$\vec{p} = m\vec{v}$ $\boxed{I\vec{\omega}\ ?}$

Dynamics of Rigid Bodies IV
Angular Momentum

PHYS1112

Lecture 11

m	I
$\vec{r}$	$\vec{\theta}$
$\vec{v} = \frac{d\vec{r}}{dt}$	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$
$\vec{a} = \frac{d^2\vec{r}}{dt^2}$	$\vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2}$
$\vec{F}$	$\vec{\tau} = \vec{r} \times \vec{F}$
$= m\vec{a}$	$= I\vec{\alpha}$
$K = \frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$

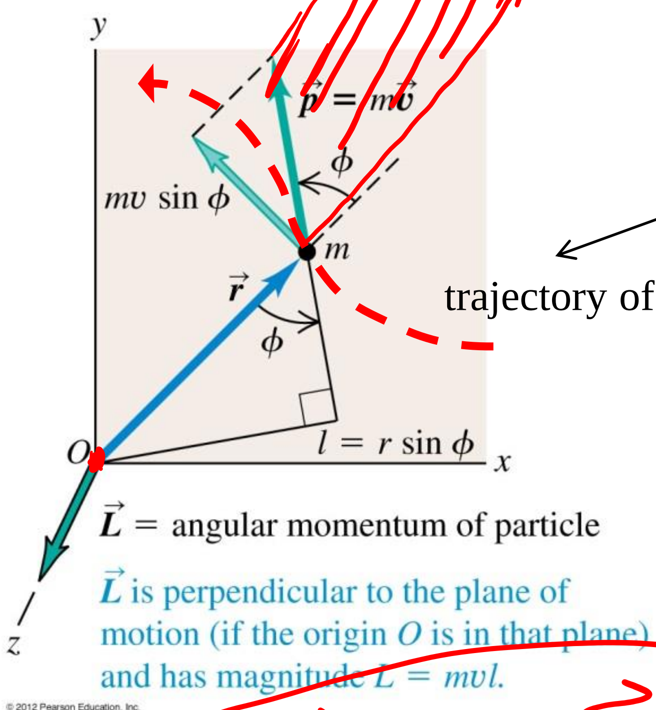
Intended Learning Outcomes

- After this lecture you will learn:
  1) the angular momentum of a system of particles and rigid body.
  2) how to describe dynamics of a system using its angular momentum.
  3) conservation of angular momentum.
  4) precession of angular momentum vector in a gyroscope.

For a point particle, define its **angular momentum** about the origin O by

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

$\vec{\tau} = \vec{r} \times \vec{F}$?



?

trajectory of m

the particle need not be rotating about any axis, can be travelling in a straight line

$\vec{r}(t)$

$\vec{p}(t)$

$$L = mvr \sin \phi = (mv \sin \phi) r$$
$$= mv(r \sin \phi)$$

$\frac{d\vec{r}}{dt} = \vec{v} \neq 0 \quad \vec{p} = m\vec{v}$

$$\frac{d\vec{L}}{dt} = \boxed{\left(\frac{d\vec{r}}{dt} \times \vec{p}\right)}_{=0} + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) = \vec{r} \times \vec{F} = \vec{\tau}$$

