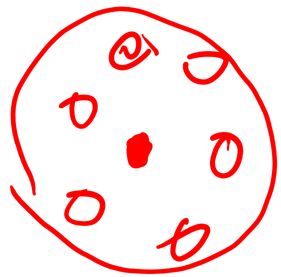
CM



$$\boxed{\Sigma F = Ma}$$

"model"

Dynamics of Rigid Bodies I

PHYS1112

Lecture 8

translation

rotation

Intended Learning Outcomes

- After this lecture you will learn:
    1) radian as a measure of angle
    2) angular displacement, velocity and acceleration and their vector representation
    3) angular motion as compared to rectilinear motion
    4) rotational kinetic energy and moment of inertia

Measuring angles in radian

$$\begin{cases} \vec{r} \quad \Delta\vec{r} \quad \vec{v} \quad \vec{a} \\ \theta \quad \Delta\theta \quad \vec{\omega} \quad \vec{\alpha} \end{cases}$$

Define the value of an angle θ in **radian**

as $\qquad \theta = \frac{s}{r}, \cancel{\frac{m}{m}}$

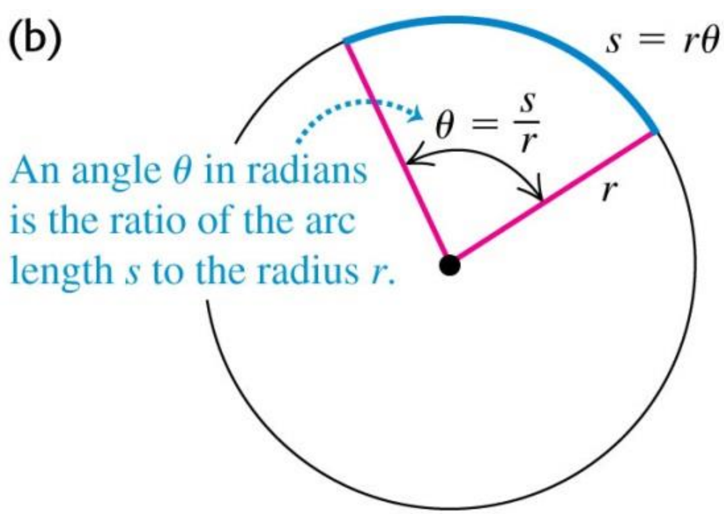
or arc length $\qquad s = r\theta$

- a pure number, without dimension
- independent of radius r of the circle
- one complete circle

$$\theta = \frac{2\pi r}{r} = 2\pi \text{ (in radian)} \leftrightarrow 360°$$

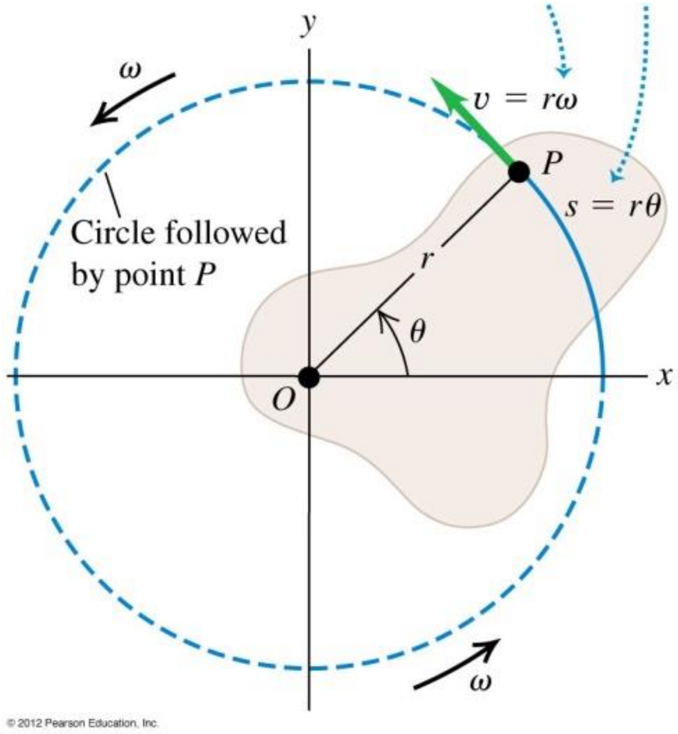
$$\pi \text{ (in radian)} \leftrightarrow 180°$$

$$\pi/2 \text{ (in radian)} \leftrightarrow 90°$$



Consider a rigid body rotating about a fixed axis

Convention: θ measured from x axis in counterclockwise direction



angular displacement: $\Delta\theta = \theta_2 - \theta_1$

angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t} \xrightarrow{\Delta t \to 0} \frac{d\theta}{dt}$$

(average) (instantaneous)

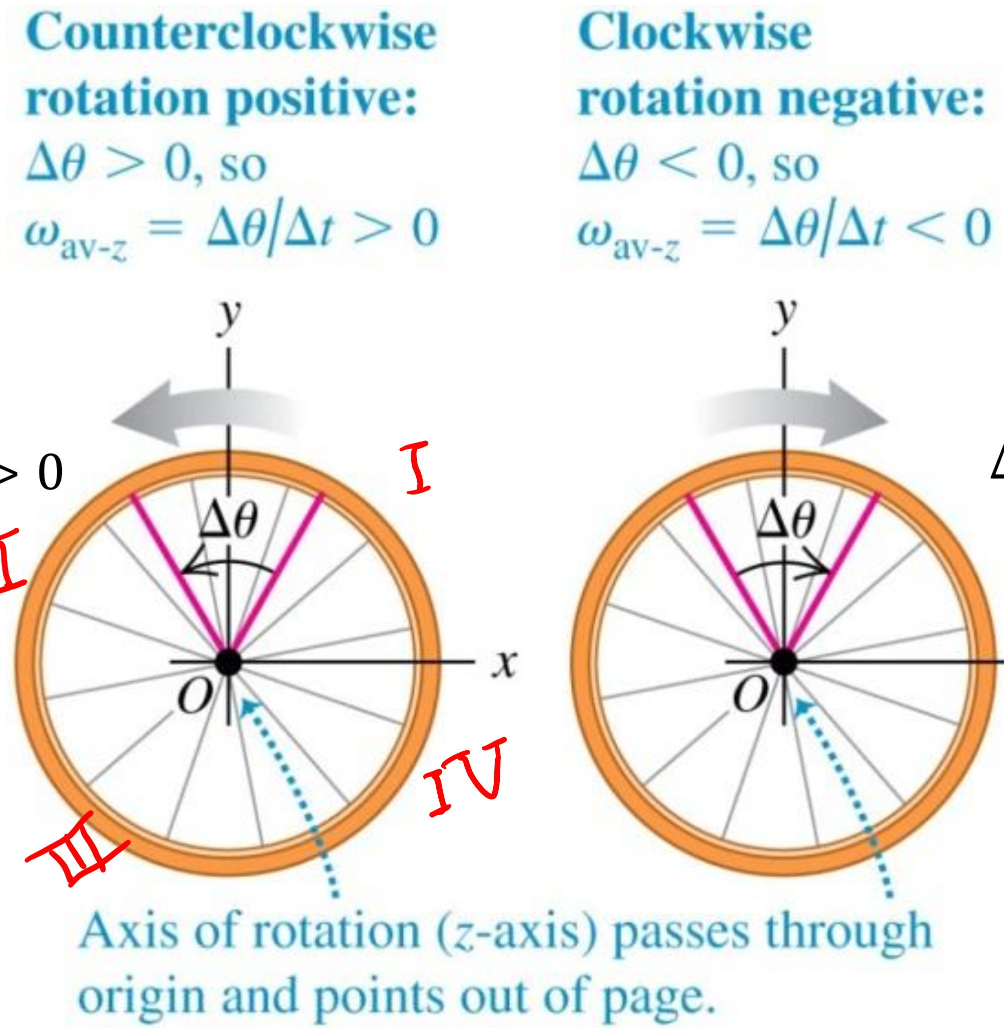
angular acceleration:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Convention: θ measured from x axis in counterclockwise direction

