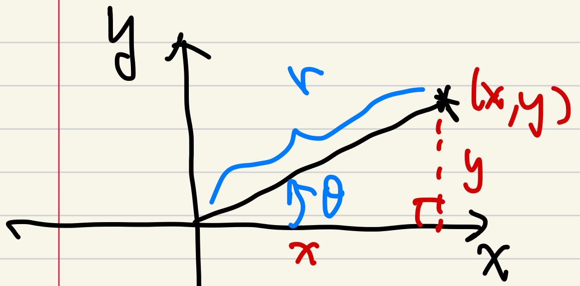
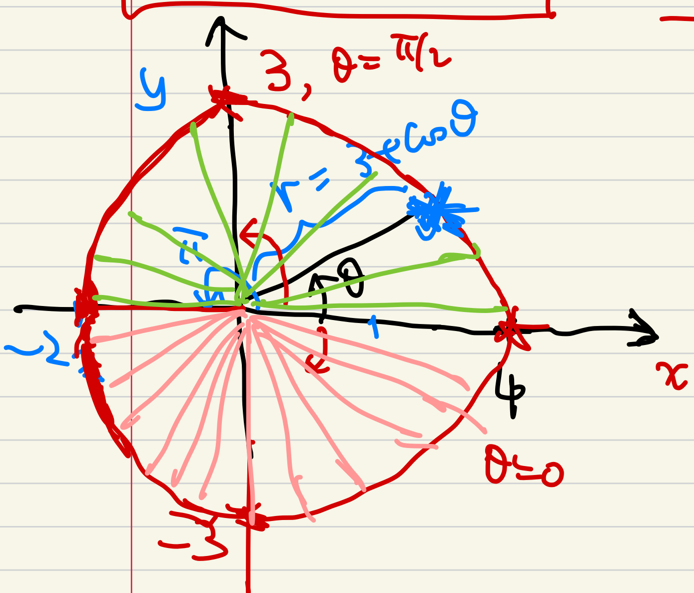
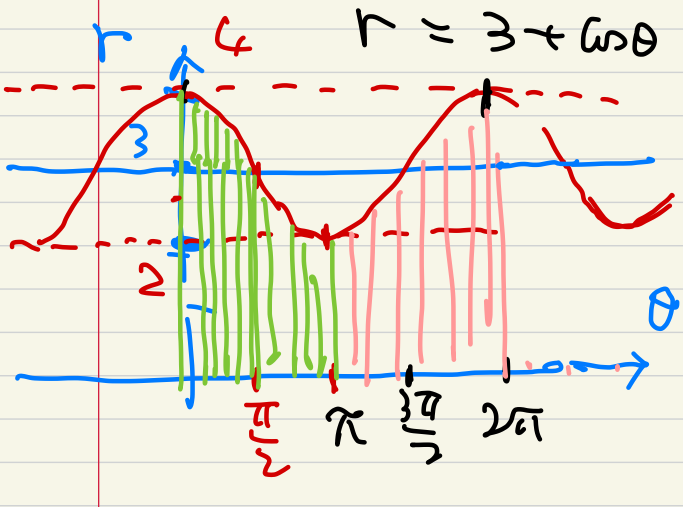
Polar Equations :



$$\boxed{\begin{cases} r^2 = x^2 + y^2 \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}}$$





Example :

$$\boxed{r = 3 + \cos\theta}$$

$$r^2 = 3r + r\cos\theta$$

$$x^2 + y^2 = 3\sqrt{x^2+y^2} + x$$

$$(x^2 + y^2 - x)^2 = 9(x^2 + y^2)$$

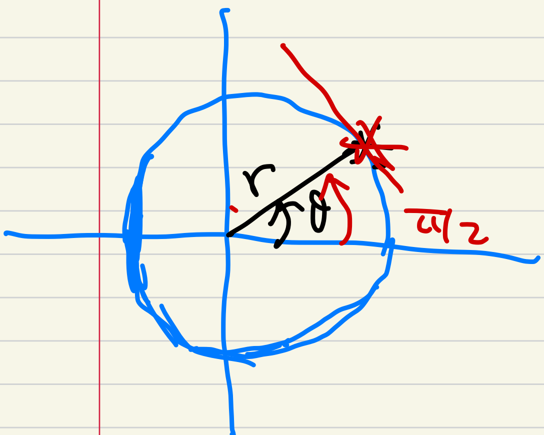
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$r = 3 + \cos\theta$	$3+1 = 4$	$3 + \frac{\sqrt{2}}{2}$	3	$3 - \frac{\sqrt{2}}{2}$	$3-1 = 2$

$\cos\theta$ is decreasing

$\cos\theta = \cos(-\theta)$

$$2 \leq 3 + \cos\theta \leq 4$$

Slope of Tangent Line.



