MATH2111 Tutorial 4

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1 Linear Independence

1. **Definition (Linear Independence):**
   (a) An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation
   $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$
   has only the trivial solution.
   (b) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that
   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$
2. **Theorem (Linear Independence of Columns of Matrix):**
   The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has **only** the trivial solution.
3. **Theorem (Characterization of Linearly Dependent Sets):**
   An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.
4. **Theorem (Conditions For Linear Dependence):**
   (a) If a set contains more vectors than the entries in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
   (b) If a set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

2 Transformations

1. **Definition (Transformation):**
   A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
   The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T.
   The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
   For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the range of T.

3 Exercises

1(a). Express the general solutions of the following non-homogeneous systems in terms of the given particular solutions.

$$\begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 5 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 4 \\ -x_1 - x_2 + x_3 - x_5 = -1 \end{cases}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the above linear system.

1(a). For $A\vec{x}=\vec{b}$, let $\vec{p}$ be a particular solution,

$\vec{x}_h$ is any solution of $A\vec{x}=\vec{0}$

Then solution set of $Ax=b$ is of form: $\vec{w} = \vec{p} + \vec{x}_h$

① Solve $A\vec{x}=\vec{0}$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 2 & 2 & 0 & 2 & 4 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_3+R_1 \to R_3]{R_2-2R_1 \to R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & -6 & -6 & -2 & 0 \\ 0 & 0 & 4 & 4 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3 \to R_3]{-\frac{1}{6}R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow[R_3-R_2 \to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \end{array}\right]$$

$$\xrightarrow[6R_3 \to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1-3R_3 \to R_1]{R_2-\frac{1}{3}R_3 \to R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \leftarrow \text{RREF}$$

Thus
$$\begin{cases} x_1 + x_2 + x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

x_2, x_4 are free.

$$\therefore \vec{x}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 & -x_4 \\ x_2 & \\ & -x_4 \\ & x_4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_4 \qquad x_2, x_4 \in \mathbb{R}$$

② Check $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$

Thus, solution set for $A\vec{x} = \vec{b}$ is

$$\vec{w} = \vec{p} + \vec{x}_h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_4 \qquad x_2, x_4 \in \mathbb{R}$$

1(b). Denote the coefficient matrix as $\boldsymbol{A}$. Use as many columns of $\boldsymbol{A}$ as possible to construct a matrix $\boldsymbol{B}$ with the property that the equation $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. (Solve $\boldsymbol{Bx} = \boldsymbol{0}$ to verify your work.)

1(b). According to the RREF of A,
the first third and fifth columns are pivot columns

If use the corresponding columns of A to form B,
we have:

$$B = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 4 \\ -1 & 1 & -1 \end{bmatrix},$$

If do row reduction on $[B|\vec{0}]$, we get $\left[\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{array}\right]$.

Since $B\vec{x}=\vec{0}$ has no free variable, it has only trivial solution.

If use any additional columns of A to form B, then $B\vec{x}=\vec{0}$ will have free variable, so $B\vec{x}=\vec{0}$ has nontrivial solution.

2. Find conditions on p and q such that the set of vectors

$$\left\{ \underset{=\vec{a_1}}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}}, \underset{=\vec{a_2}}{\begin{bmatrix} 2 \\ 5 \\ 5 \\ 1 \end{bmatrix}}, \underset{=\vec{a_3}}{\begin{bmatrix} 1 \\ 2 \\ 4 \\ p \end{bmatrix}}, \underset{=\vec{a_4}}{\begin{bmatrix} 3 \\ 8 \\ 9 \\ q \end{bmatrix}} \right\}$$

is linearly independent.

