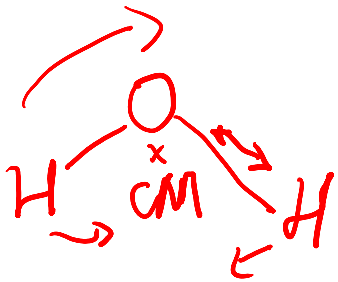


Periodic Motion II

PHYS1112

Lecture 15

Intended Learning Outcomes

- After this lecture you will learn:
  1) various types of harmonic oscillators
  2) damping effect on a harmonic oscillator
  3) forced oscillator and the phenomenon of resonance

Recap from last lecture

A mechanical system with equation of motion

$$d^2x/dt^2 = -\omega^2 x$$

is said to execute simple harmonic motion in the coordinate x, with frequency

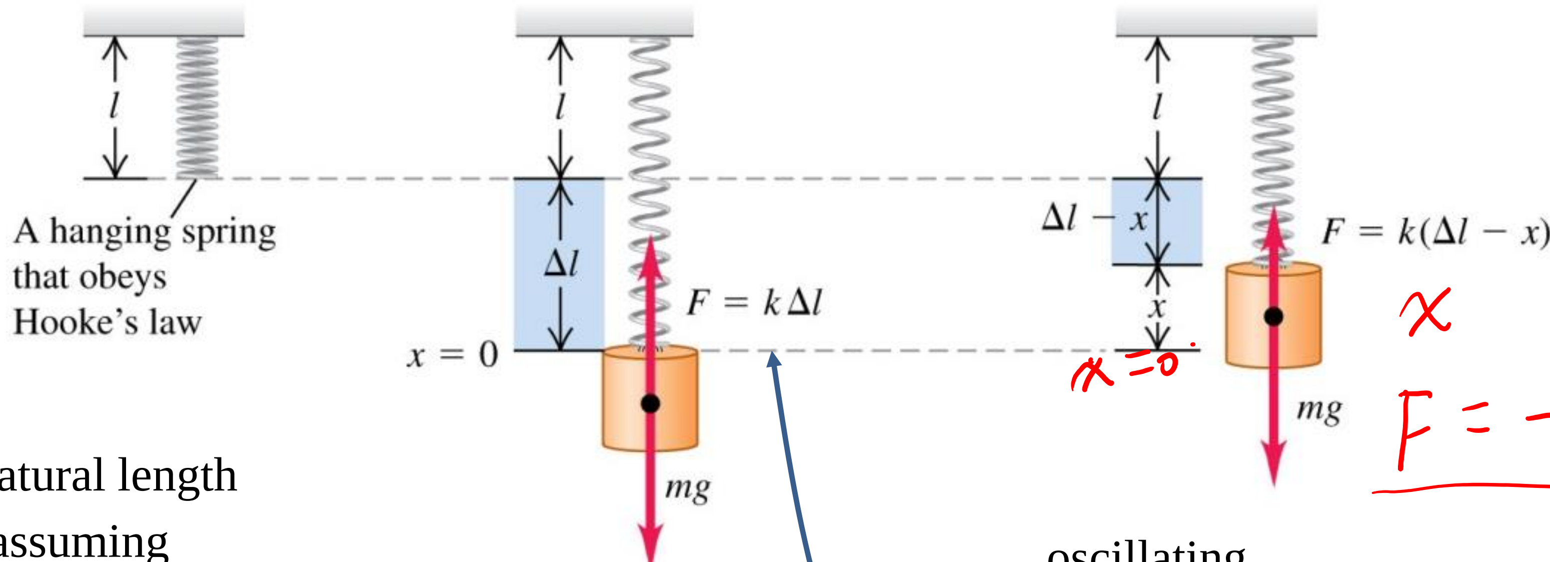
$$f = \omega/2\pi$$

and period

$$T = 1/f = 2\pi/\omega$$

$$\begin{cases} \cos t \\ \sin t \\ e^{i\omega t} \end{cases}$$

Vertical Spring and Mass System



↑ +

x

$F = -kx$

natural length (assuming spring massless)

in equilibrium,

$mg = k\Delta l$

$k\Delta l - mg = 0$

oscillating,

$F_{\text{net}} = k(\Delta l - x) - mg = -kx$

x measured from equilibrium position

Implication: same as horizontal spring and mass system *if x* measured from equilibrium position, *not* from natural (unextended) position.

