Lecture 14 Periodic Motion I

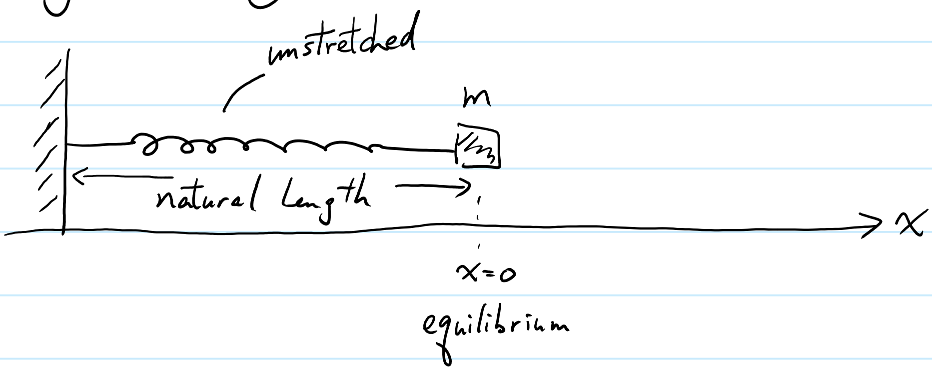
Periodic Motion = motion repeating in a fixed period of time.

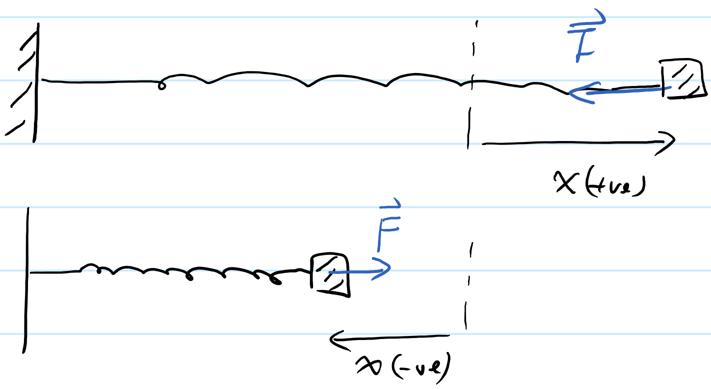
e.g. planetary motion, clock, pendulum, vibration, wood floating on water surface, even chemical reactions!

Simple Harmonic Oscillation (oscillator) — SHM

- Restoring force is linearly proportional to displacement from the equilibrium position.

Spring-Mass system.





Hooke's Law

$$\vec{F} = -kx\hat{i}$$

or $F_x = -kx$.

restoring force: push the mass back to equilibrium.

Equation of motion: $\vec{F} = m\vec{a}$

$$-kx = F_x = ma_x = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$a_x = -\frac{k}{m}x \quad \text{or} \quad \ddot{x} = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Second Order Differential Equation of the form.

$$\boxed{\ddot{x} = -\omega^2 x \quad , \quad \omega^2 > 0} \qquad (*)$$

Simple harmonic motion (SHM)

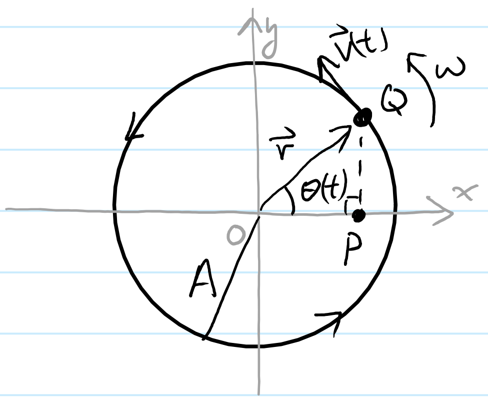
Solving $(*)$ means to find a function $x(t)$ such that it satisfies $(*)$.

Solution of diff. eqn is unique. i.e. once you find a $x(t)$ that satisfies $(*)$, then this $x(t)$ is the only solution.

How to find $x(t)$?

i.e. What $x(t)$ would have a 2nd derivative equal to a negative number times itself?

Consider a particle, Q, in uniform circular motion.

