¡b¿ ¡H3¿ Math1012 Calculus IA¡br¿

Homework-7 ¡br¿ Deadline: 11/21/2021 at 11:59pm HKT ¡/H3¿

¡br/¿ ¡/b¿

Give 4 or 5 significant digits for numerical answers.

When entering numerical answers, you can if you wish enter mathematical expressions such as $3^2 or 3**2 instead of 9, sin(3* pi/2) instead of -1, etc.. In other words, WeBWorK can work as a calcu$ [...] $Tompute the value of the expression entered.$

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¡li¿¡a href="https://webwork.math.ust.hk/webwork$_q$uick$_s$tart.htm "Quick$_W$ebwork$_G$uide" > QuickWeBWorKStudentGuide < /a >< /li >< li >< ahref = "https : //www.math.ust.hk/ support" "Math$_S$upport$_C$enter" > MathSupportCenter(Room3011 − 3013) < /a >< /li >< /ul >

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1. (8 points) A right circular cone is to be inscribed in another right circular cone of volume 10m^3 and altitude 8m, with the same axis and with the vertex of the inner cone touching the base of the outer cone.

Draw a picture of the cones.

What must be the altitude of the inscribed cone in order to have the largest possible volume? _________

Correct Answers:

- `2.66666666666667`

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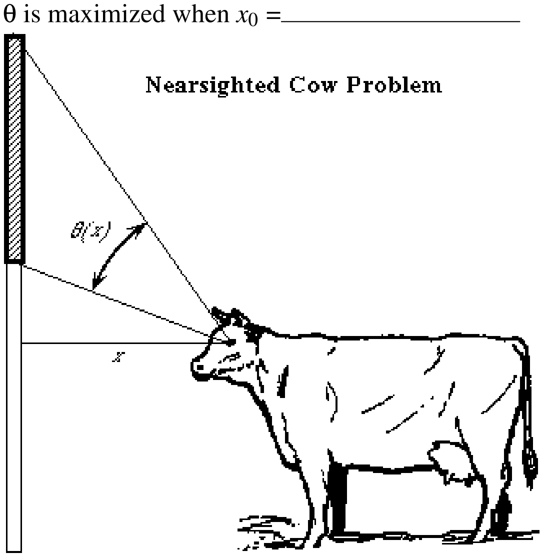
2. (8 points) **The Nearsighted Cow Problem: A Calculus Classic.**

A rectangular billboard 6 feet in height stands in a field so that its bottom is 7 feet above the ground. A nearsighted cow with eye level at 4 feet above the ground stands x feet from the billboard. Express θ, the vertical angle subtended by the billboard at her eye, in terms of x. Then find the distance x_0 the cow must stand from the billboard to maximize θ.

$\theta(x)$ = ___________________

θ is maximized when x_0 = ___________________



(**Click on image for a larger view**)

Correct Answers:

- `atan((6+3)/x)-atan(3/x)`
- `5.19615242270663`

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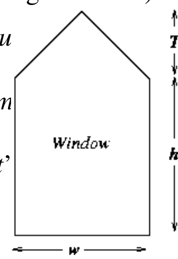
3. (8 points) Consider a window the shape of which is a rectangle of height h surmounted by a triangle having a height T that is 0.5 times the width w of the rectangle (as shown in the figure below).



If the cross-sectional area is A, determine the dimensions of the window which minimize the perimeter.

$h =$ ___________________________

$w =$ ___________________________

Solution:

SOLUTION

The area of the rectangular part of the window is hw, and the area of the triangle is $\frac{1}{2}wT = 0.25w^2$. Thus the cross-sectional area, A is

$$A = hw + 0.25w^2.$$

The perimeter of the window is $P = w + 2h + 2d$, where d is the length of the angled top edges of the triangle. We can find d by looking at half of the triangle and using the Pythagorean theorem: $d = \sqrt{(\frac{1}{2}w)^2 + T^2} = \sqrt{\frac{1}{4}w^2 + 0.5^2w^2} = \sqrt{\frac{1}{4} + 0.5^2}w \approx 0.707w$. Thus

$$P = w + 2h + 2(0.707)w = 2.414w + 2h.$$

From $A = hw + 0.25w^2$ we get $h = \frac{A}{w} - 0.25w$. Thus,

$$P = \frac{2A}{w} - 0.5w + 2.414w = \frac{2A}{w} + 1.914w.$$

We now have the perimeter in terms of w and the constant A. Differentiating, we obtain

$$\frac{dP}{dw} = -\frac{2A}{w^2} + 1.914.$$

To find the critical points we set $P' = 0$ and solve for w, finding $w = \sqrt{2A/1.914} \approx 1.022\sqrt{A}$. Substituting this back into our expression for h, we have $h = \frac{A}{1.022\sqrt{A}} - \frac{1}{2}(0.5)1.022\sqrt{A} \approx 0.723\sqrt{A}$.

Since $P \to \infty$ as $w \to 0^+$ and as $w \to \infty$, this critical point must be a global minimum. We can also see this because it is the only critical point and the second derivative, $P'' = 4A/w^3 > 0$, indicating that P is concave up.

Correct Answers:

- `A/(sqrt(2*A/(1-0.5+2*sqrt(0.5*0.5+1/4)))) - 0.5*(sqrt(2*A/`
- `sqrt(2*A/(1-0.5+2*sqrt(0.5*0.5+1/4)))`

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4. (8 points) A fence 5 feet tall runs parallel to a tall building at a distance of 4 feet from the building. We want to find the the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.