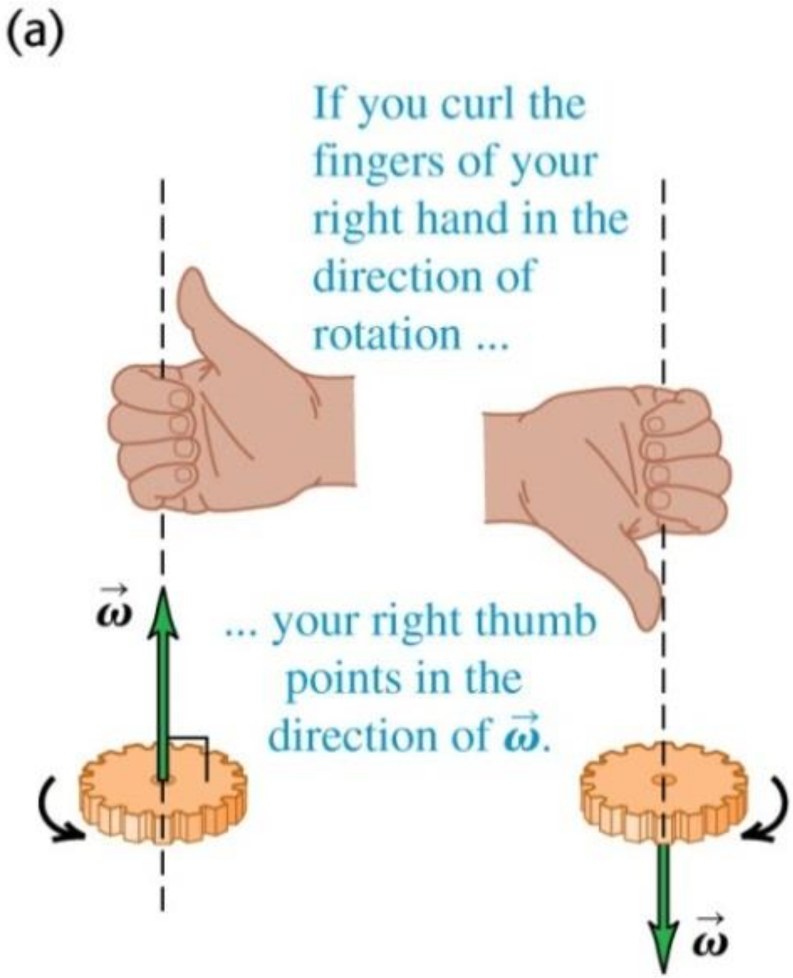
$\Delta\theta = \theta_f - \theta_i > 0$

$\Delta\theta = \theta_f - \theta_i < 0$



linear

Angular velocity is a vector, direction defined by the **right hand rule**



direction of $\vec{\boldsymbol{\omega}}$ represents sense of rotation

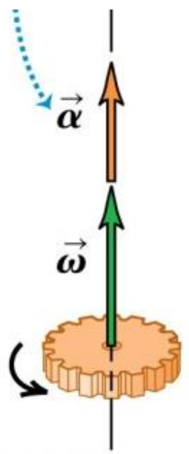
Angular acceleration is defined as $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$

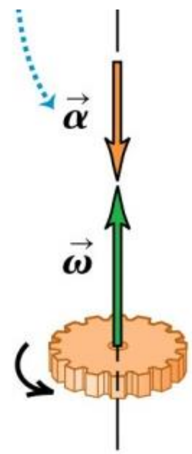
$\vec{\omega} = \frac{\Delta \vec{\theta}}{\Delta t}$

$= \frac{d\vec{\theta}}{dt}$

⚠ if rotation axis is fixed, $\vec{\boldsymbol{\alpha}}$ along the direction of $\vec{\boldsymbol{\omega}}$

Rotation speeding up, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the same direction



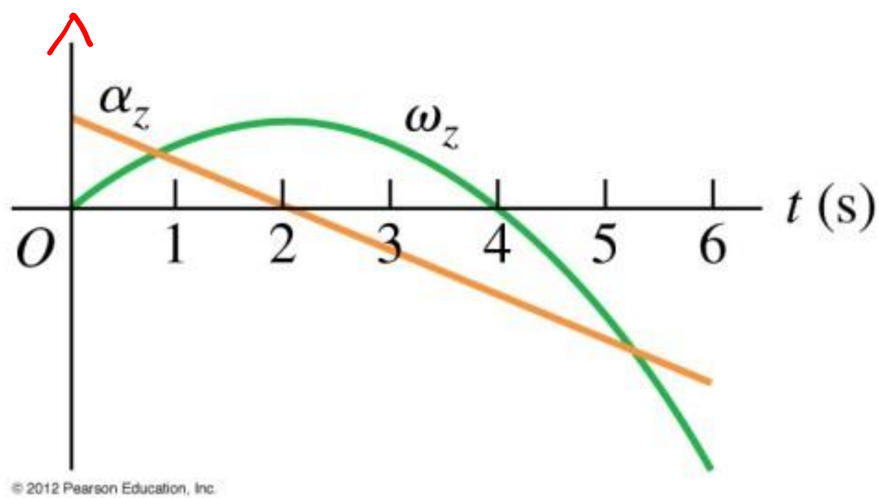


Rotation slowing down, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the opposite direction

Question

- The figure shows a graph of ω and α versus time. During which time intervals is the rotation speeding up?

(i) $0 < t < 2$ s; (ii) 2 s $< t <$ 4 s; (iii) 4 s $< t <$ 6 s.



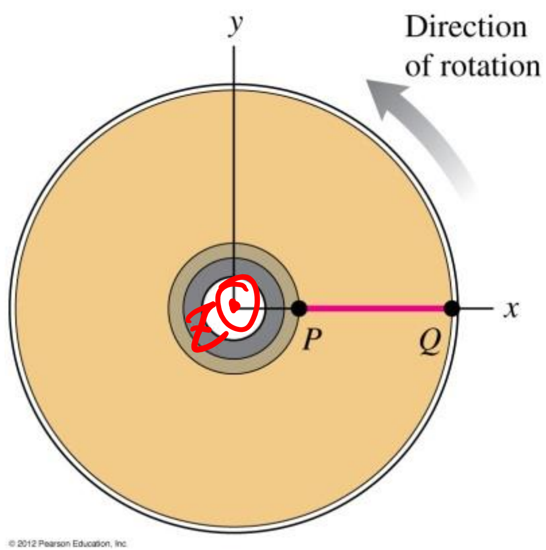
Rotation with constant angular acceleration

Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

$$\begin{cases} x \to \theta \\ v_x \to \omega_z \\ a_x \to \alpha_z \end{cases}$$

Example

A Blu-ray disc is slowing down to a stop with constant angular acceleration $\alpha = -10.0\ \text{rad/s}^2$. At $t = 0$, $\omega_0 = 27.5$ rad/s, and a line *PQ* marked on the disc surface is along the *x* axis.



angular velocity at $t = 0.300$ s:

$$\omega = \omega_0 + \alpha t$$

$$= 27.5\ \text{rad/s} + \left(-10.0\ \text{rad/s}^2\right)(0.300\ \text{s})$$

$$= 24.5\ \text{rad/s}$$

Suppose θ is the angular position of PQ at $t = 0.300$ s

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = 7.80\ \text{rad}$$

$$= (7.8\ \cancel{\text{rad}})\left(\frac{360°}{2\pi\ \cancel{\text{rad}}}\right) = 447° = 87°$$

What are the directions of $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$?

