§1.8 Introduction to Linear Transformations

Def: A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ is a rule that assigns to each vector in $\mathbb{R}^n$ to a vector $T(\vec{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$: domain of T

$\mathbb{R}^m$: codomain of T

For $\vec{x} \in \mathbb{R}^n$, $T(\vec{x})$ is called the image of $\vec{x}$. The set of all images $T(\vec{x})$ is called the range of T.

A: matrix $m \times n$, $\vec{x} \in \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{x}) \triangleq A\vec{x}$.

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

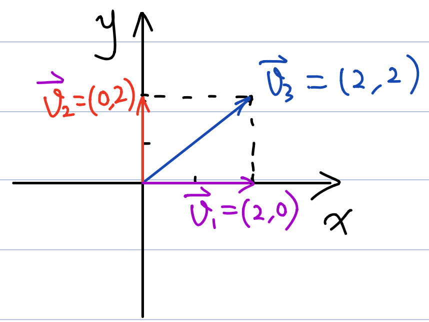
$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

Thus T is the projection from $\mathbb{R}^3$ to x_1x_2-plane

Ex. Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^2$.

Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

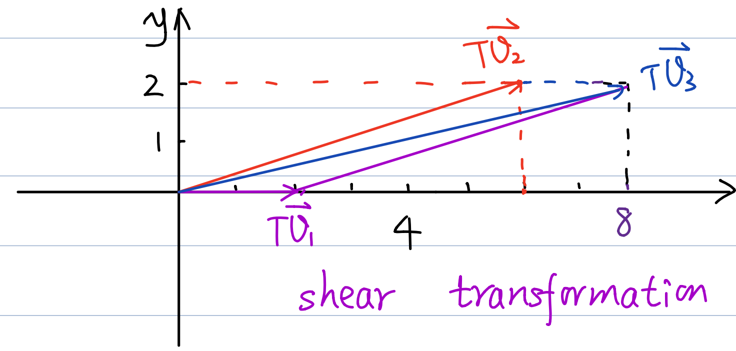


$$T\vec{v}_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$T\vec{v}_2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$\downarrow T$

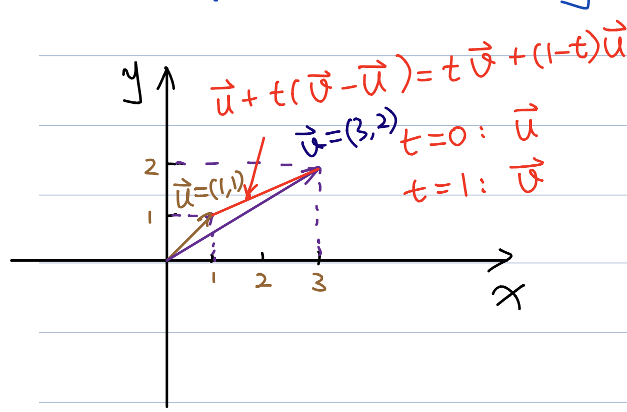
$$T\vec{v}_3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

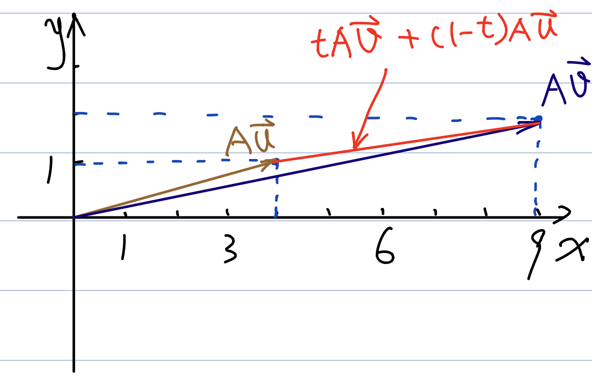


shear transformation

T deforms the square as if the top of the square we pushed to the right while the base is held fixed

T maps a line segment to a line segment





$$A(t\vec{v}+(1-t)\vec{u}) = A(t\vec{v}) + A((1-t)\vec{u}) = tA\vec{v} + (1-t)A\vec{u}$$

Def: A transformation (or mapping) T is linear if

i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in D(T)$ domain of T

ii) $T(c\vec{u}) = c\,T(\vec{u})$ for all scalars c and $\vec{u} \in D(T)$

i.e. Linear transformation preserve the operations of vector addition and scalar multiplication.

