An Introduction To Bayesian Statistics

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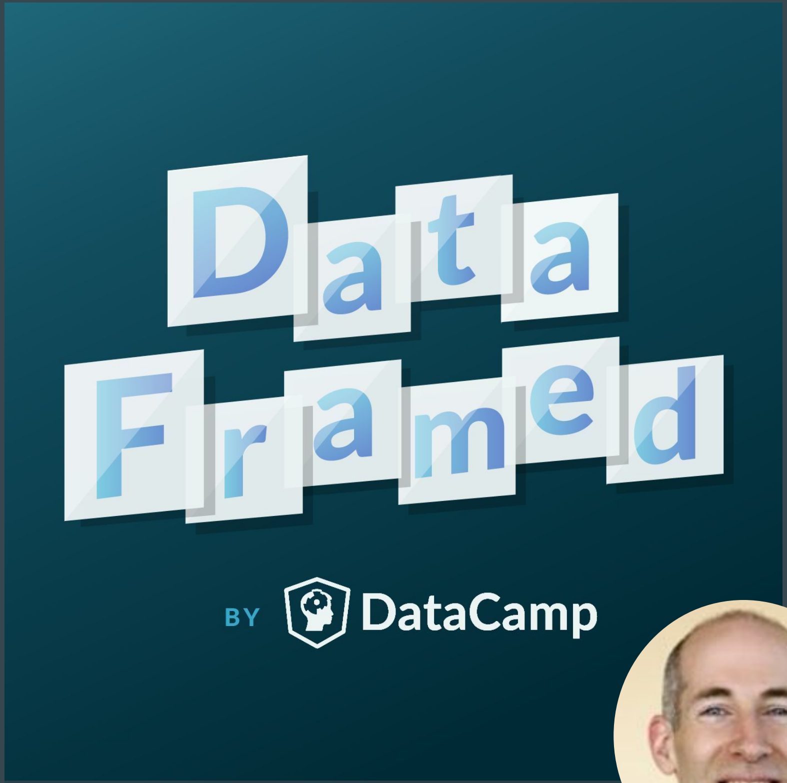
“[Bayesian Stats.] is one of those ideas that seems hard when you first encounter it. Then, at some point there is a breakthrough and then it seems obvious.

Once you’ve got it, it’s such a beautiful idea that it changes how you see everything.”

- Prof. Allan Downey,
  Data Framed, 2018



How do we estimate probability?

Quick Revision

- Classical: By considering equal outcomes
- Frequentist: Relative Frequency over time
- Bayesian: By quantifying our uncertainties

Coin Toss: Classical Est.

```
    Coin
     |
  -------
  |     |
  H     T

0.5     0.5
```

Dice: Classical Est.

```
                 Dice
                  |
   ------------------------------
   |     |     |     |     |     |
   1     2     3     4     5     6

  0.16  0.16  0.16  0.16  0.16  0.16
```

Classical Stats

- Requirements
  - All Outcomes are known
  - Outcomes are assumed to be equally likely

- Advantages
  - Fast Estimation
  - Easy to understand

- Disadvantages
  - Outcomes must be known
  - Even if outcomes are finite, doing the combinatorics can still be relatively difficult
  - Often created overly simplified models when applied to complex phenomena

How do we estimate probability?

- ~~Classical~~
- Frequentist
- Bayesian

Frequentist Est.

- Check to see if thermometer is properly calibrated

Frequentist Approach:

Take many readings and use the expectation value (mean) and std for sample

Calculate the probability of your data given your data following some parameter.

Frequentist Stats

- Requirements
  - Possibility to perform experiments indefinitely
  - Parameters are assumed to be fixed
  - Able to estimate params given enough experiments

- Advantages
  - Works well with simulations
  - “Objective”

- Disadvantages
  - Requires large sample size to be meaningful
  - Does not allow for integration of domain knowledge
  - P-values and confidence intervals are unintuitive
  - Difficult to communicate to non-statisticians

Thermometer Calibration: Test

1. If P-value = 0.001 (highly significant), is the probability of getting this result or a more extreme one given our data 0.001? ❌

2. For a given confidence interval, does the parameter lie within it 95% of the time? ❌

Thermometer Calibration: Test

1. If P-value = 0.001 (highly significant), is the probability of getting this result or a more extreme one given our data 0.001?

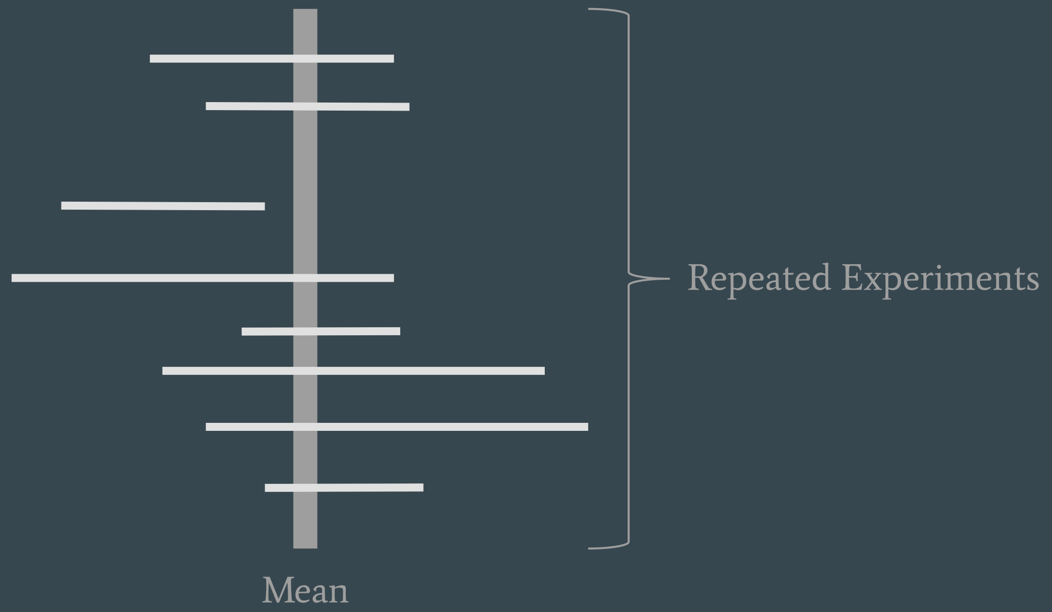
   Prob. of this result or a more extreme one given a certain mean/std (i.e. Means of two groups are the same, H_0)