Question

- In the above example, suppose the initial angular velocity is doubled to $2\omega_0$, and the angular acceleration (deceleration) is also doubled to 2α, it will take (more / less / the same amount of) time for the disc to come to a stop compared to the original problem.

$$t = \frac{\cancel{2}\omega_0 - 0}{\cancel{2}\alpha}$$

Q9.2

A DVD is initially at rest so that the line PQ on the disc's surface is along the $+x$-axis. The disc begins to turn with a constant $\alpha_z = 5.0\ \text{rad/s}^2$. At $t = 0.40$ s, what is the angle between the line PQ and the $+x$-axis?

0

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$= \tfrac{1}{2} \times 5.0 \times (0.40)^2\ \text{rad}$$
$$\text{rad/s}^2 \times s^2$$
$$=$$

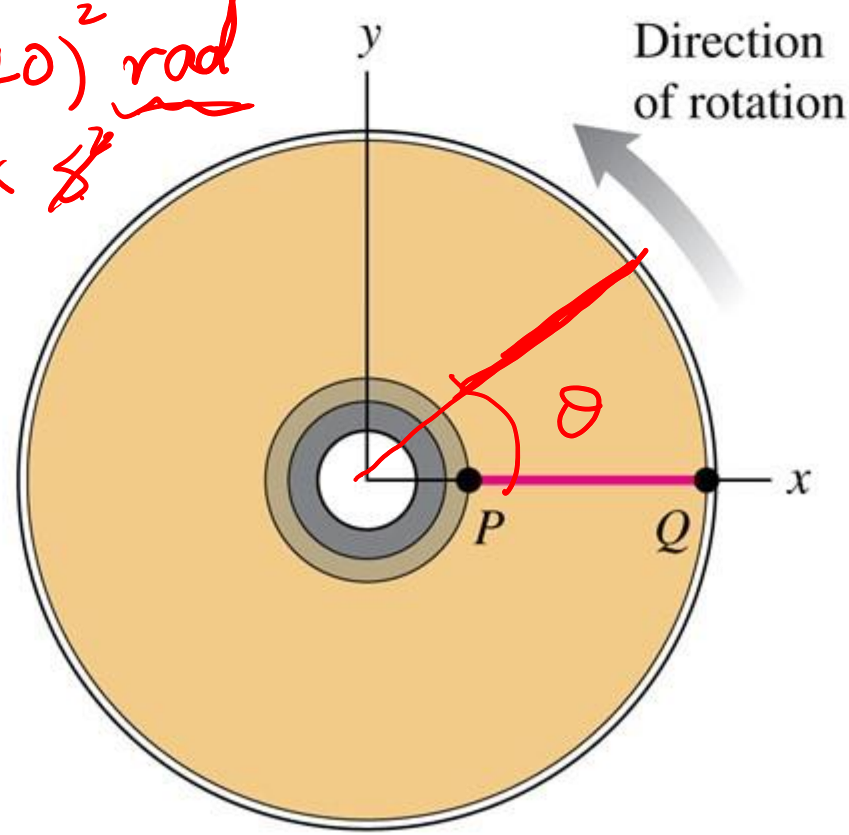
A. 0.40 rad

B. 0.80 rad

C. 1.0 rad

D. 1.6 rad

E. 2.0 rad



A9.2

A DVD is initially at rest so that the line *PQ* on the disc's surface is along the +*x*-axis. The disc begins to turn with a constant $\alpha_z = 5.0\ \text{rad/s}^2$. At $t = 0.40$ s, what is the angle between the line *PQ* and the +*x*-axis?

