§5.7 Applications to Differential Equations

System of Linear Differential Equations

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) = a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) \end{cases}$$

Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

Then $\vec{x}'(t) = A\vec{x}(t)$

* When A is a diagonal matrix

Example: $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ decoupled

That is $\begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = -5x_2(t) \end{cases}$

Solution: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-5t} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-5t}$

* For general equation $x' = Ax$

The solution is $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A

and $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are eigenfunctions of A.

$\vec{v}_i e^{\lambda_i t}$ are called eigenfunction of the differential equation.

Example: The circuit in Figure 1 can be described by the differential equation

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2C_2) \\ 1/(R_2C_2) & -1/(R_2C_2) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

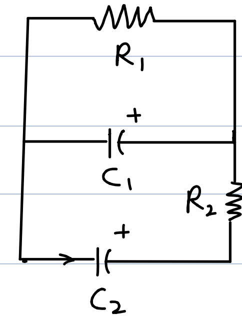


Figure 1

Suppose $R_1 = 1$ ohm, $R_2 = 2$ ohm,

$C_1 = 1$ farad, $C_2 = 0.5$ farad

$x_1(0) = 5$ volts, $x_2(0) = 4$ volts

Find formulas for $x_1(t)$ and $x_2(t)$ that describe how the voltages change over time.

Solution: Let

$$A = \begin{pmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2C_2) \\ 1/(R_2C_2) & -1/(R_2C_2) \end{pmatrix} = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}.$$

$$\vec{x}(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = -0.5$, $\lambda_2 = -2$ with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The eigenfunctions $\vec{x}^{(1)}(t) = \vec{v}_1 e^{\lambda_1 t}$, $\vec{x}^{(2)}(t) = \vec{v}_2 e^{\lambda_2 t}$ both satisfy $\vec{x}' = A\vec{x}$.

$$\vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t)$$

$$= C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

Let $t=0$,

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

we get $c_1 = 3$, $c_2 = -2$.

Hence $\vec{x}(t) = 3\begin{pmatrix}1\\2\end{pmatrix}e^{-0.5t} - 2\begin{pmatrix}-1\\1\end{pmatrix}e^{-2t}$

Example: Suppose a particle is moving in a planar force field and its position vector $\vec{x}$ satisfies $\vec{x}' = A\vec{x}$ and $\vec{x}(0) = \vec{x}_0$, where

$$A = \begin{pmatrix}4 & -5\\-2 & 1\end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix}2.9\\2.6\end{pmatrix}.$$

Solution: The eigenvalues of A is $\lambda_1 = 6$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix}-5\\2\end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.

For any constants c_1 and c_2, the function

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

$$= c_1\begin{pmatrix}-5\\2\end{pmatrix}e^{6t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-t} \quad \cdots \quad (*)$$

is a solution of $\vec{x}' = A x$.

Since $\vec{x}(0) = \begin{pmatrix}2.9\\2.6\end{pmatrix}$, we let $t=0$ in $(*)$ and get

$$\begin{pmatrix}2.9\\2.6\end{pmatrix} = c_1\begin{pmatrix}-5\\2\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}.$$

$c_1 = -\frac{3}{70}$, $c_2 = \frac{188}{70}$.

Hence $\vec{x}(t) = -\frac{3}{70}\begin{pmatrix}-5\\2\end{pmatrix}e^{6t} + \frac{188}{70}\begin{pmatrix}1\\1\end{pmatrix}e^{-t}$.

* Decoupling a Dynamical System

Consider a dynamical system $\vec{x}\,'(t) = A\vec{x}(t)$,

If A is $n \times n$ and has n linearly independent eigenvectors, then A is diagonalizable.

Suppose $\lambda_1, \cdots, \lambda_n$ are eigenvalues with corresponding eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$.

Let $P = [\vec{v}_1, \cdots, \vec{v}_n]$

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Define $\vec{y}(t) = P^{-1}\vec{x}(t)$.

Then $x(t) = P\vec{y}(t)$.

$$\frac{d}{dt} P\vec{y}(t) = A(P\vec{y}) = (PDP^{-1})P\vec{y}(t) = PD\vec{y}(t)$$

Left-multiply both sides by P^{-1}, we have

$$y'(t) = Dy(t).$$

or
$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

We know $y_1(t) = c_1 e^{\lambda_1 t}$, $\cdots$ $y_n(t) = c_n e^{\lambda_n t}$.

$$y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{y}(0) = P^{-1}\vec{x}(0) = P^{-1}\vec{x}_0$$

$$x(t) = Py(t) = [\vec{v}_1, \cdots, \vec{v}_n]\vec{y}(t)$$
$$= c_1\vec{v}_1 e^{\lambda_1 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$$

This is consistent with the eigenfunction expansion discussed earlier.