2. For a given confidence interval, does the parameter lie within it 95% of the time?

   Intervals of repeated experiments will contain the parameter 95% of the time

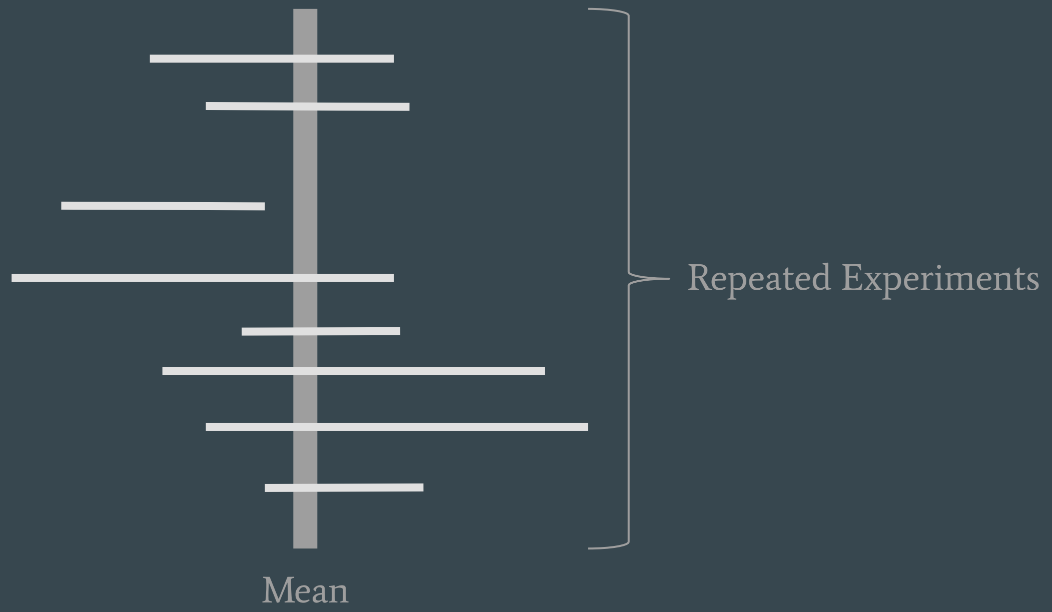
Thermometer Calibration: Test Learnings

Frequentism expects that parameters exist and are fixed, the probabilities are the likelihood of our data given these expectations

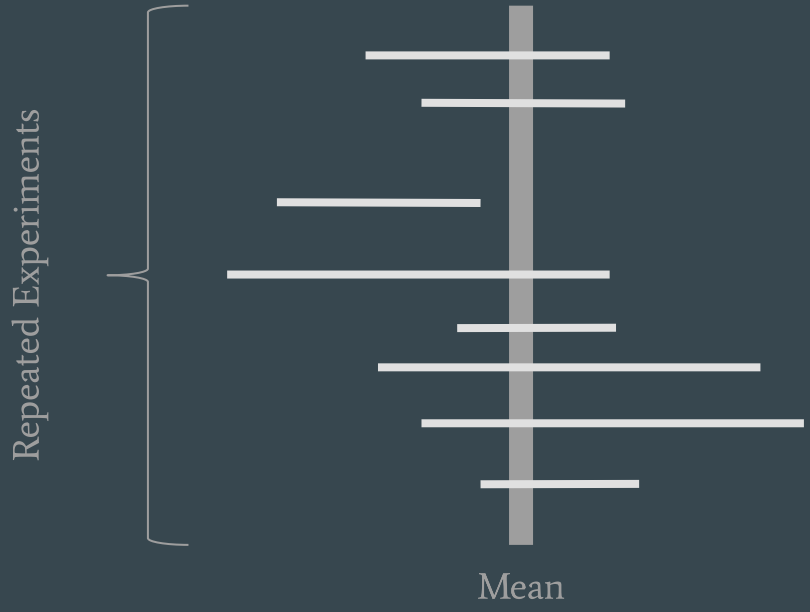


Thermometer Calibration: Test Learnings

Frequentism expects that parameters exist and are fixed, the probabilities are the likelihood of our data given these expectations



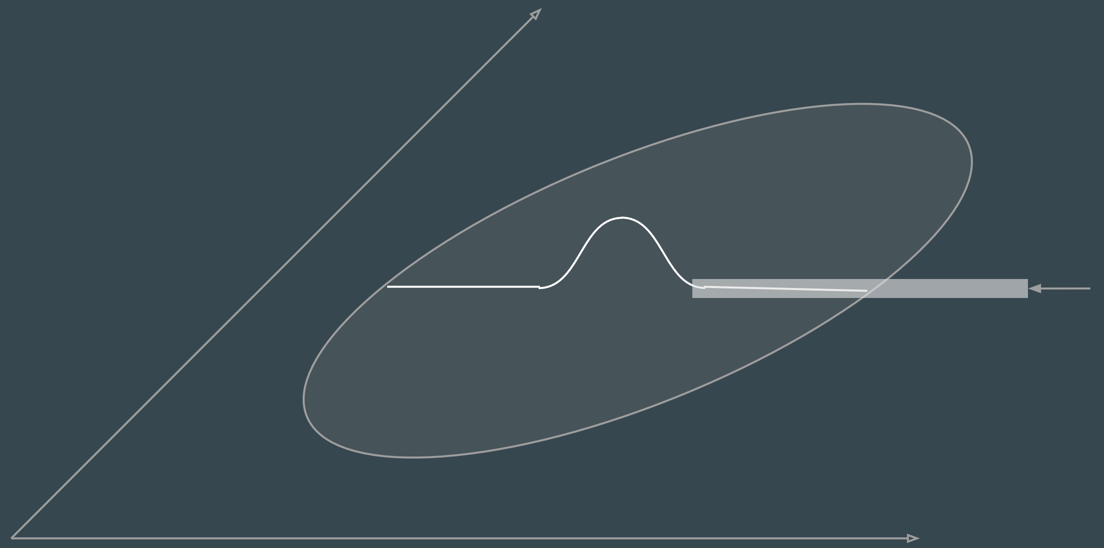
“Wait, wasn’t this was we were doing with frequentism?”



