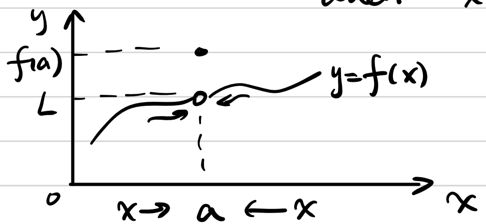
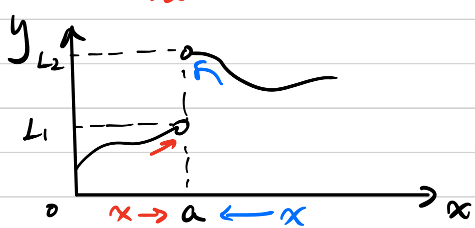
MATH1012

1. Review

Limit: $\lim_{x \to a} f(x) = L \xrightarrow{\text{means}}$ The value of $f(x)$ can be arbitrary close to L when x tends to a.



One-sided Limit: $\lim_{x \to a^-} f(x) = L_1$ $\lim_{x \to a^+} f(x) = L_2$



Notice: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

Sometimes, $\lim_{x\to a} f(x)$ does not exist.

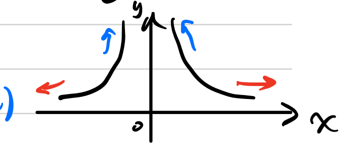
$\Downarrow$

when $x \to a$, $f(x)$ can not be arbitrary close to any real number.

Example 1. $f(x) = \frac{1}{x^2}$.

$\lim_{x\to 0} f(x)$ does not exist.

(As $x\to 0$, $f(x)$ can be arbitrary large)



$\Rightarrow$ We write: $\lim_{x\to 0} f(x) = +\infty$. (Notice: $\lim_{x\to 0} f(x)$ does not exist because $+\infty$ is not a number.)

$x=0$: vertical asymptote of f. $y=0$: horizontal asymptote of f.

In general:

vertical asymptote $x=a$: $\lim_{x\to a^-} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = +\infty$

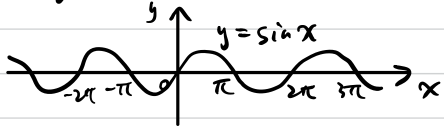
or $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^+} f(x) = +\infty$.

or $\lim_{x\to a} f(x) = -\infty$ or $\lim_{x\to a} f(x) = +\infty$.

horizontal asymptote $y=L$: $\lim_{x\to +\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.

Example 2 : $f(x) = \sin\frac{1}{x}$. $(x \neq 0)$. $\lim_{x\to 0} f(x)$ does not exist.

Recall :



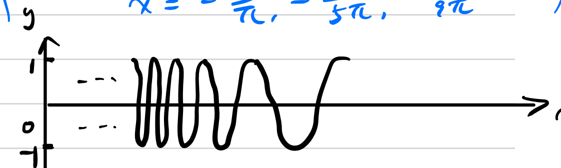
We can observe that :

$$\sin\frac{1}{x} = 0 \iff \frac{1}{x} = \underbrace{k}_{\text{a nonzero integer}}\pi \iff x = \frac{1}{k\pi}.\ k \neq 0 \quad \left(\text{e.g. } x = \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \cdots\right)$$

$$\sin\frac{1}{x} = 1 \iff \frac{1}{x} = 2\underbrace{k}_{\text{an integer}}\pi + \frac{\pi}{2} \iff x = \frac{1}{2k\pi + \frac{\pi}{2}} \quad \left(\text{e.g. } x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \cdots;\ x = -\frac{2}{3\pi}, -\frac{2}{7\pi}, -\frac{2}{11\pi}, \cdots\right).$$

$$\sin\frac{1}{x} = -1 \iff \frac{1}{x} = 2\underline{k}\pi + \frac{3\pi}{2} \iff x = \frac{1}{2k\pi + \frac{3\pi}{2}} \quad \left(\text{e.g. } x = \frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi}, \cdots;\ x = -\frac{2}{\pi}, -\frac{2}{5\pi}, -\frac{2}{9\pi}, \cdots\right)$$

x :	$\frac{2}{\pi}$	$\frac{1}{\pi}$	$\frac{2}{3\pi}$	$\frac{1}{2\pi}$	$\frac{2}{5\pi}$	$\frac{1}{3\pi}$	$\frac{2}{7\pi}$	$\frac{1}{4\pi}$	$\frac{2}{9\pi}$	$\cdots$
$f(x)$:	1	0	-1	0	1	0	-1	0	1	$\cdots$

