075 = \mathbf{0.262m}$

c) From Bernoulli's equation, the static pressure at the nose is equal to the total pressure so

$$p_0 = p_\infty + \frac{\rho}{2}U_\infty^2$$

At the surface, directly above the origin $(x = 0, y = 0.075)$ we have, from Bernoulli, the static pressure as

$$p_s = p_\infty + \frac{\rho}{2}U_\infty^2 - \frac{\rho}{2}V^2 = p_\infty + \frac{\rho}{2}(U_\infty^2 - u^2 - v^2) = p_\infty - \frac{\rho}{2}v^2 = p_\infty - \frac{\rho}{2}U_\infty^2\left(\frac{0.3}{2\pi y}\right)^2$$

$$p_s = p_\infty - \frac{\rho}{2}U_\infty^2\frac{4}{\pi^2}$$

The probe needs

$$\alpha(p_0 - p_s) = \frac{\rho}{2}U_\infty^2$$

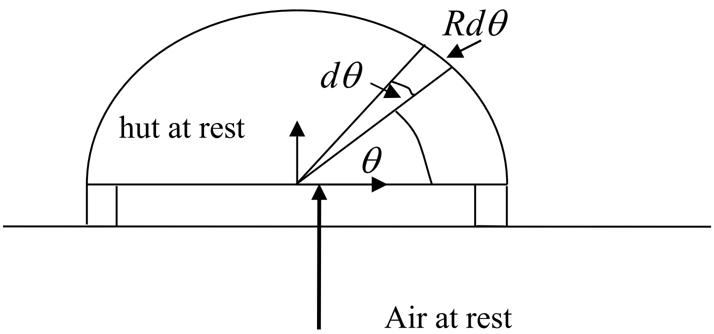
So

$$\alpha(p_0 - p_s) = \alpha\left(\frac{\rho}{2}U_\infty^2 + \frac{\rho}{2}U_\infty^2\frac{4}{\pi^2}\right) = \alpha\frac{\rho}{2}U_\infty^2\left(1+\frac{4}{\pi^2}\right) = \frac{\rho}{2}U_\infty^2 \qquad \rightarrow \qquad \alpha = \frac{\pi^2}{4+\pi^2} = \mathbf{0.712}$$

10. (a) To find the net lift on the hut we need the lift forces on upper and lower surfaces. Now as the flow is at rest under the hut, from Bernoulli's equation the static pressure is equal to the total pressure far upstream. This pressure produces a total upwards force on the lower surface of

$$l_l = 2p_0R$$

Need to find l_u

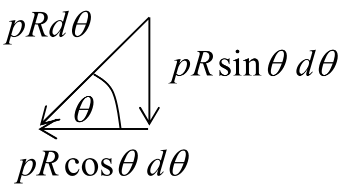


UPPER SURFACE

Consider a small arc of the surface as shown in the sketch of size $Rd\theta$. The force acting on this element is given by

$$pRd\theta$$

this acts normal to the surface and must be resolved to get the components



So the force in the vertical direction over the entire upper surface of the hut is given by

$$l_u - \int_0^\pi p(\theta)R\,sin\,\theta\,d\theta$$

The net lift is therefore given by

$$l = 2p_0R - \int_0^\pi p(\theta)R\,sin\,\theta\,d\theta$$

(b) Now using

$$p(\theta) = p_\infty + \frac{1}{2}\rho U_\infty^2(1 - 4\,sin^2\,\theta)$$

and the fact that from Bernoulli's equation

$$p_0 = p_\infty + \frac{1}{2}\rho U_\infty^2$$

We find that

$$l = 2p_\infty R + \rho U_\infty^2 R - \int_0^\pi [p_\infty + \frac{1}{2}\rho U_\infty^2(1 - 4\,sin^2\,\theta)]R\,sin\,\theta\,d\theta$$

