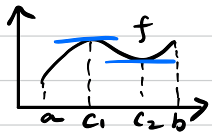
Review: Applications of differentiation.

1. Use Fermat's Theorem to find absolute maximum and minimum of $f(x)$.

(1) Fermat's Theorem: If $f'(c)$ exists and $f(c)$ is a local maximum or minimum, then $f'(c)=0$. $\Rightarrow$ tangent line at $(c, f(c))$ is horizontal.

Example:



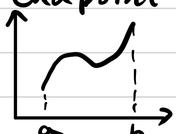
$f(c_1)$: local maximum.
$f(c_2)$: local minimum. $\Big\} \Rightarrow f'(c_1)=f'(c_2)=0$.

Notice: $f'(c)=0$ does not imply c is a local maximum or minimum. (e.g. $f(x)=x^3$, $f'(0)=0$)

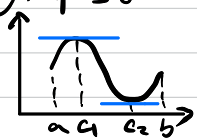
(2). Suppose $f(x)$ is continuous on a closed interval $[a, b]$.

Three cases for location of absolute maximum and minimum.

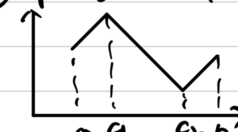
① endpoint



②. $f'=0$.



③ f' does not exist.



	①	②	③
absolute maximum:	$f(b)$	$f(c_1)$ → also a local maximum	← $f(c_1)$
absolute minimum:	$f(a)$	$f(c_2)$ → also a local minimum	← $f(c_2)$

($f'(c_1)$ and $f'(c_2)$ does not exist)

Summary: To find absolute maximum and minimum of f on $[a, b]$, we only need to compare the values of f at critical numbers and end points.

↙ definition.

c is called a critical number of f if $f'(c)=0$ or $f'(c)$ does not exist.

Example 1: Find absolute maximum and minimum of $f(x) = x^3 - 6x$ on $[-2, 6]$.

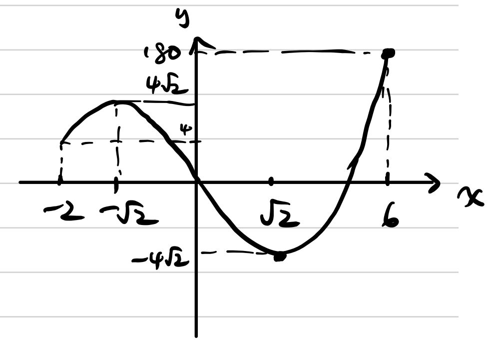
Step 1: Find values of f at all critical numbers.

$$f'(x) = 3\cdot x^2 - 6 = 3(x - \sqrt{2})(x + \sqrt{2}).$$

$$f'(-\sqrt{2}) = f'(\sqrt{2}) = 0 \Rightarrow \text{critical numbers : } -\sqrt{2} \text{ and } \sqrt{2}.$$

$$\boxed{f(-\sqrt{2})} = -2\sqrt{2} + 6\cdot\sqrt{2} = 4\sqrt{2}. \qquad \boxed{f(\sqrt{2})} = 2\cdot\sqrt{2} - 6\sqrt{2} = -4\sqrt{2}.$$

Step 2: Calculate $f(-2)$ and $f(6)$.



$$\boxed{f(-2)} = -8 + 12 = 4.$$

$$\boxed{f(6)} = 6^3 - 36 = 180.$$

$\Rightarrow$ absolute maximum : $f(6) = 180$.

absolute minimum : $f(\sqrt{2}) = -4\sqrt{2}$.

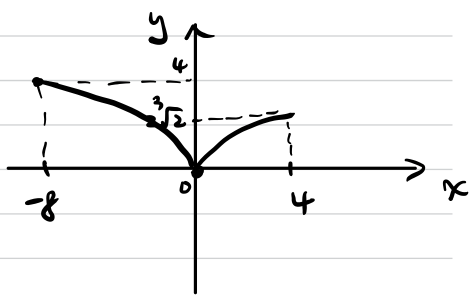
Example 2: Find absolute maximum and minimum of $y = f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2}$ on $[-8, 4]$.

Step 1: Find the values of f at all critical numbers.

$$f'(x) = \frac{2}{3} \cdot x^{-\frac{1}{3}} = \frac{2}{3} \cdot \frac{1}{\sqrt[3]{x}} .$$

critical number: 0.

$\boxed{f(0)} = 0$.



Step 2: Calculate $f(-8)$ and $f(4)$.