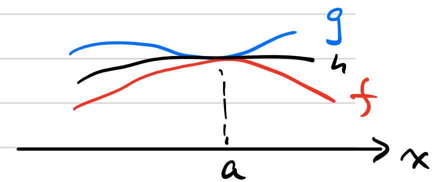


$\Rightarrow$ When $x \to 0^+$, $f(x)$ oscillates infinitely between -1 and 1. $\Rightarrow \lim_{x\to 0^+} f(x)$ does not exist.

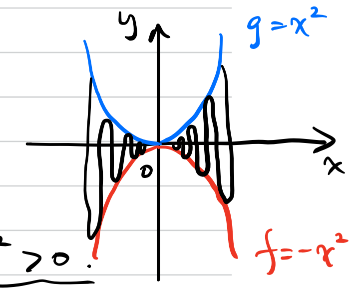
2. The Squeeze Theorem (The Sandwich Theorem).

Suppose $f(x) \le h(x) \le g(x)$ when x is near a.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then $\lim_{x\to a} h(x) = L$.



Example 1: Compute $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$



We can observe that: (1) $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

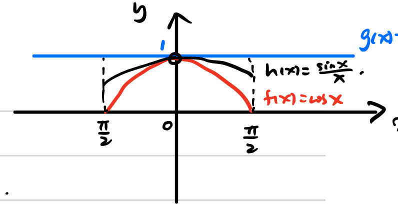
(2). For any $x \neq 0$, we have $-1 \le \sin\frac{1}{x} \le 1$. and $x^2 > 0$.

so we have $\boxed{-x^2 \le x^2 \sin\frac{1}{x} \le x^2}$ for all $x \neq 0$.

Define $f(x) = -x^2$. $h(x) = x^2 \cdot \sin\frac{1}{x}$. $g(x) = x^2$.

Then $f(x) \le h(x) \le g(x)$ and $\lim_{x\to 0} f(x) = 0$. $\lim_{x\to 0} g(x) = 0 \Rightarrow \lim_{x\to 0} h(x) = 0$.

Example 2 : Compute $\lim_{x\to 0} \frac{\sin x}{x} = 1$ → $\frac{0}{0}$ type.



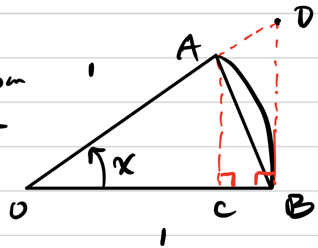
We have $\boxed{\cos x \le \frac{\sin x}{x} \le 1}$ when $x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Define $f(x) = \cos x$. $h(x) = \frac{\sin x}{x}$. $g(x) = 1$. $\Rightarrow f(x) \le h(x) \le g(x)$ $\Rightarrow \lim_{x\to 0} h(x) = 1$

$\lim_{x\to 0} f(x) = 1$. $\lim_{x\to 0} g(x) = 1$

Proof of $\cos x \le \frac{\sin x}{x} \le 1$ when $x \in (0, \frac{\pi}{2}]$:

a sector from the unit circle. ←



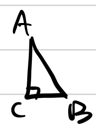
$OA = OB = 1$. arc $AB = x$ → $\frac{x}{2\pi} = \frac{\text{arc}AB}{2\pi \cdot 1}$

$\sin x = \frac{AC}{AO} = AC$.

$\frac{x}{2\pi} = \frac{\text{area of } \triangle}{\pi \cdot 1^2}$.

$\tan x = \frac{BD}{OB} = BD$.

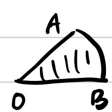
proof of $\frac{\sin x}{x} \le 1$: For



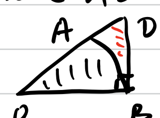
, we have $AC \le AB \Rightarrow \sin x \le AB$.

