More on Using Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin A$$

$$2 \sin A \cos B = \sin(A-B) + \sin(A+B)$$

Product to Sum

Example

$$\int \sin 4x \cos 2x \, dx$$

$$= \int \frac{1}{2}\left[\sin(4x-2x) + \sin(4x+2x)\right] dx$$

$$= \frac{1}{2}\int (\sin 2x + \sin 6x)\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2} - \frac{\cos 6x}{6}\right] + C$$

$\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$ can be handled similarly by these identities!

Example

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

(1) $\displaystyle\int \cos 4x \cos 3x \, dx$

$$= \frac{1}{2}\int \left[\cos(\underbrace{4x-3x}_{x}) + \cos(\underbrace{4x+3x}_{7x})\right] dx$$

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$= \frac{1}{2}\left[\sin x + \frac{\sin 7x}{7}\right] + C$$

(2) $\displaystyle\int \sin 7x \sin 5x \, dx$

$$= \frac{1}{2}\int \left[\cos(\underbrace{7x-5x}_{2x}) - \cos(\underbrace{7x+5x}_{12x})\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 12x}{12}\right] + C$$

Example,

period = 2π

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$\boxed{\cos mx \;\; \cos nx}$ same

where m, n are +ve integers

If $m = n$, $\displaystyle\int_{-\pi}^{\pi} \sin^2 mx \, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2mx}{2} \, dx$

double angle formula

$\cos 2mx = 1 - \sin^2 mx$

$$= \left[\frac{x}{2} - \frac{\sin 2mx}{2 \cdot 2m} \right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

If $m \neq n$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left[\cos(m-n)x - \cos(m+n)x \right] dx$$

$$= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi}$$

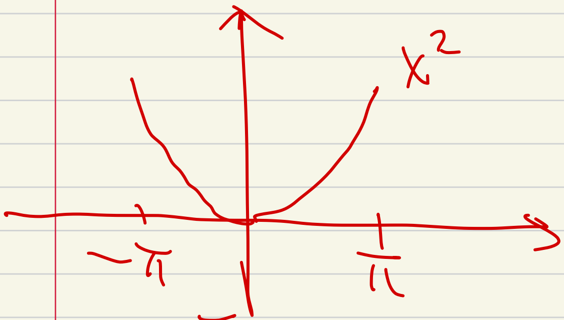
$$= 0$$

Also $\displaystyle\int_{-\pi}^{\pi} \boxed{\sin nx \cos mx \, dx} = 0$ for all n, m (+ve integers)

$$\frac{\sin(n-m)x + \sin(n+m)x}{2}$$

integrate and check it!!

"Fourier Series"



$$x^2 = a_0 + a_1\cos x + a_2\cos 2x + \cdots$$
$$+ b_1\sin x + b_2\sin 2x + \cdots$$

What is a_2 ?

$$\int_{-\pi}^{\pi} x^2\cos 2x\,dx = \int_{-\pi}^{\pi}\Big(a_0\cos 2x + a_1\cos x\cos 2x + a_2\cos^2 2x + a_3\cos 2x\cos 3x + \cdots$$
$$+ b_1\sin x\cos 2x + b_2\sin 2x\cos 2x + \cdots\Big)dx$$

$$= a_2\int_{-\pi}^{\pi}\cos^2 2x\,dx$$

$$\underset{\uparrow\, n}{a_2} = \frac{\int_{-\pi}^{\pi} x^2\cos \overset{n}{2}x\,dx}{\int_{-\pi}^{\pi}\cos^2 2x\,dx}$$

$$b_n = \frac{\int_{-\pi}^{\pi} x^2\sin nx\,dx}{\int_{-\pi}^{\pi}\sin^2 nx\,dx}$$

$\sin^2\theta + \cos^2\theta = 1$, $\boxed{1+\tan^2\theta} = \sec^2\theta$

Could be helpful for integrals involving a^2-x^2, a^2+x^2, or x^2-a^2

Example.

$$\int \frac{1}{\sqrt{4-x^2}} \boxed{dx}$$

What if $4-x^2 = 4-4\sin^2\theta$?

