HKUST – Department of Computer Science and Engineering

COMP2711: Discrete Math Tools for CS – Fall 2016

Midterm Examination

Date: Oct 14, 2016 (Friday)
Time: 19:00–21:00
Venues: LT B (L1 students) and LT L (L2 and L3 students)

Surname (e.g., Chan): ______________________

Given Name (e.g., Tai Man): ______________________

Student ID: ______________ Seat Number: ______________

Tutorial Session (Please tick only one of the following boxes):

☐ L1 (T1A) (Friday 12:30pm - 1:20pm) ☐ L1 (T1B) (Monday 5:00pm - 5:50pm)
☐ L2 (T2A) (Friday 10:30am - 11:20am) ☐ L2 (T2B) (Tuesday 6:00pm - 6:50pm)
☐ L3 (T3A) (Monday 11:00am - 11:50am) ☐ L3 (T3B) (Tuesday 5:00pm - 5:50pm)

Instructions

- You are allowed to bring a calculator and an A4-sized cheating sheet hand-written on only one side.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work.
- **Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.**
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60.
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	Total
Points	10	10	15	10	10	15	15	15	100
Score									

Seat Number: ____________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name: ____________________________

Student's Signature: _______________________
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Problem 1: [10 pts] Consider a web system for online shopping that generates an order id for each business. The order id is a sequence of 7 characters where each character could be either one of the 26 Upper Case English Alphabets (A to Z) or one of the 10 digits (0 to 9).

(a) How many possible order id's are there?

(b) How many possible order id's are there if no two characters are the same? (E.g., A034NXD is valid and A034NAD is invalid)

(c) How many possible order id's are there if the leftmost character must be a digit and no two consecutive characters are the same? (E.g., 3ABA3C6 is valid. C3ARX7B and 3ARRX7B are invalid)

(d) How many possible order id's are there if the order id is alternative between alphabet and digit? (E.g., A3R4B7Q and 4V7X3V7 are valid, B7XB4Q3 is invalid)

Solution: *(a)* 36^7

(b) $36^{\underline{7}}$ *or* $7!\binom{36}{7}$

(c) $10 \cdot 35^6$. *The leftmost character has 10 choices, then the second leftmost character has 35 choices because it cannot be the same as it's adjacent character, and so on.*

(d) $10^4 \cdot 26^3 + 10^3 \cdot 26^4$. *It can either be four digits and three alphabets or three digits and four alphabets. The position of digits and alphabets are fixed in either case.*

Suggested Marking:
Part (a) (2pts)
Part (b) (2pts)
Part (c) (3pts)
Part (d) (3pts)

Seat Number: ___________

Problem 2: [10 pts] Consider a university with 300 faculties, 500 teaching assistants, 600 postgraduate students and 10,000 undergraduate students. A tennis club would like to invite a group of guests to join an activity. The group of guests must consist of 4 faculties, 10 teaching assistants, 10 postgraduate students and 40 undergraduate students. Assumes that each person initially has only one role (i.e. no one person can be both a postgraduate student and a teaching assistant, etc).

(a) How many possible groups of guests are there?

(b) Now, one postgraduate student has a special treatment. This postgraduate student could also be accepted to be a teaching assistant (i.e., this particular person has two roles, a postgraduate student and a teaching assistants). The tennis club can invite this guy as a role of either a teaching assistant or a postgraduate student (but not both), or does not invite this guy at all. How many possible groups of guests are there?

Solution: *(a)* $\binom{300}{4}\binom{500}{10}\binom{600}{10}\binom{10000}{40}$

(b) The special guy can be in the group as a postgraduate student or not in the group at all, this is the same as part (a). In addition, this special guy can be in the group as a teaching assistant. In this case, we only need 9 more teaching assistants, and the total number of possible postgraduate students becomes 599 because the special guy already in the group as teaching assistant. Therefore, we have $\binom{300}{4}\binom{500}{10}\binom{600}{10}\binom{10000}{40} + \binom{300}{4}\binom{500}{9}\binom{599}{10}\binom{10000}{40}$ *possible groups of guests. Another possible solution is* $\binom{300}{4}\binom{500}{10}\binom{599}{9}\binom{10000}{40}+\binom{300}{4}\binom{501}{10}\binom{599}{10}\binom{10000}{40}$.