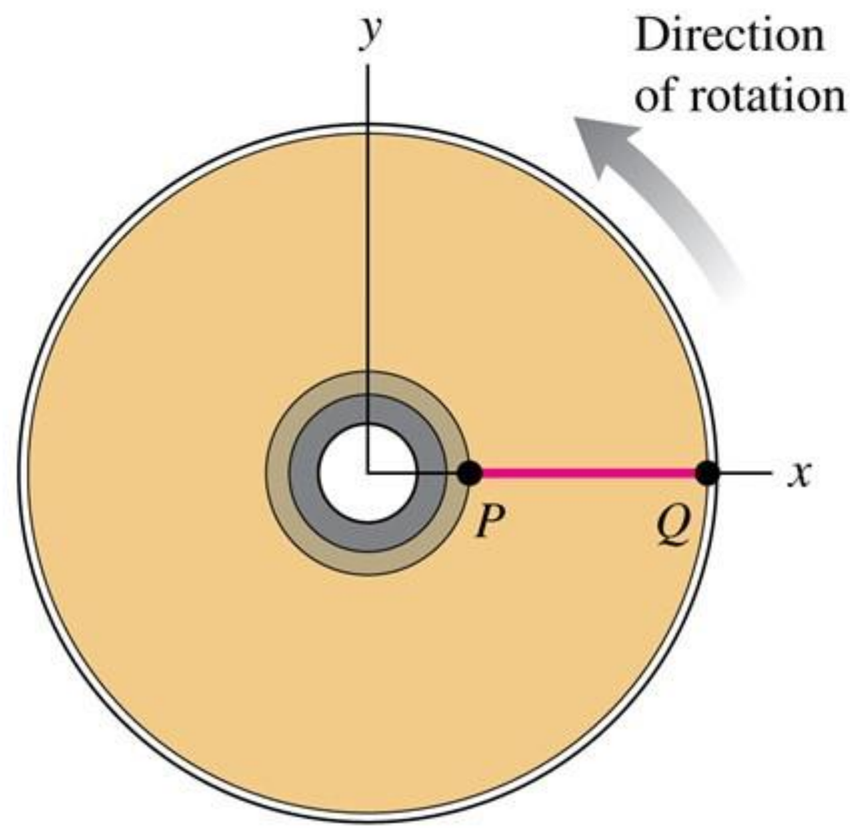
A. 0.40 rad

B. 0.80 rad

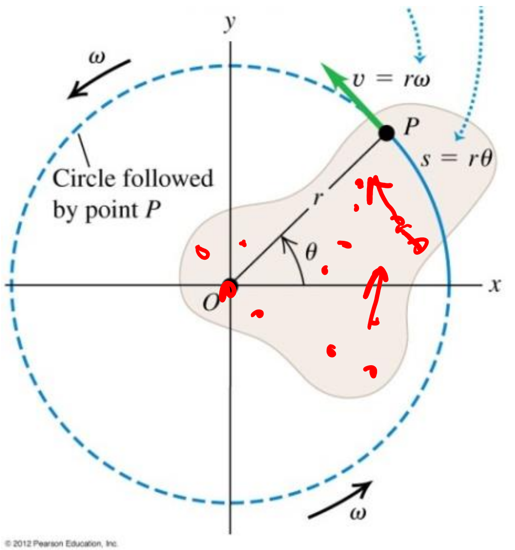
C. 1.0 rad

D. 1.6 rad

E. 2.0 rad



Rigid body rotation



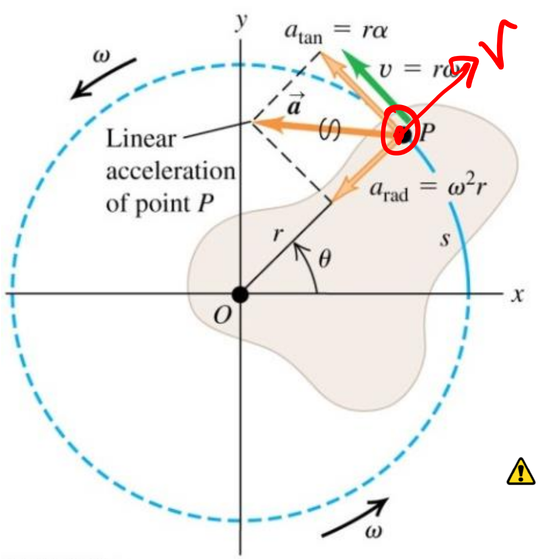
In time Δt, angular displacement is $\Delta\theta$,
tangential displacement is $\Delta s = r\Delta\theta$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

∴ tangential velocity

$$v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \rightarrow r\frac{d\theta}{dt} = r\omega$$

$$\boxed{\text{Linear velocity of point } P,\ \vec{\boldsymbol{v}}, \text{ is tangential and has magnitude } v = |r\omega|}$$

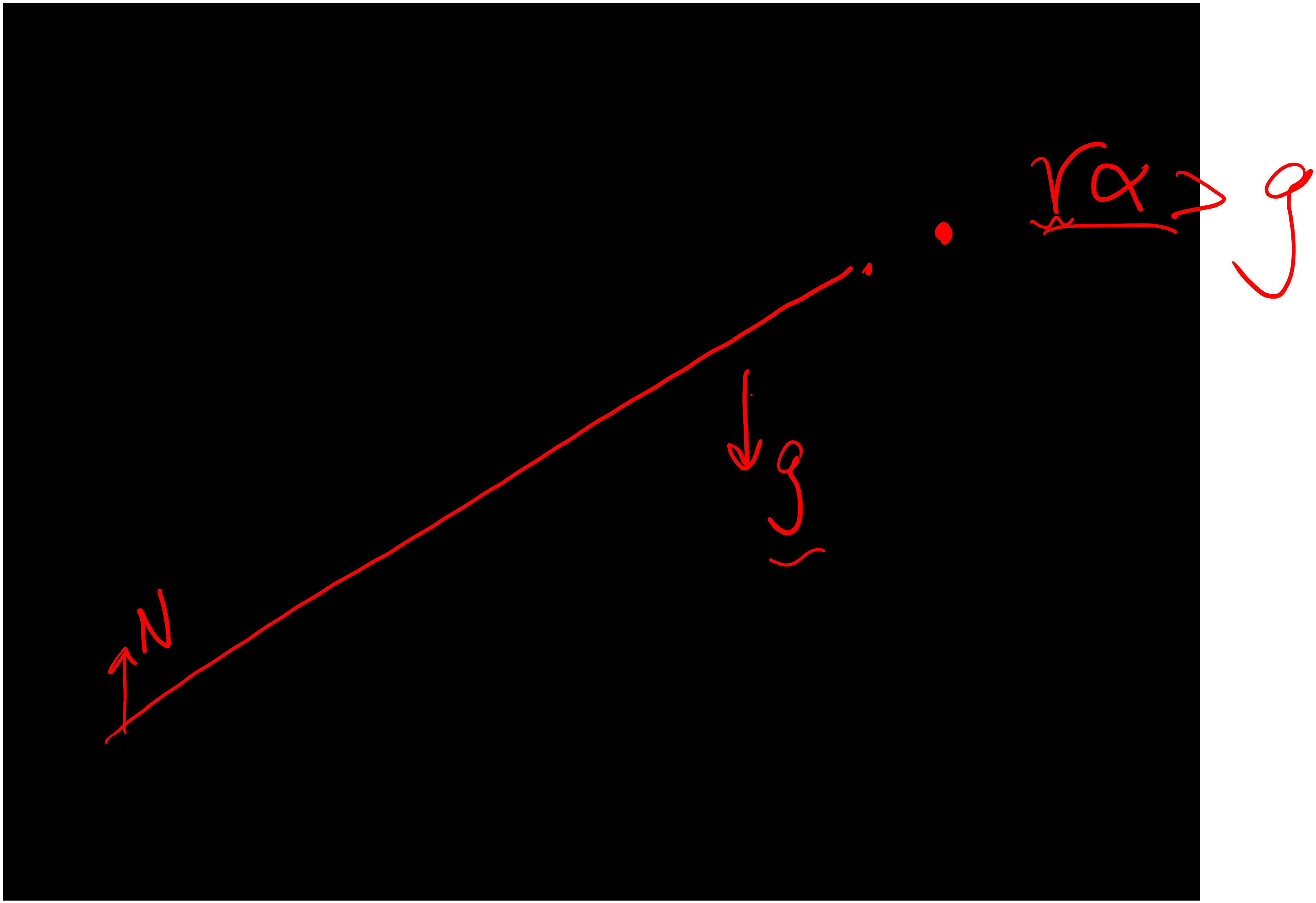


tangential acceleration $$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

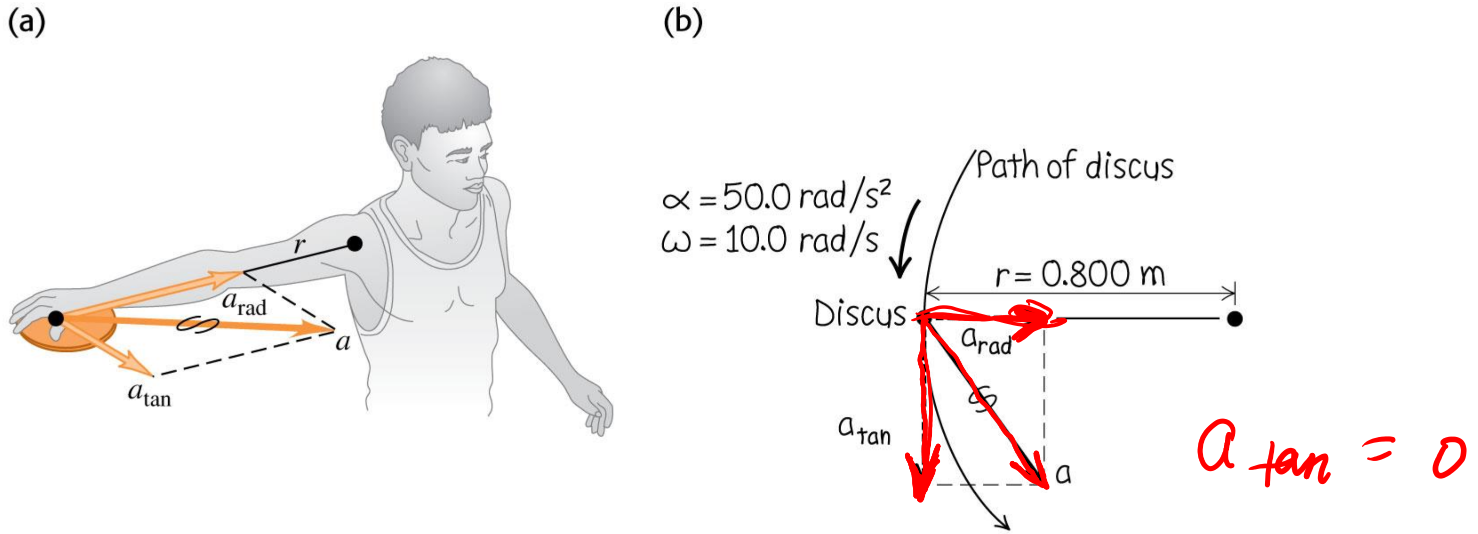
radial acceleration $$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r = \frac{(r\omega)^2}{r}$$

