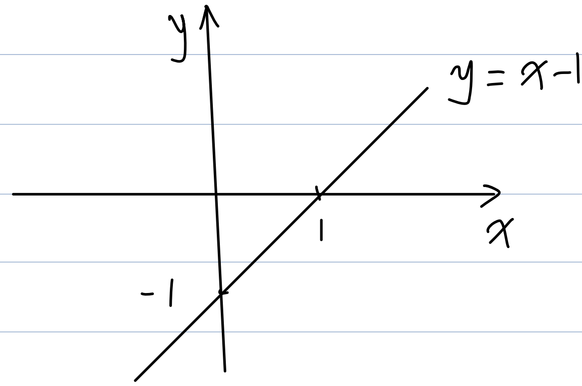
# §1.1 Systems of Linear Equations

A linear equation: $x - y = 1$ (1)

Solution of (1)




In general, a linear equation:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n, b$ are constants

$x_1, x_2, \cdots x_n$ are variables

$a_1, a_2, \cdots, a_n$ —— coefficients

Example: 1) $x - y + z = 0$ linear

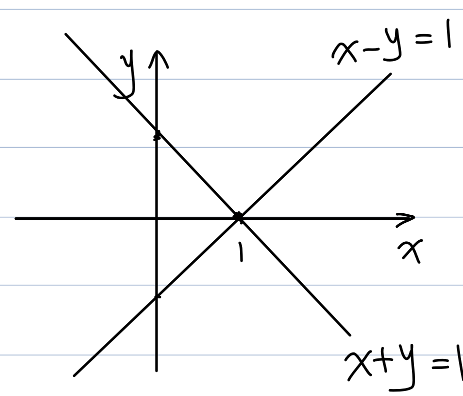
2) $4x_1 - 2x_2 + 3x_3 = 1$ linear

3) $x_1 - x_2 + x_3 - x_4 = 2$ linear

4) $x^2 + y^2 = 1$ non-linear

* A system of linear equations

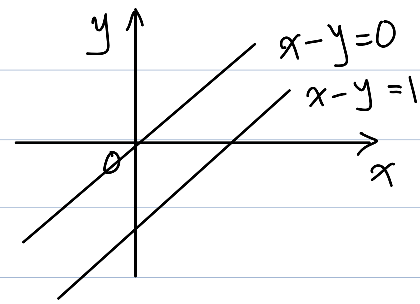
Example: $\begin{cases} x - y = 1 \\ x + y = 1 \end{cases}$ (2)




Solution of the system

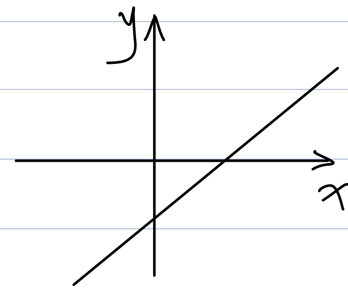
(1, 0)

$$\begin{cases} x-y=1 \\ x-y=0 \end{cases} \qquad (3)$$




No solution

$$\begin{cases} x-y=1 \\ 2x-2y=2 \end{cases} \qquad (4)$$




infinitly many solutions

Remark: A system of linear equations has

1) no solution, or —— inconsistent
2) exactly one solution, or } consistent
3) infinitely many solutions

Def: If there is no solution exist, we say the linear system is inconsistent, otherwise, the linear system is consistent.

We need to find a systematic way to solve the system.

- Matrix notation: (This notation will simplify the method)

1) $\begin{cases} x-y=1 \\ x+y=1 \end{cases}$ $\qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ : coefficient matrix

$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ : augmented matrix

2) $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ 5x_1 \quad\quad - 5x_3 = 10 \end{cases}$ $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix}$ coefficient matrix

$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$ augmented matrix

Def: Size of a matrix: $m$ rows $n$ columns

called $m \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Example: Solving the system:

$\begin{cases} x - y = 1 & ① \\ x + y = 1 & ② \end{cases}$ $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$② - ①$ $\begin{cases} x - y = 1 & ①' \\ 2y = 0 & ②' \end{cases}$ $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$

$\frac{1}{2}②'$: $\begin{cases} x - y = 1 & ①'' \\ y = 0 & ②'' \end{cases}$ $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$①'' + ②''$: $\begin{cases} x = 1 \\ y = 0 \end{cases}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

We get the solution $x = 1$, $y = 0$

* Elementary row operations:

1) Interchange  Interchange the i-th row and the j-th row

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots & \vdots & \cdot & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

2) Scaling  Multiply all entries in a row by a nonzero constant.

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{in} \\ \vdots & \vdots & \vdots \end{pmatrix} \rightarrow \begin{pmatrix} \vdots & \vdots & \vdots \\ ca_{i1} & ca_{i2} & ca_{in} \\ \vdots & \vdots & \vdots \end{pmatrix}$$

3) Replacement  Replace one row by the sum of itself and the multiple of another row

$$\begin{pmatrix} \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots & \vdots & & \vdots \\ a_{i1}+ca_{j1} & a_{i2}+ca_{j2} & \cdots & a_{i}+ca_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

Def: Two matrices are row equivalent if there is a sequence of elementary row operations that transform one matrix into the other

**Remark:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example: Solve the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & ① \\ 2x_2 - 8x_3 = 8 & ② \\ 5x_1 - 5x_3 = 10 & ③ \end{cases} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$$

$$\xrightarrow{③ - 5\times①} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 10x_2 - 10x_3 = 10 \end{cases} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{pmatrix}$$

$$\xrightarrow{\frac{②}{2} \ \& \ \frac{③}{10}} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_2 - x_3 = 1 \end{cases} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{③ - ②} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ 3x_3 = -3 \end{cases} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow{\frac{③}{3}} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{cases} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[②+4③]{①-③} \begin{cases} x_1 - 2x_2 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases} \qquad \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{①+2②} \begin{cases} x_1 \qquad\qquad = 1 \\ \qquad x_2 \qquad = 0 \\ \qquad\qquad x_3 = -1 \end{cases} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Question: 1) Is the system consistent (existence)

$\Leftrightarrow$ Does at least one solution exist?

2) If a solution exists, is it the only one? (Uniqueness)

Example:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \qquad 2x_2 - 8x_3 = 8 \\ 5x_1 \qquad\quad - 5x_3 = 10 \end{cases}$$

has a unique solution

Example:

$$\begin{cases} \qquad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{cases}$$

$$\begin{pmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{pmatrix} \xrightarrow[①\ \&\ ②]{\text{Interchange}} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{pmatrix}$$

$$\xrightarrow{③-①\times 2} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{pmatrix} \xrightarrow{③+2②} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ \qquad x_2 - 4x_3 = 8 \\ \qquad\qquad 0 = 15 \end{cases}$$

no solution.

/eʃəlaːn/

## §1.2 Row Reduction and Echelon forms

(step like)

**Def:** A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1) All nonzero rows are above any rows of all zeros
2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3) All entries in a column below a leading entry are zeros

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

4) The leading entry in each nonzero row is 1.
5) Each leading 1 is the only nonzero entry in its column.

**Def:** a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Example:

$$\begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

echelon form $\qquad$ reduced echelon form

$$\begin{pmatrix} 3 & -1 & 0 & 8 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form　　　　reduced echelon form