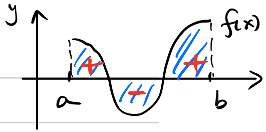
Review : Calculation of definite integral.

$\int_a^b f(x)\,dx \underset{\text{meaning}}{\overset{\text{geometric}}{=\!=}}$ sum of the signed areas between the graph of f and x-axis over $[a, b]$

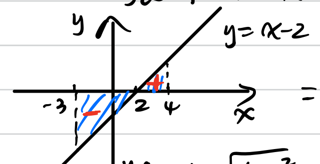


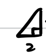
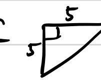
$=$ (sum of areas above the x-axis) $-$ (sum of areas below the x-axis)

We have introduced 4 ways to calculate $\int_a^b f(x)\,dx$:

① We can directly calculate $\int_a^b f(x)\,dx$ if the area is Δ, $\square$ or $\bigcirc$

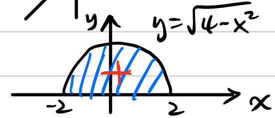
Example : $\int_{-3}^{4} (x-2)\,dx =$

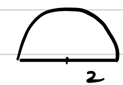


$=$ area of  $-$ area of  $= \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 5\cdot 5$

$= -\frac{21}{2}$.

Example : $\int_{-2}^{2} \sqrt{4-x^2}\,dx =$



$=$ area of  $= \frac{1}{2}\cdot(\pi\cdot 2^2) = 2\pi$.

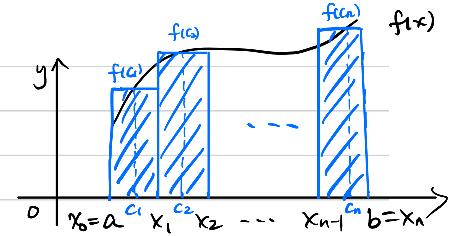
② use the symmetry of $f(x)$:

If $f(x)$ is an odd function on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$ Example : $\int_{-2}^{2} \underbrace{x^2\sin x}_{\text{odd function}}\,dx = 0$

If $f(x) = g(x) + h(x)$ and $g(x)$ is odd on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{a} g(x)\,dx}_{=0} + \int_{-a}^{a} h(x)\,dx = \int_{-a}^{a} h(x)\,dx$

Example : $\int_{-2}^{2} \left(\sqrt{4-x^2} + \underbrace{x^2\sin x}_{\text{odd function}}\right) dx = \int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$.

③ use the Riemann sum



step 1: Divide $[a,b]$ into n subintervals evenly:
$[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$

Step 2: Choose $c_1 \in [x_0, x_1]$, $\cdots$, $c_n \in [x_{n-1}, x_n]$.

$\Rightarrow$ Riemann sum $\frac{b-a}{n}[f(c_1)+\cdots+f(c_n)]$

a sum of signed areas of these rectangles.
(an approximation to $\int_a^b f(x)\,dx$)

Step 3: $\boxed{\lim\limits_{n\to+\infty} \frac{b-a}{n}[f(c_1)+\cdots+f(c_n)] = \int_a^b f(x)\,dx}$

④ use Fundamental Theorem of Calculus (FTC)

If $F(x)$ is an antiderivative of $f(x)$. then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Step 1: Find $F(x)$. (Find $\int f(x)\,dx$)

We can use the Substitution Rule:

$$\int f(g(x))\,g'(x)\,dx \overset{u=g(x)}{=} \int f(u)\,du.$$

step 2: Calculate $F(b)-F(a)$

Two applications of definite integral

① define some new functions by using definite integral

Examples: $G(x)=\int_0^{x^4} e^{t^2}\,dt$ $\quad G(x)=\int_{\sqrt{x}}^{3x+1} \ln(t^2+1)\,dt$. In general: $G(x)=\int_{h(x)}^{f(x)} g(t)\,dt$.

The derivatives of these functions are easy to find:

Example: $G(x)=\int_0^{x^4} e^{t^2}\,dt$ $\quad F(x)$ is an antiderivative of e^{x^2}.

