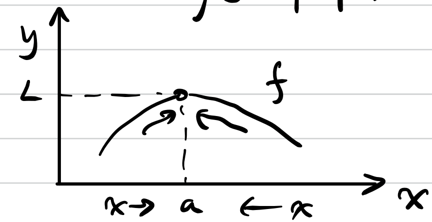
Sometimes, $f(a)$ is not defined.
Sometimes, $L$ is not in the range of $f$.

**Review:**

Limit: $\lim_{x\to a} f(x) = L$.

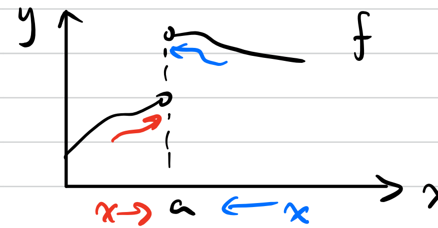
⇓ means

$f(x)$ can be arbitrary close to $L$ when $x$ tends to $a$.




One-sided limit: $\lim_{x\to a^-} f(x) = L_1$

$\lim_{x\to a^+} f(x) = L_2$



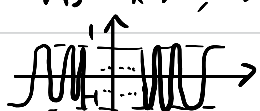

Notice: $\lim_{x\to a} f(x)$ exists $\Leftrightarrow$ $L_1 = L_2$.

Sometimes, $\lim_{x\to a} f(x)$ does not exist.

$\lim_{x\to 0} \frac{1}{x^2}$ $\left( x\to 0,\ \frac{1}{x^2} \to +\infty \right)$.

$\lim_{x\to 0} \frac{|x|}{x}$. $\left( \begin{array}{l} x\to 0^+,\ \frac{|x|}{x} = 1 \to 1 \\ x\to 0^-,\ \frac{|x|}{x} = -1 \to -1 \end{array} \right)$

$\lim_{x\to 0} \sin\frac{1}{x}$ : As $x\to 0$, $\sin\frac{1}{x}$ oscillates between $-1$ and $1$.



Computation of limits:

1). Limit laws: Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

$\Rightarrow \lim_{x\to a} f(x) \pm g(x)$, $\lim_{x\to a} f(x)\cdot g(x)$, $\lim_{x\to a} \frac{f(x)}{g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$)

2) Algebraic tricks:

① $\frac{0}{0}$ type ($\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x-2}$) ② $\frac{\infty}{\infty}$ type ($\lim_{x\to+\infty} \frac{\sqrt{x^4+1}+x^2}{3x^2+x}$). ③ $\infty-\infty$ type ($\lim_{x\to+\infty} \sqrt{x+2}-\sqrt{x}$).

(rationalization, cancel the common factors, reduce the highest power of $x$).

3). The Squeeze Theorem
(The Sandwich Theorem)

$$\left.\begin{array}{l} f(x) \le h(x) \le g(x) \\ \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L \end{array}\right\} \lim_{x\to a} h(x) = L.$$

Example: $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ $\quad -x^2 \le x^2 \sin\frac{1}{x} \le x^2$ $(x\neq 0)$

$\lim_{x\to 0} x \sin\frac{1}{x} = 0$ $\quad -|x| \le x\sin\frac{1}{x} \le |x|$ $(x\neq 0)$.

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\quad \cos x \le \frac{\sin x}{x} \le 1$ $(x\in[-\frac{\pi}{2},\frac{\pi}{2}].\ x\neq 0)$.

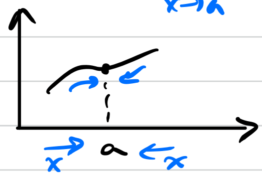
$\Rightarrow \lim_{x\to 0} \frac{1-\cos x}{\frac{1}{2}x^2}$, $\lim_{x\to 0} \frac{x}{x+\sin x}$, $\lim_{x\to 0} \frac{\sin 3x}{2x}$.

