Substitution Rule ~ Chain Rule.

$$\int f(\overset{u}{g(x)})\overbrace{g'(x)\,dx}^{du} = \int f(u)\,du$$

Simpler after the substitution?

by letting $\boxed{u = g(x)}$

$\frac{du}{dx} = g'(x)$

$du = g'(x)dx$

If you could find this one

$F(u) + C$

i.e. $F'(u) = f(u)$

$F(g(x)) + C$

Checking: $\frac{d}{dx}[F(g(x)) + C]$

$= F'(g(x))\,g'(x)$

$= f(g(x))\,g'(x)$

Example :

1) $$\int x^2 \sqrt{x^3+4}\, dx$$

Let $u = x^3+4$

$$\frac{du}{dx} = 3x^2$$

$$du = 3\boxed{x^2\, dx}$$

$$\frac{1}{3}du = x^2 dx$$

$$= \int \frac{1}{3}\sqrt{u}\, du$$

$$= \frac{1}{3}\int u^{1/2}\, du$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{1}{2}+1} u^{1/2+1} + C = \frac{1}{3}\cdot\frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{9}(x^3+4)^{3/2} + C$$

2) $$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt$$

$$u = \sqrt{t} = t^{1/2}$$

$$\frac{du}{dt} = \frac{1}{2}t^{-1/2}$$

$$\boxed{2du = \frac{dt}{\sqrt{t}}}$$

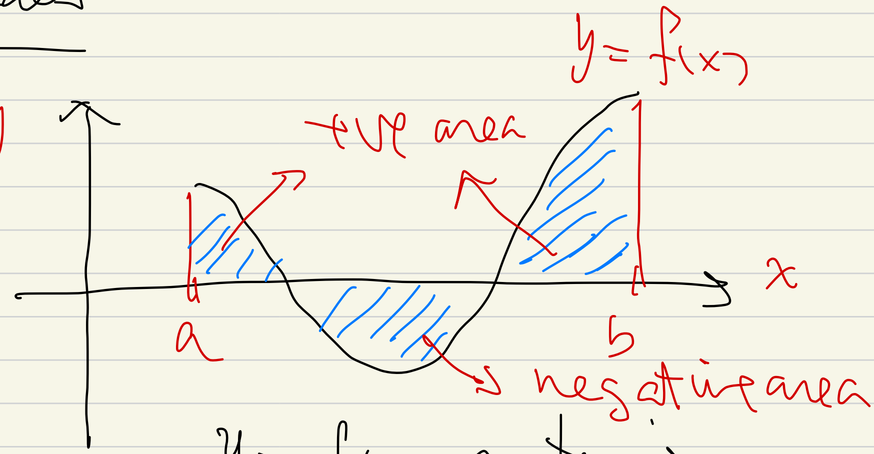
$$= \int 2\cos u\, du$$

$$= 2\sin u + C$$

$$= 2\sin\sqrt{t} + C$$

Definite Integrals

$$\int_a^b f(x)\,dx =$$

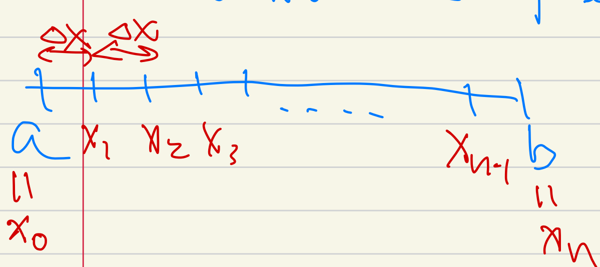


$y = f(x)$ continuous on $[a, b]$.

$=$ Sum of +ve/-ve areas

$$= \lim_{n\to\infty} \underbrace{\left[f(c_1) + f(c_2) + \cdots + f(c_n)\right]}_{\text{a Riemann Sum}} \Delta x, \qquad \Delta x = \frac{b-a}{n}$$

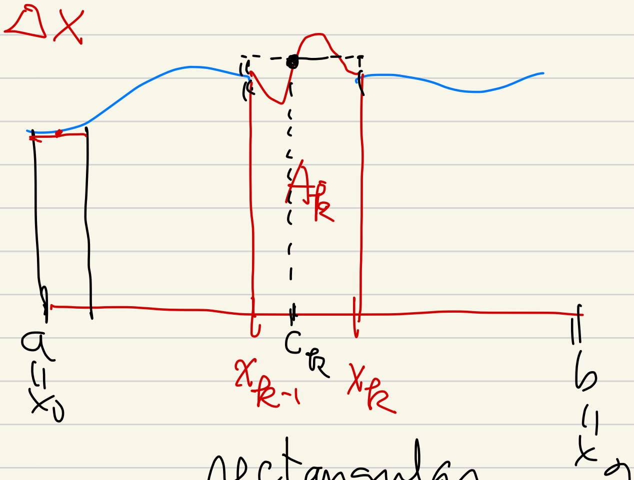
A subdivision of $[a, b]$ into n subintervals of equal length Δx



