Dynamics of Rigid Bodies III

PHYS1112

Lecture 10

Intended Learning Outcomes

- After this lecture you will learn:
  1) how to deal with a rigid body rotating about a moving axis, e.g. a yo-yo
  2) the ideal case of rolling without slipping.
  3) rolling friction in realistic cases.
  4) work and power in rotation motion.

We know how to deal with:

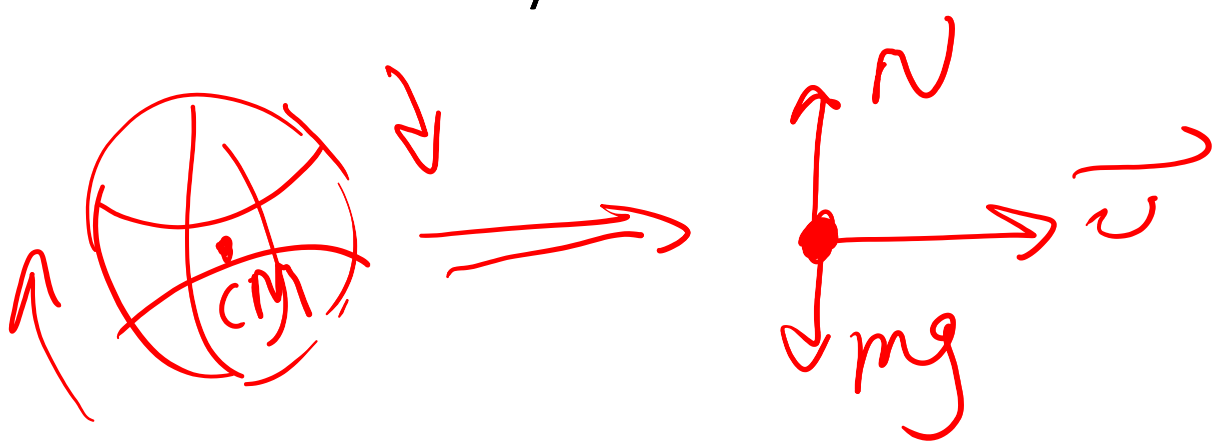
translation of a point particle (or CM of a rigid body):

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = m\vec{\boldsymbol{a}}$$

rotation of a rigid body about a *fixed* axis:

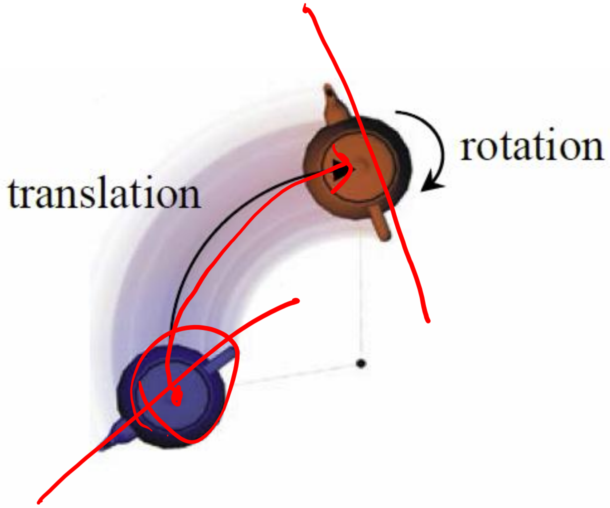
$$\sum \vec{\tau}_{\text{ext}} = I\vec{\alpha}$$

In general, a rigid body is rotating about a *moving* axis, i.e., has both types of motion simultaneously.



Theorem: (stated without proof here)
Every possible motion of a rigid body can be represented as

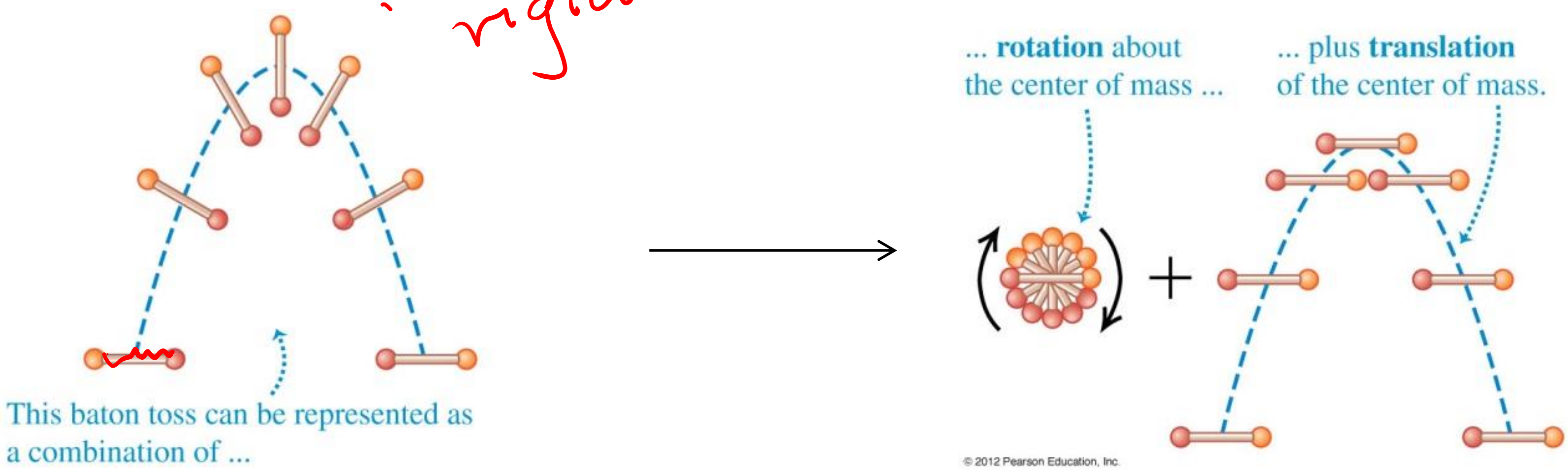
i. a translational motion of the CM (as if no rotation), plus
ii. a rotation about an axis through its CM (as if no translation)