Example

Suppose a car's shock absorbers are worn out so that it provides no damping to oscillations. Its mass is 1000 kg. A 100 kg person sits in it and its center of gravity lowers by 2.8 cm. It then hits a bump and start oscillating.

center of mass

Spring constant $F = -kx$

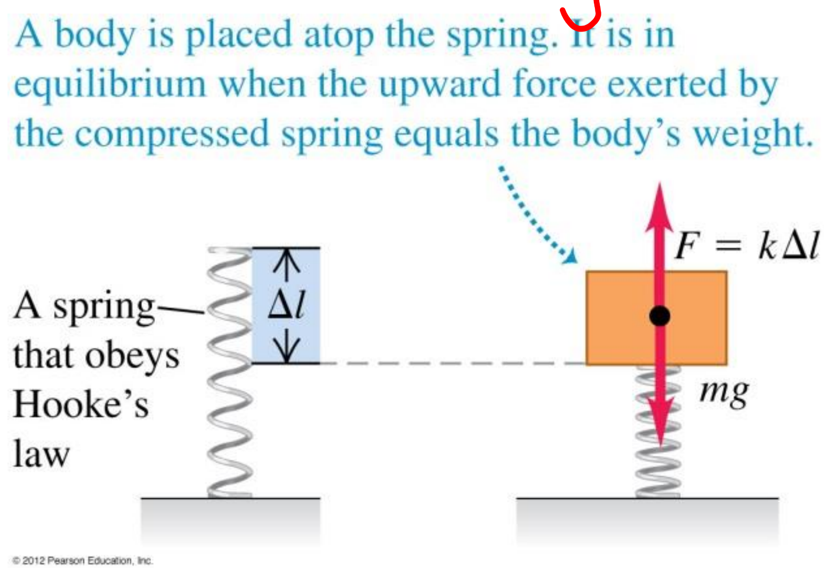
$$k = -\frac{F}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}}$$
$$= 3.5 \times 10^4\ \text{kg/s}^2$$

Period of the oscillation

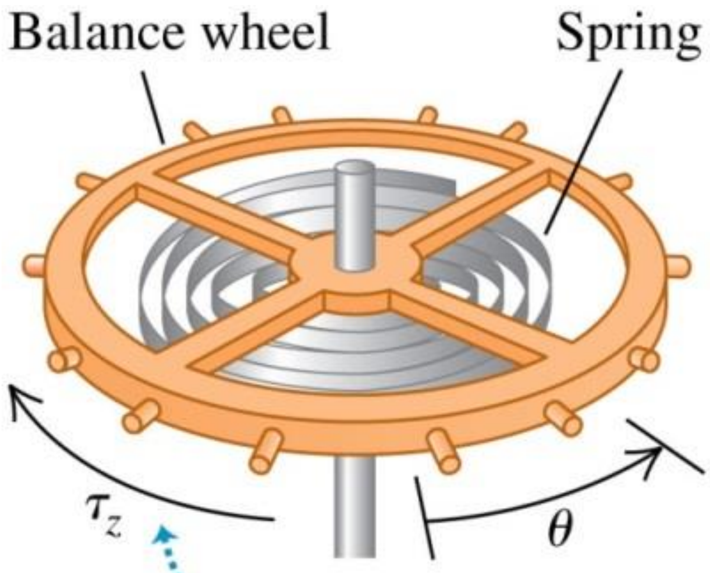
$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

g



Angular Oscillation



The spring torque τ_z opposes the angular displacement θ.

$F = -kx \qquad F = ma$

$\tau = -k\theta \qquad \tau = I\alpha$

Assumption: restoring *torque* proportional to angular displacement

$x(t)$

$\theta(t)$

$$\tau = -\kappa\theta = I\alpha \quad \Rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta}$$

$\alpha =$

$$\omega = \sqrt{\frac{\kappa}{I}}, \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}},$$

