Application of Derivatives

4.1 Maximum and Minimum of Functions

Definition 4.1 *Let f be a function and c be a number. f has an absolute maximum (minimum) at c if*

$$f(x) \leq (\geq) f(c) \text{ for all } x.$$

Definition 4.2 *Let f be a function and c be a number. f has a relative maximum (minimum) at c if*

$$f(x) \leq (\geq) f(c) \text{ for all } x \text{ near to } c.$$

Remark 4.3 *Sometimes, absolute and relative maximum (minimum) of a function are called global and local maximum (minimum) respectively also.*

Definition 4.4 *Let f be a function and c be a number. c is a critical point of f if $f'(c) = 0$.*

Theorem 4.5 *Suppose that f is defined on (a, b) and $a < c < b$. If f has a relative maximum or minimum at c and f is differentiable at c, then c is a critical point of f.*

proof:
Suppose that f has a relative maximum at c. Then

$$f(x) \leq f(c) \text{ for all } x \text{ near to } c.$$

Then, if $h > 0$ is sufficiently small,

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

So that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

On the other hand, if $h < 0$ is sufficiently small,

$$\frac{f(c+h) - f(c)}{h} \geq 0.$$

So that

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

Now, $\lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists by hypothesis. We have

$$0 \geq \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0.$$

Therefore, $f'(c) = \lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = 0$.

Remark 4.6 *The converse of the previous theorem is false. That is, even if we know that $f'(a) = 0$, we may not conclude that f has a relative maximum or minimum at a. For instance, consider $f(x) = x^3$. 0 is a critical point of f, but 0 is neither a relative maximum nor minimum of f.*

Example 4.7 *Let $f(x) = x^4$. Find the possible relative maximum and minimum of f.*

solution:
$f'(x) = 4x^3$ so that $f'(x) = 0$ ONLY when $x = 0$. In other words, those nonzero numbers are not critical points of f, they are impossible to be either a relative maximum or minimum of f. Whether 0 is a relative maximum or minimum of f or neither remains to be seen.

4.2 Increasing and Decreasing Functions

Definition 4.8 *f is an increasing (decreasing) function if*

$$f(x) \geq (\leq) f(y) \text{ whenever } x > y$$

Remark 4.9 *Sometimes, a function is increasing in an interval but decreasing in another interval. For instance, the function cosine is decreasing on $[0, \pi]$ but it is increasing on $[\pi, 2\pi]$.*

Theorem 4.10 *If f is differentiable, then f is increasing (decreasing) on $[a, b]$ if and only if $f'(x) \geq (\leq) 0$ whenever $a < x < b$.*

proof ($\Rightarrow$):
Assume that f is increasing on $[a, b]$. Let $a < x < b$, and h is any small number so that $a < x + h < b$. Then, by considering the cases when h is positive or negative, we have

$$\frac{f(x+h)-f(x)}{h} \geq 0.$$

Thus, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} \geq 0$.
($\Leftarrow$):
The so-called "mean value theorem" is needed. See MATH1013 for detail.

Example 4.11 *Prove that $e^x \geq 1 + x$ for $x \geq 0$.*

solution:
Let $f(x) = e^x - x - 1$. Then, $f'(x) = e^x - 1 \geq 0$ for $x \geq 0$. Hence, f is increasing on $[0, x]$ for all $x > 0$. In particular, $f(x) \geq f(0) = 0$ for all $x \geq 0$. In other words, $e^x - x - 1 \geq 0$ for $x \geq 0$.

Example 4.12 *Prove that $e^x \geq 1 + x + x^2/2$ for $x \geq 0$.*

solution:
Let $f(x) = e^x - 1 - x - x^2/2$. Then, $f'(x) = e^x - 1 - x \geq 0$ for $x \geq 0$ by the previous example. Hence, f is increasing on $[0, x]$ for all $x > 0$. In particular, $f(x) \geq f(0) = 0$ for all $x \geq 0$. In other words, $e^x - 1 - x - x^2/2 \geq 0$ for $x \geq 0$.