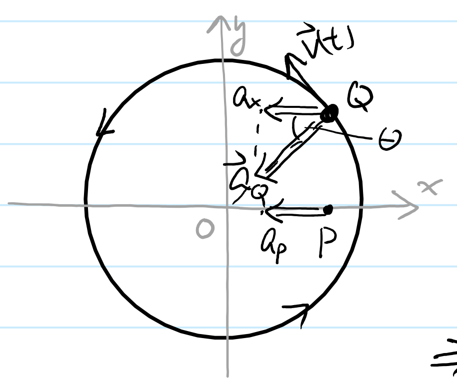


P is the projection of Q along x-axis.

i.e. x-component of $\vec{r}$

we know $\vec{r} = \begin{cases} x(t) = A\cos(\theta(t)) = x_P \\ y(t) = A\sin(\theta(t)) \end{cases}$

where $\theta(t) = \omega t$



for acceleration of Q : $\vec{a}_Q$

$$a_Q = \frac{V^2}{A} = A\omega^2 \quad , \quad V = A\omega$$

Acceleration of P : point to left.

$$a_P = a_x = -a_Q\cos\theta = -\omega^2 \underbrace{A\cos\theta}_{x_P}$$

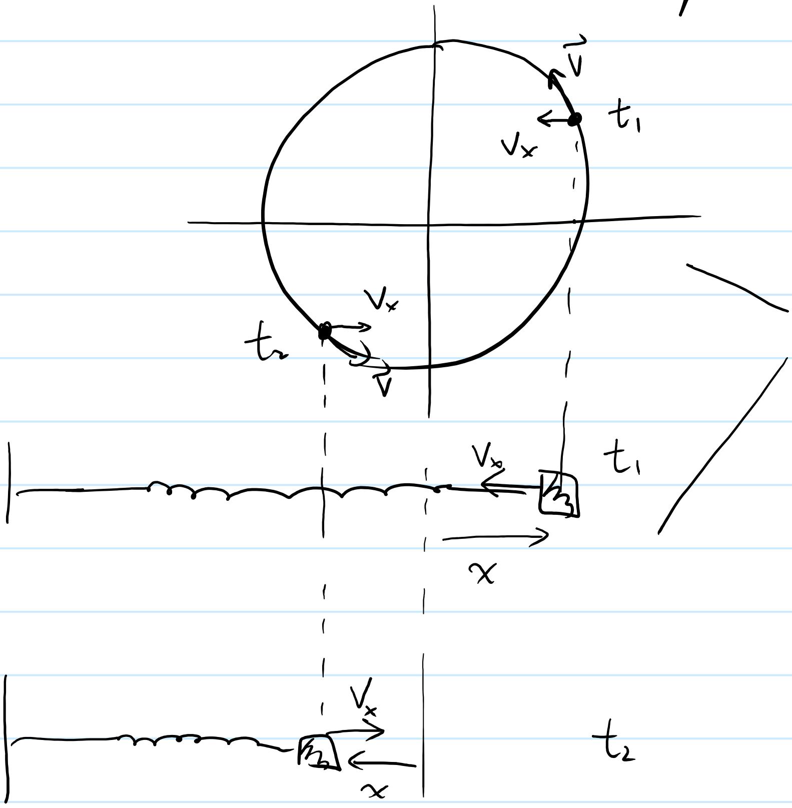
$$\Rightarrow \quad a_P = \ddot{x}_P = -\omega^2 x_P$$

Now we find a $x(t)$ such that $\ddot{x} = -\omega^2 x$

That is the position of the shadow of a uniform circular motion.

$$x(t) = A \cos \omega t \qquad \theta(t) = \omega t$$

To describe SHM, one just needs to consider a uniform circular motion with $R = A$ & $\omega^2 = \frac{k}{m}$, and take its horizontal component.



1-to-1 correspondence between circular motion and SHM

Compare $a_x = -\frac{k}{m}x$ to $a_x = -\omega^2 x$

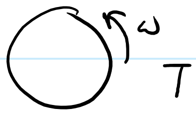
$$\boxed{\omega^2 = \frac{k}{m}.},$$

<u>In circular motion</u>

$$\omega = \frac{2\pi}{T}$$

ω: angular speed.