Suggested Marking:
Part (a) (4pts)
Part (b) (6pts)

Seat Number: ___________

Problem 3: [15 pts] Consider the following identity. For any two integers n and k where $0 < k \leq n$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

(a) Prove the identity above by means of a combinatorial proof.

(b) Prove the identity above by means of an algebraic proof.

Solution: *(a)*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

could be re-written as

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

That is,

$$\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}$$

Consider n distinct objects.

Suppose that we want to distribute these n distinct objets into 3 distinguishable buckets. The first bucket contains 1 object, the second bucket contains $k-1$ objects, and the third bucket contains the remaining objects (i.e., $n-k$ objects).

The total number of ways of distributing the objects into these 3 buckets can be interpreted in the following two methods.

- *The first method is to select k objects out of n objects first and to keep the remaining objects (i.e., $n-k$ objects) in the third bucket. Then, select 1 object out of the selected k objects and place it to the first bucket and keep the remaining objects (i.e., $k-1$ objects) in the second bucket. Here, the total number of ways in this method is $\binom{n}{k}\binom{k}{1}$.*
- *The second method is to select 1 object out of n objects first and put it to the first bucket. Then, we select $k-1$ objects out of the remaining objects (i.e., $n-1$ objects) and put it to the second bucket. All remaining objects (i.e., $n-k$ objects) are put to the third bucket. Here, the total number of ways in the second method is $\binom{n}{1}\binom{n-1}{k-1}$.*

Since the first method has the same physical meaning as the second method, we have

$$\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}$$

(b)

$$
\begin{aligned}
RHS &= \frac{n}{k}\binom{n-1}{k-1} \\
&= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k} \\
&= LHS
\end{aligned}
$$

Suggested Marking:
Part (a) (9pts)
Part (b) (6pts)

Problem 4: [10 pts]

(a) Construct a truth table for the statement $(p \oplus q) \Leftrightarrow r$.

(b) Are the following two statements logically equivalent? If yes, prove that the two statements are logically equivalent. Otherwise, give a counter example to show that the statements are not logically equivalent.

(i) $(p \Rightarrow q) \Rightarrow r$

(ii) $(p \vee r) \wedge (q \Rightarrow r)$

Solution: *(a) Truth Table:*

p	q	r	$p \oplus q$	$(p \oplus q) \Leftrightarrow r$
T	T	T	F	F
T	T	F	F	T
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	T

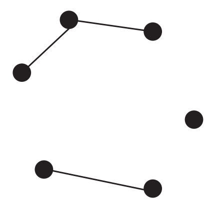
(b) Yes. They are logically equivalent.

$(p \Rightarrow q) \Rightarrow r$
$(\neg p \vee q) \Rightarrow r$
$\neg(\neg p \vee q) \vee r$
$(p \wedge \neg q) \vee r$
$(p \vee r) \wedge (\neg q \vee r)$
$(p \vee r) \wedge (q \Rightarrow r)$

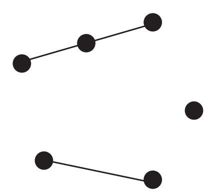
Suggested Marking:
Part (a) (4pts)
Part (b) (6pts)

Seat Number: ___________

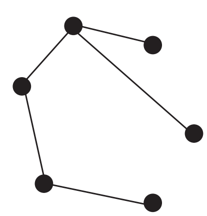
Problem 5: [10 pts] There are n dots painted on a paper. A kid draws some line segments on the paper and each line segment connects exactly two dots as the two end-points of this line segment. For example, the following figure shows 3 line segments and 6 dots.



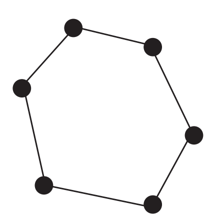
Note that if one line segment (connecting two dots) is an extension of another line segment (connecting two dots where one of them is the dot connected by the previous line segment mentioned), we still refer them as two separate line segments (not one line segment). For example, the following figure shows 3 line segments and 6 dots.



