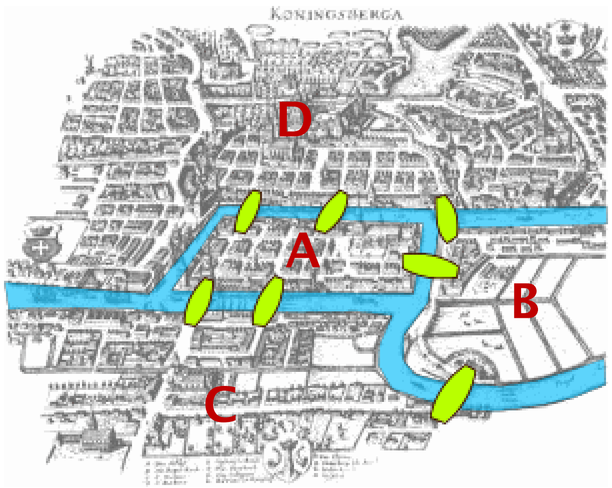
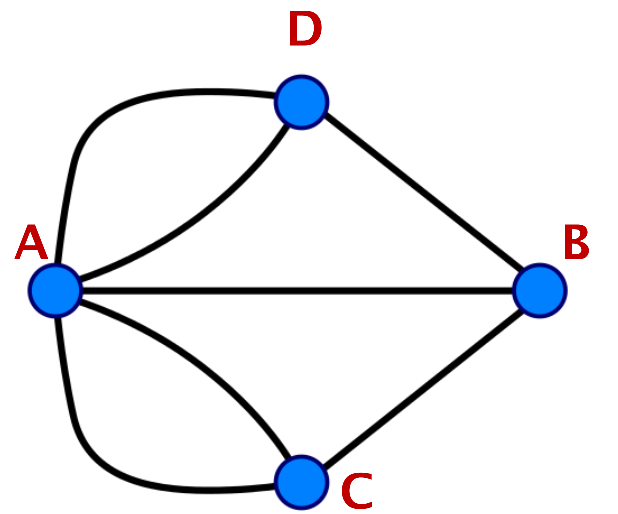
Part VI: Graphs

- Reading: Rosen 10.1, 10.2, 10.3, 10.4, 10.5

The Seven Bridges of Königsberg

- Q: Can you find a path to cross all seven bridges, each exactly once?

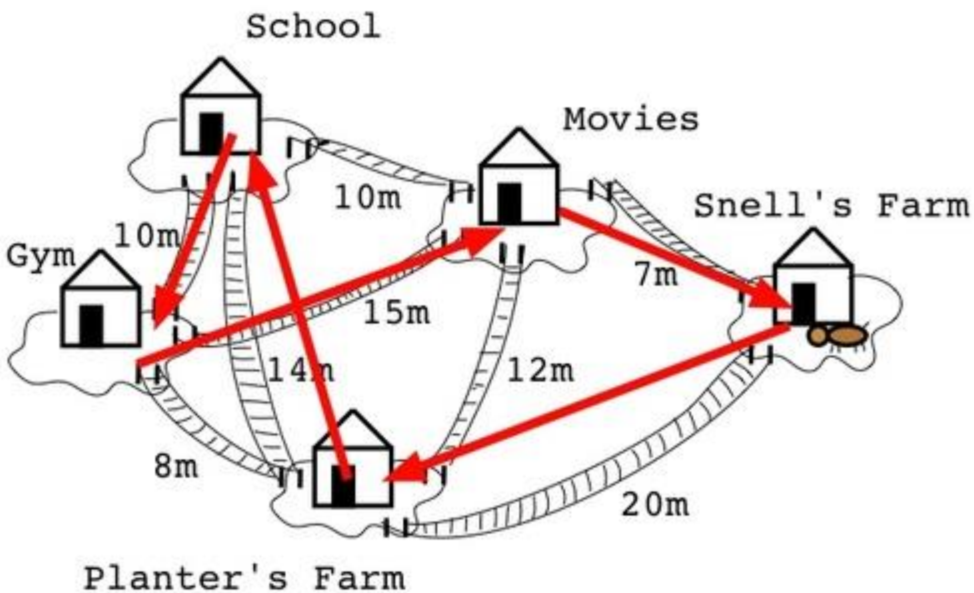




- Q: (Reformulated as a graph problem) Can you find a path in the graph that includes every edge exactly once?

