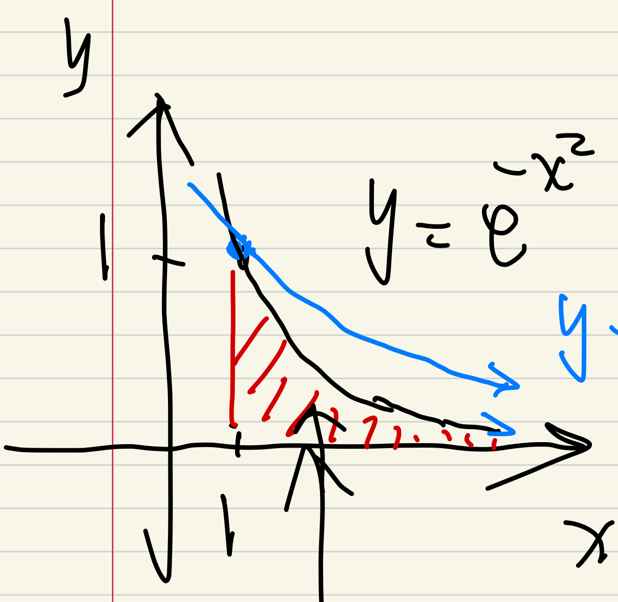
Convergence / Divergence of Improper Integrals by comparison

Example $\int_1^\infty e^{-x^2}\,dx$ — Convergent ? ~~divergent~~ ?

$$= \lim_{L\to\infty}\int_1^L \boxed{e^{-x^2}}\,dx$$

has no easy antiderivative !

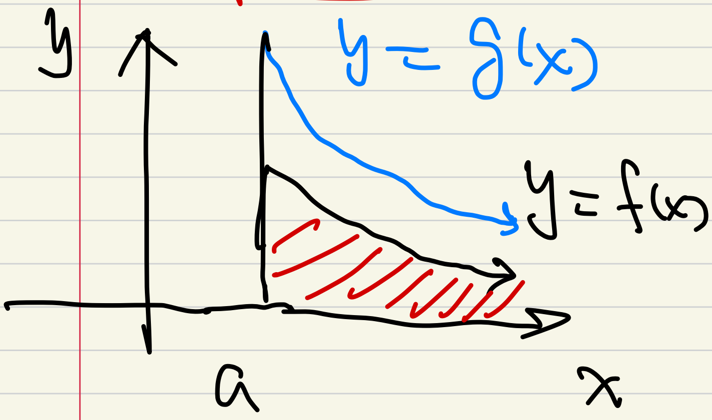


i.e. finite red area !

$e^{-x^2} \le e^{-x}$ for all $x \ge 1$.

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty = -\lim_{x\to\infty} e^{-x} + 1 = 1 < \infty.$$

Comparison Test



Let $f(x)$ be a positive function on $[a,\infty)$ and $g(x) \geq f(x)$.

$g(x)$

① If $\boxed{f(x) \leq g(x)}$ on $[a,\infty)$ and $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is also convergent.

② If $\boxed{f(x) \leq g(x)}$ on $[a,\infty)$ and $\int_a^\infty f(x)\,dx$ is divergent, then $\int_a^\infty g(x)\,dx$ is also divergent.

Example.

$$\int_1^\infty \frac{1}{x^3+\sqrt[3]{x}}\,dx$$ convergent? ~~divergent~~?

We know already that

$$\int_1^\infty \frac{1}{x^3}\,dx \text{ is convergent.}$$

$$= \left[\frac{x^{-2}}{-2}\right]_1^\infty = \frac{1}{2}$$

$$= \lim_{x\to\infty} \frac{x^{-2}}{-2} + \frac{1}{2}$$

For $x \geq 1$

$$0 < \frac{1}{x^3+\sqrt[3]{x}} \leq \frac{1}{x^3}$$

$$\int_1^\infty \frac{1}{x^3+\sqrt[3]{x}}\,dx < \int_1^\infty \frac{1}{x^3}\,dx < \infty$$

i.e. the integral is convergent.

$\rightarrow \int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } 0<p\le 1 \end{cases}$

Example

$$\int_1^\infty \boxed{\frac{\sqrt{x}}{x^2+2x+3}}\,dx$$ is convergent!

$$\frac{\sqrt{x}}{x^2+2x+3} \underset{\text{x very large}}{\approx} \frac{x^{\frac{1}{2}}}{x^2} = \frac{1}{x^{3/2}},$$

We also know $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent.

$$\frac{\sqrt{x}}{x^2+2x+3} \le \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

i.e. $\int_1^\infty \frac{\sqrt{x}}{x^2+2x+3}\,dx$ is convergent

since $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent.

Example.

$$\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx \quad \begin{cases} \text{convergent?} \\ \text{divergent?} \end{cases}$$

Note that

$$\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$$

Hence

$\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$ is divergent

Since $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ is divergent.

