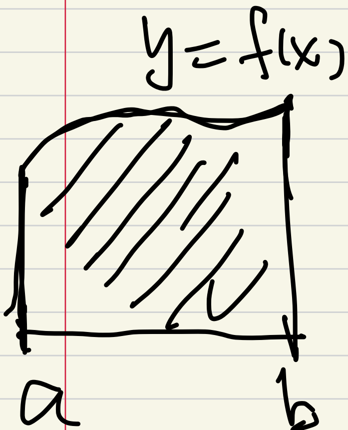
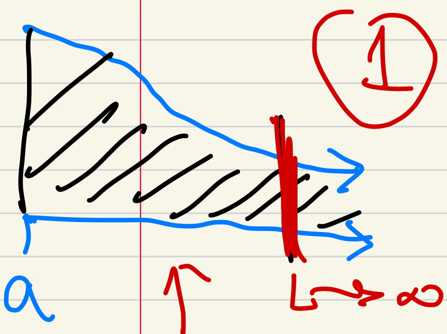
Improper Integrals



$$\int_a^b f(x)\,dx \longleftrightarrow$$

① Continuous functions over $[a,\infty)$, $(-\infty,a]$ ✓ or $(-\infty,\infty)$ ✓



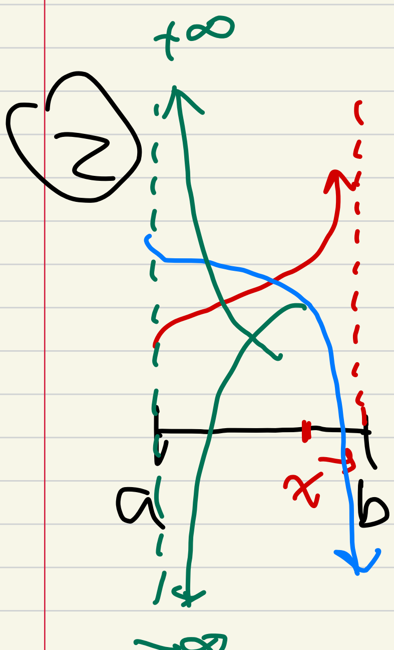
$$\int_a^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{a} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$$

② Unbounded functions on finite intervals

③ A mixture of ① and ②



$f(x) \to +\infty$ as $x \to b^-$

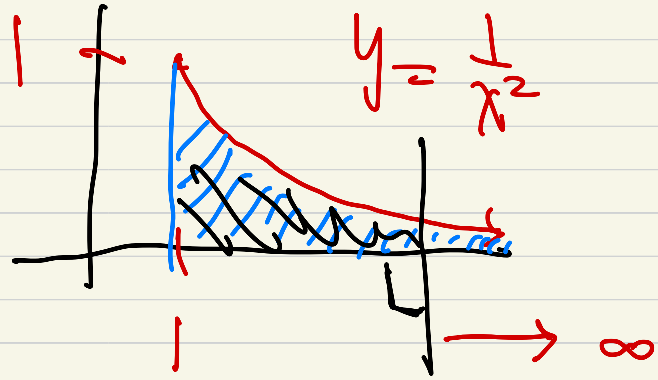
$f(x) \to +\infty$ as $x \to a^+$

$f(x) \to -\infty$ as $x \to b^-$

$$\int_a^b f(x)\,dx$$

Example

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$



means

$$= \lim_{L\to\infty} \underbrace{\int_1^{L} \frac{1}{x^2}\,dx}_{\text{a well-known definite integral}}$$

$$\int x^{-2}\,dx = -x^{-1} + C$$

$$= \lim_{L\to\infty} \Big[-x^{-1} \Big]_1^{L}$$

(A convergent improper integral)

$$= \lim_{L\to\infty} \left(-\frac{1}{L} + 1 \right) = \underline{1}.$$

$$\int_a^{\infty} f(x)\,dx \overset{\text{means}}{=} \lim_{L\to\infty} \int_a^{L} f(x)\,dx$$

The improper integral is "convergent" if the limit exists as a finite value.

