MATH2111 Tutorial 10

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1 Eigenvectors and Eigenvalues

1. **Definition**. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

2. **Theorem**. The eigenvalues of a triangular matrix are the entries on its main diagonal.

3. **Theorem**. If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

4. **Definition**. The **eigenspace** of A corresponding to the eigenvalue λ (or the λ-eigenspace of A, sometimes written $E_\lambda(A)$) is the solution set to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

5. **Theorem**. Eigenspaces are subspaces. (Because λ-eigenspace of A is the null space of $A - \lambda I$)

2 The Characteristic Equation

1. **Theorem**. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation of A
$$\det(A - \lambda I) = 0$$

2. **Definition**. The **(algebraic) multiplicity** of an eigenvalue λ_k is its multiplicity as a root of the characteristic equation, i.e. it is the number of times the linear factor $(\lambda - \lambda_k)$ occurs in $\det(A - \lambda I)$.

3. **Definition**. The **geometric multiplicity** of an eigenvalue λ_k is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the null space of $A - \lambda_k I$.

4. **Theorem (The Invertible Matrix Theorem)**. Let A be an $n \times n$ matrix. Then A is invertible if and only if:

   (a) The number 0 is not an eigenvalue of A.

   (b) The determinant of A is not zero.

5. **Definition**. If A and B are $n \times n$ matrices, then A and B are similar if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

6. **Theorem**. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

7. **Procedures to find the Eigenvalues and Eigenvectors of** A

   (a) Solve the characteristic equation $\det(A - \lambda I) = 0$, the solutions are the eigenvalue(s) of A with the corresponding multiplicity(ies).

   (b) For each of the eigenvalue(s) found in (a), find the basis of the solution space of $(A - \lambda I)\mathbf{v} = \mathbf{0}$, these are the corresponding eigenvector(s).

3 Exercises

1. Let λ be an eigenvalue of A. Find an eigenvalue of the following matrices.
(1) A^2
(2) $A^3 + A^2$
(3) $A^3 + 2I$
(4) If A is invertible, A^{-1}
(5) If $p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$, define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A (with $A^0 = I$). That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

let $\vec{v}$ be the corresponding eigenvector of A.

(1) Since $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v})$

$= \lambda\cdot(\lambda\vec{v}) = \lambda^2\vec{v}$

$\therefore$ λ^2 is an eigenvalue of A^2.

(2) Since $(A^3+A^2)\vec{v} = A^3\vec{v} + A^2\vec{v}$

$= A\cdot A^2\vec{v} + A^2\vec{v}$

$= A(\lambda^2\vec{v}) + \lambda^2\vec{v}$

$= \lambda^2(A\vec{v}) + \lambda^2\vec{v}$

$= \lambda^3\vec{v} + \lambda^2\vec{v}$

$= (\lambda^3+\lambda^2)\vec{v}$

$\therefore$ $\lambda^3+\lambda^2$ is an eigenvalue of A^3+A^2.

Actually λ^m is an eigenvalue of A^m for positive integer m.
(can be proved by mathematical induction.)

(3) Since $(A^3+2I)\vec{v} = A^3\vec{v} + 2\vec{v}$

$= \lambda^3\vec{v} + 2\vec{v}$

$$= (\lambda^3+2)\vec{v}$$

$\therefore$ λ^3+2 is an eigenvalue of A^3+2I.

(4) Since A is invertible, so $\lambda \neq 0$, otherwise for eigenvector $\vec{v}$,

$$\vec{v} = A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v}) \underset{\substack{\uparrow \\ \text{assume} \\ \lambda=0}}{=} A^{-1}\vec{0} = \vec{0},$$

however, this is impossible, because eigenvector is nonzero.

We know that $A\vec{v} = \lambda\vec{v}$ with $\lambda \neq 0$, left multiply A^{-1},

$$A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v})$$

$$\Rightarrow \vec{v} = \lambda(A^{-1}\vec{v})$$

$$\Rightarrow A^{-1}\vec{v} = \lambda^{-1}\vec{v}.$$

$\therefore$ λ^{-1} is an eigenvalue of A^{-1}.

(5) We know that $A\vec{v} = \lambda\vec{v}$,

$$\begin{aligned} p(A)\vec{v} &= (c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n)\vec{v} \\ &= c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} + \cdots + c_n A^n\vec{v} \\ &= c_0\vec{v} + c_1\lambda\vec{v} + c_2\lambda^2\vec{v} + \cdots + c_n\lambda^n\vec{v} \\ &= (c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n)\vec{v} \\ &= p(\lambda)\vec{v} \end{aligned}$$

$\therefore$ $p(\lambda)$ is an eigenvalue of $p(A)$.

2. Let

$$A = \begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix}$$

Determine whether the following vectors are eigenvectors of A.

(1) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(1)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of A.

(corresponds to $\lambda = 1$)

(2)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is an eigenvector of A.

(corresponds to $\lambda = 2$)

(3)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 8 \\ 1 & -2 & -2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}+\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}+2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\neq c\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}+\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}+\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is NOT an eigenvector of A.

