COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Fall Semester – Tutorial 5

Question 1: How many permutations of the 10 digits (0 through 9) either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

Solution : We need to use inclusion-exclusion with three sets. There are 7! permutations that begin 987, since there are 7 digits free to be permuted among the last 7 spaces (we are assuming that it is meant that the permutations are to start with 987 in that order, not with 897, for instance). Similarly, there are 8! permutations that have 45 in the fifth and sixth positions, and there are 7! that end with 123. (We assume that the intent is that these digits are to appear in the order given.) There are 5! permutations that begin with 987 and have 45 in the fifth and sixth positions; 4! that begin with 987 and end with 123; and 5! that have 45 in the fifth and sixth positions and end with 123. Finally, there are 2! permutations that begin with 987, have 45 in the fifth and sixth positions, and end with 123 (since only the 0 and the 6 are left to place). Therefore the total number of permutations meeting any of these conditions is $7! + 8! + 7! - 5! - 4! - 5! + 2! = 50,138$.

Question 2: There are ten groups and each group has two people. They sit down in a row of twenty seats. How many ways are there that at least one group sits together? Your answer could be a summation of at most ten terms.

Solution : Let E_i denote the set of ways that the i-th group sits together. The number of ways that at least one group sits together is then $|\bigcup_{i=1}^{10} E_i|$.

By the inclusion-exclusion principle, we have

$$|\bigcup_{i=1}^{10} E_i| = \sum_{k=1}^{10} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 10} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|.$$

Note that, $\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 10} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}| = \binom{10}{k} |E_1 \cap E_2 \cap \cdots \cap E_k|$. $|E_1 \cap E_2 \cap \cdots \cap E_k|$ means that the first k groups must sit together. So,

$$|E_1 \cap E_2 \cap \cdots \cap E_k| = (20-k)!2^k$$

Substituting back, we have

$$\bigcup_{i=1}^{10} |E_i| = \sum_{k=1}^{10} (-1)^{k+1} \binom{10}{k} (20-k)!2^k.$$

Question 3: This problem was posed by the Chevalier de Méré and was solved by Blaise Pascal and Pierre de Fermat.

(a) Find the probability of rolling at least one six when a fair die is rolled four times.

(b) Find the probability that a double six comes up at least once when a pair of dice is rolled 24 times. Answer the query the Chevalier de Méré made to Pascal asking whether this probability was greater than 1/2.

(c) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of dice is rolled 24 times?

Solution : (a) There are 6^4 possible outcomes when a die is rolled four times. There are 5^4 outcomes in which a 6 does not appear, so the probability of not rolling a 6 is $5^4/6^4$. Therefore the probability that at least one 6 does appear is $1 - 5^4/6^4 = 671/1296$, which is about 0.518.

(b) There are 36^{24} possible outcomes when a pair of dice is rolled 24 times. There are 35^{24} outcomes in which a double 6 does not appear, so the probability of not rolling a double 6 is $35^{24}/36^{24}$. Therefore the probability that at least one double 6 does appear is $1 - 35^{24}/36^{24}$, which is about 0.491. No, the probability is not greater than 1/2.

(c) From our answers above we see that the former is more likely, since $0.518 > 0.491$.

Question 4: There is a type of cereal that contains a toy in each box. There are 10 types of toys, $T_1, T_2, \ldots, T_{10}$ and every box has probability $\frac{1}{10}$ of containing each possible toy, independently of every other box.

Your little brother buys a box of cereal each week for 20 weeks and keeps all of the toys that he finds.

(a) What is the probability that your brother has at least one copy of toy T_1 after 20 weeks?

(b) What is the probability that after 20 weeks your brother has collected *all* of the 10 different toys?

Solution : (a)

$$\begin{aligned}
& p(\text{at least one copy of } T_1 \text{ in 20 weeks}) \\
= \; & 1 - p(\text{no copy of } T_1 \text{ in 20 weeks}) \\
= \; & 1 - p(\text{no } T_1 \text{ in week 1 AND no } T_1 \text{ in week 2 AND } \ldots \text{no } T_1 \text{ in week 20}) \\
= \; & 1 - p(\text{no } T_1 \text{ in week 1})p(\text{no } T_1 \text{ in week 2}) \cdots p(\text{no } T_1 \text{ in week 20}) \\
= \; & 1 - \left(1 - \frac{1}{10}\right)^{20}
\end{aligned}$$

(b) Let E_i be the event that he has no copy of toy T_i. Then $A = \bigcup_{i=1}^{10} E_i$ is the event that there is at least one toy that he has no copy of.

- $p(\text{all 10 types of toys in 20 weeks}) = 1 - p(A)$
- Use inclusion-exclusion principal to calculate $p(A)$

Recall the Inclusion-Exclusion formula:

$$p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$$

$p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$ is the probability that all of the toys he finds are one of the $10 - k$ unspecified ones,

$$p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \left(\frac{10-k}{10}\right)^{20}$$

There are $\binom{10}{k}$ ways of choosing such k-tuples, so

$$\sum_{\substack{i_1, i_2, \ldots, i_k \\ 1 \le i_1 < i_2 < \cdots < i_k \le 10}} p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \binom{10}{k} \left(\frac{10-k}{10}\right)^{20}$$

Therefore,

$$\begin{aligned}
p(A) &= \sum_{k=1}^{10} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le 10}} p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) \\
&= \sum_{k=1}^{10} (-1)^{k+1} \binom{10}{k} \left(\frac{10-k}{10}\right)^{20}
\end{aligned}$$

Therefore, the answer is $1 - \sum_{k=1}^{10} (-1)^{k+1} \binom{10}{k} \left(\frac{10-k}{10}\right)^{20}$.