MATH2111 Tutorial 5

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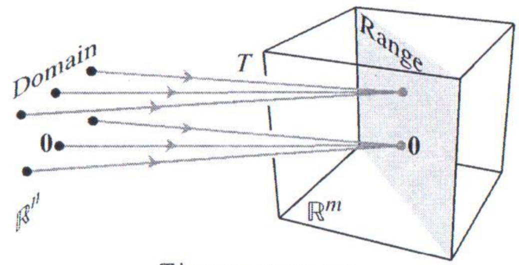
1 Linear Transformation

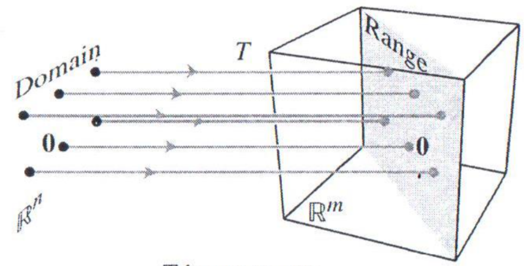
1. **Definition (Linear Transformation):**
   A transformation (or mapping) T is linear if:
   (a) $T(\boldsymbol{u}+\boldsymbol{v})=T(\boldsymbol{u})+T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T.
   (b) $T(c\boldsymbol{u})=cT(\boldsymbol{u})$ for all scalars c and all $\boldsymbol{u}$ in the domain of T.
2. **Theorem 4 (Properties of Linear Transformation):**
   If T is a linear transformation, then
   (a) $T(\mathbf{0})=\mathbf{0}$
   (b) $T(c\boldsymbol{u}+d\boldsymbol{v})=cT(\boldsymbol{u})+dT(\boldsymbol{v})$ for all vectors $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T and all scalars c, d
   (c) $T\left(c_1\boldsymbol{v}_1+\ldots+c_p\boldsymbol{v}_p\right)=c_1T\left(\boldsymbol{v}_1\right)+\ldots+c_pT\left(\boldsymbol{v}_p\right)$ for all vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ in the domain of T and all scalars $c_1, \ldots, c_p$.

2 The Matrix of a Linear Transformation

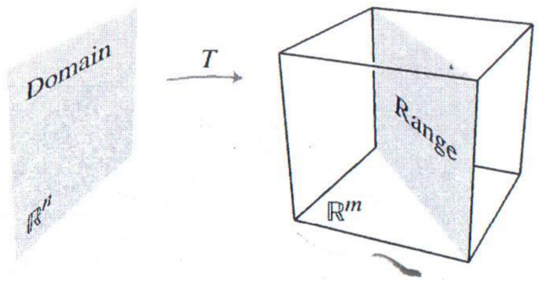
1. **Theorem:**
   Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that
   $$T(\boldsymbol{x})=A\boldsymbol{x} \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n$$
   A is the $m \times n$ matrix whose jth column is the vector $T\left(\boldsymbol{e}_j\right)$, where $\boldsymbol{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$:
   $$A=\left[T\left(\boldsymbol{e}_1\right) \cdots T\left(\boldsymbol{e}_n\right)\right]$$
   This matrix A is called the **standard matrix** for the linear transformation T.
2. **Definition (One-To-One):**
   A mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be one-to-one if each $\boldsymbol{b}$ in $\mathbb{R}^m$ is the image of at most one $\boldsymbol{x}$ in $\mathbb{R}^n$.



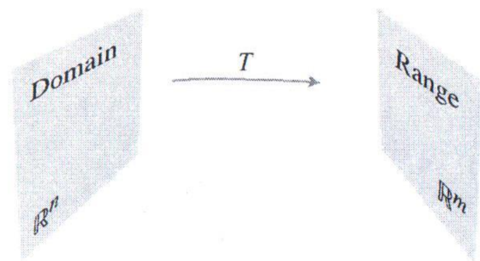
T is *not* one-to-one



T *is* one-to-one



T is *not* onto $\mathbb{R}^m$



T *is* onto $\mathbb{R}^m$

3. **Definition (Onto):**
   A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if each $\boldsymbol{b}$ in $\mathbb{R}^m$ is the image of at least one $\boldsymbol{x}$ in $\mathbb{R}^n$.

4. **Theorem:**
   Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\boldsymbol{x}) = \mathbf{0}$ has only the trivial solution.

5. **Theorem:**
   Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

   (a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

   (b) T is one-to-one if and only if the columns of A are linearly independent.

3 Matrix Operations

1. **Theorem (Property of Matrix):**
   Let A, B and C be matrices of the same size, and let r and s be scalars. Then

   (a) $A + B = B + A$

   (b) $(A + B) + C = A + (B + C)$

   (c) $A + 0 = A$

   (d) $r(A + B) = rA + rB$

   (e) $(r + s)A = rA + sA$

   (f) $r(sA) = (rs)A$

2. **Definition (Matrix Multiplication):**
If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix, then the product AB is the $m \times p$ matrix with entry

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

3. **Theorem (Properties of Matrix Multiplication):**
Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

   (a) $A(BC) = (AB)C$ (associative law of multiplication)

   (b) $A(B + C) = AB + AC$ (left distributive law)

   (c) $(B + C)A = BA + CA$ (right distributive law)

   (d) $r(AB) = (rA)B = A(rB)$ for any scalar r

   (e) $I_m A = A = AI_n$ (identity for matrix multiplication)

   WARNINGS:
   1. In general, $AB \neq BA$.
   2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.
   3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

4. **Definition (Powers of a Matrix):**
If A is an $n \times n$ matrix, k is a positive integer,

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}, \qquad A^0 = I_n$$

5. **Definition (Transpose of a Matrix):**
Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by $A^\top$, whose columns are formed from the corresponding rows of A.

