A Macro Disequilibrium Model

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1. Introduction

What follows is a simplified version of the macroeconomic model developed in the paper "'The total movement of this disorder is its order': Investment and utilization dynamics in long-run disequilibrium", published in *Metroeconomica*. In the paper itself, the model is explained in much more detail, various results are proved formally, rationales are given for the assumptions, and different possibilities are explored. The relevant theoretical literature, as well as various empirical studies that provide support for the model, are also discussed at length. Readers interested in those issues should read the paper, which can be found at https://onlinelibrary.wiley.com/doi/10.1111/meca.12377.

Here, the model is simplified and the presentation is very brief. There are only two sectors: a private sector (which consists of businesses, business owners, and workers), and a public sector. The private sector purchases consumption goods and fixed capital equipment, which are both produced by the private sector itself using capital and labor. The public sector also makes purchases from the private sector. Firms adjust capacity utilization to meet changes in demand, and prices are fixed. Moreover, all commodities (whether consumption goods or capital equipment) are assumed to be produced with the same technology. Thus all commodities are effectively identical, and I normalize the price to one so that real and nominal quantities are interchangeable.

In what follows, I also use the following notational conventions. If x is any variable depending on time, then $\dot{x}$ denotes the time derivative, $\hat{x}$ denotes the growth rate, and $\bar{x}$ denotes the long-run average value (starting at time $t = 0$). Thus

$$\dot{x} = \frac{dx}{dt}, \qquad \hat{x} = \frac{\dot{x}}{x}, \qquad \text{and} \qquad \bar{x} = \lim_{t\to\infty} \frac{1}{t}\int_0^t x[s]ds\,.$$

2. Basic equations

Aggregate demand consists of private consumption C, gross fixed-capital investment I, and government spending A:

$$Y = C + I + A. \tag{1}$$

Output, Q, adjusts to meet changes in demand:

$$Q = Y. \tag{2}$$

The private sector's net financial wealth, M, evolves according to the equation

$$\dot{M} = Y + iM - C - I - T \tag{3}$$

where T denotes aggregate tax revenue and i is the interest rate. The public sector's debt, D, evolves according to

$$\dot{D} = A + iD - T. \tag{4}$$

I assume that there is a fixed tax rate τ on all private income, so total tax revenue is:

$$T = \tau Q + \tau i M. \tag{5}$$

3. Investment decisions

The rate of change in the capital stock, K, is

$$\dot{K} = I - \delta K. \tag{6}$$

where I is gross investment and δ is the (positive and constant) depreciation rate. Capacity utilization is

$$u = \frac{Q}{\rho K} \tag{7}$$

where ρ is a parameter describing the output-capital ratio when productive capacity is fully utilized. Total profits are

$$R = (1-\tau)(1-\psi)Q, \tag{8}$$

where ψ is a parameter (strictly between zero and one) that represents unit labor costs. The rate of profit on fixed capital is

$$r = \frac{R}{K} = (1-\tau)(1-\psi)\rho u. \tag{9}$$

Finally, g denotes the gross accumulation rate, and we have

$$I = gK. \tag{10}$$

For the investment function, I will adopt the basic form

$$g = b + \theta r. \tag{11}$$

Here, θ is a positive constant strictly less than one, which quantifies the way current profitability influences the rate of accumulation. I also assume that the investment function tends to shift in response to changes in u. This means b exhibits a tendency to rise when capacity utilization is above its target u^d, and fall when capacity utilization is below u^d, reflecting firms' efforts to keep capacity utilization at the desired rate.

There is an upper limit to how high b can rise in this model. To understand why, we need to keep in mind a few important points. First, investment decisions made at a particular point in time will often entail a long-run commitment; if capitalists decide to increase the rate of accumulation, they will not necessarily be able to quickly reverse that decision later if there is a sudden fall in profitability. Second, businesses are constantly adjusting their investment plans in an effort to keep u near u^d, and, to the extent that these efforts are successful, businesses can reasonably expect that the profit rate will fluctuate around the value $(1-\tau)(1-\psi)u^d$ in the long run. Finally, businesses will experience a financial deficit if g is too high in relation to the profit rate. For these reasons, if b is too high for too long in relation to $(1-\tau)(1-\psi)u^d$, businesses put themselves at risk of financial difficulties.

