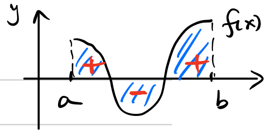
Review of definite integral :

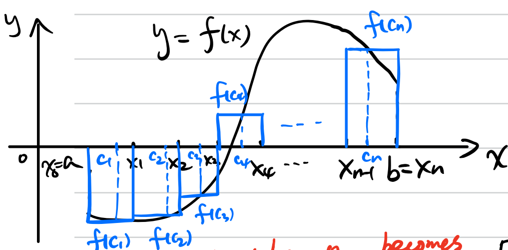
$\underbrace{\int_a^b f(x)\,dx}_{\text{a number independent with } x.}$ $\underset{\text{meaning}}{\overset{\text{geometric}}{=\!=}}$ sum of the signed areas between the graph of f and x-axis over $[a,b]$



$=$ (sum of areas above the x-axis) $-$ (sum of areas below the x-axis)

Notice : ① $\int_a^a f(x)\,dx = 0$ for all $f(x)$. ② $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds$.

$\int_a^b f(x)\,dx$ can be calculated by using the Riemann sum :

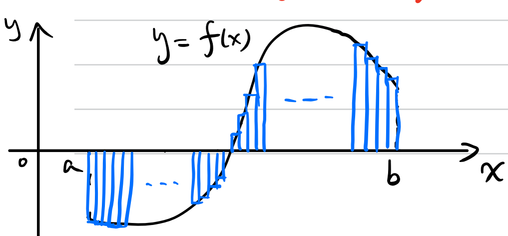


Step 1 Divide $[a,b]$ into n small subintervals evenly.
(width of each subinterval $= \frac{b-a}{n}$)

$x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + 2\cdot\frac{b-a}{n}$, $\cdots$, $x_n = a + n\cdot\frac{b-a}{n} = b$.

$\Downarrow$ when n becomes larger and larger

Step 2 Choose $c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2]$, $\cdots$, $c_n \in [x_{n-1}, x_n]$



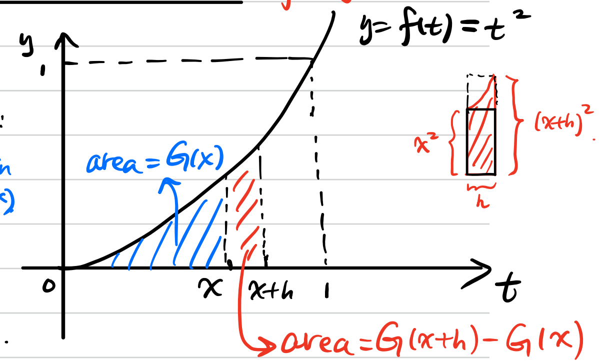
Riemann sum $\frac{b-a}{n} f(c_1) + \cdots + \frac{b-a}{n} f(c_n)$: an approximation of $\int_a^b f(x)\,dx$.

Step 3 $\lim\limits_{n\to+\infty} \frac{b-a}{n}\left[f(c_1) + f(c_2) + \cdots + f(c_n) \right] = \int_a^b f(x)\,dx$

Recall: $F(x)$ is an antiderivative of $f(x)$ if $\frac{d}{dt}F(t) = f(t)$.

Another way to calculate $\int_a^b f(t)\,dt$: use antiderivatives of $f(t)$.

Example: Find $\int_0^1 t^2\,dt$. $= \frac{1}{3}$



To calculate $\int_0^1 t^2\,dt$, we define a new function:

area function ← $\boxed{G(x) = \int_0^x t^2\,dt}$ for $x \in [0,1]$ → domain of $G(x)$

$\Rightarrow$ $G(0) = \int_0^0 t^2\,dt = 0$. $G(1) = \int_0^1 t^2\,dt$.

Therefore, if we find $G(x)$, then we find $\int_0^1 t^2\,dt$.

Trick to find $G(x)$: We first find $G'(x)$.