Example 4.13 *Evaluate $\lim\limits_{x \to +\infty} \dfrac{x}{e^x}$ if it exists.*

solution:
By the previous example, $e^x > x^2/2$ for $x \geq 0$. Thus,

$$0 < \frac{x}{e^x} < \frac{2}{x} \quad \text{for } x \geq 0.$$

Note that $\lim\limits_{x \to +\infty} \dfrac{2}{x} = 0$. Hence $\lim_{x \to +\infty} \dfrac{x}{e^x} = 0$ exists and is 0 by sandwich.

Example 4.14 *Evaluate $\lim\limits_{x \to +\infty} \dfrac{x^2}{e^x}$ if it exists.*

solution:
Since

$$\frac{x^2}{e^x} = 4 \frac{x/2}{e^{x/2}} \frac{x/2}{e^{x/2}}$$

and $\lim\limits_{x \to +\infty} \dfrac{x/2}{e^{x/2}}$ exists and is 0 by the previous example, we see that $\lim\limits_{x \to +\infty} \dfrac{x^2}{e^x}$ exists and is 0.

Remark 4.15 *In general, for every positive integer n, we have $\lim\limits_{x \to +\infty} \dfrac{x^n}{e^x} = 0$.*

Remark 4.16 *We may generalize the previous remark further and see that for every polynomial P,*

$$\lim_{x \to +\infty} \frac{P(x)}{e^x} = 0.$$

Example 4.17 *Find all the relative maximum and minimum of the function $f(x) = 2x^3 + 3x^2 - 12x + 7$.*

solution:
Since

$$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1) \begin{cases} > 0 & x > 1 \\ = 0 & x = 1 \\ < 0 & -2 < x < 1 \\ = 0 & x = -2 \\ > 0 & x < -2 \end{cases}$$

we see that f is increasing on $[1, +\infty)$ and $(-\infty, -2]$; and decreasing on $[-2, 1]$. Thus,

$$\begin{array}{ll} f(x) \geq f(1) & \text{when } x \geq 1, \\ f(-2) \geq f(x) \geq f(1) & \text{when } -2 \leq x \leq 1, \\ f(-2) \geq f(x) & \text{when } x \leq -2, \end{array}$$

so that

$$\begin{array}{l} f(x) \geq f(1) \text{ when } x \geq -2, \text{ and} \\ f(x) \leq f(-2) \text{ when } x \leq 1. \end{array}$$

Hence f has a relative minimum at 1 and a relative maximum at -2.

Example 4.18 *Find all the relative maximum and minimum of the function* $f(x) = \dfrac{x}{x^2+1}$.

solution:
Since

$$f'(x) = \frac{(1-x)(1+x)}{(x^2+1)^2} \begin{cases} < 0 & x > 1 \\ = 0 & x = 1 \\ > 0 & -1 < x < 1 \\ = 0 & x = -1 \\ < 0 & x < -1 \end{cases}$$

f has a relative minimum at -1 and a relative maximum at 1. Moreover $\lim\limits_{x \to \pm\infty} f(x) = 0$. f has a horizontal asymptote $y = 0$.

Example 4.19 *Find all the relative maximum and minimum of the function* $f(x) = xe^x - e^x - x^2$.

solution:
Since

$$f'(x) = x(e^x - 2) \begin{cases} > 0 & x > \ln 2 \text{ or } x < 0 \\ = 0 & x = 0 \text{ or } \ln 2 \\ < 0 & 0 < x < \ln 2 \end{cases}$$

f has a relative minimum at $\ln 2$ and a relative maximum at 0.

Example 4.20 *Locate the absolute minimum of f if*

$$f(x) = \sec x + 2\csc x \quad \textit{for } 0 < x < \pi/2.$$

solution:

$$f'(x) = \frac{\sin^3 x - 2\cos^3 x}{\sin^2 x \cos^2 x} \begin{cases} > 0 & \tan^{-1}\sqrt[3]{2} < x < \pi/2 \\ = 0 & x = \tan^{-1}\sqrt[3]{2} \\ < 0 & 0 < x < \tan^{-1}\sqrt[3]{2}. \end{cases}$$

Now f is increasing on $[\tan^{-1}\sqrt[3]{2}, \pi/2)$, we see that $f(x) \geq f(\tan^{-1}\sqrt[3]{2})$ for $\tan^{-1}\sqrt[3]{2} \leq x < \pi/2$. Moreover f is decreasing on $(0, \tan^{-1}\sqrt[3]{2}]$, we see that $f(x) \geq f(\tan^{-1}\sqrt[3]{2})$ for $0 < x \leq \tan^{-1}\sqrt[3]{2}$. Hence,

$$f(x) \geq f(\tan^{-1}\sqrt[3]{2}) \quad \text{for } 0 < x < \pi/2.$$

In other words, f attains absolute minimum at $\tan^{-1}\sqrt[3]{2}$.