We say that the n dots are all connected if for every pair of two dots x and y, there exists a sequence of line segments, namely $(d_1, d_2), (d_2, d_3), (d_3, d_4), ..., (d_{l-1}, d_l)$ where d_i is a dot for $i \in [1, l]$, such that $d_1 = x$ and $d_l = y$. For example, the following figure shows that the 6 dots are all connected but the above 2 figures shows that 6 dots are not all connected.



The following shows an example that the line segments form a *loop*.



(a) Prove by **contraposition** that if the kid only draws $n-2$ line segments, then the n dots are not all connected.

(b) Prove by **contradiction** that if the kid draws a loop that connects the n dots, then s/he needs at least n line segments to draw the loop.

Solution: *(a) Let p be the kid only draws $n-2$ line segments.*
Let q be the n dots are not all connected.
To prove $p \Rightarrow q$ using contraposition, we state $\neg q$ which is "n dots are all conntected". Since all n dots are connected, each dot must have a least one line segment attached to it. We use a highlighter pen to highlight one arbitrary line segment and the two dots connected by the line segment. Then we highlight another line segment such that it connects one of the "highlighted" dots and one of the "unhighlighted" dots. Thus we will have two line segments and three dots highlighted and $n-3$ dots unhighligthed. The process repeats until all n dots are highlighted. The process is repeatable since every dot is connected. That means there must be some unhighlighted dots connecting with a highlighted dot at any state of the process. After n dots are highlighted, there are $n-1$ line segments highlighted. That implies the kid has drawn at least $n-1$ line segments ($\neg p$).

(b) Assume that it is possible to draw a loop with smaller than n line segments for n dots. Since each dot needs to be connected with exactly two line segments in a loop, we count the total number of connections between a line segment and a dot, there are $2n$ connections. If the loop has smaller than n line segment, say, there are only k line segments where $k < n$, and thus each line segment on average needs to connect $2n/k$ dots (more than 2 dots). However, each line segment can exactly connect two dots. By contradiction, it requires at least n line segments to draw a loop that connects n dots.

Suggested Marking:
Part (a) (5pts). 1pt partial credit if $\neg p$ is correct. 1pt partial credit if $\neg q$ is correct.
Part (b) (5pts).

Problem 6: [15 pts] Let $q(x)$ be "x is even". Let Z be a set of all integers. Let E be a set of all even integers.

For each of the following pairs, say whether (i) is equivalent to (ii). If they are equivalent, all you have to do is to say that they are equivalent. If they are not equivalent, give a counter example. A counter example should involve a specification of $p(x)$ and an explanation as to why the resulting statement is false.

(a) (i) $\forall x \in E(p(x))$
(ii) $\forall x \in Z(q(x) \Rightarrow p(x))$

(b) (i) $\forall x \in E(p(x))$
(ii) $\forall x \in Z(q(x) \wedge p(x))$

(c) (i) $\exists x \in E(p(x))$
(ii) $\exists x \in Z(q(x) \Rightarrow p(x))$

(d) (i) $\exists x \in E(p(x))$
(ii) $\exists x \in Z(q(x) \wedge p(x))$

Solution: *(a) Yes.*

(b) No.
Let $p(x)$ be "x is even".
"$\forall x \in E(p(x))$" is true but "$\forall x \in Z(q(x) \wedge p(x))$" is false.

(c) No.
Let $p(x)$ be "x is odd".
"$\exists x \in E(p(x))$" is false but "$\exists x \in Z(q(x) \Rightarrow p(x))$" is true (because when $x = 1$ (where $x \in Z$), $q(x)$ is false and $p(x)$ is true (which means that "$q(x) \Rightarrow p(x)$" is true)).

(d) Yes.

Suggested Marking:
Part (a) (3pts)
Part (b) (4pts)
Part (c) (5pts)
Part (d) (3pts)

Seat Number: ____________

Problem 7: [15 pts] There are n families joining a party. Each family consists of 3 people, a mother, a father and a child. In the party, there are $3n$ seats. If we say that three people in each family must sit together, this means that one member in this family sits at a seat between the two other members in the family.

(a) What is the probability that k ($1 \leq k \leq n$) specified families end up sitting together (regardless of whether the other $n-k$ families sit together or not)?

(b) What is the probability that at least one family sits together?
You may use the summation ($\sum$) sign and $\binom{n}{k}$ to express your answer.