On the other hand, we can observe $AB \le \text{arc}AB \Rightarrow AB \le x$ } $\sin x \le x$.

proof of $\cos x \le \frac{\sin x}{x}$: area of



≤ area of



$\Rightarrow \frac{x}{2} \le \frac{1}{2} \cdot 1 \cdot \tan x \Rightarrow x \le \frac{\sin x}{\cos x}$

$\Rightarrow \cos x \le \frac{\sin x}{x}$.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

$\frac{\sin u}{u}$. $u=3x$.

Application : $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 3 = 3 \cdot \lim_{x\to 0} \boxed{\frac{\sin 3x}{3x}} = 3\cdot 1 = 3$

$$\lim_{x\to 0} \frac{x}{x+\sin x} = \lim_{x\to 0} \frac{1}{1+\frac{\sin x}{x}} = \frac{1}{1+\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{1-\cos x}{\frac{1}{2}x^2} = \lim_{x\to 0} \frac{1-1+2\left(\sin\frac{x}{2}\right)^2}{\frac{1}{2}\cdot x^2} = \lim_{x\to 0} \frac{2\cdot\left(\sin\frac{x}{2}\right)^2}{\frac{1}{2}\cdot x^2}$$

Recall the double-angle formula:

$$\cos x = 1 - 2\cdot\left(\sin\frac{x}{2}\right)^2.$$

$$= \lim_{x\to 0} \frac{\left(\sin\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2} = \left(\lim_{x\to 0} \boxed{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}\right)\cdot\left(\lim_{x\to 0} \boxed{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}\right)$$

$\frac{\sin u}{u}$, $u=\frac{x}{2}$.

$$= 1.$$

3. Continuity

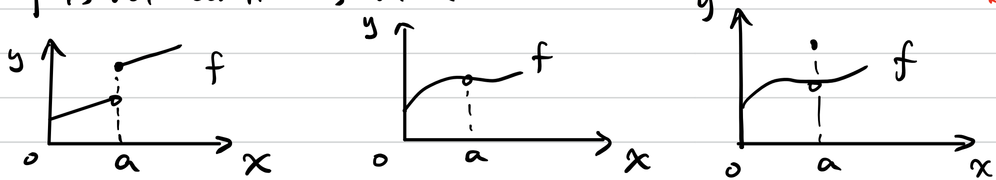
(1). $f(x)$ is continuous at $a \iff \lim\limits_{x \to a} f(x) = f(a)$ $\xRightarrow{\text{means}}$

① $\lim\limits_{x\to a} f(x)$ exists

② $f(a)$ is defined.

③ $\lim\limits_{x\to a} f(x) = f(a)$

Examples: f is not continuous at a:



(2) $f(x)$ is continuous on $I \iff f$ is continuous at all points in I.

Example: $y = \frac{1}{x}$ is continuous on $(-\infty, 0) \cup (0, +\infty)$.

Example: The following funtions are all continuous in their domain:

Polynomials $(x^{10} + x^2 - 6)$. Rational function $\left(\frac{x^3+3}{x-6}\right)$. root function $(\sqrt{x})$

exponential function (3^x). logarithmic function $(\log_3 x)$.

trigonometric function $(\tan x)$. inverse trigonometric function $(\tan^{-1} x)$.

3). rules of continuity :

① f and g are continuous at a.

$\Rightarrow f+g,\ f-g,\ \underset{=}{c}\cdot f,\ f\cdot g,\ \frac{f}{g}\ (g(a)\neq 0)$ are continuous at a.

a constant

Example : $y = \sin x + 3^x - x^3$ is continuous on $(-\infty, +\infty)$.

② g is continuous at a.
f is continuous at $g(a)$

$\Rightarrow f\circ g = f(g(x))$ is continuous at a.

③ $\lim\limits_{x\to a} g(x) = b$
$f(x)$ is continuous at b

$\Rightarrow \lim\limits_{x\to a} f(g(x)) = f(b)$.

Example: Take $f(x) = \log_2 x$. Then $\lim\limits_{x\to a} \log_2(g(x)) = \log_2\left(\lim\limits_{x\to a} g(x)\right)$

if $\lim\limits_{x\to a} g(x) > 0$.