Given a region
Our parameter will be more or less likely

Example from J. Vanderplas’s 2014 Scipy Talk

“Wait, wasn’t this was we were doing with frequentism?”



P-val:
Probability of seeing result
At least this extreme given
Null-hypothesis (i.e. specific
param)

Thermometer Calibration: Test Learnings

Child doesn’t move, your repeated photos contain them 95% of the time

Frequentist Stats Disav. Cont.

What if?

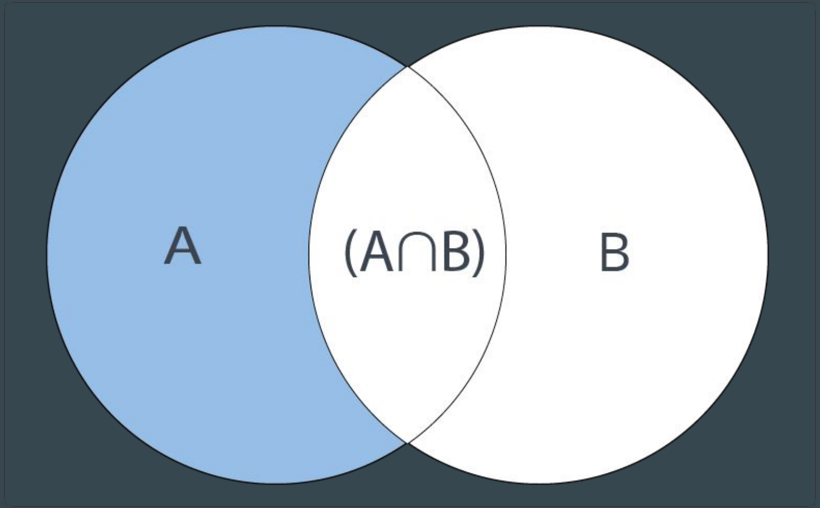
- Amount of data you have is limited? ✓
- You have relevant and applicable prior information ✓
- “Infinite” experiments are not possible? (Cost, feasibility) ✓
- Stakeholders have a hard time understanding frequentist logic? ✓
- Children never stay still and assuming they do is blasphemy ✓

How do we estimate probability?

- ~~Classical~~
- ~~Frequentist~~
- Bayesian

Bayes Theorem: Derivation

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Bayes Theorem: Derivation

The Same

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\therefore P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes Theorem: Alternate View

θ = Parameter,

X = Data

posterior, likelihood, prior

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)}$$

normalisation

Bayes Theorem: Alternate View

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)}$$

Problem:

- How to calculate p(X)
- How to calculate p(θ)

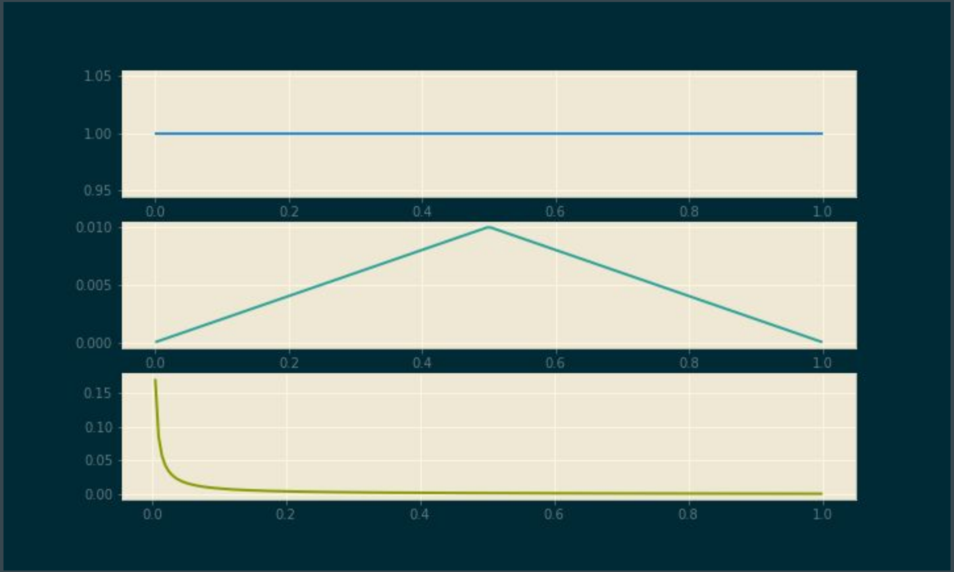
Bayes Theorem: How to Calculate P(θ)?

1. Create Your Own
2. Take Previous posterior

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)}$$

Bayes Theorem: How to Calculate P(θ)?

1. Create Your Own
2. Take Previous posterior



Bayes Theorem: How to Calculate P(θ)?

Problem: Are you Baking your biases into your model?

Bayes Theorem: How to Calculate P(θ)?

Might as well have your explicit and tangible biases.

As the sample size increases, priors get washed out. (As long as you are "reasonable")

- Low Sample Size: Frequentist Stats is börked anyway, so why not?
- High Sample Size: Prior Doesn't matter
- "Reasonable" = Cromwell's Rule

Bayes Theorem: How to Calculate P(X)?

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)}$$

1. Sum over all possible hypotheses
2. This is the hard part
3. Can be ignored if comparing inside of the same distribution

$$\boxed{\begin{aligned} p(X) &= \sum_{i=0}^{n} p(X|\theta_i)p(\theta_i) \\ p(X) &= \int p(X|\theta)p(\theta)d\theta \end{aligned}}$$

How!?: Doctor's Diagnosis

- Doctor's Diagnosis:
  - You are suspicious you have a rare disease
  - Disease affects 0.1% of the population
  - Test is 99% accurate
  - What is the probability you have this disease given that you tested positive?