The Traveling Salesman Problem

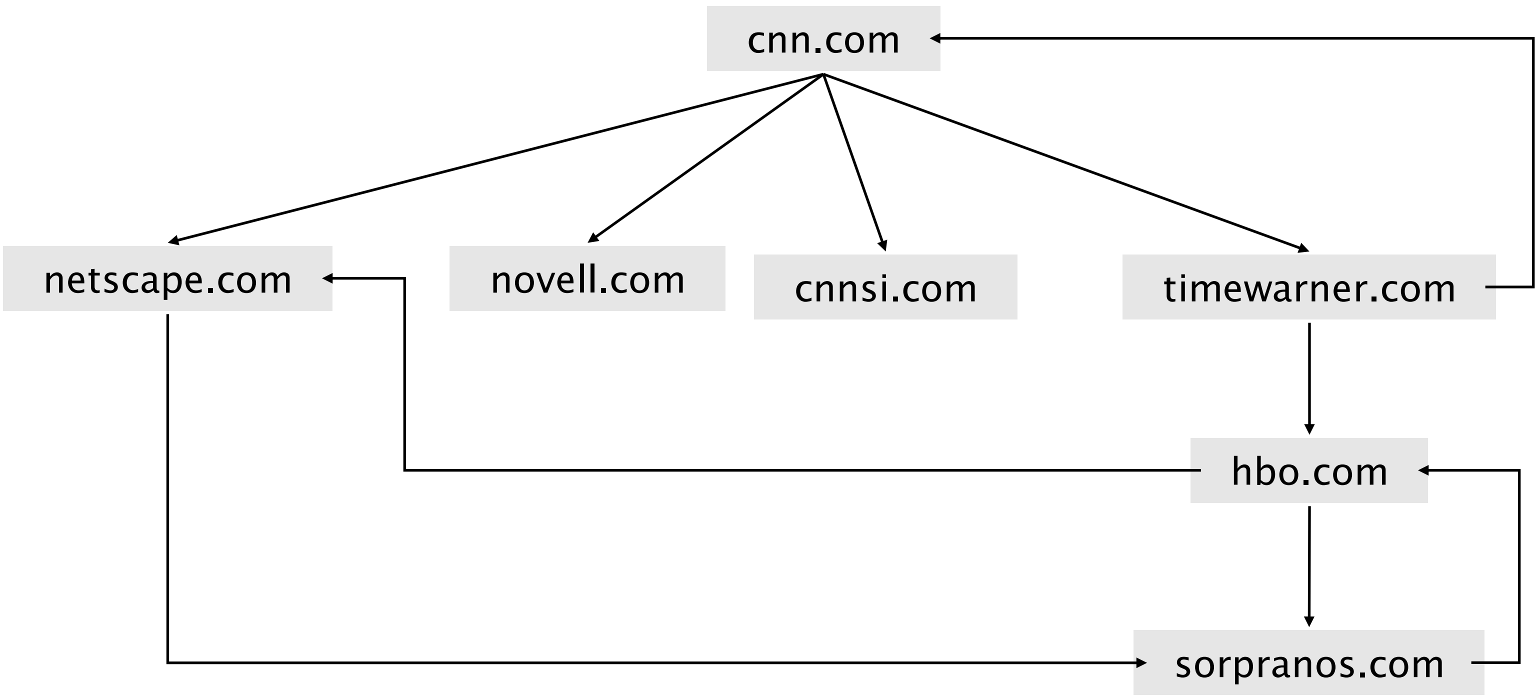
- Q: How to visit all places with the shortest total distance, and come back to origin?



- Q: (Reformulated as a graph problem) Given a graph where edges have weights (lengths), how to find a cycle with minimum total weight that includes all vertices?

World Wide Web

- Web graph.
  - Node: web page.
  - Edge: hyperlink from one page to another (directed).

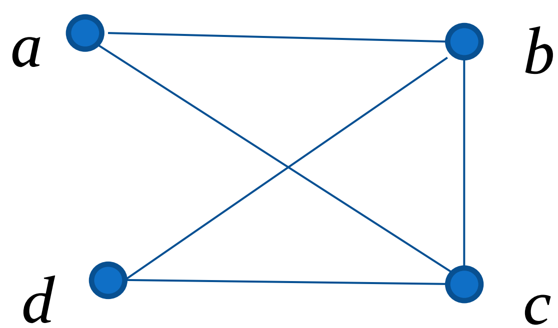


Outline

- **Definition of graphs**
- Representing Graphs
- Connectivity
- Euler Paths and Circuits

Graphs

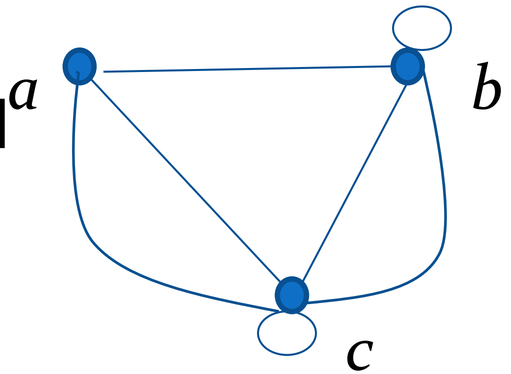
- **Definition**
  A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. An edge is said to connect its two endpoints. The endpoints connected by an edge are called adjacent (or neighbors), and the edge is incident to its endpoints.

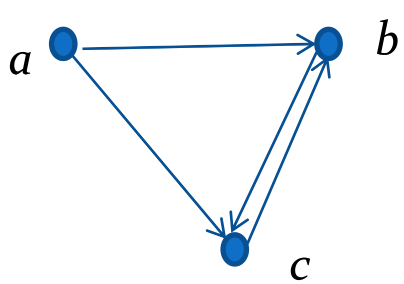


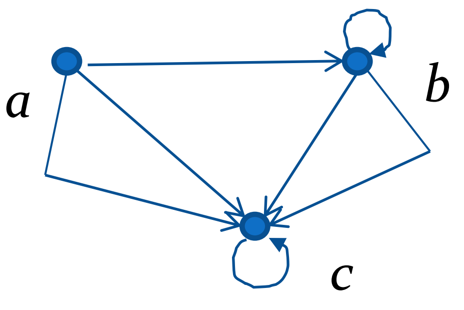
This is a graph with four vertices and five edges.

Types of Edges

- Loop: An edge connecting a vertex to itself
- Multiple edges: Edges connecting the same two vertices.
- Simple graph: Graphs without loops and multiple edges.
- Edges can be directed or undirected
  - Undirected edge $e = \{u, v\}$
  - Directed edge $e = (u, v)$

