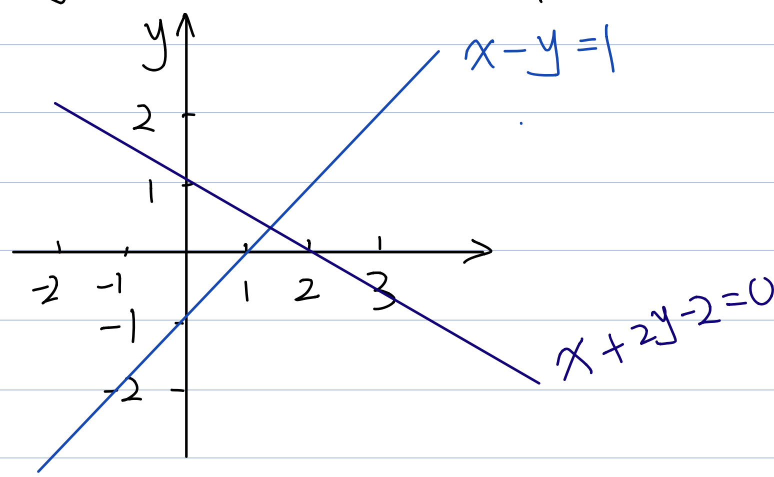
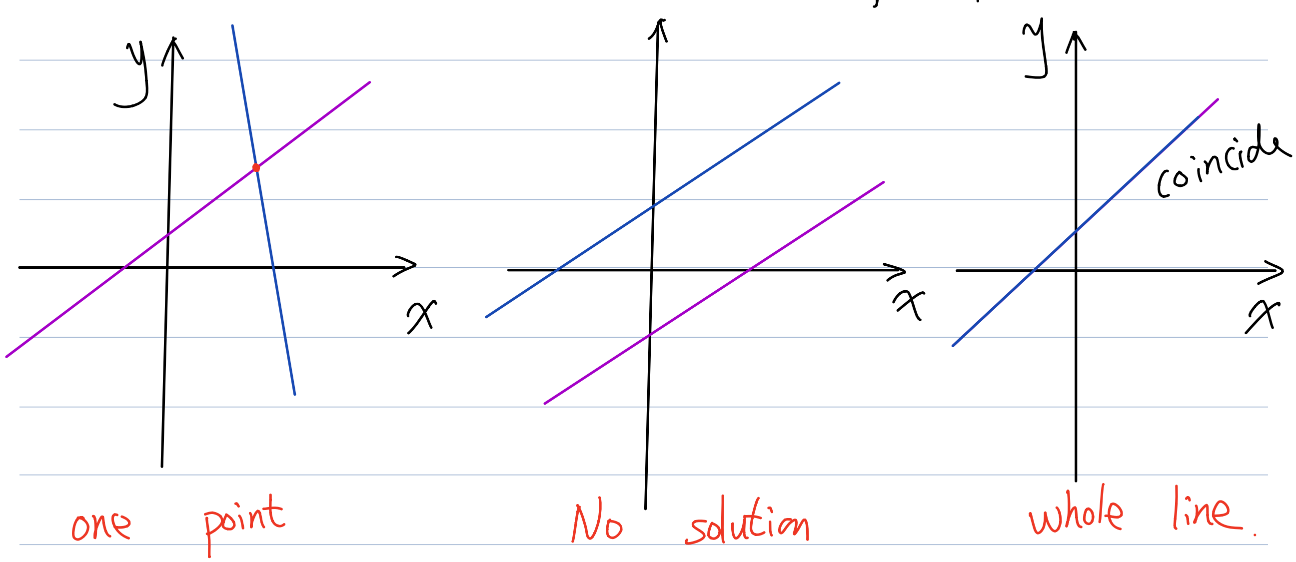
Geometric visualization of linear equation of 2 or 3 variables