2. Denote $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \\ 0 & 1 & p & q \end{bmatrix}$,

$$x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3} + x_4\vec{a_4} = \vec{0}$$
$$\Updownarrow$$
$$A\vec{x} = \vec{0}$$

By thm, we need to gaurantee $A\vec{x} = \vec{0}$ has only trivial solution.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 5 & 2 & 8 & 0 \\ 3 & 5 & 4 & 9 & 0 \\ 0 & 1 & p & q & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & p & q & 0 \end{array}\right]$$

$$\xrightarrow[R_4 - R_2 \to R_4]{R_3 + R_2 \to R_3} \left[\begin{array}{cccc|c} \underline{\underline{1}} & 2 & 1 & 3 & 0 \\ 0 & \underline{\underline{1}} & 0 & 2 & 0 \\ 0 & 0 & \underline{\underline{1}} & 2 & 0 \\ 0 & 0 & p & q-2 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 - pR_3 \to R_4} \left[\begin{array}{cccc|c} \underline{\underline{1}} & 2 & 1 & 3 & 0 \\ 0 & \underline{\underline{1}} & 0 & 2 & 0 \\ 0 & 0 & \underline{\underline{1}} & 2 & 0 \\ 0 & 0 & 0 & q-2-2p & 0 \end{array}\right] \leftarrow \text{REF}$$

The matrix already has pivots in columns 1, 2, 3, to have a pivot in column 4, we must have $\underline{q-2-2p \neq 0}$.

3. Consider matrix $\boldsymbol{A}$,

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \end{bmatrix},$$

Find a vector which is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and also in Span $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$, or explain why such a vector cannot exist.

(Given $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.)

3. $\vec{v} \in \text{Span}\{\vec{a_1}, \vec{a_2}\}$ if $\vec{v} = c_1\vec{a_1} + c_2\vec{a_2}$ for some c_1, c_2.

$\vec{v} \in \text{Span}\{\vec{a_3}, \vec{a_4}\}$ if $\vec{v} = c_3\vec{a_3} + c_4\vec{a_4}$ for some c_3, c_4.

$\therefore c_1\vec{a_1} + c_2\vec{a_2} = \vec{v} = c_3\vec{a_3} + c_4\vec{a_4}$

$\therefore c_1\vec{a_1} + c_2\vec{a_2} - c_3\vec{a_3} - c_4\vec{a_4} = \vec{0}$

$\therefore A\begin{bmatrix} c_1 \\ c_2 \\ -c_3 \\ -c_4 \end{bmatrix} = \vec{0}$

Since $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.

$\therefore$ take $c_1 = 3$, $c_2 = -2$, $c_3 = 2$, $c_4 = -1$

$\therefore \vec{v} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -1 \end{bmatrix}$, which is also $2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$

4. State whether each of the following statement is true or false. (If it is true, give a brief justification; if it is false, give a counterexample.)

(a) If $\boldsymbol{A}\begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \mathbf{0}$, then $\boldsymbol{A}\boldsymbol{e}_4$ is a linear combination of the first three columns of $\boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a 4×3 matrix with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, and suppose $\boldsymbol{b}$ is a vector in $\mathbb{R}^4$ such that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}\}$ is linearly dependent. Then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

4 (a). True.

Let $\vec{a}_i$ be the i-th column of A.

$A\begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \vec{0}$ means $4\vec{a}_1 + 2\vec{a}_3 - 3\vec{a}_4 = \vec{0}$,

$\therefore A\vec{e}_4 = \vec{a}_4 = \frac{4}{3}\vec{a}_1 + \frac{2}{3}\vec{a}_3 + 0\cdot\vec{a}_2$

$\vec{e}_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith row.

$\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$

4(b). False.

Take $\vec{a}_1 = \vec{e}_1$, $\vec{a}_2 = \vec{e}_2$, $\vec{a}_3 = \vec{e}_1 + \vec{e}_2$, and $\vec{b} = \vec{e}_3$

$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}\}$ Easy to check these 4 vectors are linearly dependent ($\vec{a}_3 = \vec{a}_1 + \vec{a}_2$).

However, $\vec{b}$ can't be expressed as a linear combination of $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$.

Remark: linear dependence only implies NOT necessarily all some vector can be expressed as a linear combination of rest vectors.

5. Consider

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ 0 \\ 3x_1 - x_2 \end{bmatrix}$$

(a) What is the domain of F?

(b) Find the image of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ under F.

5(a). domain: $\mathbb{R}^3$

(b). $F\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0 \\ 0 \\ 3\times 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$