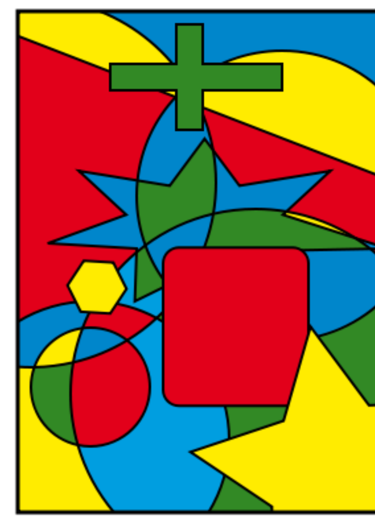
Part I Logic

- L01 Propositional Logic
- L02 Predicate Logic
- L03 Inference Rules and Proof Techniques

What is Logic?

- Puzzles
- Prerequisite of law schools (LSAT)
  - Riders A, B, C, D, E, F, and G take busses 1, 2, 3 to work.
  - Neither E nor G takes bus 1 on a day when B does.
  - G does not take bus 2 on a day when D does.
  - When A and F take the same bus, it is always bus 3.
  - C always takes bus 3.
  - Who takes which bus?
- Rules of mathematical reasoning
- Automated reasoning and artificial intelligence
  - Automated theorem proving: Four color theorem
  - Proof verification



L01: Propositional Logic

- Outline
  - Propositions
  - Compound Propositions
  - Propositional Equivalences

- Reading
  - Kenneth Rosen, section 1.1-1.3

Proposition

- **Definition**

  A **proposition** is a declarative statement (i.e., a sentence that declares a fact) that is either true or false, but not both

- **Remark**

  Propositions are the basic building blocks of logic. The area of logic that deals with propositions is called **propositional logic**.

- **Definition**

  The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and false, denoted by F, if it is a false proposition.

Examples

- Each of the following declarative statements is a proposition:
  (a) Beijing is the capital of China.
  (b) COMP 2711 is an elective course for the COMP program.
  (c) $2 + 2 = 2^2$
  (d) $1 + 1 = 3$

Propositions (a) and (c) are true but (b) and (d) are false.

Examples

- These statements are not propositions:

  (a) No parking

  (b) Who has an iPad?

  (c) $y = \log(x+1)$

  (d) $x^2 - 3x + 1 = 0$

  (c) and (d) can become propositions if x and y are assigned values

- Logical paradoxes are not propositions

  (a) This sentence is false. (Liar paradox)

  (b) A male barber shaves all and only those men who do not shave themselves. (Barber paradox)

Outline

- Introduction to Propositions
- **Compound Propositions**
- Propositional Equivalence

Logical Operator and Truth Table

- **Propositional variable:** We use lowercase letters $p, q, r, \ldots$ to represent propositions.
- **Logical operators** or **logical connectives** can be used to turn existing propositions into new propositions.
- The definition of a logical operator can be given in the form of a **truth table** by enumerating all possible truth values of the proposition(s) involved.

Negation

- **Definition**
  Let p be a proposition. The **negation** of p, denoted by $\neg p$ or $\bar{p}$ and read as "not p", is the statement "it is not the case that p". The truth value of $\neg p$ is the opposite of the truth value of p.

p	$\neg p$
T	F
F	T

- Example:
  - p: "Tom's iPhone has at least 32GB of memory"
  - $\neg p$: "It is not the case that Tom's iPhone has at least 32GB of memory"
  - $\neg p$: "Tom's iPhone has less than 32GB of memory"

Conjunction

- **Definition**

  Let p and q be propositions. The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p and q".

  The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- **Definition**

  The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q".

  The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

- **Definition**

  The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement

- **Definition**
  The **conditional statement** $p \rightarrow q$ is the proposition "if p, then q".
  p is called the **hypothesis** (or **premise**) and q is called the **conclusion** (or **consequence**).
  Also called an **implication**.
- Note: There does not need to be any connection between the hypothesis and the conclusion. The true value of $p \rightarrow q$ depends only on the truth values of p and q.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples

- “If Beijing is the capital of China, then Beijing is a city of China”.
- True.
- “If Beijing is a city of China, then Beijing is the capital of China.”
- True.
- “If the moon is made of green cheese, then I have more money than Bill Gates. ”
- True.
- “If the moon is not made of green cheese, then I have more money than Bill Gates. ”
- False.

Necessary and sufficient conditions

- If $p \rightarrow q$ is true, then p is a **sufficient condition** of q, and q is a **necessary condition** of p
- Example:
  - "If I am elected, then taxes will be lowered."
  - "I am elected" is a sufficient condition for "lower taxes".
    - Other candidate may also lower taxes.
  - "Lower taxes" is a necessary condition for "I am elected"
    - If taxes are not lowered, then I am definitely not elected.

Other Ways to Express conditional statements in English

- Equivalent ways of expressing $p \to q$:
  - if p, then q
  - q if p
  - q when p
  - p implies q
  - q follows from p
  - p is a sufficient condition for q
  - q is a necessary condition for p
  - p only if q (falsity of q implies falsity of p)
    - "The company can succeed only if it has sufficient funding."

Converse, Contrapositive

- **Definition**

  The **converse** of $p \rightarrow q$ is $q \rightarrow p$.

  The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

  The **inverse** of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

  $p \rightarrow q$ and its contrapositive are **equivalent**.

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Biconditional Statement

- **Definition**

  The **biconditional statement** $p \leftrightarrow q$ is the proposition "p if and only if q".

  The statement is true when p and q have the same truth value and is false otherwise.

