Math1014 Calculus II
Week 3-4: Definite Integrals: Work, Arc Length, Surface Area, ...

1. A swimming pool is 20 m long and 10 m wide, with a bottom that slopes uniformly from a depth of 1 m at one end to a depth of 2 m at the other end. Assuming the pool if full, how much work is required to pump the water to a level 0.2 m above the top of the pool?

   Set up an upward y-axis from a corner at the bottom of the pool. Then for $0 \le y \le 1$, the cross section area has width 10 m, length L which satisfies $\frac{L}{20} = \frac{y}{1}$; i.e. $L = 20y$. For $1 \le y \le 2$, the cross section area is $20 \cdot 10$ m^2.

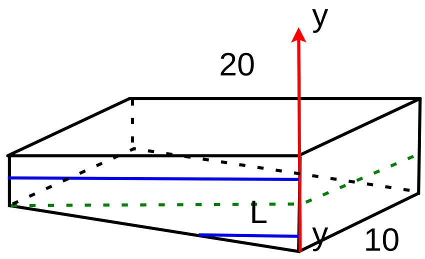
   So the work to lift a thin layer of water at depth y with tiny height Δy up $2.2 - y$ meters is approximately

$$\Delta W \approx \begin{cases} \underbrace{[\rho(10 \cdot 20y)\Delta y]}_{density \cdot volume = mass} \cdot g \cdot (2.2 - y) & \text{if } 0 \le y \le 1 \\ \underbrace{[\rho(10 \cdot 20)\Delta y]}_{density \cdot volume = mass} \cdot g \cdot (2.2 - y) & \text{if } 1 \le y \le 2 \end{cases}$$

   where $\rho = 1000\ kg/m^3$ is the density of water, and $g = 9.8\ m/s^2$.

   The work required is thus

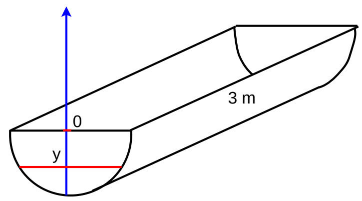
$$\begin{aligned} W &= \int_0^1 \rho g \cdot 10 \cdot 20y \cdot (2.2 - y)dy + \int_1^2 \rho g \cdot 20 \cdot 10(2.2 - y)dy \\ &= 20\rho g \left(\int_0^1 (22y - 10y^2)dy + \int_1^2 (22 - 10y)dy \right) \\ &= 20\rho g \left(\left[11y^2 - \frac{10}{3}y^3 \right]_0^1 + \left[22y - 5y^2 \right]_1^2 \right) \\ &= \frac{880}{3}\rho g = 2.87467 \times 10^6 \text{ J.} \end{aligned}$$



2. A water trough has a semicircular cross section with a radius of 0.25m and a length of 3 m. How much work is required to pump the water out of the trough when it is full?

   Water density: $\rho = 1000$ kg/m^3, gravity acceleration $= 9.8$ m/s^2.

$$\begin{aligned} \text{work required} &= \int_{-0.25}^{0} (-y) \cdot g \cdot \rho \cdot \underbrace{3 \cdot 2\sqrt{0.25^2 - y^2}\, dy}_{\approx \text{“volume of a thin layer or water”}} \\ &= \rho g \left[2(0.25^2 - y^2)^{3/2} \right]_{-0.25}^{0} = \frac{\rho g}{32} \text{ J.} \end{aligned}$$



3. Find the arc length of the curve.

   (i) $y^2 = 4(x + 4)^3$, $0 \le x \le 2$, $y > 0$.   (ii) $x = \frac{y^3}{6} + \frac{1}{2y}$, $\frac{1}{2} \le y \le 1$.