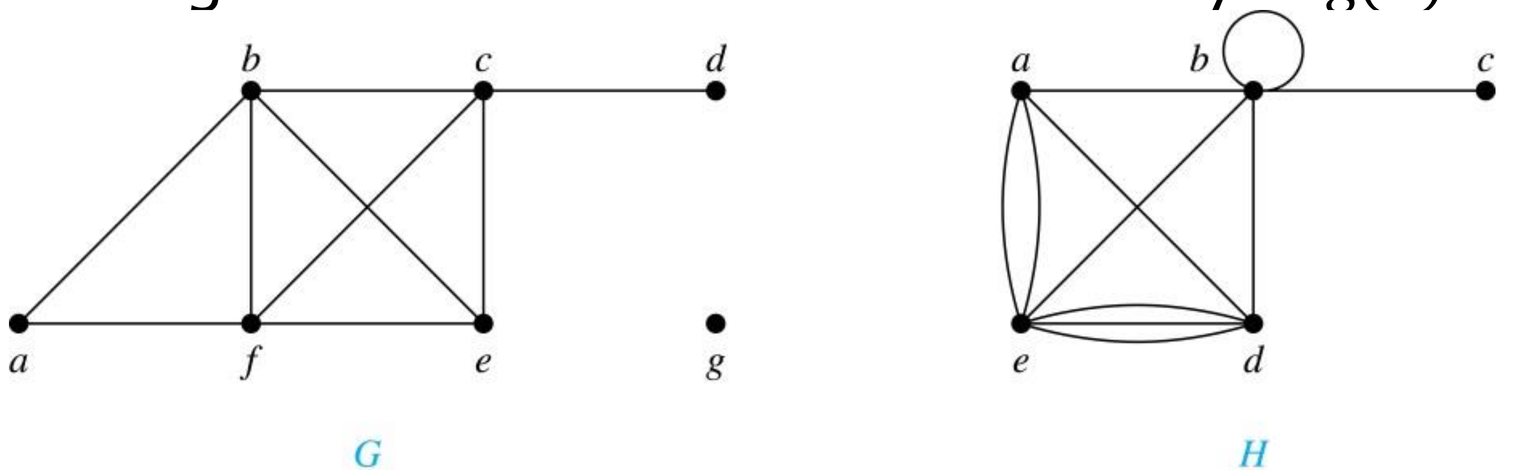






Degrees

- **Definition**
  The degree of a vertex in a undirected graph is the number of edges incident to it, except that a loop at a vertex contributes two to the degree of that vertex.
  The degree of the vertex v is denoted by $\deg(v)$.



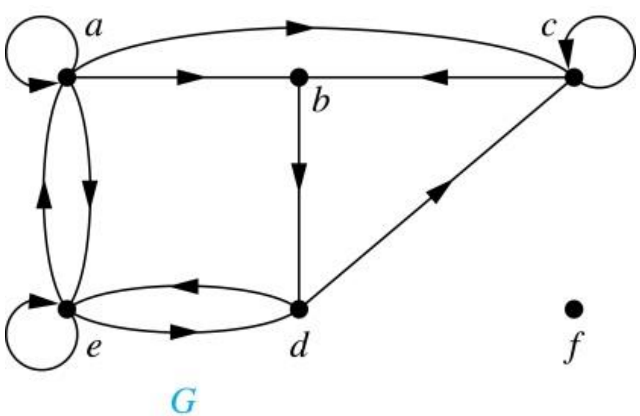
G: $\deg(a) = 2, \deg(b) = \deg(c) = \deg(f) = 4, \deg(d) = 1,$
$\deg(e) = 3, \deg(g) = 0.$
H: $\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1, \deg(d) = 5.$

Property of Degrees

- **Theorem**
  If $G = (V, E)$ is an undirected graph with m edges, then
  $$2m = \sum_{v \in V} \deg(v)$$
- **Proof**
  Each edge contributes two to the total degree of all vertices.
- **Example**
  How many edges are there in a graph with 10 vertices, all of which have degree 6?
- **Example**
  Show that at a party, the number of people who shake hands with an odd number of people must be even.

Directed Graphs

- **Definition**
  The in-degree of a vertex v, denoted $\deg^{-}(v)$, is the number of edges which terminate at v. The out-degree of v, denoted $\deg^{+}(v)$, is the number of edges with v as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.



$\deg^{-}(a) = 2, \deg^{-}(b) = 2, \deg^{-}(c) = 3,$
$\deg^{-}(d) = 2, \deg^{-}(e) = 3, \deg^{-}(f) = 0.$

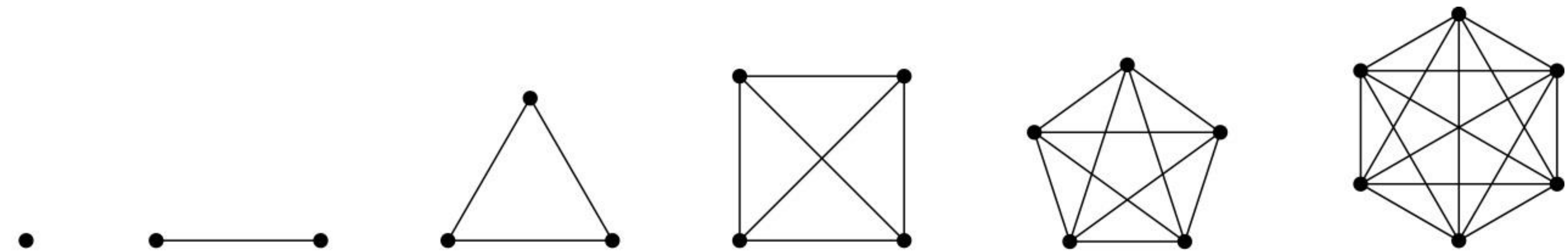
$\deg^{+}(a) = 4, \deg^{+}(b) = 1, \deg^{+}(c) = 2,$
$\deg^{+}(d) = 2, \deg^{+}(e) = 3, \deg^{+}(f) = 0.$

Property of Degrees

- **Theorem**
  If $G = (V, E)$ is a directed graph with m edges, then
$$m = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$
- **Proof**
  Each edge contributes one to the out-degree of its starting vertex and one to the in-degree of its terminal vertex

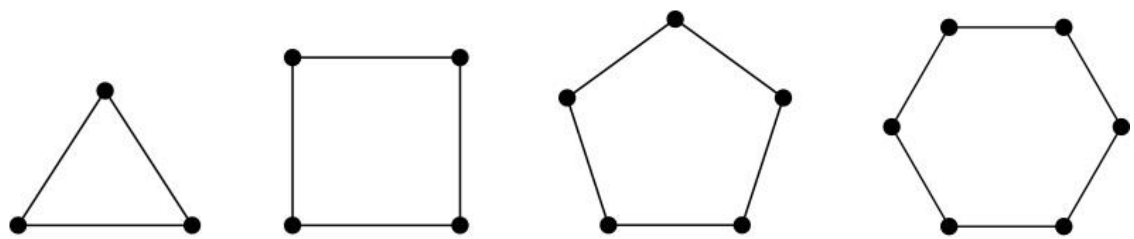
Special Graphs

- Complete graphs: K_n
  - A simple graph that contains exactly one edge between each pair of distinct vertices.



