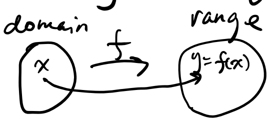
MATH1012 Calculus IA

$\boxed{\text{Review}}$

input $x \xrightarrow{f} y = f(x)$ output

※ One input of x will only give one output of y.
(y is uniquely given by x).

1. Domain and range



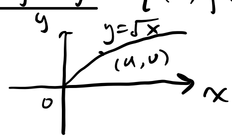
※ Usually, the domain of f is the set where f is well-defined.

Example: $y=\sqrt{x+1}$. domain: $[-1,+\infty)$, range: $[0,+\infty)$.

2. The graph of $f = \{(x, f(x)) \mid x \in D\}$

D → the domain of f.

Example:



The point (u,v) is on the graph of f if and only if $v = f(u)$.

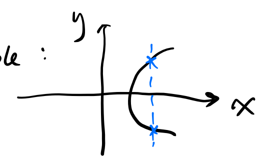
3. Two tests

Given a curve —vertical line test→ a function / not a function

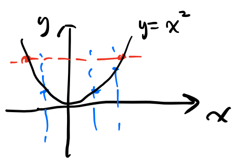
a function —horizontal line test→ one-to-one / not one-to-one.

For any output y, there exists an unique x such that $y=f(x)$

Example:

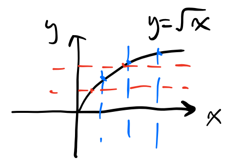


not a function



$y=x^2$ is a function.

$y=x^2$ is not one-to-one.



$y=\sqrt{x}$ is a function

$y=\sqrt{x}$ is one-to-one.

4. Operations on functions

Given f and g.

D_f: the domain of f.

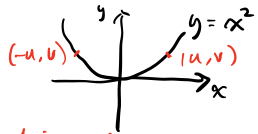
D_g: the domain of g.

operations	domain
$(f+g)(x) = f(x)+g(x)$	$D_{f+g} = D_f \cap D_g$.
$(f-g)(x) = f(x)-g(x)$	$D_{f-g} = D_f \cap D_g$.
$(f\cdot g)(x) = f(x)\cdot g(x)$	$D_{f\cdot g} = D_f \cap D_g$.
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$D_{\frac{f}{g}} = D_f \cap D_g \cap \{x \mid g(x) \neq 0\}$.
composition: $(f\circ g)(x) = f(g(x))$.	$D_{f\circ g} = D_g \cap \{x \mid g(x) \in D_f\}$.
composition: $(g\circ f)(x) = g(f(x))$.	$D_{g\circ f} = D_f \cap \{x \mid f(x) \in D_g\}$.

Notice: In general, $f\circ g \neq g \circ f$.

5. Some special properties of functions.

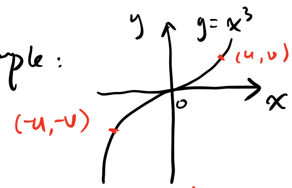
(1). f is called "even" : $f(x) = f(-x)$. for any $x \in D_f$.

Example :


$f(-x) = (-x)^2 = x^2 = f(x)$.

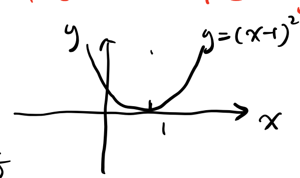
→ symmetric about y axis.

f is called "odd" : $f(x) = -f(-x)$. for any $x \in D_f$.

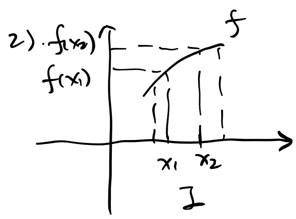
Example :


$f(-x) = (-x)^3 = -x^3 = -f(x)$.

→ symmetric about the origin.