Example.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx \quad \hookrightarrow \quad \int \frac{1}{x^p}\,dx$$

$$= \int_1^\infty \frac{\ln x}{x^{1/2}\cdot x^{3/2}}\,dx$$

$$= \int_1^\infty \frac{2\ln x^{\frac{1}{2}}}{x^{1/2}}\,\frac{1}{x^{3/2}}\,dx$$

$$\leq \int_1^\infty \frac{2}{x^{3/2}}\,dx$$

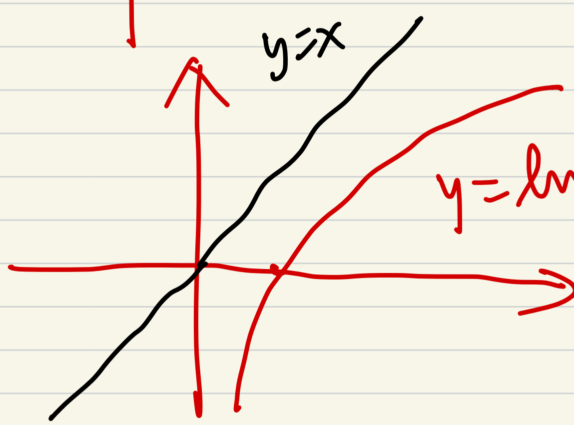
which is convergent.

e.g.

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx$$

is convergent.

$$\frac{\ln x}{x^{1/2}\,x^{3/2}} \overset{?}{\leq} \frac{1}{x^{3/2}}$$



$$\frac{\ln x}{x} < 1 \quad \text{for } x \geq 1$$

$$\frac{\ln\sqrt{x}}{\sqrt{x}} < 1$$

Therefore,

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

is also convergent.

Note that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

$$= \int_1^\infty -\ln x \; d\,x^{-1}$$

$\int u\,dv = uv - \int v\,du$

$$= -\frac{\ln x}{x}\Big|_1^\infty + \int_1^\infty \frac{1}{x}\cdot \boxed{\frac{1}{x}\,dx}$$

$d\ln x$

$$= -\lim_{x\to\infty} \frac{\ln x}{x} + \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{Convergent}}$$

$\frac{\infty}{\infty}$

L'Hopital's Rule

$$= -\lim_{x\to\infty} \frac{\frac{1}{x}}{1} + \int_1^\infty \frac{1}{x^2}\,dx$$

$$= 0$$

$$= \left[-x^{-1}\right]_1^\infty$$

$$= 1.$$

$$\int_1^\infty \frac{\ln x}{\cancel{\sqrt{x}} + x^2}\,dx$$ is also convergent since

$$\leq \int_1^\infty \frac{\ln x}{x^2}\,dx < \infty$$

Example. Same idea for the other kinds of improper integrals

$$\int_3^7 \underbrace{\frac{1}{\sqrt{(x-3)(x^2+x+1)}}}_{f(x)}\,dx$$ convergent? ✓ / ~~divergent~~?

$f(x) \to \infty$ as $x \to 3^+$

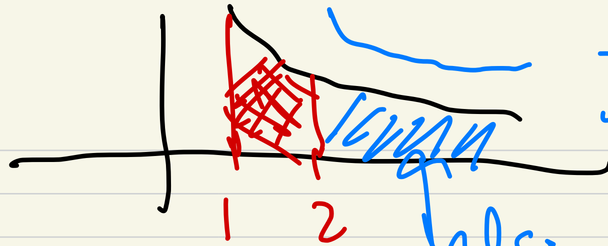
Just note that

$$0 \leq \frac{1}{\sqrt{(x-3)\underbrace{(x^2+x+1)}_{\geq 1}}} \leq \frac{1}{\sqrt{x-3}}$$

$$\int_3^7 \frac{1}{\sqrt{(x-3)(x^2+x+1)}}\,dx \leq \int_3^7 (x-3)^{-1/2}\,dx = \left[2(x-3)^{\frac{1}{2}}\right]_3^7$$

$$= 4 < \infty$$

Example:



$\frac{2}{3x^3}$

also finite

$$\int_1^\infty \boxed{\frac{x^2+3x+1}{2x^5-x^2-x+4}}\,dx$$

is "convergent",

$\rightarrow$ x large!