How!?: Doctor's Diagnosis

$$p(d \mid pos.) = \frac{p(pos. \mid d)p(d)}{p(pos.)}$$

$$p(d \mid pos.) = \frac{p(pos. \mid d)p(d)}{p(pos. \mid d)p(d) + p(pos. \mid \neg d)p(\neg d))}$$

$$p(d \mid pos.) = \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.01 \times 0.999}$$

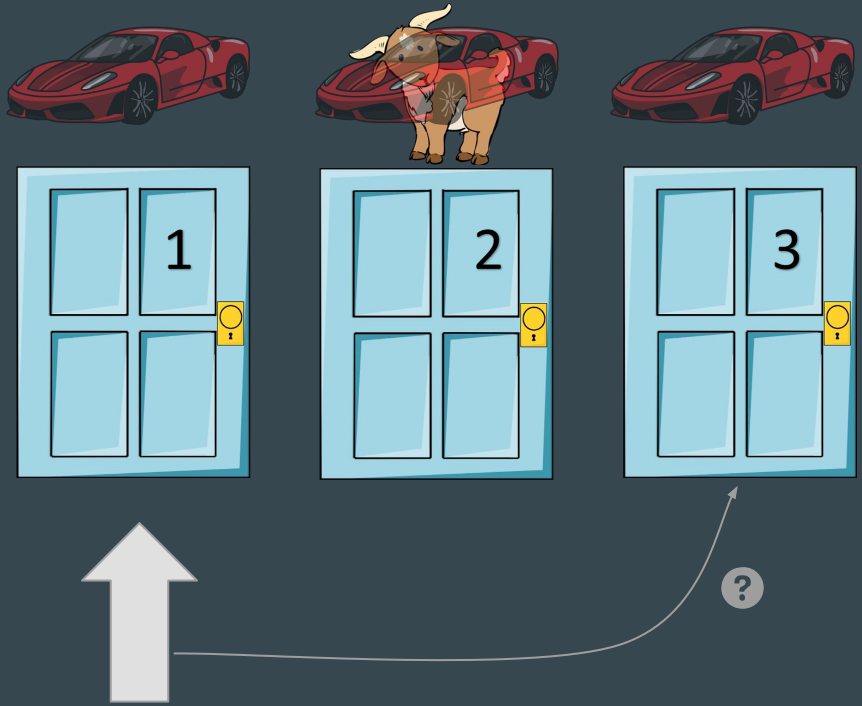
($9.9\ 10^{-4}$) ($9.99\ 10^{-3}$)

How!?: Doctor's Diagnosis

- Doctor's Diagnosis:
  - Works out to be roughly 9% chance of having disease
  - Rate is lower than expected because disease is so rare
  - 99% is a high accuracy, but not *that* high
  - Application: Think about this example next time you build a classifier

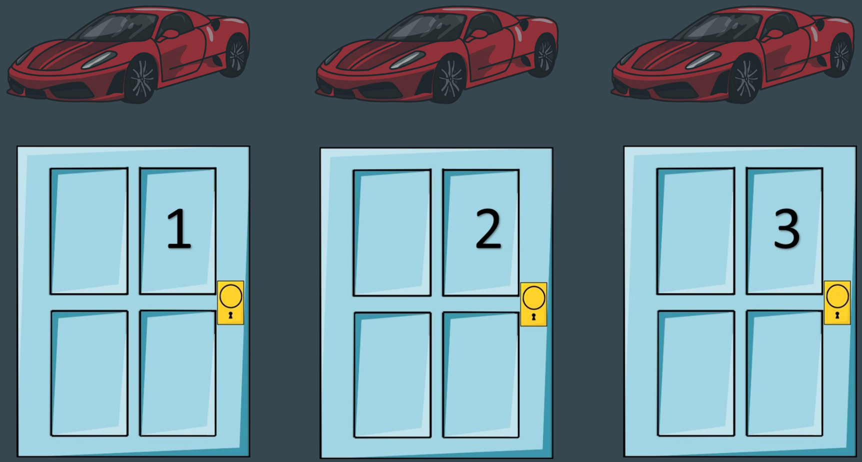
How!?: Monty Hall Problem

- The Monty Hall Problem:
  - You Pick Door 1
  - Monty opens door 2 to reveal a goat
  - Should you switch to door 3?