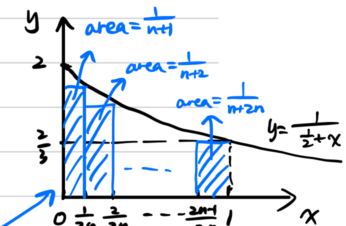
$\boxed{\text{key idea}}$ consider $G(x)$ as a composite function of $F(x)$ and x^4: $G(x)\overset{FTC}{=}F(x^4)-F(0)$

$$\Rightarrow \frac{d}{dx}G(x)=\frac{d}{dx}F(x^4)-\underbrace{\frac{d}{dx}F(0)}_{=0\text{ (because }F(0)\text{ is a constant)}}=\frac{d}{dx}F(\underbrace{x^4}_{=u})\overset{\text{chain rule}}{=}\frac{d}{du}F(u)\cdot\frac{du}{dx}=e^{u^2}\cdot 4x^3=e^{x^8}\cdot 4x^3$$

② calculate the limit of a sum of real numbers.

Example: $\lim_{n\to+\infty}\sum_{k=1}^{2n}\frac{1}{n+k}=\lim_{n\to+\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+2n}\right)=\ln 3$ $\quad$ a sum of $\boxed{2n}$ real numbers.

$\boxed{\text{key idea}}$ consider $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+2n}$ as a Riemann sum



Let's take $[a,b]=[0,1]$ and divide $[0,1]$ into $\boxed{2n}$ subintervals evenly (width $=\frac{1}{2n}$)

Let each $c_k=\frac{k}{2n}$: $c_1=\frac{1}{2n}$, $c_2=\frac{2}{2n}$, $c_3=\frac{3}{2n}$, $\cdots$, $c_{2n}=\frac{2n}{2n}$. $\quad$ Let $f(x)=\frac{1}{\frac{1}{2}+x}$.

Then $\frac{1}{n+k}=\frac{1}{2n}\cdot\frac{1}{\frac{1}{2}+\frac{k}{2n}}=\frac{b-a}{2n}\cdot\frac{1}{\frac{1}{2}+c_k}=\frac{b-a}{2n}f(c_k)$ $\Rightarrow \sum_{k=1}^{2n}\frac{1}{n+k}=\sum_{k=1}^{2n}\frac{b-a}{2n}f(c_k)$. $\quad$ a sum of areas of $\boxed{2n}$ rectangles

$$\Rightarrow \lim_{n\to+\infty}\sum_{k=1}^{2n}\frac{1}{n+k}=\int_0^1\frac{1}{x+\frac{1}{2}}\,dx \underset{F(x)=\ln(x+\frac{1}{2})}{\overset{FTC}{=}} F(1)-F(0)=\ln\left(\tfrac{3}{2}\right)-\ln\left(\tfrac{1}{2}\right)=\ln 3.$$

Overview of Functions : $y=f(x)$

y is uniquely determined by x.

1. definition : A function $y=f(x)$ is a rule assigning a unique number y to a number x.

2. domain of f : the set of numbers where f is well-defined.
 range of f : the set of numbers to which f maps.

Example : $y=\ln(x^2-2x-3)$.

Note : $x^2-2x-3=(x-3)(x+1)$

domain : $x^2-2x-3>0 \Rightarrow (x-3)(x+1)>0$
$\Rightarrow (-\infty,-1)\cup(3,+\infty)$

3. $y=f(x)$ is a one-to-one function if $\begin{cases} y \text{ is uniquely determined by } x \\ x \text{ is also uniquely determined by } y \end{cases}$

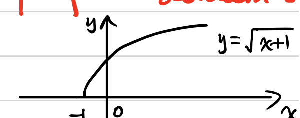
We can define the inverse function for a one-to-one function.

$$x \underset{g}{\overset{f}{\rightleftarrows}} y$$

$f(x)=y \iff g(y)=x$.

domain of f = range of g, range of f = domain of g

Example : Find inverse function of $y=f(x)=\sqrt{x+1}$



Step 1 : express x in terms of y

$y=\sqrt{x+1} \Rightarrow y^2=x+1 \Rightarrow x=y^2-1$

Step 2 : interchange x and y : $y=x^2-1$.

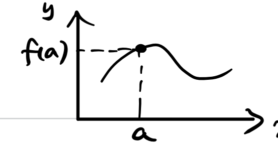
$\Rightarrow g(x)=x^2-1$ is the inverse function of $f(x)=\sqrt{x+1}$

check :

$f(1)=\sqrt{2}$, $g(\sqrt{2})=1$.

$f(2)=\sqrt{3}$, $g(\sqrt{3})=2$.

$f(3)=\sqrt{4}$, $g(\sqrt{4})=3$.

