Lecture 9 Rigid Body II

- moment of inertia
- parallel axis theorem
- cross product
- torque and Newton's 2^{nd} Law in rotational dynamics.

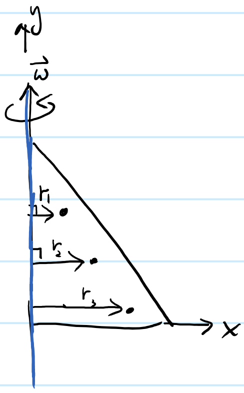
Moment of inertia

Recall for a rigid body rotating about a fixed axise, the kinetic energy is

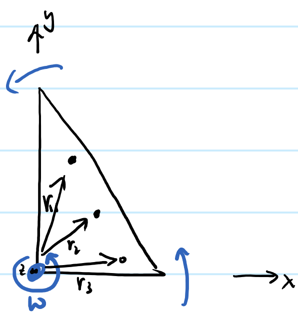
$$K = \frac{1}{2} I \omega^2$$

where $I = \sum_i m_i r_i^2$ ← moment of inertia ("laziness" in rotation)

r_i is the perpendicular distance from the rotation axis to the mass m_i.



rotating about y-axis

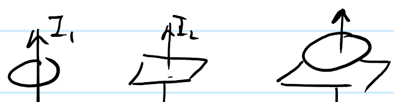


rotating about z-axis.

Properties of I: – Shape dependent. (not just mass)

– Axis dependent. (r_i depends on the rotation axis)

– Additive $I = I_1 + I_2$

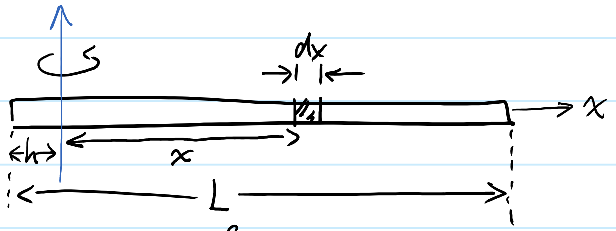


$I_1 + I_2 = I_{tot}$

$\boxed{\text{Same axis}}$

Example - Rod with uniform linear density $\rho = \frac{M}{L}$.

Rod.



total mass $= M$.

mass per unit length $= \frac{M}{L}$

$$: dm = \rho\, dx = \frac{M}{L} \cdot dx$$

$$I = \sum m_i r_i^2 \longrightarrow \int dm \cdot x^2$$

$$I = \int_{-h}^{L-h} \frac{M}{L}\, dx \cdot x^2$$

$$= \frac{M}{L} \frac{1}{3} x^3 \Big|_{-h}^{L-h}$$

$$= \frac{M}{3L}\left[(L-h)^3 - (-h)^3\right]$$

$$= \frac{M}{3}\left(L^2 - 3Lh + 3h^2\right)$$

$$(L-h)^3 = L^3 - 3L^2 h + 3Lh^2 - h^3$$

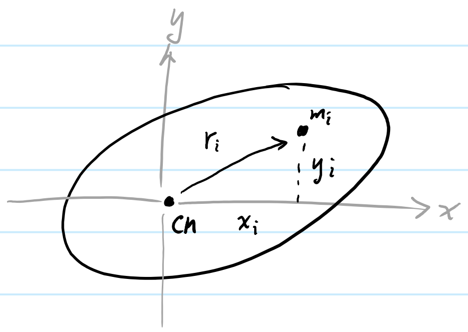
- Rotating about the center of the rod

$$h = L/2 \Rightarrow I = I_{cm} = \frac{M}{3}\left(1 - \frac{3}{2} + \frac{3}{4}\right)L^2 = \frac{1}{12} ML^2$$

- Rotating about the end of the rod.

$$h = 0 \Rightarrow I_{end} = \frac{1}{3} ML^2$$

Parallel axis theorem.

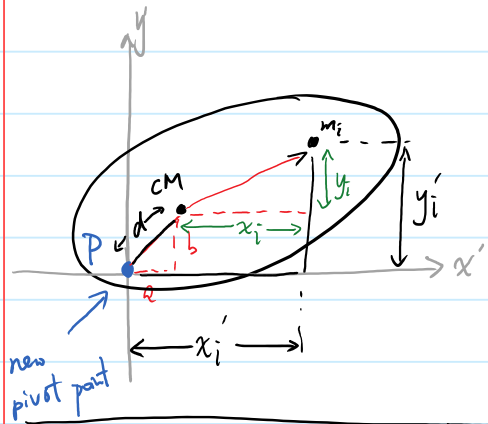


at origin.

