Math2001 Answer to Homework 3

Exercise 2.37

(3),(4),(5),(6),(7) are maps, but (1),(2) are not.

Exercise 2.39.(1)

$f(A_1) = \{(x, 2x) : x \in [0,1]\}$, $f(A_2) = \{(x, 2x) : x \in \mathbb{Z}\}$;

Exercise 2.39.(2)

$f(A_1) = \{(x, \sqrt{1-x^2}) : x \in [-1,1]\}$ is the upper half unit circle, $f(A_2) = \{(1,0), (-1,0)\}$;

Exercise 2.39.(3)

$f(A_1) = (-1,1)$, $f(A_2) = \mathbb{R}$;

Exercise 2.39.(4)

$f(\mathbb{Z}) = \{1\}$,$f([0,1]) = \{0,1\}$,$f(\{\sqrt{2}, \sqrt{3}\}) = \{0\}$.

Exercise 2.40.(1)

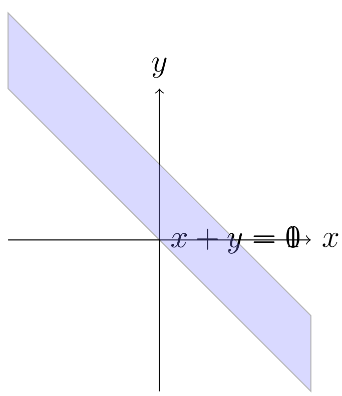
$f^{-1}(B_1) = \left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$, $f^{-1}(B_2) = \left(-\frac{1}{3}, \frac{1}{3}\right)$;

Exercise 2.40.(2)

$f^{-1}(B_1) = \mathbb{R}$, $f^{-1}(B_2) = \left\{x : -\frac{\pi}{2} + 2k\pi < x < 2k\pi, \forall k \in \mathbb{Z}\right\}$;

Exercise 2.40.(3)

$f^{-1}(B_1) = \{(x,y) : 0 \leq x + y \leq 1\}$ as following figure, $f^{-1}(B_2) = \mathbb{R}^2$;



Exercise 2.40.(4)

$f(B_1) = \mathbb{R} - \mathbb{Q} = \{\text{irrational numbers}\}$,$f(B_2) = \mathbb{Q}$,$f(B_3) = \emptyset$.

Exercise 2.42

The preimage of $A \times B$ under the diagonal map is $A \cap B$.

Exercise 2.43

The preimage of B under the inclusion map is $A \cap B$.

Exercise 2.47

Given any $y \in f(A_1 \cup A_2)$, there exists $x \in A_1$ or $x \in A_2$ such that $f(x) = y$, then $y \in f(A_1)$ or $y \in f(A_2)$, hence $y \in f(A_1) \cup f(A_2)$. Then we have $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$. Conversely, $A_1 \subseteq A_1 \cup A_2$ implies $f(A_1) \subseteq f(A_1 \cup A_2)$, similarly $f(A_2) \subseteq f(A_1 \cup A_2)$, hence $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. Above all, $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$.

Given any $x \in f^{-1}(B_1 \cup B_2)$, then $f(x) \in B_1 \cup B_2$, which means $f(x) \in B_1$ or $f(x) \in B_2$, hence $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$, then $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$, $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$. Conversely, $x \in f^{-1}(B_1)$ implies $f(x) \in B_1 \subseteq B_1 \cup B_2$, then $x \in f^{-1}(B_1 \cup B_2)$, which forces

$f^{-1}(B_1) \subseteq f^{-1}(B_1 \cup B_2)$; similarly we have $f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$; so $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$. Above all, $f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2)$.

Exercise 2.48

Given any $y \in f(A_1) - f(A_2)$, then $y \in f(A_1)$, there exists $x \in A_1$ such that $f(x) = y$. We claim that $x \notin A_2$, otherwise $x \in A_2$ implies $f(x) \in f(A_2)$, $y = f(x) \notin f(A_1) - f(A_2)$, contradiction. So $x \in A_1 - A_2$, $y = f(x) \in f(A_1 - A_2)$, which means $f(A_1) - f(A_2) \subseteq f(A_1 - A_2)$.

Consider $f(x) = x^2$, $A_1 = [-1, 1], A_2 = [0, 1]$.

Exercise 2.49

Given any $x \in f^{-1}(B_1) - f^{-1}(B_2)$, then $x \in f^{-1}(B_1)$ but $x \notin f^{-1}(B_2)$, equivalently $f(x) \in B_1$ but $f(x) \notin B_2$, hence $f(x) \in B_1 - B_2$, $x \in f^{-1}(B_1 - B_2)$; so we have $f^{-1}(B_1) - f^{-1}(B_2) \subseteq f^{-1}(B_1 - B_2)$.

Conversely, given any $x \in f^{-1}(B_1 - B_2)$, we have $f(x) \in B_1 - B_2$, $f(x) \in B_1$ but $f(x) \notin B_2$, equivalently $x \in f^{-1}(B_1)$ but $x \notin f^{-1}(B_2)$, hence $x \in f^{-1}(B_1) - f^{-1}(B_2)$; so we have $f^{-1}(B_1 - B_2) \subseteq f^{-1}(B_1) - f^{-1}(B_2)$.

Above all, we have $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

Exercise 2.50

Given any $y \in B \cap f(X)$, $y \in f(X)$ implies there exists $x \in X$ such that $f(x) = y \in B$, then $x \in f^{-1}(B)$, so $y = f(x) \in f(f^{-1}(B))$, which means $B \cap f(X) \subseteq f(f^{-1}(B))$.

Conversely, given any $y \in f(f^{-1}(B))$, there exits $x \in f^{-1}(B) \subseteq X$ such that $y = f(x) \in f(X)$; according to the definition of preimage, $x \in f^{-1}(B)$ implies $y = f(x) \in B$, so $y \in B \cap f(X)$, which means $f(f^{-1}(B)) \subseteq B \cap f(X)$.

Above all, $f(f^{-1}(B)) = B \cap f(X)$.