Math1014 Calculus II
A Short Summary of Basic Calculus

Here is a quick review of some basic calculus from Math1013.

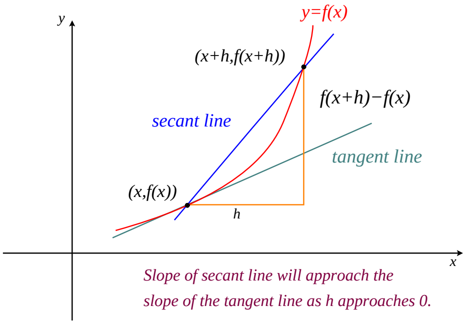
Derivatives

Notation: $\dfrac{dF}{dx}$, $F'(x)$, y' or $\dfrac{dy}{dx}$, etc.

Meaning: $$F'(x) = \frac{dF}{dx} = \lim_{h\to 0} \underbrace{\frac{F(x+h)-F(x)}{h}}_{\substack{\text{slope of the secant line through} \\ (x,F(x)) \text{ and } (x+h,F(x+h))}}$$

$= \text{slope of tangent line at } (x, F(x))$

$= \text{rate of change of } F$



Basic derivatives:

$$\frac{d(\text{constant})}{dx} = 0, \quad \frac{dx^p}{dx} = px^{p-1}, \quad \frac{d\ln|x|}{dx} = \frac{1}{x}, \quad \frac{de^x}{dx} = e^x$$

$$\frac{d\sin x}{dx} = \cos x, \quad \frac{d\cos x}{dx} = -\sin x, \quad \frac{d\tan x}{dx} = \sec^2 x, \quad \frac{d\sec x}{dx} = \sec x \tan x$$

$$\frac{d\sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d\cos^{-1} x}{dx} = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d\tan^{-1} x}{dx} = \frac{1}{1+x^2},$$

(Each derivative is a limit!)

Differentiation Rules:

"<u>Term by Term</u>": $[af(x)+bg(x)]' = af'(x)+bg'(x)$ for any constants a, b

<u>Product Rule</u>: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

<u>Quotient Rule</u>: $$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

<u>Chain Rule:</u> $$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Antiderivatives/Indefinite Integrals

Notation: $\displaystyle\int f(x)dx$

Meaning: $\displaystyle\int f(x)dx = F(x) + C$ is just another way to say $\dfrac{dF}{dx} = f(x)$.

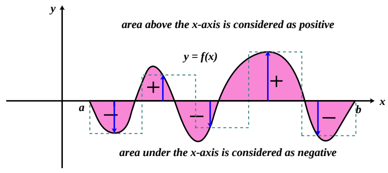
Example: $\displaystyle\int e^{2x}dx = \frac{1}{2}e^{2x} + C$, because $\dfrac{d}{dx}\left(\dfrac{1}{2}e^{2x}\right) = \dfrac{1}{2}e^{2x}\dfrac{d(2x)}{2x} = e^{2x}$

(By Chain Rule, $\dfrac{d}{dx}e^{g(x)} = e^{g(x)}\dfrac{dg}{dx}$)

Definite Integrals

Notation: $\displaystyle\int_a^b f(x)dx$, where f is a continuous function on $[a, b]$.

Meaning: $$\int_a^b f(x)dx = \lim_{n\to\infty} \underbrace{\frac{b-a}{n}\Big[f(c_1)+f(c_2)+\cdots+f(c_n)\Big]}_{\text{"Riemann sum": sum of +ve/-ve "rectangular areas"}} =$$



sum of +ve/-ve areas

Example: $$\int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right] = \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}$$

where $c_1 = \frac{1}{n}$, $c_2 = \frac{2}{n}, \ldots, c_n = \frac{n}{n}$.

(This is of course just the area of a triangle!)

Fundamental Theorem of Calculus

The FUNDAMENTAL THEOREM OF CALCULUS (FTC) gives you the connection between the two kinds of limit processes:

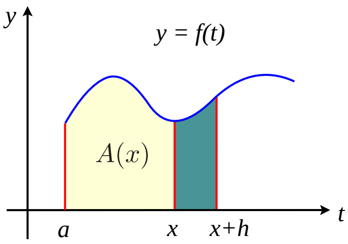
$$\int_a^b \longleftrightarrow \frac{d}{dx} \longleftrightarrow \int$$

Consider a function f continuous on the interval $[a, b]$.

FTC Version I: $\displaystyle \frac{d}{dx}\int_a^x f(t)dt = f(x)$

i.e., the "area function" $A(x) = \displaystyle\int_a^x f(t)dt$ is an antiderivative of f;

or, the rate of change of area under the graph of f is given by f.