CM is at $(x_{cm}, y_{cm}) = \left(\frac{1}{M_{tot}}\sum_i m_i x_i , \frac{1}{M_{tot}}\sum_i m_i y_i\right) = (0,0)$

Moment of inertia about the CM. $= I_{cm} = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$

Now we want to rotate the object at P. and define a new coordinate system $x'-y'$ plane. in where the CM is located at (a, b).



Moment of inertia about P.

$$I_P = \sum_i m_i \left(x_i'^2 + y_i'^2\right)$$

$$= \sum_i m_i \left[(a + x_i)^2 + (b + y_i)^2\right]$$

$$= \underbrace{\sum_i m_i}_{M_{tot}} (a^2 + b^2) + \underbrace{\sum_i m_i (x_i^2 + y_i^2)}_{I_{cm}}$$

$$+ 2\left(\cancel{\sum_i m_i x_i}\right)^{0} a + 2\left(\cancel{\sum_i m_i y_i}\right)^{0} b.$$

Relation between two coordinate sys

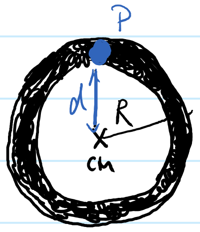
$$\begin{cases} x_i' = a + x_i \\ y_i' = b + y_i \end{cases}$$

$\sum_i m_i x_i = M_{tot} X_{CM}$. but $X_{cm} = 0$ so as y_{cm}

$\Rightarrow \sum_i m_i x_i = 0 , \quad \sum_i m_i y_i = 0$

$$\Rightarrow \boxed{I_P = I_{cm} + M_{tot} \cdot d^2}$$ parallel axis theorem.

Example. Ring . mass = M



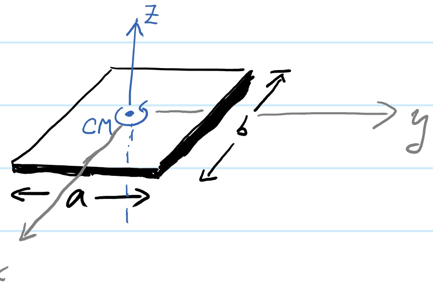
$I_{cm} = \sum_i m_i r_i^2 = \sum m_i R^2 = M R^2$

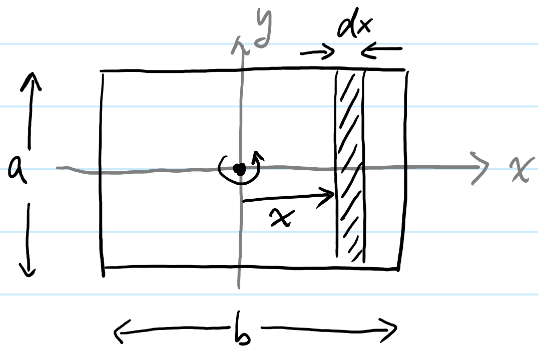
$I_P = I_{cm} + M \cdot d^2$ $\qquad d = R$

$= MR^2 + M R^2$

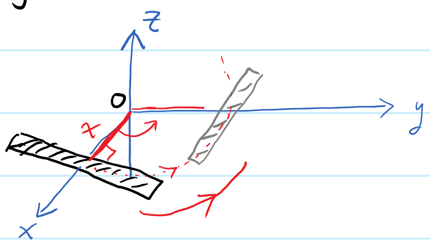
$= 2MR^2$ harder to rotate a ring about P

Example Rectangular plate.
about z-axis through its CM





Consider the mass element dm as a rod at $\perp$ distance x from the z-axis.