$$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}$$

$x_1 = x_0 + \Delta x, \quad x_2 = x_0 + 2\Delta x, \ \ldots$

$x_k = x_0 + k\Delta x$

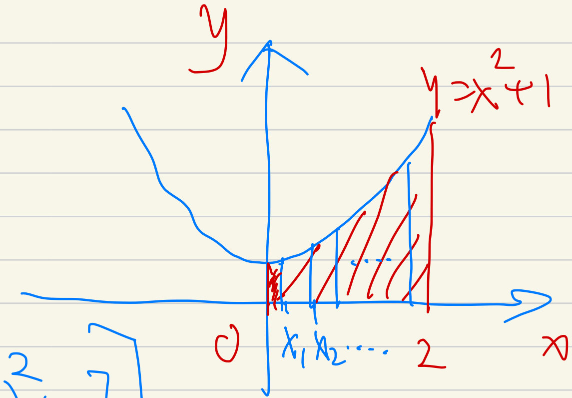


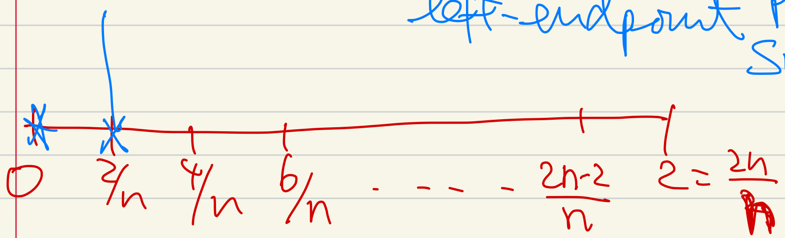
rectangular area $\simeq f(c_i)\Delta x$

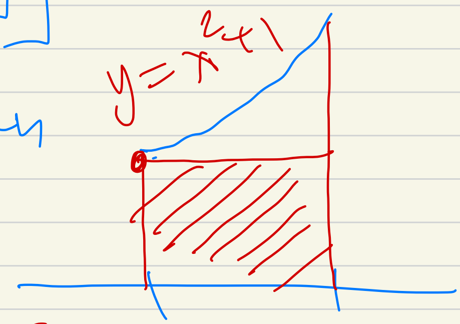
Example $y = x^2 + 1$

$$\int_0^2 (x^2+1)\,dx$$

$$= \lim_{n\to\infty} \frac{2}{n} \underbrace{\left[(0^2+1) + \left(\left(\frac{2}{n}\right)^2+1\right) + \cdots + \left(\left(\frac{2n-2}{n}\right)^2+1\right)\right]}_{\text{left-endpoint Riemann Sum}}$$







$c_k = x_{k-1}$ $\quad x_k$

$$\Delta x = \frac{2-0}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[2^2 \frac{(1^2+2^2+\cdots+(n-1)^2)}{n^2} + n\right]$$

$$f(x_{k-1})\cdot \Delta x$$

$$= \lim_{n\to\infty} \frac{8(1^2+2^2+\cdots+(n-1)^2)}{n^3} + 2 \qquad = (x_{k-1}^2+1)\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8(n-1)(n)(2n-1)}{6n^3} + 2$$

$$x_1 = x_0 + \Delta x = 0 + 2/n$$

$$= \lim_{n\to\infty} \frac{4}{3}\left(1-\cancel{\tfrac{1}{n}}^{\,0}\right)\left(2-\cancel{\tfrac{1}{n}}^{\,0}\right) + 2$$

$$1^2+2^2+\cdots+N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{8}{3} + 2$$

$N = 2$

$1+4=5$, $\quad N = n-1$

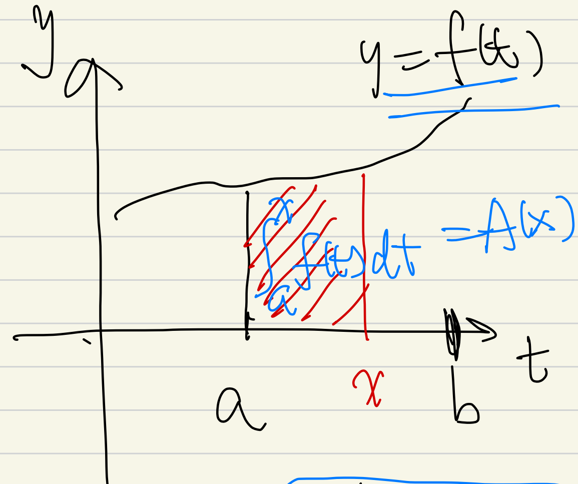
$$\frac{2\cdot 3\cdot(4+1)}{6} = 5$$

Fundamental Theorem of Calculus

1) $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$A'(x) = f(x)$

$\boxed{A(x) = \int f(x)\,dx}$



$\Downarrow$

2) $$\int_a^b f(t)\,dt = F(b) - F(a) \quad \text{if } \boxed{F'(x) = f(x)}$$

$= A(b) - A(a)$

$A(x) = F(x) - F(a)$

$A(x) = F(x) + C$

$0 = A(a) = F(a) + C$

$C = -F(a)$

Example.