⚠ This does not only apply to the position and orientation of the rigid body. It applies to the dynamics, e.g. torque and energy, even when the CM is accelerating.

e.g. tossing a baton

"rigid"



translation + rotation

+ vibrational

rotation about a fixed axis through CM

translation of CM (considered as a particle)

KE measured in lab frame (both translation and rotation)	=	$\frac{1}{2}I\omega^2$ as if no translation	+	$\frac{1}{2}Mv_{cm}^2$ as if no rotation

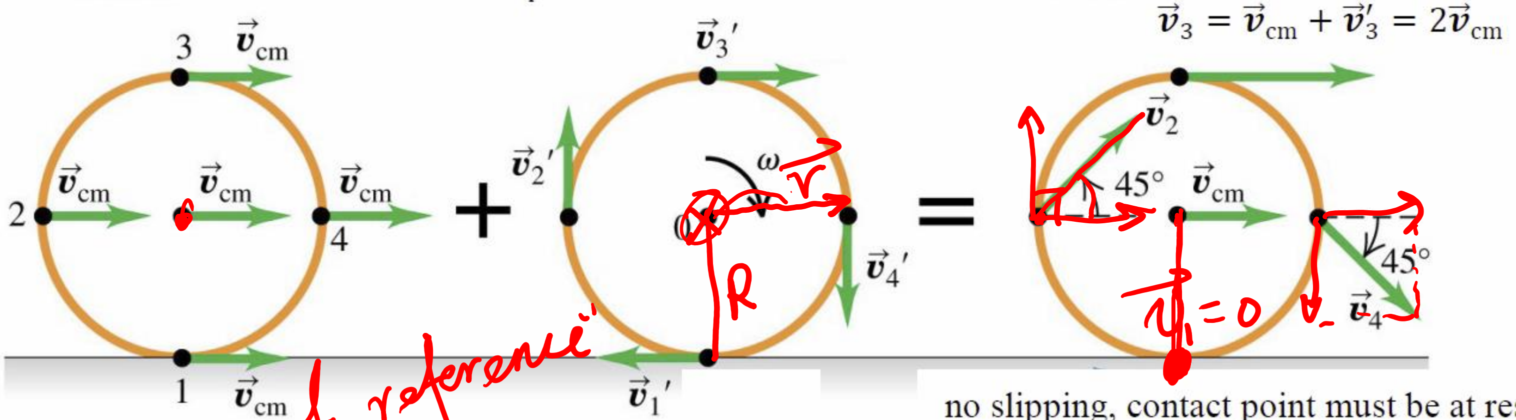
Rolling without slipping

Translation of center of mass: velocity $\vec{v}_{cm}$

Rotation around center of mass: magnitudes of $\vec{v}_1'$, $\vec{v}_2'$, $\vec{v}_3'$, and $\vec{v}_4'$ are $R\omega$

Combined motion

$\vec{v}_3 = \vec{v}_{cm} + \vec{v}_3' = 2\vec{v}_{cm}$



"frame of reference"

$\vec{v}_{tan} = \vec{\omega} \times \vec{r}$

no slipping, contact point must be at rest (instantaneously)

$\vec{v}_1 = \vec{v}_1' + \vec{v}_{cm} = 0 \Rightarrow \boxed{v_{cm} = R\omega}$?

$|\vec{v}_{cm}| = |\vec{v}_1'| = R\omega$

$= |\vec{v}_4'|$

Two possible ways to look at this problem:

translation of CM (as if no rotation)+ rotation about CM (as if no translation)