$r = 3 + \cos\theta$

Find the slope of the tangent line at the point with $\theta = \pi/4$.

(x, y) - coordinates

$$\begin{cases} x = r\cos\theta = \left(3+\cos\frac{\pi}{4}\right)\cos\frac{\pi}{4} = \left(3+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} \\ y = r\sin\theta = \left(3+\cos\frac{\pi}{4}\right)\sin\frac{\pi}{4} = \left(3+\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} \end{cases}$$

$$(x, y) = \left(\frac{1}{2} + \frac{3\sqrt{2}}{2}, \frac{1}{2} + \frac{3\sqrt{2}}{2}\right)$$

① $\left[(x^2+y^2) - x\right]^2 = 9(x^2+y^2)$ $\Leftrightarrow$ Implicit Differentiation

Math 1012/1013.

$$\left.\frac{dy}{dx}\right|_{\left(1+\frac{\sqrt{3}}{2}, 1+\frac{\sqrt{3}}{2}\right)} = ?$$

② Use Chain Rule !!

$$\begin{cases} x = (3+\cos\theta)\cos\theta \\ y = (3+\cos\theta)\sin\theta \end{cases} \Rightarrow \frac{dy}{d\theta} \overset{\text{Chain Rule}}{=} \frac{dy}{dx}\cdot\frac{dx}{d\theta}$$

$$\frac{dx}{d\theta} = -3\sin\theta - 2\cos\theta\sin\theta$$

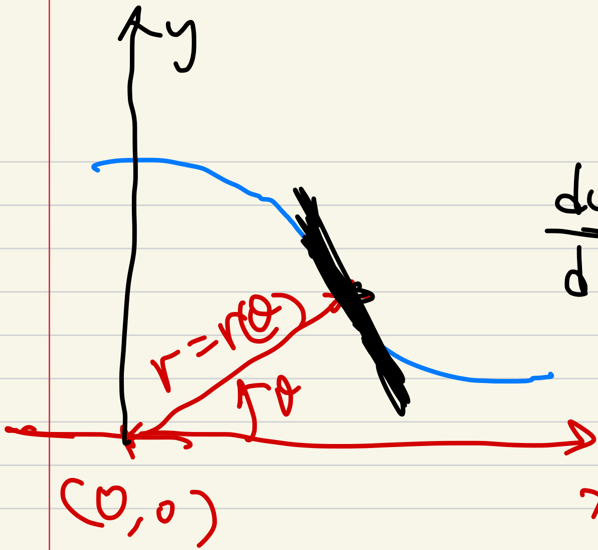
$$\frac{dy}{d\theta} = 3\cos\theta + \cos^2\theta - \sin^2\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\pi/4}$$

$\theta = \pi/6$

$$= \frac{-3\cdot\frac{\sqrt{2}}{2} - \cancel{2}\cdot\frac{\sqrt{2}}{2}\cdot\cancel{\frac{\sqrt{2}}{2}}}{3\cdot\frac{\sqrt{2}}{2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-3\sqrt{2} - 2}{3\sqrt{2}}$$



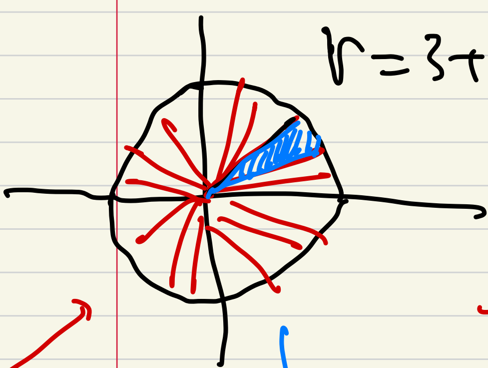
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Downarrow$$

$$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases} \Rightarrow \frac{dy}{dx} = \frac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{-r(\theta)\sin\theta + r'(\theta)\cos\theta}$$

Area Problems in Polar Coordinates.

$r = 3+\cos\theta$



What is the area enclosed by the curve?

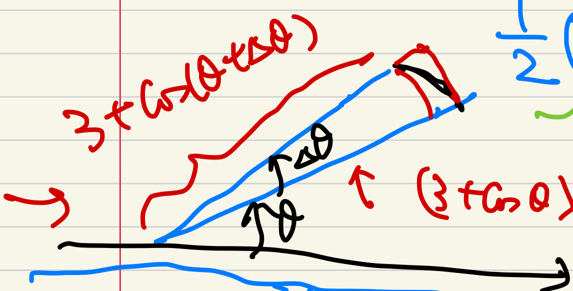
$$\int_0^{2\pi} \frac{1}{2}(3+\cos\theta)^2 d\theta \le A \le \int_0^{2\pi} \frac{1}{2}(3+\cos\theta)^2 d\theta$$

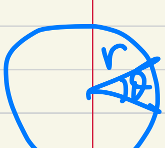
look at a tiny sector of the area.