T: period in circular motion.



<u>Period in SHM</u> = Period in circular motion $= T$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

<u>Frequency</u>

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$ # of cycles per unit of time.

<u>Angular frequency</u>

$$\omega = 2\pi f.$$ instead of angular velocity.

<u>General Solution</u>.

$$x(t) = \underbrace{A}_{\text{Amplitude}}\cos(\omega t + \underbrace{\phi}_{\text{phase angle}})$$

(A, ϕ) are determined by initial condition. (x_0, v_0)

As a result: $v(t) = \dot{x}(t) = -\omega A \sin(\omega t + \phi) = -V_{max}\sin(\omega t + \phi)$

$a(t) = \ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x(t)$

ω depends on the parameter of the system.

$\omega = \sqrt{\frac{k}{m}}$ for spring-mass system.

To determine A and ϕ for a given initial condition, (x_0, V_0)

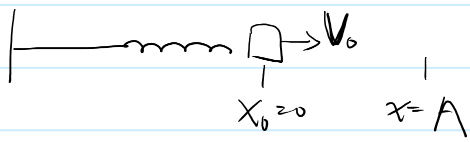
At $t=0$, $x(t=0) = x_0$, $V(t=0) = V_0$

$$\Rightarrow \begin{cases} x_0 = x(0) = A\cos\phi \\ V_0 = \dot{x}(0) = -\omega A \sin\phi \end{cases}$$

$$\Rightarrow \begin{cases} x_0^2 + \frac{V_0^2}{\omega^2} = A^2\cos^2\phi + A^2\sin^2\phi = A^2 \\ \tan\phi = -\frac{V_0}{\omega x_0} \end{cases}$$

$$\begin{cases} A = \sqrt{x_0^2 + \frac{V_0^2}{\omega^2}} \\ \phi = \begin{cases} \tan^{-1}\left(\frac{-V_0}{\omega x_0}\right) & \text{for } x_0 \geq 0 \\ \tan^{-1}\left(-\frac{V_0}{\omega x_0}\right) + \pi & \text{for } x < 0 \end{cases} \end{cases}$$

Example: at $t=0$

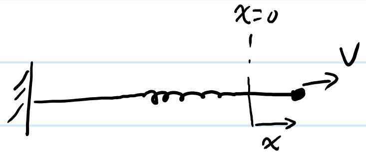


$$\Rightarrow \begin{cases} A = \frac{V_0}{\omega} \\ \phi = -\frac{\pi}{2} \end{cases} \Rightarrow x(t) = \frac{V_0}{\omega}\cos\left(\omega t + \frac{\pi}{2}\right)$$

Energy

$$E = K + U$$

$$K = \frac{1}{2} m v^2$$



$$U = \frac{1}{2} k x^2 , \qquad \omega^2 = \frac{k}{m}$$

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} \underline{m\,\omega^2} A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

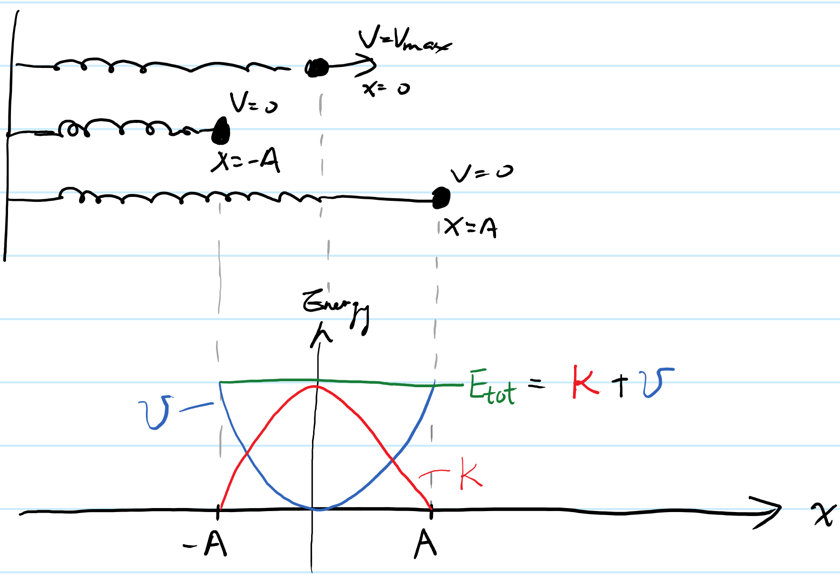
$$E = \frac{1}{2} k A^2 \quad \overset{\text{or}}{=} \quad \frac{1}{2} m \underbrace{\omega^2 A^2}_{V_{max}^2} = \frac{1}{2} m V_{max}^2$$