$= \vec{\tau}$

$m\frac{d\vec{r}}{dt}$

$\vec{F}$

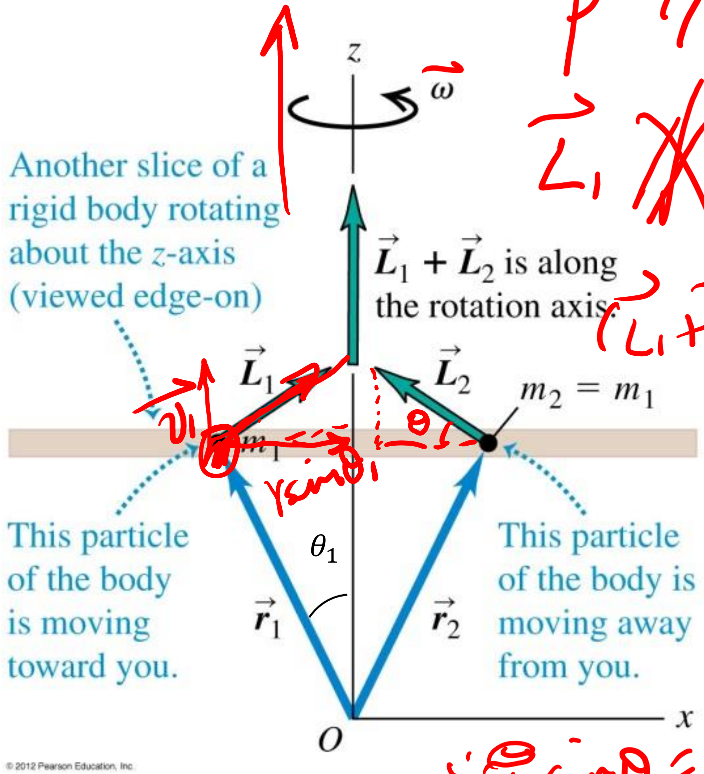
$\vec{v} \times m\vec{v}$

$$\boxed{\frac{dL}{dt} = \vec{r} \times \vec{F}}$$

i.e. $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$ ✓

c.f. $\boxed{\frac{d\vec{P}}{dt} = \vec{F}}$ ✓

For a rigid body



$\vec{p} \parallel \vec{v}$

$\vec{L}_1 \nparallel \vec{\omega}$

$(\vec{L}_1 + \vec{L}_2) \parallel \vec{\omega}$

"$L_1 \sin\theta = I\omega$"

Take the rotation axis as the z axis,
m_1 is a small mass of the rigid body

$$L_1 = mv_1 r_1 = m(\omega r_1 \sin\theta_1) r_1$$?

If rotation axis is a symmetry axis, then there exist m_2 on the opposite side whose x-y components of angular momentum cancel those of m_1.
Therefore only z component of any $\vec{L}_i$ is important.

Total angular momentum $\vec{L} = \sum \vec{L}_i = \sum L_i \sin\theta_i \,\hat{k}$, points along rotation axis with magnitude

$$L = \sum [m_i(\omega r_i \sin\theta_i) r_i] \sin\theta_i = \left(\sum m_i \underbrace{(r_i \sin\theta_i)}_{\perp \text{ distance of } m_i \text{ to rotation axis}}{}^2 \right) \omega$$