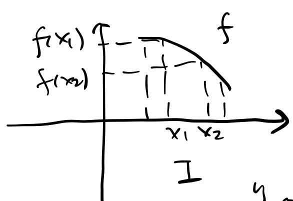
Example :


$y = (x-1)^2$ is not even and is not odd.

2).


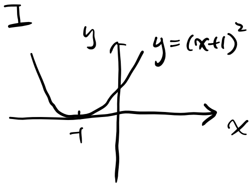
f is increasing on the interval I.

$\Leftrightarrow$ $f(x_1) < f(x_2)$ for any $x_1, x_2 \in I$ with $x_1 < x_2$.

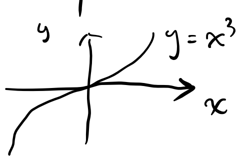


f is decreasing on the interval I.

$\Leftrightarrow$ $f(x_1) > f(x_2)$ for any $x_1, x_2 \in I$ with $x_1 < x_2$.

Example :


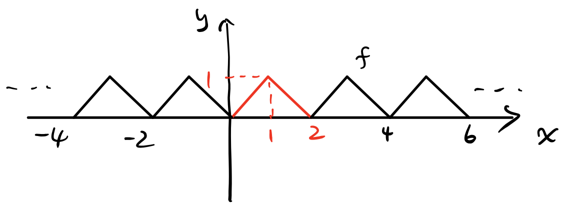
f is increasing on $[-1, +\infty)$.
f is decreasing on $(-\infty, -1]$.



f is increasing on $(-\infty, +\infty)$.
(f is an increasing function).

3). f is periodic : $f(x) = f(x+p)$ for any $x \in D_f$.

The smallest such p is called the period of f.



← The period of f is 2.

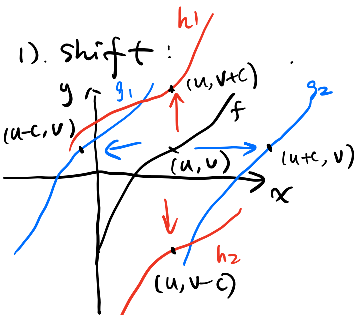
To conclude: Some basic information about a function:

1) domain and range.
2). y-intercept. x-intercept.
3) one-to-one? (using the horizontal line test).
4). odd? even? neither?
5). perodic?
6). the increasing / decreasing intervals.

6. Transformations of functions.

$\begin{cases} \text{shift} \\ \text{stretch / shrink.} \\ \text{reflect.} \end{cases}$

1). shift:



$c > 0$.

$g_1(x) = f(x+c)$. → horizontally shift to the left

$g_2(x) = f(x-c)$. → horizontally shift to the right.

$h_1(x) = f(x) + c$ → vertically shift upward.

$h_2(x) = f(x) - c$. → vertically shift downward.

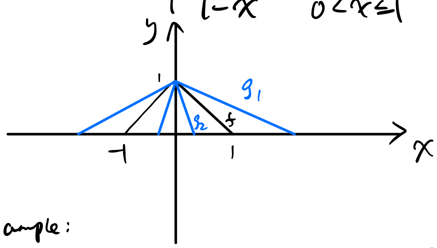
Example: If (u, v) is on the graph of f → $\underbrace{v = f(u)}$.

then $(u-c, v)$ is on the graph of g_1 (because $g_1(x) = f(x+c)$

→ $\underbrace{g_1(u-c)} = \underbrace{f(u-c+c)} = f(u) = v$)

2). stretch / shrink.

$y = f(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 < x \le 1 \end{cases}$



$c > 1$

$g_1(x) = f(\frac{1}{c}x)$. → horizontally stretching.

$g_2(x) = f(cx)$. → horizontally shrinking.

$h_1(x) = c \cdot f(x)$ → vertically stretching

$h_2(x) = \frac{1}{c} \cdot f(x)$. → vertically shrinking.