a) A linear equation of two variables gives a straight line on a plane R^2

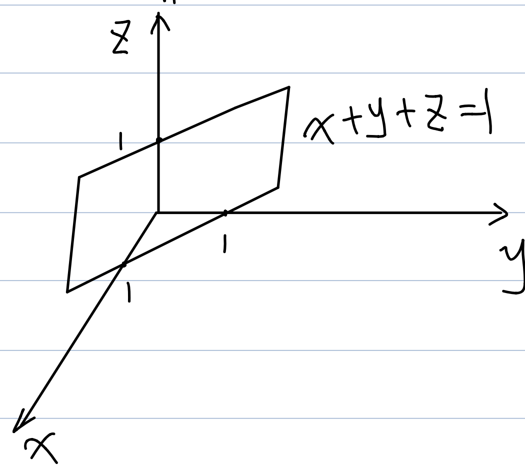
Example



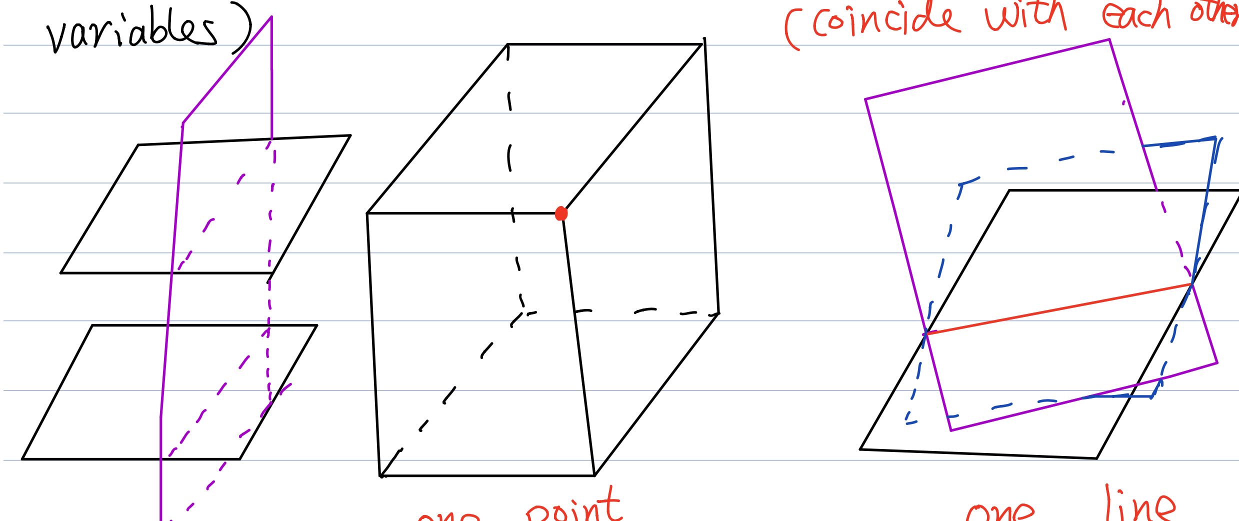
A system of several linear equations of two variables is a collection of lines and the solution means the intersection of all the lines



b) A linear equation of three variables gives a plane in $\mathbb{R}^3$.



A system of linear equations of three variables is a collection of planes, and the solution means the intersection of all the planes. The intersection can be empty (no solution), single point (unique solution), a line (one free variable), or a plane (two free variables) (coincide with each other)



No solution one point one line

* The general language for describing those solutions are Euclidean spaces $\mathbb{R}^n$.

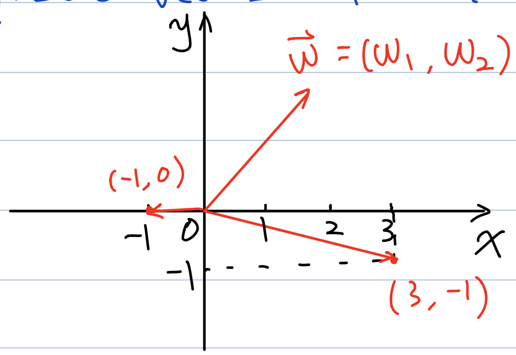
§1.3 Vector Equations

* Vectors in $\mathbb{R}^2$

$\vec{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

are called column vectors

Represent vectors in (x, y)-plane



* Two vectors are equal if the corresponding entries are equal.