$$\int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^2$$

antiderivative of x^2+1

$\frac{d}{dx}(?) = x^2+1$

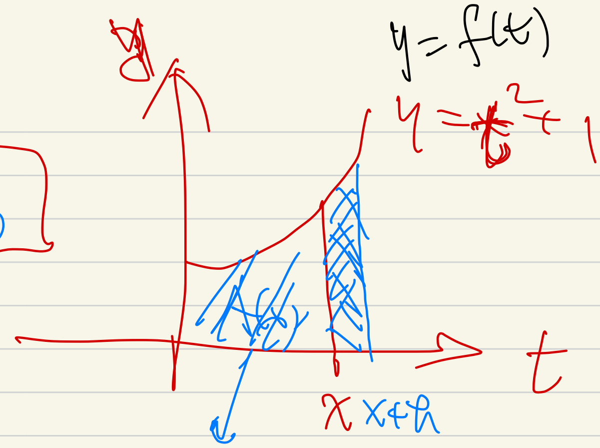
$? = \int (x^2+1)\,dx$

$= \frac{x^3}{3} + x + C$

$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right)$$

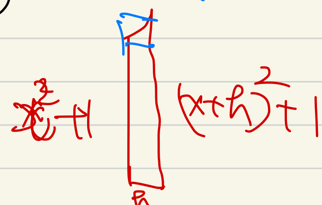
$$= \frac{8}{3} + 2$$

$$A'(x) = \lim_{h\to 0} \frac{\boxed{A(x+h) - A(x)}}{h}$$

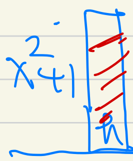


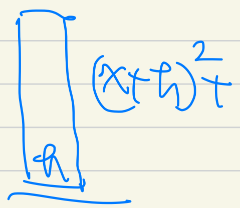
$A(x)$ = area function

$$= \lim_{h\to 0} \frac{\;\;}{h}$$



$$\lim_{h\to 0} \frac{\;\;}{h} \leq A'(x) \leq \lim_{h\to 0} \frac{\;\;}{h}$$





$$\begin{array}{ccccc} \| & & & & \| \\ x^2+1 & \leq & A'(x) & \leq & \lim_{h\to 0}\left[(x+h)^2+1\right] \\ & & & & \| \\ & & & & x^2+1 \end{array}$$

$$A'(x) = x^2+1$$

$$\int_a^b f(x)\,dx = \left[\begin{matrix}\text{antiderivative} \\ \text{of } f\end{matrix}\right]_a^b$$

Example

(1) $$\int_0^{\pi} \sin x \, dx = \Big[-\cos x \Big]_0^{\pi}$$

antiderivative of $\sin x$

$$\frac{d\cos x}{dx} = -\sin x$$

$$= (-\cos\pi) - (-\cos 0)$$

$$= 1 - (-1) = 2.$$

(2) $$\int_0^{\pi} x \sin x^2 \, dx = \Big[\int x \sin x^2 \, dx \Big]_0^{\pi}$$

Let $u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$x=0,\ u=0$

$x=\pi,\ u=\pi^2$

$$\int_0^{\pi^2} \frac{1}{2} \sin u \, du = \Big[-\frac{1}{2}\cos u \Big]_0^{\pi^2}$$

$$= -\frac{1}{2}\cos(\pi^2) + \frac{1}{2}$$

u integration limits

$$= \Big[-\frac{1}{2}\cos x^2 \Big]_0^{\pi}$$

$$\boxed{\frac{d}{dx}\cos x^2 = -\sin x^2 \cdot 2x}$$

$$\int_a^b f(g(x))g'(x)\,dx \overset{\text{substitutions}}{=} \int_{u(a)}^{u(b)} f(u)\,du$$

Example:

1) $$\int_0^5 x e^{x^2}\,dx$$

$u = e^{x^2}$

$\frac{du}{dx} = 2x e^{x^2}$

$\frac{1}{2}\,du = x e^{x^2}\,dx$

$x=0, \quad u=1$

$x=5, \quad u=e^{25}$

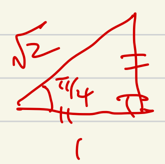
$$= \int_1^{e^{25}} \frac{1}{2}\,du$$

$$= \left[\frac{u}{2}\right]_1^{e^{25}}$$

$$= \frac{e^{25}-e}{2}$$

2) $$\int_0^{\pi/4} \sec x\,dx$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$



$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\underbrace{\sec 0}_{=1} + \underbrace{\tan 0}_{=0}|$$

$$= \ln(\sqrt{2}+1)$$

Example

$$\int \sin^3 x \boxed{\cos x \, dx}$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$$= \int u^3 \, du$$

$$= \frac{u^4}{4} + C = \frac{\sin^4 x}{4} dx$$

Exercise. Try $u = \cos x$!

$$\left\{ \begin{array}{l} u = \sin x \\ u = \cos x \end{array} \right. \Longrightarrow \int \left(\text{Some trigonometric function} \right) dx$$

Next time !