Review: Applications of $f'(x)$.

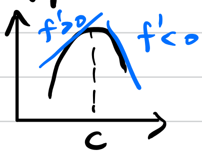
1. Use $f'(x)$ to find intervals where f is increasing and decreasing.

① $f'(x) > 0$ on (a,b) $\Rightarrow$ f is increasing on (a,b).

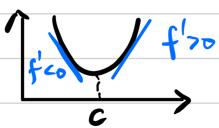
② $f'(x) < 0$ on (a,b) $\Rightarrow$ f is decreasing on (a,b).

2. Use $f'(x)$ to find local maximum and local minimum.

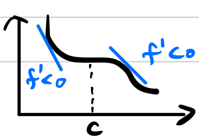
Suppose that c is a critical number. ($f'(c)=0$ or $f'(c)$ does not exist).

① 

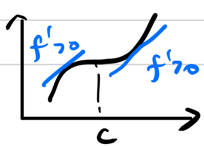
$$\left.\begin{array}{l} f'(x)>0 \text{ when } x<c \\ f'(x)<0 \text{ when } x>c \end{array}\right\} \Rightarrow f(c) \text{ is a local maximum.}$$

② 

$$\left.\begin{array}{l} f(x)<0 \text{ when } x<c \\ f(x)>0 \text{ when } x>c \end{array}\right\} \Rightarrow f(c) \text{ is a local minimum.}$$

③ 

or



$f'(x)$ does not change sign $\Rightarrow$ $f(c)$ is not a local maximum or a local minimum.

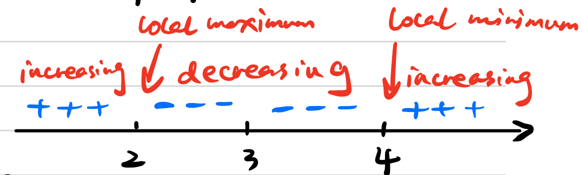
Example: Suppose that the derivative of f is $f'(x) = (x^2+1)(x-2)(x-3)^2(x-4)^3$.

Find all local maximum and local minimum of f.

$f'(x)$ has 3 roots: 2, 3, 4.

$\Rightarrow$ critical numbers: 2, 3, 4.

sign of $f'(x)$:



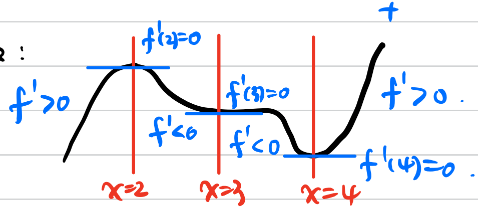
1) when $x > 4$, $f'(x) = \underbrace{(x^2+1)}_{+}\underbrace{(x-2)}_{+}\underbrace{(x-3)^2}_{+}\underbrace{(x-4)^3}_{+} > 0$

2) when $3 < x < 4$, $f'(x) = \underbrace{(x^2+1)}_{+}\underbrace{(x-2)}_{+}\underbrace{(x-3)^2}_{+}\underbrace{(x-4)^3}_{-} < 0$

3). when $2 < x < 3$. $f'(x) = \underbrace{(x^2+1)}_{+}\underbrace{(x-2)}_{+}\underbrace{(x-3)^2}_{+}\underbrace{(x-4)^3}_{-} < 0$

4) when $x < 2$, $f'(x) = \underbrace{(x^2+1)}_{+}\underbrace{(x-2)}_{-}\underbrace{(x-3)^2}_{+}\underbrace{(x-4)^3}_{-} > 0$

f looks like:



local maximum: $f(2)$

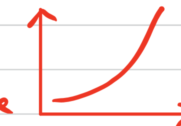
local minimum: $f(4)$.

second derivative of f.

1. How $\boxed{f''(x)}$ affects the shape of the graph of $y=f(x)$.

(1). Suppose that f is increasing on (a,b).

