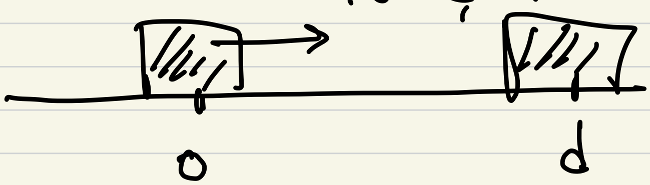
Work Done

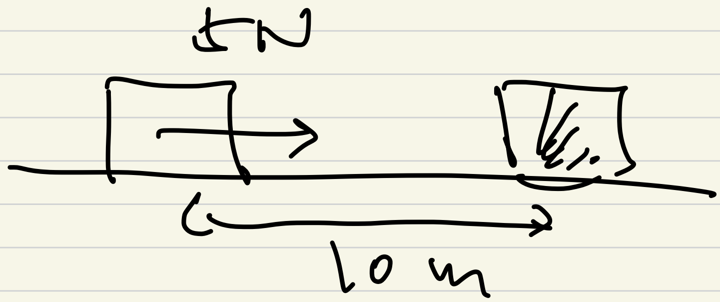
mass

Force: F = a constant force (N)



Work done $= F \cdot d$ N·m = J

Joule.

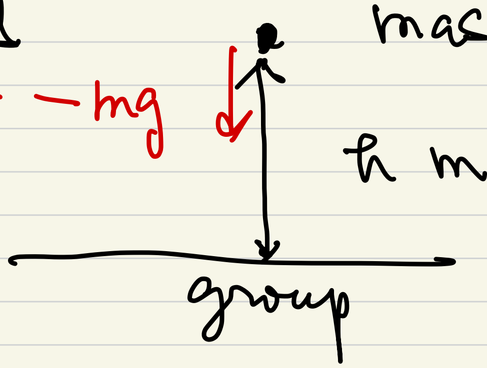


Work done $W = 5 \cdot 10 = 50$ J

Example

mass = m kg

gravity force $= -mg$



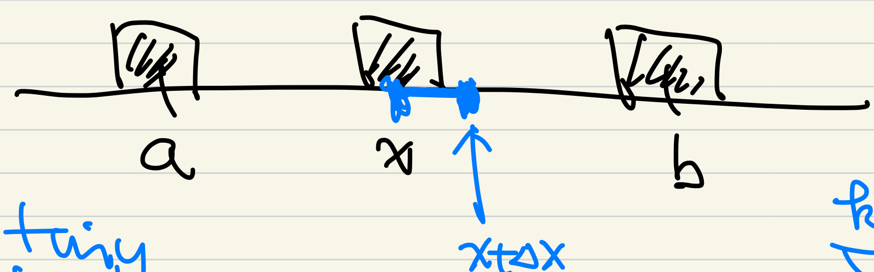
$g \approx 9.8\ m/s^2$

$F \cdot h$

Work to lift a mass of m kg to a height of h m $= (mg)h$

How about changing forces?

Force $= F(x) =$ a continuous function



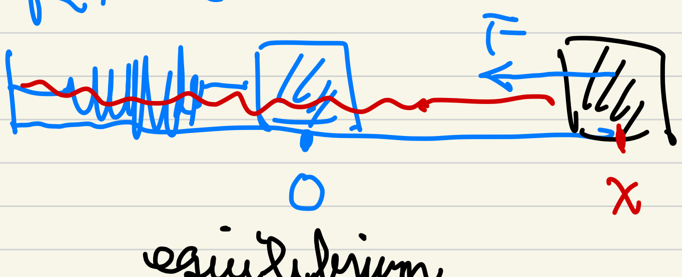
tiny work done for x to $x+\Delta x$ $\approx F(x)\Delta x$ $\Leftrightarrow$ $\sum_{i=1}^{k} F(x_i)\Delta x$

How should we define work done in this case?

