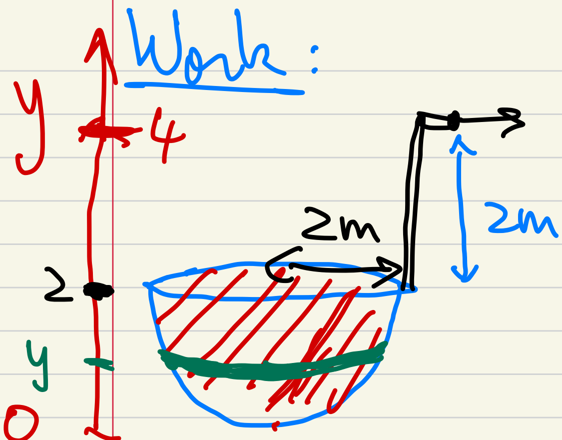
Work:



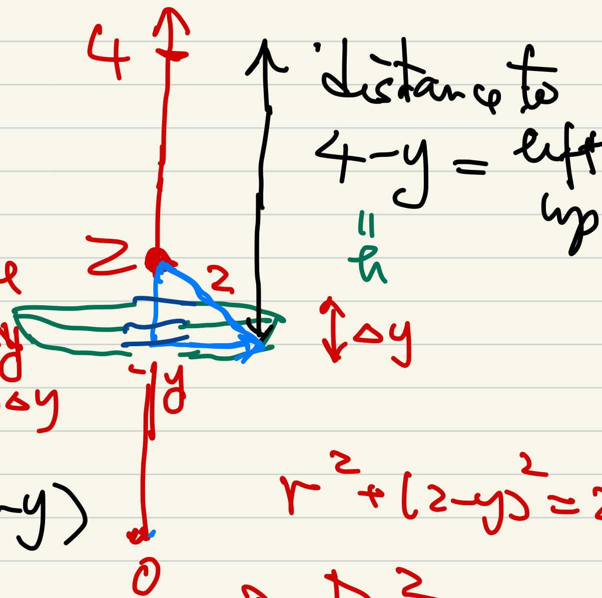
A tank in the shape of a hemi-sphere is filled with water

(density of water $\rho = 1000\ kg/m^3$)

"a continuous distribution" of mass

Find the required work for pumping all water to the outlet 2m above the top of the tank.

Look at a thin slice of water at y



thin volume $= \pi(r)^2\Delta y$
$= \pi(4-(2-y)^2)\Delta y$

$m \cdot g \cdot h$

$= \underbrace{(\pi(4-(2-y)^2)\Delta y \cdot \rho)}_{\text{volume} \cdot \text{density}} g(4-y)$

$r^2 + (2-y)^2 = 2^2$

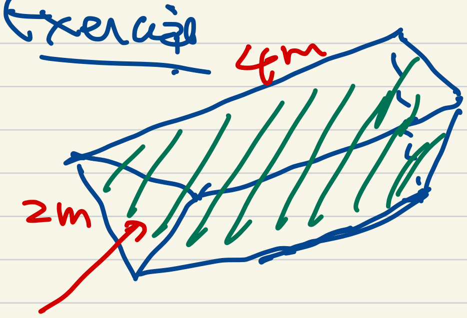
Work required $= \displaystyle\int_0^2 \pi\rho g\,(4-(2-y)^2)(4-y)\,dy$

$ = \pi\rho g \displaystyle\int_0^2 4(4-y) - (2-y)^2(4-y)\,dy$

$16 - 4y - 8 + 2y + 4y^2 - y^3$

$= \pi\rho g\left[8y - y^2 + \frac{4}{3}y^3 - \frac{y^4}{4}\right]_0^2$ (J)

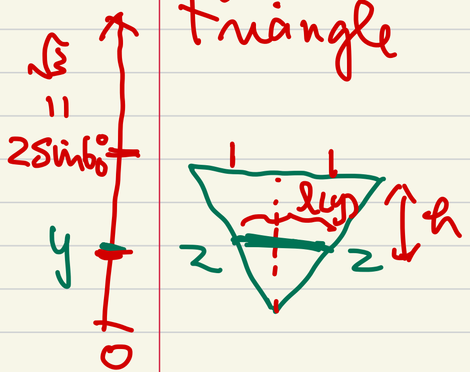
Exercise

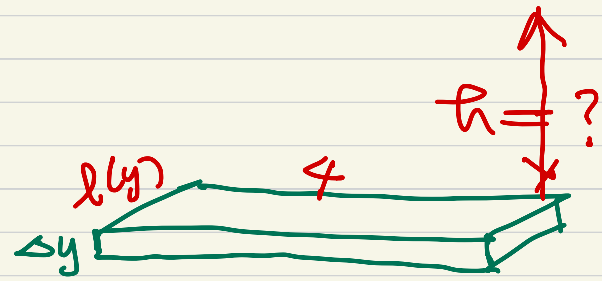


equilateral triangle

full of water inside the tank.