Here are some hints for finding a solution:
Use the angle that the ladder makes with the ground to define the position of the ladder and draw a picture of the ladder leaning against the wall of the building and just touching the top of the fence.

If the ladder makes an angle 0.83 radians with the ground, touches the top of the fence and just reaches the wall, calculate the distance along the ladder from the ground to the top of the fence.

____

The distance along the ladder from the top of the fence to the wall is

____

Using these hints write a function $L(x)$ which gives the total length of a ladder which touches the ground at an angle x, touches the top of the fence and just reaches the wall.
$L(x)$ = ______________.

Use this function to find the length of the shortest ladder which will clear the fence.

The length of the shortest ladder is ______________ feet.

Correct Answers:

- `6.77569784366157`
- `5.92701684763074`
- `5/sin(x)+4/cos(x)`
- `12.7016576360847`

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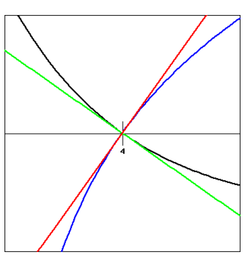
5. (3 points) The functions f and g and their tangent lines at $(4,0)$ are shown in the figure below.



f is shown in blue, g in black, and the tangent line to f is $y = 1.8(x-4)$, and is graphed in red, and the tangent line to g is $y = -0.9(x-4)$, and is graphed in green.

Find the limit
$\lim_{x\to 4} \frac{f(x)}{g(x)} =$ ___

Solution:

SOLUTION

Observe that both $f(4)$ and $g(4)$ are zero. Also, $f'(4) = 1.8$ and $g'(4) = -0.9$, so by l'Hopital's rule,

$$\lim_{x\to 4} \frac{f(x)}{g(x)} = \frac{f'(4)}{g'(4)} = \frac{1.8}{-0.9} = -2.$$

Correct Answers:

- `1.8/-0.9`

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6. (3 points) Evaluate the limit using L'Hospital's rule if necessary

$$\lim_{x\to 0} \frac{x^3}{\sin x - x}$$

Answer: ________

Solution:

SOLUTION

Since $\lim_{x\to 0} x^3 = 0$ and $\lim_{x\to 0}(\sin x - x) = 0$ we can apply l'Hospital's rule:

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(\sin x - x)} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1}$$

Since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}(\cos x - 1) = 0$ we have again an indeterminate form of type $\frac{0}{0}$, so we apply l'Hospital's rule again:

$$\lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x}$$

A third application of l'Hospital's rule gives

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = \frac{6}{-1} = -6$$

Correct Answers:

- `-6`

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7. (3 points) Find the limit: $\lim_{x\to 3} \frac{\ln(x/3)}{x^2-9} =$ ______________

(Enter **undefined** *if the limit does not exist.)*

Solution:

SOLUTION

Let $f(x) = \ln(x/3)$ and $g(x) = x^2 - 9$, so $f(3) = 0$ and $g(3) = 0$ and l'Hopital's rule can be used. To apply l'Hopital's rule, we first find $f'(x) = 1/x$ and $g'(x) = 2x$, then

$$\lim_{x\to 3} \frac{\ln(x/3)}{x^2-9} = \lim_{x\to 3} \frac{1/x}{2x} = \lim_{x\to 3} \frac{1}{2x^2} = \frac{1}{18}.$$

Correct Answers:

- `1/18`

8. (3 points) Evaluate the following limit:

$$\lim_{x\to\infty} 5xe^{\frac{1}{x}} - 5x$$

Enter **-inf** if your answer is $-\infty$, enter **inf** if your answer is ∞, and enter **DNE** if the limit does not exist.

Limit = ____________

Correct Answers:

- `5`

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9. (3 points) Evaluate the limit below, given that

$$f(t) = \left(\frac{2^t + 5^t}{3}\right)^{1/t} \qquad \text{for } t \neq 0.$$

$\lim_{t\to+\infty} f(t) =$ ________________

Solution:

SOLUTION

This limit is of the form ∞^0 so we apply l'Hopital's rule to

$$\ln f(t) = \frac{\ln\left((2^t+5^t)/3\right)}{t}.$$

We have

$$\lim_{t\to+\infty} \ln f(t) = \lim_{t\to+\infty} \frac{\left((\ln 2)2^t + (\ln 5)5^t\right)/\left(2^t+5^t\right)}{1} = \lim_{t\to+\infty} \frac{(\ln 2)2^t + (\ln 5)5}{2^t+5^t}$$

$$\lim_{t\to+\infty} \frac{(\ln 2)(2/5)^t + \ln 5}{(2/5)^t + 1} = \lim_{t\to+\infty} \frac{0+\ln 5}{0+1} = \ln 5.$$

Thus

$$\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} e^{\ln f(t)} = e^{\lim_{t\to+\infty} \ln f(t)} = e^{\ln 5} = 5.$$

Correct Answers:

- `5`

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10. (3 points) If f' is continuous, $f(4) = 0$, and $f'(4) = 13$, evaluate

$$\lim_{x\to 0} \frac{f(4+3x) + f(4+5x)}{x}.$$

Limit: __________

Correct Answers:

- `8*13`