Recall the definition of derivative: $G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h \to 0} \frac{\text{area of}}{h}$

$h \cdot x^2 \le$ area of $\le h \cdot (x+h)^2$

$\Rightarrow$ $x^2 \le \frac{\text{area of}}{h} \le (x+h)^2$

$\Rightarrow$ $\lim_{h\to 0} x^2 \le \lim_{h \to 0} \frac{\text{area of}}{h} \le \lim_{h\to 0} (x+h)^2$

$\Rightarrow$ $x^2 \le G'(x) \le x^2$ $\Rightarrow$ $G'(x) = x^2$

$\Rightarrow$ $G(x)$ is an antiderivative of x^2.

Recall $\int x^2\,dx = \frac{1}{3}x^3 + C$.

$\Rightarrow$ $G(x) = \frac{1}{3}x^3 + C$ for some constant C.

Recall: $G(0) = 0$ $\xRightarrow{\text{solve for } C}$ $C = 0$.

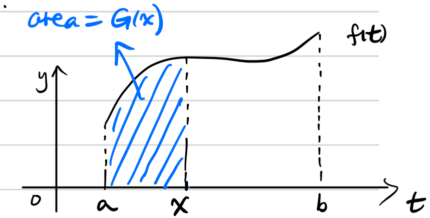
$G(x) = \frac{1}{3}x^3$.

$\Rightarrow$ $G(1) = \int_0^1 x^2\,dx = \frac{1}{3}$.

called "FTC" for short.

Fundamental Theorem of Calculus : use antiderivaties of $f(t)$ to calculate $\int_a^b f(t)dt$.

Suppose $f(t)$ is a continuous function on $[a, b]$.



We define the area function of $f(t)$:

$$\boxed{G(x) = \int_a^x f(t)\, dt} \quad \text{for } x \in [a, b].$$

→ domain of $G(x)$

(1). 1st version of FTC : we have $\boxed{G'(x) = f(x).}$

In other words : ① $G(x)$ is an antideriatie of $f(x)$

② $\int f(x)\, dx = G(x) + C.$

③ $G(x) = \int_a^x G'(t)\, dt.$

(2). 2nd version of FTC : If $F(x)$ is an antiderivatie of $f(x)$,

then $\boxed{\int_a^b f(t)\, dt = F(b) - F(a).}$

because $F(x) = G(x) + C$ for some constant C

$$\Rightarrow F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a)$$
$$= \int_a^b f(t)\, dt - \int_a^a f(t)\, dt = \int_a^b f(t)\, dt.$$

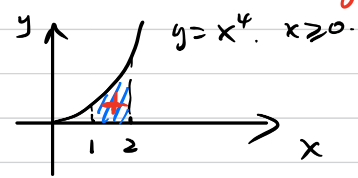
Summary: If we use Fundamental Theorem of Calculus (FTC) to calculate $\int_a^b f(x)\,dx$, then we take the following 2 steps:

Step 1: Find an antiderivative $F(x)$ of $f(x)$. (Find $\int f(x)\,dx$)

Step 2: Calculate $F(b) - F(a) \overset{FTC}{=} \int_a^b f(x)\,dx$

Problems of finding $\int_a^b f(x)\,dx \xrightarrow[FTC]{\text{becomes}}$ Problems of finding $\int f(x)\,dx$.

Example 1: Find $\int_1^2 x^4\,dx = \frac{31}{5}$.



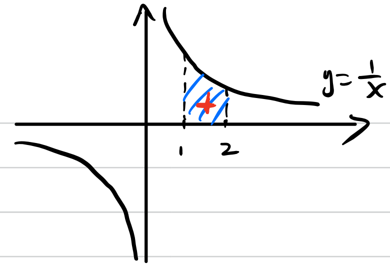
Step 1: Find $\int x^4\,dx$.