# Continuity of functions

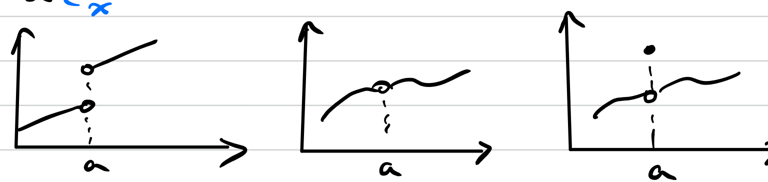
1. $f(x)$ is continuous at $x=a$ $\Leftrightarrow$ $\lim_{x\to a} f(x) = f(a)$

① $\lim_{x\to a} f(x)$ exists ② $f(a)$ is defined. ③ $\lim_{x\to a} f(x) = f(a)$

continuous at $x=a$




not continuous at $x=a$:




$f(x)$ is continuous on $I$ $\Leftrightarrow$ $f(x)$ is continuous at each point in $I$.

Example: $y=\frac{1}{x}$ is continuous on $(-\infty, 0) \cup (0, +\infty)$.

Notice: For continuous functions, the value of the limit is given by the value of functions.

2. Two basic facts about continuous functions { The Extreme Value Theorem
The Intermediate Value Theorem

Let $f(x)$ be continuous on $[a,b]$.

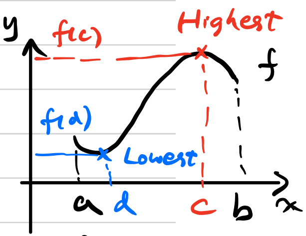
(1) The Extreme Value Theorem.

called "absolute maximum value of $f$"

There is a number $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$.

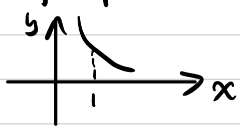
called "absolute minimum value of $f$"

There is a number $d \in [a,b]$ such that $f(d) \leq f(x)$ for all $x \in [a,b]$.

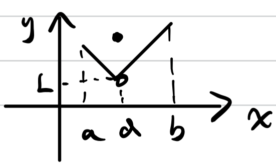



Notice: The existence of $c$ and $d$ is guarateened by the continuity of $f$ on $[a,b]$.

If $f$ is not continuous on $[a,b]$:




$f(x) = \frac{1}{x}$ is continuous on $(0,1]$.
$f(x)$ does not have the absolute maximum value. ($x \to 0^+$, $\frac{1}{x} \to +\infty$)

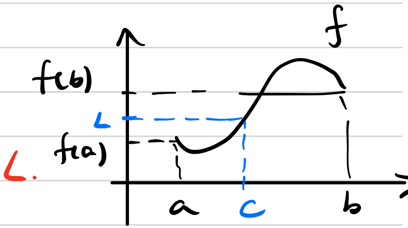



$f(x)$ is not continuous on $[a,b]$.
$f(x)$ does not have the absolute minimum value. ( $f(x) > L$, $x \in [a,b]$
But $L$ is not in the range of $f$.

## (2) The Intermediate Value Theorem.

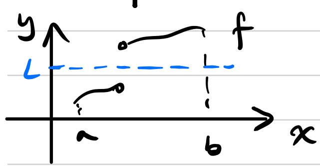
For any number $L$ between $f(a)$ and $f(b)$,

there exists a number $c \in [a, b]$, such that $f(c) = L$.



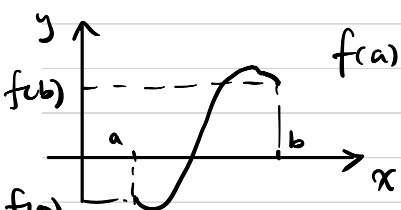

Notice: The existence of $c$ is guaranteed by the continuity of $f$ on $[a, b]$.

If $f$ is not continuous on $[a, b]$:




There does not exist a number $c \in [a, b]$ such that $f(c) = L$.

Application: Find the roots of $f(x)$.




$f(a) \times f(b) < 0$ $\Rightarrow$ There exists a number $c \in [a, b]$ such that $f(c) = 0$. $\xRightarrow{\text{means}}$ $f$ has a root between $a$ and $b$.

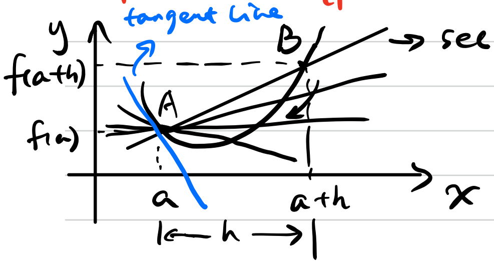
( a special case of the Intermediate Value Theorem).