Therefore, there is a ceiling, B, for the possible values of b. To ensure that gross investment can never become negative in the model, I also assume that b cannot go below zero. To formalize these ideas, I adopt the equation

$$\dot{b} = \phi[b](u - u^d), \tag{12}$$

where

$$\phi[b] = \eta b(B - b). \tag{13}$$

The parameter η is positive and describes the speed with which capitalists react to changes in u. Thus b fluctuates in a Harrodian fashion, but equation (15) implies that its rate of change will slow down if it approaches either its upper limit ($b = B$) or lower limit ($b = 0$).

4. Consumption decisions

The variable

$$m = \frac{M}{K}, \tag{14}$$

measures the private sector's net financial wealth in relation to the size of the economy. I assume that consumption expenditures tend to grow when $M > \sigma Q$, and decay when $M < \sigma Q$, where σ is a positive constant corresponding to the idea of a *stock-flow norm*. Scaling M and Q to the size of the economy, we obtain the equation

$$\hat{C} = \xi(m - \sigma\rho u) \tag{15}$$

For the growth rate $\hat{C}$ of private consumption expenditures.

5. Government spending

Government spending in this model can fluctuate in complex, and potentially destabilizing, ways. For example, if policymakers target a particular ratio of government debt to potential GDP (increasing spending when the ratio is lower than the target and decreasing spending if the ratio is above the target), then this can be destabilizing. On the other hand, policymakers could also employ counter-cyclical (stabilizing) policy rules. This model is developed to be general enough to cover these different scenarios as special cases. Rather than specifying the exact path of government spending, I assume A (aggregate government spending at a given time) satisfies

$$\hat{A} = \Gamma \tag{16}$$

where Γ is a differentiable function of time with a bounded derivative. To make concrete the idea that A fluctuates around the long-run trend rate of growth γ, I also assume there are positive constants κ_1 and κ_2 such that

$$\kappa_1 e^{\gamma t} < A[t] < \kappa_2 e^{\gamma t} \tag{17}$$

for all t.

6. Parameter values and initial conditions

I will assume that the long-run average growth rate of government spending, γ, is higher than the after-tax interest rate:

$$(1-\tau)i < \gamma. \tag{18}$$

We will see below that the long-run trend rate of output growth for the model is γ. In this sense, the above condition is similar to the usual assumption that the (after-tax) interest rate does not exceed the growth rate of the economy.

Second, we need to make an assumption regarding the parameter B. The reader should recall that this parameter represents the upper limit for the term b in the accumulation function (11), and reflects the existence of financing constraints on investment. I will assume that

$$B < \big(\tau + (1-\theta)(1-\tau)(1-\psi)\big)\rho u^d. \tag{19}$$

Note that when capacity utilization is at the rate targeted by businesses, this means $u = u^d$, and in that case the condition (19) is equivalent to the assumption that $g < \tau\rho u^d + (1-\tau)(1-\psi)\rho u^d$. Since $(1-\tau)(1-\psi)\rho u^d$ is the profit rate when capacity is utilized at its normal rate, this means that the financing constraint on investment may be relatively lax; the inequality (19) can be satisfied even for solution trajectories in which investment perpetually exceeds profits. The assumption (19) simply imposes an upper limit on the *degree to which* investment can exceed profits in the long run. Although the financing constraint on investment could be modeled in different ways, I will adopt (19) as a reasonable starting point.

Third, we need the initial conditions to be chosen so that they are consistent with the economic interpretation of the model. Since the model describes a closed system with two sectors (the private sector and the government), I will assume that initially the net financial claims held by those two sectors sum to zero, which means

$$M[0] = D[0]. \tag{20}$$

If (20) did not hold, it would imply the existence of a financial relationship with some entity outside the model, thus violating the assumption that the equations above describe a closed system. I also assume that the initial values $C[0]$ (private consumption) and $K[0]$ (the capital stock) are positive. I also assume $b[0]$ is strictly between its limiting values 0 and B.