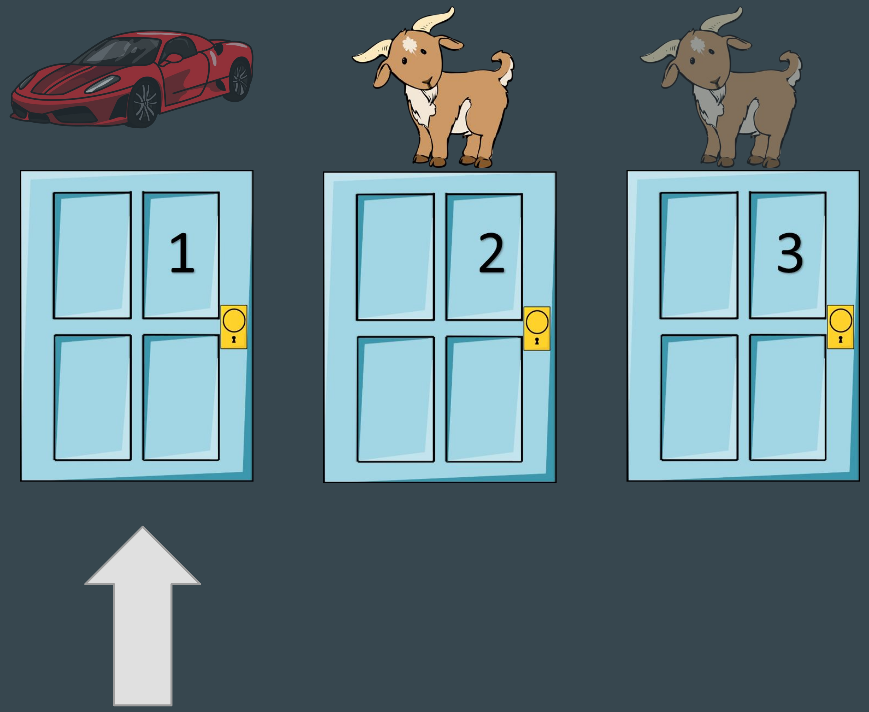
How!?: Monty Hall Problem

Hypothesis i	Prior P(θ_i)
Car Behind 1	1/3
Car Behind 2	1/3
Car Behind 3	1/3



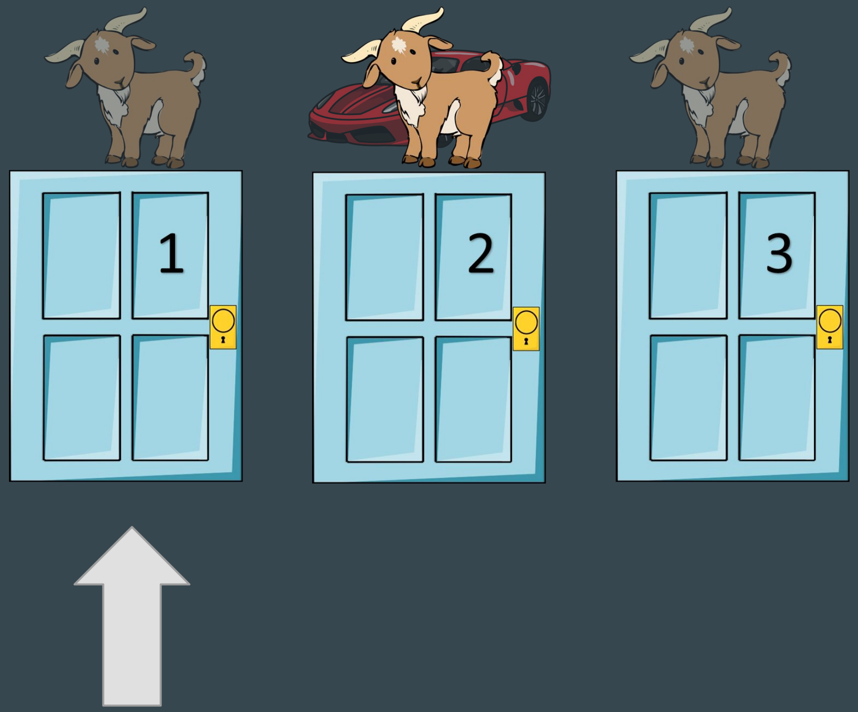
How!?: Monty Hall Problem

Hypothesis i	Likelihood $p(X \mid \theta_i)$
Car Behind 1	
Car Behind 2	
Car Behind 3	



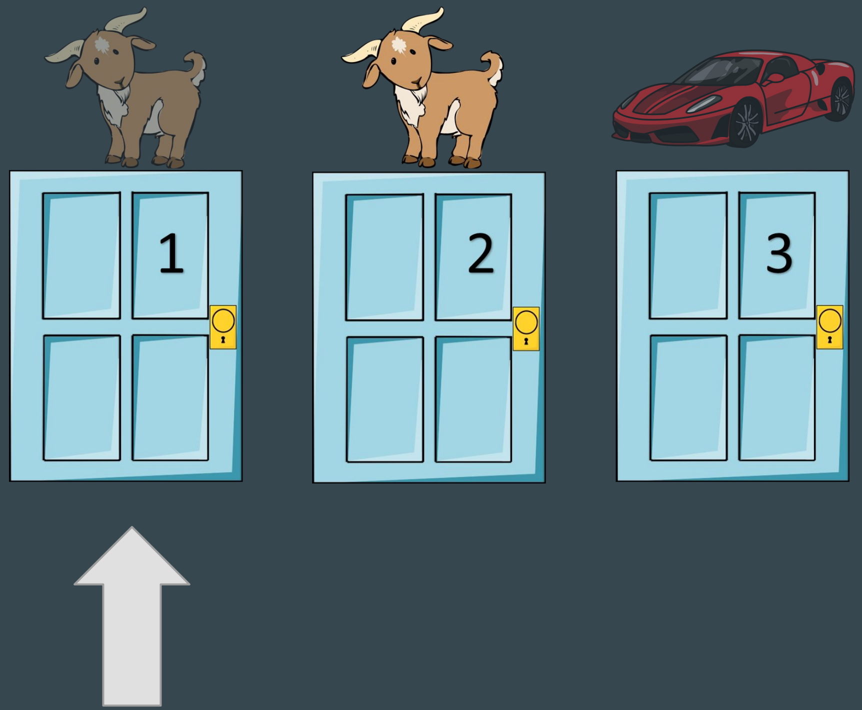
How!?: Monty Hall Problem

Hypothesis i	Likelihood p(X | θ_i)
Car Behind 1	1/2
Car Behind 2	
Car Behind 3	



How!?: Monty Hall Problem

Hypothesis i	Likelihood $p(X \mid \theta_i)$
Car Behind 1	1/2
Car Behind 2	0.0
Car Behind 3	



How!?: Monty Hall Problem

Hypothesis i	Prior $p(\theta_i)$	Likelihood $p(X \mid \theta_i)$
Car Behind 1	1/3	1/2
Car Behind 2	1/3	0.0
Car Behind 3	1/3	1.0

How!?: Monty Hall Problem

Hypothesis i	Prior $p(\theta_i)$	Likelihood $p(X \mid \theta_i)$	Prior * Likelihood
Car Behind 1	1/3	1/2	1/6
Car Behind 2	1/3	0.0	0.0
Car Behind 3	1/3	1.0	2/6

$$P(X) = \sum_{i=0}^{n} p(X|\theta_i)p(\theta_i) = 1/6 + 0 + 2/6$$
$$= 3/6$$
$$= 1/2$$

* This is the Dot Product of $p(\theta_i)$ and $p(X \mid \theta_i)$

How!?: Monty Hall Problem

Hypothesis i	Prior $p(\theta_i)$	Likelihood $p(X \mid \theta_i)$	Prior * Likelihood	Posterior
Car Behind 1	1/3	1/2	1/6	1/3
Car Behind 2	1/3	0.0	0.0	0.0
Car Behind 3	1/3	1.0	2/6	2/3

Key to problem:

Monty does not choose doors at random and so opening a door provides you with information

"Table Method" from A. Downey's *Think Bayes*

How!?: Train Analysis

- You see a train labeled 60 (labels in ascending order of creation)

- How many trains does this company own given that you've seen #60?

