MATH2111 Tutorial 9

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1 Coordinate System

1. **Theorem (The Unique Representation Theorem)**. Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \cdots + c_n\boldsymbol{b}_n$$

2. **Definition**. Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis B (or the B-coordinates of $\mathbf{x}$) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$. And the coordinate vector of $\mathbf{x}$ relative to B, or the B-coordinate vector of $\mathbf{x}$ is written as

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

3. **Change of coordinates matrix**.
Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis of $\mathbb{R}^n$. Let $P_B = [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n]$.
$\boldsymbol{v} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \ldots + c_n\boldsymbol{b}_n$ if and only if

$$\boldsymbol{v} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = P_B[\boldsymbol{v}]_B.$$

P_B is called the **change-of-coordinates matrix** from B to the standard basis in $\mathbb{R}^n$.
Since $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is linearly independent, P_B is invertible. Thus

$$[\boldsymbol{v}]_B = P_B^{-1}\boldsymbol{v}.$$

4. **Theorem (the coordinate mapping)**. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

2 Dimension of a Vector Space

1. **Theorem**. If a vector space V has a basis $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

2. **Theorem**. If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

3. **Definition**. If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is **defined to be zero**. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

4. **Theorem (Basis Extension Theorem).**. Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finitedimensional and
$$\dim H \leq \dim V$$

5. **Theorem (The Basis Theorem)**. Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

6. **Fact**.
   (a) dim Nul A = number of free variables in $A\mathbf{x} = \mathbf{0}$.
   (b) dim Col A = number of pivot colums of A.

7. **Definition**. Let A be an $m \times n$ matrix.
   (a) The dimension of Nul A is called the nullity of A.
   (b) The dimension of Col A is called the column rank of A.
   (c) The dimension of Row A is called the row rank of A.

3 Rank of a Matrix

3.1 Row Space

1. **Definition (Row Space)**. The row space of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A.
$$\text{Row } A = \text{Span}\,\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$$
where $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the row vectors of the matrix A.

2. **Theorem**. The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

3. **Theorem**.
   (a) Row A = Col $A^\top$
   (b) Suppose A is row equivalent to B, then Row A = Row B.

3.2 Rank

1. **Definition**. The **rank** of A is the dimension of the column space of A.

2. **Theorem (Rank Theorem)**. Let A be an $m \times n$ matrix. Suppose A has p pivot positions. Then

   (a) nullity of $A = n - p$

   (b) column rank of $A = p$

   (c) row rank of $A = p$

   Therefore, by defining Rank A = column rank of A = row rank of A, we have

   $$\text{nullity } A + \text{rank } A = n, \text{ the number of columns.}$$

3. **Theorem (Dimension Theorem)**. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then

   $$\dim \ker T + \dim \operatorname{range} T = n$$

The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent.
(a) A is an invertible matrix.
(b) A is row equivalent to the $n \times n$ identity matrix.
(c) A has n pivot positions.
(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A form a linearly independent set.
(f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
(j) There is an $n \times n$ matrix C such that $CA = I$.
(k) There is an $n \times n$ matrix D such that $AD = I$.
(l) A^T is an invertible matrix.
(m) The columns of A form a basis of R^n
(n) $\text{Col } A = \mathbb{R}^n$
(o) $\dim \text{Col } A = n$
(p) $\text{rank } A = n$
(q) $\text{Nul } A = \{\mathbf{0}\}$
(r) $\dim \text{Nul } A = 0$
(s) $\det(A) \neq 0$

4 Exercises

1. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

(1) $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(2) $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

Q1 (1). $c_1\vec{b_1} + c_2\vec{b_2} = \vec{x}$

$$\begin{cases} c_1 + 5c_2 = 4 \\ -2c_1 - 6c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -6 \\ c_2 = 2 \end{cases}$$

$$\therefore [\vec{x}]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

(2) $c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3} = \vec{x}$

i.e. $\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{x}$

$[\vec{x}]_B$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right] \xrightarrow[R_3 - 3R_1 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_2 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3 \to R_2]{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\therefore [\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

2.

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$$

(1). Show that the set $\mathcal{B}$ is a basis of $\mathbb{R}^3$
(2). Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
(3). Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$

Q2. (1) $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 8 \\ 4 & -5 & 7 \end{bmatrix}$

$\xrightarrow[R_3+4R_1 \to R_3]{R_2+3R_1 \to R_2} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 2 \\ 0 & -5 & -1 \end{bmatrix}$

$\xrightarrow[R_3+\frac{5}{2}R_2 \to R_3]{} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is linearly independent.

And there are 3 vectors in B,

Thus, B is a basis of $\mathbb{R}^3$.

(2) $P_B = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$

(3) $\vec{x} = P_B[\vec{x}]_B$

3. For each subspace, find a basis, and state the dimension.

(1) $\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$

(2) $\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

Q3. (1) $\begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} = s\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

basis : $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ ← prove by yourself.

dimension: 2

(2) $\begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} = a\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} + b\begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

Check if basis:

check the linear independence firstly.

$$\begin{bmatrix} 3 & 6 & -1 \\ 6 & 2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3+3R_1\to R_3 \\ R_4+R_1\to R_4}]{R_2-2R_1\to R_2} \begin{bmatrix} 3 & 6 & -1 \\ 0 & -10 & 0 \\ 0 & 23 & 0 \\ 0 & 7 & 0 \end{bmatrix} \xrightarrow[\substack{R_3+\frac{23}{10}R_2\to R_3 \\ R_4+\frac{7}{10}R_2\to R_4}]{} \begin{bmatrix} 3 & 6 & -1 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{ basis : } \left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \right\}$$

dimension: 2

4. Determine the dimensions of Nul A and Col A for the matrices.

(1)

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q4. (1)

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

Two pivots in column 1 and 3, thus $\dim \operatorname{Col} A = 2$.

Two free variables x_2, x_4, thus $\dim \operatorname{Nul} A = 2$.

(2)

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivots in column 1, 3, 4, thus $\dim \operatorname{Col} A = 3$.

Three free variables x_2, x_5, x_6, thus $\dim \operatorname{Nul} A = 3$.

5. Assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q5. Since A, B are row equivalent,

$\text{rank}\,A = \text{rank}\,B = 3$.

By Rank thm, $\dim \text{Nul}\,A = 6 - \text{rank}\,A = 6-3 = 3$.

Looking at B, the pivot columns are column 1, 2, 4,

the pivot rows are row 1, 2, 3,

thus, look back to A,

$$\text{Col}A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$$

$$\text{Row}A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 7 \\ 9 \\ -9 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}^T \right\}.$$

To find Nul A, we need $\text{rref}(A)$.

$A \sim B. \quad \therefore A\vec{x} = \vec{0} \Leftrightarrow B\vec{x} = \vec{0}.$

actually, B is one possible form of $\text{REF}(A) := C$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -3 & 7 & 9 & -9 & 0 \\ 0 & 1 & -1 & 3 & 4 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 - 3R_3 \to R_2]{R_1 - 7R_3 \to R_1} \left[\begin{array}{cccccc|c} 1 & 1 & -3 & 0 & 16 & 5 & 0 \\ 0 & 1 & -1 & 0 & 7 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 9 & 2 & 0 \\ 0 & 1 & -1 & 0 & 7 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\| $$

$$[C \mid \vec{0}]$$

$$\uparrow$$

$$\mathrm{RREF}(A) = \mathrm{RREF}(B)$$

x_3, x_5, x_6 are free.

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $\mathrm{Nul}A = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$