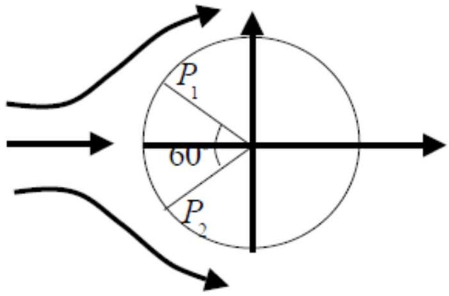
$$= 2p_\infty R + \rho U_\infty^2 R - R\int_0^\pi [(p_\infty + \frac{1}{2}\rho U_\infty^2)\, sin\,\theta - 2\rho U_\infty^2\, sin^3\,\theta)]d\theta$$
$$= 2p_\infty R + \rho U_\infty^2 R - R(p_\infty + \frac{1}{2}\rho U_\infty^2)\int_0^\pi sin\,\theta\, d\theta + 2\rho U_\infty^2 R \int_0^\pi sin^3\,\theta\, d\theta$$
$$= 2p_\infty R + \rho U_\infty^2 R - 2R(p_\infty + \frac{1}{2}\rho U_\infty^2) + 2\rho U_\infty^2 R \int_0^\pi sin^3\,\theta\, d\theta$$
$$= 2\rho U_\infty^2 R \int_0^\pi sin^3\,\theta\, d\theta = 2\rho U_\infty^2 R \int_0^\pi (1 - cos^2\,\theta)\, sin\,\theta\, d\theta$$
$$= 2\rho U_\infty^2 R \left[- cos\,\theta + \frac{1}{3} cos^3\,\theta \right]_0^\pi$$
$$= 2\rho U_\infty^2 R \frac{4}{3}$$

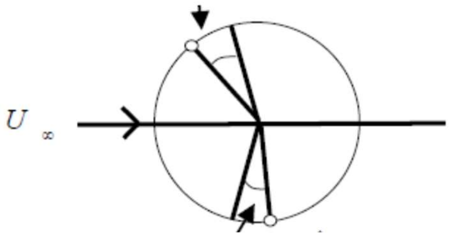
So finally, the net lift force acting on the hut is equal to

$$l = \frac{8}{3}\rho_\infty U_\infty^2 R$$

11. Following the notes, (where we have derived things using polar coordinates) we have the flow around a cylinder and we derive

$$c_p = \left(1 - \frac{V^2}{U_\infty^2}\right) = 1 - 4\sin^2\theta$$





Method 1

If we consider that the sensors are placed at $\pm\theta$, and the cylinder is rotated by a small angle β, then the difference in c_p between the two sensors will be

$$\Delta c_p = c_{p1} - c_{p2} = 4(-\sin^2(\theta + \beta) + \sin^2(\theta - \beta))$$

$$\Delta c_p = 4(-(\sin\theta\cos\beta + \cos\theta\sin\beta)^2 + (\sin\theta\cos\beta - \cos\theta\sin\beta)^2)$$
$$\Delta c_p = -16\sin\theta\cos\theta\cos\beta\sin\beta$$

In the limit of infintesimally small β then the sensitivity scaled by the freestream dynamic pressure, k, is

$$k = \lim_{\beta\to 0}\frac{\Delta c_p}{\beta} = -16\sin\theta\cos\theta \times \lim_{\beta\to 0}\left(\cos\beta\frac{\sin\beta}{\beta}\right) = -16\sin\theta\cos\theta\left(1\frac{\beta}{\beta}\right) = -16\sin\theta\cos\theta$$

We can find the same answer by directly differentiating

Method 2

For a small rotation angle β, then the rise in c_p in one sensor can be approximated by

$$\Delta c_{p1} = \frac{dc_p}{d\theta}\beta$$

By symmetry, the c_p in the other sensor will fall by the same amount. Therefore, again taking the limit of infintesimally small β

$$k = 2\frac{dc_p}{d\theta} = 2\frac{d(1 - 4\sin^2\theta)}{d\theta} = -16\sin\theta\cos\theta$$