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$$

same

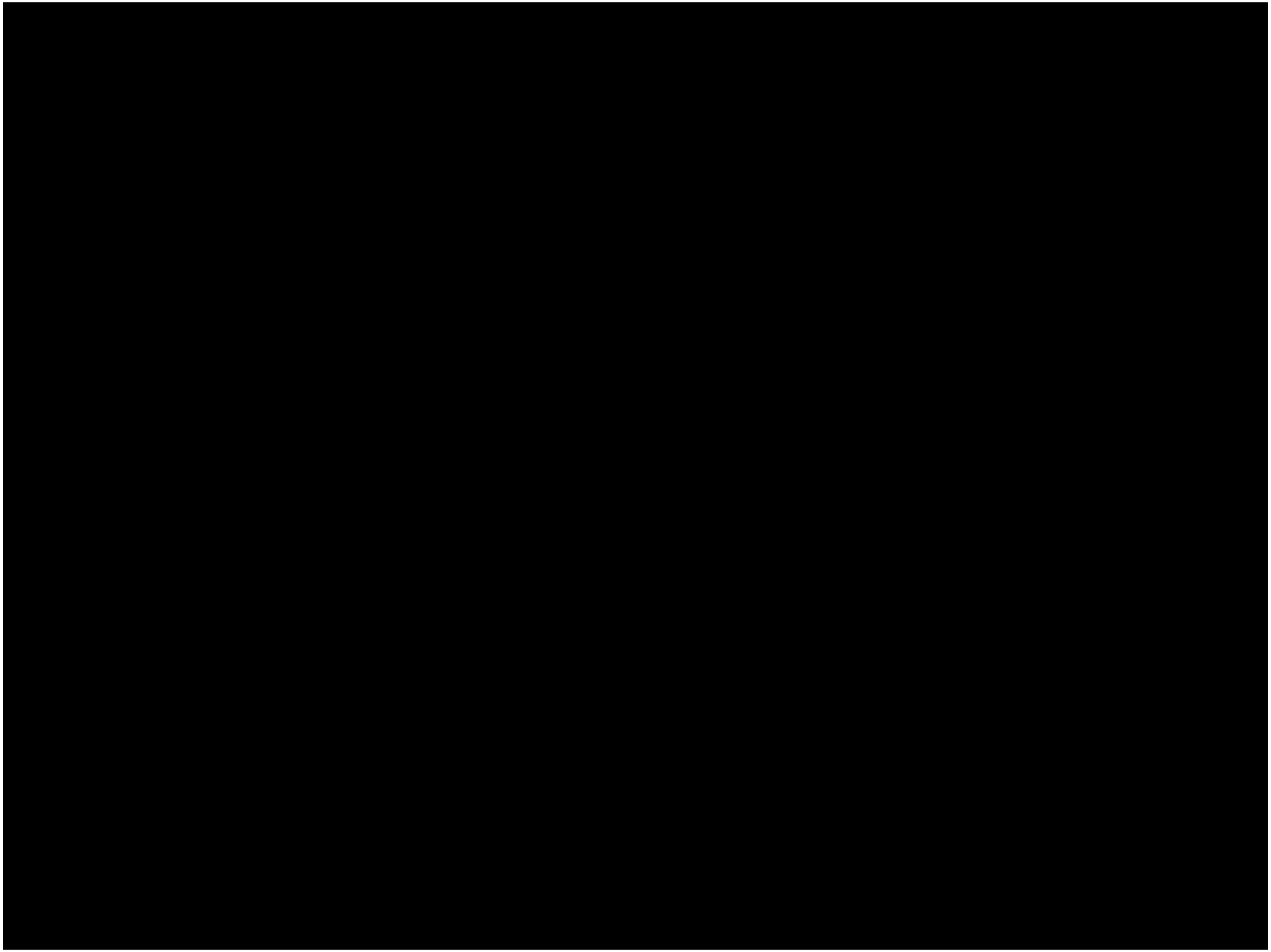
a single rotation about an **instantaneous axis of rotation** (as if no translation)

$$K = \frac{1}{2}(I + MR^2)\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$$

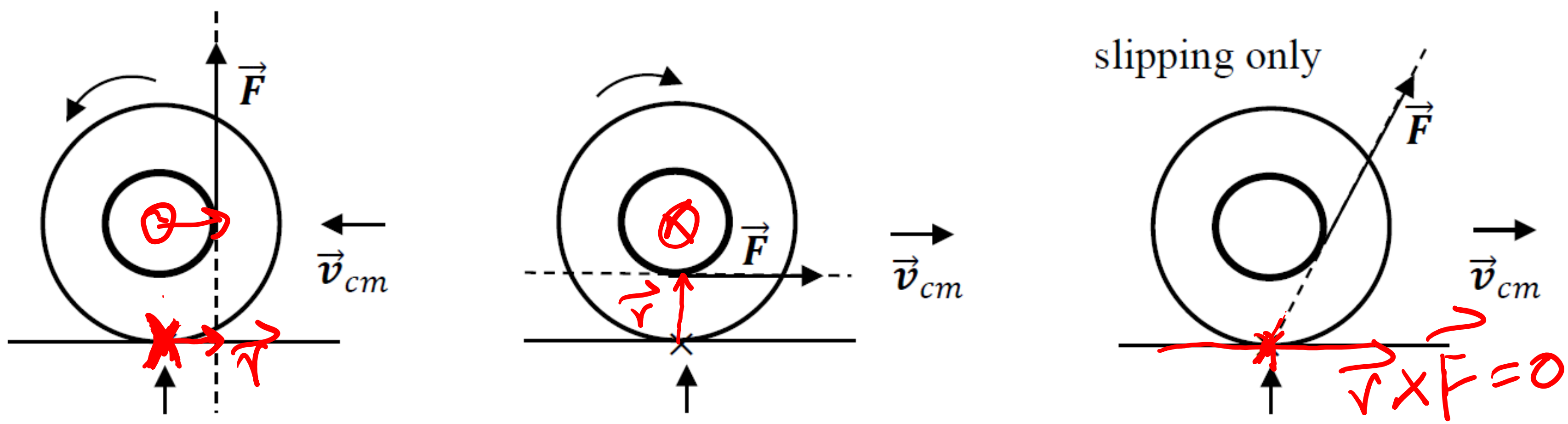
parallel axis theorem

$v_{cm} = R\omega$

Demonstration: pulling a spool



Demonstration: a rolling spool to show that the contact point with the floor is an instantaneous axis of rotation

