MATH 2111: Tutorial 11: Eigenvalue, Eigenspace, Similarity and DIagona

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Review

- Eigenspace
- Characteristic Function
- Similarities & Diagonalization

Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix. Given λ and ρ are two distinct eigenvalues of A. Show that eigenspaces of λ and ρ are orthogonal. Namely, for any vectors $x_1 \in \mathcal{E}_\rho(A)$, $x_2 \in \mathcal{E}_\lambda(A)$, it has $x_1^\top x_2 = 0$.

Proof: For any $x_1 \in \mathcal{E}_\rho(A)$, it has

$$Ax_1 = \rho x_1$$

For any $x_2 \in \mathcal{E}_\lambda(A)$, it has

$$Ax_2 = \lambda x_2.$$

$\lambda x_1^T x_2 = x_1^T A x_2$ is some scalar

$\rho x_2^T x_1 = x_2^T A x_1.$ is some scalar

Also, $(x_1^T A x_2)^T = x_2^T A^T x_1 = x_2^T A x_1.,$

$(x_1^T x_2)^T = x_2^T x_1$

$\Rightarrow \lambda x_1^T x_2 = \rho x_2^T x_1$

$\Rightarrow x_1^T x_2 = 0.$

Given $A \in \mathbb{R}^{n\times n}$ and its characteristic function $f(\lambda) = \lambda^2(\lambda+1)(\lambda-1)(3-\lambda)^{n-4}$.

(1) Write down eigenvalues and their multiplicities.

(2) What is characteristics function of matrix $A + 2I$?

(1). It has eigenvalues

$\lambda = 0$, multiplicities 2

$\lambda = 1$, $\lambda = -1$ multiplicities 1

$\lambda = 3$ multiplicities $n-4$

(2) By definition, $|A-\lambda I| = f(\lambda)$.

$$|A+2I-\lambda I| = |A-(\lambda-2)I| = f(\lambda-2)$$

$$= (\lambda-2)^2\cdot(\lambda-1)(\lambda-3)(5-\lambda)^{n-4}$$

Actually, with eigenvalues of A :

$\lambda=0$, multiplicities 2

$\lambda=1$, $\lambda=-1$ multiplicities 1

$\lambda=3$ multiplicities $n-4$,

we know eigenvalues of $A+2I$:

$\lambda=2$, multiplicities 2

$\lambda=3$, $\lambda=1$ multiplicities 1

$\lambda=5$ multiplicities $n-4$

Suppose $A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(1) Find out characteristics function of A.

(2) Determine whether A is diagonalizable.

(1) $\det(A-\lambda I) = (\lambda-4)\cdot(\lambda-1)^3$.

(2). No. Since $\dim \text{eignspace}_{\lambda=1}(A) = 4 - \operatorname{rank}(A-I) = 1$.

$\neq$ multiplicites of 1

Diagonalize the following matrix, if possible,

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

First, compute eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 4-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda)\cdot[(4-\lambda)(2-\lambda)-1] + 1\cdot[0-(4-\lambda)]$$

$$= (4-\lambda)\cdot(\lambda^2 - 6\lambda + 6) = 0.$$

$\lambda_1 = 4, \quad \lambda_2 = 3+\sqrt{3}, \quad \lambda_3 = 3-\sqrt{3}$

Second, compute eigenvectors.

☆ $\lambda_1 = 4, \quad A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$

One of eigenvectors: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

☆ $\lambda_2 = 3+\sqrt{3}, \quad A - \lambda_2 I = \begin{pmatrix} 1-\sqrt{3} & 0 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -1-\sqrt{3} \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 0 & 1+\sqrt{3} \\ 0 & -2 & 1+\sqrt{3} \\ 1 & 1 & -1-\sqrt{3} \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & 0 & -\frac{1+\sqrt{3}}{2} \\ 0 & 1 & -\frac{1+\sqrt{3}}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

One of eigenvectors: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{pmatrix}.$

☆ $\lambda_3 = 3-\sqrt{3}. \quad A - \lambda_3 I = \begin{pmatrix} 1+\sqrt{3} & 0 & 1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & \sqrt{3}-1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & \sqrt{3}-1 \\ 0 & 2 & \sqrt{3}-1 \\ 1 & 1 & \sqrt{3}-1 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & 0 & \frac{\sqrt{3}-1}{2} \\ 0 & 1 & \frac{\sqrt{3}-1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

One of the eigenvectors: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}-1}{2} \\ 1 \end{pmatrix}$

Step 3: Diagonal.

$$A \cdot \begin{pmatrix} -1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3+\sqrt{3} & 0 \\ 0 & 0 & 3-\sqrt{3} \end{pmatrix}$$

Denote $P \triangleq \begin{pmatrix} -1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}+1}{2} & -\frac{\sqrt{3}-1}{2} \\ 0 & 1 & 1 \end{pmatrix}$

$$\Rightarrow \quad P^{-1} A P = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3+\sqrt{3} & 0 \\ 0 & 0 & 3-\sqrt{3} \end{pmatrix}$$

Determine range of α such that the following matrix is similar to some real diagonal matrix,

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

First compute eigenvalues: $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & \alpha \\ \alpha & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 - \alpha^2$

when $\alpha \neq 0$, it has two distinct eigenvalues

$\lambda = 1 \pm \alpha$.

In this case, A is diagonalizable.

when $\alpha = 0$, A is a diagonal matrix.

Remark

Given λ and ρ are two distinct eigenvalues of matrix $A \in \mathbb{R}^{n\times n}$. Suppose x_1 is an eigenvector corresponding to λ and x_2 is an eigenvector corresponding to ρ, namely,

$$Ax_1 = \lambda x_1, \quad Ax_2 = \rho x_2.$$

Then $x_1 + x_2$ is not eigenvector of A.