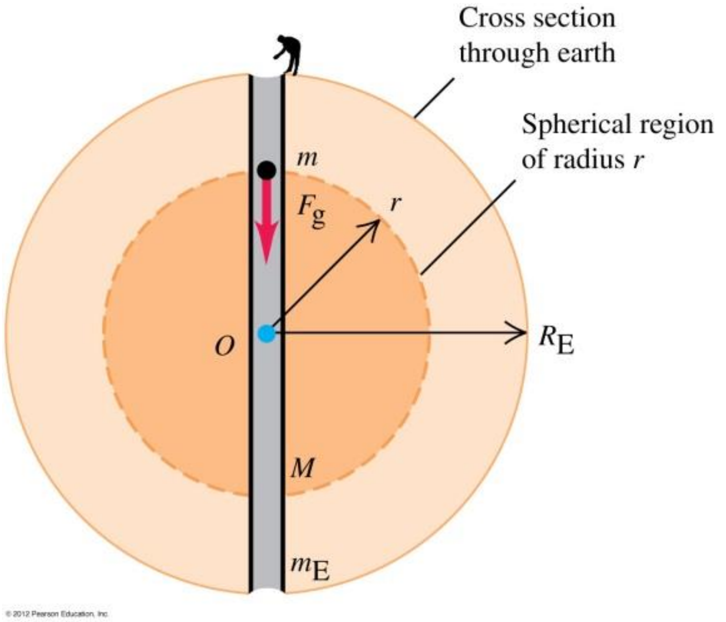
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

An imaginary case – A Tunnel Through the Earth

From lecture 13, a mass in the tunnel experience a force

$$F_g = -\frac{Gm_E m}{R_E^3} r \quad \Rightarrow \quad \frac{d^2 r}{dt^2} = \frac{F_g}{m} = -\underbrace{\frac{Gm_E}{R_E^3}}_{\omega^2} r$$

F_g opposite to r

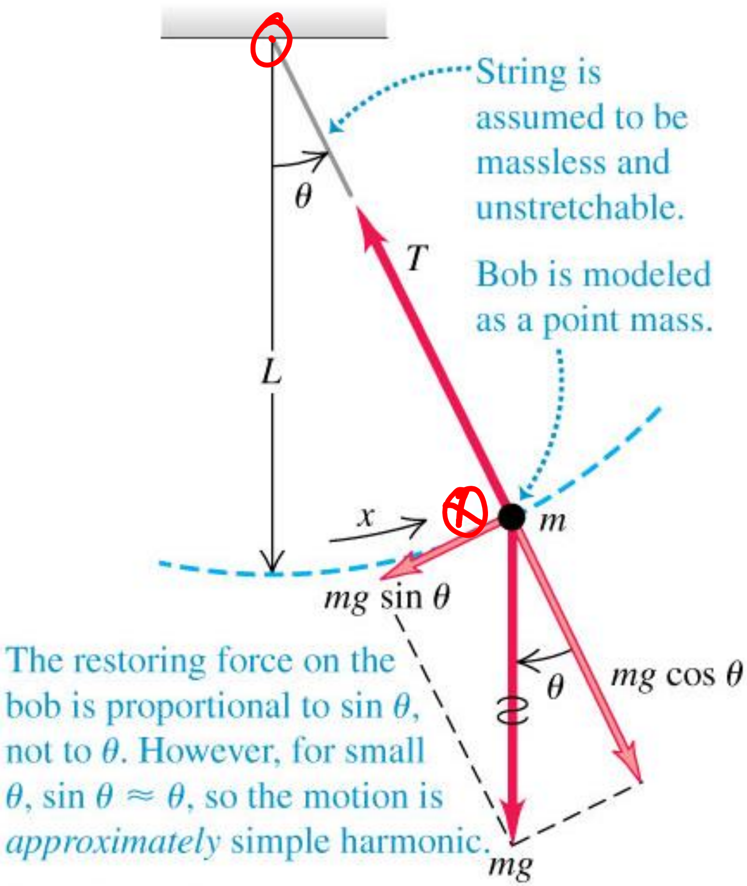


Execute SHM with period with

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{Gm_E}}$$

Simple Pendulum

Galilei observed that a lamp hung from the ceiling of a church swung with constant period



x is the arc length

If θ small enough, $\sin\theta \approx \theta = x/L$

In the tangential direction, the restoring force

$$F_\theta = -mg\sin\theta \approx -mg\frac{x}{L} = ma_{\tan}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{g}{L}x}$$

$$-mg\sin\theta = m\frac{d^2x}{dt^2}$$

$$-\cancel{m}g\frac{x}{L} = \cancel{m}\frac{d^2x}{dt^2}$$

$$\omega = \sqrt{\frac{g}{L}}, \qquad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}},$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Q14.9

