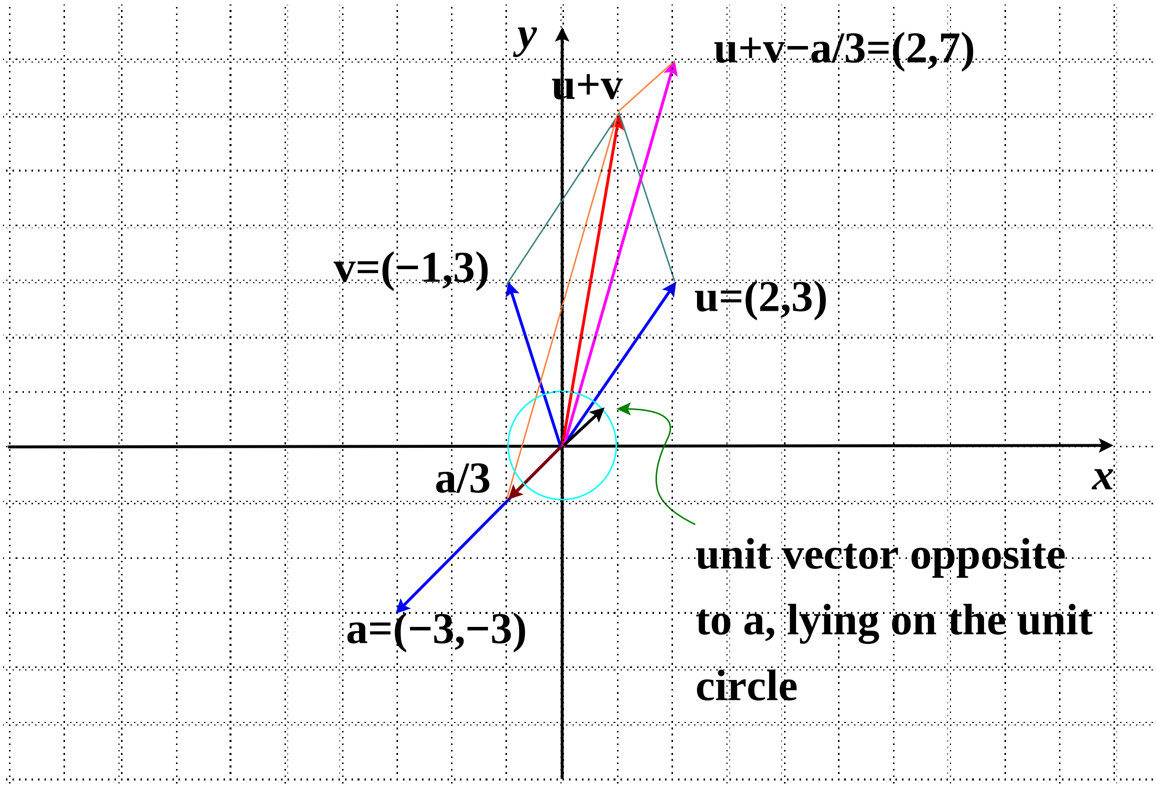
Math1014 Calculus II
Basic Problems on Vectors in the Plane and Space

1. Let $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -1, 3\rangle$, $\mathbf{a} = \langle -3, -3\rangle$.

   (a) Draw arrows with initial point at the origin to represent these vectors in the plane.

   (b) Draw arrows to represent the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}+\mathbf{v}-\frac{1}{3}\mathbf{a}$ without algebraic calculation.

   (c) Find and draw a unit vector in the opposite direction of $\mathbf{a}$.



   (d) Calculate $\mathbf{u}+\mathbf{v}-\frac{1}{3}\mathbf{a}$, then write it as a linear combination of the standard basis vector $\mathbf{i}$, $\mathbf{j}$.

$$\mathbf{u}+\mathbf{v}-\frac{1}{3}\mathbf{a} = \langle 2,3\rangle + \langle -1,3\rangle - \frac{1}{3}\langle -3,-3\rangle = \langle 2-1, 3+3\rangle - \langle -1,-1\rangle$$
$$= \langle 1,6\rangle + \langle 1,1\rangle = \langle 2,7\rangle$$

   As a linear combination of $\mathbf{i} = \langle 1, 0\rangle$, $\mathbf{j} = \langle 0, 1\rangle$,

$$\mathbf{u}+\mathbf{v}-\frac{1}{3}\mathbf{a} = \langle 2,7\rangle = 2\langle 1,0\rangle + 7\langle 0,1\rangle = 2\mathbf{i} + 7\mathbf{j}$$

   (e) Find constants α and β such that $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v}$.