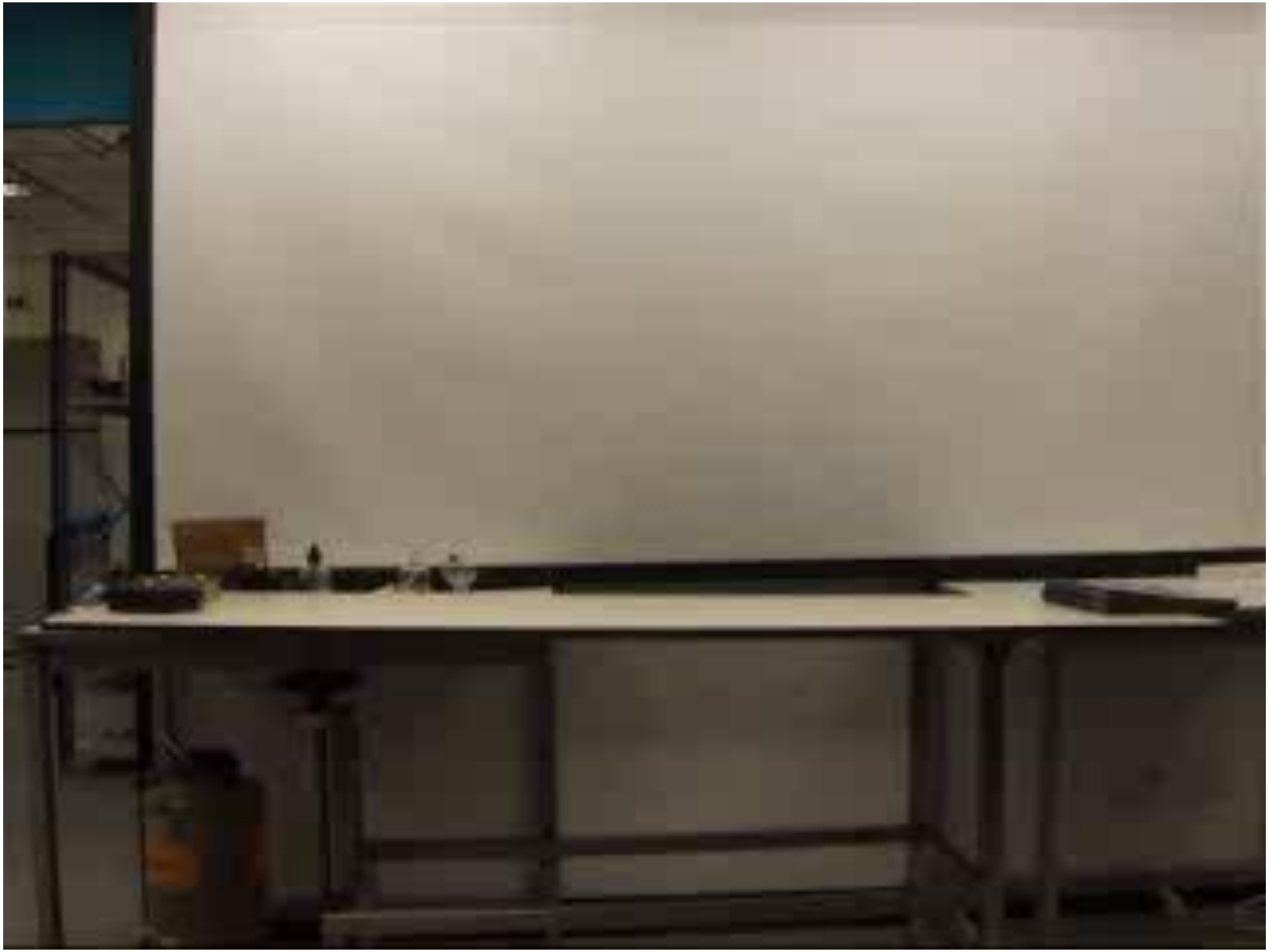


Line of $\vec{F}$ to the right of rotation axis, counterclockwise

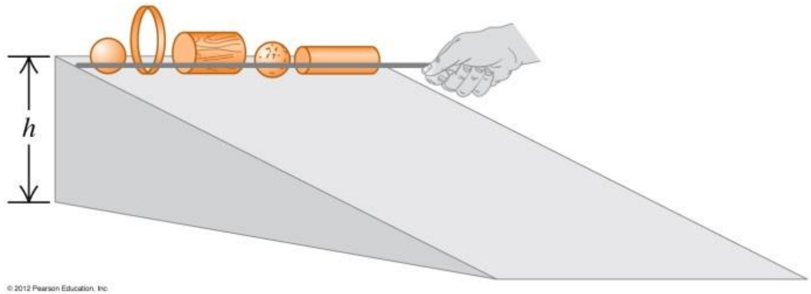
Line of $\vec{F}$ above rotation axis, clockwise

Line of $\vec{F}$ through rotation axis, no rotation

Demonstration: Ring and disk



Example



⚠ Rolling without slipping, friction does no work

: $c = \frac{1}{2}$

: $c = 1$

What determines which body rolls down the incline fastest?

