MATH2111 Tutorial 2

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1 Procedures to solve a system of linear equations

1. Write the system as augmented matrix $[\boldsymbol{M} \mid \boldsymbol{b}]$;
2. Use EROs to reduce $[\boldsymbol{M} \mid \boldsymbol{b}]$ into RREF $[\boldsymbol{M}' \mid \boldsymbol{b}']$;
3. Locate the pivot columns of $[\boldsymbol{M}' \mid \boldsymbol{b}']$;
4. If $\boldsymbol{b}'$ is a pivot column, the system is inconsistent (has 0 solution);
   otherwise, the system is consistent, locate the free columns of $\boldsymbol{M}'$.
   (a) If there is a free column, then the system has infinitely many solutions;
   (b) otherwise the system has a unique solution.

2 Vectors

1. A **column vector** is a matrix with one column. We add and subtract vectors of the same size by doing operations component-wise:
$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \pm \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ \vdots \\ u_n \pm v_n \end{bmatrix}, \text{ and } \quad c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} \text{for } c \in \mathbb{R}.$$
2. Let n be a positive integer and define $\mathbb{R}^n$ to be the set of vectors with n rows.
3. **Algebraic Properties of Vectors in $\mathbb{R}^n$:**
   For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:
   (1). $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
   (2). $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
   (3). $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
   (4). $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$ where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
   (5). $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
   (6). $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
   (7). $c(d\mathbf{u}) = (cd)\mathbf{u}$
   (8). $1\mathbf{u} = \mathbf{u}$

4. **Linear Combination and Span**
   Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots, \boldsymbol{u}_k\}$ be a collection of vectors in $\mathbb{R}^n$.

   (a) Another vector $\boldsymbol{v} \in \mathbb{R}^n$ is a **linear combination** of S if

   $$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$$

   for some scalars $c_1, c_2, \cdots, c_k \in \mathbb{R}$.

   (b) The **span** of S, $\mathrm{Span}(S)$, is the collection of all vectors of the form $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$, i.e.

   $$\mathrm{Span}(S) := \{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \mid c_1, c_2, \cdots, c_k \in \mathbb{R}\}$$

   So $\mathrm{Span}(S)$ contains all possible linear combinations of S.

5. **Theorem** A vector $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k$ if and only if there exists a solution to the corresponding linear system with the augmented matrix $\left[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_k \;\middle|\; \mathbf{b}\right]$.

3 Exercises

1. Suppose $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & a & 3 \\ 1 & 1 & b & 4 \end{array}\right)$ is an augmented matrix. Determine a and b such that the linear system
(1) is inconsistent,
(2) has a unique solution,
(3) has infinite many solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & a & 3 \\ 1 & 1 & b & 4 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & b-1 & 4 \end{array}\right]$$

① When $a=2$ or $b=1$, inconsistent

② When $a-2 \neq 0$, $b-1 \neq 0$ i.e. $a \neq 2$, $b \neq 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & b-1 & 4 \end{array}\right] \xrightarrow[R_3 - \frac{b-1}{a-2} R_2 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & 0 & 4 - \frac{b-1}{a-2} \cdot 3 \end{array}\right]$$

if $4 - \frac{b-1}{a-2} \cdot 3 \neq 0$, i.e. $4(a-2) \neq 3(b-1)$, inconsistent

if $4 - \frac{b-1}{a-2} \cdot 3 = 0$, i.e. $4(a-2) = 3(b-1)$, infinitely many solutions.

2. Suppose $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & a & 3 & 3 \\ 1 & 2 & b & 4 \end{array}\right)$ is an augmented matrix. Determine a and b such that the linear system
(1) is inconsistent,
(2) has a unique solution,
(3) has infinite many solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & a & 3 & 3 \\ 1 & 2 & b & 4 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & a-2 & 1 & 3 \\ 0 & 1 & b-1 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & b-1 & 4 \\ 0 & a-2 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 - (a-2)R_2 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & b-1 & 4 \\ 0 & 0 & 1-(a-2)(b-1) & 3-4(a-2) \end{array}\right]$$

① if $1-(a-2)(b-1)=0$ and $3-4(a-2) \neq 0$,

i.e. $a \neq \frac{11}{4}$, $b \neq \frac{7}{3}$, it's inconsistent

② if $1-(a-2)(b-1) \neq 0$, i.e. $(a-2)(b-1) \neq 1$, unique solution

③ if $1-(a-2)(b-1)=0$ and $3-4(a-2)=0$,

$\therefore (a-2)(b-1)=1$, and $a=\frac{11}{4}$,

i.e. $a=\frac{11}{4}$, $b=\frac{7}{3}$, infinitely many solutions.