Example 4.21 *Locate all relative maximums and minimums of f if*

$$f'(x) = x(x-1)(x-2)(x-3)(x-4)(x-5) \quad \textit{for all } x.$$

solution:
Since

$$f'(x) \begin{cases} > 0 & x > 5 \text{ or } 3 < x < 4 \text{ or } 1 < x < 2 \text{ or } x < 0, \\ = 0 & x = 0 \text{ or } 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5, \\ < 0 & 4 < x < 5 \text{ or } 2 < x < 3 \text{ or } 0 < x < 1, \end{cases}$$

f has relative maximums at 0, 2, 4 and relative minimums at 1, 3, 5.

Example 4.22 *Locate all relative maximums and minimums of f if*

$$f'(x) = (x^2+1)(x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5 \quad \textit{for all } x.$$

solution:
Since

$$f'(x) \begin{cases} > 0 & x > 5 \text{ or } 2 < x < 3 \text{ or } 1 < x < 2, \\ = 0 & x = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5, \\ < 0 & 4 < x < 5 \text{ or } 3 < x < 4 \text{ or } x < 1, \end{cases}$$

f has relative maximums at 3 and relative minimums at 1, 5. Note that f has neither a relative maximum nor minimum at the critical points 2, 4.

Remark 4.23 *We say that a function f has a saddle point at a if f has a critical point at a but neither a relative maximum nor minimum at a. In the last example, the function f had saddle points at 2, 4.*

4.3 Optimization

Example 4.24 *Find the biggest possible area of a rectangle whose perimeter is L.*

solution:
Define

$$f(x) = x(\frac{L}{2} - x) \text{ for } 0 \le x \le \frac{L}{2}.$$

The issue is to find the absolute maximum of f. Now,

$$f'(x) = -2x + \frac{L}{2} \begin{cases} < 0 & \frac{L}{2} > x > \frac{L}{4} \\ = 0 & x = \frac{L}{4} \\ > 0 & \frac{L}{4} > x > 0 \end{cases}$$

Thus, f is increasing on $[0, L/4]$. So,

$$f(x) \le f(L/4) \quad \text{for } 0 \le x \le L/4.$$

f is decreasing on $[L/4, L/2]$. So,

$$f(x) \le f(L/4) \quad \text{for } L/4 \le x \le L/2.$$

Therefore $f(x) \le f(L/4)$ for $0 \le x \le L/2$.

Now let the length of one edge of this rectangle be x. Then, the length of the adjacent edge is $\frac{L}{2} - x$. The area of this rectangle is

$$\begin{aligned} & x(\tfrac{L}{2} - x) \\ & = f(x) \\ & \le f(L/4) \text{ as } 0 < x < L/2 \\ & = \text{area of the rectangle with dimension } \tfrac{L}{4} \times \tfrac{L}{4}. \end{aligned}$$

Thus the maximum possible area of such a rectangle is $f(L/4) = L^2/16$.

Example 4.25 *Let P be the parabola defined by $y = x^2$. Find the point in P which is closest to* $(3,0)$.

solution:
Define

$$f(x) = (x-3)^2 + x^4 \quad \text{for all } x.$$

Then

$$f'(x) = 4x^3 + 2x - 6 = 4(x-1)[(x+\frac{1}{2})^2 + 5/4] \begin{cases} > 0 & \text{if } x > 1, \\ = 0 & \text{if } x = 1, \\ < 0 & \text{if } x < 1. \end{cases}$$

Hence f is increasing on $[1, +\infty)$, so that $f(x) \ge f(1)$ for $x \ge 1$. On the other hand, f is decreasing on $(-\infty, 1]$, so that $f(x) \ge f(1)$ for $x \le 1$. Therefore $f(x) \ge f(1)$ for all x.