Suppose a rigid body's moment of inertia about its symmetry axis is $I = cMR^2$

$$0 + \cancel{M}gh = \underbrace{\tfrac{1}{2}\cancel{M}v_{\text{cm}}^2}_{\text{translation KE of CM}} + \underbrace{\tfrac{1}{2}c\cancel{M}\cancel{R^2}\Big(\overbrace{\frac{v_{\text{cm}}}{\cancel{R}}}^{\omega}\Big)^2}_{\text{rotation KE about a fixed axis}} + 0$$

$$gh = \frac{v_{cm}^2}{2} + \frac{1}{2}c\,v_{cm}^2$$

initial KE

initial PE

final PE

$$\Rightarrow \uparrow v_{cm} = \sqrt{\frac{2gh}{1+c\downarrow}}$$

⚠ depends on c only, independent of M and R

Rigid body with smaller c rolls faster :

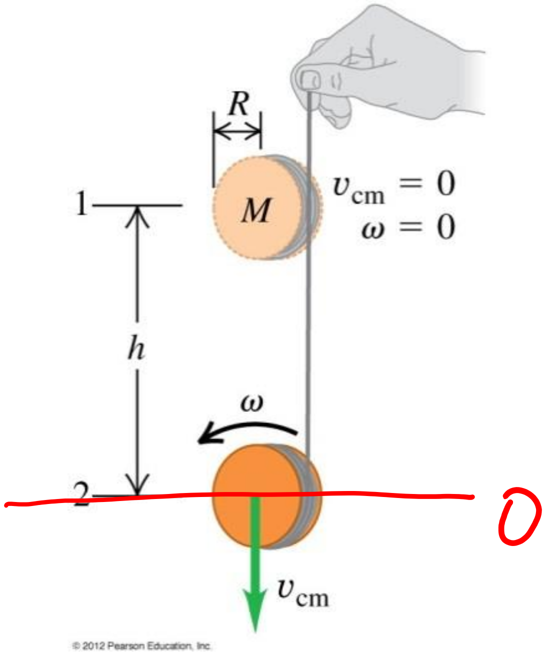
solid sphere ($c = \frac{2}{5}$) ✓

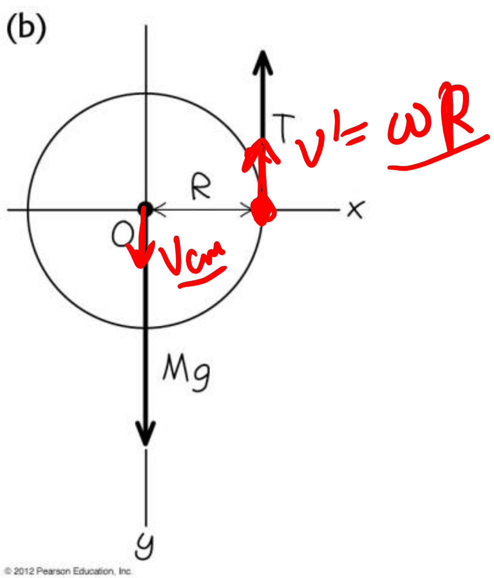
> solid cylinder ($c = \frac{1}{2}$) $I = \frac{1}{2}MR^2$

> thin walled hollow sphere ($c = \frac{2}{3}$)

> thin walled hollow cylinder ($c = 1$) $I = MR^2$

A Yo-yo





To find v_{cm} at point 2, need energy conservation

$$\underset{\text{initial KE}}{0} + \underset{\text{initial PE}}{Mgh} = \underbrace{\tfrac{1}{2}Mv_{cm}^2}_{\text{translation KE of CM}} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2}_{\text{rotation KE about a fixed axis}} + \underset{\text{final PE}}{0}$$