3. For the given matrix A and the given eigenvalue λ, find the corresponding collection of eigenvectors.

$$A = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix},\ \lambda = 1$$

$$A - I = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 7 \\ 4 & 9 & 7 \\ -8 & -18 & -14 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 9 & 7 & 0 \\ 4 & 9 & 7 & 0 \\ -8 & -18 & -14 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 9/4 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore\ x_1 = -\frac{9}{4}x_2 - \frac{7}{4}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Collection of eigenvectors is :

$$\left\{ s \begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \ \middle|\ s, t \in \mathbb{R},\ \text{NOT both zero} \right\}$$

Remark:

- if question asks the eigenspace:

$$\left\{ s\begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

- if asks a basis for eigenspace:

$$\left\{ \begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Suppose that λ and ρ are two different eigenvalues of the square matrix A. Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{\mathbf{0}\}$

① To show $\{\vec{0}\} \subseteq \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$

Choose $\vec{x} \in \{\vec{0}\}$, then $\vec{x} = \vec{0}$.

Since eigenspaces are subspaces,

$\mathcal{E}_A(\lambda)$ and $\mathcal{E}_A(\rho)$ contain the zero vector.

So, $\vec{0} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$.

② To show $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) \subseteq \{\vec{0}\}$

Suppose $\vec{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$,

then $\vec{x}$ is the eigenvector for A corresponds to both λ and ρ.

i.e. $A\vec{x} = \lambda\vec{x}$, $A\vec{x} = \rho\vec{x}$.

$$\begin{aligned} \therefore \vec{x} &= 1 \cdot \vec{x} \\ &= \frac{1}{\lambda-\rho}(\lambda-\rho)\vec{x} \end{aligned}$$

since λ and ρ are different. $\lambda - \rho \neq 0$

$$\begin{aligned} &= \frac{1}{\lambda-\rho}(\lambda\vec{x} - \rho\vec{x}) \\ &= \frac{1}{\lambda-\rho}(A\vec{x} - A\vec{x}) \\ &= \frac{1}{\lambda-\rho}\vec{0} \\ &= \vec{0} \end{aligned}$$

$\therefore \vec{x} = \vec{0}$. $\Rightarrow \vec{x} \in \{\vec{0}\}$.

5. Find the eigenvalues eigenvalues, eigenspaces, algebraic and geometric multiplicities of the following matrices.

(1) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(1) Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \xlongequal{R_2+R_1 \to R_2} \begin{vmatrix} 1-\lambda & -1 & 1 \\ -\lambda & -\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} \leftarrow$$

$$= -(-\lambda)\begin{vmatrix} -1 & 1 \\ -1 & 1-\lambda \end{vmatrix} + (-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= \lambda\left((\lambda-1)+1\right) - \lambda\left[(1-\lambda)^2-1\right]$$

$$= \lambda^2 - \lambda(\lambda^2-2\lambda)$$

$$= \lambda^2 - \lambda^3 + 2\lambda^2$$

$$= 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda)$$

let $\det(A-\lambda I) = 0$, we have

$$-\lambda^2(\lambda-3) = 0$$

$\therefore$ eigenvalues $\lambda_1 = \lambda_2 = 0$, algebraic multiplicity 2.

$\lambda_3 = 3$, algebraic multiplicity 1.

① For $\lambda = 0$:

$$A-\lambda I = A - 0\cdot I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solve $(A-\lambda I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ x_1 = x_2 - x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 2.

② For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

Solve $(A - \lambda I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & -2 & -1 & 0 \\ -2 & -1 & 1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} -1 & -2 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} -1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \to R_3]{} \left[\begin{array}{ccc|c} -1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+2R_2 \to R_1} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

(2) Characteristic polynomial :

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)^2 .$$

let $\det(A-\lambda I) = 0$, we have

$$(3-\lambda)(1-\lambda)^2 = 0$$

$\therefore$ eigenvalues $\lambda_1 = \lambda_2 = 1$, algebraic multiplicity 2.

$\lambda_3 = 3$, algebraic multiplicity 1.

① For $\lambda=1$,

$$A-I=\begin{bmatrix}2&0&0\\0&0&1\\0&0&0\end{bmatrix}$$

Solve $(A-\lambda I)\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}2&0&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$$\therefore \begin{cases}x_1=0\\x_3=0\end{cases}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ s\begin{bmatrix}0\\1\\0\end{bmatrix} \middle| s\in\mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

② For $\lambda=3$:

$$A-3I=\begin{bmatrix}0&0&0\\0&-2&1\\0&0&-2\end{bmatrix}$$

Solve $(A-\lambda I)\vec{x}=\vec{0}$

$$\left[\begin{array}{ccc|c}0&0&0&0\\0&-2&1&0\\0&0&-2&0\end{array}\right]\xrightarrow{R_2\leftrightarrow R_1}\left[\begin{array}{ccc|c}0&-2&1&0\\0&0&0&0\\0&0&-2&0\end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_2 \to R_2]{-\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + \frac{1}{2}R_2 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

Remark:

Geometric multiplicity can NEVER EXCEED the algebraic multiplicity.