$\boxed{f(-8)} = \sqrt[3]{64} = 4$.

$\boxed{f(4)} = \sqrt[3]{16} = 2\sqrt[3]{2}$.

$\Rightarrow$ absolute maximum: $f(-8) = 4$

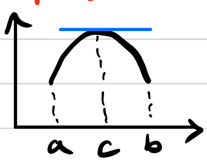
absolute minimum: $f(0) = 0$.

2. Rolle's Theorem and Mean Value Theorem.

(1). Rolle's Theorem : (a special case of Mean Value Theorem).

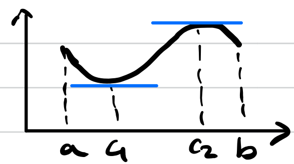
$$\left.\begin{array}{l} f \text{ is continuous on } [a,b] \\ f \text{ is differentiable on } (a,b). \\ f(a) = f(b) \end{array}\right\} \Rightarrow \text{There exists a number } \underbrace{c \in (a,b)}_{\to \text{ may not be unique}} \text{ such that } f'(c)=0.$$

Examples:



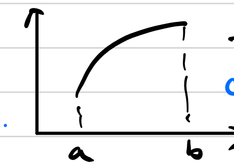
$f(a) = f(b)$.

$f'(c) = 0$.



$f(a) = f(b)$

$f'(c_1) = f'(c_2) = 0$.



$f(a) \neq f(b)$

c does not exist.

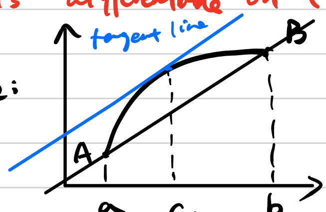
(2). Mean Value Theorem

We can always find a number $c \in (a,b)$ such that the tangent line at $(c, f(c))$ is parallel to the secant line AB.

$\Uparrow$ means

$$\left.\begin{array}{l} f \text{ is continuous on } [a,b] \\ f \text{ is differentiable on } (a,b) \end{array}\right\} \Rightarrow \text{There exists a number } \underbrace{c \in (a,b)}_{\to \text{ may not be unique.}} \text{ such that } f'(c) = \frac{f(b)-f(a)}{b-a}.$$

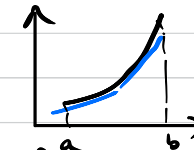
Example:



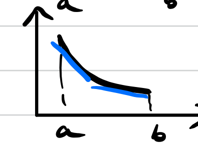
the slope of secant line AB $= \dfrac{f(b) - f(a)}{b - a}$

the slope of tangent line at $(c, f(c))$ is $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Now we can use the Mean Value Theorem to prove :



① If $f'(x) > 0$ on (a,b), then f is increasing on $(a.b)$.



② If $f'(x) < 0$ on (a,b), then f is decreasing on $(a.b)$.

proof of ① : Let x_1 and x_2 be two numbers satisfying $a < x_1 < x_2 < b$.

Our aim is to prove $f(x_2) > f(x_1)$.

By the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = \underbrace{f'(c)}_{\text{positive by assumption}} \cdot \underbrace{(x_2 - x_1)}_{\text{positive because } x_1 < x_2} .$$

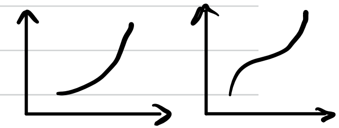
Therefore, $f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_2) > f(x_1)$. $\Rightarrow$ f is increasing on (a,b).

proof of ② is similar.

3. How $f'(x)$ affects the shape of the graph of $f(x)$.

(1) $f'(x) > 0$ on (a,b) $\Rightarrow$ f is increasing on $(a.b)$. examples:

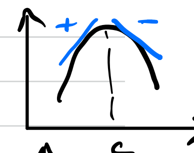


$f'(x) < 0$ on (a,b) $\Rightarrow$ f is decreasing on $(a.b)$ examples:

(2). Suppose that c is a critical number. ($f'(c)=0$ or $f'(c)$ does not exist).

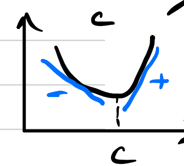
① $f'(x)$ changes sign from "+" to "−" at c $\Rightarrow$ $f(c)$ is a local maximum.

→ f is increasing when $x<c$ and decreasing when $x>c$.

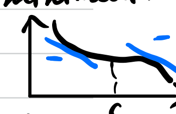


②. $f'(x)$ changes sign from "−" to "+" at C $\Rightarrow$ $f(c)$ is a local minimum.