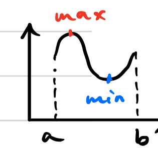


4. $y=f(x)$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

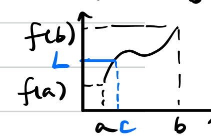
$y=f(x)$ is continuous on I if f is continuous at each $x \in I$.

Two basic facts about a continuous function $f(x)$ on $[a,b]$:

(1) The Extreme Value Theorem : $f(x)$ must attain an absolute maximum and an absolute minimum on $[a,b]$.



(2) $\boxed{\text{The Intermediate Value Theorem}}$: For any number L between $f(a)$ and $f(b)$, we can find $c \in [a,b]$ such that $f(c)=L$.

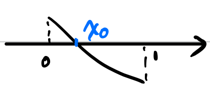


$\Downarrow$

A very important application : Suppose that $f(x)$ is continuous on $[a,b]$.

If $f(a)$ and $f(b)$ have opposite signs, then $f(x)=0$ has a root $x_0 \in [a,b]$.

Example : Show that $f(x) = 2^x - 4x^2$ has a root x_0 between 0 and 1.

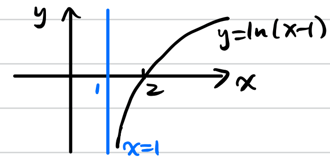


$$\left.\begin{array}{l} f(0) = 2^0 - 0 = 1 > 0 \\ f(1) = 2^1 - 4 = -2 < 0 \end{array}\right\} \Rightarrow f(x)=0 \text{ has a root } x_0 \text{ between } 0 \text{ and } 1.$$

5. vertical asymptotes and horizontal asymptotes of $y=f(x)$.

$x=a$ is a vertical asymptote if $\lim_{x\to a} f(x)=\pm\infty$ or $\lim_{x\to a^{\pm}} f(x)=\pm\infty$.

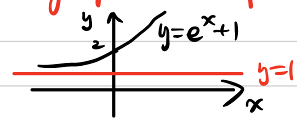
Example: $y=\ln(x-1)$



$\lim_{x\to 1^+} \ln(x-1)=-\infty$.

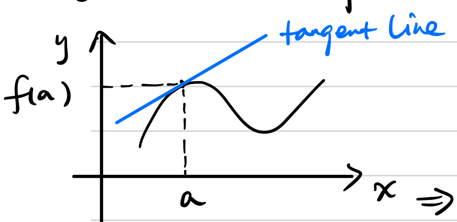
$y=L$ is a horizontal asymptote if $\lim_{x\to+\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$.

Example: $y=e^x+1$.



$\lim_{x\to-\infty}\left(e^x+1\right)=1$.

6. How to find the tangent line of f at $(a, f(a))$?



$\Rightarrow$

$f'(a)=\left.\frac{dy}{dx}\right|_{x=a}$ = slope of the tangent line at $(a, f(a))$

The point-slope form of the tangent line at $(a, f(a))$:

$y=f'(a)\cdot(x-a)+f(a)$.

Problem of finding the tangent line $\Rightarrow$ Problem of finding $f'(a)=\left.\frac{dy}{dx}\right|_{x=a}$.

7. How to find the intervals where f is increasing and decreasing?

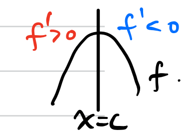
$f'(x) > 0 \Rightarrow f(x)$ is increasing $\qquad f'(x) < 0. \Rightarrow f(x)$ is decreasing

8. How to find the local maximum (relatie maximum) and local minimum (relatie minimum) of f?

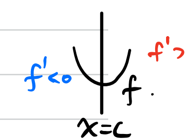
Step 1: Find all critical numbers. $\longrightarrow$ c is a critical number if $f'(c) = 0$ or $f'(c)$ does not exist.

Step 2: Calulete the sign of $f'(x)$

① sign of $f'(x)$: increasing $+ + +$ | c | $- - -$ decreasing. $\to x$ $\Rightarrow f(c)$ is a local maximum.



② sign of $f'(x)$: decreasing $- - -$ | c | $+ + +$ increasing. $\to x$ $\Rightarrow f(c)$ is a local minimum



③ sign of $f'(x)$: increasing $+ + +$ | c | $+ + +$ $\to x$

decreasing. $- - -$ | c | $- - -$ $\to x$

$\Rightarrow f(c)$ is neither a local maximum nor a local minimum.