How!?: Train Analysis

Hypothesis i	Prior $p(\theta_i)$	Likelihood $p(X \mid \theta_i)$	Prior * Likelihood	Posterior
1 Train	1/1000	0.0	0.0	$post_1$
2 Trains	1/1000	0.0	0.0	$post_2$
...	...	...	...	...
60 Trains	1/1000	1/60	$1/(6*10^4)$	$post_{60}$
...	...	...	...	...
1000 Trains	1/1000	1/1000	$1/10^6$	$post_{1000}$

Σ = P(Train)

How!?: Train Analysis

- What if we change priors?
  - Posterior changes

- What if we increase the max number of trains
  - Posterior changes

Example from A. Downey's "Think Bayes"

How!?: Part 1 - Discrete Case Recap

- Remember the table calculation!
- Steps:
  - Pick Prior (Often Uniform)
  - Multiply by Frequentist Likelihood
  - Divide by Normalisation constant
- Similar things are true for the cont. case

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{\sum_{i=0}^{n} p(X|\theta_i)p(\theta_i)}$$

How!?: Part 2 - Continuous Case

AB Testing:

a. Two variants
b. What is the probability of A being better than B?

How!?: Part 2 - Continuous Case

Challenges w/ Freq. AB Tests:

- Test needs to reach pre-defined sample size
- Need to adjust α for multiple tests (Bonferroni Corrections)
- People start accepting null hypotheses (Logically illegal)
- Can only reject/fail to reject null hypothesis, (leads to p-hacking)
- People peek at tests before tests are over, (more p-hacking)

How!?: Part 2 - Continuous Case

Bayesian AB Tests:

- Tests can be started and stopped as necessary
- No need for adjustments, everything is included in calculation
- Probabilistic statements give better indication of risk
- No p-hacking

How!?: Part 2 - Continuous Case

$$P(\theta|X) = Beta(\alpha, \beta)$$

$$\alpha = 1 + \sum y_i,$$
$$\beta = n - 1 + \sum y_i$$

Conjugate Priors

Derivation:

- Choose Prior: Uniform
- Choose Likelihood: Bernoulli
- Do some math magic
- Get Beta distribution out

$$P(\theta|X) = Beta(\alpha, \beta)$$

See more Conj. Priors in U. of Santa Cruz's Coursera Course

Conjugate Priors

Likelihood	Model parameters	Conjugate prior distribution	Prior hyperparameters	Posterior hyperparameters	Interpretation of hyperparameters[note 1]	Posterior predictive[note 2]
Bernoulli	p (probability)	Beta	$\alpha,\ \beta$	$\alpha + \sum_{i=1}^{n} x_i,\ \beta + n - \sum_{i=1}^{n} x_i$	$\alpha - 1$ successes, $\beta - 1$ failures[note 1]	$p(\tilde{x} = 1) = \frac{\alpha'}{\alpha' + \beta'}$

https://en.wikipedia.org/wiki/Conjugate_prior#Table_of_conjugate_distributions
(Just Google “conjugate priors table wikipedia”)

Conjugate Priors

Poisson	λ (rate)	Gamma	k, θ	$k + \sum_{i=1}^{n} x_i,\ \frac{\theta}{n\theta + 1}$	k total occurrences in $\frac{1}{\theta}$ intervals	$\operatorname{NB}(\tilde{x} \mid k', \theta')$ (negative binomial)
			$\alpha,\ \beta$[note 3]	$\alpha + \sum_{i=1}^{n} x_i,\ \beta + n$	α total occurrences in β intervals	$\operatorname{NB}\left(\tilde{x} \mid \alpha', \frac{1}{1+\beta'}\right)$ (negative binomial)

Exponential	λ (rate)	Gamma	$\alpha,\ \beta$[note 3]	$\alpha + n,\ \beta + \sum_{i=1}^{n} x_i$	α observations that sum to β[6]	$\operatorname{Lomax}(\tilde{x} \mid \beta', \alpha')$ (Lomax distribution)

https://en.wikipedia.org/wiki/Conjugate_prior#Table_of_conjugate_distributions
(Just Google "conjugate priors table wikipedia")

Demo:
Conjugate Priors

https://seeing-theory.brown.edu/bayesian-inference/index.html

Posteriors - What Now?

Do we calculate P-values now?

- No need, just calculate $P(\theta_2 > \theta_1)$
- Takes some calculation, but the result is nicer

Can we test multiple versions

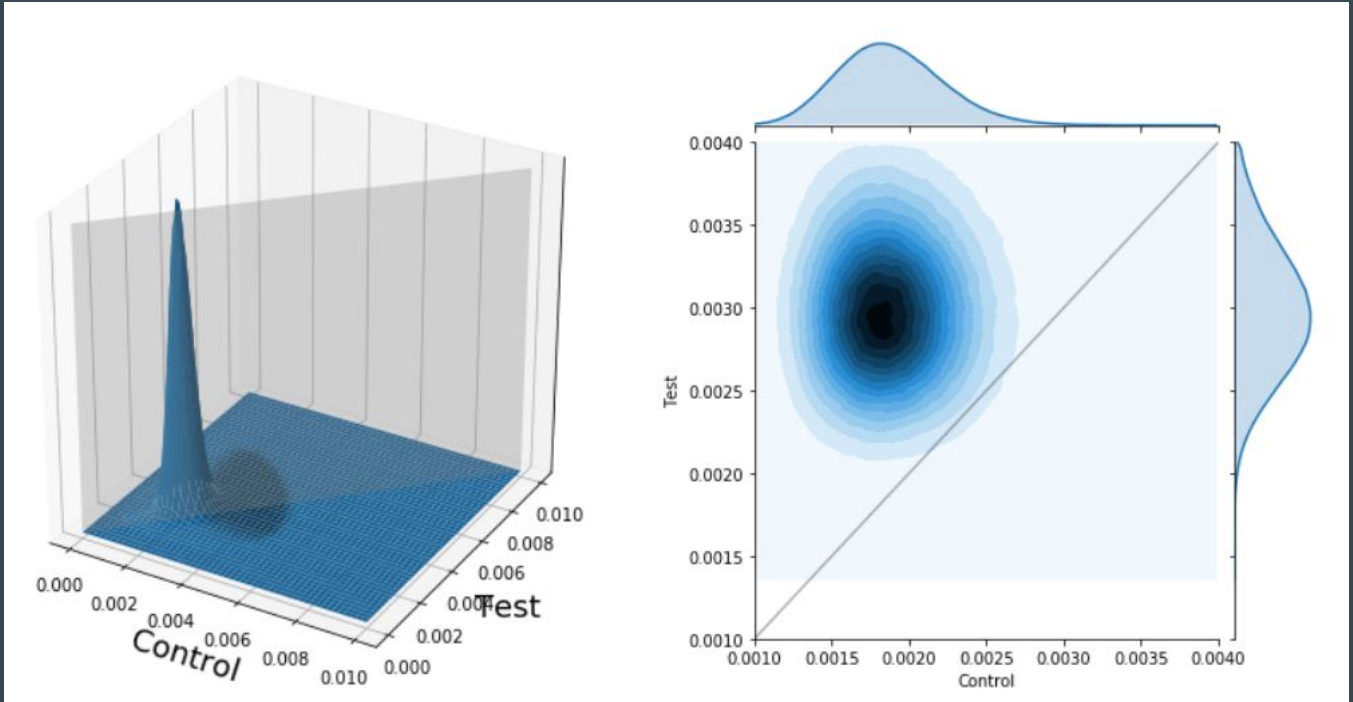
- Absolutely (Still, try and be reasonable)