If $f''(x)>0$ on (a,b), then $f'(x)$ is increasing on (a,b) and f looks like

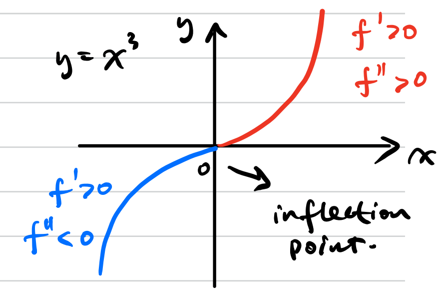


If $f''(x)<0$ on (a,b), then $f'(x)$ is decreasing on (a,b) and f looks like

Example: $f(x)=x^3$.

$f'(x)=3x^2\geq 0 \Rightarrow f$ is increasing.

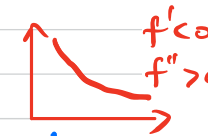
$f''(x)=(3x^2)'=6x \Rightarrow \begin{cases} f''<0 & \text{if } x<0 \\ f''>0 & \text{if } x>0. \end{cases}$



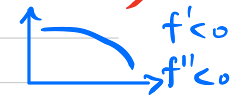
Definition: A point P on the curve of $y=f(x)$ is called an inflection point if f is continuous at P and $f''(x)$ changes sign at P.

(2) Suppose that f is decreasing on (a,b).

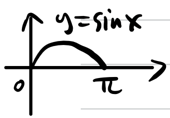
If $f''(x)>0$ on (a,b), then $f'(x)$ is increasing on (a,b) and f looks like



If $f''(x)<0$ on (a,b), then $f'(x)$ is decreasing on (a,b) and f looks like

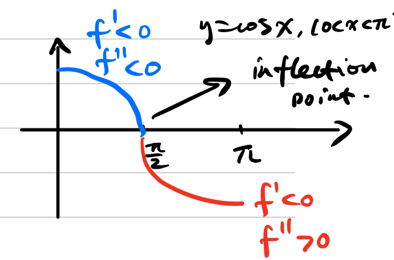


Example: $f(x)=\cos x$. $(0<x<\pi)$.



$f'(x)=-\sin x<0$ for $x\in(0,\pi)$. $\Rightarrow$ f is decreasing on $(0,\pi)$.

$$f''(x)=-\cos x\begin{cases}<0 & \text{for } x\in(0,\frac{\pi}{2}).\\ >0 & \text{for } x\in(\frac{\pi}{2},\pi).\end{cases}$$

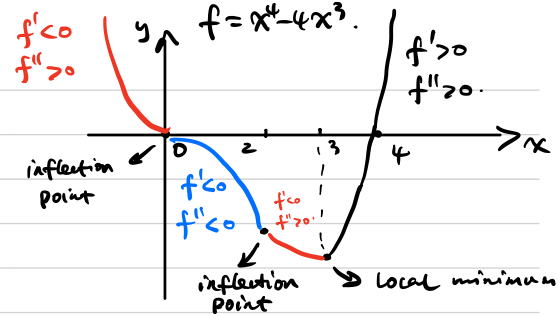


$(\frac{\pi}{2},0)$ is an inflection point. $f'(\frac{\pi}{2})=-\sin\frac{\pi}{2}=-1$

Notice: In general, $\underbrace{\text{an inflection point}}_{\Downarrow \text{ where } f'' \text{ changes sign}}$ is not a $\underbrace{\text{critical point}}_{\Downarrow \text{ where } f'=0 \text{ or } f' \text{ does not exist}}$.

2. How to sketch the graph of $y=f(x)$.

Example 1: $f(x)=x^4-4\cdot x^3$.



Step 1: Find the domain : $(-\infty,+\infty)$.

Step 2: Find vertical and horizontal asymptotes.

① Does f tend to ∞ at some point? No. $\Rightarrow$ f does not have vertical asymptote.