Summing as integral

$$W = \int_a^b F(x)\,dx$$

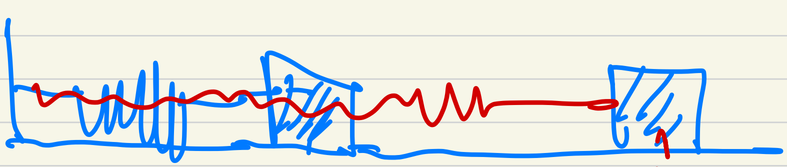
Example (Hooke's Law) Spring-Mass



Spring force $\propto$ x

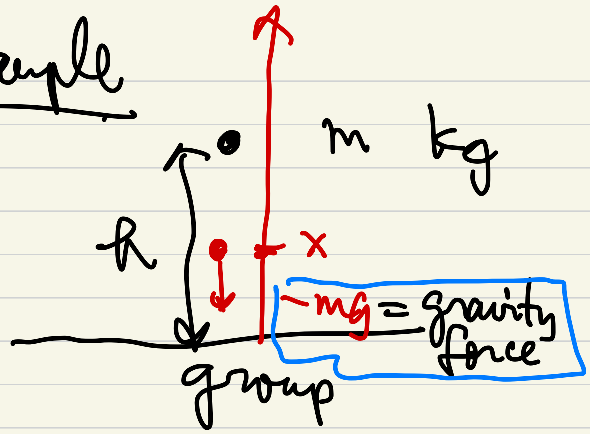
proportional to

i.e. $F = -kx$ $k > 0$ (spring constant)



Work required to push the mass to position b $= \int_0^b kx\,dx = \frac{1}{2}kb^2.$

Example



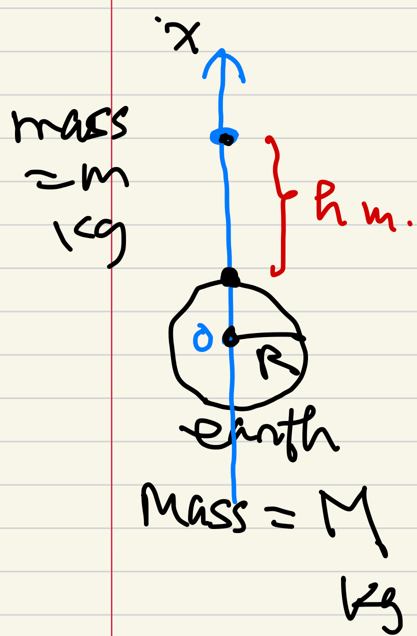
Work required to lift the mass to h m above the ground

$$\| $$

$$W = \int_0^h \underbrace{F(x)}_{\text{force.}}\, dx$$

$$= \int_0^h mg\, dx = mgh$$

Example. A more accurate physical law is "Newton's Gravitional Law"



The attraction force between the two masses is

$$F = \frac{-GmM}{x^2} \qquad G = \text{gravitional constant.}$$

Work required to lift a mass of the surface of earth to a height of h meters $= \displaystyle\int_R^{R+h} \frac{GmM}{x^2}\, dx$