U at $x = A$ where $K = 0$ depends only on A.

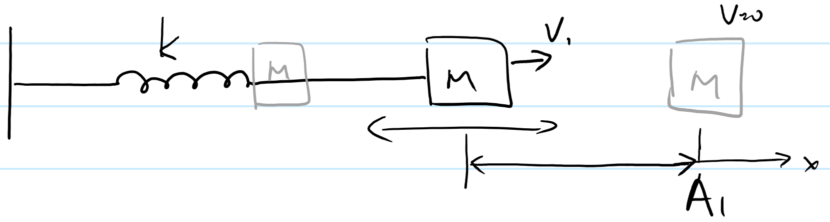
K at $x = 0$ where $U = 0$ depends only on V_{max}.

Since A and $V_{max} = \omega A$ are constant,

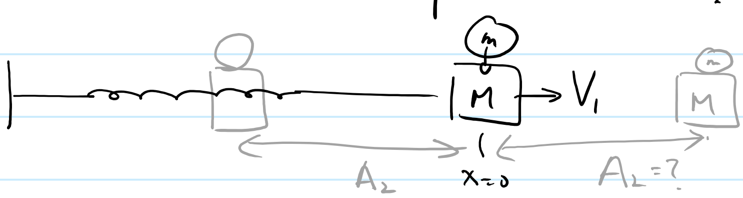
total mechanical energy in SHM is <u>conserved</u>.



Example



When the block is at $x=0$, a bubble gum of mass m is dropped and sticks on the block. Then they oscillate together. Find the new amplitude A_2.



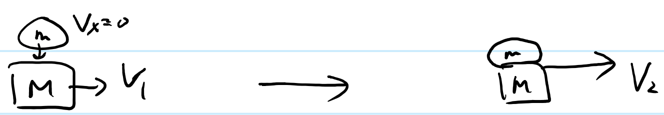
Before collision

By conservation of Energy

$$E = \underset{\text{at } x=0}{\frac{1}{2}MV_1^2} = \underset{\text{at } x=A}{\frac{1}{2}kA^2}$$

$$V_1 = \sqrt{\frac{k}{M}}\,A_1$$

Right after collision



$$P_i = P_f$$

$$MV_1 = (m+M)V_2$$

$$V_2 = \frac{M}{m+M}V_1 = \frac{M}{m+M}\cdot\sqrt{\frac{k}{M}}\,A_1$$

Oscillation after collision

$$E = \frac{1}{2}(m+M)V_2^2 = \frac{1}{2}kA_2^2$$

still the same spring, k,

$$\Rightarrow \quad A_2 = \sqrt{\frac{(M+m)}{k}}\,V_2 = \sqrt{\frac{(M+m)}{k}}\cdot\frac{M}{(m+M)}\sqrt{\frac{k}{M}}\,A_1$$

$$A_2 = \sqrt{\frac{M}{m+M}}\,A_1 < A_1$$

is ω the same?

What if the bubble gum is dropped when the block is at $x = A_1$?

Would $\begin{pmatrix} \text{amplitude} \\ \text{frequency} \\ V_{max} \\ E_{tot} \end{pmatrix}$ changes?

Hint: Both objects has zero velocity when they stick.
$\Rightarrow$ No relative velocity.

No kinetic friction $\Rightarrow$ No energy loss during collision.

$\Rightarrow$ Energy remains unchanged.