Let $x = 2\sin\theta$, $\frac{dx}{d\theta} = 2\cos\theta$, $dx = 2\cos\theta\, d\theta$

$\sin\theta = \frac{x}{2}$

$$= \int \frac{2\cos\theta\, d\theta}{\sqrt{4(1-\sin^2\theta)}}$$

$$= \int 1 \cdot \frac{\cancel{2\cos\theta}}{\cancel{2\cos\theta}}\, d\theta$$

$$= \int 1\, d\theta = \theta + C = \sin^{-1}\frac{x}{2} + C$$

Example

$$\int \frac{dx}{4+x^2}$$

" $4+x^2 = 4+4\tan^2\theta$ "

Let $x = 2\tan\theta$, $\tan\theta$ " $= x/2$ "

$\frac{dx}{d\theta} = 2\sec^2\theta$

$$= \int \frac{2\,\cancel{\sec^2\theta}}{4\underbrace{(1+\tan^2\theta)}_{\cancel{\sec^2\theta}}}\, d\theta = \int \frac{1}{2}\, d\theta$$

$$= \frac{1}{2}\theta + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

Example: $\leftarrow$ Corrected after class

Get rid of the $\sqrt{}$!

$$\int \underline{x^2} \underbrace{\sqrt{9-x^2}}_{\sqrt{9\cos^2\theta}} \, dx$$

$9 - x^2 = 9 - 9\sin^2\theta$

$\boxed{x = 3\sin\theta}$

$\frac{dx}{d\theta} = 3\cos\theta$

$$= \int 3^2 \sin^2\theta \sqrt{9\cos^2\theta} \cdot 3\cos\theta \, d\theta$$

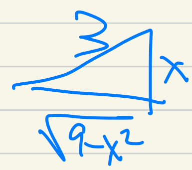
$$= 81 \int \sin^2\theta \cos^2\theta \, d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{81}{4} \int \sin^2 2\theta \, d\theta$$

$\sin\theta = \frac{x}{3}$

$$= \frac{81}{4} \int \frac{1+\cos 4\theta}{2} \, d\theta$$



$$= \frac{81}{8}\left[\theta + \frac{\sin 4\theta}{2}\right] + C$$

$$= \frac{81}{8}\left[\sin^{-1}\frac{x}{3} + \frac{\cancel{2}\sin 2\theta\cos 2\theta}{\cancel{2}}\right] + C$$

$$= \frac{81}{8}\left[\sin^{-1}\frac{x}{3} + 2\sin\theta\cos\theta\left[2\cos^2\theta - 1\right]\right] + C$$

$$= \frac{81}{8}\sin^{-1}\frac{x}{3} + \frac{81}{4} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3}\left(\frac{2(9-x^2)}{9} - 1\right) + C$$

Example

$1+\tan^2\theta = \sec^2\theta$

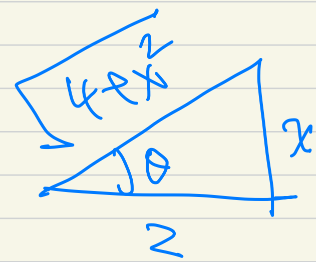
Trigonometric Substitution

$$\int \frac{dx}{(4+x^2)^{3/2}}$$

$x = 2\tan\theta$

$\frac{dx}{d\theta} = 2\sec^2\theta$

$$= \int \frac{2\sec^2\theta \, d\theta}{(4\sec^2\theta)^{3/2}}$$



$$= \int \frac{2\sec^2\theta \, d\theta}{8\sec^3\theta}$$

$\tan\theta = \frac{x}{2}$

$$= \frac{1}{4}\int \cos\theta \, d\theta = \frac{1}{4}\sin\theta + C$$

$$= \frac{x}{4\sqrt{4+x^2}} + C$$