$$W = \int_R^{R+h} \frac{GmM}{x^2}\,dx$$

$$= \left[-\frac{GmM}{x}\right]_R^{R+h}$$

$$= \frac{GmM}{R} - \frac{GmM}{R+h}.$$

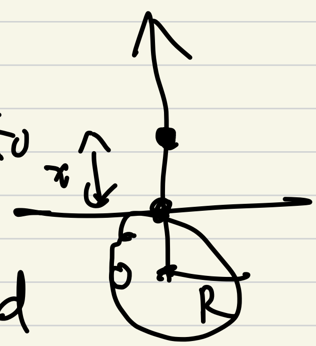
$$F = \frac{GmM}{x^2}$$

$$\approx \frac{GmM}{R^2}$$

$$\shortparallel$$

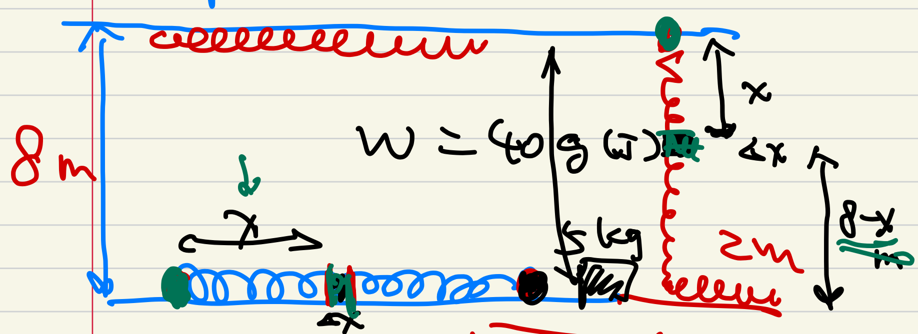
$$mg \implies g = \frac{GM}{R^2}$$

if not too long when compared with R.



$X \approx R$

Example. (Work done on a continuous distribution of mass).



Hang an end of the chain to the ceiling. What is the required work?

a uniform: 10 m, $\boxed{5\ \text{kg}}$ chain

$$mgh = 5 \cdot g \cdot 8 = 40g$$
$$(m \cdot g \cdot h)$$

mass of the tiny piece of chain with length Δx $= \Delta x \cdot \frac{5}{10}$ kg

$$(m \cdot \text{kg}/m) = \text{kg}$$

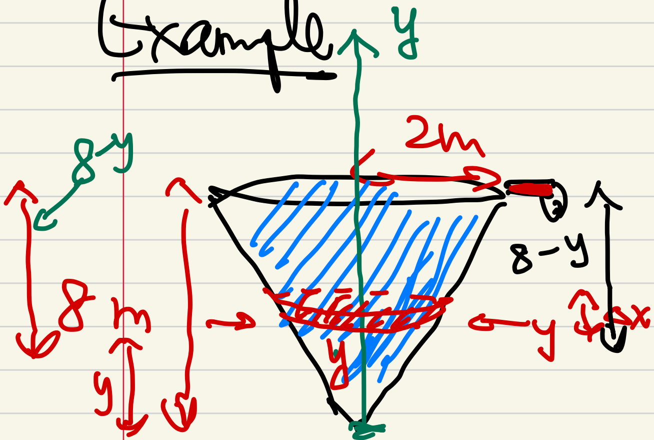
Work for this tiny piece $\approx (8-x)(g)\frac{5}{10}\Delta x$

$$\underbrace{h \cdot g \cdot \text{"m"}}$$

Summing

$$W = \int_0^8 \frac{5}{10} g \cdot (8-x)\,dx$$
$$= \frac{g}{2}\left[-\frac{(8-x)^2}{2}\right]_0^8 = 16g\ (J)$$

Example

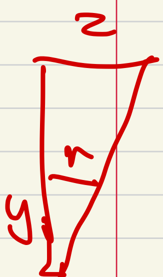


Given a cone container filled with water.

Find the work required to pump all water to an outlet at the top of the container.

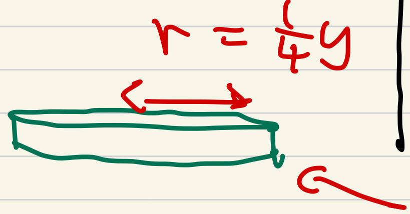
$$\boxed{\text{"}mgh\text{"} = \overset{h}{(8-y)} \cdot \overset{g}{g} \cdot \overset{m}{\rho\pi\left(\tfrac{1}{4}y\right)^2 \Delta y}}$$

lift it up by $8-y$ meters up



$\frac{r}{y} = \frac{2}{8}$

A tiny slice of the water at height y



volume $\approx \pi\left(\tfrac{1}{4}y\right)^2 \Delta y$

mass $\approx \pi\left(\tfrac{1}{4}y\right)^2 \Delta y \cdot \rho$

tiny mass ρ = density of water $= 1000\ \text{kg}/\text{m}^3$

Work required

$$W = \int_0^8 \overset{1000\,\text{kg/m}^3}{\rho}\ \overset{9.8\,\text{m/s}^2}{g}\ \pi\left(\tfrac{1}{4}y^2\right)(8-y)\,dy$$

$=$ an easy integral.