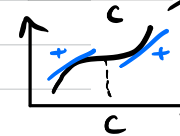
→ f is decreasing when $x<C$ and increasing when $x>C$.



③. $f'(x)$ does not change sign at C $\Rightarrow$ $f(c)$ is not a local maximum or minimum.



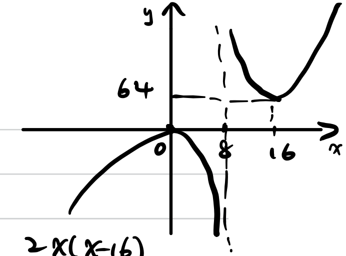
Summary: $f'(x) \Rightarrow \begin{cases} \text{Find the interval where } f \text{ is increasing or decreasing.} \\ \text{Find the local maximum and minimum.} \end{cases}$



Example 1: Let $f(x) = \frac{2x^2}{x-8}$. Domain of $f = \{x \mid x \neq 8\}$.

Find all intervals where f is increasing or decreasing.

Find all local maximum and minimum.



$$f'(x) = \frac{(2x^2)' \cdot (x-8) - 2 \cdot x^2 \cdot (x-8)'}{(x-8)^2} = \frac{4x \cdot (x-8) - 2x^2 \cdot 1}{(x-8)^2} = \frac{2x^2 - 32x}{(x-8)^2} = \frac{2x(x-16)}{(x-8)^2}.$$

① . f is increasing where $f'(x) > 0 \Rightarrow$ We need to solve: $\frac{2x(x-16)}{\underbrace{(x-8)^2}_{\text{positive}}} > 0$.

solve: $\frac{2x(x-16)}{(x-8)^2} > 0 \overset{\text{equivalent}}{\Longleftrightarrow}$ solve: $\underbrace{2x(x-16) > 0}_{x<0 \text{ or } x>16}$ and $x \neq 8$. $\Rightarrow x \in (-\infty, 0) \cup (16, +\infty)$.

Therefore, f is increasing on $(-\infty, 0) \cup (16, +\infty)$.

② f is decreasing where $f'(x) < 0 \Rightarrow$ We need to solve $\frac{2x(x-16)}{\underbrace{(x-8)^2}_{\text{positive}}} < 0$.

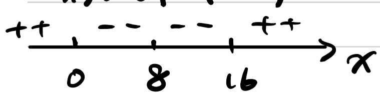
$\Rightarrow$ solve $\underbrace{2x(x-16) < 0}_{0<x<16}$ and $x \neq 8$. $\Rightarrow x \in (0, 8) \cup (8, 16)$.

Therefore, f is decreasing on $(0, 8) \cup (8, 16)$.

③. We first find all critical numbers. $\Rightarrow$ Find x such that $f'(x) = 0$ or $f'(x)$ does not exist.

$\Rightarrow x = 0, 16$ are critical numbers.

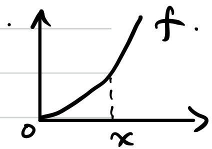
sign of $f'(x)$



$f(0)$ is a local maximum. $f(16)$ is a local minimum.

Example 2: Prove: $e^x > 1 + x$ for all $x > 0$. ← use the derivative to prove inequalities.

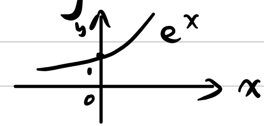
Define $f(x) = e^x - (1+x)$. We have $f(0) = e^0 - 1 = 1 - 1 = 0$.



Now we only need to prove f is increasing on $(0, +\infty)$.

(Because if f is increasing on $(0,+\infty)$, then for any $x>0$ we have $f(x) > f(0) = 0$).

$f'(x) = e^x - 1 > 0$ for all $x > 0$.



Therefore, f is increasing on $(0, +\infty)$. $\Rightarrow$ $f(x) > f(0)$ for all $x > 0$.

$\Rightarrow$ $e^x > 1+x$ for all $x>0$.

Similarly, we can also prove: $e^x > 1 + x + \frac{1}{2}x^2$ for all $x > 0$.

Define $f(x) = e^x - (1 + x + \frac{1}{2} \cdot x^2)$. $f(0) = 0$.

$f'(x) = e^x - (1+x) > 0$ for all $x>0$ $\Rightarrow$ f is increasing on $(0, +\infty)$.

$\Rightarrow$ $f(x) > f(0) = 0$ for all $x > 0$.