A simple pendulum consists of a point mass suspended by a massless, unstretchable string. If the mass is doubled while the length of the string remains the same, the period of the pendulum

A. becomes four times greater.

B. becomes twice as great.

C. becomes greater by a factor of $\sqrt{2}$.

D. remains unchanged.

E. decreases.

A14.9

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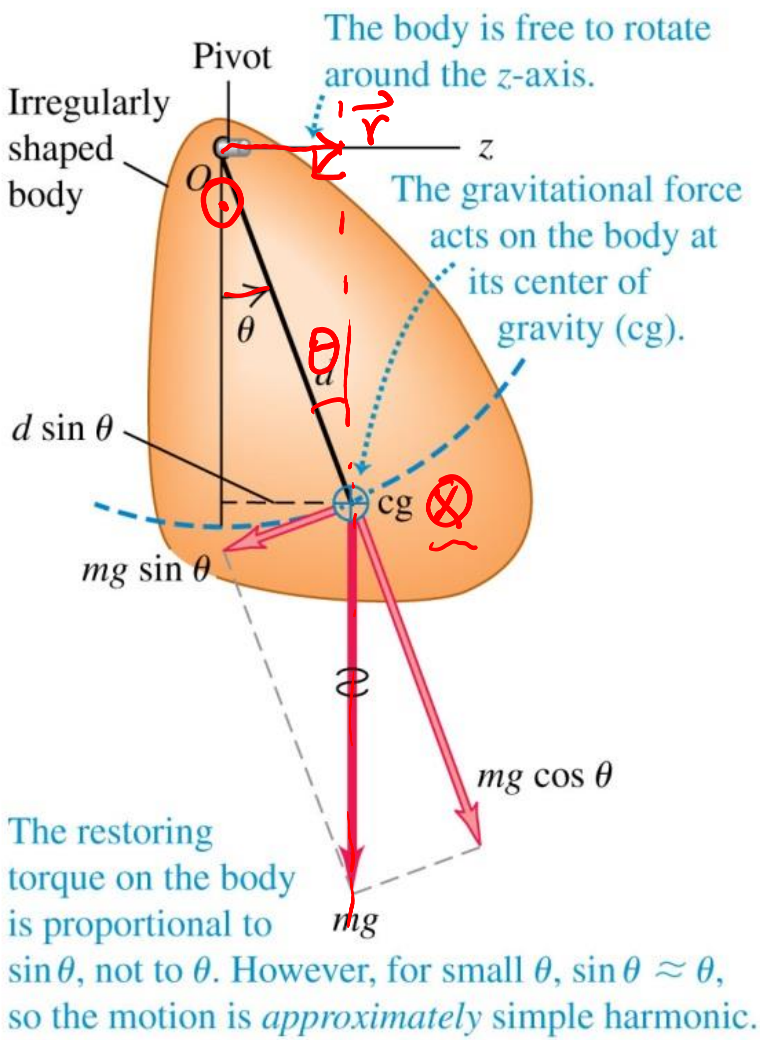
Physical Pendulum

$\tau = I\alpha$

$\tau = \vec{r} \times \vec{F}$

$= -d \sin\theta \, mg$ (with $\sin\theta \approx \theta$)



In small θ limit:

$$\tau = -(mg)d\sin\theta \approx -(mgd)\theta$$

$$= I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta}$$

$$\omega = \sqrt{\frac{mgd}{I}},$$

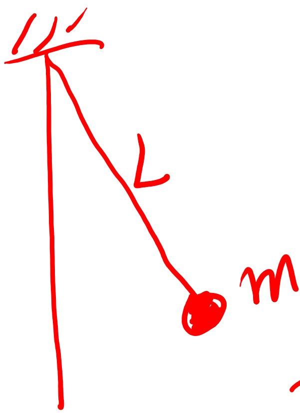
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgd}{I}},$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Demonstration:
Simple Pendulum vs. Meter Stick (same center of mass, different periods)

Question:

The CG of a simple pendulum of mass *m* and length *L* is at a distance *L* from its pivot. The CG of a uniform rod of mass *m* and length 2*L*, pivoted at one end, is also at a distance *L* from its pivot. The period of the rod is (longer / shorter / the same) as the period of the pendulum.