$$dI_{cm} = \frac{1}{12} dm \cdot a^2$$

using parallel axis theorm

$$dI_0 = dI_{cm} + dm \cdot x^2$$

$$= \left(\frac{a^2}{12} + x^2\right) dm$$

where $dm = \frac{M}{b} \cdot dx$

Moment of inertia of the plate : sum of all dI_0

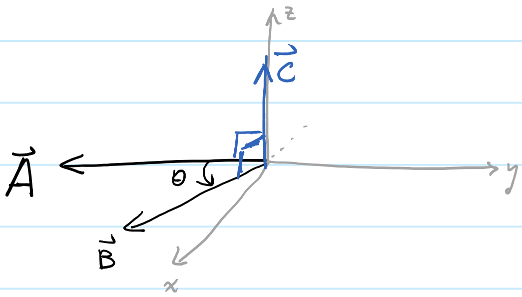
$$I = \int_{x=-b/2}^{x=b/2} dI_0 = \frac{M}{b}\int_{-b/2}^{b/2}\left(\frac{a^2}{12} + x^2\right)dx$$

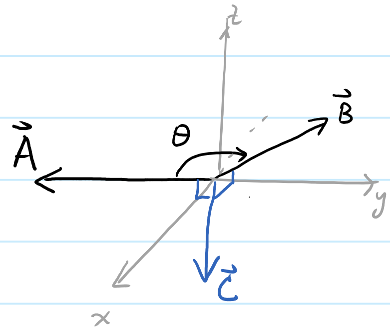
$$= \frac{M}{b}\left[\frac{a^2}{12}b + \frac{1}{3}\left(\frac{b}{2}\right)^3 \cdot 2\right]$$

$$= \frac{1}{12} M(a^2+b^2)$$

Cross Product.

$$\vec{C} = \vec{A} \times \vec{B}$$





magnitude : $|\vec{C}| = A \cdot B \cdot \sin\theta$

direction : using Right Hand Rule. , $\vec{c} \perp \vec{B}$ & $\vec{c} \perp \vec{A}$.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \qquad \text{anti-commutative.}$$

Special cases: If $\vec{A} \parallel \vec{B}$, $\vec{A} \times \vec{B} = \vec{0}$. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

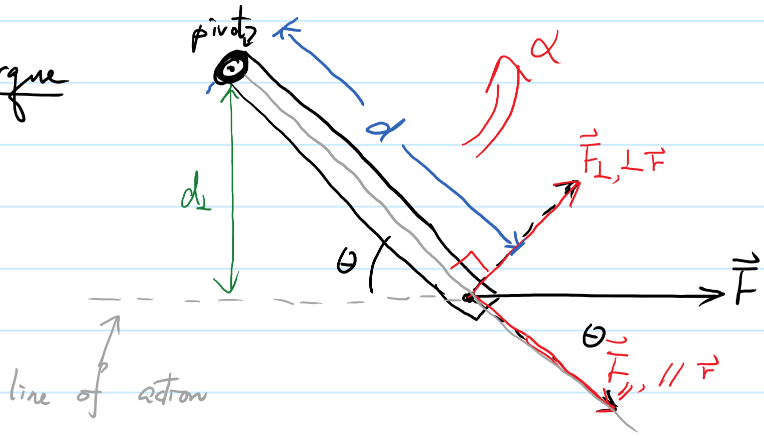
If $\vec{A} \perp \vec{B}$, $|\vec{A} \times \vec{B}| = A \cdot B$.

RHR.: $\begin{cases} \hat{i} \times \hat{j} = \hat{k} \\ \hat{j} \times \hat{k} = \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} \end{cases}$

In general,

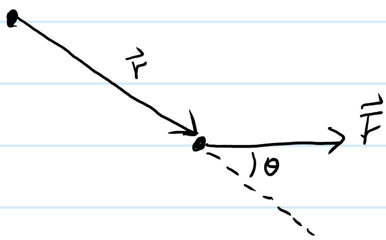
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

Torque



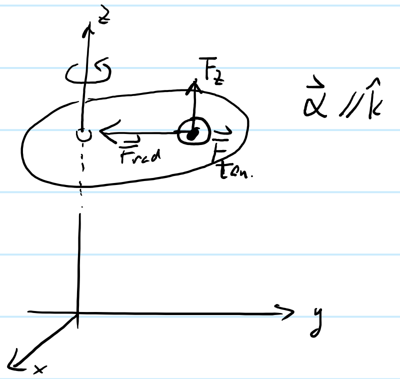
Torque $\propto d$, $F_\perp$ only.

$$\tau = d \cdot F_\perp$$
$$= d \cdot F \cdot \sin\theta$$
$$= F \cdot d_\perp$$