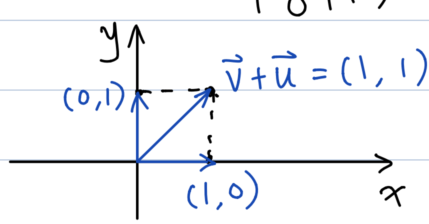
Example: $\begin{pmatrix} 4 \\ 7 \end{pmatrix} \neq \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

* Sum of two vectors

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then $\vec{v} + \vec{u} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix}$

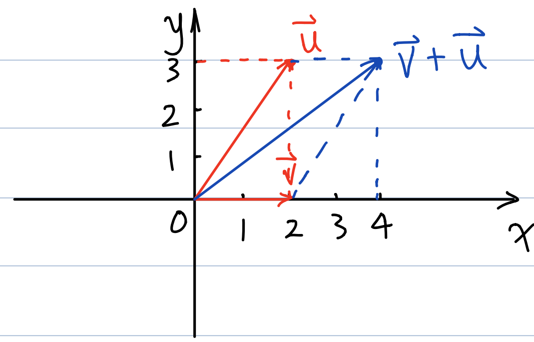
Example: $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $\vec{v} + \vec{u}$

Solution: $\vec{v} + \vec{u} = \begin{pmatrix} 1+0 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



Example: $\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, Find $\vec{v} + \vec{u}$.

Solution: $\vec{v} + \vec{u} = \begin{pmatrix} 2+2 \\ 0+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



Parallelogram rules for addition: If $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\vec{u} + \vec{v}$ corresponds to the fourth vertex of the parallelogram whose other vectors are $\vec{u}$, $\vec{0}$ and $\vec{v}$.

* Scalar multiple of $\vec{u}$ by c, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$c\vec{u} = c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$$

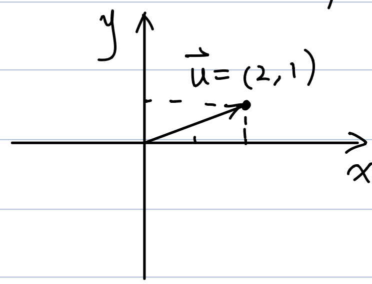
Example: Given $\vec{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, find $4\vec{u}$, $(-3)\vec{v}$

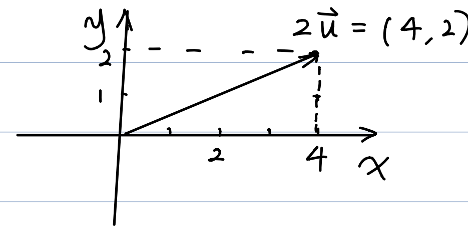
and $4\vec{u} + (-3)\vec{v}$.

Solution: $4\vec{u} = 4\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \times 1 \\ 4 \times (-2) \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

$$(-3)\vec{v} = -3\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -6 \\ 15 \end{pmatrix}$$

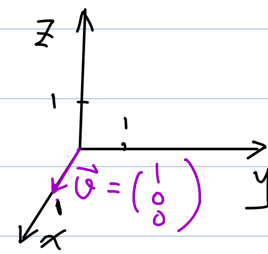
$$4\vec{u} + (-3)\vec{v} = \begin{pmatrix}4\\-8\end{pmatrix} + \begin{pmatrix}-6\\15\end{pmatrix} = \begin{pmatrix}4-6\\-8+15\end{pmatrix} = \begin{pmatrix}-2\\7\end{pmatrix}$$