§6.1 Inner Product, Length and Orthogonality

$\mathbb{R}^2, \mathbb{R}^3$: length, distance, perpendicularity

* The Inner Product

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$\vec{u}^T\vec{v} = [u_1, u_2, \cdots, u_n]\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \vec{u}\cdot\vec{v}$$

The number $\vec{u}^T\vec{v}$ $(\vec{u}\cdot\vec{v})$ is called the inner product (or dot product) of $\vec{u}$ and $\vec{v}$.

Example: Compute $\vec{u}\cdot\vec{v}$ and $\vec{v}\cdot\vec{u}$ for $\vec{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$.

Solution: $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = [2 \ \ -5 \ \ -1]\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$

$$= 2\cdot 3 + (-5)\cdot 2 + (-1)\cdot(-3)$$
$$= -1$$

$$\vec{v}\cdot\vec{u} = \vec{v}^T\vec{u} = [3 \quad 2 \quad -3]\begin{pmatrix}2\\-5\\-1\end{pmatrix}$$

$$= 3\cdot 2 + 2\cdot(-5) + (-3)\cdot(-1)$$

$$= -1$$

Theorem: Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

b) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

c) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$

d) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ iff $\vec{u} = \vec{0}$.

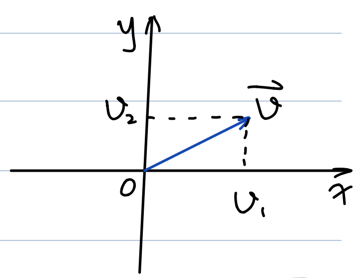
Based on b) and c), we have

$$(c_1\vec{u}_1 + \cdots + c_p\vec{u}_p)\cdot\vec{w} = c_1(\vec{u}_1\cdot\vec{w}) + \cdots + c_p(\vec{u}_p\cdot\vec{w})$$

* The length of a vector

Def: The length (or norm) of $\vec{v}$ is the nonnegative scalar $\|\vec{v}\|$ defined by

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$



$\|\vec{v}\| = \sqrt{v_1^2+v_2^2}$

For any scalar c,

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|$$

Proof: $\|c\vec{v}\|^2 = c\vec{v}\cdot(c\vec{v}) = c^2\vec{v}\cdot\vec{v} = c^2\|\vec{v}\|^2$

So $\|c\vec{v}\| = \sqrt{c^2\|\vec{v}\|^2} = |c|\,\|\vec{v}\|$

Unit vector: $\vec{u}$ with $\|\vec{u}\| = 1$

For a vector $\vec{v} \neq \vec{0}$, $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector parallel to $\vec{v}$. This process is called normalizing $\vec{v}$ and we say $\vec{u}$ is in the same directions as $\vec{v}$.

Example: Let $\vec{v} = (1, -2, 2, 0)$. Find a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

Solution: $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{9} = 3$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(1, -2, 2, 0)}{3} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, 0\right)$$

Example: Let W be the subspace of $\mathbb{R}^2$ spanned by $\vec{x} = (\frac{2}{3}, 1)$. Find a unit vector $\vec{z}$ that is a basis for W.

Solution: $\vec{W} = \text{Span}\{\vec{x}\} = \{c\vec{x} \mid c \in \mathbb{R}\}$

$$\vec{z} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{(\frac{2}{3}, 1)}{\sqrt{(\frac{2}{3})^2 + 1^2}} = \frac{(\frac{2}{3}, 1)}{\sqrt{\frac{2^2+3^2}{3^2}}} = \frac{3\times(\frac{2}{3}, 1)}{\sqrt{13}}$$

$$= \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$$

* Distance in $\mathbb{R}^n$

Def: For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$, written as $\text{dist}(\vec{u}, \vec{v})$, is the length of the vector $\vec{u} - \vec{v}$. That is,

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Example: Compute the distance between the vectors $\vec{u} = (7,1)^T$ and $\vec{v} = (3,2)^T$.

Solution: $\vec{u} - \vec{v} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$$\|\vec{u} - \vec{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

Example: If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}$$
$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

* Orthogonal Vectors (perpendicular)

Def: Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal (to each other) if $\vec{u} \cdot \vec{v} = 0$.