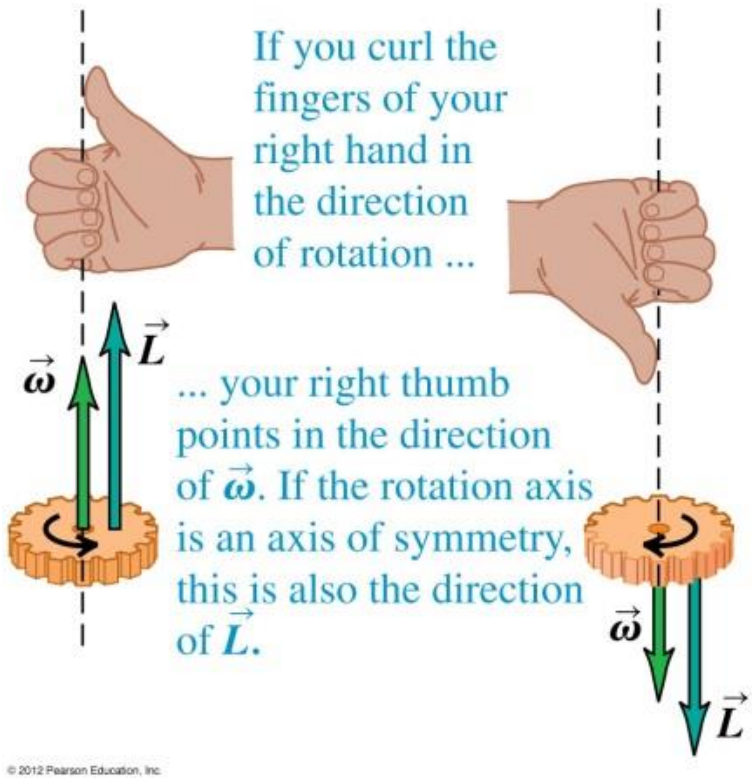
$I = m r^2$

lever arm

Conclusion: if rotation axis is a symmetry axis, then

$$\boxed{\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}},$$

$$c.f.\ \vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$$

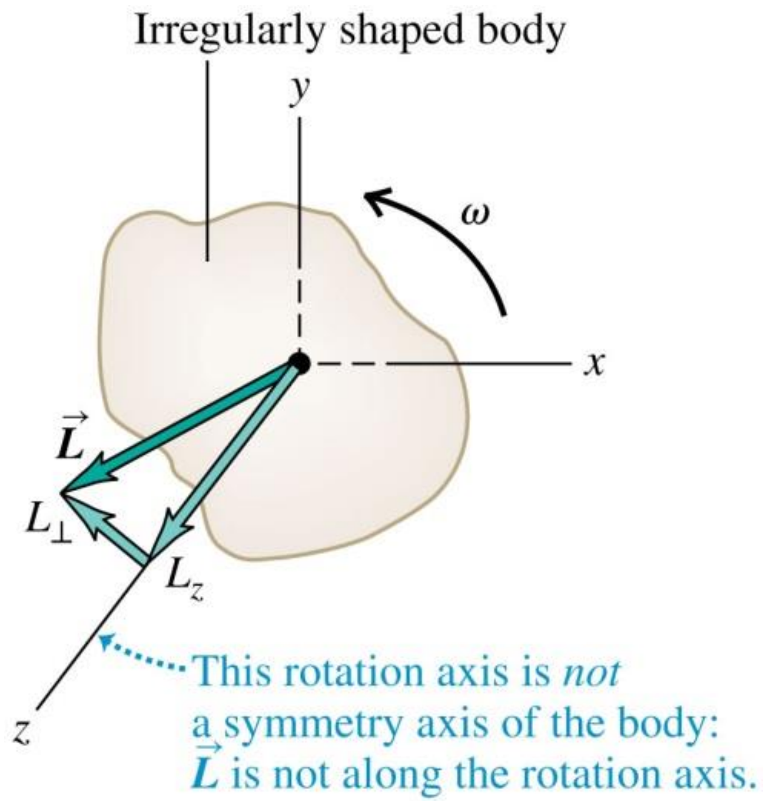


$\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{L}}$ have the same direction

What if the rotation axis is not a symmetry axis? $\vec{L} \neq I\vec{\omega}$

$\vec{p} = m\vec{v}$

$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$



If the rotation axis (z axis) is fixed,

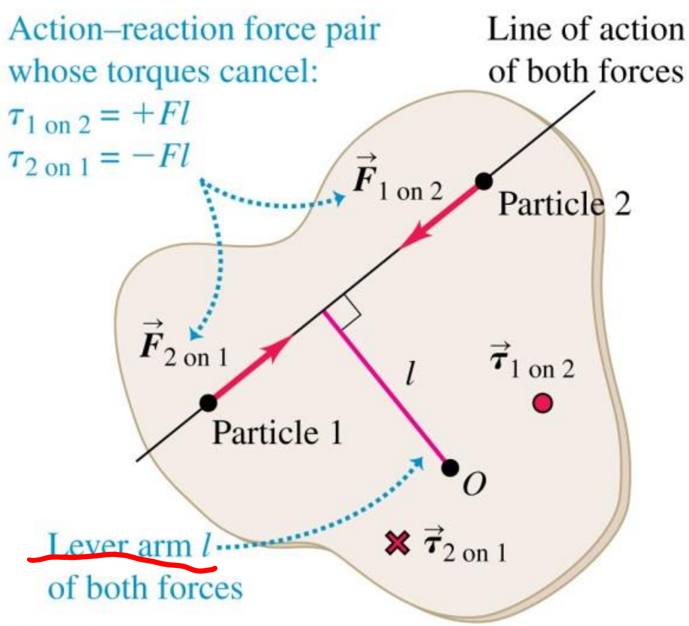
a) $\vec{L}$ changes, i.e., there exist a finite torque to keep the body rotating.

b) $L_\perp$ not important since it does not produce physical motion, "angular momentum" may refer to the component of $\vec{L}$ along the axis of rotation, i.e. $L_z = I\omega$, but not $\vec{L}$ itself in this case.

" $\vec{L} = I\vec{\omega}$ " for principal axis

Internal forces (action and reaction pairs) have the same line of action $\rightarrow$ no net torque.

