Integration By Parts

$$\int u v' \, dx = uv - \int v u' \, dx$$

or $$\int u \, dv = uv - \int v \, du$$

① $$\int \sin^n \theta \, d\theta = -\frac{\sin^{n-1}\theta \cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta \, d\theta$$

$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\longrightarrow \cdots \longrightarrow$ $\int \sin\theta \, d\theta$, $\int 1 \, d\theta$

② $$\int \cos^n \theta \, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta \, d\theta$$

③ $$\int \sec^n \theta \, d\theta = \frac{\sec^{n-2}\theta \tan\theta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\theta \, d\theta$$

$$\frac{d}{d\theta}\sec^{n-2}\theta = (n-2)\sec^{n-3}\theta \underline{\frac{d\sec\theta}{d\theta}}$$

$$\int \sec^n\theta \, d\theta = \int \sec^{n-2}\theta \boxed{\sec^2\theta \, d\theta}$$

$$d\tan\theta = \sec^2\theta \, d\theta$$

$$= \int \sec^{n-2}\theta \, d\tan\theta$$

$$= \sec^{n-2}\theta\tan\theta - \int \tan\theta \, d\sec^{n-2}\theta$$

$$= \sec^{n-2}\theta\tan\theta - \int \tan\theta \cdot (n-2)\sec^{n-3}\theta \cdot \sec\theta\tan\theta \, d\theta$$

$$= \sec^{n-2}\theta\tan\theta - (n-2)\int \sec^{n-2}\theta \cdot \underbrace{\tan^2\theta}_{(\sec^2\theta - 1)} \, d\theta$$

$$\boxed{\int \sec^n\theta \, d\theta} = \sec^{n-2}\theta\tan\theta - (n-2)\boxed{\int \sec^n\theta \, d\theta}$$

$$+ (n-2)\int \sec^{n-2}\theta \, d\theta$$

$$\cancel{(n-1)}\int \sec^n\theta \, d\theta = \frac{\sec^{n-2}\theta\tan\theta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\theta \, d\theta$$

$$\cancel{n-1}$$

$$\rightarrow \cdots \rightarrow \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\rightarrow \int \sec^2\theta \, d\theta = \tan\theta + C$$

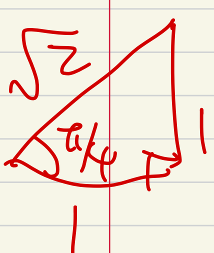
Example

① $\displaystyle\int \sec^4\theta \, d\theta$

$$= \frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}\int \sec^2\theta \, d\theta$$

$$= \frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}\tan\theta + C$$

② $\displaystyle\int_0^{\pi/4} \sec^5\theta \, d\theta = \left.\frac{\sec^3\theta\tan\theta}{4}\right|_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4} \sec^3\theta \, d\theta$



$$= \frac{2\sqrt{2}}{4} + \frac{3}{4}\left[\left.\frac{\sec\theta\tan\theta}{2}\right|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\theta \, d\theta\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right]$$

Exercise:

$u = 2\theta$

$$\int \cos^6 2\theta \, d\theta \longleftrightarrow \int \frac{1}{2} \cos^6 u \, du$$

$= ?$

Example

$$I = \frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x + C$$

a, b, a^2+b^2

$\parallel$

$$I = \int e^{2x} \sin 3x \, dx$$

$u = \sin 3x, \; v' = e^{2x}$

$u = e^{2x}, \; v' = \sin 3x$

$$= \int \sin 3x \, d\left(\frac{e^{2x}}{2}\right)$$

$$= \frac{e^{2x}}{2} \sin 3x - \int \frac{e^{2x}}{2} \, d \sin 3x$$

$3 \cos 3x \, dx$

$\boxed{\cos 3x \, d \frac{e^{2x}}{2}}$

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{2} \int \boxed{e^{2x} \cos 3x \, dx}$$

$$= \frac{e^{2x}}{2} \sin 3x - \frac{3}{2} \left[\frac{e^{2x}}{2} \cos 3x - \int \frac{e^{2x}}{2} \, \underbrace{d \cos 3x}_{-3 \sin 3x \, dx} \right]$$

$$I = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} I$$

$$\underbrace{\left(1 + \frac{9}{4}\right)}_{\frac{13}{4}} I = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x + C_1$$

$\times \frac{4}{13}$

Example.

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

$\nearrow$ $u = 2\sin\theta$!