$T = 2\pi\sqrt{\frac{L}{g}}$

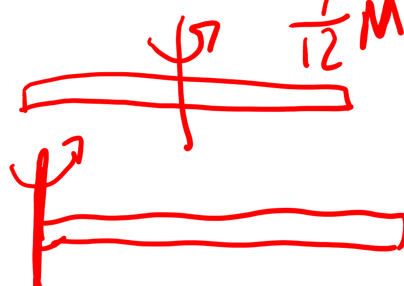
$I = mL^2$

$T = 2\pi\sqrt{\frac{\cancel{m}L^{\cancel{2}}}{\cancel{m}g\cancel{L}}}$

$T = 2\pi\sqrt{\frac{I}{mgL}}$

$I = ?$

$\frac{1}{12}M(2L)^2$



$I = \frac{1}{3}m(2L)^2 = \frac{4}{3}mL^2$

$T = 2\pi\sqrt{\frac{\frac{4}{3}\cancel{m}L^{\cancel{2}}}{\cancel{m}g\cancel{L}}}$

$= 2\pi\sqrt{\frac{4}{3}\frac{L}{g}}$

$\sqrt{\frac{4}{3}}$

Damped Harmonic Oscillator

Suppose oscillator experience fluid resistance at low speed, $f = -bv$ (oppose to v)

$$ma = -kx - bv,$$

or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

2nd order *differential equation*

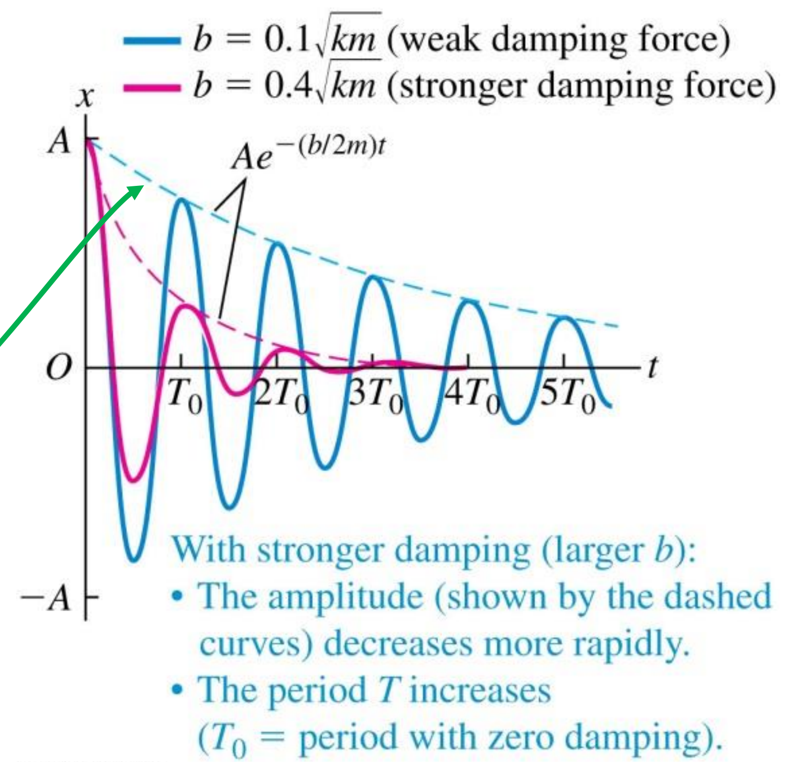
Solution:

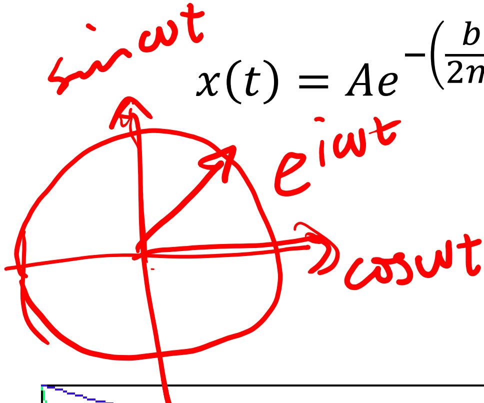
$$x(t) = \underbrace{Ae^{-\left(\frac{b}{2m}\right)t}}_{\text{exponentially decaying amplitude, or envelop, damp out the oscillation}} \underbrace{\cos(\omega' t + \phi)}_{\text{periodic oscillation with angular frequency } \omega'}$$

exponentially decaying amplitude, or envelop, damp out the oscillation

periodic oscillation with angular frequency ω'