$c = \frac{1}{2}$

$$\Rightarrow \quad v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

c.f. for free falling $v_{cm} = \sqrt{2gh}$

To find the downward acceleration of the yo-yo, need dynamic equations

T

Translation of CM: $\quad Mg - T = Ma_{cm}$

mg

Rotation of cylinder about its axis:

$$TR = I_{cm}\alpha = \left(\tfrac{1}{2}MR^2\right)(a_{cm}/R)$$

$\omega = \frac{V_{cm}}{R}$

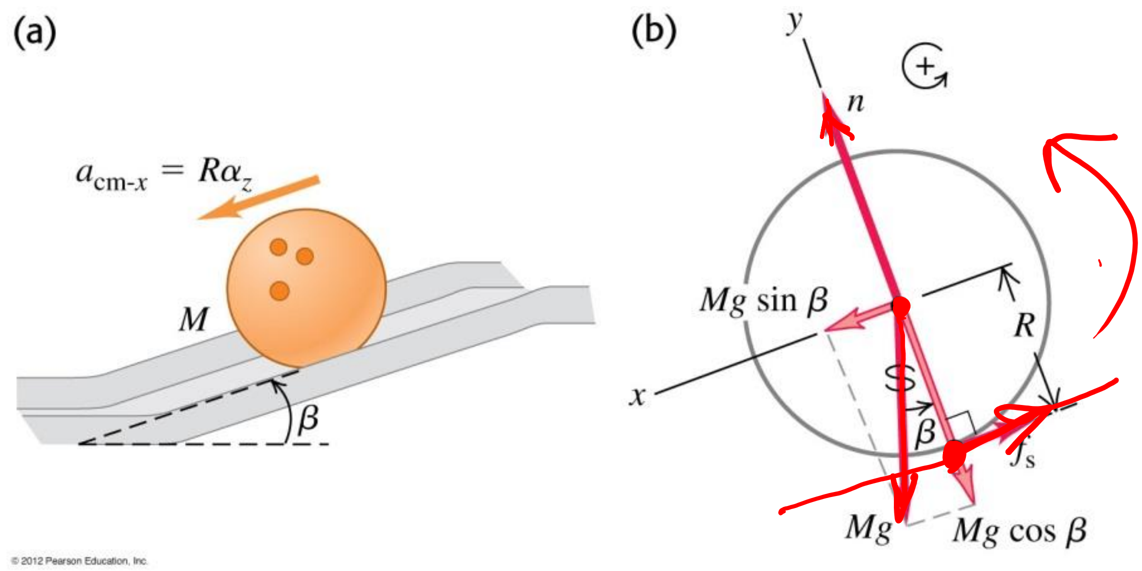
$\alpha \quad a$

$\frac{d\omega}{dt} = \frac{dV_{cm}}{dt}\frac{1}{R}$

Get

$$a_{cm} = \tfrac{2}{3}g$$
$$T = \tfrac{1}{3}Mg$$

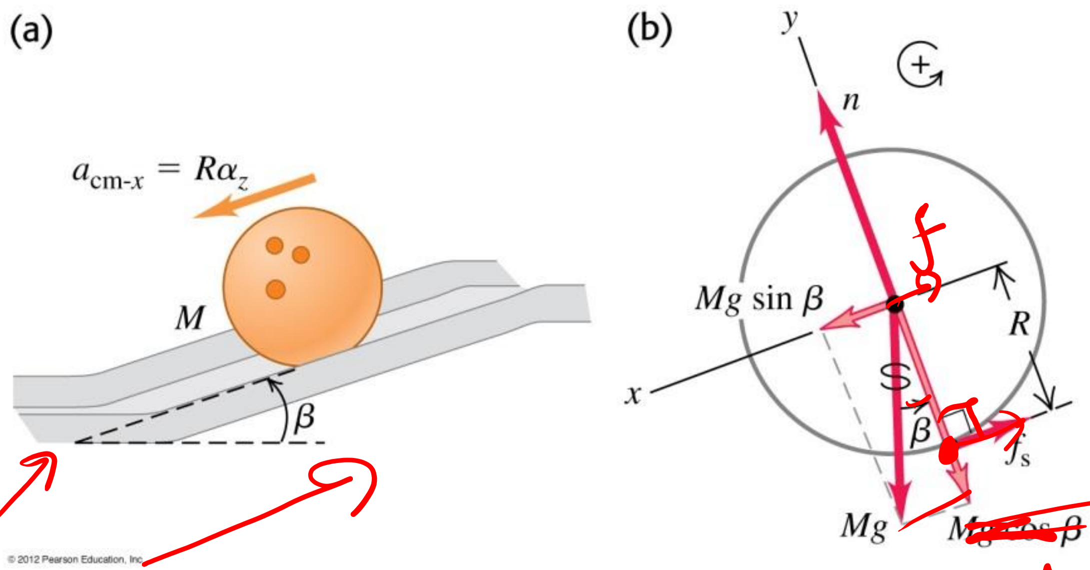
Role of friction: Example



Rolling without slipping is not possible without friction.
Consider a rigid sphere going freely down an inclined plane. If no friction, no torque about the center and the sphere slides down the plane.