⚠ Every point of the rigid body has identical $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$, but different $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$

Demonstration: falling faster than *g* – same angular acceleration (same rod), the far end of the rod has linear acceleration larger than *g*.



Example



An athlete whirls a discus in a circle of radius 80.0 cm. At some instant $\omega =$ 10.0 rad/s, and $\alpha = 50.0$ rad/s². Then

$$a_{\text{tan}} = r\alpha = (0.800\ \text{m})\left(50.0\ \ \text{rad/s}^2\right) = 40.0\ \text{m/s}^2$$

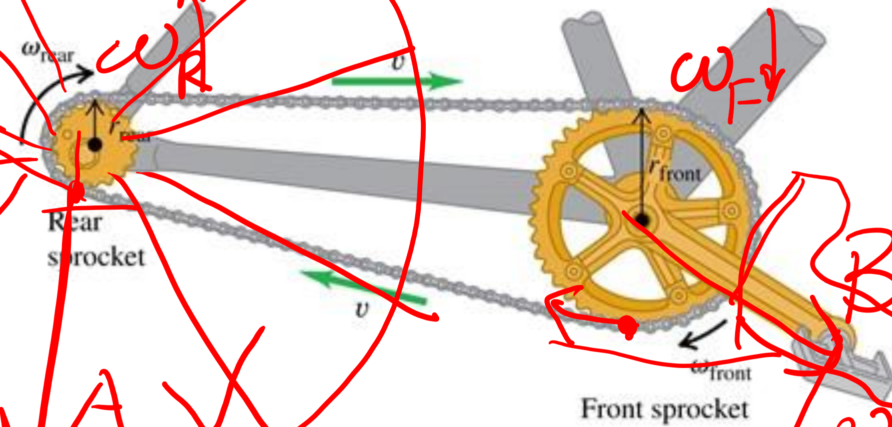
$$a_{\text{rad}} = \omega^2 r = (10.0\ \ \text{rad/s})^2(0.800\ \text{m}) = 80.0\ \text{m/s}^2$$

Magnitude of the linear acceleration is

$$a = \sqrt{a_{\text{tan}}^2 + a_{\text{rad}}^2} = 89.4\ \text{m/s}^2$$

Q9.5

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



A. a faster linear speed and a faster angular speed.

B. the same linear speed and a faster angular speed.

C. a slower linear speed and the same angular speed.

D. the same linear speed and a slower angular speed.

E. none of the above.

$v = r\omega$

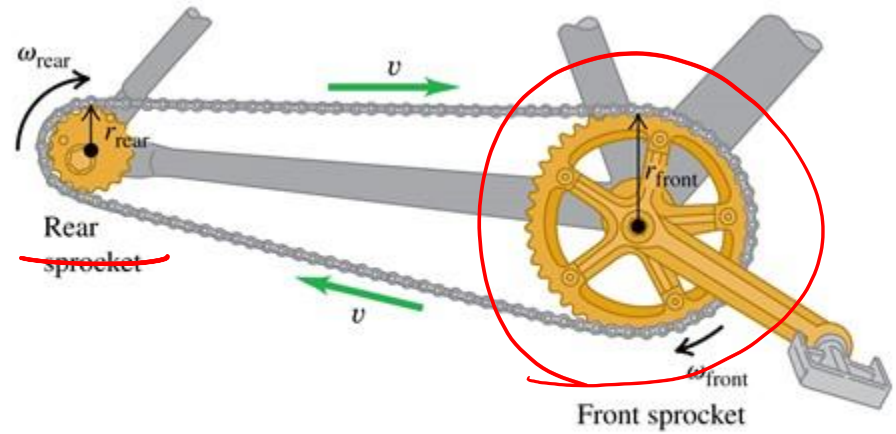
$\omega = \frac{v}{r}$

$v_B = \omega_F r_B$

$v_A = \omega_R r_A$

A9.5

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



A. a faster linear speed and a faster angular speed.

B. the same linear speed and a faster angular speed.

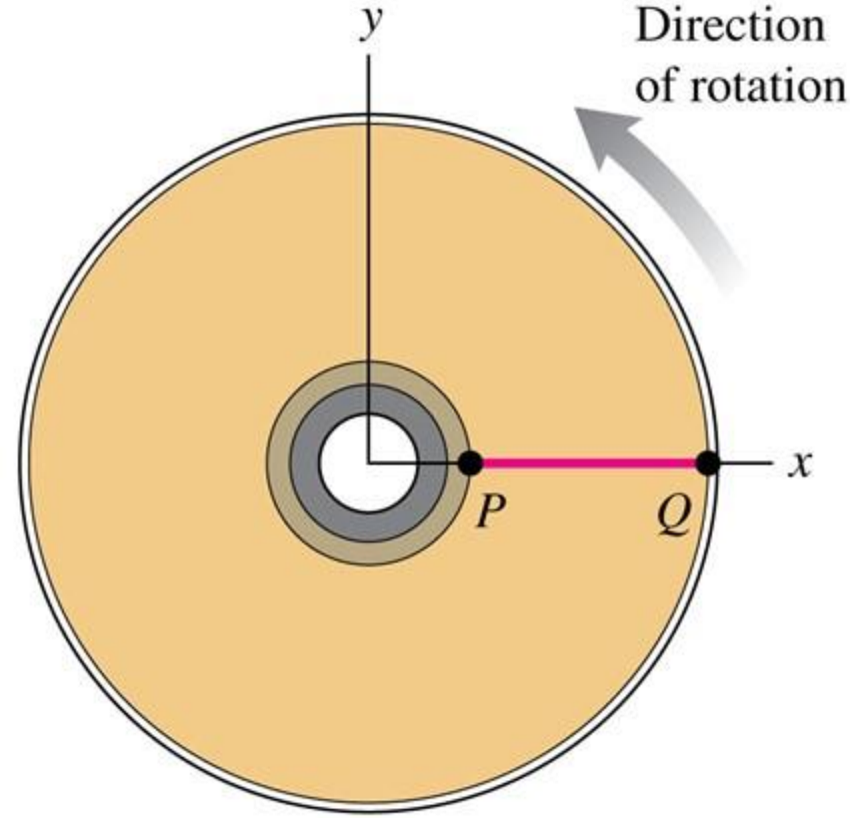
C. a slower linear speed and the same angular speed.

D. the same linear speed and a slower angular speed.

E. none of the above.

Q9.4

A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration a_{rad} and tangential acceleration a_{tan} compare at points P and Q?