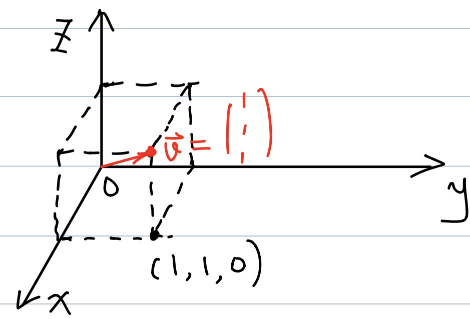


* Vectors in $\mathbb{R}^3$

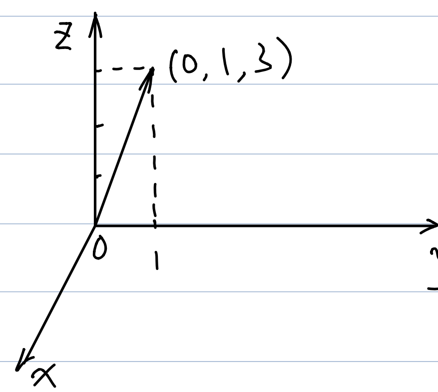
$$\vec{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \qquad \vec{v} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$



$$\vec{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$



$$\vec{w} = \begin{pmatrix}0\\1\\3\end{pmatrix}$$



* Vectors in $\mathbb{R}^n$: an ordered list of n numbers.

$$\vec{u} = \begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix} \qquad \text{zero vector } \vec{0} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

* Algebraic properties of operations on vectors

For all vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ in $\mathbb{R}^n$ and all scalars c and d, we have

(i) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(ii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(iii) $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$

(iv) $\vec{u}+(-\vec{u}) = 0 = (-\vec{u})+\vec{u}$

(v) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

(vi) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

(vii) $c(d\vec{u}) = (cd)\vec{u}$

(viii) $1\cdot\vec{u} = \vec{u}$

$-\vec{u}$ denotes $(-1)\vec{u}$.

* Linear combinations

Example: Let $\vec{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$. $\vec{w} = \begin{pmatrix}2\\3\\0\end{pmatrix}$.

Then $\vec{w} = \begin{pmatrix}2\\3\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} = 2\vec{u}+3\vec{v}$

So $\vec{w}$ can be written as a linear combination of $\vec{u}$ and $\vec{v}$.

* In general, given vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ in $\mathbb{R}^n$, and given scalars $c_1, c_2, \cdots, c_p$, the vector $\vec{w}$

$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ is called a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ with weights coefficients $c_1, \cdots, c_p$.

Example: $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$.

Example: Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$

Determine whether $\vec{b}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Solution: It is equivalent to the existence of x_1 and x_2 such that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2$.

$$\begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$$

i.e. solve the system $\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3 \end{cases}$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{b} \end{array}$$

$$\begin{pmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{pmatrix} \xrightarrow[③+5①]{②+2①} \begin{pmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{pmatrix} \xrightarrow[\frac{③}{16}]{\frac{②}{9}} \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{③-②} \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{①-2②} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The system of linear equations is equivalent to $\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$

Then $\vec{b}$ can be a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

* In general, a vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n, \vec{b}] \qquad (*)$$

In particular, $\vec{b}$ can be generated by a linear combination of $\vec{a}_1, \cdots, \vec{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix $(*)$.

Def: If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ are vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ is denoted by $\text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \cdots, \vec{v}_p$.

That is, $\text{span}\{\vec{v}_1, \cdots \vec{v}_p\}$ is the collection of all vectors that can be written in the form $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$

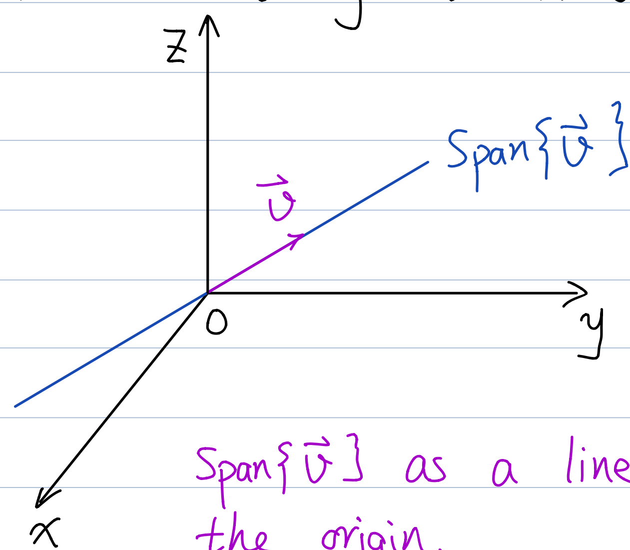
Example: $\text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}a\\b\\0\end{pmatrix} \middle| \; a\in\mathbb{R},\ b\in\mathbb{R}\right\}$

the xy-plane in xyz coordinate space $\mathbb{R}^3$.

