Math1014 Midterm Exam Solution, Spring 2013

Part I: MC Answers

White Version

Question	1	2	3	4	5	6	7	8	9	10	11	12
Answer	b	a	c	d	a	a	d	b	c	b	e	e

Part II: Long Questions

1. [*14 pts*] The shape of a container is the same as the surface of revolution obtained by rotating the curve $x = y^2$ about the y-axis, where $0 \le x \le 100$ (in meters). Suppose water flows into the container and stops flowing in just when 50% of the volume is filled. (Water density =1000 kg/m^3, and $g = 9.8$ m/s^2.)

   (a) Find the minimum work required to pump the water back to the top of the container. [9 pts]
   Answer: Let k be the depth of water in the container when it is 50% filled. Then

   $$\int_0^k \pi y^4 dy = \frac{1}{2}\int_0^{10} \pi y^4 dy$$

   $$\pi \cdot \frac{k^5}{5} = \frac{1}{2}\pi\frac{10^5}{5}$$

   $$k^5 = \frac{10^5}{2} \Longleftrightarrow k = \frac{10}{\sqrt[5]{2}}$$

   The work required to pump the water to the top of the container is

   $$\int_0^{\frac{10}{\sqrt[5]{2}}} \rho g \pi y^4 (10 - y) dy$$

   $$= 9800\pi \int_0^{\frac{10}{\sqrt[5]{2}}} (10y^4 - y^5) dy$$

   $$= 9800\pi \left[2y^5 - \frac{1}{6}y^6\right]_0^{\frac{10}{\sqrt[5]{2}}}$$

   $$= 9.8 \cdot 10^8 \pi \left(1 - \frac{10}{12\sqrt[5]{2}}\right)$$

   (b) Express by a definite integral the hydrostatic force on the inside surface of the container when 50% of its volume is filled. [5 pts]
   Answer: The hydrostatic force on the surface of the container is

   $$\int_0^k \rho g (k - y) 2\pi y^2 \sqrt{1 + (\frac{dy^2}{dy})^2}\, dy$$

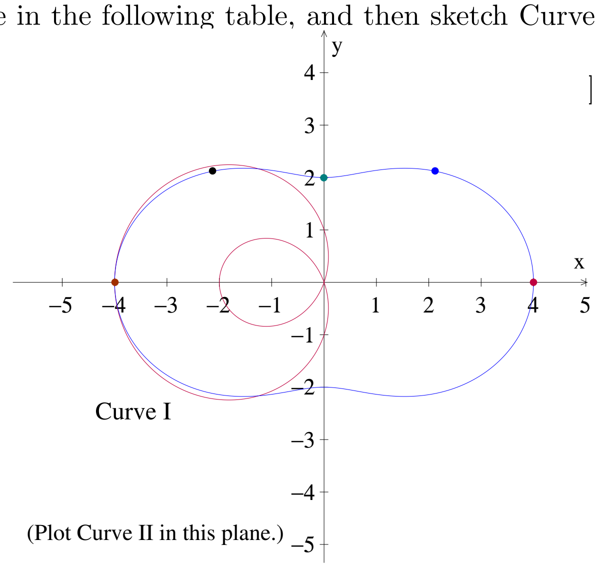
   $$= \int_0^{\frac{10}{\sqrt[5]{2}}} (9.8 \cdot 1000)(2\pi)(\frac{10}{\sqrt[5]{2}} - y) y^2 \sqrt{1 + 4y^2} dy$$

2. [*14 pts*] Curve I on the xy-plane is defined by the polar equation $r = 1 - 3\cos\theta$ whose graph is given below.

(a) Consider polar Curve II defined by the polar equation $r = 3 + \cos(2\theta)$. Fill in the exact radial coordinates of some points on this curve in the following table, and then sketch Curve II together with Curve I in the given figure.

$r = 3 + \cos(2\theta)$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	4	3	2	3	4
θ	$-\pi/4$	$-\pi/2$	$-3\pi/4$	$-\pi$	
r	3	2	3	4	



(b) Find the area of the region which lies inside Curve II, but does not overlap with any part of the region enclosed by Curve I. [8 pts]

Answer: The curves intersects when $1 - 3\cos\theta = 3 + \cos 2\theta$; i.e.,

$$\cos 2\theta + 3\cos\theta + 2 = 0$$

$$2\cos^2\theta + 3\cos\theta + 1 = (2\cos\theta + 1)(\cos\theta + 1) = 0$$

$$\theta = \pm\frac{2\pi}{3}, \pi.$$

Note that the curves are symmetric with respect to the x-axis. Curve I hits the origin when $\cos\theta = \frac{1}{3}$; e.g., when $\theta = \cos^{-1}\frac{1}{3}$.

The area of the region wanted is:

$$2\int_0^{\frac{2\pi}{3}} \frac{1}{2}(3 + \cos 2\theta)^2 d\theta - 2\int_{\cos^{-1}\frac{1}{3}}^{\frac{2\pi}{3}} \frac{1}{2}(1 - 3\cos\theta)^2 d\theta$$

$$= \int_0^{\frac{2\pi}{3}} (9 + 6\cos 2\theta + \cos^2 2\theta) d\theta - \int_{\cos^{-1}\frac{1}{3}}^{\frac{2\pi}{3}} (1 - 6\cos\theta + 9\cos^2\theta) d\theta$$

$$= \left[\frac{19}{2}\theta + 3\sin 2\theta\right]_0^{\frac{2\pi}{3}} + \frac{1}{2}\int_0^{\frac{2\pi}{3}} \cos 4\theta d\theta - \left[\frac{11}{2}\theta - 6\sin\theta\right]_{\cos^{-1}\frac{1}{3}}^{\frac{2\pi}{3}} - \frac{9}{2}\int_{\cos^{-1}\frac{1}{3}}^{\frac{2\pi}{3}} \cos 2\theta d\theta$$

$$= \frac{8\pi}{3} + 3\sin\frac{4\pi}{3} + 6\sin\frac{2\pi}{3} + \frac{11}{2}\cos^{-1}\frac{1}{3} - 6\sin\cos^{-1}\frac{1}{3} + \left.\frac{\sin 4\theta}{8}\right|_0^{\frac{2\pi}{3}} - \left.\frac{9}{4}\sin 2\theta\right|_{\cos^{-1}\frac{1}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{8\pi}{3} + \frac{43}{16}\sqrt{3} + \frac{11}{2}\cos^{-1}\frac{1}{3} - 6\sin\cos^{-1}\frac{1}{3} + \frac{9}{4}\sin 2\cos^{-1}\frac{1}{3}$$

$$= \frac{8\pi}{3} + \frac{43}{16}\sqrt{3} + \frac{11}{2}\cos^{-1}\frac{1}{3} - 3\sqrt{2}$$

Note that $\sin\cos^{-1}\frac{1}{3} = \frac{2}{3}\sqrt{2}$.