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$





$$x(t) = Ae^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t + \phi)$$

$$\boxed{e^{i\omega t}}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\frac{k}{m} > \frac{b^2}{4m^2}$$

When $b < 2\sqrt{km}$, $\omega' > 0$, called **underdamping**.

Stronger damping (larger b), oscillation dies off faster

$$\frac{k}{m} < \frac{b^2}{4m^2}$$

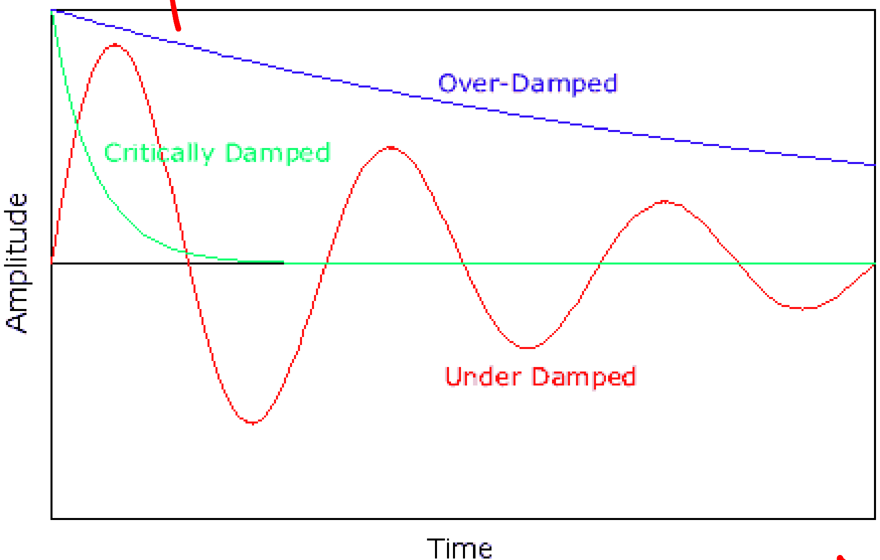
When $b > 2\sqrt{km}$, ω' imaginary (only if you know complex numbers), no oscillation, solution becomes

$$\omega' = ia$$

$$x(t) = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}$$

When $b = 2\sqrt{km}$, $\omega' = 0$, no oscillation, return to equilibrium position in shortest time, called **critical damping**

