COMP 2711 Discrete Math Tools for Computer Science

2022 Fall Semester - Homework 2

Question 1: Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set, or give a counting method. For those that are uncountable, give a simple proof.

(a) the set $A \times B \times \mathbb{N}$ where $A = \{1, 2\}$, $B = \{3, 4\}$.

(b) all bit strings not containing the bit 1.

(c) all bit strings that is finite.

(d) all bit strings that is infinite

(e) the real numbers containing a finite number of 1s in their decimal representation (reminder: the real numbers can contain a infinite number of 2s or 3s)

(f) the real numbers containing only 1s in their decimal representation

Answer:

(a) The set is countable. We can list its elements in the order $(1,3,0),(1,4,0),(2,3,0),(2,4,0),(1,3,1),(1,4,1),(2,3,1,),(2,4,1),$ $(1,3,2),\cdots$, giving us the one-to-one correspondence $1 \leftrightarrow (1,3,0)$, $2 \leftrightarrow (1,4,0), 3 \leftrightarrow (2,3,0), 4 \leftrightarrow (2,4,0), 5 \leftrightarrow (1,3,1), 6 \leftrightarrow (1,4,1),$ $7 \leftrightarrow (2,3,1), 8 \leftrightarrow (2,4,1), 9 \leftrightarrow (1,3,2), \cdots$

(b) The bit strings not containing 1 are just the bit strings consisting of all 0s, so this set is $\{\lambda, 0, 00, 000, 0000, \cdots\}$, where λ denotes the empty string (the string of length 0). Thus this set is countably infinite, where the correspondence matches the positive integer n with the string of $n-1$ 0's.

(c) This set is countably infinite, since any integer can be expressed in binary. The one-to-one correspondence is just the conversion between decimal number and binary number. Decimal number $\{0, 1, 2, 3, 4, \cdots\}$ can be converted to $\{0, 1, 10, 11, 100, \cdots\}$. Thus the correspondence matches the positive integer n with the binary format of $n-1$

(d) This set is uncountable. Two ways to prove:

  (i) Apply the diagonalization argument in lecture note.

  (ii) Forming an infinite bit string is the same as selecting numbers from $\mathbb{N}$. Each digit can either be 1 or 0. Therefore, when we label the digits starting from 0 in order, we can consider a digit 1 as selecting the labeled number from $\mathbb{N}$. So an infinite bit string is corresponding to a subset of $\mathbb{N}$. We denote the set of all infinite bit strings to be S, $\|S\| = \|P(\mathbb{N})\|$. We know $P(\mathbb{N})$ is uncountable from the lecture note. Thus, S is uncountable.

(e) This set is uncountable, as can be shown by applying the diagonalization argument in lecture note.

(f) This set is countable. We can list its elements in the order $1, 11, 1.1, 111, 1.11, 11.1, 1111, 1.111, 11.11, 111.1, \cdots$, giving us the one-to-one correspondence $1 \leftrightarrow 1, 2 \leftrightarrow 11, 3 \leftrightarrow 1.1, 4 \leftrightarrow 111, 5 \leftrightarrow 1.11, 6 \leftrightarrow 11.1, 7 \leftrightarrow 1111, 8 \leftrightarrow 1.111, 9 \leftrightarrow 11.11, 10 \leftrightarrow 111.1, \cdots$

Question 2: In COMP2711 T5, there are totally 50 students. For HW1, each grade (A, B, C, D, F(didn't submit)) has 10 students. TA wants to select students from COMP2711 T5 based on the grade of HW1.

(a) There are 4 different tasks, each needs one student for help. A student might be assigned to multiple tasks. Task 1 and Task 2 only accept students with A grade or B grade. Students with F grade can only be assigned to Task 4 and cannot be assigned to any other tasks. Among these 4 tasks, at least one task should be done by student with C grade. How many different ways to assign students in these 4 tasks?

(b) TA wants to randomly select students from COMP2711 T5, how many students must he select to be sure of having at least four students with A grade?

