MATH 2111: Tutorial 5 Linear Transformations and Matrix Operations

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Review

- Linear Transformation Definition
- Matrix of a linear transformation, Standard Matrix
- Onto Map & One-to-one Map
- Matrix Operation: sums, scalar multiples, matrix multiples, matrix transpose

Given transformation $T(x_1, x_2, x_3) = (x_2 + 1, x_3 + 1)$.

(1) What is $T(1, 2, 1)$?

(2) Is $T(\cdot)$ a linear transformation?

(1) Here, $x_1=1$, $x_2=2$, $x_3=1$. so $T(1,2,1) = (2+1, 1+1) = (3, 2)$.

(2). No. According to definition of linear transformation,

$$T(x_1+y_1, x_2+y_2, x_3+y_3) = (x_2+y_2+1, x_3+y_3+1)$$

$$\neq T(x_1, x_2, x_3) + T(y_1, y_2, y_3).$$

So $T(\cdot)$ is not a linear transformation

Or you could also show that,

for scalars $c \neq 1$,

$$T(cx_1, cx_2, cx_3) = T(cx_1+1, cx_2+1, cx_3+1)$$

$$= c\,T(x_1, x_2, x_3) + (1-c)\cdot(1,1,1)$$

$$\neq T(x_1, x_2, x_3).$$

So $T(\cdot)$ is not a linear transformation

Example 2

Standard Matrix

(1) Find the standard matrix of the following linear transformation

$$T(x_1, x_2, x_3, x_4) = (5x_1 - x_2, 5x_2 - x_3, 5x_3 - x_4, 5x_4 - x_1).$$

(2) Find the linear transformation of the following standard matrix

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

(1). $\begin{pmatrix} 5 & -1 & 0 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 5 & -1 \\ -1 & 0 & 0 & 5 \end{pmatrix}$.

(2). $T(x_1, x_2, x_3) = (5x_1 + x_2 + x_3, \ x_1 + 5x_2 + x_3, \ x_1 + x_2 + 5x_3)$

Example 3

Onto Map & One-to-One Map

Given linear transformation
$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)$, determine whether

(1) T is a one-to-one map,

(2) T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

Standard Matrix of T is $A \triangleq \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array} \xrightarrow{③-①} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{array}{c} ①' \\ ②' \\ ③' \end{array} \xrightarrow{③'+②'} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Pivot

$\Rightarrow$ T is a one-to-one map & maps $\mathbb{R}^3$ to $\mathbb{R}^3$

Suppose α is an angle. Given linear transformation
$T(x_1, x_2) = (\cos\alpha \cdot x_1 + \sin\alpha \cdot x_2, -\sin\alpha \cdot x_1 + \cos\alpha \cdot x_2)$.
Determine whether

(1) T is a one-to-one map,

(2) T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Standard Matrix of T is $A \triangleq \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

Case One: when $\sin\alpha = 0$, we have $|\cos\alpha| = 1$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow$ T is one-to-one & maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Case Two: when $\cos\alpha = 0$, we have $|\sin\alpha| = 1$.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$ T is one-to-one & maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Case Three: when $\sin\alpha \neq 0$, $\cos\alpha \neq 0$,

$$A = \begin{array}{c} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \end{array} \begin{array}{c} ① \\ ② \end{array} \xrightarrow{② + ① \cdot \tan\alpha} \begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha} \end{pmatrix}$$

$$\| $$

pivot!

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ 0 & \frac{1}{\cos\alpha} \end{pmatrix}$$

$\Rightarrow$ T is one-to-one & maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Anyway, T is one-to-one & maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Remark: T is a rotation transformation.

Example 5

Matrix Operations

Given $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(1) Compute AB.

(2) Compute A^2, A^3.

(3) Compute $A^\top B$.

(1) $$AB = \begin{pmatrix} 0\times1+1\times4+0\times7 & 0\times2+1\times5+0\times8 & 0\times3+1\times6+0\times9 \\ 0\times1+0\times4+1\times7 & 0\times2+0\times5+1\times8 & 0\times3+0\times6+1\times9 \\ 1\times1+0\times4+0\times7 & 1\times2+0\times5+0\times8 & 1\times3+0\times6+0\times9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{pmatrix}$$

(2). $$A^2 = A\cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 = A \cdot A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3) $A^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$A^T B = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Remark: AB switches rows of B;

$A^3 = I_3$.