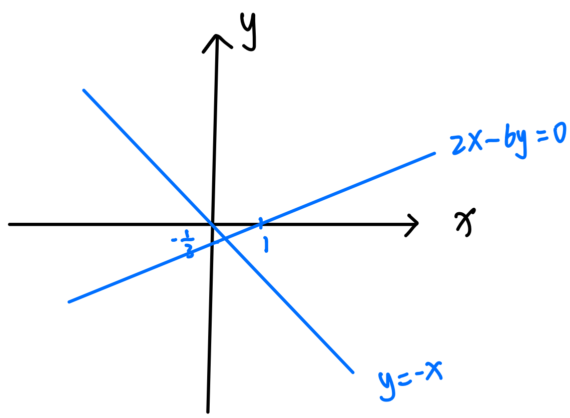
3. Plot the following linear systems:

(1) Two variables: $\begin{cases} x + y = 0, \\ 2x - 6y = 2 \end{cases}$ $x-3y=1$

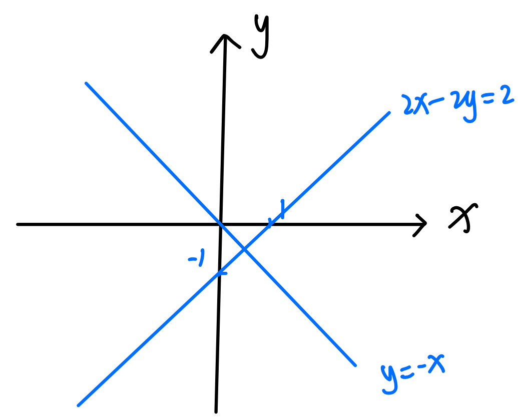
(2) Two variables: $\begin{cases} x + y = 0, \\ 2x - 2y = 2. \end{cases}$

(3) Three variables: $\begin{cases} x + y = 0 \\ y + z = 2 \end{cases}$

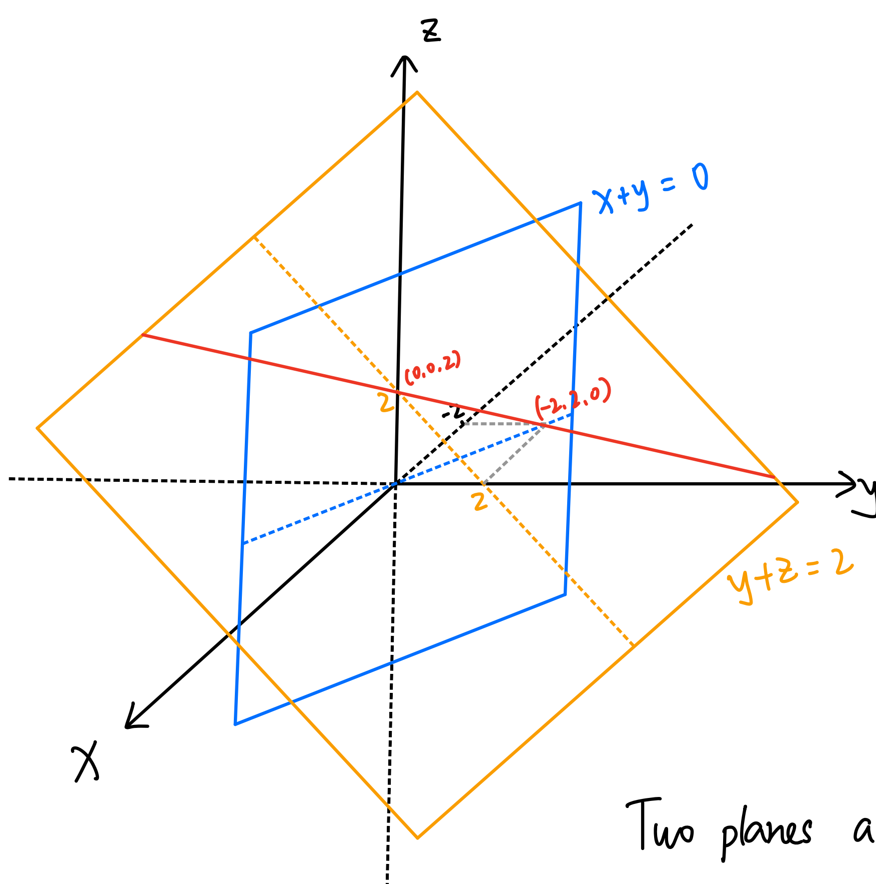
(1)