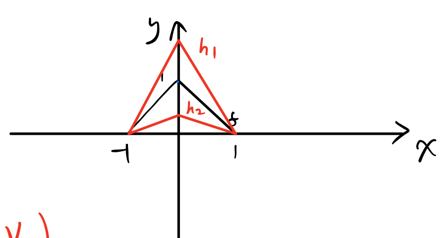
Example:

If (u, v) is on the graph of f.

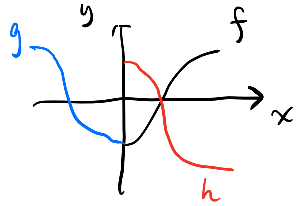
then (cu, v) is on the graph of g_1.

(because $g_1(x) = f(\frac{1}{c}x)$

→ $g_1(cu) = f(\frac{1}{c} \cdot cu) = f(u) = v$)



3). reflect.



$g(x) = f(-x)$. obtained by reflecting f about y-axis.

$h(x) = -f(x)$ obtained by reflecting f about x-axis.

7. Polynomial. $y = \underline{\underline{a_n}} \cdot x^n + \underline{\underline{a_{n-1}}} \cdot x^{n-1} + \cdots + \underline{\underline{a_1}} \cdot x + \underline{\underline{a_0}}$.

$a_0, a_1, \cdots, a_n$: coefficients of y

a_n : the leading coefficient.

n : nonnegative integer called the "degree" of y.

$n = 0$, $y = a_0 \rightarrow$ constant function.

$n = 1$. $y = a_1 x + a_0 \rightarrow$ linear function.

$n = 2$. $y = a_2 x^2 + a_1 x_1 + a_0 \rightarrow$ quadratic function.

Notice : domain of y is $(-\infty, +\infty)$

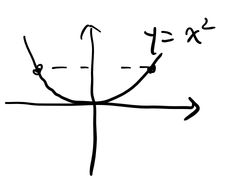
Rational functions : $y = \dfrac{P(x)}{Q(x)}$ → polynomial.

Notice: domain of y is $\{x \mid Q(x) \neq 0\}$.

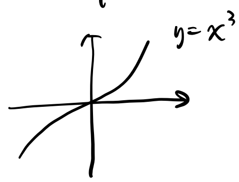
8. Inverse functions. $\begin{cases} \text{only defined for one-to-one functions} \\ \text{only one-to-one functions have the inverse functions.} \end{cases}$

Recall : f is one-to-one $\Leftrightarrow$ for any output y, there exists an unique input x such that $y = f(x)$.

Example :



not one-to-one.



one-to-one.

Given a one-to-one function. $f(x)$, we define the rule g as :

output y of $f \xrightarrow{g}$ the unique input x of f. that satisfying $f(x) = y$.

1) Observation :

① g is a function (because one input of g only give one output of g).

$\to$ called "the inverse function" of f, and denoted by f^{-1}

②. $x = g(y)$. (x is expressed in terms of y).

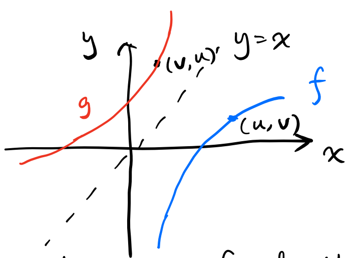
③. domain of f = range of g.

range of f = domain of g.

2). the graph of f and g.

If (u,v) is on the graph of f, then we have $v = f(u)$.

By the definition of g, we have $u = g(v)$, so (v,u) is on the graph of g



Notice: The points (u,v) and (v,u) are symmetric about the line $y=x$.

$\to$ The graphs of f and g are symmetric about the line $y=x$.

3). how to find the inverse function?

Our goal is to express x in terms of y.

Example: $y = f(x) = \sqrt{x+1}$.

domain of f: $[-1, +\infty)$.

range of f: $[0, +\infty)$.

$y^2 = x+1$.