   (i) Note that $y = 2(x + 4)^{3/2}$, hence $\frac{dy}{dx} = 3(x + 4)^{1/2}$.

$$\text{arc length} = \int_0^2 \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^2 \sqrt{1 + 9(x + 4)} dx$$

$$= \int_0^2 (9x + 37)^{1/2} dx = \left[\frac{2}{27}(9x + 37)^{3/2} \right]_0^2 = \frac{2}{27}(55\sqrt{55} - 37\sqrt{37})$$

(ii) $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$, hence $\dfrac{dx}{dy} = \dfrac{y^2}{2} - \dfrac{1}{2y^2} = \dfrac{y^4-1}{2y^2}$.

$$\text{arc length } = \int_{\frac{1}{2}}^{1} \sqrt{1+(\frac{dx}{dy})^2}dy = \int_{\frac{1}{2}}^{1} \sqrt{1+\frac{y^8-2y^4+1}{4y^4}}dy = \int_{\frac{1}{2}}^{1} \sqrt{\frac{y^8+2y^4+1}{4y^4}}dy$$

$$= \int_{\frac{1}{2}}^{1} \frac{y^4+1}{2y^2}dy = \int_{\frac{1}{2}}^{1} [\frac{y^2}{2} + \frac{1}{2y^2}]dy = \Big[\frac{y^3}{6} - \frac{1}{2y}\Big]_{\frac{1}{2}}^{1} = \frac{1}{6} - \frac{1}{2} - \frac{1}{48} + 1 = \frac{31}{48}$$

4. Find the area of the surface of revolution obtained by rotating the curve $y = 1 - x^2$, $0 \le x \le 1$, about: (i) the y-axis; (ii) the line $x = -1$. $\quad$ $(\int \sqrt{a^2+u^2}du = \frac{x}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+u^2}) + C)$

(i) $\dfrac{dy}{dx} = -2x$, and hence the surface area is given by

$$S = \int_0^1 2\pi x ds = \int_0^1 2\pi x\sqrt{1+4x^2}dx = 2\pi\Big[\frac{2}{3}\cdot\frac{1}{8}(1+4x^2)^{3/2}\Big]_0^1 = \frac{(5\sqrt{5}-1)\pi}{6}$$

Or consider $x = \sqrt{1-y}$, $0 \le y \le 1$. Then $\dfrac{dx}{dy} = -\dfrac{1}{2}(1-y)^{-1/2}$, and the surface area is given by

$$A = \int_0^1 2\pi x\sqrt{1+(\frac{dx}{dy})^2}dy = 2\pi\int_0^1 \sqrt{1-y}\sqrt{\frac{5-4y}{4(1-y)}}\,dy$$

$$= \pi\int_0^1 \sqrt{5-4y}dy = \pi\Big[-\frac{2}{3}\cdot\frac{1}{4}(5-4y)^{3/2}\Big]_0^1 = \frac{\pi}{6}(5\sqrt{5}-1)$$

(ii) The surface area is given by

$$A = \int_0^1 2\pi(1+x)ds = \int_0^1 2\pi(1+x)\sqrt{1+4x^2}dx$$

$$= \int_0^1 2\pi\cdot 2\sqrt{\frac{1}{4}+x^2}dx + \int_0^1 2\pi x\sqrt{1+4x^2}dx$$

$$= 4\pi\Big[\frac{x}{2}\sqrt{\frac{1}{4}+x^2} + \frac{\frac{1}{4}}{2}\ln\big(x+\sqrt{\frac{1}{4}+x^2}\big)\Big]_0^1 + \frac{(5\sqrt{5}-1)\pi}{6}$$

$$= \sqrt{5}\pi + \frac{1}{2}\pi\ln(2+\sqrt{5}) + \frac{(5\sqrt{5}-1)\pi}{6} = \frac{11\sqrt{5}-1}{6}\pi + \frac{1}{2}\pi\ln(2+\sqrt{5})$$

5. Find the surface area of the torus obtained by rotating the circle $x^2 + (y-3)^2 = 1$ about the x-axis.

Note that $y = 3 \pm \sqrt{1-x^2}$, where $-1 \le x \le 1$. Moreover,

$$\frac{dy}{dx} = \pm\frac{-x}{\sqrt{1-x^2}}$$

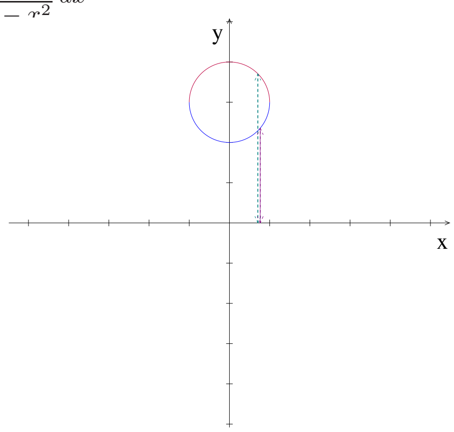
$$ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+\frac{x^2}{1-x^2}}\,dx = \frac{1}{\sqrt{1-x^2}}\,dx$$