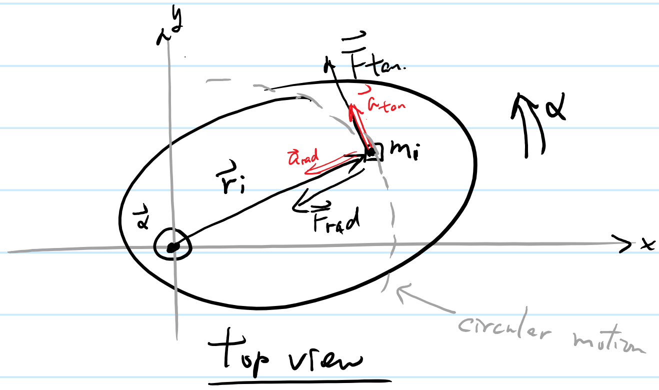


$\vec{\tau} = \vec{r} \times \vec{F}$ $\odot$

$\vec{\tau}$ points out of the page.



$\vec{\alpha} \,//\, \hat{k}$



Top view

for every mass in the object,

tangential component of $\vec{a}$ $\qquad a_{i,tan} = r_i \alpha$

Newton's 2nd Law :

$$m_i a_{i,tan} = F_{i,tan}$$

$$r_i \times \quad m_i r_i \alpha = F_{i,tan.} \quad \times r_i$$

$$m_i r_i^2 \alpha = r_i F_{i,tan}$$

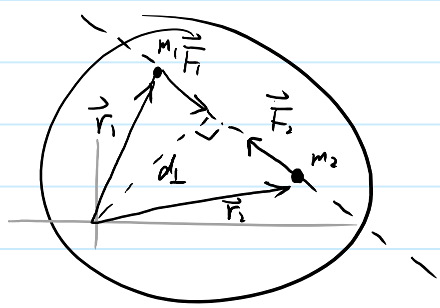
$$I_{iz} \alpha = \tau_{iz}$$

$$\Rightarrow \quad \sum_i \tau_{iz} = \sum_i I_{iz} \alpha$$

$$\Rightarrow \quad \boxed{\tau_{tot} = \tau_{ext} = I_z \alpha}$$

Only external forces are included.

What about torques due to internal forces?



$$\left.\begin{array}{ll} \vec{\tau}_1 \otimes & \text{into the page} \\ \vec{\tau}_2 \odot & \text{out of the page} \end{array}\right\} \text{opposite in direction}$$