Find the work required to pump all water to the top of the tank.

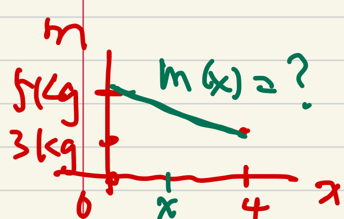




$m \cdot g \cdot h = ?$

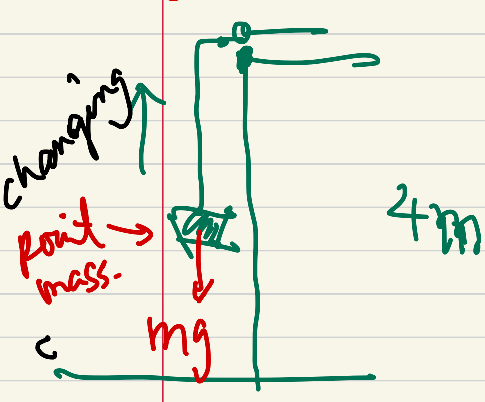
$\downarrow \int$

$$W = \int_0^{\sqrt{3}} \_\_\_\_\_\_ \, dy$$



Point mass

$$W = \int_0^4 m(x)\, g \, dx$$



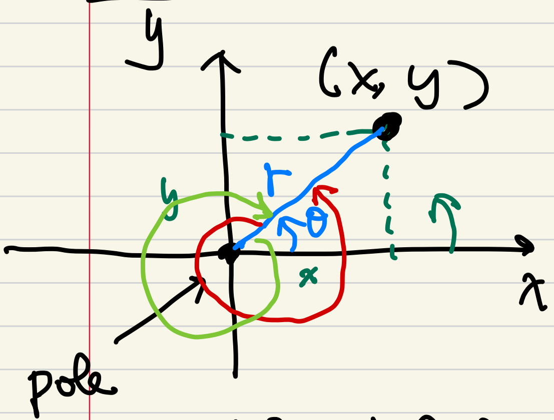
$m = m(x)$

(Ignore the weight of the bucket and the rope)

Initially 5 kg of water is in the bucket on ground, but water is leaking at a constant rate when it is pulled up.

If only 3 kg of water is left when the bucket reach the platform what is the work required to lift this bucket of water?

Polar Coordinates



$(r, \theta) = ($radial coordinates, angular coordinates$)$

$=$ Polar coordinates of the point where $r \neq 0$.

choice of θ is not unique

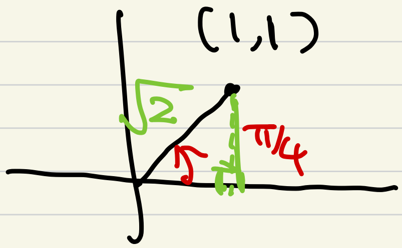
pole $= r=0$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ r = \sqrt{x^2+y^2} \end{cases}$$

$\theta + 2k\pi$ for any integer k.

$\tan\theta = \frac{y}{x}$ if $x \neq 0$. $\theta+2\pi,\ \theta-2\pi,\ \dots$

e.g.



Polar Coordinates:

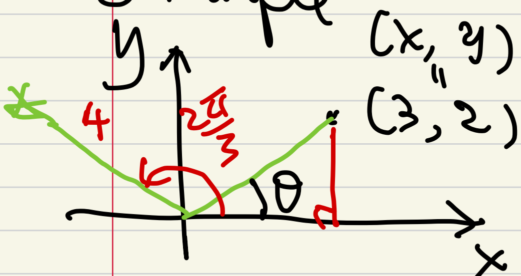
$(r, \theta) = (\sqrt{2}, \pi/4)$

Could also choose $= (\sqrt{2}, \pi/4 + 2\pi)$

$= (\sqrt{2}, \frac{9\pi}{4})$

...