Now, if (x, x^2) is a point in P, the square of the distance from $(3,0)$ to (x, x^2) is

$$\begin{aligned} & (x-3)^2 + x^4 \\ & = f(x) \\ & \ge f(1) \\ & = \text{the square of the distance from } (3,0) \text{ to } (1,1). \end{aligned}$$

Consequently, the point $(1,1)$ is the point in P which is closest to $(3,0)$.

Example 4.26 *A football goal of width W meters is put on the byline in the usual way. A striker is running in a direction perpendicular to the byline so that the perpendicular distance from the near post to the line he is running is D. Where should he shoot in order that the shooting angle is maximized?*

solution:
For any $k > 0$, we define the function

$$f(x) = \frac{x}{k^2 + x^2} \text{ for } x >\geq 0.$$

Then,

$$f'(x) = \frac{k^2 - x^2}{(k^2 + x^2)^2} \begin{cases} > 0 & 0 < x < k \\ = 0 & x = k \\ < 0 & x > k \end{cases}$$

Hence, f is increasing on $[0, k]$. $f(x) \leq f(k)$ for $0 \leq x \leq k$. f is decreasing on $[k, +\infty)$. $f(x) \leq f(k)$ for $x \geq k$. Consequently, f has an absolute maximum at k.

Now we let the striker be x meters away from byline. The shooting angle at that moment is

$$\tan^{-1} \frac{x}{D} - \tan^{-1} \frac{x}{W + D}.$$

In other words, tangent of the shooting angle at that moment is, by the substraction formula for tan,

$$\frac{\frac{x}{D} - \frac{x}{W+D}}{1 + \frac{x}{D}\frac{x}{W+D}}$$
$$= \frac{xW}{D(W + D) + x^2}$$
$$= Wf(x) \text{ where } k^2 = D(W + D)$$
$$\leq Wf(k)$$
$$= \text{tangent of the shooting angle when the striker is } k = \sqrt{D(W + D)} \text{ meters away from the byline.}$$

So the striker should shoot when he is $\sqrt{D(W + D)}$ meters away from the byline.

Example 4.27 *Design a soft drink can having a volume V and its surface area is minimized.*

solution:
Define

$$f(x) = 2\pi x^2 + 2V/x \text{ for } x \geq 0.$$

Then,

$$f'(x) = 2\frac{2\pi x^3 - V}{x^2} \begin{cases} > 0 & x > \sqrt[3]{V/2\pi} \\ = 0 & x = \sqrt[3]{V/2\pi} \\ < 0 & 0 < x < \sqrt[3]{V/2\pi} \end{cases}$$

Thus, f has an absolute minimum at $\sqrt[3]{V/2\pi}$.

Let the soft drink be cylindrical with radius r^2. Then, $\pi r^2 h = V$ so that $h = \frac{V}{\pi r^2}$. Now, the surface are of the can is

$$\begin{aligned}&2\pi r^2 + 2\pi r h\\ &= 2\pi r^2 + 2V/r\\ &= f(r)\\ &\geq f(\sqrt[3]{V/2\pi})\\ &= \text{surface area of the can when the radius of its base is } \sqrt[3]{V/2\pi}.\end{aligned}$$

That is, we make a can with radius $\sqrt[3]{V/2\pi}$, height $\frac{V}{\pi \sqrt[3]{V/2\pi}^2} = \sqrt[3]{4V/\pi}$. Then its volume is V and its surface area is the smallest possible.

4.4 L'Hôpital Rule

Theorem 4.28 (L'Hôpital Rule) *Let f, g be functions differentiable at c and $f(c) = g(c) = 0$. If the limit*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists or is either $+\infty$ or $-\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

sloppy proof:
For every $h \neq 0$,

$$\frac{f(c+h)}{g(c+h)} = \frac{\frac{f(c+h)-f(c)}{h}}{\frac{g(c+h)-g(c)}{h}}$$

By hypothesis, the limit of RHS exists and is $\dfrac{f'(c)}{g'(c)}$ as $h \to 0$. We see that the limit of LHS exists also and

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Remark 4.29 *L'Hôpital rule is still applicable for limits at infinity, as well as the case when both the numerator and denominator tend to $\pm\infty$.*

Example 4.30 *Evaluate*

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x}.$$

if it exists.

solution:
Let $f(x) = e^x - e^{-x}$ and $g(x) = x$. Then, $f(0) = g(0) = 0$. We may apply

L'Hôpital rule in this problem. Now, we have the derivatives $f'(x) = e^x + e^{-x}$ and $g'(x) = 1$.