* If T is a linear transformation, then

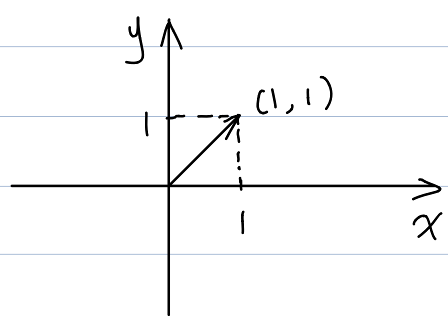
$$T(\vec{0}) = \vec{0}$$

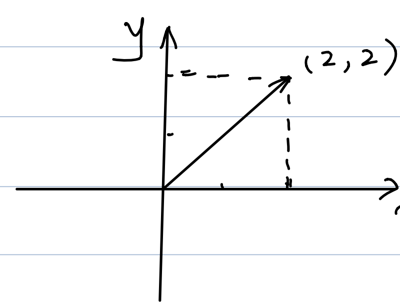
and
$$T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$$

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_p T(v_p)$$

Ex: Given a scalar, define $T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$. T is called contraction if $0 \le r \le 1$ and a dilation when $r > 1$

For example $r=2$



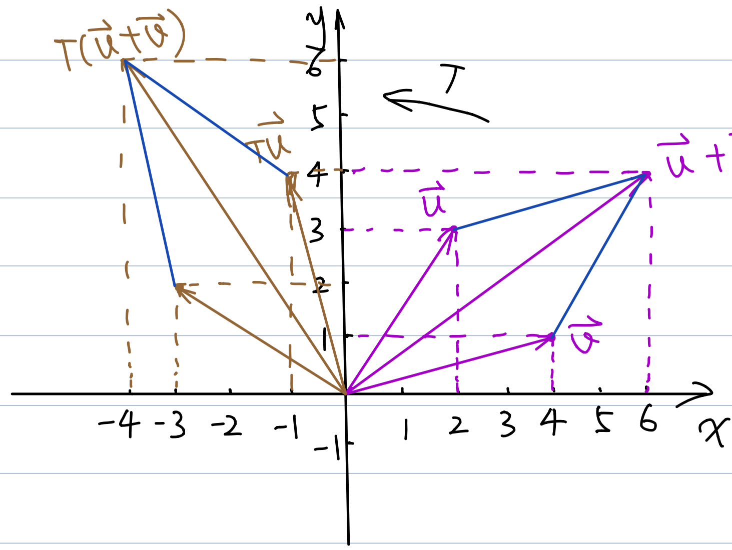


Ex: Define $T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$ by $T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

Find the image of $\vec{u}, \vec{v}$ and $\vec{u}+\vec{v}$.

$\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{u} + \vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Solution: $T\vec{u} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $T\vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $T(\vec{u}+\vec{v}) = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$



§1.9 The matrix of a linear Transformation

Ex. Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be vectors in $\mathbb{R}^2$

Suppose $T: \mathbb{R}^2 \longmapsto \mathbb{R}^3$ is a linear transformation such that $T(\vec{e}_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$ and $T(\vec{e}_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$

Find a formula for the image of any $\vec{x} \in \mathbb{R}^2$.

Solution: $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T\left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

$$= T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus, $T\vec{x} = A\vec{x}$ where $A = (T(\vec{e}_1), T(\vec{e}_2))$

Thm: Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n.$$

In fact, $A = (T\vec{e}_1, \cdots, T\vec{e}_n)$ ← standard matrix for the linear transformation

where $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Example: Find the standard matrix A for the dilation transformation $T(\vec{x}) = 3\vec{x}$.

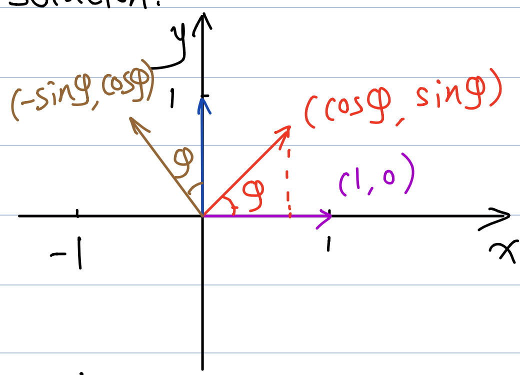
Solution: $T(\vec{e}_1) = 3\vec{e}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$T(\vec{e}_2) = 3\vec{e}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

So $A = (T(\vec{e}_1), T(\vec{e}_2)) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

Ex: Let $T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. Find the standard matrix A of this transformation.