Finally, we need to assume that that the investment function (11) is able to sustain growth at the rate γ when productive capacity is utilized at the normal rate. This means:

$$b^* + \theta(1-\tau)(1-\psi)\rho u^d = \gamma + \delta, \qquad \text{for some } b^* \text{ strictly between 0 and } B. \tag{21}$$

Note that the inequality (19) imposes an upper limit on the possible values for b^*.

7. Theorems

This model is a somewhat simplified version of the one developed in my paper, but the proofs in the paper establish the following facts:

Theorem 1. *Given a set of parameters and initial conditions meeting the assumptions above, the model will have a unique solution. This solution exists for all time.*

Theorem 2. *The long-run trend rates of output growth and capital accumulation are equal to the trend growth rate of government spending, γ. In other words, there are positive constants π_1, π_2, π_3 and π_4 such that, for all $t \geq 0$,*

$$\pi_1 e^{\gamma t} \leq K[t] \leq \pi_2 e^{\gamma t} \tag{22}$$

and

$$\pi_3 e^{\gamma t} \leq Q[t] \leq \pi_4 e^{\gamma t}. \tag{23}$$

Moreover, the long-run average values for Q and K will both be γ.

Theorem 3: *The long-run average rate of capacity utilization is $\overline{u} = u^d$, and the long-run average rate of profit is $\overline{r} = (1-\tau)(1-\psi)u^d$.*

8. The simulations

To illustrate some of the possibilities for this model, I have created simulations in which Γ, the growth rate of government spending, is a stochastic process. The graphs use smooth interpolation to display the solutions, in solving the model I actually implement Γ

as a piecewise continuous function. This is simply done for the purposes of illustration; some more realistic fiscal policy rules are analyzed in my paper.

To solve the model, I reduce it to a system of four nonlinear differential equations, and then apply the Runge-Kutta method. The system can be written as follows:

$$\dot{a} = a(\Gamma - g + \delta) \tag{24}$$
$$\dot{b} = \phi[b](u - u^d) \tag{25}$$
$$\dot{c} = c(\xi(m - \sigma\rho u) - g + \delta) \tag{26}$$
$$\dot{m} = (1 - \tau)(\rho u + im) - c - g - m(g - \delta) \tag{27}$$

where

$$a = \frac{A}{K} \tag{28}$$

and

$$c = \frac{C}{K}. \tag{29}$$

We also can rewrite u (the rate of capacity utilization) and r (the rate of profit) as follows:

$$u = \Lambda(a + b + c) \tag{30}$$
$$\Lambda = \frac{1}{\rho(1 - \theta(1 - \tau)(1 - \psi))} \tag{31}$$
$$r = (1 - \tau)(1 - \psi)\rho u \tag{32}$$

Note that in equation (25), the expression $\phi[b] = \eta b(B - b)$ regulates how quickly investment reacts to changes in capacity utilization. Given the parameter values η and B, the maximum possible value for $\phi[b]$ is $\eta B^2/2$. I call this value the *scaled adjustment speed* for investment. The *adjustment speed* for consumption is the parameter ξ. The *target financial assets/income ratio* is σ. The *target capacity utilization rate* is u^d. The default parameter values I use for the model are $\tau = 0.3$, $\theta = 0.6$, $\psi = 0.5$, $i = 0.01$, $u^d = 0.7$, $\delta = 0.06$, $\eta = 1000$, $B = 0.04$, $\xi = 25$, $\sigma = 0.4$.

The GitHub repo for my code is available here:
https://github.com/SThompsonChicago/macro-disequilibrium

The deployed application can be found here:
https://sthompsonchicago.github.io/macro-disequilibrium/

Please feel free to send any questions/comments to me at drsgthompson@gmail.com.