The angle of 60° given in the question translates into $\theta = 150^o$, hence

$$k = 6.928$$

This is the sensitivity per radian, therefore the sensitivity per degree is

$$k_{deg} = 6.928 * \frac{\pi}{180} = \mathbf{0.1209}$$

For maximum sensitivity we need to find the θ where k is a maximum. So

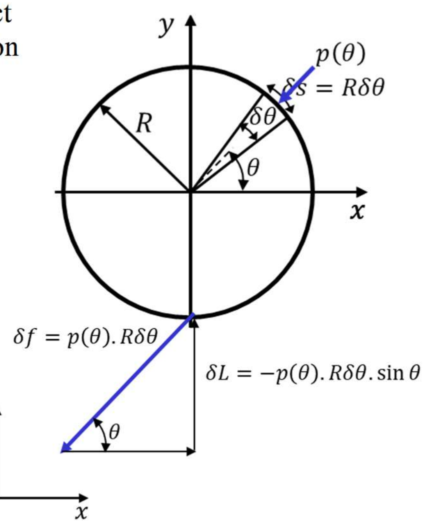
$$k_{max} \quad \rightarrow \quad \frac{dk}{d\theta} = 0 \quad \rightarrow \quad -16(-\sin^2\theta + \cos^2\theta) = 0 \qquad \rightarrow \qquad \sin\theta = \pm\cos\theta$$

$k = k_{max}$ when $\theta = 135^o,\ 45^o$

Take the value when $\theta = 135^o$ as this places the static tappings in the front of the cylinder. The separation of the two tappings for maximum sensitivity is therefore $\mathbf{90^o}$, and the sensitivity is

$$k_{deg} = -16\sin 135^o \cos 135^o * \frac{\pi}{180} = \mathbf{0.1396}$$

12. Although the net force on a cylinder is zero, this question examines the fact that this net zero force in the cross-stream direction is due to the cancelation of cross stream forces on each semi-cylinder.



From notes, we have that the c_p is given by

$$c_p = \left(1 - \frac{V^2}{U_\infty^2}\right) = 1 - 4\sin^2\theta$$

The force on the top semi-cylinder is given by (just as we did for the spinning cylinder) but we have an internal pressure as well p_i, so we integrate the pressure difference.

$$L_u = -\int_0^\pi (p(\theta) \ - p_i)R\sin\theta\, d\theta$$

$$L_u = -\int_0^\pi (p(\theta) - p_\infty + p_\infty - p_i)R\, sin\,\theta\, d\theta$$

$$L_u = -\int_0^\pi (p(\theta) - p_\infty)R\sin\theta\, d\theta \ - \int_0^\pi (p_\infty - p_i)R\, sin\,\theta\, d\theta$$

Unlike when we integrate around the whole cylinder the second integral is non-zero.

$$L_u = -\tfrac{1}{2}\rho U_\infty^2 R\int_0^\pi c_p \sin\theta\, d\theta - (p_\infty - p_i)R\int_0^\pi sin\,\theta\, d\theta$$

$$L_u = -\tfrac{1}{2}\rho U_\infty^2 R\int_0^\pi (1 - 4\sin^2\theta)\sin\theta\, d\theta - (p_\infty - p_i)R\int_0^\pi sin\,\theta\, d\theta$$

$$L_u = \tfrac{1}{2}\rho U_\infty^2 R\left\{4\int_0^\pi \sin^3\theta d\theta - \int_0^\pi \sin\theta\, d\theta\right\} - (p_\infty - p_i)R\int_0^\pi sin\,\theta\, d\theta$$

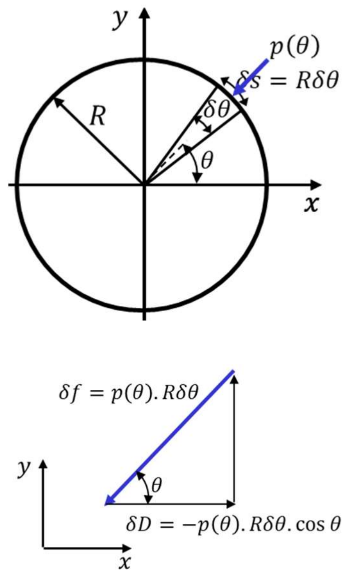
$$L_u = \tfrac{1}{2}\rho U_\infty^2 R\left\{4\left[-\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta\right]_0^\pi + [cos\,\theta]_0^\pi\right\} + (p_\infty - p_i)R[cos\,\theta]_0^\pi$$

$$L_u = \frac{5}{3}\rho U_\infty^2 R - 2(p_\infty - p_i)R$$

$$L_u = \frac{5}{3}1.2 \times 25^2 \times 1 - 2 \times 1.01 \times 10^5(1 - 0.995) \times 1 = 1250 - 1010 = 240N$$