K_1 K_2 K_3 K_4 K_5 K_6

- A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

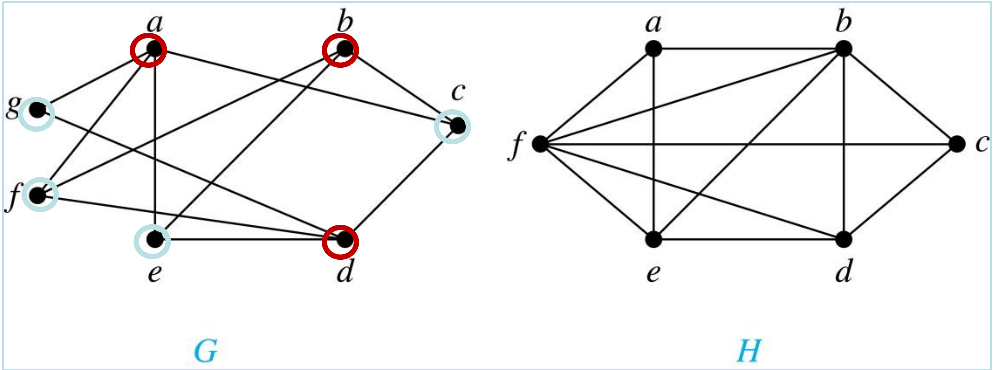


C_3 C_4 C_5 C_6

Bipartite Graphs

- **Definition**
  A simple graph $G = (V, E)$ is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.
  - Equivalently, a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

G is bipartite

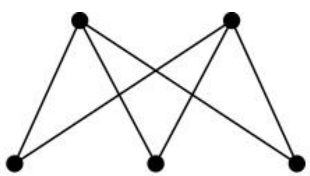


H is not bipartite

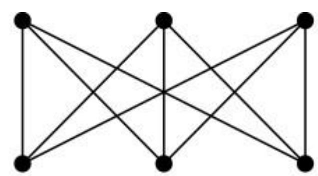
- **Example**
  Show that C_3 is not bipartite.

Complete Bipartite Graphs

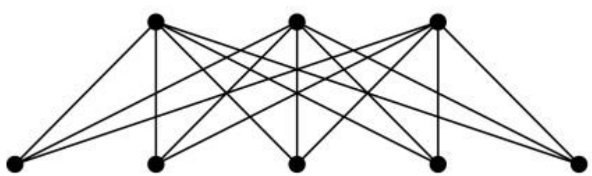
- **Definition**
  A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.



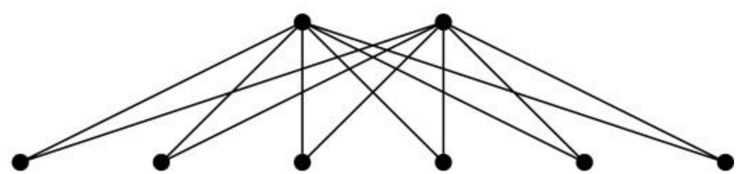
$K_{2,3}$



$K_{3,3}$



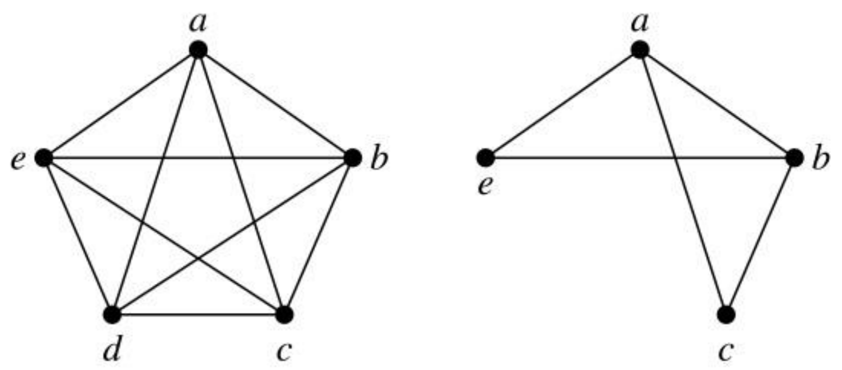
$K_{3,5}$



$K_{2,6}$

Subgraphs

- **Definition**
  A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$, such that for any $e \in F$, both of its endpoints must be in W.



Outline

- Definition of graphs
- **Representing Graphs**
- Connectivity
- Euler Paths and Circuits

Adjacency Lists

- **Definition**: An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.
- **Examples:**

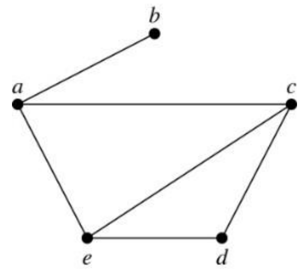


TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

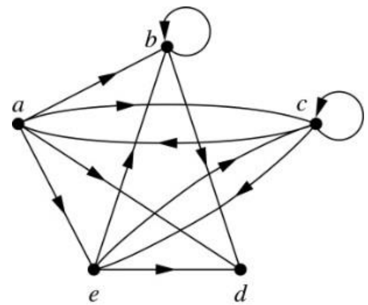


TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

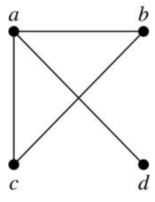
Adjacency Matrices

- **Definition:** Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The adjacency matrix $\boldsymbol{A_G}$ of G, with respect to the listing of vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.
- In other words, if the graphs adjacency matrix is $\boldsymbol{A_G} = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

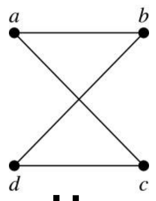
Adjacency Matrices

- **Example**:



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.



