Math1014 Calculus II
Improper Integrals

- Evaluating *improper integrals* by taking suitable limits;
- determining *convergence or divergence of improper integrals* by comparison.

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(i) $\int_{-\infty}^{0} \frac{1}{2x-5}dx$ (ii) $\int_{0}^{\infty} \frac{x}{(x^2+2)^2}dx$ (iii) $\int_{-\infty}^{1} e^{-2t}dt$

(iv) $\int_{0}^{\infty} \frac{1}{z^2+3z+2}dz$ (v) $\int_{-\infty}^{6} re^{r/3}dr$ (vi) $\int_{2}^{3} \frac{1}{\sqrt{3-x}}dx$

(vii) $\int_{6}^{8} \frac{1}{(x-6)^3}dx$ (viii) $\int_{0}^{2} \frac{e^{1/x}}{x^3}dx$ (ix) $\int_{0}^{1} \frac{\ln x}{\sqrt{x}}dx$

(i) $\int_{-\infty}^{0} \frac{1}{2x-5}dx = \lim_{L\to-\infty}\int_{L}^{0} \frac{1}{2x-5}dx$

$= \lim_{L\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_L^0$

$= \frac{1}{2}\ln 5 - \lim_{L\to-\infty}\frac{1}{2}\ln|2L-5| = -\infty$

Divergent.

(ii) $\int_{0}^{\infty} \frac{x}{(x^2+2)^2}dx = \lim_{L\to\infty}\int_{0}^{L} \frac{x}{(x^2+2)^2}dx$

$= \lim_{L\to\infty}\left[-\frac{1}{2}(x^2+2)^{-1}\right]_0^L$

$= -\lim_{L\to\infty}\frac{1}{2(L^2+2)} + \frac{1}{4} = \frac{1}{4}$

Convergent.

(iii) $\int_{-\infty}^{1} e^{-2t}dt = \lim_{L\to-\infty}\int_{L}^{1} e^{-2t}dt$

$= \lim_{L\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_L^1$

$= -\frac{1}{2}e^{-2} + \lim_{L\to-\infty}\frac{1}{2}e^{-2L} = \infty$

Divergent.

(iv) $\int_{0}^{\infty} \frac{1}{z^2+3z+2}dz = \int_{0}^{\infty}\frac{1}{(z+1)(z+2)}dz$

$= \lim_{L\to\infty}\int_0^L\left[\frac{1}{z+1} - \frac{1}{z+2}\right]dz$

$= \lim_{L\to\infty}\Big[\ln|z+1| - \ln|z+2|\Big]_0^L = \lim_{L\to\infty}\left[\ln\frac{|z+1|}{|z+2|}\right]_0^L$

$= \lim_{L\to\infty}\ln\left|\frac{L+1}{L+2}\right| - \ln\frac{1}{2} = \ln 1 + \ln 2 = \ln 2$

Convergent.

(v) $\int_{-\infty}^{6} re^{r/3}dr = \int_{-\infty}^{6} 3r\,de^{r/3}$

$= 3re^{r/3}\Big|_{-\infty}^{6} - \int_{-\infty}^{6} 3e^{r/3}dr = 3re^{r/3}\Big|_{-\infty}^{6} - 9e^{r/3}\Big|_{-\infty}^{6}$

$= 9e^2 - \lim_{r\to-\infty}(3re^{r/3} - 9e^{r/3}) = 9e^2$

Convergent.

(vi) $\int_{2}^{3} \frac{1}{\sqrt{3-x}}dx = \lim_{L\to 3^-}\int_2^L (3-x)^{-1/2}dx$

$= \lim_{L\to 3^-}\left[-2(3-x)^{1/2}\right]_2^L$

$= -\lim_{L\to 3^-} 2(3-L)^{1/2} + 2 = 2$

Convergent.

(vii) $\int_{6}^{8} \frac{1}{(x-6)^3}dx = \lim_{L\to 6^+}\int_L^8 (x-6)^{-3}dx$

$= \lim_{L\to 6^+}\left[-\frac{1}{2}(x-6)^{-2}\right]_L^8$

$= -\frac{1}{8} + \lim_{L\to 6^+}\frac{1}{2(L-6)^2} = \infty$

Convergent.

(viii) $\int_{0}^{2} \frac{e^{1/x}}{x^3}dx \geq \int_0^2 \frac{e^{1/2}}{x^3}dx$

$= \lim_{L\to 0^+}\left[-\frac{1}{2}e^{1/2}x^{-2}\right]_L^2$

$= -\frac{1}{8}e^{1/2} + \lim_{L\to 0^+}\frac{e^{1/2}}{2L^2} = \infty$

Divergent.