Example: Show that $f(x) = x - \cos x$ has a root in $(0, \frac{\pi}{2})$.

$$\left.\begin{aligned} f(0) &= 0 - \cos 0 = -1 < 0. \\ f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0. \end{aligned}\right\} \Rightarrow \text{There exists a number } c \in \left(0, \frac{\pi}{2}\right) \text{ such that } f(c) = 0.$$

# Differentiation of functions

1. $f(x)$ is differentiable at $x=a$ $\Longleftrightarrow$ $\lim_{h\to 0} \boxed{\frac{f(a+h)-f(a)}{h}}$ exists.




|| (if exists)

the slope of the tangent line of $f$ at $x=a$.

||

the rate of change of $f$ at $x=a$.

(describe how fast $f(x)$ decreases/increases at $x=a$)

Slope of the secant line AB:

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{a+h-a} = \boxed{\frac{f(a+h)-f(a)}{h}} \overset{\text{means}}{\Longrightarrow}$$ the average rate of change of $f$ over $[a, a+h]$.

$\Delta$ (called "delta") $\overset{\text{means}}{\Longrightarrow}$ difference, change.

As $h \to 0$, secant line AB $\to$ tangent line.

the slope of secant line AB $\to$ the slope of the tangent line.

To conclude: $f$ is differentiable at $x=a$ $\overset{\text{means}}{\Longrightarrow}$ the rate of change of $f$ at $x=a$ is computable

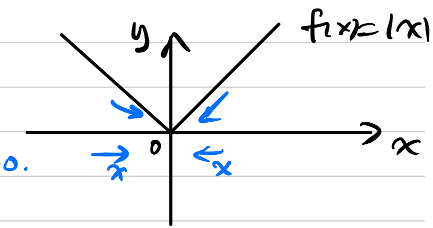
## 2. Comparison between "continuous" and "differentiable".

$f(x)$ is continuous at $x=a$ $\Leftrightarrow$ $\lim\limits_{x\to a} f(x) = f(a)$

$\Uparrow \quad \not\Downarrow$

$f(x)$ is differentiable at $x=a$ $\Leftrightarrow$ $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists.

Example: $f(x)=|x|$.




$\lim\limits_{x\to 0} f(x) = 0 = f(0) \Rightarrow f$ is continuous at $x=0$.

$\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$ does not exist. $\Rightarrow f$ is not differentiable at $x=0$.

(The tangent line also does not exist at $x=0$)

$\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ does not exist.

This example shows that: $f$ is continuous at $x \neq a$ $\not\Rightarrow$ $f$ is differentiable at $x=a$

Theorem: Suppose that $f$ is differentiable at $x=a$.

Then $f$ is continuous at $x=a$.

Proof: Our aim is to prove $\lim_{x\to a} f(x) - f(a) = 0$.

we can consider $\underline{\underline{x}}$ as $a+\underline{\underline{h}}$

Notice that $\lim_{x\to a} f(x) - f(a) = \lim_{h\to 0} f(a+h) - f(a)$

$$= \lim_{h\to 0}\left(h \cdot \frac{f(a+h)-f(a)}{h}\right)$$

because $f$ is differentiable at $x=a$

$$= \underbrace{\left(\lim_{h\to 0} h\right)}_{=0} \times \underbrace{\left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right)}_{= \text{a finite number.}}$$

$$= 0.$$

Therefore, $f$ is continuous at $x=a$.

To conclude: $f$ is differentiable at $x=a \Longrightarrow f$ is continuous at $x=a$.

$f$ is continuous at $x=a \not\Longrightarrow f$ is differentiable at $x=a$.

3. Derivative.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \xlongequal[\text{denoted by}]{\text{if exists}} f'(a).$$

Let a not be fixed $\longrightarrow$

$$\underbrace{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} \xlongequal[\text{denoted by}]{\text{if exists}} f'(x) \xrightarrow{\text{called}} \text{the derivative of } f.$$

we consider this limit as a function of $x$.