A. P and Q have the same a_{rad} and a_{tan}.

B. Q has a greater a_{rad} and a greater a_{tan} than P.

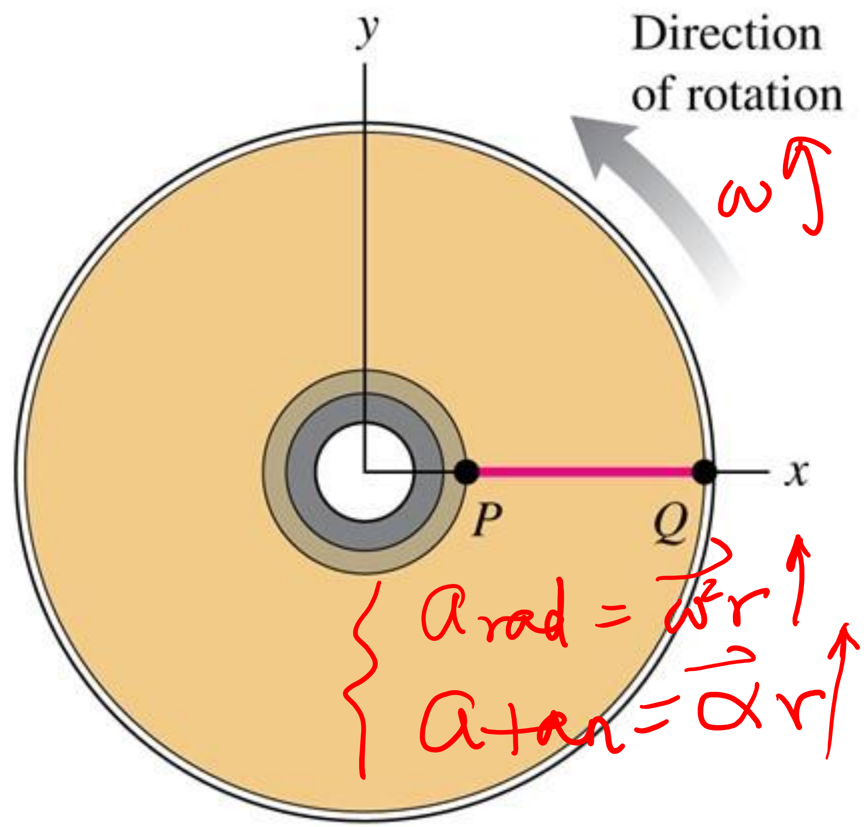
C. Q has a smaller a_{rad} and a greater a_{tan} than P.

D. Q has a greater a_{rad} and a smaller a_{tan} than P.

E. P and Q have the same a_{rad}, but Q has a greater a_{tan} than P.

A9.4

A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration a_{rad} and tangential acceleration $a_{\tan}$ compare at points *P* and *Q*?



A. *P* and *Q* have the same a_{rad} and $a_{\tan}$.

B. *Q* has a greater a_{rad} and a greater $a_{\tan}$ than *P*.

C. *Q* has a smaller a_{rad} and a greater $a_{\tan}$ than *P*.

D. *Q* has a greater a_{rad} and a smaller $a_{\tan}$ than *P*.

E. *P* and *Q* have the same a_{rad}, but *Q* has a greater $a_{\tan}$ than *P*.

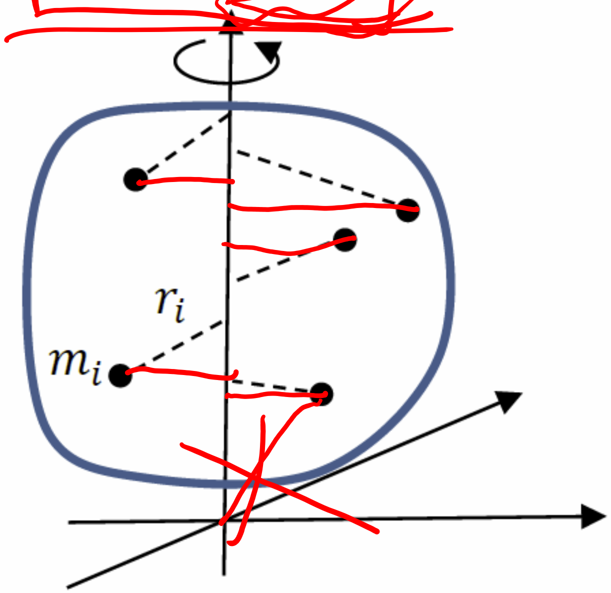
Rotational kinetic energy of a rigid body

Consider a rigid body as a collection of particles, the kinetic energy due to rotation is

$$\boxed{K = \sum \tfrac{1}{2} m_i v_i^2} = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2}\underbrace{\left(\sum m_i r_i^2\right)}_{} \omega^2$$

c.f. in rectilinear motion, $K = \frac{1}{2} m v^2$

moment of inertia I, analogous to *mass* in rectilinear motion



$$\boxed{K = \tfrac{1}{2} I \omega^2, I = \sum m_i r_i^2}$$

⚠ When defining I, must specify a rotation axis. r_i is the ⊥ distance to the rotation axis, not the distance from the origin.

Q9.6

You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the sphere must be

A. four times its initial value.

B. twice its initial value.

C. the same as its initial value.

D. half of its initial value.

E. one-quarter of its initial value.

A9.6

You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the sphere must be

$$2^2 \left(\frac{1}{2}\right)^2 = 1$$

$$K = \frac{1}{2} \sum_i m_i r_i^2 \cdot \omega^2$$

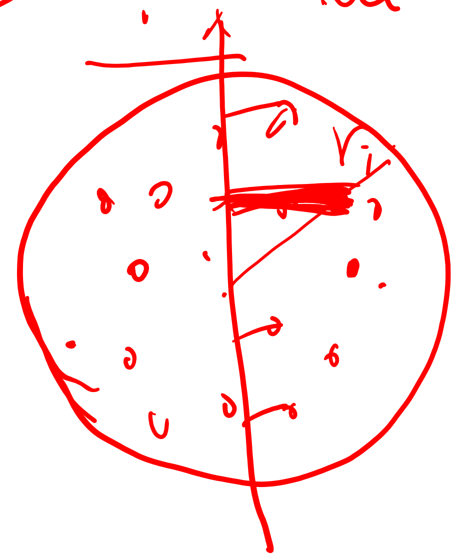
A. four times its initial value.

B. twice its initial value.

C. the same as its initial value.

D. half of its initial value.

E. one-quarter of its initial value.



Q9.7

The three objects shown here all have the same mass and the same outer radius. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which object is rotating *fastest*?

A. Object A is rotating fastest.

B. Object B is rotating fastest.

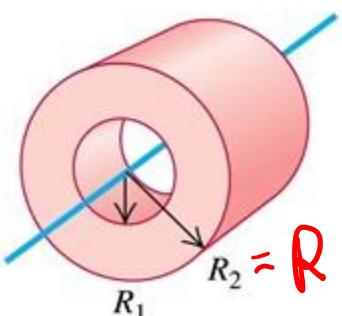
C. Object C is rotating fastest.