The same force is pulling the lower semi-cylinder down, so the total force per metre equals the force on each bolt equals $\mathbf{480N}$

13. From notes, we have that the c_p is given



$$c_p = \left(1 - \frac{V^2}{U_\infty^2}\right) = 1 - 4\sin^2\theta$$

The drag force on the cylinder is given by (similar to what we did for the spinning cylinder)

$$D = -\int_0^{2\pi} (p(\theta) - p_\infty)R\cos\theta\, d\theta - \int_0^{2\pi} p_\infty R\cos\theta\, d\theta$$

$$D = -\tfrac{1}{2}\rho U_\infty^2 R\int_0^{2\pi} c_p\cos\theta\, d\theta$$

$$D = -\tfrac{1}{2}\rho U_\infty^2 R\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} c_p\cos\theta\, d\theta \ - \tfrac{1}{2}\rho U_\infty^2 R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} c_p\cos\theta\, d\theta$$

Remembering that the c_p is constant for the rear portion of the cylinder i.e.

$c_p\left(\pm\frac{\pi}{2}\right) = c_p\left(-\frac{\pi}{2} \rightarrow \frac{\pi}{2}\right) = -3$

And that otherwise

$$c_p = 1 - 4\sin^2\theta = -3 + 4\cos^2\theta$$

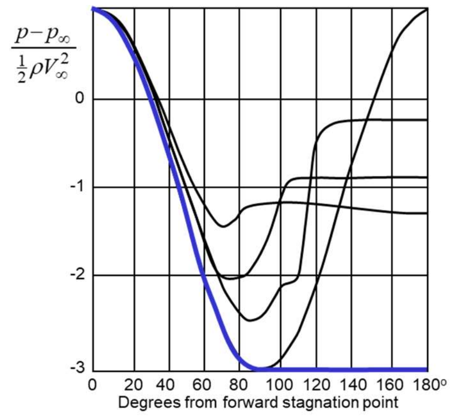
$$D = -\tfrac{1}{2}\rho U_\infty^2 R \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-3 + 4\cos^2\theta)\cos\theta\, d\theta \; + \tfrac{3}{2}\rho U_\infty^2 R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta$$

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 \times 2R} = \frac{1}{2}\left\{-4\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3\theta d\theta + 3\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta\, d\theta + 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta\right\}$$

$$C_D = \frac{1}{2}\left\{-4\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3\theta d\theta + 3\int_0^{2\pi} \cos\theta\, d\theta\right\} = -2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3\theta d\theta$$

$$C_D = -2\left[\frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\sin\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -2\left[-\frac{4}{3}\right] = \mathbf{\frac{8}{3}}$$

If we look at the notes for fluid "Behaviour" in engineering science, we will see that this level of drag coefficient is very high. However, consider the figure to the right. We are approximating the drag coefficient based on a model of the pressure coefficient given by the blue line.



14. (a) The total velocity is made up from a doublet a freestream and a vortex. To calculate the strength of the vortex we will need to know the strength of the doublet. From notes the doublet strength is given by

$$\kappa = 2\pi U_\infty R^2$$

However, we can work out the doublet strength by considering that the strength of the doublet needed to produce a cylinder of a set size in an onset flow is unchanged by the presence of the vortex (spinning). So, we can take the non-spinning cylinder look at the position of the stagnation point at the leading edge and use that to calculate the strength of the doublet in the spinning cylinder case.