Otherwise, the improper integral is "divergent"

Example For any $p > 0$

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{convergent?} & \text{Yes if } p > 1 \\ \text{divergent?} & \text{Yes } \boxed{\text{if } 0 < p \leq 1} \end{cases}$$

$\parallel$

$$\lim_{L\to\infty} \int_1^L \frac{1}{x^p}\,dx \begin{cases} p = 1 \\ p \neq 1 \end{cases}$$

$p = 1$,
$$\lim_{L\to\infty} \int_1^L \frac{1}{x}\,dx \leftarrow \text{divergent}$$

$$= \lim_{L\to\infty} \Big[\ln|x| \Big]_1^L$$

$$= \lim_{L\to\infty} \ln L = \infty$$

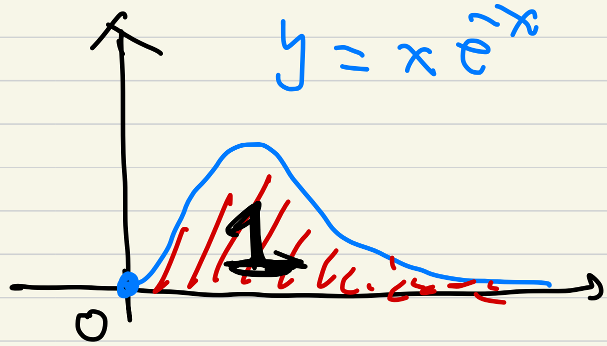
$p \neq 1$,
$$\lim_{L\to\infty} \int_1^L \frac{1}{x^p}\,dx$$

$$= \lim_{L\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^L$$

$$= \lim_{L\to\infty} \boxed{\frac{L^{-p+1}}{-p+1}} - \frac{1}{-p+1} \begin{cases} \infty & \text{if } -p+1 > 0 \\ \frac{1}{p-1} & \text{if } -p+1 < 0 \end{cases}$$

Example

$$\int_0^\infty x e^{-x}\,dx$$



$$= \lim_{L\to\infty} \int_0^L x e^{-x}\,dx \qquad \int x\, d e^{-x}$$

$$= \lim_{L\to\infty} \left[-xe^{-x}\Big|_0^L + \int_0^L e^{-x}\,dx \right]$$

$$= \lim_{L\to\infty} \left[-\frac{L}{e^L} + \left[-e^{-x}\right]_0^L \right]$$

$$= -\lim_{L\to\infty} \frac{L}{e^L} - \lim_{L\to\infty} e^{-L} + 1$$

$\frac{\infty}{\infty}$-type

|| L'Hopital's Rule

0

$$-\lim_{L\to\infty} \frac{1}{e^L} = 0$$

$$= 1$$

$\boxed{\text{L'Hôpital Rule !!}}$

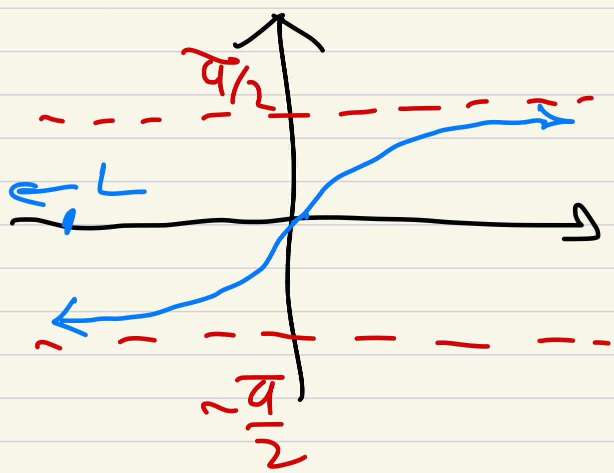
Example $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

$$= \lim_{L \to -\infty} \int_{L}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{L \to -\infty} \Big[\tan^{-1} x \Big]_{L}^{0} \qquad \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}$$

$$= \lim_{L \to -\infty} \left(-\tan^{-1} L \right)$$

$$= \frac{\pi}{2}$$



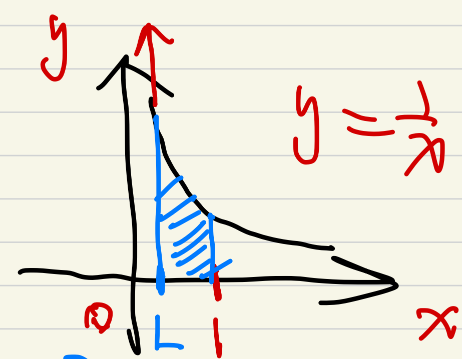
$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \pi$$

Same idea for improper integrals of unbounded functions on finite intervals!