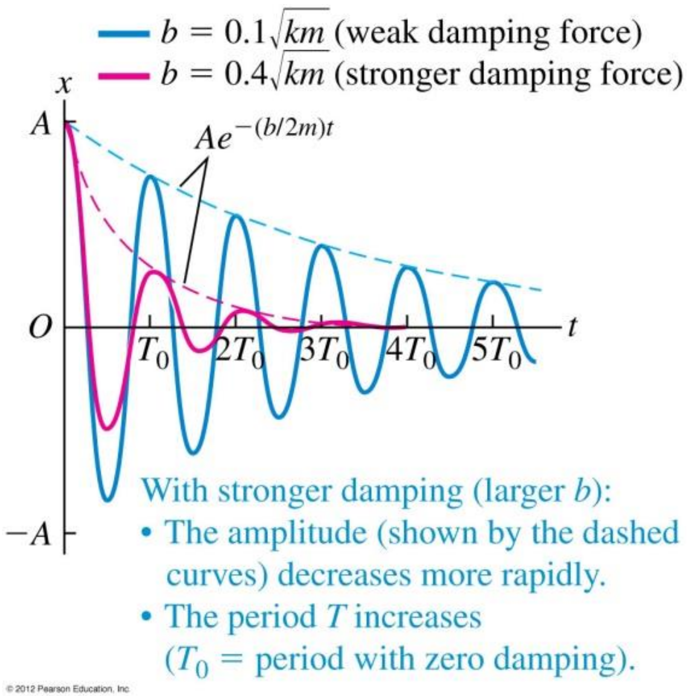
$$\frac{k}{m} = \frac{b^2}{4m^2}$$



$$e^{i\omega t} = e^{i \cdot i a t} = e^{-at}$$

$$\boxed{= 1}$$

Rate of energy dissipation



$$ma = -kx - bv$$

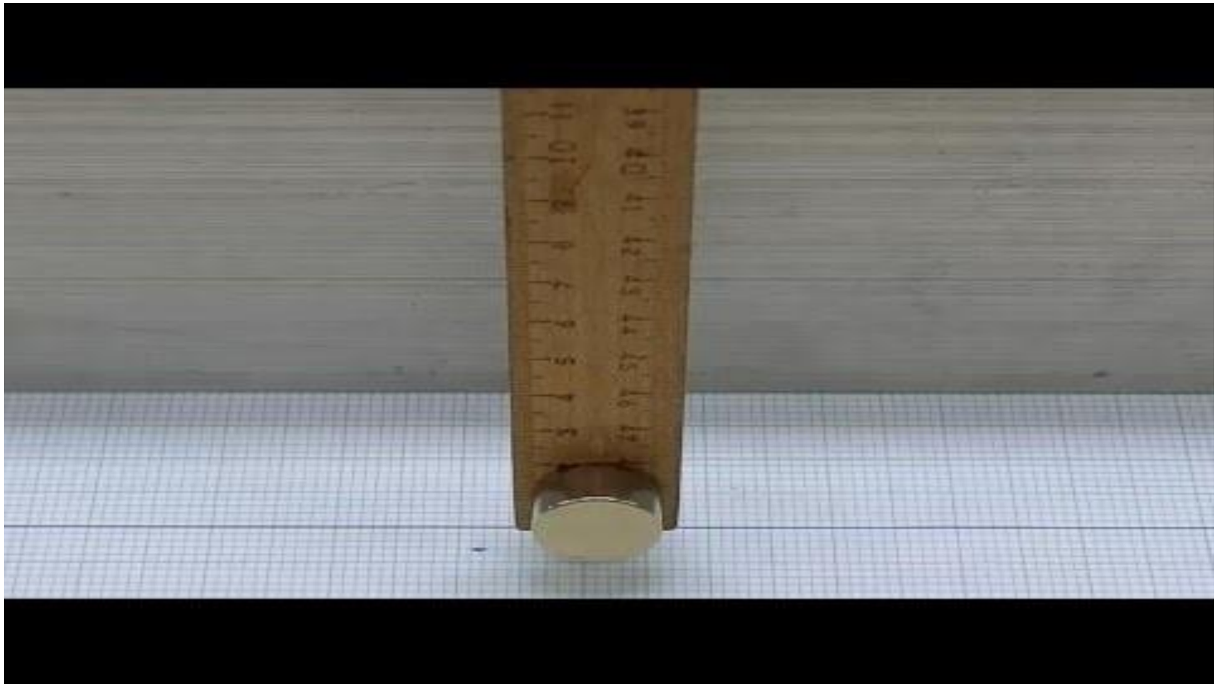
$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\underbrace{\frac{dv}{dt}}_{a} + kx\underbrace{\frac{dx}{dt}}_{v} = v\underbrace{(ma + kx)}_{-bv} = \underline{-bv^2}$$