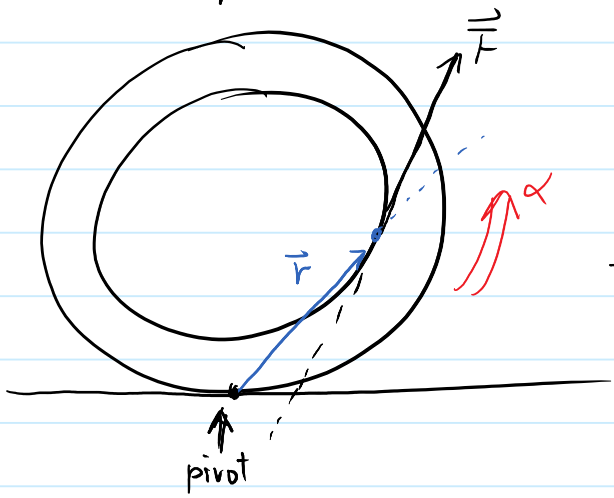
$$|\vec{\tau}_1| = F_1 \cdot d_\perp$$

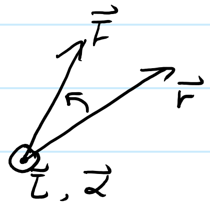
$$|\vec{\tau}_2| = F_2 \, d_\perp$$

$$|\vec{F}_1| = |\vec{F}_2| \Rightarrow |\vec{\tau}_1| = |\vec{\tau}_2|$$

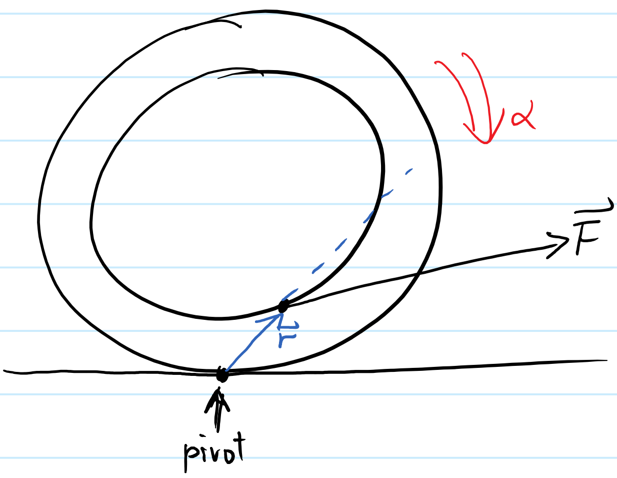
$$\Rightarrow \vec{\tau}_1 = -\vec{\tau}_2 , \quad \sum \vec{\tau}_{int} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{0} .$$

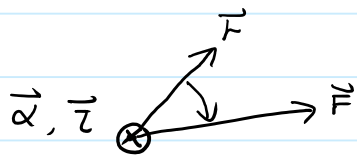
Demo. spool.





$\vec{r} \times \vec{F}$: out of page $\Rightarrow \vec{\alpha}$ out of page

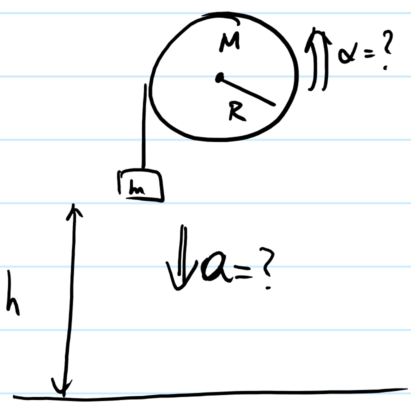




$\vec{r} \times \vec{F}$: into the page, $\Rightarrow \vec{\alpha}$ into the page

α

Review

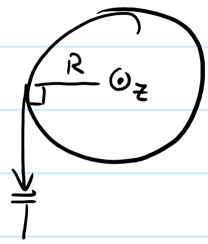


① $\tau_z = I_z \alpha_z$

② $\vec{F} = m\vec{a}$

③ $V_{tan} = r\omega \Rightarrow a_{tan} = r\alpha$.

For the cylinder, $\odot z$ as +ve.



$$\tau_z = I_z \alpha_z$$

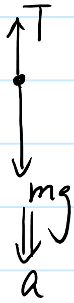
$$+RT = I_{cm} \alpha$$

$$T = I_{cm} \frac{\alpha}{R}$$

$$\alpha = \frac{a}{R} \qquad T = \frac{\frac{1}{2}MR^2 a}{R^2}$$

$$\Rightarrow \quad T = \frac{1}{2}Ma$$

For the mass, $F_{net} = ma$



$$mg - T = ma$$

$$mg - \frac{1}{2}Ma = ma$$

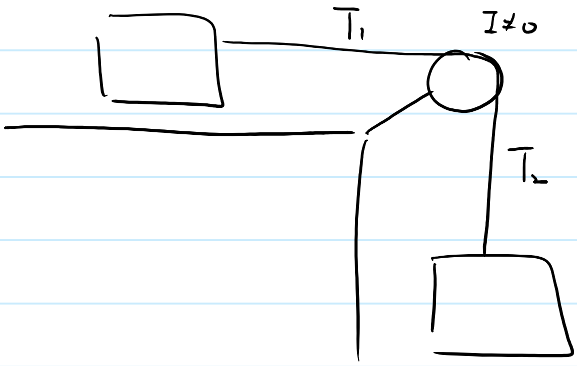
$$a = \frac{mg}{m + \frac{1}{2}M} = \frac{g}{1 + \frac{M}{2m}} < g.$$

<u>check.</u> $v_f^2 = v_i^2 + 2ah$

$$v_f = \sqrt{\frac{2gh}{1 + \frac{M}{2m}}} = \sqrt{\frac{2mgh}{m + \frac{I_{cm}}{R^2}}}$$

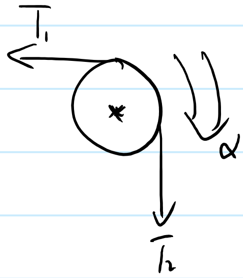
$I_{cm} = \frac{1}{2}MR^2$

$\frac{1}{2}m =$



$T_1 \neq T_2$

$T_2 > T_1 \because$



$\tau_{net} = (T_2 - T_1)R > 0$

for non-zero α.
