MATH2111 Tutorial 3

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1 The Matrix Equation

1. **Matrix-Vector Products**. We can multiply an $m \times n$ matrix $\boldsymbol{A}$ by a vector $\boldsymbol{v} \in \mathbb{R}^n$. The result, written $\boldsymbol{Av}$, belongs to $\mathbb{R}^m$. If $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n \in \mathbb{R}^m$ are the columns of $\boldsymbol{A}$ and $v_1, v_2, \cdots, v_n \in \mathbb{R}$ are the entries of $\boldsymbol{v}$, then

$$\boldsymbol{Av} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \cdots + v_n\boldsymbol{a}_n$$

2. **Property of Matrix-Vector Products**. If $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then:
   (a) $\boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{Au} + \boldsymbol{Av} \in \mathbb{R}^m$
   (b) $\boldsymbol{A}(c\boldsymbol{v}) = c(\boldsymbol{Av}) \in \mathbb{R}^m$

3. **Theorem**. If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$, and if $\boldsymbol{b}$ is in $\boldsymbol{R}^m$, the matrix equation
$$\boldsymbol{Ax} = \boldsymbol{b}$$
has the same solution set as the vector equation
$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$
which, in turn, has the same solution set as the system of linear equations whose augmented matrix is
$$\left[\begin{array}{cccc|c} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \end{array}\right]$$

4. **Theorem**. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then the following statements are logically equivalent:
   (a) For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution.
   (b) Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $\boldsymbol{A}$.
   (c) The columns of $\boldsymbol{A}$ span $\mathbb{R}^m$. i.e. Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} = \mathbb{R}^m$
   (d) A has a pivot position in every row.

   Warning: The above theorem is about a coefficient matrix, not an augmented matrix. If an augmented matrix $[\boldsymbol{A} \mid \boldsymbol{b}]$ has a pivot position in every row, then the equation $\boldsymbol{Ax} = \boldsymbol{b}$ may or may not be consistent.

   Example 1.1. Could a set of n vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$ if $n < m$? Explain.

2 Solution Sets of Linear Systems

1. **Homogeneous Linear Systems**. A system of linear equations is said to be **homogeneous** if it can be written in the form

   $$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

   where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{0}$ is the zero vector in $\mathbb{R}^m$.

   Note:
   The system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ always has at least one solution, namely, $\boldsymbol{x} = \boldsymbol{0}$ (the zero vector in $\mathbb{R}^n$), and
   (a) this zero solution is called the **trivial solution**.
   (b) the other non-zero solution are called the **nontrivial solution**.

2. **Theorem**. The homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

3. **Theorem**. Suppose $\boldsymbol{A}$ has k free columns, then the homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has k free variables, and the general solution can be written as **parametric vector form**

   $$\boldsymbol{x} = s_1\boldsymbol{x}_1 + s_2\boldsymbol{x}_2 + \ldots + s_k\boldsymbol{x}_k$$

   In other words, the solution set of the homogeneous system is

   $$\text{Span}\,\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$$

   Note: If there are no non-pivot columns (i.e. no free variables), the solution set is just $\{\boldsymbol{0}\}$.

4. **Non-Homogeneous Linear Systems**. Suppose the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent for some given $\boldsymbol{b}$ ($\boldsymbol{b} \neq \boldsymbol{0}$), and let $\boldsymbol{p}$ be a particular solution. Then the solution set of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is the set of all vectors of the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is any solution of the homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

5. **Procedures of Writing a Solution Set (of a consistent system) in Parametric Vector Form**

   (a) Row reduce the augmented matrix to reduced row echelon form (RREF).

   (b) Express each basic variable in terms of any free variables appearing in an equation.

   (c) Write a typical solution $\boldsymbol{x}$ as a vector whose entries depend on the free variables, if any.

   (d) Decompose $\boldsymbol{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

3 Exercises

1. Write the matrix equation as a vector equation, or vice versa.

(a) $$\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

(b) $$x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

2. Suppose $\boldsymbol{A}$ is a 3×3 matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$ with the property that $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution. Explain why the columns of $\boldsymbol{A}$ must span $\mathbb{R}^3$.

3. Determine if the columns of the matrix span $\mathbb{R}^4$

$$\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$$

4. Determine if the system has a nontrivial solution.

(1) $$\begin{cases} 2x_1 - 5x_2 + 8x_3 = 0 \\ -2x_1 - 7x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + 7x_3 = 0 \end{cases}$$

(2) $$\begin{cases} x_1 - 3x_2 + 7x_3 = 0 \\ -2x_1 + x_2 - 4x_3 = 0 \\ x_1 + 2x_2 + 9x_3 = 0 \end{cases}$$

5. Describe all solutions of $\boldsymbol{Ax} = \mathbf{0}$ in parametric vector form.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. (1) Suppose $\boldsymbol{w}$, $\boldsymbol{p}$ are two solutions of the equation $\boldsymbol{Ax} = \boldsymbol{b}$ and define $\boldsymbol{v}_h = \boldsymbol{w} - \boldsymbol{p}$.Show that $\boldsymbol{v}_h$ is a solution of $\boldsymbol{Ax} = \mathbf{0}$.

(2) Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution. Explain why the solution is unique precisely when $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.