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} e^x + e^{-x}$$

exists and is 2. Therefore,

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{x} = 2.$$

Example 4.31 *Evaluate*

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

if it exists.

solution:
Let $f(x) = \sin x - x$ and $g(x) = x^3$. Since $f(0) = g(0) = 0$, we may apply L'Hôpital rule to evaluate $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}$. Now,

$$f'(x) = \cos x - 1, \quad g'(x) = 3x^2$$

so that $f'(0) = g'(0) = 0$. To evaluate $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$, we apply L'Hôpital rule again. Note that

$$f''(x) = -\sin x, \quad g''(x) = 6x$$

so that

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}.$$

We conclude that

$$-\frac{1}{6} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f(x)}{g(x)}.$$

Example 4.32 *Evaluate*

$$\lim_{x\to 0} \frac{(e^x - 1)\sqrt{\cos x}}{(1 + x^2)\sin x}$$

if it exists.

solution:
First of all, we evaluate

$$\lim_{x\to 0} \frac{e^x - 1}{x}.$$

Both $e^x - 1$ and x evaluate at 0 to be zero. We apply L'Hôpital rule to evaluate this limit. The derivatives of $e^x - 1$ and x are e^x and 1 respectively and $\lim\limits_{x\to 0} e^x = 1$. Thus,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Its obvious that

$$\lim_{x\to 0}\frac{x}{\sin x}=1 \text{ and } \lim_{x\to 0}\frac{\sqrt{\cos x}}{1+x^2}=1,$$

Consequently,

$$\begin{aligned}&\lim_{x\to 0}\frac{(e^x-1)\sqrt{\cos x}}{(1+x^2)\sin x}\\ &=\lim_{x\to 0}\frac{(e^x-1)}{x}\frac{x}{\sin x}\frac{\sqrt{\cos x}}{(1+x^2)}\end{aligned}$$

exists and is 1.

Remark 4.33 *L'Hôpital rule is not applicable to the limit* $\lim_{x\to a}\frac{f(x)}{g(x)}$ *when* $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *does not exist. For instance if* $f(x)=x+x^2\sin\frac{1}{x}$ *and* $g(x)=x$ *for* $x\neq 0$. *Then,* $\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}1+2x\sin\frac{1}{x}-\cos\frac{1}{x}$ *fails to exist. However,* $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}1+x\sin\frac{1}{x}$ *exists and is* 1 *(why?).*

Example 4.34 *Evaluate*

$$\lim_{x\to 0^+}(1+x)^{1/x}.$$

if it exists.

solution:
First of all we study $\lim_{x\to 0^+}\frac{\ln(1+x)}{x}$. Now, both $\ln(1+x)$ and x evaluate at 0 to be zero, and since the derivative of $\ln(1+x)$ is $\frac{1}{1+x}$, the derivative of x is 1 and

$$\lim_{x\to 0^+}\frac{1}{1+x}=1.$$

By L'Hôpital rule,

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x}=1.$$

Therefore, by the fact that the exponential function is continuous,

$$\lim_{x\to 0^+}e^{(\ln(1+x)/x}=\lim_{x\to 0^+}(1+x)^{1/x}$$

exists and is $e^1=e$.

Remark 4.35 *If \$1 is put into an account with interest rate* r *per year. The previous example said that the sum would be* $\$e^{rT}$ *after* T *years if interest is paid continuously.*

Example 4.36 *Evaluate*

$$\lim_{x\to+\infty}\sqrt[x]{x}$$

if it exists.

solution:
First of all we evaluate $\lim_{x \to +\infty} \frac{\ln x}{x}$. Now, both $\ln x$ and x tend to $+\infty$ as x tends to $+\infty$, and $\lim_{x \to +\infty} \frac{\frac{1}{x}}{1}$ exists and is 0. Thus we apply L'Hôpital rule to see that

$$\lim_{x \to +\infty} \frac{\ln x}{x} \text{ exists and is } 0.$$

By the continuity of the exponential function,

$$\lim_{x \to +\infty} e^{\ln x / x} \text{ exists and is } 1.$$

Consequently $\lim_{x \to +\infty} x^{1/x}$ exists and is 1.