MATH 2111: Tutorial 12
Inner Product and Orthogonality

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- Inner product, length, and orthogonality
- Orthogonal sets
- Orthogonal projections

Determine which pairs of vectors are orthogonal.

$$\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

$$a\cdot b = a^T b = \begin{bmatrix} 8 & -5 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = 8\cdot(-2)+(-5)\cdot(-3) = -1 \neq 0$$

orthogonal

$$u\cdot v = u^T v = \begin{bmatrix} 12 & 3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12\cdot 2+3\cdot(-3)+(-5)\cdot 3 = 0$$

$$u\cdot v = u^T v = \begin{bmatrix} 3 & 2 & -5 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix} = 3\cdot(-4)+2\cdot 1+(-5)(-2)+0\cdot 6 = 0$$

$$y\cdot z = y^T z = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = -3\cdot 1+7\cdot(-8)+4\cdot 15+0\cdot(-7) = 1 \neq 0$$

(1) Verify the parallelogram law for vectors $\mathbf{u}$ and $\mathbf{v}$ $\mathbb{R}^{n}$:

$$\| \mathbf{u}+\mathbf{v} \|^2+\| \mathbf{u}-\mathbf{v} \|^2= 2 \| \mathbf{u} \|^2 +2 \| \mathbf{v} \|^2$$

(2) Let $\mathrm{W}=\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{\mathrm{p}}\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_{\mathrm{j}}$, then for $1 \leq \mathrm{j} \leq \mathrm{p}$, then $\mathbf{x}$ is orthogonal to every vector in W.

(1) $\|u+v\|^2+\|u-v\|^2 = (u+v)\cdot(u+v)+(u-v)\cdot(u-v)$
$= (u\cdot u+u\cdot v+v\cdot u+v\cdot v)+(u\cdot u-u\cdot v-v\cdot u+v\cdot v)$
$= 2u\cdot u+2v\cdot v$
$= 2\|u\|^2+2\|v\|^2$

(2) For $\forall v \in W=\text{Span}\{v_1,\cdots,v_p\}$, $v=\sum_{i=1}^{p} a_i v_i$

$$x\cdot v = x^T v = x^T \sum_{i=1}^{p} a_i v_i = \sum_{i=1}^{p} a_i (x^T v_i) = \sum_{i=1}^{p} a_i 0 = 0$$

Then x is orthogonal to every vector in W.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

$u_1 \cdot u_2 = u_1^T u_2 = [3\ -3\ 0]\begin{bmatrix}2\\2\\-1\end{bmatrix} = 3.2+(-3)+0\cdot(-1)=0$

$u_1 \cdot u_3 = u_1^T u_3 = [3\ -3\ 0]\begin{bmatrix}1\\1\\4\end{bmatrix} = 3\cdot1+(-3)\cdot1+0\cdot4=0$

$u_2 \cdot u_3 = u_2^T u_3 = [2\ 2\ -1]\begin{bmatrix}1\\1\\4\end{bmatrix} = 2\cdot1+2\cdot1+(-1)\cdot4 = 0$

Then $\{u_1, u_2, u_3\}$ is an orthogonal set, then is linearly indepedent

Since $\dim \mathbb{R}^3 = 3$, $\{u_1, u_2, u_3\}$ is an orthogonal basis

$[u_1\ u_2\ u_3]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = x$

$$\left[\begin{array}{ccc|c}3 & 2 & 1 & 5\\-3 & 2 & 1 & -3\\0 & -1 & 4 & 1\end{array}\right] \xrightarrow{②+①} \left[\begin{array}{ccc|c}3 & 2 & 1 & 5\\0 & 4 & 2 & 2\\0 & -1 & 4 & 1\end{array}\right] \xrightarrow{③+\frac{1}{4}②} \left[\begin{array}{ccc|c}3 & 2 & 1 & 5\\0 & 4 & 2 & 2\\0 & 0 & \frac{9}{2} & \frac{3}{2}\end{array}\right]$$

$a_1 = \frac{4}{3},\ a_2 = \frac{1}{3},\ a_3 = \frac{1}{3}$ $\qquad x = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$

Find the closest point to y in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{6}{12}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{30}{10}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26}\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$,

and let $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that dim $\mathbf{dim}\, W + \mathbf{dim}\, W^\perp = n$.

a. $w_i \in W$ $(i=1,\cdots,p)$, $v_j \in W^\perp$ $(j=1,\cdots,q)$

By the definition of $W^\perp$, $w_i \cdot v_j = 0$

Since $\{w_1,\cdots,w_p\}$, $\{v_1,\cdots,v_q\}$ are orthogonal sets,

$w_i \cdot w_j = 0$ $(i \neq j)$, $v_i \cdot v_j = 0$ $(i \neq j)$,

Then $\{w_1,\cdots,w_p,v_1,\cdots,v_q\}$ is an orthogonal set

b. By the Orthogonal Decomposition Theorem,

for each y in $\mathbb{R}^n$, $y = \hat{y} + z$, where $\hat{y}$ is in W, z in $W^\perp$

Assume $\hat{y} = \sum_{i=1}^{p} a_i w_i$, $z = \sum_{i=1}^{q} b_i v_i$

Then $y = \sum_{i=1}^{p} a_i w_i + \sum_{i=1}^{q} b_i v_i$ is a linear combination of $\{w_1,\cdots,w_p,v_1,\cdots,v_q\}$

Then $\{w_1,\cdots,w_p,v_1,\cdots,v_q\}$ spans $\mathbb{R}^n$.

c. Due to b, $p+q=n$.

Since $\dim W = \dim\{w_1,\cdots,w_p\} = p$, $\dim W^\perp = \dim\{v_1,\cdots,v_q\} = q$.

We have $\dim W + \dim W^\perp = n$.