Example



$\longleftrightarrow$ Polar Coordinates:

$r = \sqrt{3^2+2^2}$ $\quad \tan\theta = \frac{2}{3}$

$r = \sqrt{13}$

$(r, \theta) = (4, \frac{2\pi}{3})$

$(x, y) = (4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3})$

$= (-2, 2\sqrt{3})$

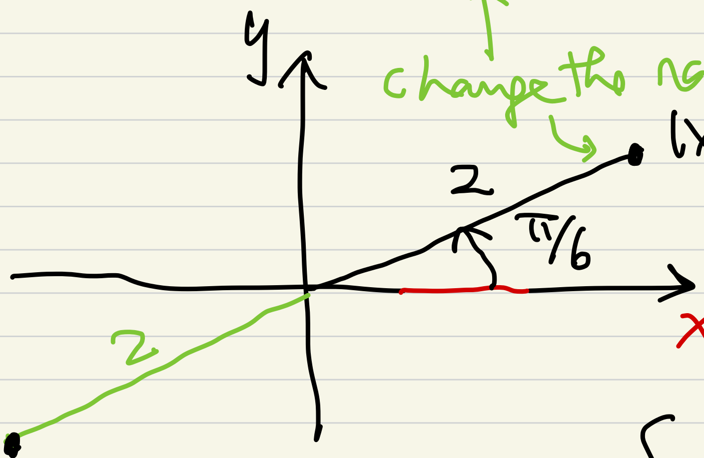
$\theta = \tan^{-1}\frac{2}{3}$, $\quad \theta = \tan^{-1}\frac{2}{3} + 2k\pi$

$k =$ integer

Extending to negative radial coordinates!

$(r, \theta) = (-2, \frac{\pi}{6})$? Which point is this?

change the radial direction



$(x,y) = (2\cos\pi/6, 2\sin\pi/6)$
$= (\sqrt{3}, 1)$

$(-x,-y) = (-\sqrt{3}, -1)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$x = -2\cos\frac{\pi}{6} = -\sqrt{3}$$

$$y = -2\sin\frac{\pi}{6} = -1$$

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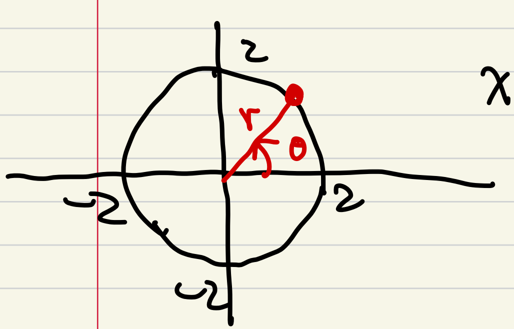
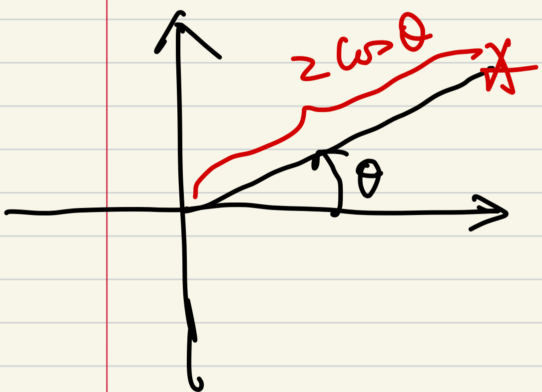
xy-equation vs. polar equations

r, θ



$x^2 + y^2 = 2^2 \iff r = 2$

Example: What is the curve given by the polar equation $r = 2\cos\theta$.



A circle centered at $(1,0)$

when $\theta = 0$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\pi/2$
$2\cos\theta$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

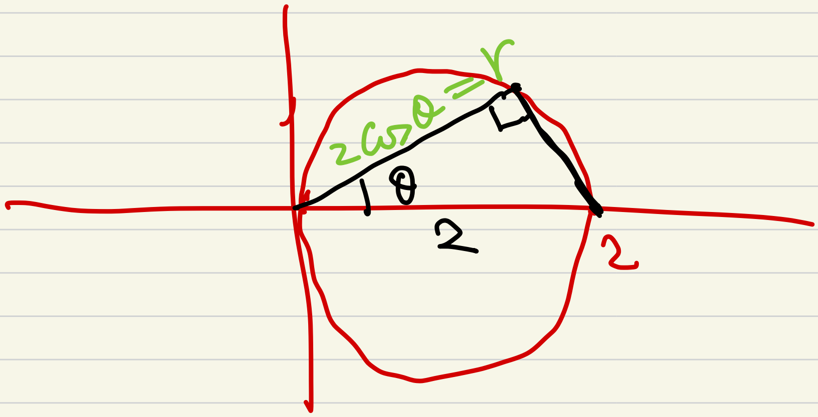
$r = 2\cos\theta \longleftrightarrow xy\text{-equation}.$

$$r^2 = 2r\cos\theta$$

$$x^2 + y^2 = 2x$$

$$(x^2 - 2x + 1) + y^2 = 1$$

$$(x-1)^2 + y^2 = 1^2$$



Example. $r = 3 + \cos\theta$ (Exercise).

Graph this curve !!

Next time !