(c) What is the probability that no family sits together?
You may use the summation ($\sum$) sign and $\binom{n}{k}$ to express your answer.

Solution: *(a) no. of ways that k specified families sit together*
$= (3n-2k)!(3!)^k$
no. of ways of seatings
$= (3n)!$
P(k specified families sit together)
$= \frac{(3n-2k)!(3!)^k}{(3n)!}$

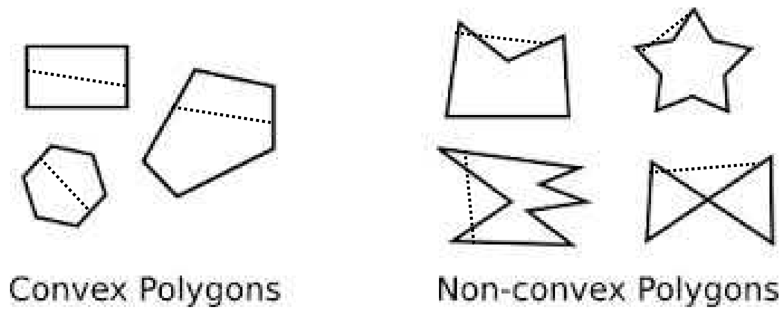
(b) Let E_i be the event that family i sits together for each $i \in [1, n]$.
P(at least one family sits together)
$= P(E_1 \cup E_2 \cup ... \cup E_n)$
$= \sum_{k=1}^{n}(-1)^{k+1} \sum_{i_1,i_2,...,i_k:1\leq i_1<i_2<...<i_k\leq n} P(E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_k})$
$= \sum_{k=1}^{n}(-1)^{k+1} \sum_{i_1,i_2,...,i_k:1\leq i_1<i_2<...<i_k\leq n} \frac{(3n-2k)!(3!)^k}{(3n)!}$
$= \sum_{k=1}^{n}(-1)^{k+1} \binom{n}{k} \frac{(3n-2k)!(3!)^k}{(3n)!}$

(c) P(no family sits together)
= 1 - P(at least one family sits together)
$= 1 - \sum_{k=1}^{n}(-1)^{k+1} \binom{n}{k} \frac{(3n-2k)!(3!)^k}{(3n)!}$

Suggested Marking:
Part (a) (5pts)
Part (b) (5pts)
Part (c) (5pts)

Seat Number: ____________

Problem 8: [15 pts] A convex polygon is a polygon in which no line segment between two points on the boundary ever goes outside the polygon. Refer to the examples below for convex and non-convex polygons.



Assume the coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$ are the vertices of a convex polygon arranged in counterclockwise. Use induction to prove that the area of the polygon is as follows. For $n \geq 3$,

$$\text{Area} = \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + \cdots + x_{n-1}y_n + x_ny_1)$$
$$-(y_1x_2 + y_2x_3 + y_3x_4 + \cdots + y_{n-1}x_n + y_nx_1)]$$

(Note, for the sake of simplicity, you can always assume that $x_1 < x_i, \forall i \in [2, n]$. That is, (x_1, y_1) is the leftmost vertice of the polygon.)

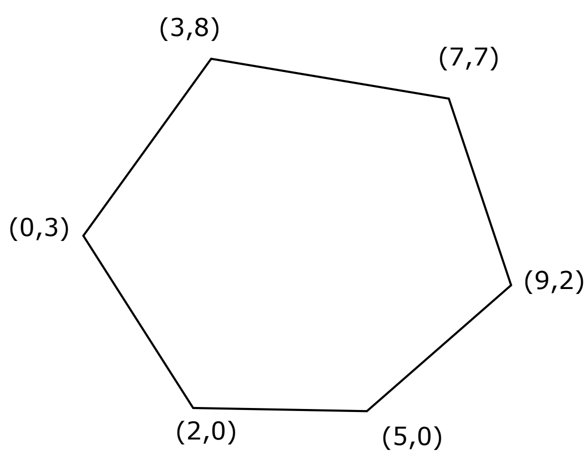


Figure 1: According to the formula, the area of this polygon is $\frac{1}{2}[(0 \cdot 0 + 2 \cdot 0 + 5 \cdot 2 + 9 \cdot 7 + 7 \cdot 8 + 3 \cdot 3) - (3 \cdot 2 + 0 \cdot 5 + 0 \cdot 9 + 2 \cdot 7 + 7 \cdot 3 + 8 \cdot 0)] = 48.5$ unit squares.

