MATH 2111: Tutorial 12
Inner Product and Orthogonality

T1A&T1B QUAN Xueyang
T1C&T2A SHEN Yinan
T2B&T2C ZHANG Fa

Department of Mathematics, HKUST

- Inner product, length, and orthogonality
- Orthogonal sets
- Orthogonal projections

Determine which pairs of vectors are orthogonal.

$$\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

(1) Verify the parallelogram law for vectors $\mathbf{u}$ and $\mathbf{v}$ $\mathbb{R}^{\mathrm{n}}$:

$$\| \mathbf{u}+\mathbf{v} \|^2+\| \mathbf{u}-\mathbf{v} \|^2= 2 \| \mathbf{u} \|^2 +2 \| \mathbf{v} \|^2$$

(2) Let $\mathrm{W} = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{\mathrm{p}}\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_{\mathrm{j}}$, then for $1 \leq \mathrm{j} \leq \mathrm{p}$, then $\mathbf{x}$ is orthogonal to every vector in W.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

Find the closest point to y in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$,

and let $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that dim $\mathbf{dim}\, W + \mathbf{dim}\, W^\perp = n$.