MATH 2111: Tutorial 6
Inverse and Determinant of a Matrix

T1A&T1B QUAN Xueyang
T1C&T2A SHEN Yinan
T2B&T2C ZHANG Fa

Department of Mathematics, HKUST

- Algorithm for finding the inverse of a matrix
- The invertible matrix theorem, invertible linear transformations
- Definition of determinant
- Properties of determinant

Find the inverses of the matrices below, if they exist

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

1. $$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[③-2①]{②+3①} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{③+3②} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \xrightarrow[③\div 2]{①+③,\ ②+③} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$= \left[\, I \mid A^{-1} \,\right]$$

2. $$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[③+2①]{②-4①} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{③-2②} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$$

not exists

Suppose T and U are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$? Why or why not?

Let A, B be the standard matrices of T, U

Then the standard matrix of $T(Ux)$ is AB

Since $T(Ux) = x$, $\forall x \in R^n$, we have $AB = I$

$A, B \in R^{n \times n}$, then by invertible matrix theorem

A, B are invertible, $B = A^{-1}$, $BA = I$

Since BA is the standard matrix of $U(Tx)$,

then we have $U(Tx) = x$, $\forall x \in R^n$

Compute the determinants by cofactor expansions.

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$$

1. $(2,3)$, $\det A = -3 \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix} = -3 \cdot 2 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$

$\downarrow$

$(2,2)$

$$= -6\left(4 \cdot \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right)$$

$$= -6(4 \cdot (4-3) - 5(6-5)) = 6$$

2. $(3,5)$, $\det A = 1 \cdot \begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 2 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix} = 1 \cdot 2 \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

$\downarrow$

$(3,1)$

$$= 2 \cdot \left(3 \cdot \begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right)$$

$$= 2(3 \cdot (-8-3) + 2(2+16)) = 6$$

Find the determinant by row reduction to echelon form.

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$$

$$\xrightarrow[④-①]{②+2①}\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 2 & -5 & 2 & -1 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix} \xrightarrow[⑤-②]{④-②} \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$$

$$\xrightarrow{③\leftrightarrow④} - \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix} \xrightarrow{⑤-\frac{7}{3}②} - \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -1 \cdot 2 \cdot (-1) \cdot 3 \cdot 1 = 6$$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A+B) = \det A + \det B$

if and only if $a + d = 0$.

$$\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - bc = 1+ad+a+d-bc$$

$$\det A + \det B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1+ad-bc$$

$$\det(A+B) = \det A + \det B \iff$$

$$1+ad+a+d-bc = 1+ad-bc$$

$$\iff a+d=0$$