Example. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$, where $p > 0$



avoid the trouble at 0!

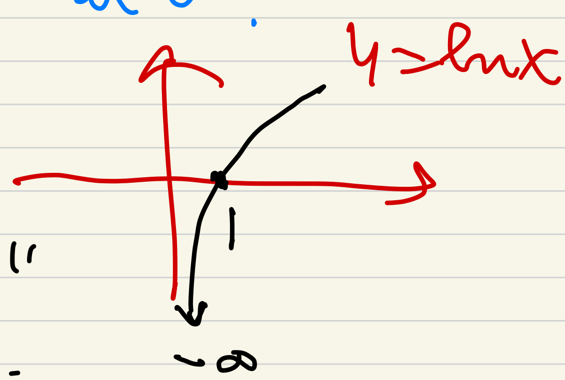
$p = 1$ means:

$$= \lim_{L\to 0^+}\int_L^1 \frac{1}{x}\,dx$$

$$= \lim_{L\to 0^+}\big[\ln x\big]_L^1$$

$$= \lim_{L\to 0^+} -\ln L$$

$$= +\infty$$



"The improper integral" is divergent for $p = 1$.

If $p \neq 1$, $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{L\to 0^+}\int_L^1 x^{-p}\,dx$

$$= \lim_{L\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_L^1 = \frac{1}{-p+1} - \lim_{L\to 0^+}\frac{L^{\boxed{-p+1}}}{-p+1}$$

① convergent if $-p+1 > 0$ i.e. $p < 1$

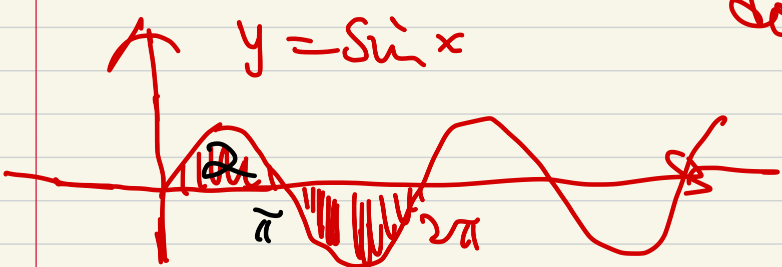
② divergent if $-p+1 < 0$, $p > 1$

So,
$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{if } 0<p<1 \text{ (convergent)} \\ \text{divergent (to } \infty) & \text{if } p \geq 1. \end{cases}$$

Remark:

$$\int_0^\infty \sin x\,dx = \lim_{L\to\infty}\Big[-\cos x\Big]_0^L$$
$$= \boxed{-\lim_{L\to\infty}\cos L} + 1$$

does not exist



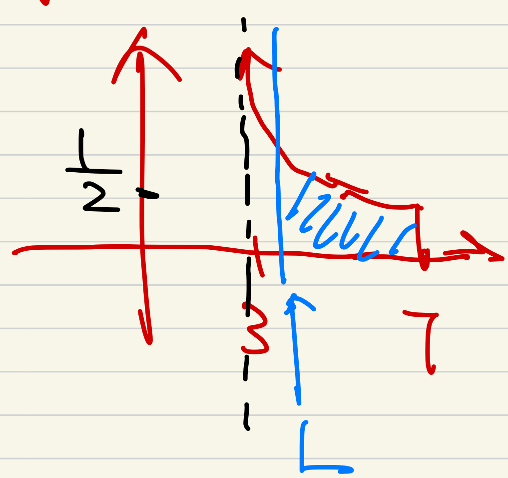
oscillating between 0 and 2.