$$\frac{x^2}{2x^5} = \frac{1}{2x^3}$$

"Inequality"

$$\int_1^\infty \frac{1}{x^3}\,dx < \infty.$$

$$\frac{x^2+3x+1}{2x^5-x^2-x+4}$$

added after lecture

$$< \frac{x^2+3x^2+x^2}{2x^5-x^2-x+4} = \frac{5x^2}{2x^5\left(1-\frac{1}{x^3}-\frac{1}{x^5}+\frac{4}{x^5}\right)}$$

$$< \frac{5}{2x^3\left(1-\frac{2}{x^3}\right)}$$

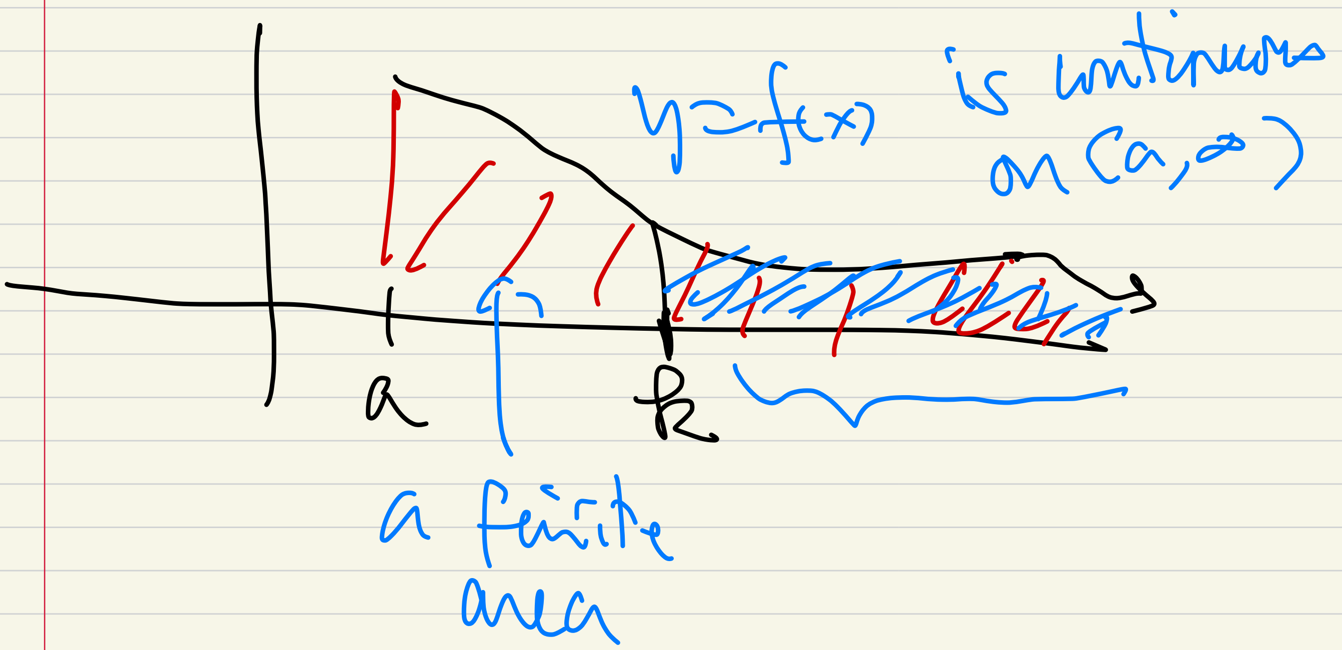
$x \geq 2$

$$< \frac{5}{\cancel{2}x^3\cdot\frac{3}{\cancel{4}\,2}}$$

$$1-\frac{2}{x^3} > 1-\frac{2}{8} = \frac{3}{4}$$

Remark.

Note that $\int_a^{\infty} f(x)\,dx$ is convergent if and only if $\int_k^{\infty} f(x)\,dx$ is convergent for some $k \geq a$.



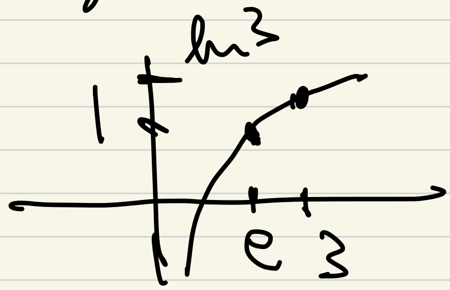
① $\int_a^\infty (f(x)+g(x))\,dx$
$$= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$
whenever the integrals on the right hand side converge.

② $$\int_a^\infty k f(x)\,dx = k\int_a^\infty f(x)\,dx$$
for any constant k.

③ If $\int_a^\infty |f(x)|\,dx$ is convergent, so is $\int_a^\infty f(x)\,dx$.

Example: $\int_2^\infty \frac{1}{x^p \ln x}\,dx$ $\to$ $p=1$ divergent!!

for which p is the integral convergent.



e.g. $p=1$

$$\int_2^\infty \frac{1}{x^1 \ln x}\,dx$$

$$= \int_2^3 \frac{1}{x^1 \ln x}\,dx + \boxed{\int_3^\infty \frac{1}{x \ln x}\,dx}$$

$$\int_2^\infty \frac{1}{\ln x}\,d\ln x$$

$$= \ln|\ln x| \Big|_2^\infty$$

$$= \infty$$