COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Spring Semester – Final Exam (Part 2)

Note 1: For all non-proof questions, showing your steps is not necessary unless required otherwise. However, in case your answer is wrong, showing your steps may earn you some partial credits.
Note 2: You can express your answers using binomial coefficients, factorials, and D_n (derangement number). However, you should not have summation $\sum$ in your final answers. For example, $\binom{10}{3}D_9+4!$ and $1!+2!+3!+4!$ are valid, but $\sum_{i=0}^{n}\binom{n}{i}$ or $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}$ is not. The latter has to be simplified to 2^n.

Question 1: (8 pts) Prove that in an undirected graph $G=(V,E)$, if $|E|>\binom{|V|-1}{2}$, then G is connected.

Solution: Before we give a correct proof here, let us look at an incorrect proof that students often give for this exercise. It goes something like this. "Suppose that the graph is not connected. Then no vertex can be adjacent to every other vertex, only to at most $n-2$ other vertices. One vertex connected to $n-2$ other vertices creates a component with $n-1$ vertices in it. To get the most edges possible, we must use all the edges in this component. The number of edges in this component is thus $C(n-1,2)=(n-1)(n-2)/2$, and the other component (with only one vertex) has no edges. Thus we have shown that a disconnected graph has at most $(n-1)(n-2)/2$ edges, so every graph with more edges than that has to be connected." The fallacy here is in assuming-without justification-that the maximum number of edges is achieved when one component has $n-1$ vertices. What if, say, there were two components of roughly equal size? Might they not together contain more edges? We will see that the answer is "no", but it is important to realize that this requires proof-it is not obvious without some calculations.

Here is a correct proof, then. Suppose that the graph is not connected. Then it has a component with k vertices in it, for some k between 1 and $n-1$, inclusive. The remaining $n-k$ vertices are in one or more other components. The maximum number of edges this graph could have is then $C(k,2)+C(n-k,2)$, which, after a bit of algebra, simplifies to $k^2-nk+(n^2-n)/2$. This is a quadratic function of k. It is minimized when $k=n/2$ and maximized at two extreme values of the range, namely $k=1$ and $k=n-1$. In the latter two cases its value is $(n-1)(n-2)/2$. Therefore the largest number of edges that a disconnected graph can have is $(n-1)(n-2)/2$, so every graph with more edges than this must be connected.

This contradicts $|E|>\binom{v-1}{2}$, so the assumption is wrong, G is connected.

Question 2: (10 pts) Consider the following 5-door version of the Monty Hall problem. There are 5 doors, behind one of which (randomly with equal probability) there is a car (which you want), and behind the rest of which there are goats (which you don't want). You choose a door. Monty then opens 2 goat doors. Assume that Monty knows which door has the car, and will choose with equal probabilities from all his choices of which goat doors to open. Without loss of generality, assume your first guess is door 1. Let W be the random variable denoting the door with the car and M the random variable denoting the two doors that Monty opens. Note that M is a random variable whose value is a subset of $\{1, 2, 3, 4, 5\}$.

(a) (2 pts) What is $P(M = \{2, 3\} \mid W = 1)$? What is $P(M = \{3, 5\} \mid W = 1)$?

(b) (2 pts) What is $P(M = \{2, 3\} \mid W = 2)$?

(c) (2 pts) What is $P(M = \{3, 4\} \mid W = 2)$? What is $P(M = \{3, 5\} \mid W = 4)$?

(d) (2 pts) What is $P(M = \{3, 4\})$?

(e) (2 pts) What is $P(W = 2 \mid M = \{3, 4\})$?

Solution:

(a) $P(M = \{2, 3\} \mid W = 1) = P(M = \{3, 5\} \mid W = 1) = 1/\binom{4}{2} = 1/6.$

(b) $P(M = \{2, 3\} \mid W = 2) = 0$

(c) $P(M = \{3, 4\} \mid W = 2) = P(M = \{3, 5\} \mid W = 4) = 1/\binom{3}{2} = 1/3.$

(d) $P(M = \{3, 4\}) = P(M = \{3, 4\} \mid W = 1) \cdot P(W = 1) + P(M = \{3, 4\} \mid W = 2) \cdot P(W = 2) + P(M = \{3, 4\} \mid W = 5) \cdot P(W = 5) = 1/6 \cdot 1/5 + 1/3 \cdot 1/5 + 1/3 \cdot 1/5 = 1/6.$