(ix) $\int_{0}^{1} \frac{\ln x}{\sqrt{x}}dx = \int_0^1 2\ln x\,dx^{1/2} = 2x^{1/2}\ln x\Big|_0^1 - \int_0^1 2x^{1/2}d\ln x = -\lim_{x\to 0^+}2x^{1/2}\ln x - \int_0^1 2x^{1/2}\cdot\frac{1}{x}dx$

$= -\lim_{x\to 0^+}\frac{2\ln x}{x^{-1/2}} - \int_0^1 2x^{-1/2}dx \overset{L'Hospital}{=} -\lim_{x\to 0^+}\frac{\frac{2}{x}}{-\frac{1}{2}x^{-3/2}} - 4x^{1/2}\Big|_0^1 = 0 - 4 = -4$

2. The average speed of molecules in an ideal gas is
$$\bar{v} = \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}dv$$
where M is the molecular weight of the gas, R is the gas constant, T is the temperature, and v is the molecular speed. Show that $\bar{v} = \sqrt{\dfrac{8RT}{\pi M}}$.

Integration by parts!
$$\bar{v} = \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}dv = \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\int_0^\infty -\frac{RT}{M}v^2 de^{-Mv^2/(2RT)}$$
$$= -\frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\frac{RT}{M}v^2 e^{-Mv^2/(2RT)}\Big|_0^\infty + \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\int_0^\infty \frac{RT}{M}e^{-Mv^2/(2RT)}dv^2$$
$$= \lim_{v\to\infty} -\frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\frac{RT}{M}\frac{v^2}{e^{Mv^2/(2RT)}} + \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\Big[-\frac{RT}{M}\frac{2RT}{M}e^{-Mv^2/(2RT)}\Big]_0^\infty$$
$$= \frac{4}{\sqrt{\pi}}\Big(\frac{M}{2RT}\Big)^{3/2}\cdot\frac{RT}{M}\frac{2RT}{M} = \sqrt{\frac{8RT}{\pi M}}$$
(By L'Hospital's rule, $\lim_{v\to\infty}\dfrac{v^2}{e^{Mv^2/(2RT)}} = \lim_{v\to\infty}\dfrac{2v}{\frac{2Mv}{2RT}e^{Mv^2/(2RT)}} = 0$.)

3. (§7.8, Q 78.) Find the value of the constant C for which the integral
$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx$$
converges. Evaluate the integral for this value of C.
$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx = \lim_{L\to\infty}\int_0^L \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx$$
$$= \lim_{L\to\infty}\Big[\frac{1}{2}\ln(x^2+1) - \frac{C}{3}\ln|3x+1|\Big]_0^L = \lim_{L\to\infty}\Big[\frac{1}{2}\ln(L^2+1) - \frac{C}{3}\ln|3L+1|\Big]$$
$$= \lim_{L\to\infty}\ln\frac{(L^2+1)^{1/2}}{(3L+1)^{C/3}} = \lim_{L\to\infty}\ln\frac{L(1+1/L^2)^{1/2}}{L^{C/3}(3+1/L^{C/3})^{C/3}}$$
So the limits exist if and only if $C/3 = 1$, i.e. ,$C = 3$. Moreover, for $C = 3$,
$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx = \lim_{L\to\infty}\ln\frac{L(1+1/L^2)^{1/2}}{L(3+1/L)} = \ln\frac{1}{3}$$

4. (§7.8, Q80.) Show that if $a > -1$, and $b > a+1$, then the following integral is convergent.
$$\int_0^\infty \frac{x^a}{1+x^b}dx$$
Note that for $x \geq 1$, $0 \leq \dfrac{x^a}{1+x^b} \leq \dfrac{x^a}{x^b} = x^{a-b}$. Hence for $a-b+1 < 0$,
$$\int_1^\infty \frac{x^a}{1+x^b}dx \leq \int_1^\infty x^{a-b}dx = \lim_{L\to\infty}\int_1^L x^{a-b}dx$$
$$= \lim_{L\to\infty}\Big[\frac{1}{a-b+1}x^{a-b+1}\Big]_1^L = \lim_{L\to\infty}\frac{1}{a-b+1}L^{a-b+1} - \frac{1}{a-b+1} = -\frac{1}{a-b+1}$$
Note that
$$\int_0^\infty \frac{x^a}{1+x^b}dx = \int_0^1 \frac{x^a}{1+x^b}dx + \int_1^\infty \frac{x^a}{1+x^b}dx$$
but we have to be careful with $x = 0$, since $x^a \longrightarrow \infty$ as $x \to 0^+$ if $a < 0$. Now, if $a+1 > 0$,
$$\int_0^1 \frac{x^a}{1+x^b}dx = \lim_{L\to 0^+}\int_L^1 \frac{x^a}{1+x^b}dx \leq \lim_{L\to 0^+}\int_L^1 x^a dx = \lim_{L\to 0^+}\Big[\frac{1}{a}x^{a+1}\Big]_L^1 = \frac{1}{a} - \lim_{L\to 0^+}\frac{1}{a}L^{a+1} = \frac{1}{a}$$
Hence the improper integral converges if $a > -1$ and $b > a+1$.