summing Σ

$$\frac{1}{2}(3+\cos(\theta+\Delta\theta))^2\Delta\theta \le \Delta A \le \underbrace{\frac{1}{2}(3+\cos\theta)^2\Delta\theta}_{\text{area of a circular sector}}$$

tiny area of the sector





circular sector area $= \frac{1}{2}r^2\theta$

$$A = \int_0^{2\pi} \frac{1}{2}(3+\cos\theta)^2 \, d\theta$$

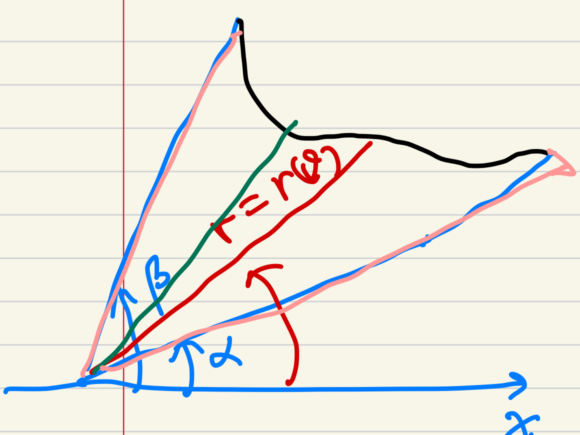
$$= \frac{1}{2}\int_0^{2\pi} \left[9 + 6\cos\theta + \cos^2\theta \right] d\theta$$

$$= \frac{1}{2}\left[9\theta + 6\sin\theta \right]_0^{2\pi} + \frac{1}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi}$$

$$\int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} \, d\theta$$

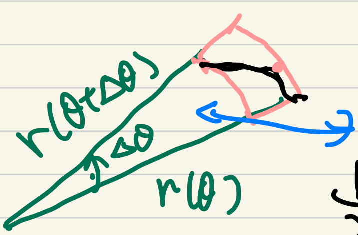
$$= 9\pi + \frac{1}{2}\pi$$

$$= \frac{19\pi}{2} \quad \text{(sq. units).}$$



$$\boxed{\text{area}\left(\quad \right) = \frac{1}{2}\int_\alpha^\beta r^2(\theta) \, d\theta}$$

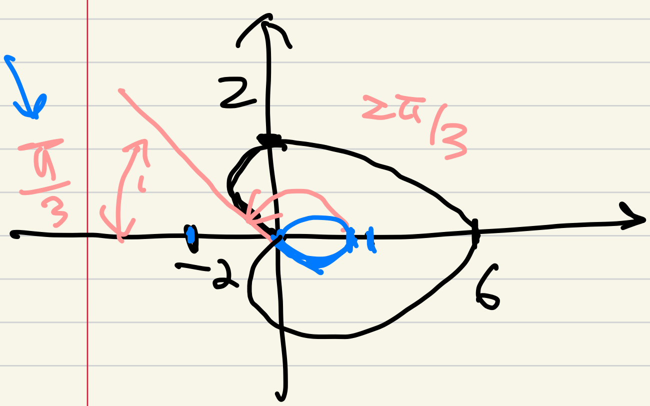
integration



$$\frac{1}{2} r_{min}^2(\theta)\Delta\theta \le \Delta A \le \frac{1}{2} r_{max}^2(\theta)\Delta\theta$$

Example.

$r = 2 + 4\cos\theta$, $\left(-2 \le 2+4\cos\theta \le 6\right)$.

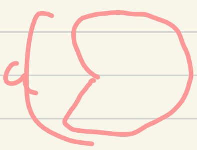


$2+4\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

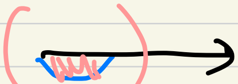
$\theta = \frac{2\pi}{3}$

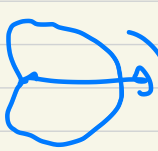
$\theta = \pi,\quad r = -2$

$$\int_0^{2\pi} \frac{1}{2}(2+4\cos\theta)^2\, d\theta =$$ area() + area()

$$= 2\int_0^{\pi} \frac{1}{2}(2+4\cos\theta)^2\, d\theta$$

$$= 2\underbrace{\boxed{\int_0^{2\pi/3} \frac{1}{2}(2+4\cos\theta)^2\, d\theta}}_{\text{area}} + 2\underbrace{\boxed{\int_{2\pi/3}^{\pi} \frac{1}{2}(2+4\cos\theta)^2\, d\theta}}_{\text{area}}$$

area    area ()