Can we peek at results or stop at any time?

- Yes!

More from Nora Neumann @ Pydata 2016 on YouTube

Posteriors - What Now?



More from Vincenzo Lavorini @ “Bayesian A/B Testing with Python: the easy guide” Towards Data Science

Demo:
Bayesian AB Testing

Bayesian Stats

- Advantages
  a. Estimation in the case of little data (specific circumstances)
  b. Parameters are distributions
  c. Probabilistic answers answer questions people tend to have

- Disadvantages
  a. Steeper learning curve
  b. Integrals are hard (MCMC computationally expensive)
  c. Criticisms of being less “objective” due to use of priors
  d. Point estimates become the same as frequentist estimations with high sample sizes

How do we estimate probability?

- Classical: By considering equal outcomes
- Frequentist: Relative Frequency over time
- Bayesian: By quantifying our uncertainties

How do we estimate probability?

- ~~Classical: By considering equal outcomes~~
- ~~Frequentist: Relative Frequency over time~~
- ~~Bayesian: By quantifying our uncertainties~~

- There is only one field of statistics

Conclusion and “Call to Action”

Understanding Bayes vs Freq. is key to understanding a lot of the field stats

- Beginner scientists: Check my sources as a jump off point
- Experienced scientists: Come and tell me how you applied this knowledge
- Decision-makers: Be aware of these kinds of analyses and when the strengths benefit your use-case.
- General Advice: Try to think about conditional probability more often (doctor’s diagnosis) in relation to drawing conclusions

Resources for further learning

- Mathematical Understanding:

  “Bayesian Stats: From Concept to Data Analysis”,
  U of Santa Cruz

- Intuition between Bayesianism & Frequentism:

  “Frequentism and Bayesianism”,
  Scipy - Jake Vander Plas

- Examples of Real World Applications:

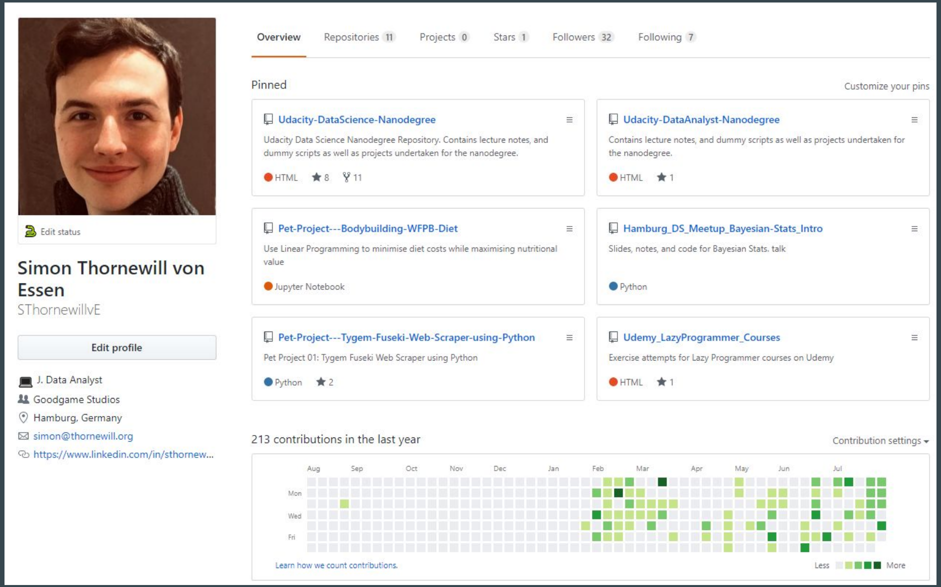
  “Think Bayes”,
  Allan Downey

- Further reading into MCMC and pyMC:

  “Bayesian Methods for Hackers”,
  Cameron Davidson-Pilon

Find Slides on Github

https://cutt.ly/zGqux9



Fin!

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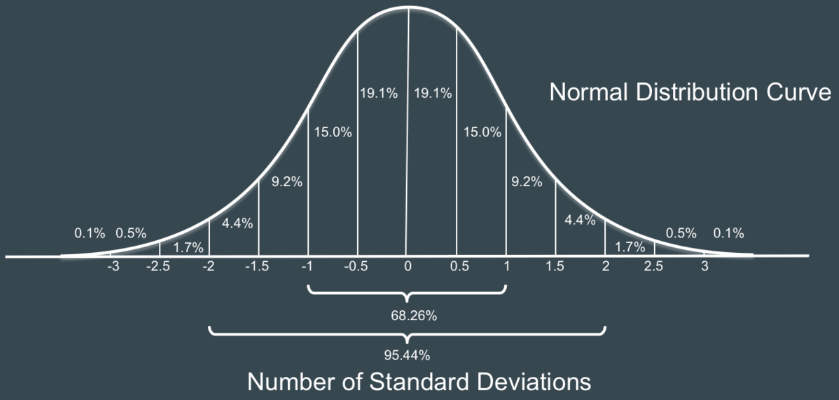
`@sthornewillve`