D. Two of these are tied for fastest.

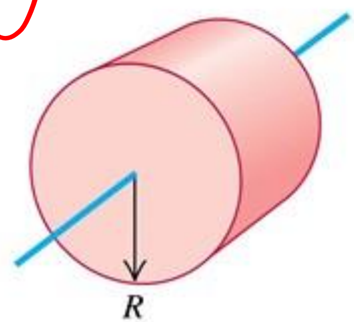
E. All three rotate at the same speed.

$K = \frac{1}{2} I \omega^2$

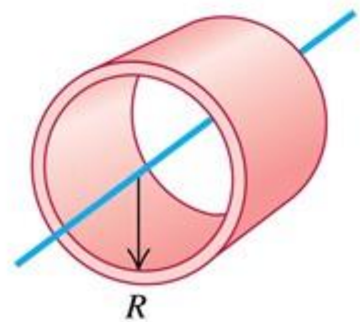
A. $I = \frac{1}{2}M(R_1^2 + R_2^2)$



B. $I = \frac{1}{2}MR^2$



C. $I = MR^2$



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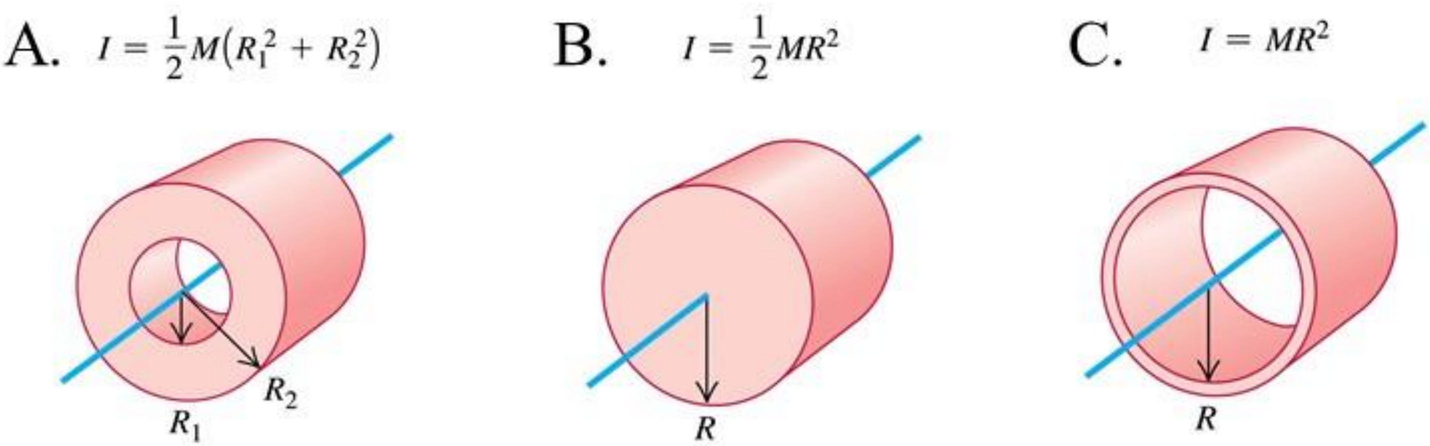
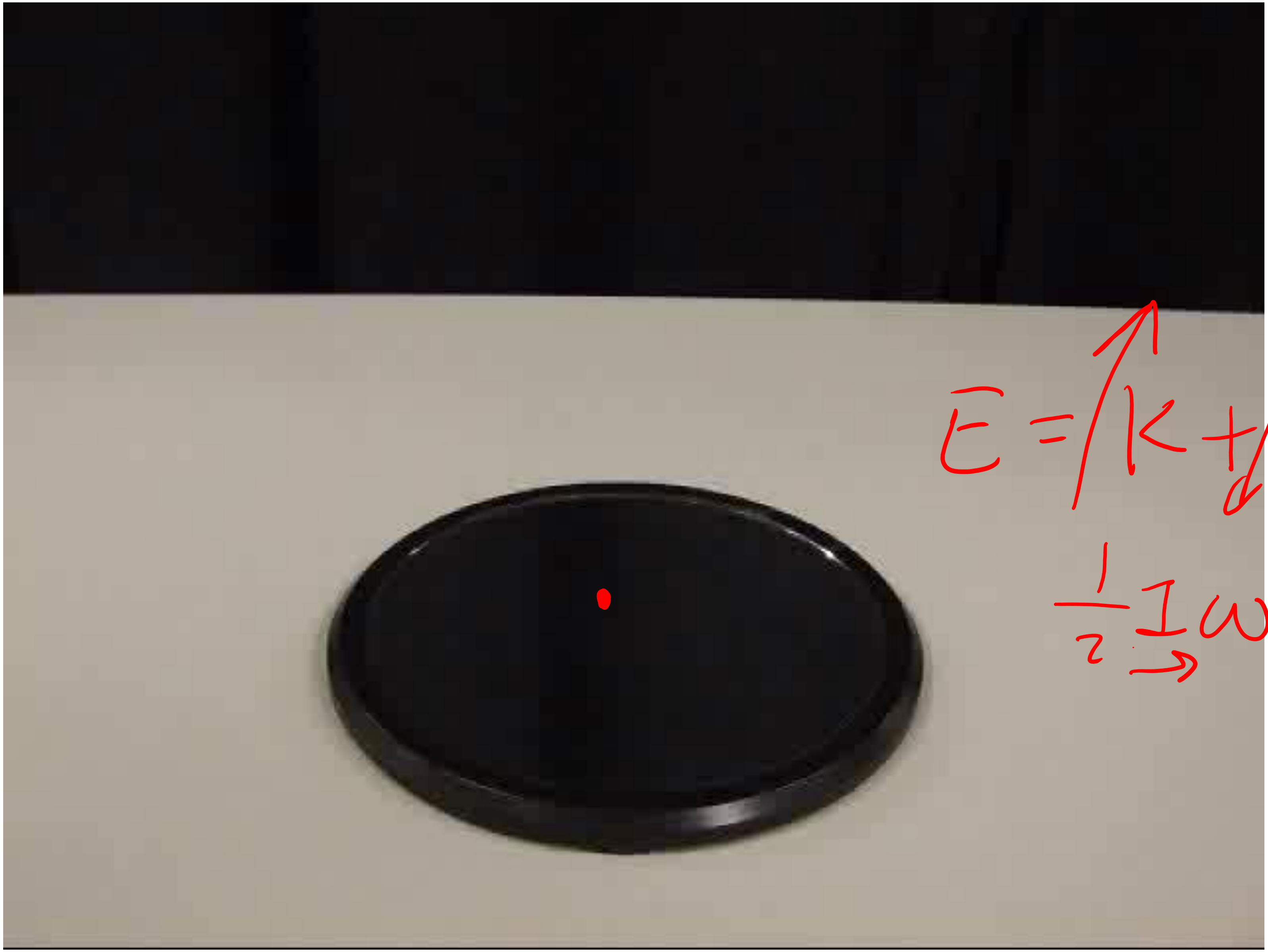


Gravitational potential energy of a rigid body

$$U = m_1 g y_1 + m_2 g y_2 + \cdots$$
$$= (m_1 y_1 + m_2 y_2 + \cdots)g = Mgy_{\text{cm}}$$

Gravitational PE is as if all the mass is concentrated at the CM.

Demonstration: Euler’s disk to demonstrate the conservation of energy – the lower the CM of the disk, the faster it spins.