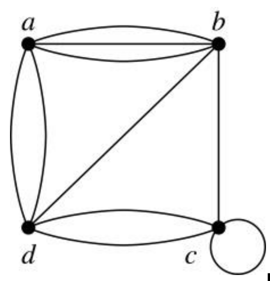
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.

- The adjacency matrix of a simple graph
  - is symmetric, i.e., $a_{ij} = a_{ji}$
  - each diagonal entry $a_{ii} = 0$
- Note:
  - Adjacency list is better for a sparse graph
  - Adjacency matrix is better for a dense graph

Adjacency Matrices

- Non-simple graphs
  - A self-loop at the vertex v_i is represented by a 1 at the (i, i)-th position of the matrix
  - When multiple edges connect the same pair of vertices v_i and v_j, (or if multiple self-loops are present at the same vertex), the (i, j)th entry equals the number of edges connecting the pair of vertices.
- **Example** (using the ordering of vertices *a*, *b*, *c*, *d*.)



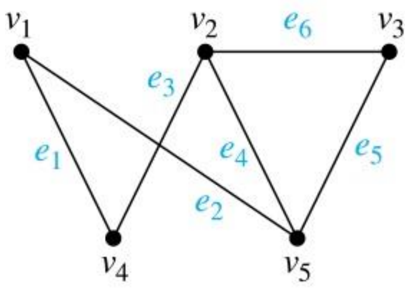
$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

- Can also represent directed graphs (the matrix may not be symmetric)

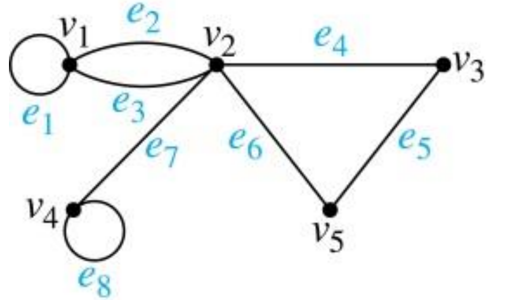
Incidence Matrices

- **Definition**: Let $G = (V, E)$ be an undirected graph with vertices where $v_1, v_2, \dots v_n$ and edges $e_1, e_2, \dots e_m$. The incidence matrix with respect to the ordering of V and E is the $n \times m$ matrix $\boldsymbol{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$



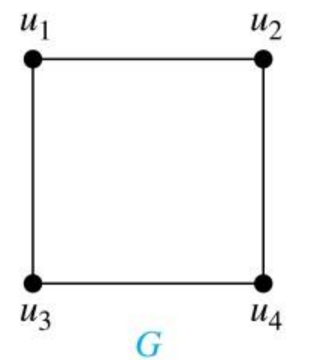
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

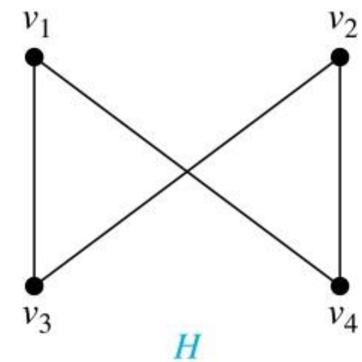


$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Graph Isomorphism

- **Definition**: Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f: V_1 \to V_2$ with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such an f is called an *isomorphism.*
- **Example**: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.
- **Solution**: Using the bijection f with
  - $f(u_1) = v_1$
  - $f(u_2) = v_4$
  - $f(u_3) = v_3$
  - $f(u_4) = v_2$





Algorithms for Graph Isomorphism

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity (in the number of vertices of the graphs).
- However, there are algorithms with linear average-case time complexity.

Outline

- Definition of graphs
- Representing Graphs
- **Connectivity**
- Euler Paths and Circuits

Paths

- **Definition**
  Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, \dots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \dots, x_{n-1}, x_n = v$ of vertices such that e_i connects x_{i-1} and x_i, for $i = 1, \dots, n$.
- For directed graphs, replace "e_i connects x_{i-1} and x_i" with "$e_i = (x_{i-1}, x_i)$".
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$
- The path is a circuit (or a cycle) if $u = v$.
- A path or circuit is simple if it does not contain the same edge more than once.

Connectivity in Undirected Graphs

- **Definition**
  In an undirected graph, two vertices u, v are connected if there is a path from u to v.
- **Example**
  Give a recursive definition of connectedness of two vertices without using the concept of a path
- **Solution**
  - For any edge $e = \{u, v\}$, u and v are connected;
  - For any u, v, w, if u and v are connected, and there is an edge between v and w, then u and w are connected.
- An undirected graph is called connected if there is a path between every pair of vertices.