(c) TA wants to randomly select students from COMP2711 T5, how many students must he select to be sure of having at least four students with the same grade?

Answer: (a) Task 1 and 2 can only be done by students with grade A or B. Each has $10 + 10$ ways to choose students. Task 3 can be done by any student who did not get F, which has $50 - 10$ different choices. Task 4 can be done by students with any grade, which has 50 different choices. For the four tasks, we have $(10+10)\cdot(10+10)\cdot(50-10)\cdot(50)$ possible ways to choose. However, since at least one task should be done by student with C grade. We need to deduct the cases where no task is done by student with C grade. A student with C grade will only be assigned to Task 3 or Task 4. Thus we still have $10+10$ ways for Task 1 and Task 2 to choose students. When students with C grade are missing in Task 3 and Task 4, there are correspondingly $50 - 10 - 10$ and $50 - 10$ cases. When no task is done by student with C grade, there are $(10+10)\cdot(10+10)\cdot(50-10-10)\cdot(50-10)$ ways to choose. Therefore the answer is $(10 + 10) \cdot (10 + 10) \cdot (50 - 10) \cdot (50) - (10+10) \cdot (10+10) \cdot (50 - 10 - 10) \cdot (50 - 10) = 320,000$.

(b) He needs to select 44 students in order to insure at least four students with grade A. If he does so, then at most 40 of them got non-A grades, so at least four got A grade. On the other hand, if he selects 43 or fewer students, then 40 of them may got non-A grades, and he might not get four students with A grade. This time the number of students did matter.

(c) There are five different grades: these are the pigeonholes. We want to know the least number of pigeons needed to ensure that at least one of the pigeonholes contains four pigeons. By the generalized pigeonhole principle, the answer is 16. If 16 students are selected, at least $\lceil 16/5 \rceil = 4$ students must got the same grade. On the other hand 15 students is not enough, because 3 might got A, 3 might got B, 3 might got C, 3 might got D, and 3 might got F.

Question 3: In how many ways can a dozen books be placed on four distinguishable shelves

(a) if the books are indistinguishable copies of the same title?

(b) if no two books are the same, and the positions of the books on the shelves matter? [Hint: Place the books on the shelves one by one. How many ways can you put each book?]

Answer: (a) All that matters is the number of books on each shelf, so the answer is the number of solutions to $x_1+x_2+x_3+x_4 = 12$, where x_i is being viewed as the number of books on shelf i. The answer is therefore $\binom{4+12-1}{12} = \binom{15}{12} = 455$.

(b) No generality is lost if we number the books $b_1, b_2, \ldots, b_{12}$ and think of placing book b_1 , then placing b_2 , and so on. There are clearly 4 ways to place b_1 , since we can put it as the first book (for now) on any of the shelves. After b_1 is placed, there are 5 ways to place b_2 , since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are 6 ways to place b_3 (to the right of b_1 , to the right of b_2 , or as the first book on any of the shelves), 7 ways to place $b_4, \ldots, 15$ ways to place b_{12}. Therefore the answer is the product of these numbers $4 \cdot 5 \cdots 15 = 217,945,728,000$.

Question 4: This semester for COMP2711, there are 81 students in L1, 86 students in L2, 74 students in L3. Assume that

(a) in L3, students may also take COMP2011, MATH1012, or COMP1021. Notice that COMP2011 and COMP1021 cannot take together. There are exactly 16 students that take none of these three courses. There are 26 students take COMP2011, 38 take MATH1012, 24 take COMP1021. And there are 10 students take both COMP2011 and MATH1012. How many students take both MATH1012 and COMP1021?