A Geometric Description of Span$\{\vec{v}\}$ and Span$\{\vec{u}, \vec{v}\}$.

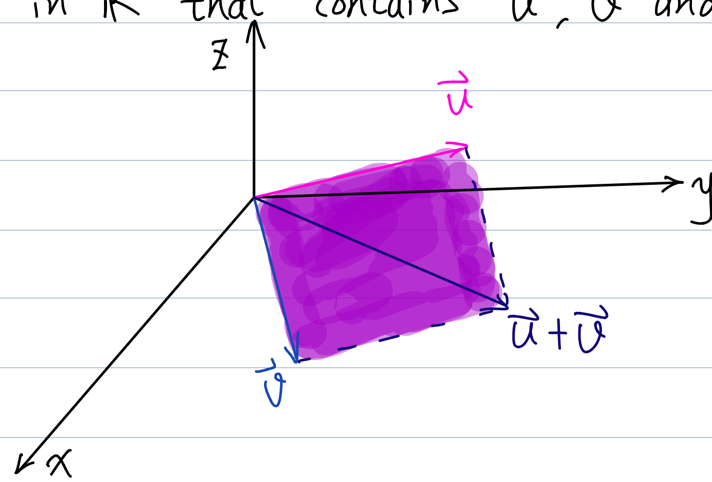
1) Let $\vec{v}$ be a nonzero vector in $\mathbb{R}^3$.

Span$\{v\}$: the set of all scalar multiples of $\vec{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\vec{v}$ and $\vec{0}$.



Span$\{\vec{v}\}$ as a line through the origin.

2) Let $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\vec{v}$ not a multiple of $\vec{u}$, then Span$\{\vec{u}, \vec{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\vec{u}$, $\vec{v}$ and $\vec{0}$.



Example: Let $\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}$. Then

span $\{\vec{a}_1, \vec{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\vec{b}$ in that plane?

Solution: We need to find if the equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

have a solution.

We do row reduction on the augmented matrix $(\vec{a}_1, \vec{a}_2, \vec{b})$:

$$\begin{pmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ has no solution, and so $\vec{b}$ is not in Span$\{a_1, a_2\}$.

Example: A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on materiables, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product

C, the company spends \$0.40 on materials, \$0.30 on labor, and \$0.15 on overhead. Let

$$\vec{b} = \begin{pmatrix} 0.45 \\ 0.25 \\ 0.15 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} 0.40 \\ 0.30 \\ 0.15 \end{pmatrix}.$$

Then $\vec{b}$ and $\vec{c}$ represent the "costs per dollar of income" for the two products.

a). What economic interpretation can be given to the vector $100\vec{b}$?

b) Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

Solution: a)

$$100\vec{b} = 100\begin{pmatrix} 0.45 \\ 0.25 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 45 \\ 25 \\ 15 \end{pmatrix}$$

The vector $100\vec{b}$ lists the various costs for producing \$100 worth of product B. Namely, \$45 for materials, \$25 for labor, and \$15 for overhead.

b) The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\vec{b}$, and the costs of manufacturing x_2 dollars worth of C are given by $x_2\vec{c}$. Hence the total costs for both products are given by the vector $x_1\vec{b} + x_2\vec{c}$.

Exercise: For what value(s) of h will $\vec{y}$ be in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}.$$

Solution: The vector $\vec{y}$ belongs to $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix}$$

The system is consistent if and only if $h-5=0$
i.e. $h=5$.