Example

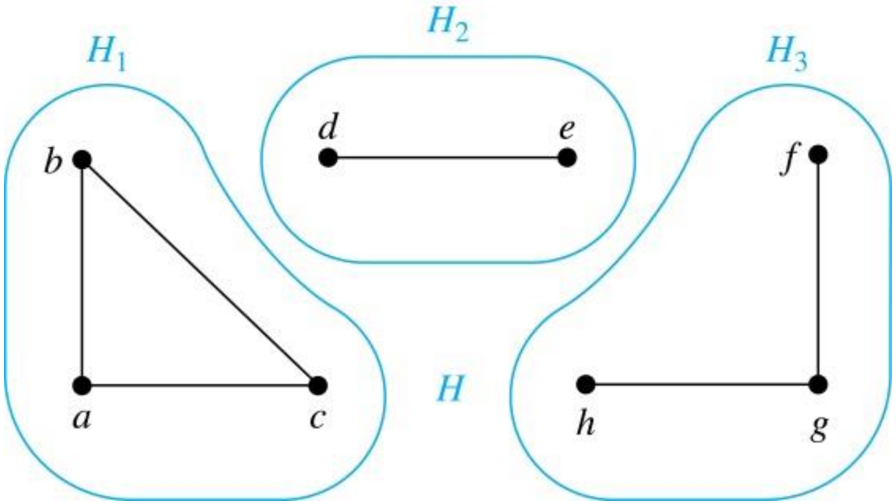
- Suppose in a wireless network of n mobile devices, each device is within communication range with at least $n/2$ other devices (assuming n is an even number). Show that all devices are connected.
- Reformulated as a graph problem: Let G be an undirected graph where each node has degree $\geq n/2$. Show that G is connected.

Proof

- Consider any two nodes u and v in G. There are two cases:
- If there is an edge $\{u, v\}$, then u and v are connected.
- If there is no direct edge between u and v, then they must have a common neighbor, say w, because
  - There are $n - 2$ nodes other than u and v.
  - u and v each have $\geq n/2$ neighbors among these nodes.
- Thus there is a path between u and v.
- The above argument holds for any two nodes u, v, so the graph G is connected.
- Q: If the threshold $n/2$ is changed to $n/2 - 1$, does the claim still hold?

Connected Components

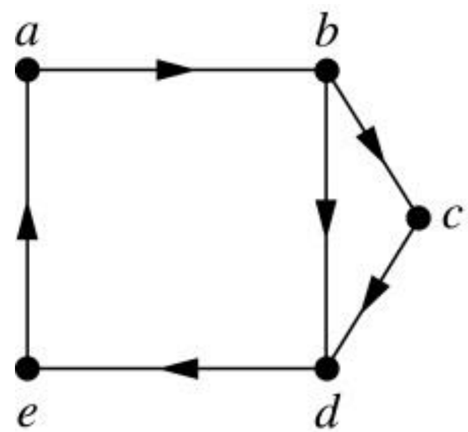
- **Definition**
  A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.



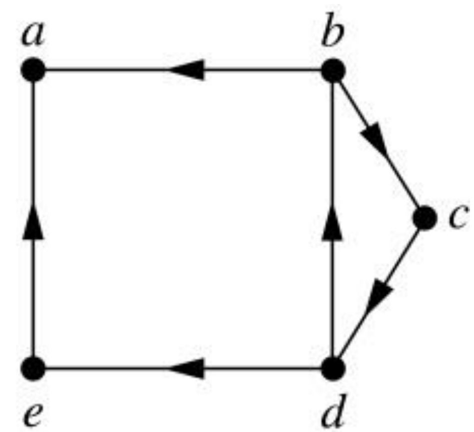
Connectivity in Directed Graphs

- **Definition**
  A directed graph G is weakly connected if the undirected graph obtained by ignoring the directions of the edges of G is connected.
- **Definition**
  A directed graph $G = (V, E)$ is strongly connected if for any $u, v \in V$, there is a path from u to v.

Strongly connected



G

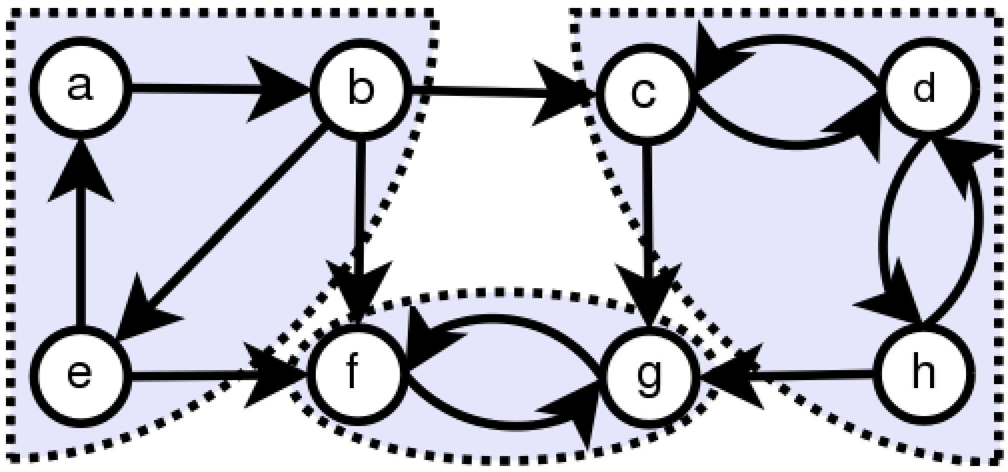


H

Weakly connected by not strongly connected

Strongly Connected Components

- **Definition**
  The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs are called the strongly connected components of G.