- Equivalent ways of expressing $p \leftrightarrow q$:
  - p if and only if q
  - p iff q
  - p is necessary and sufficient for q
  - if p then q, and conversely

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ambiguity in natural language

- Sometimes, “if … then …” in natural language actually means “if and only if”.
- “If you finish your meal, then you can have dessert”
- What if you don’t finish your meal?
  - In natural language: then you can’t have dessert.
  - In logic: unspecified
- In this course, we should avoid such ambiguities.

Precedence of Logical Operators

- Multiple logical operators can be used to construct **compound propositions**.
- Parentheses can be used for clarity

$$p \lor \neg q \to p \land q,$$

may be written more clearly as

$(p \lor \neg q) \to (p \land q)$.

Operator	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\to$	4
$\leftrightarrow$	5

- Remark
  - Boolean operations in programming languages (e.g., &&, ||, !, == in C++) correspond to $\lor$, $\land$, $\neg$, $\leftrightarrow$
  - "**if** … **then** …" statements in programming languages do not correspond to $\to$

Outline

- Introduction to Propositions
- Compound Propositions
- **Propositional Equivalence**

Tautology and Contradiction

- **Definition**

A **tautology** is a compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.

A **contradiction** is a compound proposition that is always false.

A **contingency** is a compound proposition that is neither a tautology nor a contradiction.

Tautology and Contradiction

- **Example**

  $p \lor \neg p$ is a tautology

  $p \land \neg p$ is a contradiction

  $p \to \neg p$ is a contingency

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$	$p \to \neg p$
T	F	T	F	F
F	T	T	F	T

Logical Equivalence

- Compound propositions that always have the same truth values are called **logically equivalent**
- **Equivalent definition**
  The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.
- The notation is $p \equiv q$ (or $p \Leftrightarrow q$).
- **Remark**
  The symbol $\equiv$ is not a logical operator and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

  Sometimes $\Leftrightarrow$ is used instead of $\equiv$ to denote logical equivalence.

The use of $\equiv, \leftrightarrow, =$

- $\equiv$ ($\Leftrightarrow$) denotes logical equivalence regardless of truth values of the propositional variables
- $\leftrightarrow$ is a binary logical operator
- $=$ is not used on propositions. It is only used between two mathematical objects.
- Examples:
  - $(p \to q) \leftrightarrow (\neg p \vee q) \equiv T$
  - $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
  - $T \leftrightarrow F \equiv F$
  - $(2 = 3) \wedge (5 \neq 4) \equiv F$
  - $p: 2 + 2 = 4$
    p is true

Logical Equivalence: De Morgan's laws

- **Example**

  Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

De Morgan’s laws

- **Example:** Use De Morgan’s laws to express the negation of “today is Saturday and today is a holiday”
- p: today is Saturday
- q: today is a holiday
- By De Morgan’s laws $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- “today is not Saturday or today is not a holiday”
- **Example:** Use De Morgan's laws to express the negation of “Alice will send a secret message or Bob will send a secret message”
- “Alice will not send a secret message and Bob will not send a secret message.”

Logical Equivalence

- **Example**

  Can we express $p \rightarrow q$ using negations and disjunctions?

  Yes, $p \rightarrow q \equiv \neg p \vee q$ (Can you prove this using a truth table?)

Logical Equivalence

- **Example**

  Can we express $p \oplus q$ using negations, conjunctions and disjunctions?

  Yes, $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ (Can you prove this using a truth table)

Examples

- **Example**
  Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

$$
\begin{aligned}
\neg(p \rightarrow q) &\equiv \\
&\neg(\neg p \vee q) \equiv \\
&\neg(\neg p) \wedge \neg q \equiv \\
&\; p \wedge \neg q
\end{aligned}
$$

Propositional Equivalences

- The following table summarizes the major propositional equivalences:

Equivalence	Name
$p \wedge \mathsf{T} \equiv p$ $p \vee \mathsf{F} \equiv p$	Identity laws
$p \vee \mathsf{T} \equiv \mathsf{T}$ $p \wedge \mathsf{F} \equiv \mathsf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Propositional Equivalences (cont'd)

Equivalence	Name
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}$ $p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Propositional Equivalences (cont'd)

Equivalence	Name
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	Involving conditional statements
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$	Involving biconditional statements

Examples

- **Example**

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

- **Answer**

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge (\neg(\neg p \wedge q)) \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) \\
&\equiv \neg p \wedge (p \vee \neg q) \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \\
&\equiv \mathrm{F} \vee (\neg p \wedge \neg q) \\
&\equiv \neg p \wedge \neg q
\end{aligned}
$$

Examples

- **Example**

  Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology by developing a series of logical equivalences.

$$
\begin{aligned}
&(p \wedge q) \rightarrow (p \vee q) \equiv \\
&\qquad \neg (p \wedge q) \vee (p \vee q) \equiv \\
&\qquad (\neg p \vee \neg q) \vee (p \vee q) \equiv \\
&\qquad (p \vee \neg p) \vee (q \vee \neg q) \equiv \\
&\qquad \mathrm{T} \vee \mathrm{T} \equiv \mathrm{T}
\end{aligned}
$$

Revisit the Knight and Knave puzzle

- Let p be “A is knight” and q be “B is knight”
- A: “We are both knaves”: $\neg p \wedge \neg q$
- B says nothing
- Two possibilities
  - If A's claim is true, then $\neg p \wedge \neg q$ is true.
    If A is telling the truth, then he is knight and p is true.
    However, $\neg p \wedge \neg q$ and p cannot both be true.
  - If A's claim is false, then $\neg p \wedge \neg q$ is false, i.e.,
    $\neg(\neg p \wedge \neg q)$ is true. Using de Morgan's law,
    $\neg(\neg p \wedge \neg q) \equiv p \vee q$.
    Since A's claims is false, he is a khave, so p is false.
    Therefore, q must be true.