$E = K + U$

$\frac{1}{2} I \omega^2$

Example

Assumption: rotation of cylinder is frictionless

no slipping between cylinder and cable

$\frac{1}{2}\,\frac{1}{2} M \underbrace{R^2\omega^2}_{v^2}$

At the moment the block hits the ground, speed of block is v, angular speed of cylinder is ω

$$v = R\omega$$

$m = M$?

$$0 + \underbrace{mgh}_{\text{initial PE of block}} = \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational KE, } I = \frac{1}{2}MR^2}$$

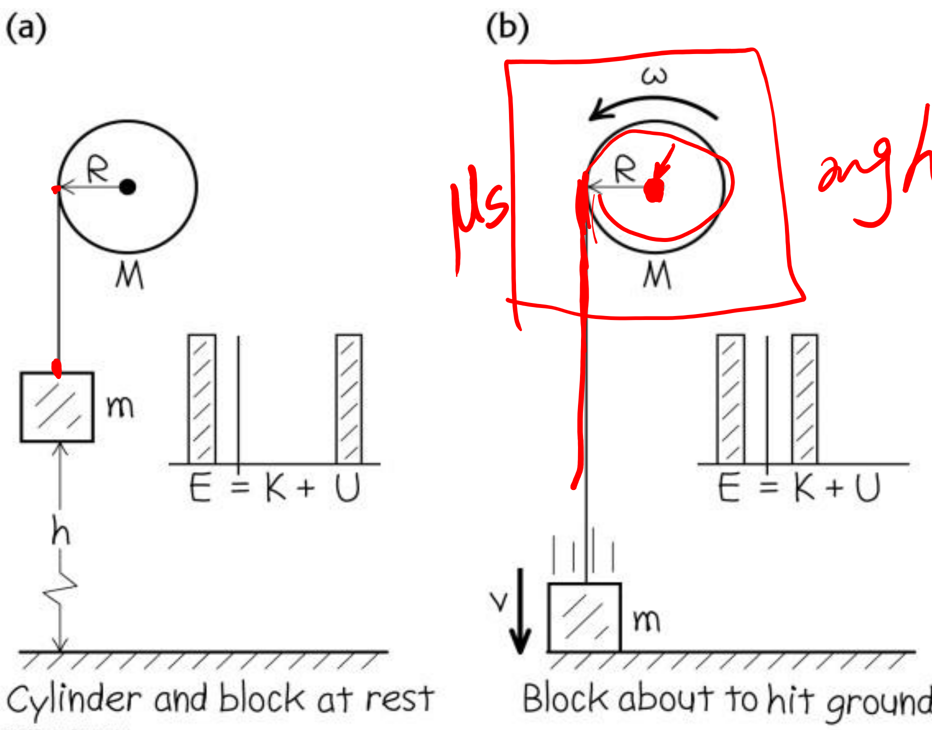
$$mgh = \frac{1}{2}mv^2 + \frac{1}{4}Mv^2$$

initial PE of block

rotational KE, $I = \frac{1}{2}MR^2$

(refer to next lecture)

$$v = \sqrt{\frac{mgh}{\frac{1}{2}m + \frac{1}{4}M}}$$



$$\Rightarrow \qquad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

if $M = 0$, $v = \sqrt{2gh}$, same as free falling

Question: Is there friction between the string and pulley? Does it dissipate energy?

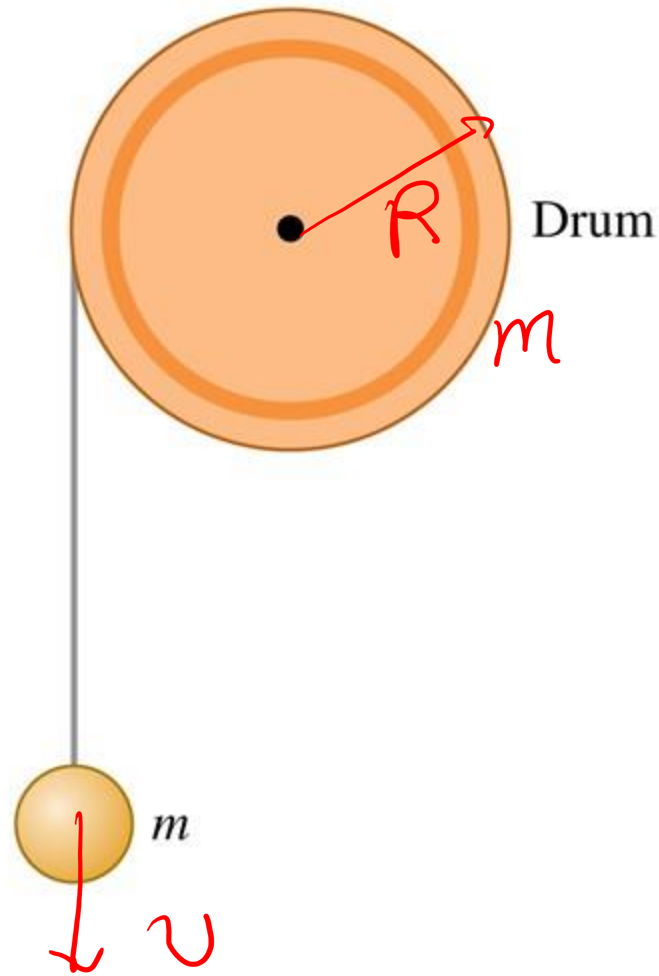
Question

- Suppose the cylinder and block have the same mass, $m = M$. Just before the block hits the floor, its KE is (larger than / less than / the same as) the KE of the cylinder.

Q9.8

A thin, very light wire is wrapped around a drum that is free to rotate. The free end of the wire is attached to a ball of mass m. The drum has the same mass m. Its radius is R and its moment of inertia is $I = (1/2)mR^2$. As the ball falls, the drum spins.

At an instant that the ball has translational kinetic energy K, what is the rotational kinetic energy of the drum?



A. K

B. $2K$

C. $K/2$

D. $K/4$

E. none of these

A9.8

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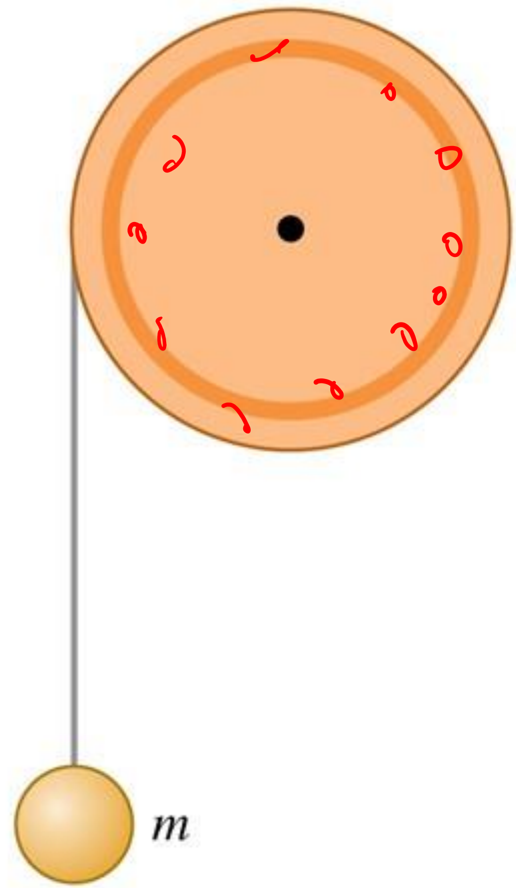

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