The area of the torus is

$$\text{volume } = \int_{-1}^{1} 2\pi y_{up}ds + \int_{-1}^{1} 2\pi y_{down}ds$$

$$= \int_{-1}^{1} 2\pi\Big[3+\sqrt{1-x^2}\Big]\frac{1}{\sqrt{1-x^2}}dx + \int_{-1}^{1} 2\pi\Big[3-\sqrt{1-x^2}\Big]\frac{1}{\sqrt{1-x^2}}dx$$

$$= 6\pi\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = 6\pi\Big[\sin^{-1}x\Big]_{-1}^{1} = 12\pi(\frac{\pi}{2}+\frac{\pi}{2}) = 12\pi^2$$



6. Find the area of the surface of revolution obtained by rotating the part of the curve

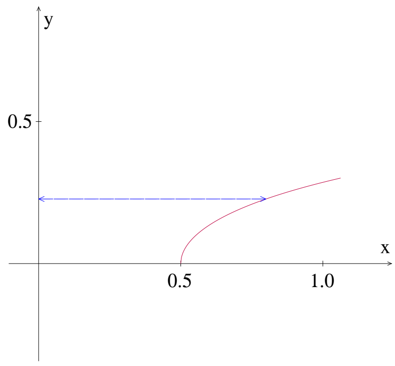
$$y = \frac{1}{2}\ln(2x + \sqrt{4x^2 - 1})$$

between the points $(\frac{1}{2}, 0)$ and $(\frac{17}{16}, \ln 2)$ about the y-axis.

Note that $e^{2y} = 2x + \sqrt{4x^2 - 1}$, and hence $4x^2 - 1 = (e^{2y} - 2x)^2 = e^{4y} - 4xe^{2y} + 4x^2$, i.e.,

$$x = \frac{e^{4y} + 1}{4e^{2y}} = \frac{e^{2y} + e^{-2y}}{4}, \qquad \frac{dx}{dy} = \frac{e^{2y} - e^{-2y}}{2}$$

$$\begin{aligned}
\text{surface area } &= \int_0^{\ln 2} 2\pi x ds = \int_0^{\ln 2} 2\pi x\sqrt{1 + (\frac{dx}{dy})^2}\, dy \\
&= \int_0^{\ln 2} 2\pi \frac{e^{2y} + e^{-2y}}{4}\sqrt{1 + (\frac{e^{2y} - e^{-2y}}{2})^2} dy \\
&= \frac{\pi}{4}\int_0^{\ln 2} (e^{2y} + e^{-2y})^2 dy = \frac{\pi}{4}\int_0^{\ln 2} (e^{4y} + 2 + e^{-4y}) dy \\
&= \frac{\pi}{4}\Big[\frac{1}{4}e^{4y} + 2y - \frac{1}{4}e^{-4y}\Big]_0^{\ln 2} = \frac{\pi}{16}\Big[\frac{255}{16} + 8\ln 2\Big]
\end{aligned}$$



7. Find the mass of a metal plate in the shape of the region bounded by the curves $y = x^3$, $x + y = 2$, $y = 0$, if the value of the density function at the coordinate point (x, y) is $\rho(x, y) = (1 + y)$ kg/m^2.

The curves $y = x^3$, $y = -x + 2$ intersect at the point $(1, 1)$.

$$\begin{aligned}
\text{mass } &= \int_0^1 \underbrace{(1 + y)}_{\text{density}}\underbrace{[(2 - y) - y^{1/3}]dy}_{\text{thin horizontal area}} \\
&= \int_0^1 (2 + y - y^2 - y^{1/3} - y^{4/3}) dy \\
&= \Big[2y + \frac{y^2}{2} - \frac{y^3}{3} - \frac{3}{4}y^{4/3} - \frac{3}{7}y^{7/3}\Big]_0^1 \\
&= 2 + \frac{1}{2} - \frac{1}{3} - \frac{3}{4} - \frac{3}{7} = \frac{83}{84} \quad \text{(kg)}
\end{aligned}$$