(b) the instructors want to recruit student helpers. There are 15 different positions, and they need to hire 3 students from L1, 8 students from L2, 4 students from L3. How many different ways for instructors to assign students to these 15 positions? (No calculation required)

(c) the instructors want to recruit a team of 90 student helpers. How many different student helper team can be formed if they require that a student helper team must contain at least one student from each lectures? (No calculation required)

(d) we already have 6 student helpers working on different tasks. After midterm, instructors want to rearrange the student helpers so that only two of them keep working on their original tasks, other student helpers will be swapped to do different tasks. How many ways can the instructors make the rearrangement?

Answer: (a) Denote the set all students in L3 to be U, students take COMP2011 to be A, students take MATH1012 to be B, and students take COMP1012 to be C. We know that $|U| = 74, |\overline{A} \cap \overline{B} \cap \overline{C}| = 16, |A| = 26, |B| = 38, |C| = 24, |A \cap B| = 10, |A \cap C| = 0$.
Since $|A \cap C| = 0$, $|A \cap B \cap C| = 0$
$|\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}| = 16$
From definition, $|A \cup B \cup C| = |U| - |\overline{A \cup B \cup C}| = 74 - 16 = 58$ By Inclusion-Exclusion Principle,
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|B \cap C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| - |A \cup B \cup C|$
$|B \cap C| = 26 + 38 + 24 - 10 - 0 + 0 - 58 = 20$

(b) One way to solve this problem is as follows.
We have $\binom{81}{3}$ ways to choose students from L1, $\binom{86}{8}$ ways to choose students from L2, and $\binom{74}{4}$ ways to choose students from L3.
The number of ways to order the chosen 15 students is $P(15, 15)$.
Therefore, there are $\binom{81}{3} \cdot \binom{86}{8} \cdot \binom{74}{4} \cdot P(15, 15)$ ways to assign students.

(c) There are $\binom{81+86+74}{90} = \binom{241}{90}$ ways to form a student helper team without restrictions.
They can be divided into 3 cases:

(i) All the students are from the same lecture
(ii) The students are from exactly two different lectures
(iii) The students are from exactly three different lectures

The answer is $\binom{241}{90}$ - case(i) - case(ii). case(i) is impossible because no lecture has 90 or more students (Pigeonhole Principle).
case(ii) can be divided into 3 subcases:

- L1 students are not in the team:

$$\binom{86+74}{90} = \binom{160}{90} \tag{1}$$

- L2 students are not in the team:

$$\binom{81+74}{90} = \binom{155}{90} \tag{2}$$

- L3 students are not in the team:

$$\binom{81+86}{90} = \binom{167}{90} \tag{3}$$

Therefore, the answer is

$$\binom{241}{90} - \binom{160}{90} - \binom{155}{90} - \binom{167}{90} \tag{4}$$

(d) We first select two students from the 6 team members that will keep working on their original tasks, which is $\binom{6}{2} = 15$. Then for the remaining 4 students, there are D_4 ways of rearrangement.

$$D_4 = 4! \cdot [1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}] = 9 \tag{5}$$

So the answer is $\binom{6}{2} \cdot D_4 = 15 \times 9 = 135$ ways.

Question 5: Prove the identity $\binom{n}{r}\binom{r}{a} = \binom{n}{a}\binom{n-a}{r-a}$, whenever n and r are nonnegative integers with $n \geq r \geq a$,

(a) using a combinatorial argument.

(b) using an argument based on the formula for the number of r-combinations of a set with n elements.

Answer: (a) Suppose that we have a set of n elements, and we wish to choose a subset S with a elements and another disjoint subset with $r - a$ elements.
The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of <u>these</u> elements are to go into the first of the subsets.
The right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

(b) On the one hand,

$$\binom{n}{r}\binom{r}{a} = \frac{n!}{r!(n-r)!} \cdot \frac{r!}{a!(r-a)!} = \frac{n!}{a!(n-r)!(r-a)!},$$

and on the other hand

$$\binom{n}{a}\binom{n-a}{r-a} = \frac{n!}{a!(n-a)!} \cdot \frac{(n-a)!}{(r-a)!(n-r)!} = \frac{n!}{a!(n-r)!(r-a)!}$$