⚠ $dE/dt < 0$ means energy decreases, i.e., dissipation

⚠ consistent with $\underset{P}{dE/dt} = Fv = (\underline{-bv})\underline{v}$ ✓

Demonstration

Under, critical, and overdamping
https://www.youtube.com/watch?v=99ZE2RGwqSM



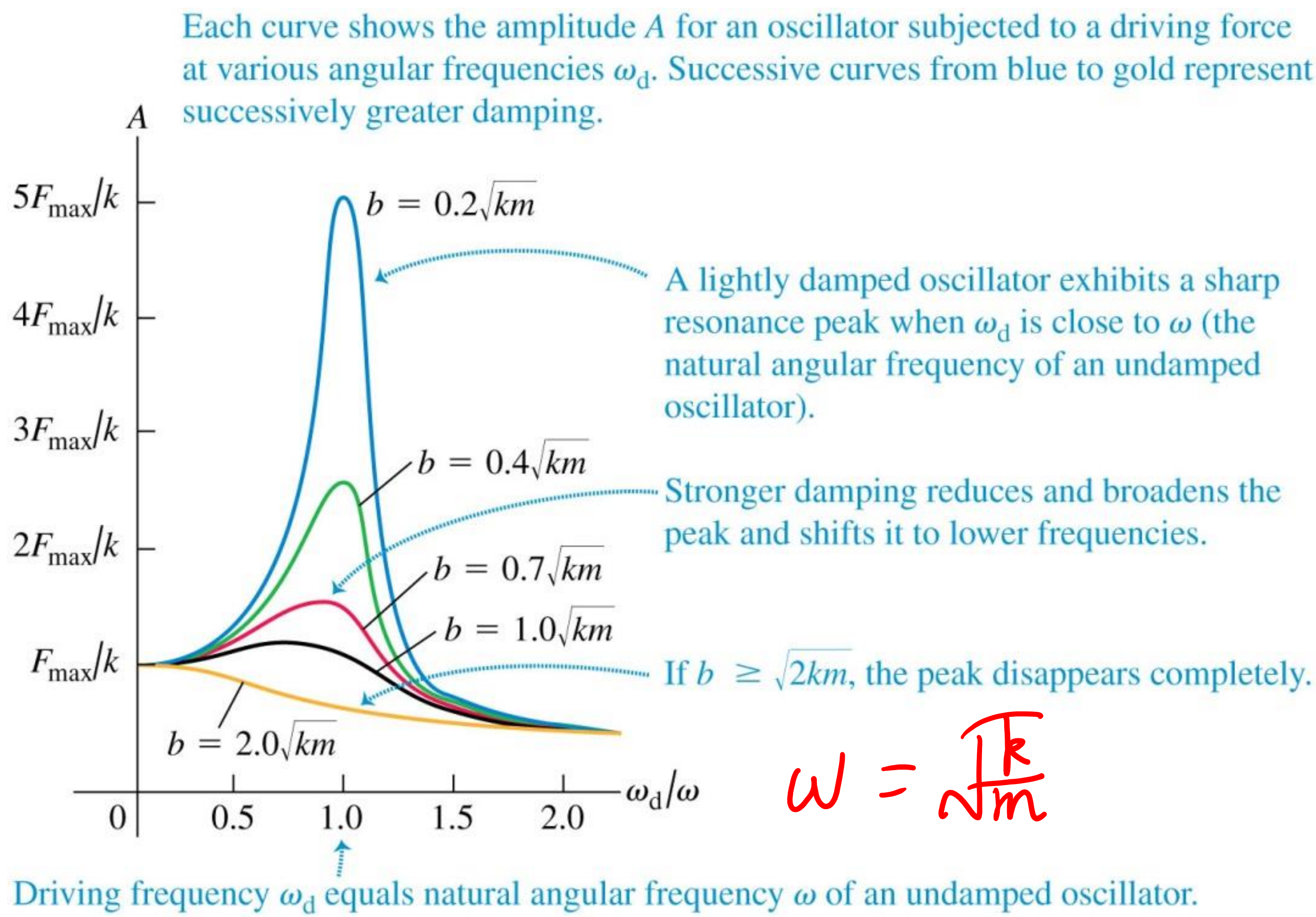
Forced Oscillations

Suppose an external periodic driving force $F(t) = F_{\text{max}} \cos \omega_d t$

$$ma = -kx - bv + F(t)$$

$$\Rightarrow \quad \boxed{m \frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_{\text{max}} \cos \omega_d t}$$

Oscillator vibrates with ω_d. What if $\omega_d \to \omega = \sqrt{k/m}$, the **natural frequency** of the free (undamped) oscillator?
Driving force deposit energy into the “natural mode” of vibration, expect amplitude to increase



$$A = \frac{F_{\max}}{\sqrt{\left(k - m\omega_d^2\right)^2 + b^2\omega_d^2}}$$

A reaches maximum when $\omega_d \approx \sqrt{k/m}$, called **resonance**

- ⚠ when damping b small, peak is higher, sharper, and closer to the **natural frequency** of the free (undamped) oscillator
- ⚠ resonance vibration can be strong enough to make bridges and buildings collapse