Example 1:

Show that
$$\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 1 & a_1+b_1 & a_2 & \cdots & a_n \\ 1 & a_1 & a_2+b_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix} = b_1 b_2 \cdots b_n.$$

Example 2:

Compute
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$
and
$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

Solution :

Example 1:

$$\begin{array}{c} ① \\ ② \\ \\ \vdots \\ ⓝ \end{array} \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 1 & a_1+b_1 & a_2 & \cdots & a_n \\ 1 & a_1 & a_2+b_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix}$$

$$\Bigg\downarrow \begin{array}{l} ⓝ - ① \\ \vdots \\ ② - ① \end{array}$$

$$\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & b_n \end{vmatrix} = b_1 \cdots b_n.$$

Example 2:

$$\begin{array}{c} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \\ \text{①} \quad \text{②} \quad \text{③} \end{array} , \xrightarrow[\text{②} - c\cdot\text{①}]{\text{③} - c\cdot\text{②}} \begin{vmatrix} 1 & a-c & a^2-ac \\ 1 & b-c & b^2-bc \\ 1 & 0 & 0 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & a-c & (a-c)\cdot a \\ 1 & b-c & (b-c)b \\ 1 & 0 & 0 \end{vmatrix} = (-1)^{1+3} \begin{vmatrix} a-c & (a-c)a \\ b-c & (b-c)b \end{vmatrix}$$

$$= (a-c)(b-c) \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}$$

$$= (a-c)(b-c)(b-a).$$

$$\begin{array}{c}\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} \\ \quad 1 \quad\ 2 \quad\ 3 \quad\ 4 \end{array} \xrightarrow[\ ]{\substack{4 - d\cdot 3 \\ 3 - d\cdot 2 \\ 2 - d\cdot 1}} \begin{vmatrix} 1 & a-d & a(a-d) & a^2(a-d) \\ 1 & b-d & b(b-d) & b^2(b-d) \\ 1 & c-d & c(c-d) & c^2(c-d) \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)^{1+4} \cdot \begin{vmatrix} a-d & a(a-d) & a^2(a-d) \\ b-d & b(b-d) & b^2(b-d) \\ c-d & c(c-d) & c^2(c-d) \end{vmatrix}$$

$$= (-1) \cdot (a-d)(b-d)(c-d) \cdot \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)(a-d)(b-d)(c-d)(a-c)(b-c)(b-a).$$