Recall: $\int x^p\,dx = \frac{1}{p+1} x^{p+1} + C$. for $p \neq -1$.

$$\Rightarrow \int x^4\,dx = \underbrace{\frac{1}{5} x^5}_{=F(x)} + C.$$

Step 2: Calculate $F(2) - F(1)$

$$\int_1^2 x^4\,dx \overset{FTC}{=} F(2) - F(1) = \left(\frac{1}{5}\cdot 2^5\right) - \left(\frac{1}{5}\cdot 1\right) = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}.$$

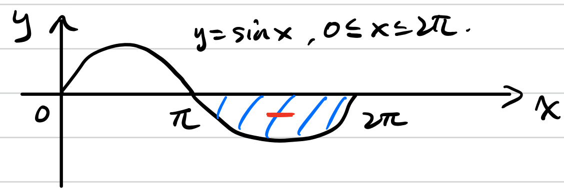
Example 2. Find $\int_1^2 \frac{1}{x}\, dx =$  $= \ln 2$

Step 1: Find $\int \frac{1}{x}\, dx$.

Recall: $\int \frac{1}{x}\, dx = \underbrace{\ln|x|}_{=F(x)} + C$.

Step 2: Calculate $F(2) - F(1)$.

$$\int_1^2 \frac{1}{x}\, dx \overset{FTC}{=} F(2) - F(1) = \ln 2 - \ln 1 = \ln 2.$$

Example 3. Find $\int_{\pi}^{2\pi} \sin x\, dx =$  $= -2$.

Step 1: Find $\int \sin x\, dx$.

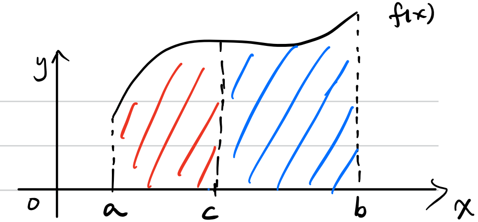
Recall: $\int \sin x\, dx = \underbrace{-\cos x}_{=F(x)} + C$

Recall: $\cos 2\pi = 1$.
$\cos \pi = -1$.

Step 2: Calculate $F(2\pi) - F(\pi)$.

$$\int_{\pi}^{2\pi} \sin x\, dx \overset{FTC}{=} F(2\pi) - F(\pi) = (-\cos 2\pi) - (-\cos\pi)$$
$$= (1) - (-(-1)) = 1 - 1 = -2$$

Properties of $\int_a^b f(x)\,dx$:



① $\int_a^a f(x)\,dx = 0$.

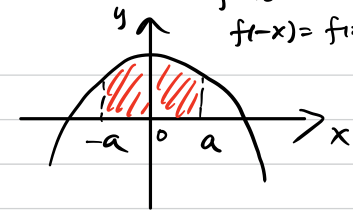
② If $a < c < b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

③. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Example: If $\int_1^2 f(x)\,dx = 3$, then $\int_2^1 f(x)\,dx = -3$.

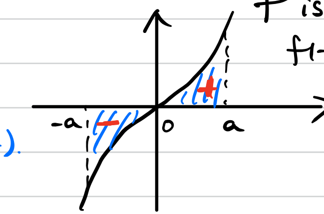
④ If $f(x)$ is an even function on $[-a,a]$,

then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



If $f(x)$ is an odd function on $[-a,a]$,

then $\int_{-a}^{a} f(x)\,dx = 0$. $\Rightarrow$ We don't need to find antiderivaties of $f(x)$.



Example: Find $\int_{-2}^{2} x^5\sqrt{4-x^2}\,dx = 0$

Let $f(x) = x^5\sqrt{4-x^2}$.

Notice : $f(-x) = (-x)^5\cdot\sqrt{4-(-x)^2} = -x^5\sqrt{4-x^2} = -f(x)$

$\Rightarrow$ $f(x)$ is odd on $[-2,2]$. $\Rightarrow$ $\int_{-2}^{2} f(x)\,dx = 0$.