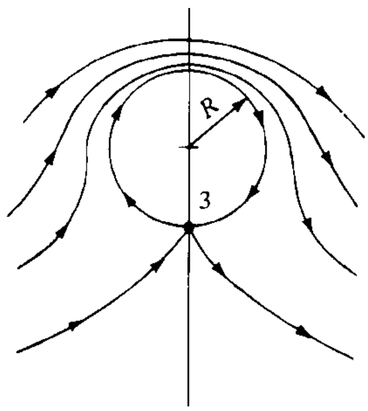
For the non-spinning cylinder, the stagnation point is at $(x = -R, y = 0)$ and the horizontal velocity is given by

$$u = U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{(x^2 + y^2)^2} = U_\infty + \frac{-\kappa}{2\pi}\frac{(R^2 - 0^2)}{(R^2 + 0^2)^2} = U_\infty - \frac{\kappa}{2\pi R^2} = 0$$

Rearranging

$$U_\infty = \frac{\kappa}{2\pi R^2} \quad \rightarrow \quad 2\pi R^2 U_\infty = \kappa$$

At all the points around the surface of the spinning cylinder we know that the radial velocity is zero (i.e. flow is always tangential to the surface). The stagnation point is at $\theta = 270^o$ or $(x = 0, y = -R)$ we know that the vertical component of velocity must be zero at this point so we only need to calculate the horizontal component. The general equation for the horizontal component is given by



$$u = U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{(x^2 + y^2)^2} + \frac{\Gamma}{2\pi}\frac{y}{(x^2 + y^2)}$$

At the stagnation point this becomes

$$u = 4 + \frac{\kappa}{2\pi R^2} - \frac{\Gamma}{2\pi R} = 0$$

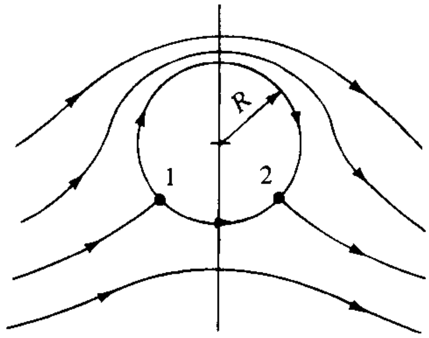
Substituting in for the strength of the doublet we have

$$4 + \frac{2\pi R^2 U_\infty}{2\pi R^2} = \frac{\Gamma}{2\pi R} \quad \rightarrow \quad \Gamma = 8\pi R + 2\pi R U_\infty = 16\pi R$$

Hence lift per unit length is

$$l = \rho U_\infty \Gamma = 1.2 \times 4 \times 16\pi \times 0.5 = 120.64N$$

(b) There are now two stagnation points



At these stagnation points we still have $u = v = 0$ and we also know that we must be on the surface of the cylinder so

$$x^2 + y^2 = R^2$$

The velocity at the stagnation points is therefore

$$u = U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{(x^2 + y^2)^2} + \frac{\Gamma}{2\pi}\frac{y}{(x^2 + y^2)}$$

$$= U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{R^4} + \frac{\Gamma}{2\pi}\frac{y}{R^2} = 0$$

$$v = \frac{-\kappa}{2\pi}\frac{2xy}{(x^2 + y^2)^2} + \frac{-\Gamma}{2\pi}\frac{x}{(x^2 + y^2)} = -\frac{\kappa}{2\pi}\frac{2xy}{R^4} - \frac{\Gamma}{2\pi}\frac{x}{R^2} = 0$$

We only need the x or the y location of the stagnation points to find the angle. So, from $v = 0$ we have

$$-\frac{\kappa}{2\pi}\frac{2xy}{R^4} = \frac{\Gamma}{2\pi}\frac{x}{R^2} \quad \rightarrow \quad -y = \frac{\Gamma}{\kappa}\frac{R^2}{2}$$

Substituting for the doublet strength, $\kappa = 2\pi U_\infty R^2$, and vortex strength, $\Gamma = l/\rho U_\infty$, gives