$\searrow$ integration by parts !

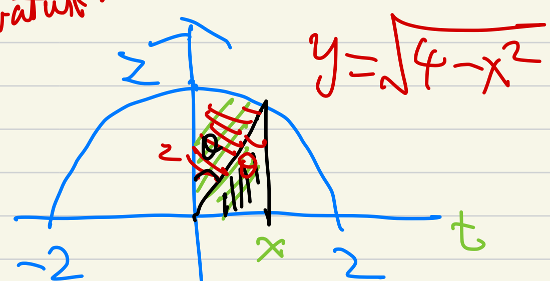
$$= \int -x \, d\,(4-x^2)^{\frac{1}{2}}$$

$$d\,(4-\cancel{x^2})^{1/2} = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x)\,dx = -\frac{x}{\sqrt{4-x^2}}\,dx$$

$$= -x(4-x^2)^{\frac{1}{2}} + \int \underline{\underline{(4-x^2)^{\frac{1}{2}}}}\,dx$$

$$= -x(4-x^2)^{1/2} + \int_0^x \sqrt{4-t^2}\,dt + C$$

area function as an antiderivative!



$$\boxed{\frac{d}{dx}\int_0^x \sqrt{4-t^2}\,dt} = \sqrt{4-x^2}$$

$$= -x(4-x^2)^{1/2} + \frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\cdot 2^2\cdot \sin^{-1}\frac{x}{2} + C$$

Example.

$$\int x \sin^{-1} x \, dx$$

$$\boxed{\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$$

$$= \int \sin^{-1} x \cdot d\frac{x^2}{2}$$

$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2}{2}\, d\sin^{-1} x$$

$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2}{2}\frac{1}{\sqrt{1-x^2}}\, dx$$

$$x = \sin\theta$$

$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta}{2\cancel{\cos\theta}} \cdot \cancel{\cos\theta}\, d\theta$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left[\frac{-\sin\theta\cos\theta}{2} + \frac{1}{2}\int 1\, d\theta\right]$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{4}\sin\theta\cos\theta - \frac{1}{4}\theta + C$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{4}\cdot x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1} x + C$$

Method of "Partial Fractions"

$\Leftrightarrow$ for integrating rational functions

Example $\displaystyle\int \frac{2x+4}{(x-1)(x-3)}\,dx$

what if $= \displaystyle\int \left(\frac{A}{x-1} + \frac{B}{x-3} \right) dx$ for some constants A, B

$A = -3$, $B = 5$

$= A \displaystyle\int \frac{dx}{x-1} + B \int \frac{dx}{x-3}$

$= A \ln|x-1| + B \ln|x-3| + C$

It's now an algebra problem.

$$\frac{2x+4}{(x-1)(x-3)} \stackrel{?}{=} \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)}$$

i.e. $2x+4 = A(x-3) + B(x-1)$

Just plug in $x=1$, $2+4 = A(1-3) + B\cdot 0$

$A = -6/2 = -3$

Let $x=3$, $6+4 = A\cdot 0 + B(3-1)$, $B = \frac{10}{2} = 5$

$$\int \frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)}\,dx$$

$p(x) \rightarrow$ a polynomial of degree $< n$

$\boxed{\text{where } a_1, \dots, a_n \text{ are distinct numbers.}}$

$\parallel$ if you can break up the rational function

$$\int \left(\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}\right) dx \Longleftrightarrow \text{easy!}$$

Example:

$$\int \frac{x^2+2x-5}{(x-1)(x-3)(x+2)}\,dx$$

$$= \int \left(\frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x+2}\right) dx = A\ln|x-1| + B\ln|x-3| + C\ln|x+2| + C$$

$1/3$ $\quad 1 \quad \frac{1}{3}$

$$\frac{x^2+2x-5}{(x-1)(x-3)(x+2)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$x^2+2x-5 = A(x-3)(x+2) + B(x-1)(x+2) + C(x-1)(x-3)$$

$x=1 \Rightarrow 1+2-5 = A\cdot(-2)(3)$, $A = \frac{-2}{-6} = \frac{1}{3}$

$x=3 \Rightarrow 9+6-5 = B\cdot 2\cdot 5$, $B = \frac{10}{10} = 1$

$x=-2$ $\quad 4-4-5 = C(-3)(-5)$, $C = \frac{-5}{+15} = -\frac{1}{3}$