Outline

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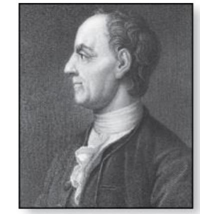
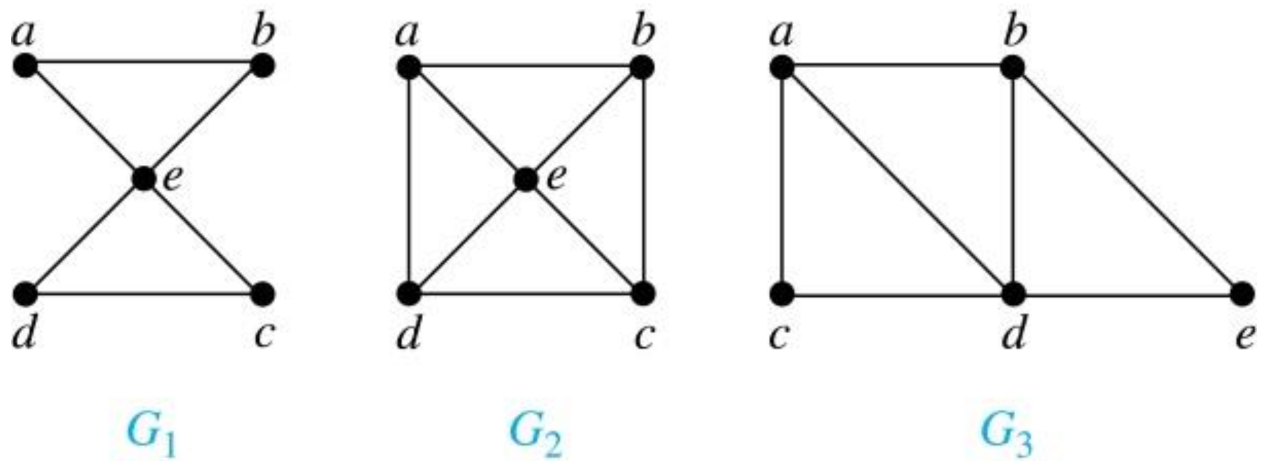
- Connectivity
- **Euler Paths and Circuits**

Euler Paths and Circuits



Leonard Euler (1707-1783)

- **Definition**
  An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.
- **Example**
  Which of the graphs below have Euler circuits? Which have Euler paths?

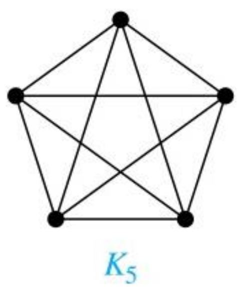


Euler Circuits

- **Theorem**
  An undirected connected graph has an Euler circuit iff all vertices have even degrees.
- **Proof**
- The “only if” direction:
  - An Euler circuit begins with a vertex a and continues with an edge incident with a, say $\{a, b\}$. The edge $\{a, b\}$ contributes one to $\deg(a)$.
  - Each time the circuit passes through a vertex it contributes two to the vertex’s degree.
  - Finally, the circuit terminates at a, contributing one to $\deg(a)$. Therefore $\deg(a)$ must be even, too.

Proof (cnt'd)

- The "if" direction: We will give an algorithm to find an Euler cycle when all degrees are even.
- First, consider the following simple algorithm:



$u \leftarrow$ **any vertex**
while u **has an edge not taken yet**
    take that edge $\{u, v\}$
    $u \leftarrow v$

- Observation: This algorithm always finds a cycle, because all degrees are even
- Problem: This algorithm may not traverse all edges.

Proof (cnt'd)

- Idea: The subgraph consisting all edges not traversed must still have even degrees at all vertices.
- Then just repeat the algorithm on those edges, and "connect" the two cycles into one.

```
c ← empty cycle
while there are still edges not taken yet
    u ← any vertex already seen
    c′ ← Find-Cycle(u)
    insert c′ into c at u

Find-Cycle(u):
while u has an edge not taken yet
    take that edge {u,v}
    u ← v
```

Euler Paths

- **Theorem**
  An undirected connected graph has an Euler path but not an Euler circuit iff there are exactly two vertices of odd degree.
- **Proof**
- The “if” direction
  - Suppose u, v have odd degrees. Add an edge between u and v. In the new graph, there are no odd degrees, so there is an Euler circuit. Remove the added edge from the Euler circuit turns it into a path, starting at u and terminating at v.

Proof (cnt'd)

- The "only if" direction
  - Suppose there is an Euler path from u to v and $u \neq v$. Add an edge between u and v in the graph as well as in the path. This turns the Euler path into a cycle. By previous theorem, all degrees in the new graph are even. Then removing the added edge from the graph introduces exactly two vertices with odd degrees.

