Math 1012 - Calculus IA

Midterm Test, Fall Semester, 2022

Time Allowed: 2 Hours | Total Marks: 100

Note: No calculators are allowed. If needed, you may use the following identities:

$$\sin(A+B) = \sin A\,\cos B + \cos A\,\sin B,$$
$$\sin(A-B) = \sin A\,\cos B - \cos A\,\sin B,$$
$$\cos(A+B) = \cos A\,\cos B - \sin A\,\sin B,$$
$$\cos(A-B) = \cos A\,\cos B + \sin A\,\sin B.$$

1. (40 Marks) Multiple Choice Questions.

(i) $\lim\limits_{x\to\infty} \dfrac{(2x-3)^{20}(3x+2)^{30}}{(x+2)^{50}} = (\quad)$.

(A) $2^{-30}\times 3^{20}$; (B) $2^{20}\times 3^{30}$;

(C) $2^{-30}\times 3^{30}$; (D) Not Exist.

(ii) $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 3x}{x^2} = (\quad)$.

(A) -2; (B) 1;

(C) 4; (D) 0.

(iii) $\lim\limits_{h\to 0} \dfrac{\sin[\ln(e+h)] - \sin 1}{\ln(e+h) - 1} = (\quad)$.

(A) $\cos 1$; (B) $\sin 1$;

(C) 1; (D) 0.

(iv) $\lim\limits_{x\to -\infty} \left(\sqrt{x^2+x} + x\right) = (\quad)$.

(A) $\frac{1}{2}$; (B) $-\frac{1}{2}$;

(C) 1; (D) $+\infty$.

2. (20 Marks) Let $f(x) = \dfrac{x^2 + e^{-\sin^2 x}}{x^2 - 1}$. Determine whether the limit $\lim_{x\to+\infty} f(x)$ exists. If it does, find its value; if it does not, explain.

Solution It is clear that

$$\lim_{x\to+\infty} \frac{x^2}{x^2-1} = \lim_{x\to+\infty} \frac{1}{1-1/x^2} = 1. \qquad \boxed{4}$$

Since $\sin^2 x \geq 0$, we have $e^{-\sin^2 x} \leq 1$. Thus, for $x > 1$,

$$0 \leq \frac{e^{-\sin^2 x}}{x^2-1} \leq \frac{1}{x^2-1}. \qquad \boxed{8}$$

Because $\lim_{x\to+\infty} \dfrac{1}{x^2-1} = 0$, by the Squeeze Rule, $\boxed{2}$

$$\lim_{x\to+\infty} \frac{e^{-\sin^2 x}}{x^2-1} = 0. \qquad \boxed{2}$$

Therefore,

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2 + e^{-\sin^2 x}}{x^2-1} = \lim_{x\to+\infty} \frac{x^2}{x^2-1} + \lim_{x\to+\infty} \frac{e^{-\sin^2 x}}{x^2-1} = 1 + 0 = 1. \qquad \boxed{4}$$

3. (20 Marks) Show that the function $x^{10} - 10x^2 + 1$ has at least one positive root.

Solution Denote $f(x) = x^{10} - 10x^2 + 1$. It is clear that f is a polynomial, so it is continuous everywhere. $\boxed{4}$

Since $f(0) = 1 > 0$, $f(1) = 1 - 10 + 1 = -8 < 0$, by the Intermediate Value Theorem, there is a number $c \in (0, 1)$ such that $f(c) = 0$. This shows that f has at least one positive root. $\boxed{4 + 12}$

4. (20 Marks) Find an equation of the tangent line to the curve $y = y(x)$ given implicitly by

$$2023 \cdot x^y + 2022 \cdot y^3 = 1$$

at the point $(1, -1)$.

Solution Denote $f(x) = x^{y(x)}$. We will use implicit differentiation to compute $f'(x)$. In fact, by taking logarithm, we have

$$\ln f(x) = y(x) \ln x. \qquad \boxed{2}$$

Differentiating both sides of the equality simultaneously, we have

$$\frac{f'(x)}{f(x)} = y'(x) \cdot \ln x + \frac{1}{x} \cdot y(x), \qquad \boxed{6}$$

so that $f'(x) = x^{y(x)} \left[y'(x) \cdot \ln x + \dfrac{1}{x} \cdot y(x) \right]$. $\boxed{3}$

Hence, by implicitly differentiating the original equation

$$2023 \cdot f(x) + 2022 \cdot [y(x)]^3 = 1,$$

we get

$$2023 \cdot f'(x) + 2022 \times 3 \cdot [y(x)]^2 \cdot y'(x) = 0, \qquad \boxed{4}$$

or

$$2023 \cdot x^{y(x)} \left[y'(x) \cdot \ln x + \frac{1}{x} \cdot y(x) \right] + 2022 \times 3 \cdot [y(x)]^2 \cdot y'(x) = 0.$$

Putting $x = 1$, $y(1) = -1$, we have

$$2023 \cdot (0 - 1) + 2022 \times 3 \cdot (-1)^2 \cdot y'(1) = 0,$$

which gives $y'(1) = \dfrac{2023}{3 \times 2022}$. $\boxed{3}$

Finally, the tangent line of the curve at $(1, -1)$ is

$$y + 1 = \frac{2023}{3 \times 2022}(x - 1). \qquad \boxed{2}$$