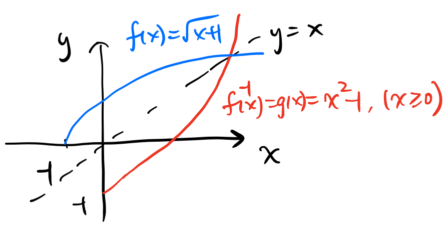
$y^2 - 1 = x$.

$x = y^2 - 1 = g(y)$.

$\downarrow$ interchange x and y (we usually use x to represent the input variable.)

$f^{-1}(x) = g(x) = x^2 - 1$.

Domain: $[0, +\infty)$.



9. exponential functions and logarithmic functions.

1). exponential function: $y = \underline{\underline{a}}^x$.

$\downarrow$

base. $a > 0$.

domain: $(-\infty, +\infty)$.

range: if $a=1$. $\{1\}$.

if $a \neq 1$ and $a > 0$. $(0, +\infty)$.

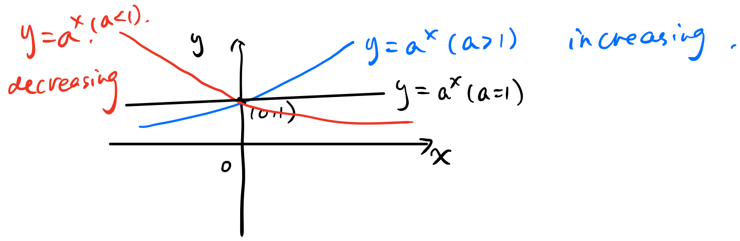
① Laws of exponents: $a>0,\ b>0$.

$a^x \cdot a^y = a^{x+y}$. $\quad \dfrac{a^x}{a^y} = a^{x-y}$. $\quad (a^x)^y = a^{xy}$.

$a^x \cdot b^x = (ab)^x$. $\quad a^{-x} = \dfrac{1}{a^x}$. $\quad a^{\frac{x}{y}} = \sqrt[y]{a^x}$.

Notice: $a^0 = 1$. $(a>0)$.

② graph of $y=a^x$.



2). logarithmic functions. $\begin{cases} y=\log_a x. \\ a^y = x. \end{cases}$ $a>0,\ a\neq 1$.

Domain: $(0,+\infty)$.

range: $(-\infty,+\infty)$.

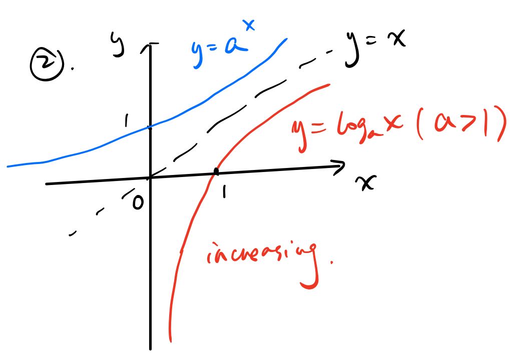
① Laws of logarithms:

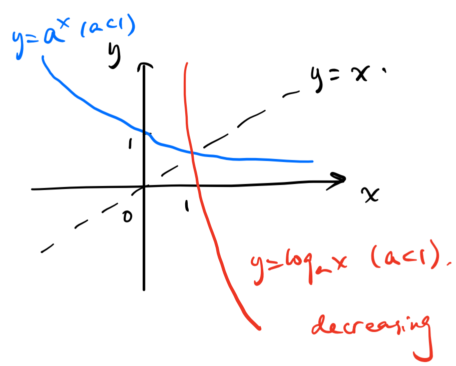
$\log_a xy = \log_a x + \log_a y$.

$\log_a \dfrac{x}{y} = \log_a x - \log_a y$.

$\log_a x^r = r \cdot \log_a x$. where r is any real number.

②.





Notice: $y=a^x$ and $y=\log_a x$ are inverse functions to each other.

Their graphs are symmetric about the line $y=x$.