Assume rolling without slipping, friction must be (static / dynamics) and must point (upward / downward) along the plane.

$$\boxed{v_{\mathrm{cm}} = R\omega \quad \Rightarrow \quad a_{\mathrm{cm}} = R\alpha}$$



f_k $\vec{v}$ f_s

$$v_{\text{cm}} = R\omega \quad \Rightarrow \quad a_{\text{cm}} = R\alpha$$

Translation of CM: $Mg\sin\beta - f = Ma_{\text{cm}}$

$f = \frac{2}{5} M a_{cm}$

Rotation of sphere about its center: $fR = I_{\text{cm}}\alpha = \left(\frac{2}{5}MR^2\right)(a_{\text{cm}}/R)$

$g \sin\beta = \frac{7}{5} a_{cm}$

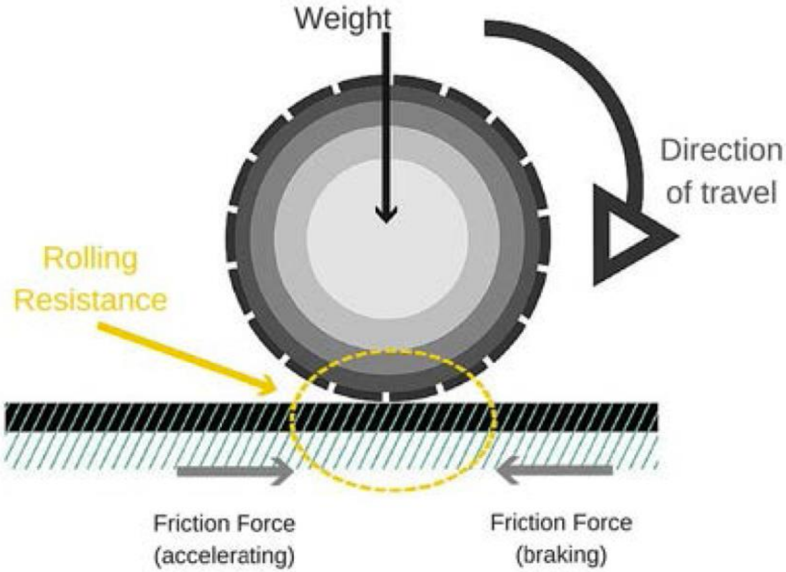
Get $\boxed{a_{cm} = \frac{5}{7} g \sin\beta}$ and $f = \frac{2}{7} Mg\sin\beta$

⚠ If the sphere is rolling uphill with no slipping, the friction will point (upward / downward) along the plane because its effect is to *decelerate* the rotation.

Demonstration: Rolling vs sliding

⚠ Rolling is slower than sliding because part of the PE is converted into rotation KE

Road friction on a moving car



To be explained in the lecture 11

Check to make sure in the accelerating (or decelerating) case, the road friction is in a direction such that it

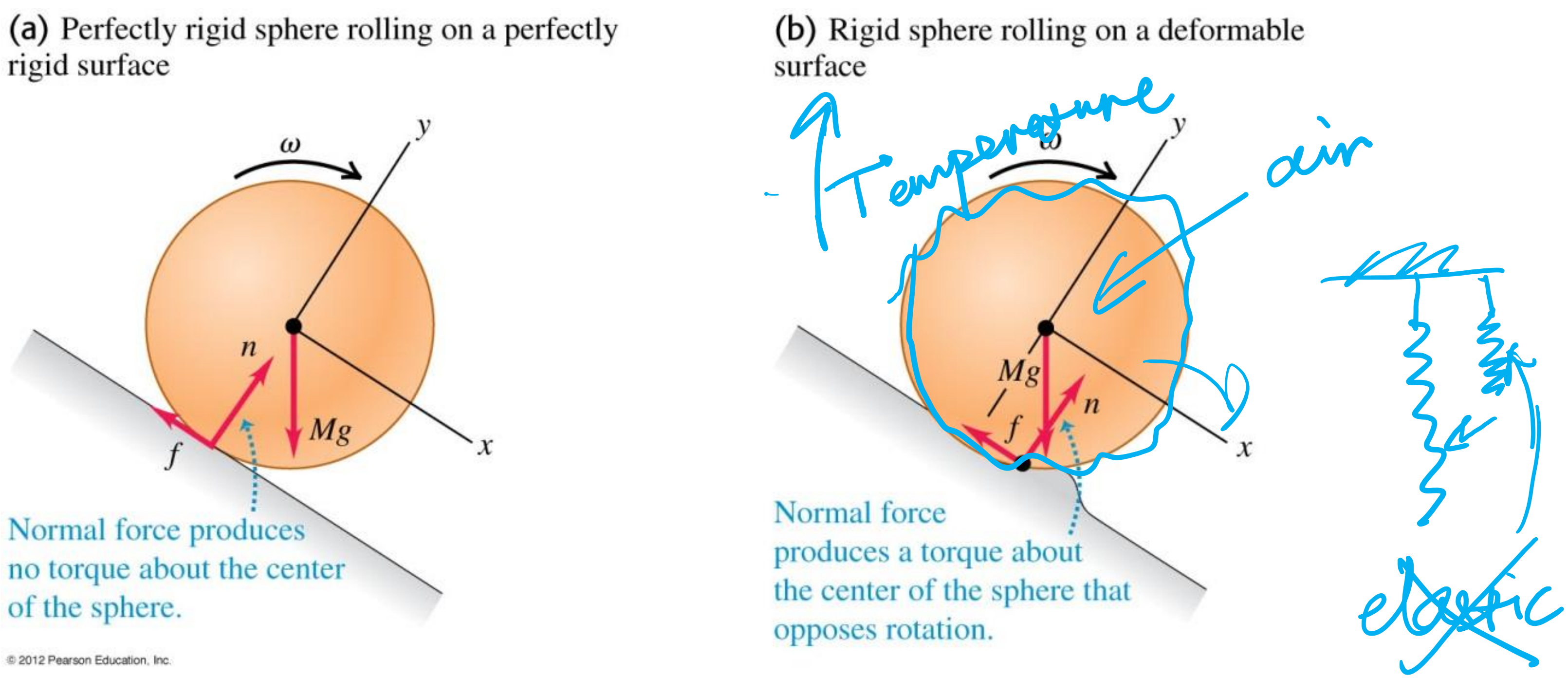
a) accelerates (or decelerates) the CM of the car, as required by $\vec{\boldsymbol{F}}_{\mathrm{ex}} = M\vec{\boldsymbol{a}}_{\mathrm{CM}}$;
b) produces a torque to oppose the torque by the engine (or the brake) that increases (or decreases) ω