6. **Theorem (Properties of Transpose of a Matrix):**
Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

   (a) $(A^\top)^\top = A$

   (b) $(A + B)^\top = A^\top + B^\top$

   (c) For any scalar r, $(rA)^\top = rA^\top$

   (d) $(AB)^\top = B^\top A^\top$

4 Exercises

1. Given transformation $T(x_1, x_2, x_3) = (x_2 + 1, x_3 + 1)$.
(1) What is $T(1, 2, 1)$?
(2) Is $T(\cdot)$ a linear transformation?

(1) Here $x_1 = 1$, $x_2 = 2$, $x_3 = 1$

$\therefore T(1,2,1) = (2+1, 1+1) = (3, 2)$

(2) No.
By def of linear transformation.

$$T(x_1+y_1, x_2+y_2, x_3+y_3) = (x_2+y_2+1, x_3+y_3+1)$$
$$\neq$$
$$T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = (x_2+1, x_3+1) + (y_2+1, y_3+1)$$
$$= (x_2+y_2+2, x_3+y_3+2)$$

$T(\cdot)$ is not a linear transformation.

Or you can show, for $c \neq 1 \in \mathbb{R}$,

$$T(cx_1, cx_2, cx_3) = (cx_2+1, cx_3+1) = cT(x_1, x_2, x_3) + (1-c)\cdot(1, 1)$$
$$\neq cT(x_1, x_2, x_3)$$

$\therefore T(\cdot)$ is not a linear transformation.

2. (1) Find the standard matrix of the following linear transformation

$$T(x_1, x_2, x_3, x_4) = (5x_1 - x_2, 5x_2 - x_3, 5x_3 - x_4, 5x_4 - x_1).$$

(2) Find the linear transformation of the following standard matrix

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

(1) $T\vec{x} = A\vec{x}$

$A = \left(T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3) \quad T(\vec{e}_4) \right)$

$T(\vec{e}_1) = (5, 0, 0, -1)$ $\qquad T(\vec{e}_2) = (-1, 5, 0, 0)$

$T(\vec{e}_3) = (0, -1, 5, 0)$ $\qquad T(\vec{e}_4) = (0, 0, -1, 5)$

$$\therefore \quad A = \begin{pmatrix} 5 & -1 & 0 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 5 & -1 \\ -1 & 0 & 0 & 5 \end{pmatrix}$$

(2) $T(\vec{e}_1) = (5, 1, 1)$, $\quad T(\vec{e}_2) = (1, 5, 1)$, $\quad T(\vec{e}_3) = (1, 1, 5)$

$T(x_1, x_2, x_3) = (5x_1 + x_2 + x_3,\ x_1 + 5x_2 + x_3,\ x_1 + x_2 + 5x_3)$

3. Given linear transformation $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)$, determine whether
(1) T is a one-to-one map,
(2) T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

(1) T is one to one.

Standard matrix of T is $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 - R_2 \to R_3]{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 + R_2 \to R_3]{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

columns of A are linearly independent.

Thus, by theorem, T is one to one.

(2) columns of A span $\mathbb{R}^3$,

thus by theorem, T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$

Rotation transformation

4. Suppose α is an angle. Given linear transformation $T(x_1, x_2) = (\cos\alpha \cdot x_1 + \sin\alpha \cdot x_2, -\sin\alpha \cdot x_1 + \cos\alpha \cdot x_2)$. Determine whether

(1) T is a one-to-one map,

(2) T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

(1) Standard atrix of T is $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

Case I: when $\cos\alpha = 0$, we have $|\sin\alpha| = 1$.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

T is a one-to-one map, and maps $\mathbb{R}^2$ onto $\mathbb{R}^2$

Case II: when $\sin\alpha = 0$, we have $|\cos\alpha| = 1$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

T is a one-to-one map, and maps $\mathbb{R}^2$ onto $\mathbb{R}^2$

Case III: when $\sin\alpha \neq 0$, $\cos\alpha \neq 0$

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \xrightarrow[R_2 + \frac{\sin\alpha}{\cos\alpha} R_1 \to R_2]{} \begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & \frac{1}{\cos\alpha} \end{pmatrix}$$

T is a one-to-one map, and maps $\mathbb{R}^2$ onto $\mathbb{R}^2$

In conclusion, T is a one-to-one map, and maps $\mathbb{R}^2$ onto $\mathbb{R}^2$

5. Given $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.
(1) Compute AB.
(2) Compute A^2, A^3.
(3) Compute $A^\top B$.

(1) $$AB = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0\times1+1\times4+0\times7 & 0\times2+1\times5+0\times8 & 0\times3+1\times6+0\times9 \\ 0\times1+0\times4+1\times7 & 0\times2+0\times5+1\times8 & 0\times3+0\times6+1\times9 \\ 1\times1+0\times4+0\times7 & 1\times2+0\times5+0\times8 & 1\times3+0\times6+0\times9 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix}$$

(2) $$A\cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 = A\cdot A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3) $$A^TB = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Remark: ① AB just switches order of rows of B.
② A: permutation matrices.
③ $A^3 = I_3$.