(2)



(3)



Two planes are NOT parallel or coincide, their intersection is a line.

4. Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(2) Determine whether vector $\boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(1) Set: $\{ \vec{x} \mid \vec{x} = c_1\vec{u} + c_2\vec{v}$ for all $c_1, c_2 \in \mathbb{R} \}$

(2) If $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$, then there exists $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\vec{u} + c_2\vec{v} = \vec{w}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right]$$

The system has no solution

$\therefore$ The vector equation $c_1\vec{u} + c_2\vec{v} = \vec{w}$ has no solution

$\therefore$ $\vec{w}$ is NOT in span$\{\vec{u}, \vec{v}\}$

5. Let $\boldsymbol{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(2) Determine h such that vector $\boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(1) Set: $\{\vec{x} \mid \vec{x} = c_1\vec{u} + c_2\vec{v},$ for all $c_1, c_2 \in \mathbb{R}\}$

(2) If $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$, then

there exists $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\vec{u} + c_2\vec{v} = \vec{w}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & h \end{array}\right] \xrightarrow[R_2 - \frac{1}{3}R_1 \to R_2]{} \left[\begin{array}{cc|c} 3 & -2 & 2 \\ 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & h \end{array}\right] \xrightarrow[R_3 - \frac{3}{2}R_2 \to R_3]{} \left[\begin{array}{cc|c} 3 & -2 & 2 \\ 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 0 & h-2 \end{array}\right]$$

when $h=2$, $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$

6. Let $\boldsymbol{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(2) Determine h such that vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(3) Determine h such that vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}, \boldsymbol{w}$.

(1) Set: $\{\vec{x} \mid \vec{x} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$ for all $c_1, c_2, c_3 \in \mathbb{R}\}$

(2) If $\vec{x}$ is spanned by $\vec{u}$ and $\vec{v}$, then

there exists $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\vec{u} + c_2\vec{v} = \vec{x}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & h \end{array}\right] \xrightarrow[R_2 - \frac{1}{3}R_1 \to R_2]{} \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & h \end{array}\right] \xrightarrow[R_3 - \frac{3}{2}R_2 \to R_3]{} \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & \frac{2}{3} & \frac{8}{3} \\ 0 & 0 & h-4 \end{array}\right]$$

when $h=4$, it's consistent

(3) Similarly, if $\vec{x}$ is spanned by $\vec{u}, \vec{v}, \vec{w}$,

then, $\exists\ c_1, c_2, c_3 \in \mathbb{R}$ s.t. $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{x}$

$$\left[\begin{array}{ccc|c} 3 & -2 & 2 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & h \end{array}\right] \xrightarrow[R_2 - \frac{1}{3}R_1 \to R_2]{} \left[\begin{array}{ccc|c} 3 & -2 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{8}{3} \\ 0 & 1 & 2 & h \end{array}\right] \xrightarrow[R_3 - \frac{3}{2}R_2 \to R_3]{} \left[\begin{array}{ccc|c} 3 & -2 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{8}{3} \\ 0 & 0 & 0 & h-4 \end{array}\right]$$

when $h=4$, it's consistent.

Q: Think about why (2) and (3) have same solution?

A: Set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. $\vec{w} = 2\vec{u} + 2\vec{v}$