MATH 2111: Tutorial 4 Linear Independence and Introduction to Linear Transformations

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Review

- Linear independence/dependence
- Linear independence of columns of matrix
- Characterization of linearly dependent sets
- Conditions for linear dependence
- Transformation

Example 1(a)

Express the general solutions of the following non-homogeneous systems in terms of the given particular solutions.

$$\begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 5 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 4 \\ -x_1 - x_2 + x_3 - x_5 = -1 \end{cases}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the above linear system.

Example 1(b)

Denote the coefficient matrix as $\boldsymbol{A}$. Use as many columns of $\boldsymbol{A}$ as possible to construct a matrix $\boldsymbol{B}$ with the property that the equation $\boldsymbol{Bx} = 0$ has only the trivial solution. (Solve $\boldsymbol{Bx} = 0$ to verify your work.)

Example 2

Find conditions on p and q such that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ p \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 9 \\ q \end{bmatrix} \right\}$$

is linearly independent.

Example 3

Consider matrix $\boldsymbol{A}$,

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \end{bmatrix},$$

Find a vector which is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and also in Span $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$, or explain why such a vector cannot exist.

(Given $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $\boldsymbol{Ax} = 0$.)

Example 4

State whether each of the following statement is true or false. (If it is true, give a brief justification; if it is false, give a counterexample.)

(a) If $\boldsymbol{A}\begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = 0$, then $\boldsymbol{Ae}_4$ is a linear combination of the first three columns of $\boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a 4×3 matrix with columns $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and suppose $\boldsymbol{b}$ is a vector in $\mathbb{R}^4$ such that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}\}$ is linearly dependent. Then $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution.

Example 5

Consider

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ 0 \\ 3x_1 - x_2 \end{bmatrix}$$

(a) What is the domain of F?

(b) Find the image of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ under F.