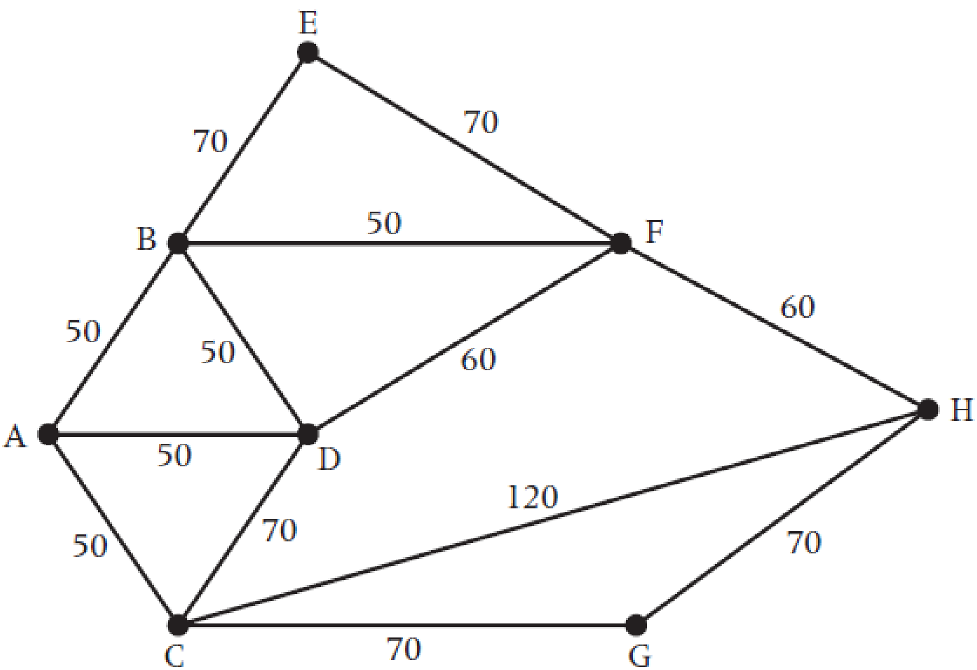
Chinese Postman Problem

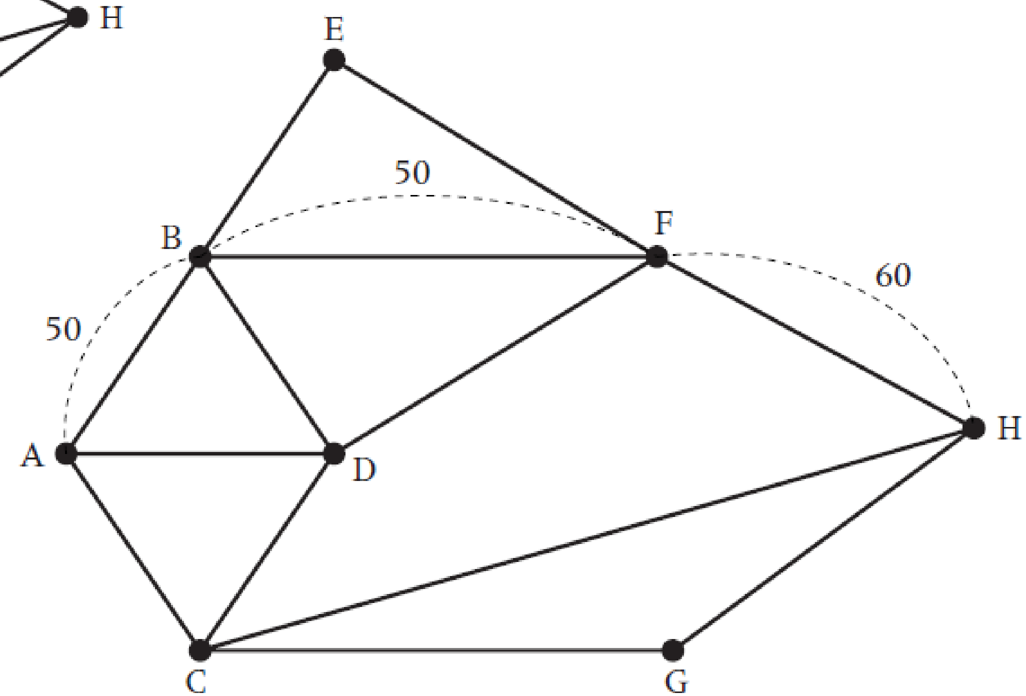
- Given a graph where the edges represent streets and vertices represent intersections, find the shortest tour so that the postman can traverse every edge at least once and return to the starting point. For each edge, its length is given.
- **Solution**
  - Easy case: If all vertices have even degree, then just use the Euler circuit.
  - What if there are odd-degree vertices
  - Note that due to the property of degrees, there must be an even number of odd-degree vertices

Solution

- Idea: Since there are odd-degree vertices, an Euler circuit does not exist, so some edges will have to be traversed more than once.
  - We want to minimize the total length of such edges.
- Medium case: There are two odd-degree vertices
  - Find the shortest path between them
  - Add all edges on this path to the graph
  - Now all vertices have even degrees and then can find an Euler circuit.

Example

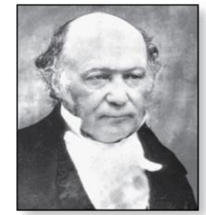




Solution (cnt'd)

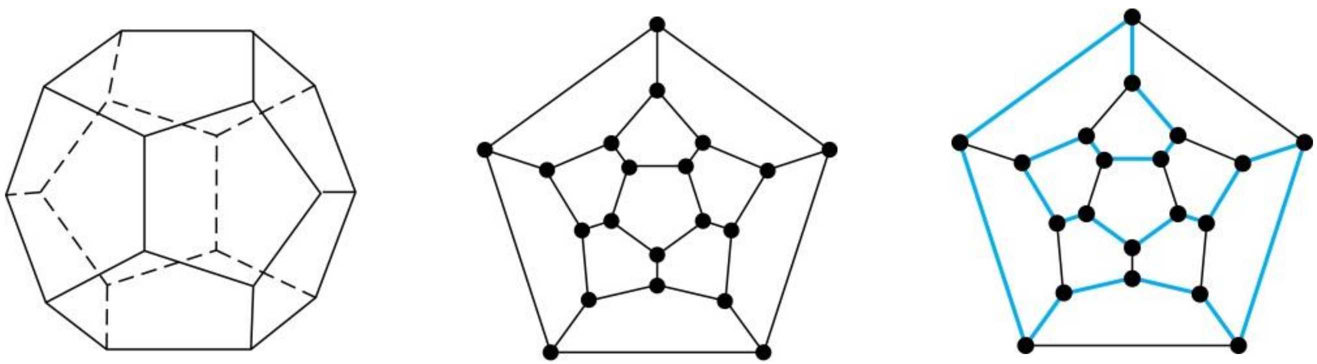
- Hard case: There are more than 2 odd-degree vertices.
- For every pair of such vertices, find their shortest path
- Find the best pairing of these vertices, such that the total length of the shortest paths between the pairs vertices is minimized
- Add the shortest paths between the paired vertices to the graph and find the Euler circuit.
- Further questions
  - How to find the shortest path?
  - How to find the best pairing (matching)?
  - Will be covered in COMP 3711

Hamilton Paths and Circuits



William Rowan Hamilton (1805- 1865)

- Euler paths and circuits contains every edge only once.
- Hamilton paths and circuits contains every vertex exactly once



(a) (b)

- Determining if a graph has a Hamilton path a circuit is NP-hard
- On weighted graphs: The *traveling salesman problem.*