Lecture 12

§4.1 Vector spaces and subspaces

Def: A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in V and for all scalars c and d.

1. The sum of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}+\vec{v}$, is in V.
2. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
3. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
4. There is a zero vector $\vec{0}$ in V such that $\vec{u}+\vec{0} = \vec{u}$.
5. For each $\vec{u}$ in V, there is a vector $-\vec{u}$ in V such that $\vec{u}+(-\vec{u}) = \vec{0}$.
6. The scalar multiple of $\vec{u}$ by c, denoted by $c\vec{u}$, is in V.
7. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u}+d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\cdot\vec{u} = \vec{u}$

Facts: For each $\vec{u}$ in V and scalar c,

1) $0\vec{u} = \vec{0}$

2) $c \cdot \vec{0} = \vec{0}$

3) $-\vec{u} = (-1)\vec{u}$

* Subspaces

Def: A subspace of a vector space V is a subset H of V that has three properties:

a) The zero vector of V is in H.

b) H is closed under vector addition. That is, for each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u}+\vec{v}$ is in H

c) H is closed under multiplication by scalars. That is, for each $\vec{u}$ in H and each scalar c, the vector $c\vec{u}$ is in H.

An equivalent definition, H is a subspace of V iff for $\vec{u}$ and $\vec{v}$ in H and any scalar a and b, $a\vec{u}+b\vec{v}$ is in H.

Ex: The set consisting of zero vector only in a vector space is a subspace V, called the zero subspace.

Ex: $H = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^3$ is a subspace

$H' = \{(x, y, 1) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^3$ is not a subspace.

Ex: $P_n = \{$all polynomials in variable x with degree $\leq n\}$

$P_m = \{$all polynomials in variable x with degree $\leq m\}$

if $m < n$, then $P_m \subset P_n$ is a subspace

$Q_n = \{$all polynomials f in $P_n \mid f(0) = 1\} \subset P_n$ is not a subspace.

* A subspace spanned by a set.

Ex: Given $\vec{v}_1$ and $\vec{v}_2$ in a vector space V, let $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$. Then H is a subspace of V.

Solution: Let $\vec{u}$ and $\vec{v}$ be two vectors in H,

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2, \quad \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2$$

then
$$c\vec{u} + d\vec{v} = c(a_1\vec{v}_1 + a_2\vec{v}_2) + d(b_1\vec{v}_1 + b_2\vec{v}_2)$$
$$= (ca_1 + db_1)\vec{v}_1 + (ca_2 + db_2)\vec{v}_2$$

is in H.

By the definition, H is a subspace of V.

Thm: If $\vec{v}_1, \cdots, \vec{v}_p$ are in a vector space V, then $\text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a subspace of V.

H is called the subspace spanned (or generated) by $\{\vec{v}_1, \cdots, \vec{v}_p\}$. $H = \text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$

Given any subspace H of V, a spanning (or generating) set for H is a set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in H such that $H = \text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$.

Ex: $H = \{(a-3b, b-a, a, b) \mid a, b \in \mathbb{R}\}$

show H is a subspace.

Solution: $$\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -3b \\ b \\ 0 \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Then $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$ is a subspace.

Ex: For what values of h will $\vec{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$, if

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$

Solution: $\vec{y} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ has a solution for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

iff $\vec{y}$ is in H.

Thus solve the system

$$\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix}$$

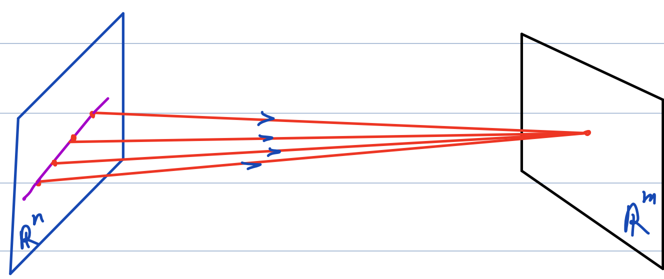
The system has a solution iff $h-5=0$.
Thus $h=5$.

§4.2 Null Spaces, Column Spaces and Linear transformations

* The Null space of a matrix

Def: The null space of an $m\times n$ matrix A, written as $\mathrm{Nul}\,A$, is the set of all solutions of the homogeneous equation $A\vec{x}=\vec{0}$. In set notation,

$$\mathrm{Nul}\,A = \{\vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x}=\vec{0}\}.$$



$\mathrm{Null}\,A$ is the set of all $\vec{x}\in\mathbb{R}^n$ that are mapped into the zero vector of $\mathbb{R}^m$ via the linear transformation $\vec{x}\mapsto A\vec{x}$.