Solution:



$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

The standard matrix is $A = \begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}$

Def: A map $T: \mathbb{R}^n \longmapsto \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.
A map $T: \mathbb{R}^n \longmapsto \mathbb{R}^m$ is said to be one-to-one if $T\vec{x} = T\vec{y}$, then $\vec{x} = \vec{y}$, i.e. each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^m$.

Thm: Let $T: \mathbb{R}^n \longmapsto \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T\vec{x} = \vec{0}$ has only the trivial solution

Proof: If T is one-to-one, then $T\vec{x} = \vec{0}$ and $T\vec{0} = \vec{0}$
$\Rightarrow \vec{x} = 0$

Conversely, if $T\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$. Let $T\vec{u} = T\vec{v}$ then $T(\vec{u}) - T(\vec{v}) = \vec{0} \Rightarrow T(\vec{u} - \vec{v}) = \vec{0} \Rightarrow \vec{u} - \vec{v} = \vec{0}$ $\Rightarrow \vec{u} = \vec{v}$. That is, T is one-to-one.

Ex: T: linear Transformation whose standard matrix is

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Solution: onto $\Leftrightarrow A\vec{x} = \vec{b}$ has solution for all $\vec{b}$.

$$(A, \vec{b}) = \begin{pmatrix} 1 & -4 & 8 & 1 & b_1 \\ 0 & 2 & -1 & 3 & b_2 \\ 0 & 0 & 0 & 5 & b_3 \end{pmatrix}$$

It is an echelon form and the last column is not a pivot column for all $\vec{b}$.
Thus T is onto.
one-to-one $\Leftrightarrow$ no free variables. The third column is not a pivot column, so x_3 is a free variable. Thus T is not one-to-one.

Thm: Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix of T.

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$.

(b) T is one-to-one iff the column vectors of A are linearly independent.

Proof: $A\vec{x} = \vec{b}$, $A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$, $\vec{x}, \vec{b} \in \mathbb{R}^m$.

(a) T is onto iff $A\vec{x} = \vec{b}$ has solution for all $\vec{b}$

$\Leftrightarrow$ any $\vec{b}$ can be written as $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$, i.e. any $\vec{b}$ is a linear combination of $\vec{a}_1, \cdots, \vec{a}_n$

$\Leftrightarrow$ span $\{\vec{a}_1, \cdots, \vec{a}_n\} = \mathbb{R}^m$.

(b) T is one-to-one iff $A\vec{x} = \vec{0}$ has only trivial solution $\Leftrightarrow$ $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$ has only trivial solution $\Leftrightarrow$ $\vec{a}_1, \cdots, \vec{a}_n$ are linearly independent.

Ex. $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

Solution: $$T(x_1, x_2) = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix} (x_1, x_2)$$

So T is a linear transformation with its standard matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}$.

The columns of A are linearly independent because they are not multiples, so T is one-to-one.

To determine if it is onto. Consider $A\vec{x} = \vec{b}$,

$$\begin{pmatrix} 3 & 1 & b_1 \\ 5 & 7 & b_2 \\ 1 & 3 & b_3 \end{pmatrix} \xrightarrow{①\leftrightarrow③} \begin{pmatrix} 1 & 3 & b_3 \\ 5 & 7 & b_2 \\ 3 & 1 & b_1 \end{pmatrix}$$

$$\xrightarrow[③-3①]{②-5①} \begin{pmatrix} 1 & 3 & b_3 \\ 0 & -8 & b_2-5b_3 \\ 0 & -8 & b_1-3b_3 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} 1 & 3 & b_3 \\ 0 & -8 & b_2-5b_3 \\ 0 & 0 & b_1-b_2+2b_3 \end{pmatrix}$$

Since $A\vec{x} = \vec{b}$ has no solution when $b_1 - b_2 + 2b_3 \neq 0$.

So the column of A does not span $\mathbb{R}^3$ and T is not onto.