   $\langle -3,-3\rangle = \alpha\langle 2,3\rangle + \beta\langle -1,3\rangle = \langle 2\alpha-\beta, 3\alpha+3\beta\rangle$ if and only if

$$\begin{cases} 2\alpha - \beta = -3 \\ 3\alpha + 3\beta = -3 \end{cases}$$

   Solving the equations, we have $\alpha = -\frac{4}{3}$, $\beta = \frac{1}{3}$, and

$$\mathbf{a} = -\frac{4}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$$

(f) Find the cosine of the angles between these vectors by using dot product.

Note that $\|\mathbf{u}\| = \sqrt{2^2+3^2} = \sqrt{13}$, $\|\mathbf{v}\| = \sqrt{10}$, $\|\mathbf{a}\| = 3\sqrt{2}$.

Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, ϕ be the angle between $\mathbf{u}$ and $\mathbf{a}$, and ψ be the angle between $\mathbf{v}$ and $\mathbf{a}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\,\mathbf{u}\,\|\|\,\mathbf{v}\,\|} = \frac{-2+9}{\sqrt{13}\sqrt{10}} = \frac{7}{\sqrt{130}}$$

The angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\frac{7}{\sqrt{130}} \approx 52.13°$.

Similarly,

$$\cos\phi = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\,\mathbf{u}\,\|\|\,\mathbf{a}\,\|} = \frac{-6-9}{\sqrt{13}\cdot 3\sqrt{2}} = -\frac{5}{\sqrt{26}}$$

$$\cos\psi = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\,\mathbf{v}\,\|\|\,\mathbf{a}\,\|} = \frac{3-9}{\sqrt{10}3\sqrt{2}} = -\frac{2}{\sqrt{20}}$$

The angle between $\mathbf{u}$ and $\mathbf{a}$ is $\phi = \cos^{-1}(-\frac{5}{\sqrt{26}}) \approx 168.69°$.

The angle between $\mathbf{v}$ and $\mathbf{a}$ is $\psi = \cos^{-1}(-\frac{2}{\sqrt{20}}) \approx 116.57°$.

(g) Find the projection of $\mathbf{u}$ on $\mathbf{a}$.

$$\text{Proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\,\mathbf{a}\,\|^2}\mathbf{a} = \frac{\langle 2,3\rangle\cdot\langle -3,-3\rangle}{18}\langle -3,-3\rangle = \frac{-15}{18}\langle -3,-3\rangle = \langle\frac{5}{2},\frac{5}{2}\rangle$$

Additional Work: *Draw the picture for this projection and see how the trigonometry is hidden in the dot product operation.*

2. Let $\mathbf{a} = \langle -2,1,-1\rangle$, $\mathbf{b} = \langle -1,0,2\rangle$, $\mathbf{c} = \langle -3,1,0\rangle$.

(a) Find a unit vector in the same direction as $\mathbf{b}$.

The unit vector required is:

$$\frac{\mathbf{b}}{\|\,\mathbf{b}\,\|} = \frac{1}{\sqrt{(-1)^2+0^2+2^2}}\langle -1,0,2\rangle = \frac{1}{\sqrt{5}}\langle -1,0,2\rangle = \langle -\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\rangle$$

(b) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\,\mathbf{a}\,\|\|\,\mathbf{b}\,\|} = \frac{2+0-2}{\sqrt{6}\sqrt{5}} = 0$$

Hence the angle between these two vectors is $\theta = \cos^{-1}0 = \frac{\pi}{2}$ rad $= 90°$; i.e., the two vectors are perpendicular (orthogonal) to each other.

(c) Find a vector perpendicular (orthogonal) to both $\mathbf{a}$ and $\mathbf{b}$.

The cross product of these two vectors is a vector orthogonal to both vectors:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 1 & -1\\ -1 & 0 & 2\end{vmatrix} = \begin{vmatrix}1 & -1\\ 0 & 2\end{vmatrix}\mathbf{i} - \begin{vmatrix}-2 & -1\\ -1 & 2\end{vmatrix}\mathbf{j} + \begin{vmatrix}-2 & 1\\ -1 & 0\end{vmatrix}\mathbf{k}\\ &= 2\mathbf{i}+5\mathbf{j}+\mathbf{k} = \langle 2,5,1\rangle\end{aligned}$$

Other vectors orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ are all scalar multiples of $\mathbf{a}\times\mathbf{b}$, including the zero vector $\mathbf{0}$.