Example

$$\int_3^7 \frac{1}{\sqrt{x-3}}\,dx$$

as $x \to 3^+$,

$$\frac{1}{\sqrt{x-3}} \to \infty$$

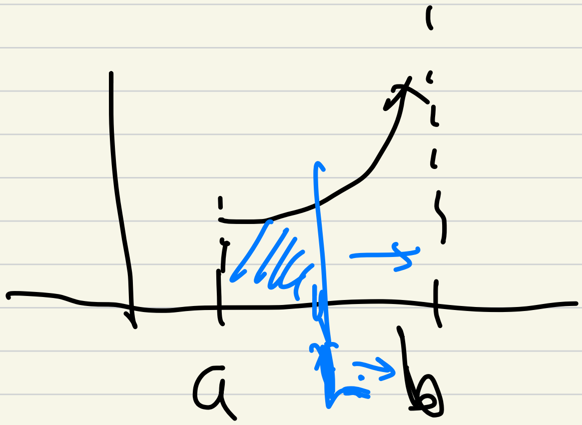


$$= \lim_{L \to 3^+} \int_L^7 (x-3)^{-\frac{1}{2}}\,dx$$

$$= \lim_{L \to 3^+} \left[2(x-3)^{\frac{1}{2}} \right]_L^7$$

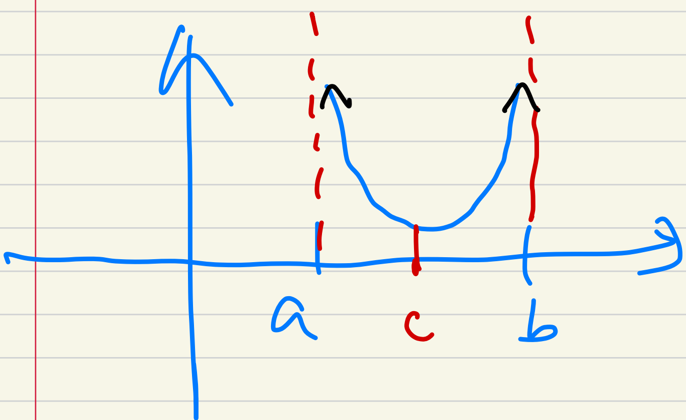
$$= \lim_{L \to 3^+} \left[2 \cdot 2 - 2(L-3)^{1/2} \right]$$

$$= 4.$$



$f(x) \to \infty$ or $-\infty$ as $x \to b^-$

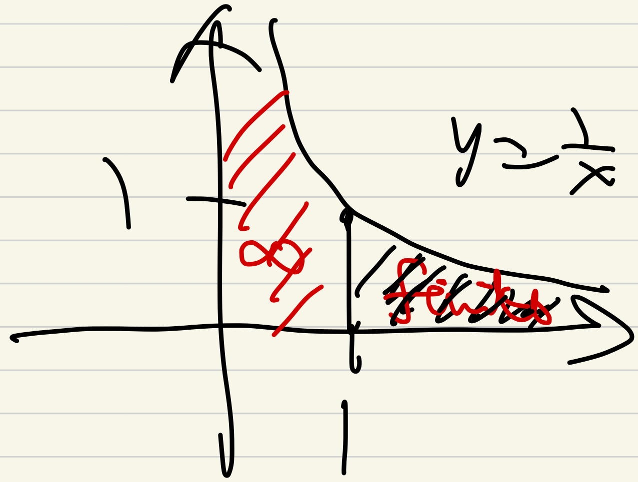
$$\int_a^b f(x)\,dx = \lim_{L \to b^-} \int_a^L f(x)\,dx$$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

$$\int_0^\infty \frac{1}{x^2}\,dx$$



$$= \underbrace{\int_0^1 \frac{1}{x^2}\,dx}_{\text{divergent}} + \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{convergent}}$$

divergent $(p = 2 > 1)$ convergent $(p > 1)$, $p = 2$

The integral $\int_0^\infty \frac{1}{x^2}\,dx$ is divergent.

$\int_{-1}^{1} \frac{1}{x}\,dx$ is divergent!

$$= \underbrace{\int_{-1}^{0} \frac{1}{x}\,dx}_{\text{divergent}} + \underbrace{\int_0^1 \frac{1}{x}\,dx}_{\text{divergent}}$$