You are given the following fact:

1. The distance between point (x_1, y_1) and point (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
2. If a point (x_0, y_0) is above the line $ax+by+c = 0$ where $b > 0$, then the (closest) distance from the point to the line is $(ax_0+by_0+c)/(\sqrt{a^2 + b^2})$

   (Note that when we say that a point (x_0, y_0) is above the line $ax + by + c = 0$ where $b > 0$, we mean that the point (x_0, y_0) has a larger y-coordinate value than the intersection point between the vertical line passing through (x_0, y_0) and the line $ax + by + c = 0$.)
3. The line which passes two points (x_1, y_1) and (x_2, y_2) can be written as $(y_1 - y_2)x + (x_2 - x_1)y + y_2x_1 - y_1x_2 = 0$

Solution: *Denote the statement by $p(n)$. We need to prove the equation for $n \geq 3$.*

Base cases: *Consider a triangle with counterclockwise vertices (x_1, y_1),(x_2, y_2),(x_3, y_3). The line passing through (x_1, y_1) and (x_2, y_2) is $(y_1 - y_2)x + (x_2 - x_1)y + y_2x_1 - y_1x_2 = 0$. Take the distance between (x_1, y_1) and (x_2, y_2) as the base of an triangle and the height be the distance between the line and (x_3, y_3) which is $((y_1 - y_2)x_3 + (x_2 - x_1)y_3 + y_2x_1 - y_1x_2)/\sqrt{(y_1 - y_2)^2 + (x_2 - x_1)^2}$ (Since (x_3, y_3) is above the line and $x_2 - x_1 > 0$). The area can be calculated as*

$$\begin{aligned}
Area &= \frac{((y_1 - y_2)x_3 + (x_2 - x_1)y_3 + y_2x_1 - y_1x_2)}{2\sqrt{(y_1 - y_2)^2 + (x_2 - x_1)^2}} \times \sqrt{(y_1 - y_2)^2 + (x_2 - x_1)^2} \\
&= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)]
\end{aligned}$$

Inductive hypothesis: *Suppose $p(n)$ is true for some $n \geq 3$, i.e.,*

$$\begin{aligned}
Area = \frac{1}{2}[&(x_1y_2 + x_2y_3 + x_3y_4 + \cdots + x_{n-1}y_n + x_ny_1) \\
&-(y_1x_2 + y_2x_3 + y_3x_4 + \cdots + y_{n-1}x_n + y_nx_1)]
\end{aligned}$$

Inductive step: *Given a convex polygon with $n+1$ vertices (x_1, y_1),(x_2, y_2), $\cdots$, (x_n, y_n), (x_{n+1}, y_{n+1}), we split the polygon into an n-vertice polygon with vertices $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ and a triangle $(x_n, y_n), (x_{n+1}, y_{n+1}), (x_1, y_1)$.*

By the inductive hypothesis, the area of the n-vertice polygon is

$$\frac{1}{2}[(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)]$$

The area of the triangle is

$$\frac{1}{2}[(x_1y_n + x_ny_{n+1} + x_{n+1}y_1) - (y_1x_n + y_nx_{n+1} + y_{n+1}x_1)]$$

The area of the $(n+1)$-vertice polygon is simply the sum of them

$$Area = \frac{1}{2}[(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)] +$$
$$\frac{1}{2}[(x_1y_n + x_ny_{n+1} + x_{n+1}y_1) - (y_1x_n + y_nx_{n+1} + y_{n+1}x_1)]$$
$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + \cdots + x_ny_{n+1} + x_{n+1}y_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_{n+1} + y_{n+1}x_1)]$$

Thus, $p(n+1)$ is also true.
Inductive conclusion:*[1pt] By the mathematical induction, we get $p(n)$ is true for all $n \geq 3$.*

Suggested Marking:
Base Case: [Suggested Marking: 5pt]
Inductive Hypothesis: [Suggested Marking: 2pt]
Inductive Step: [Suggested Marking: 8pt]

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