Ex: Let $A=\begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$ and $\vec{u}=\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$. Is $\vec{u}$ in $\mathrm{Nul}\,A$?

Solution: $A\vec{u} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Yes

Thm: The null space of an $m\times n$ matrix A is a subspace of $\mathbb{R}^n$.

Proof: If $\vec{u}, \vec{v}$ are in $\mathrm{Nul}A$, i.e. $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0}$

Then $A(a\vec{u} + b\vec{v}) = aA\vec{u} + bA\vec{v} = \vec{0}$

So $a\vec{u} + b\vec{v}$ is in $\mathrm{Nul}A$.

Thus $\mathrm{Nul}A$ is a subspace.

Ex: Let $H = \{(a, b, c, d) \mid a-2b+5c=d,\ c-a=b\}$

show that H is a subspace of $\mathbb{R}^4$.

Prof. $H = \mathrm{Nul}A$. $A = \begin{pmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$

Since $\begin{cases} a-2b+5c=d \\ c-a=b \end{cases}$

is equivalent to $\begin{cases} a-2b+5c-d=0 \\ -a-b+c=0 \end{cases}$

* An Explicit Description of Null A

Example: Find a spanning set for the null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Solution:
$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{cases}$$

The general solution is
$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$
with x_2, x_4, x_5 free

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}} + x_5 \underbrace{\begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{w}}$$

$$\triangleq x_2 \vec{u} + x_4 \vec{v} + x_5 \vec{w}$$

$\{\vec{u}, \vec{v}, \vec{w}\}$ is a spanning set for Nul A.

Remark: When Nul A contains nonzero vectors, the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\vec{x} = \vec{0}$.

* The Column Space of a Matrix

Def: The column space of an $m \times n$ matrix A, written as $\text{Col}\,A$, is the set of all linear combinations of the columns of A. If $A = [\vec{a}_1, \cdots, \vec{a}_n]$, then

$$\text{Col}\,A = \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$$
$$= \{\vec{b} \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}$$

Thm: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Ex: Find a matrix A such that $W = \text{col}\,A$.

$$W = \left\{ \begin{pmatrix} 6a-b \\ a+b \\ -7a \end{pmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

Solution: $$W = \left\{ a\begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Let $A = \begin{pmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{pmatrix}$. Then $W = \text{Col}\,A$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^m$.

The Contrast Between Nul A and Col A

Ex: Let

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

Solution: a) $k=3$

b) $k=4$

Ex: Let

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix},$$

find a nonzero vector in Col A and a nonzero vector in Nul A.

Solution: A nonzero vector in col A: $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

To find a nonzero in Nul A, we need to solve the equation $A\vec{x} = \vec{0}$.

$$[A \ \vec{0}\,] \sim \begin{pmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 9x_3 = 0 \\ x_2 - 5x_3 = 0 \\ x_4 = 0 \end{cases}$$

So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -9 \\ 5 \\ 1 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} -9 \\ 5 \\ 1 \\ 0 \end{pmatrix}$ is in Nul A.

Ex: Let

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 3 \\ -2 \\ -1 \\ 0 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}.$$

a) Determine if $\vec{u}$ is in Nul A. Could $\vec{u}$ be in Col A?

b) Determine if $\vec{v}$ is in Col A. Could $\vec{v}$ be in Nul A?

Solution: a) $A\vec{u} = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So $\vec{u}$ is not in Nul A.

$\vec{u}$ cann't be in Col A since col A is a subspace of $\mathbb{R}^3$ and $\vec{u}$ is a vector in $\mathbb{R}^4$.

b) $[A \ \vec{v}] = \begin{pmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{pmatrix}$

$$\sim \begin{pmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{pmatrix}$$

Since the last column is not a pivot column, $A\vec{x} = \vec{v}$ has a solution. Thus $\vec{v}$ is in Col$A$. $\vec{v}$ cannot be in Nul$A$ since Nul$A$ is a subspace of $\mathbb{R}^4$ and $\vec{v}$ is a vector in $\mathbb{R}^3$.