(d) Find the projection of $\mathbf{c}$ on the direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

The direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ is given by the vector $\mathbf{a}\times\mathbf{b} = \langle 2,5,1\rangle$. Hence

$$\text{Proj}_{\mathbf{a}\times\mathbf{b}}\mathbf{c} = \frac{(-3,1,0)\cdot(2,5,1)}{\|(2,5,1)\|^2}(2,5,1) = \frac{-1}{30}(2,5,1) = (-\frac{1}{15},-\frac{1}{6},-\frac{1}{30})$$

(e) Find the area of the triangle whose vertices are given by these three vectors.

Area of the triangle is one-half of the area of the parallelogram generated by $\mathbf{c}-\mathbf{a}$, $\mathbf{b}-\mathbf{a}$, hence

$$\text{Area of the triangle} = \frac{1}{2}\|(\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})\|$$

Since

$$\begin{aligned}(\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) &= \langle -1,0,1\rangle \times \langle -2,1,-2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -2 & 1 & -2 \end{vmatrix} \\ &= -\mathbf{i} - 4\mathbf{i} - \mathbf{k} = \langle -1,-4,-1\rangle \end{aligned}$$

$$\text{Area of the triangle} = \frac{1}{2}\|\langle -1,-4,-1\rangle\| = \frac{1}{2}\sqrt{18} = \frac{3\sqrt{2}}{2}$$

(f) Find the volume of the parallelopiped generated by these three vectors.

The volume of the parallelopiped generated by these three vectors is:

$$\text{Volume} = |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = |(-3,1,0)\cdot(2,5,1)| = |-1| = 1$$

3. Given three points $P:(-2,1,3)$, $Q:(-1,0,2)$, $R:(-3,1,0)$ in space.

(a) Use cross product to find a vector perpendicular to the plane generated by $\langle -1,0,2\rangle$ and $\langle -3,1,0\rangle$ (i.e., the plane containing the two arrows).

$$\langle -1,0,2\rangle \times \langle -3,1,0\rangle = \langle -2,-6,-1\rangle$$

(b) Use suitable orthogonal projection to find the distance from the point P to the plane in (a).

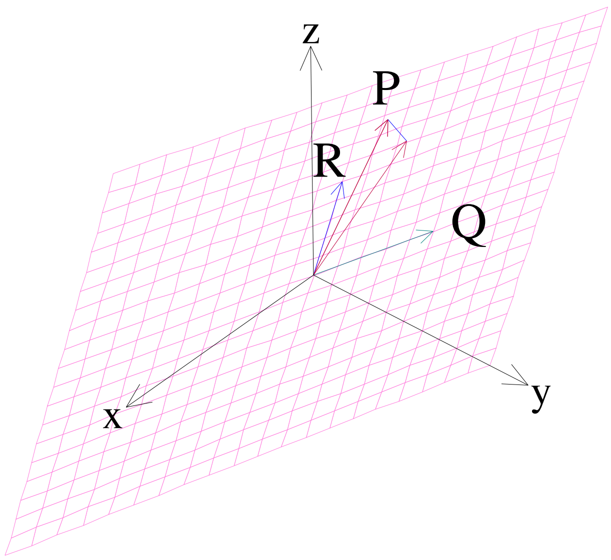
The distance is given by

$$|\text{Comp}_{\langle -2,-6,-1\rangle}\langle -2,1,3\rangle| = \frac{|\langle -2,1,3\rangle\cdot\langle -2,-6,-1\rangle|}{\|\langle -2,-6,-1\rangle\|} = \frac{5}{\sqrt{41}}$$

(c) Find the orthogonal projection of the vector $\langle -2,1,3\rangle$ on the plane in (a).

Just take

$$\begin{aligned}&\langle -2,1,3\rangle - \text{Proj}_{\langle -2,-6,-1\rangle}\langle -2,1,3\rangle \\ &= \langle -2,1,3\rangle - \frac{4-6-3}{41}\langle -2,-6,1\rangle = \langle \frac{-92}{41}, \frac{11}{41}, \frac{128}{41}\rangle\end{aligned}$$


z
P
R
Q
x
y