MATH2111 Tutorial 7

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1 Applications of Determinant

1. **Theorem (Cramer's Rule)**. Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A} \text{ for } i = 1, 2, \ldots, n$$

where $A_i(\boldsymbol{b})$ is the matrix obtained from A by replacing column i by the vector $\boldsymbol{b}$.

2. **Theorem (Inverse Formula)**. Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$\text{where } \operatorname{adj} A = (\operatorname{cof} A)^T \text{ and } \operatorname{cof} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

3. **Theorem**. If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

4. **Theorem**. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

2 Vector Spaces

1. **Definition (Vector Space)**. A vector space is a nonempty set V of objects, called **vectors**, on which are defined two operations, called **addition** and **scalar multiplication** (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

   (a) The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.

   (b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

   (c) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

   (d) There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

   (e) For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

   (f) The scalar multiple of $\mathbf{u}$ by c, denoted by c$\mathbf{u}$, is in V.

   (g) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

   (h) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{v}$

   (i) $c(d\mathbf{u}) = (cd)\mathbf{u}$

   (j) $1\mathbf{u} = \mathbf{u}$

2. **Fact**. For each $\mathbf{u}$ in V and scalar c,

   (a) $0\mathbf{u} = \mathbf{0}$

   (b) $c\mathbf{0} = \mathbf{0}$

   (c) $-\mathbf{u} = (-1)\mathbf{u}$

3. **Definition (Subspace)**. A subspace of a vector space V is a subset H of V that has three properties:

   (a) The <u>zero vector</u> of V is in H.

   (b) H is <u>closed under vector addition</u>. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

   (c) H is <u>closed under multiplication by scalars</u>. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

4. **Theorem**. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are in a vector space V, then Span $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a subspace of V.

3 Exercises

1. Use Cramer's rule to solve the following linear system.

$$\begin{cases} x_1 + x_2 = 3 \\ -3x_1 + 2x_3 = 0 \\ x_2 - 2x_3 = 2 \end{cases}$$

2. Compute the adjugate of the given matrix, and then use the inverse formula to give A^{-1}.

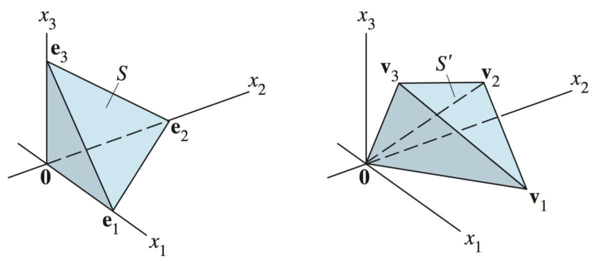
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

3. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

4. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{ \text{ area of triangle } \} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

5. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.



a. Describe a linear transformation that maps S onto S'.
b. Find a formula for the volume of the tetrahedron S' using the fact that
$\{$ volume of $S\} = (1/3)\{$ area of base $\} \cdot \{$ height $\}$

6. Let S be a set of 2×2 matrices, whose sum of all diagonal entries is zero. Verify S is a subspace of the vector space of all 2×2 matrices.