When calculating $\int_{-a}^{a} f(x)\,dx$, we first check if $f(x)$ is an odd function on $[-a,a]$.

a constant

⑤ $$\int_a^b \underline{\underline{k}} \cdot f(x)\, dx = k \cdot \int_a^b f(x)\, dx.$$

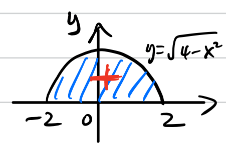
$$\int_a^b \left(f(x) + g(x) \right) dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Exaple : $$\int_{-2}^{2} (3x^5 - 2)\sqrt{4-x^2}\, dx = -4\pi.$$

Notice : $$\int_{-2}^{2} (3x^5-2)\sqrt{4-x^2}\, dx = 3\int_{-2}^{2} x^5\sqrt{4-x^2}\, dx - 2 \cdot \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

Recall : $\int_{-2}^{2} x^5 \sqrt{4-x^2}\, dx = 0$ because $f(x) = x^5\sqrt{4-x^2}$ is an odd function on $[-2, 2]$.

Recall : $\int_{-2}^{2} \sqrt{4-x^2}\, dx =$



$y = \sqrt{4-x^2}$: upper semi-circle

$= \frac{1}{2} \cdot (\pi \cdot 2^2) = 2\pi.$

if the area is easy to find, then we do not need to use Fundamental Theorem of Calculus.

$$\Rightarrow \int_{-2}^{2} (3x^5-2)\sqrt{4-x^2}\, dx = 3 \times 0 - 2 \times 2\pi = -4\pi.$$

When calculating $\int_{-a}^{a} f(x)\, dx$, we first check if a part of f(x) is an odd function on $[-a, a]$.

$\Rightarrow$ This will simplify the calculation of $\int_{-a}^{a} f(x)\, dx$. (Sometimes we don't need to find $\int f(x) dx$.)

More examples for the calculation of $\int_a^b f(x)dx$ by using Fundamental Theorem of Calculus (FTC)

Example 1. Find $\int_0^{\pi} (e^x - 3\sqrt{x} + \cos x)\, dx = e^{\pi} - 2\pi^{\frac{3}{2}} - 1$

Step 1. Find $\int (e^x - 3\sqrt{x} + \cos x)\, dx$.

Recall: $\int e^x dx = e^x + C$, $\int x^{\frac{1}{2}} dx = \frac{1}{1+\frac{1}{2}} x^{1+\frac{1}{2}} + C$, $\int \cos x\, dx = \sin x + C$

Notice: $\int (e^x - 3\sqrt{x} + \cos x)\, dx = \int e^x dx - 3\int \sqrt{x}\, dx + \int \cos x\, dx$

$$\Rightarrow \int (e^x - 3\sqrt{x} + \cos x)\, dx = e^x - 3\cdot\frac{1}{\frac{3}{2}} x^{\frac{3}{2}} + \sin x + C$$

$$= \underbrace{e^x - 2\cdot x^{\frac{3}{2}} + \sin x}_{=F(x)} + C$$

Step 2. Calculate $F(\pi) - F(0)$

$$\int_0^{\pi} (e^x - 3\sqrt{x} + \cos x)\, dx \overset{FTC}{=} F(\pi) - F(0)$$

$$= (e^{\pi} - 2\cdot\pi^{\frac{3}{2}} + \sin\pi) - (e^0 - 2\cdot 0 + \sin 0)$$

$$= e^{\pi} - 2\pi^{\frac{3}{2}} - 1.$$

Example 2: Find $\int_0^{\frac{\pi}{2}} 2x \cdot \cos(x^2)\, dx.$ $= \sin\left(\frac{\pi^2}{4}\right).$

Step 1: Find $\int 2x \cos(x^2)\, dx$

Notice: $\frac{d}{dx} \sin(x^2) \overset{\text{chain}}{\underset{\text{rule}}{=}} 2x \cdot \cos(x^2).$

$\Rightarrow \int 2x \cdot \cos(x^2)\, dx = \underbrace{\sin(x^2)}_{} + C.$ $= F(x)$

Step 2: Calculate $F(\frac{\pi}{2}) - F(0).$

$$\int_0^{\frac{\pi}{2}} 2x \cdot \cos(x^2)\, dx \overset{\text{FTC}}{=\!=} F(\tfrac{\pi}{2}) - F(0)$$

$$= \sin\left(\frac{\pi^2}{4}\right) - \sin 0$$

$$= \sin\left(\frac{\pi^2}{4}\right).$$