(e) $P(W = 2 \mid M = \{3, 4\}) = \frac{P(M=\{3,4\}|W=2)\cdot P(W=2)}{P(M=\{3,4\})} = \frac{1/3\cdot 1/5}{1/6} = 2/5.$

Question 3: (6 pts) Solve the following system of congruences:

$$\begin{aligned} 5x + 9y &\equiv 5 \pmod{11}, \\ 3x - 4y &\equiv 2 \pmod{11}. \end{aligned}$$

Solution:

$$\begin{aligned} x &\equiv 9 \pmod{11}, \\ y &\equiv 9 \pmod{11}. \end{aligned}$$

Question 4: (6 pts) A basketball player makes 80% of his free throws. We put him on the free-throw line and ask him to shoot free throws until he misses one. Let $X =$ the number of free throws the player takes until he misses (including the last one that he misses).

(a) (2 pts) What is $E[X]$?

(b) (2 pts) What is $P(X = 6)$?

(c) (2 pts) What is $P(X \leq n)$?

Solution: (a) X follows a geometric distribution with $p = 0.2$, so $E[X] = 5$.

(b) $P(X = 6) = (1 - p)^5 p = (0.8)^5 \cdot 0.2 = 0.0655$

(c) $P(X \leq n) = 1 - P(X > n) = 1 - (1 - p)^n = 1 - 0.8^n$

Question 5: (10 pts) Let G be a simple undirected graph with a set of vertices V. Let V_1 and V_2 be subsets of V so that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. Let $E(x, y)$ be the predicate representing that there is an edge from x to y. Note that the graph being undirected means that $\forall u \in V\ \forall v \in V\ (E(u, v) \leftrightarrow E(v, u))$.

(a) (6 pts) Express each of the following properties in predicate logic. You can only use $V, V_1, V_2, E(\cdot, \cdot)$, logical and mathematical operators.

(i) Every edge connects a vertex in V_1 and a vertex in V_2

(ii) For every vertex in V_1, there are edges that connect it with all vertices in V_2

(b) (2 pts) If (a)(i) is true, is G necessarily a bipartite graph? Please give a brief justification.

(c) (2 pts) If (a)(ii) is true, is G necessarily a complete bipartite graph? Please give a brief justification.

Solution: (a) (i) $(\forall u \in V_1\ \forall v \in V_1(\neg E(u, v))) \land (\forall u \in V_2\ \forall v \in V_2(\neg E(u, v)))$ or $\forall u, v \in V\ (E(u, v) \to ((u \in V_1 \land v \in V_2) \lor (u \in V_2 \land v \in V_1)))$

(ii) $\forall u \in V_1\ \forall v \in V_2(E(u, v))$.

(b) Yes. Based on the definition of bipartite graph, the graph G is bipartite because it can be partitioned into two disjoint subsets V_1 and V_2 and every edge connects a vertex in V_1 and a vertex in V_2.

(c) No. If there are edges connecting two vertices in V_1, the graph G is not a bipartite graph, so it is not a complete bipartite graph.

Question 6: (12 pts) There is a random simple undirected graph G with n vertices, where an edge exists between every pair of vertices with probability p independently. Let X be the random variable denoting the number of edges in the graph.

(a) (4 pts) What is the expectation and variance of X?

(b) (4 pts) If G has exactly m connected components $C_1, C_2, ..., C_m$, where C_i is a connected component with v_i vertices, what is the maximum possible value of X?

(c) (4 pts) What is the probability that G with $n = 4$ vertices has 3 edges and is not connected.

Solution:

(a) X_e is the indicator random variable that indicates the event of a pair of vertices $e = (u, v)$ having an edge. X_e is a Bernoulli trial, thus $E(X_e) = p$, $V(X_e) = p(1 - p)$.

There are $\binom{n}{2}$ pairs of vertices in total. Thus $E(X) = \binom{n}{2}p$ and $V(X) = \binom{n}{2}p(1 - p)$.

(b) For connected component C_i, the maximum number of edges is $\binom{v_i}{2}$, forming a complete graph.

The maximum of X is attained when all components take the maximal value at the same time.

$$\max(X) = \sum_{i=1}^{m} \max(C_i) = \sum_{i=1}^{m} \binom{v_i}{2}$$

(c) The simple graph G with 4 vertices and 3 edges is not connected if and only if three edges form a triangle and the last vertex is disconnected, which happens with the probability $\binom{4}{3}p^3(1 - p)^3$.