Puzzle: For rolling without slipping, friction does no work.

Therefore a vehicle will go on forever if there is no air resistance, just like a magnetic levitated train.

Too good to be true! ☹

In reality energy is lost because the floor and/or the rolling body are deformed, e.g. vehicle tyre.

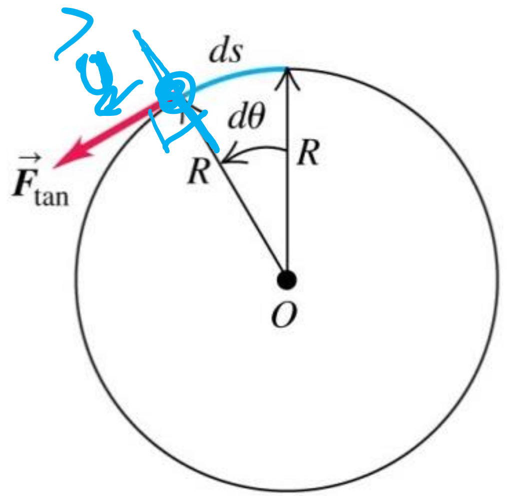


?

Energy is lost because:
•due to deformation, normal reaction produces a torque opposing the rotation.
•sliding of the deformed surfaces causes energy lost. → heat
These two effects give rise to **rolling friction**.
Consequence: trains, with metal wheels on metal tracks, are more fuel efficient than vehicles with rubber tyres.

Work and power in rotational motion

A particle or rigid body, being pushed by an external force, is undergoing circular motion about a fixed axis (such as a merry-go-round).



⚠ only the tangential component $F_{\tan}$ does work – no displacement along the radial and z directions.

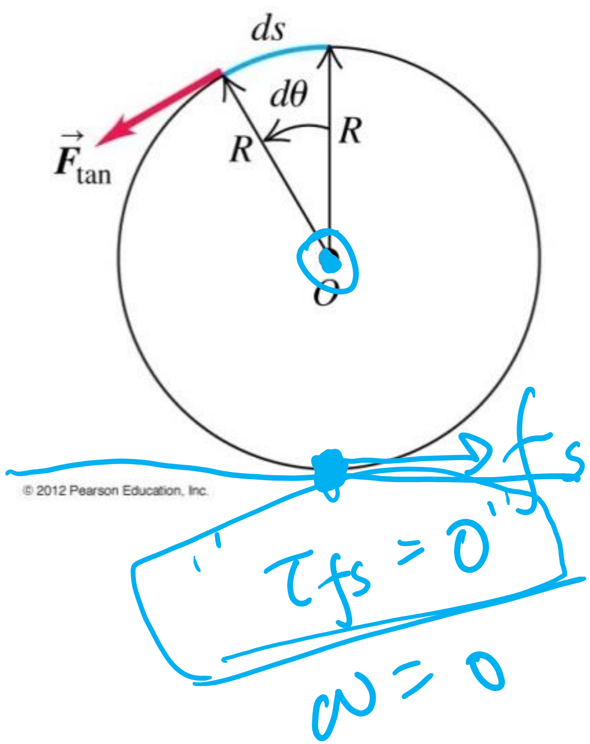
Work done after going through angle $d\theta$

$$dW = F_{\tan}(Rd\theta) = \tau d\theta$$

$$\Rightarrow \quad \boxed{W = \int \vec{\tau}\, d\vec{\theta}}$$

c.f. in translation, $W = \int \vec{F} \cdot d\vec{r}$

$$\boxed{W = \int \tau \, d\theta}$$ $\quad W = \int \underbrace{\vec{\tau}}_{\vec{r} \times \vec{F}} \cdot d\vec{\theta}$



By changing variable

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I(d\omega)\omega$$

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega d\omega = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

This is the **work-energy theorem** for rotational motion.

How about power?

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega = \vec{\tau} \cdot \vec{\omega}$$

c.f. $P = \vec{F} \cdot \vec{v}$ for translational motion.

Question

- You apply equal torques to two different cylinders, one of which has a moment of inertial twice as large as the other. Each cylinder is initially at rest. After one complete rotation, the cylinder with larger moment of inertia will have (larger / smaller / the same) kinetic energy as the other one.

$$\int_0^{2\pi} \vec{\tau} \cdot d\vec{\theta} = \Delta K =$$