② $\lim\limits_{x\to+\infty} f(x)=?$ $\lim\limits_{x\to-\infty} f(x)=?$ $f(x)=x^4\left(1-\frac{3}{x}\right)$ $\Rightarrow$ As $x\to\pm\infty$, $1-\frac{3}{x}\to 1$ and $x^4\to+\infty$, so $f(x)\to+\infty$.

$\Rightarrow$ f does not have horizontal asymptote.

Step 3: Find x-intercepts and y-intercept : $(0,0)$ and $(4,0)$.

Step 4: Calculate f' and f''. $f'=4\cdot x^3-12\cdot x^2=4\cdot x^2(x-3)$. $f''(x)=12x^2-24x=12x(x-2)$

① Find all critical numbers $x=0,3$.

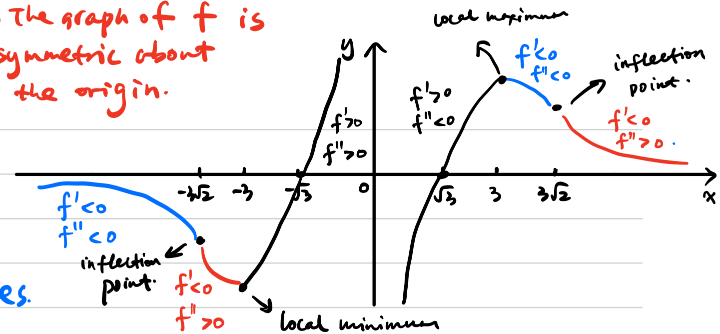
② Find intervals where f is increasing and decreasing. sign of $f'(x)$: $- -$ (0) $- -$ (3) $+++$

decreasing when $x<3$ local minimum.   increasing when $x>3$

③ Find inflection points. sign of $f''(x)$: $++$ (0) $--$ (2) $++$ $(0,f(0))$ and $(2,f(2))$.

f is odd: $f(-x) = -f(x)$. $\Rightarrow$ The graph of f is symmetric about the origin.

Example 2. Sketch the graph of $f(x) = \dfrac{x^2-3}{x^3}$.



Step 1: Find the domain: $\{x \mid x \neq 0\}$.

Step 2: Find vertical and horizontal asymptotes.

① As $x \to 0^-$, $x^2-3 \to -3$ and $x^3 \to 0^-$, so $f(x) \to +\infty$.
As $x \to 0^+$, $x^2-3 \to -3$ and $x^3 \to 0^+$, so $f(x) \to -\infty$.
$\Rightarrow$ vertical asymptote: $x=0$ (y-axis).

② $f(x) = \dfrac{1}{x} - \dfrac{3}{x^3} \Rightarrow \lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{1}{x} - \lim\limits_{x\to+\infty} \dfrac{3}{x^3} = 0-0=0$, $\lim\limits_{x\to-\infty} f(x) = 0 \Rightarrow$ horizontal asymptote: $y=0$ (x-axis).

Step 3. Find x-intercepts and y-intercept: $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$

Step 4. Calculate f' and f''.

$$f' = (x^{-1})' - 3\cdot(x^{-3})' = -x^{-2} + 9\cdot x^{-4} = \frac{-1}{x^4}\cdot(x^2-9).$$
$$f'' = (-x^{-2})' + 9\cdot(x^{-4})' = 2\cdot x^{-3} - 36\cdot x^{-5} = \frac{2}{x^5}(x^2-18).$$

① **Find all critical numbers.** $x=3$ and -3.

② **Find intervals where f is increasing and decreasing.** sign of f':

	-3		0		3	
$-\,-$		$+\,+$		$+\,+$		$-\,-$
decreasing	local minimum	increasing			local maximum	decreasing.

③ **Find inflection points.** sign of f'':

	$-3\sqrt{2}$		0		$3\sqrt{2}$	
$-\,-$		$+\,+$		$-\,-$		$+\,+$

$(3\sqrt{2}, f(3\sqrt{2}))$
$(-3\sqrt{2}, f(-3\sqrt{2}))$