Frequentist Est.: Confidence Intervals

<u>Confidence Interval</u>:

- From sample mean and standard deviation, calculate an interval

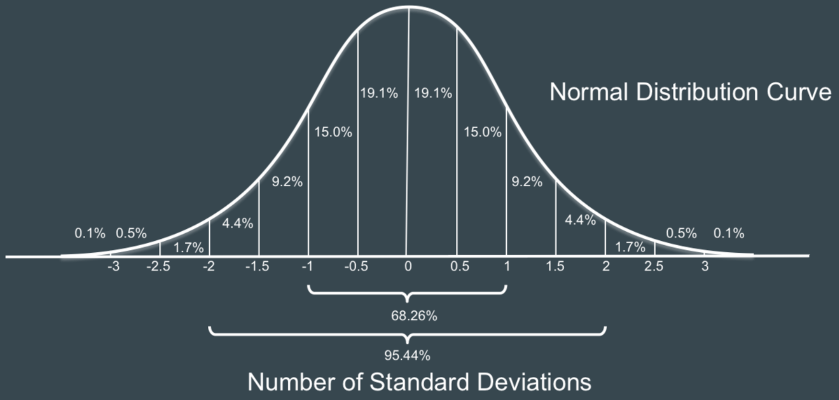
- Upon repeated experiments, intervals contain the true parameter x% of the time



Frequentist Est.: Confidence Intervals

<u>CI Intuition</u>::

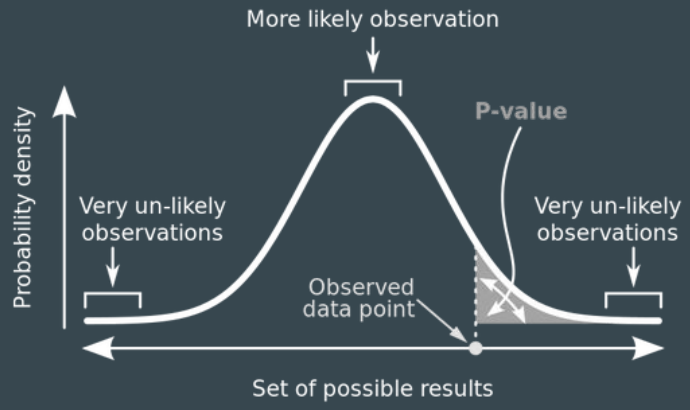
- bootstrap CI n times -> intervals would contain the mean of population 95% of the time



Frequentist Est.: P-Values

P-value:

- Probability of seeing data given a parameter



How!?: Train Analysis

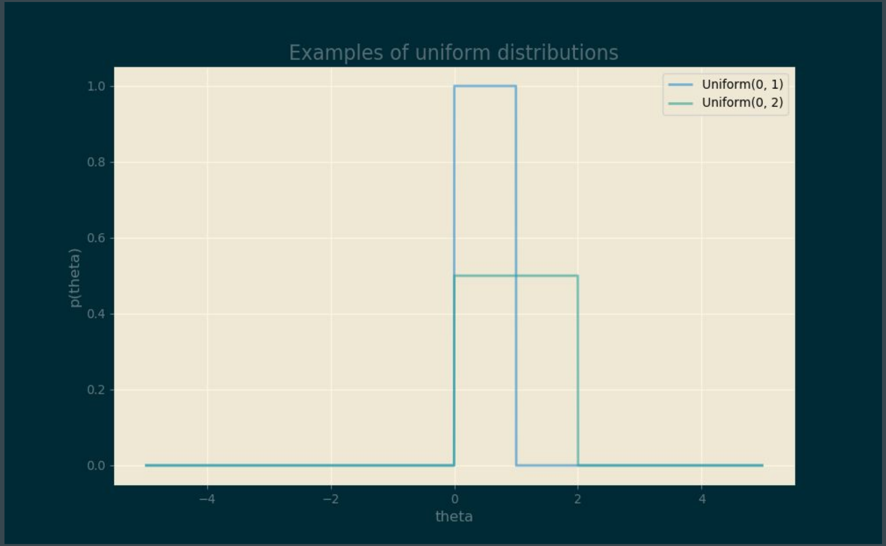
```
n = 1000

hypo = [1/n for i in range(n)]

likelihood = [1/i for i in range(n)]

post_not_norm = np.array(hypo) * np.array(likelihood)

normalisation = post_not_norm.sum()

posterior = post_not_norm / normalisation
```

How!?: Part 2 - Continuous Case

$$P(\theta) = I_{\{0 \le \theta \le 1\}}$$

Prior?:

- Uninformed Prior
- Uniform distribution
- Represented by Indicator Function



How!?: Part 2 - Continuous Case

AB Test:

- For people randomly placed in control/test
- Track conversions (1/0)
- What is our Likelihood?
  - Bernoulli

$$P(X|\theta) = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

How!?: Part 2 - Continuous Case

$$P(\theta|X) \propto [\theta^{\sum y_i}(1-\theta)^{n-\sum y_i}][I_{\{0\leq\theta\leq1\}}]$$

How!?: Part 2 - Continuous Case

$$P(\theta|X) = \frac{\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} I_{\{0\leq\theta\leq 1\}}}{\int_0^1 \theta^{\sum y_i}(1-\theta)^{n-\sum y_i} I_{\{0\leq\theta\leq 1\}} d\theta}$$

How!?: Part 2 - Continuous Case

$$P(\theta|X) = \frac{\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} I_{\{0\leq\theta\leq1\}}}{\boxed{A^{-1}} \int_0^1 \boxed{A}\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} I_{\{0\leq\theta\leq1\}} d\theta}$$

$$A = \frac{\Gamma(\sum n + 2)}{\Gamma(\sum y_i + 1)\Gamma(\sum n - y_i + 1)}$$

How!?: Part 2 - Continuous Case

$$P(\theta|X) = A(\theta^{\sum y_i}(1-\theta)^{n-\sum y_i} I_{\{0 \leq \theta \leq 1\}})$$

$$= Beta(\alpha, \beta)$$