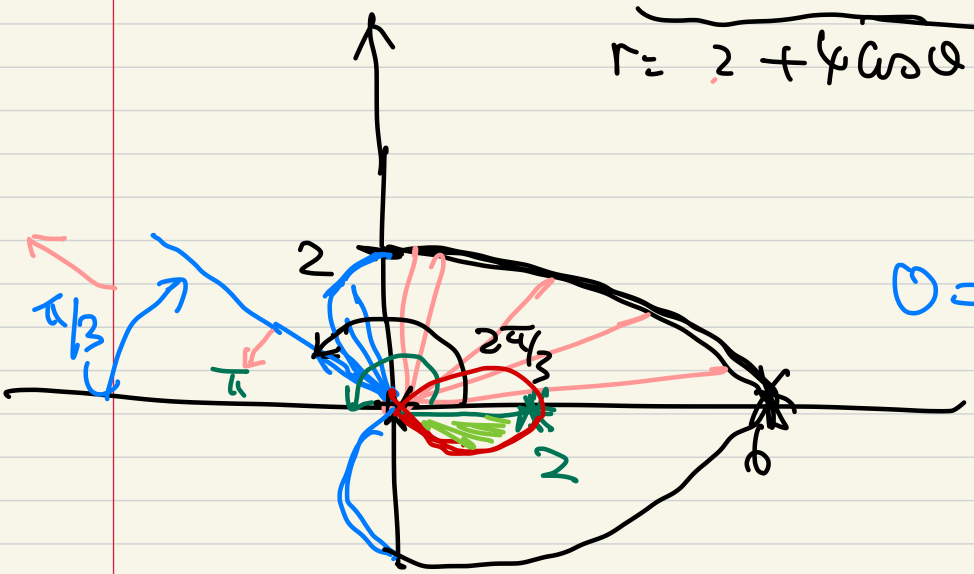
area() $= 2\cdot\int_0^{2\pi/3} \frac{1}{2}(2+4\cos\theta)^2\, d\theta$

area (o) $= 2\int_{2\pi/3}^{\pi} \frac{1}{2}(2+4\cos\theta)^2\, d\theta$

small loop inside

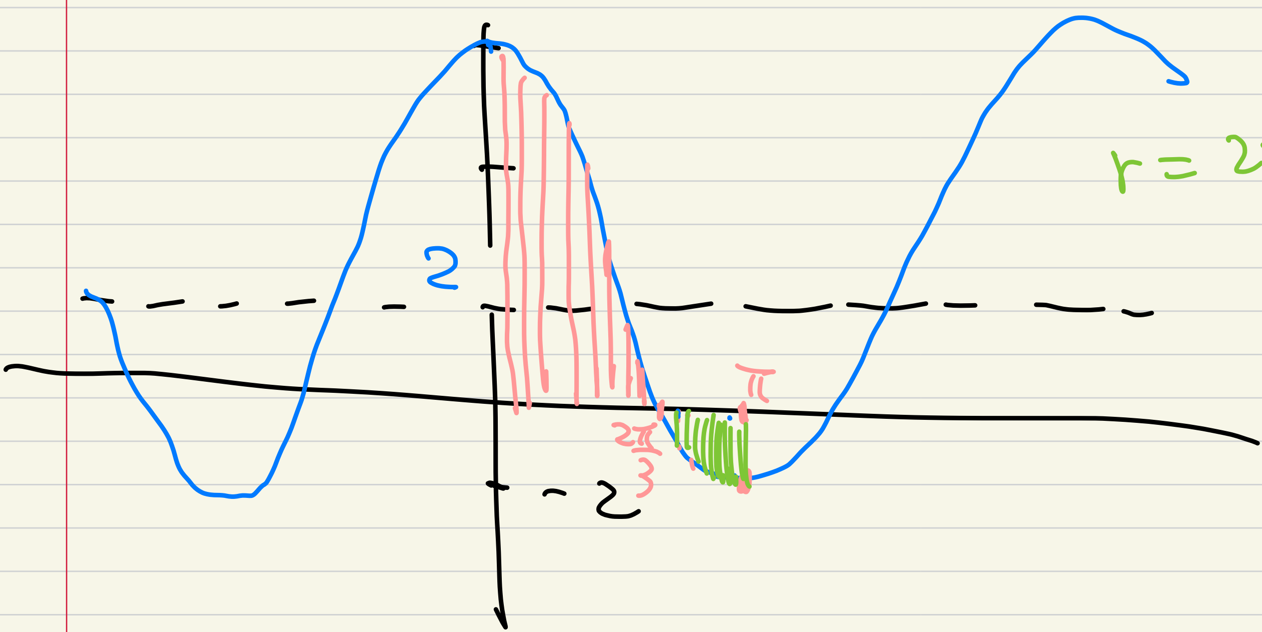
$r = 2 + 4\cos\theta$

θ	0	$\frac{\pi}{2}$	π
$r = 2 + 4\cos\theta$	$2+4 = 6$	2	-2



$0 = r = 2 + 4\cos\theta$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$



$r = 2 + 4\cos\theta.$