MATH2111 Tutorial 10

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1 Eigenvectors and Eigenvalues

1. **Definition**. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

2. **Theorem**. The eigenvalues of a triangular matrix are the entries on its main diagonal.

3. **Theorem**. If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

4. **Definition**. The **eigenspace** of A corresponding to the eigenvalue λ (or the λ-eigenspace of A, sometimes written $E_\lambda(A)$) is the solution set to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

5. **Theorem**. Eigenspaces are subspaces. (Because λ-eigenspace of A is the null space of $A - \lambda I$)

2 The Characteristic Equation

1. **Theorem**. A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation of A
$$\det(A - \lambda I) = 0$$

2. **Definition**. The **(algebraic) multiplicity** of an eigenvalue λ_k is its multiplicity as a root of the characteristic equation, i.e. it is the number of times the linear factor $(\lambda - \lambda_k)$ occurs in $\det(A - \lambda I)$.

3. **Definition**. The **geometric multiplicity** of an eigenvalue λ_k is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the null space of $A - \lambda_k I$.

4. **Theorem (The Invertible Matrix Theorem)**. Let A be an $n \times n$ matrix. Then A is invertible if and only if:

   (a) The number 0 is not an eigenvalue of A.

   (b) The determinant of A is not zero.

5. **Definition**. If A and B are $n \times n$ matrices, then A and B are similar if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

6. **Theorem**. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

7. **Procedures to find the Eigenvalues and Eigenvectors of** A

   (a) Solve the characteristic equation $\det(A - \lambda I) = 0$, the solutions are the eigenvalue(s) of A with the corresponding multiplicity(ies).

   (b) For each of the eigenvalue(s) found in (a), find the basis of the solution space of $(A - \lambda I)\mathbf{v} = \mathbf{0}$, these are the corresponding eigenvector(s).

3 Exercises

1. Let λ be an eigenvalue of A. Find an eigenvalue of the following matrices.
(1) A^2
(2) $A^3 + A^2$
(3) $A^3 + 2I$
(4) If A is invertible, A^{-1}
(5) If $p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$, define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A (with $A^0 = I$). That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

2. Let

$$A = \begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix}$$

Determine whether the following vectors are eigenvectors of A.

(1) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

3. For the given matrix A and the given eigenvalue λ, find the corresponding collection of eigenvectors.

$$A = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix}, \ \lambda = 1$$

4. Suppose that λ and ρ are two different eigenvalues of the square matrix A. Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{\mathbf{0}\}$

5. Find the eigenvalues eigenvalues, eigenspaces, algebraic and geometric multiplicities of the following matrices.

(1) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$