$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h} = f(x)$$

Note that $A(x + h) - A(x)$ is the narrow area over the interval from x to $x + h$, which is approximately $f(x)h$ when h is very close to 0. More precisely,

$$\frac{m(h)h}{h} \le \frac{A(x+h) - A(x)}{h} \le \frac{M(h)h}{h}$$

where $m(h) = \displaystyle\min_{x\le t\le x+h} f(t)$ and $M(h) = \displaystyle\max_{x\le t\le x+h} f(t)$, assuming that $h > 0$. (The interval is $x + h \le t \le x$ if $h < 0$.) Taking limit as $h \to 0$, we have $A'(x) = f(x)$, since $\displaystyle\lim_{h\to 0} m(h) = f(x) = \lim_{h\to 0} M(h)$ by the continuity of f.

FTC Version II: $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$ if $\displaystyle\int f(x)dx = F(x) + C$, i.e. $F'(x) = f(x)$.

In fact, by Version I, we have that the all other antiderivatives of f on an interval are given by

$$\int f(x)dx = \underbrace{\int_a^x f(x)dx}_{\text{one antiderivative of } f} + C$$

Now, if F is any other antiderivative of f, i.e.,

$$\int f(x)dx = F(x) + C$$

the two antiderivatives can differ from each other by at most a constant on any interval. Hence there is a constant C such that

$$\int_a^x f(x)dx = F(x) + C_1$$

By putting in $x = a$, we have $0 = \displaystyle\int_a^a f(t)dt = F(a) + C_1$, i.e., $C_1 = -F(a)$ and hence Version II follows:

$$\int_a^x f(x)dx = F(x) - F(a)\ .$$

Example: Since $\displaystyle\int e^{2x}dx = \frac{1}{2}e^{2x} + C$, we have $\displaystyle\int_1^2 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_1^2 = \frac{1}{2}(e^4 - e^2)$.

Review Exercise

See if you could complete the following:

1. (a) $\dfrac{d}{dx}\sin(5x) =$ (b) $\displaystyle\int \cos(5x)dx =$

2. (a) $\dfrac{d}{dx}\tan(4x) =$ (b) $\displaystyle\int \sec^2(4x)dx =$

3. (a) $\dfrac{d}{dx}e^{3x} =$ (b) $\displaystyle\int e^{3x}dx =$

4. (a) $\dfrac{d}{dx}e^{x^2} =$ (b) $\displaystyle\int_0^1 xe^{x^2}dx =$

5. (a) $\dfrac{d}{dx}\sqrt{4-x^2} =$ (b) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}dx =$

6. (a) $\dfrac{d}{dx}\ln|\sec x| =$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x dx =$

7. (a) $\dfrac{d}{dx}\ln|\sec x + \tan x| =$ (b) $\displaystyle\int_0^{\frac{\pi}{3}} \sec x dx =$

Basic Derivative Formulas

Basic Formula	Chain Rule Version	Other Techniques
$\dfrac{dx^p}{dx} = px^{p-1}$	$\dfrac{d\spadesuit^p}{dx} = p\spadesuit^{p-1}\dfrac{d\spadesuit}{dx}$	*Implicit Differentiation*
$\dfrac{de^x}{dx} = e^x$	$\dfrac{de^\spadesuit}{dx} = e^\spadesuit\dfrac{d\spadesuit}{dx}$	*Logarithmic Differentiation*
$\dfrac{d\ln\lvert x\rvert}{dx} = \dfrac{1}{x}$	$\dfrac{d\ln\spadesuit}{dx} = \dfrac{1}{\spadesuit}\dfrac{d\spadesuit}{dx}$	(If you know all these rules
$\dfrac{d\sin x}{dx} = \cos x$	$\dfrac{d\sin\spadesuit}{dx} = \cos\spadesuit\dfrac{d\spadesuit}{dx}$	and tricks, the derivatives of
$\dfrac{d\cos x}{dx} = -\sin x$	$\dfrac{d\cos\spadesuit}{dx} = -\sin\spadesuit\dfrac{d\spadesuit}{dx}$	$\ln x$ and $\sin x$ can give you all others.)
$\dfrac{d\tan x}{dx} = \sec^2 x$	$\dfrac{d\tan\spadesuit}{dx} = \sec^2\spadesuit\dfrac{d\spadesuit}{dx}$	
$\dfrac{d\sec x}{dx} = \sec x\tan x$	$\dfrac{d\sec\spadesuit}{dx} = \sec\spadesuit\tan\spadesuit\dfrac{d\spadesuit}{dx}$	

Mean Value Theorem

Recall that the Mean Value Theorem says:

If a function f is continuous on $[a,b]$ and differentiable on (a,b), there there is a number c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a)$$

By this theorem, if $f'(x) = 0$ for *all* x in some open interval, then f must be a constant function on this interval since for *any* two numbers a, b in the interval, say with $a < b$, we have certain number c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a) = 0 \cdot (b-a) = 0$$

i.e. $f(a) = f(b)$. In particular, the difference of any two antiderivatives F, G of f on an open interval must be a constant, since $(F(x) - G(x))' = f(x) - f(x) = 0$. Geometrically speaking, moving the graph of F up and down will not change its slope function.