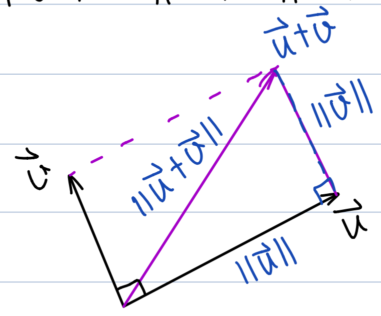
Remark: $\vec{0}$ is orthogonal to every vector in $\mathbb{R}^n$ because $\vec{0}^T \vec{v} = 0$.

/pəθæɡə'riən/

Thm: The Pythagorean Theorem

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal iff

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



Proof: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v}$$

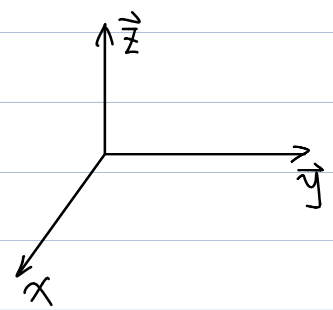
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$$

Thus $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ iff $\vec{u}\cdot\vec{v} = 0$

* Orthogonal Complements

Def: Let $z \in \mathbb{R}^n$, W a subspace of $\mathbb{R}^n$.
If $\vec{z}$ is orthogonal to every vector in W, we say $\vec{z}$ is orthogonal to $\vec{W}$.

Ex:



W = xy-plane = $\{(x, y, 0)\}$

$\vec{z} = (0, 0, 2)$

Then $\vec{z}$ is orthogonal to W.

Def: The set of all vectors $\vec{z}$ that are orthogonal to W is called the orthogonal complement of W, and is denoted by $W^\perp$ ("W perp")

Ex: W = xy-plane in $\mathbb{R}^3$

$W^\perp$ = z-axis

V = x_1x_2-plane in $\mathbb{R}^4$,

$= \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{R}\}$

$V^\perp = x_3x_4$ - plane in $\mathbb{R}^4$

$= \{(0,0,x_3,x_4) \mid x_3, x_4 \in \mathbb{R}\}$

Fact: 1) A vector $\vec{x}$ is in $W^\perp$ iff $\vec{x}$ is orthogonal to every vector in a set that spans W.

2) $W^\perp$ is a subspace of $\mathbb{R}^n$.

Proof: 1) obvious

$\vec{0} \in W^\perp$

2) Take $\vec{u}, \vec{v}$ in $W^\perp$ and $\vec{x}$ in W

Then $(a\vec{u} + b\vec{v}) \cdot \vec{x} = a\vec{u}\cdot\vec{x} + b\vec{v}\cdot\vec{x} = 0$

Thus $a\vec{u} + b\vec{v}$ is orthogonal to every vector in W. Hence $a\vec{u} + b\vec{v}$ is in $W^\perp$.

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A$$

$$(\text{Col } A)^\perp = \text{Nul } A^T$$

Proof: $A = (\vec{a}_1, \cdots \vec{a}_n) = \begin{pmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_m \end{pmatrix}$

$\text{Row } A = \text{Span}\{\vec{b}_1, \cdots, \vec{b}_m\}$

$\vec{x} \in (\text{Row} A)^\perp \Leftrightarrow \vec{x}\cdot\vec{b}_1 = 0, \cdots, \vec{x}\cdot\vec{b}_m = 0$

$\Leftrightarrow 0 = \begin{pmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_m \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} \vec{b}_1\cdot\vec{x} \\ \vdots \\ \vec{b}_m\cdot\vec{x} \end{pmatrix} \Leftrightarrow \vec{x} \in \text{Nul} A$

Since $\operatorname{Col} A = \operatorname{Row} A^T$,

$$(\operatorname{Col} A)^\perp = (\operatorname{Row} A^T)^\perp = \operatorname{Nul} A^T$$

Exercise: Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is also a subspace of $\mathbb{R}^n$. Prove that

$$\dim W + \dim W^\perp = n$$

Proof: If $W \neq \{\vec{0}\}$. Let $\{\vec{b}_1, \cdots, \vec{b}_p\}$ be a basis for W, where $1 \leq p \leq n$. Let A be the $p \times n$ matrix having rows $\vec{b}_1^T, \cdots, b_p^T$. Then $W = \operatorname{Row} A$.

Hence $W^\perp = (\operatorname{Row} A)^\perp = \operatorname{Nul} A$.

Therefore, $\dim W + \dim W^\perp$

$$= \dim(\operatorname{Row} A) + \dim(\operatorname{Nul} A)$$
$$= \operatorname{rank} A + \dim(\operatorname{Nul} A)$$
$$= n$$

If $W = \{\vec{0}\}$, then $W^\perp = \mathbb{R}^n$. The result is obviously true.