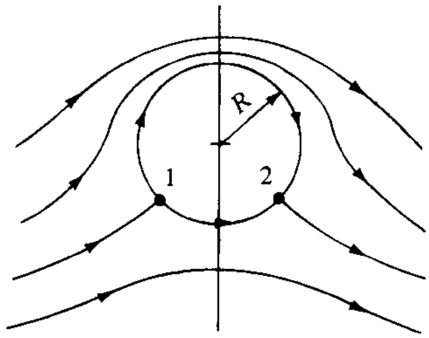
$$-y = \frac{l}{2\pi\rho U_\infty^2 R^2}\frac{R^2}{2} = \frac{l}{4\pi\rho U_\infty^2}$$

Measuring the angle anticlockwise from the x-axis

$$y = R\sin(-\theta) \quad \rightarrow \quad \sin(\theta) = -\frac{y}{R} = \frac{l}{4\pi R\rho U_\infty^2}$$

$$\theta = \sin^{-1}\left(\frac{l}{4\pi R\rho U_\infty^2}\right) = \sin^{-1}\left(\frac{19.2\pi}{4\pi \times 0.5 \times 1.2 \times 16}\right) = \sin^{-1}\left(\frac{19.2\pi}{38.4\pi}\right) = \sin^{-1}(0.5) = 30^o$$

15. We first need to find the strength of the doublet and vortex. As previously



$$\kappa = 2\pi U_\infty R^2$$

At the stagnation points

$$x = R\cos\theta, y = R\sin\theta, \quad \theta = -30^o, 210^o$$

Velocity at rear stagnation point

$$u = U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{(x^2 + y^2)^2} + \frac{\Gamma}{2\pi}\frac{y}{(x^2 + y^2)}$$

$$= U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{R^4} + \frac{\Gamma}{2\pi}\frac{y}{R^2} = 0$$

$$v = \frac{-\kappa}{2\pi}\frac{2xy}{(x^2 + y^2)^2} + \frac{-\Gamma}{2\pi}\frac{x}{(x^2 + y^2)} = -\frac{\kappa}{2\pi}\frac{2xy}{R^4} - \frac{\Gamma}{2\pi}\frac{x}{R^2} = 0$$

Using the definition of the doublet strength

$$\Gamma = -2\pi U_\infty 2y$$

using $\quad \sin(-30^o) = -0.5$

$$\Gamma = 2\pi R U_\infty$$

At the top and bottom of the cylinder there is no vertical velocity, so substituting into the u velocity equation

$$u = U_\infty + \frac{-\kappa}{2\pi}\frac{(x^2 - y^2)}{(x^2 + y^2)^2} + \frac{\Gamma}{2\pi}\frac{y}{(x^2 + y^2)} = U_\infty - U_\infty R^2\frac{(x^2 - y^2)}{(x^2 + y^2)^2} + RU_\infty\frac{y}{(x^2 + y^2)}$$

Evaluating at the top and bottom of the cylinder, where $x = 0, y = \pm R$

$$\mathrm{u}_t = U_\infty + U_\infty R^2 \frac{R^2}{R^4} + RU_\infty \frac{R}{R^2} = \mathbf{3}\boldsymbol{U}_\infty$$

$$\mathrm{u}_b = U_\infty + U_\infty R^2 \frac{R^2}{R^4} - RU_\infty \frac{R}{R^2} = \boldsymbol{U}_\infty$$

Applying Bernoulli to the pressure coefficient definition gives

$$c_p = 1 - \frac{V^2}{U_\infty^2} \qquad \rightarrow \qquad c_{pt} = 1 - \frac{u_t^2}{U_\infty^2} = \mathbf{8} \quad \& \quad c_{pb} = 1 - \frac{u_b^2}{U_\infty^2} = \mathbf{0}$$

Applying Kutta Jukowski we have

$$L = \rho U_\infty \Gamma$$

But the lift coefficient is based on a projected area (per unit span) of $2R$, therefore

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 \times A} = \frac{\rho U_\infty \Gamma}{\frac{1}{2}\rho U_\infty^2 \times 2R} = \mathbf{2}\boldsymbol{\pi}$$