Therefore for a system of particles or a rigid body



$$\boxed{\frac{d\vec{\boldsymbol{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}_{\text{ext}}}$$

c.f. $$\frac{d\vec{\boldsymbol{P}}}{dt} = \sum \vec{\boldsymbol{F}}_{\text{ext}} = 0$$

Under no external torque (⚠ *not* force)

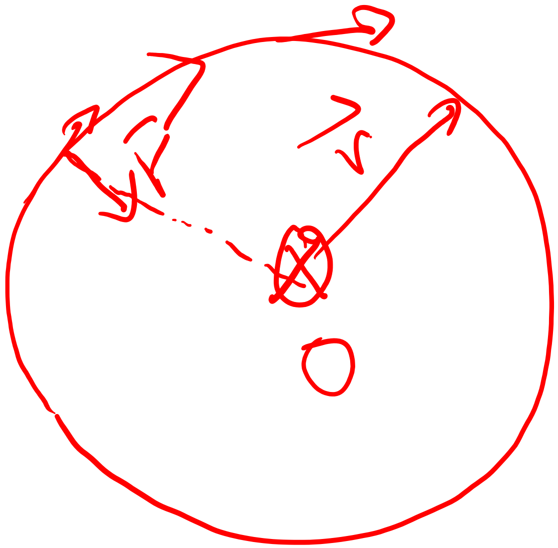
$$\frac{d\vec{\boldsymbol{L}}}{dt} = 0$$

conservation of angular momentum

"isotropic"

Question:

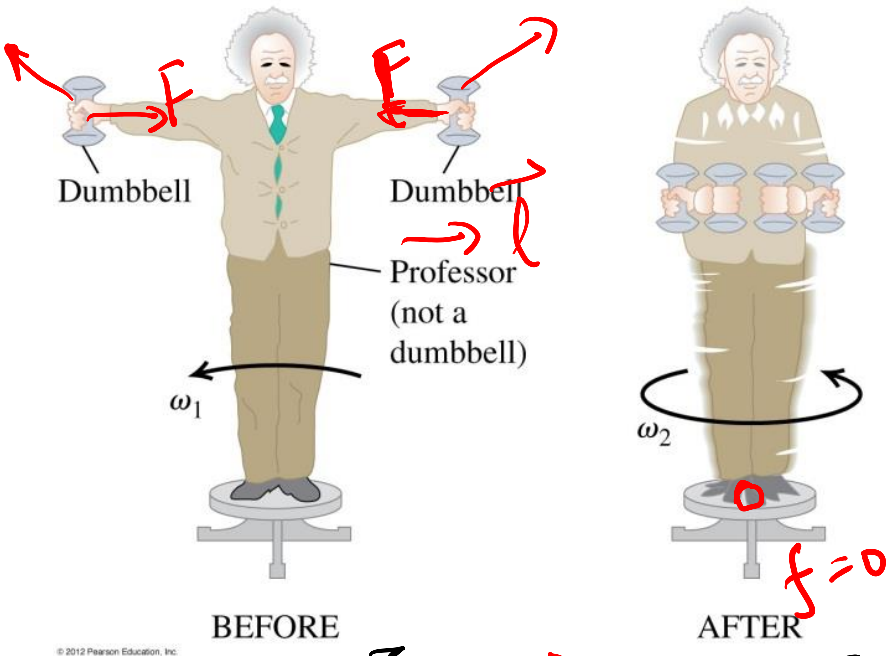
A particle is going around in uniform circular motion (constant speed). Does its linear momentum $\vec{p}$ conserve? Does its angular momentum $\vec{L}$ conserve?



"$\vec{L} = I\vec{\omega}$"

$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

Demonstration: A spinning physics professor



$I_1 > I_2$

Conservation of angular momentum

$$I_1\omega_1 = I_2\omega_2$$

If $I_2 = I_1/2$, then $\omega_2 = 2\omega_1$, and

$K_2 = \frac{1}{2} I_2 \omega_2^2 = \underline{\ 2\ } K_1$.

$\frac{1}{2} \times 2^2 K_1$

Where comes the extra energy?

And in the reverse process $I_2 \rightarrow I_1$, where goes the energy?

Example

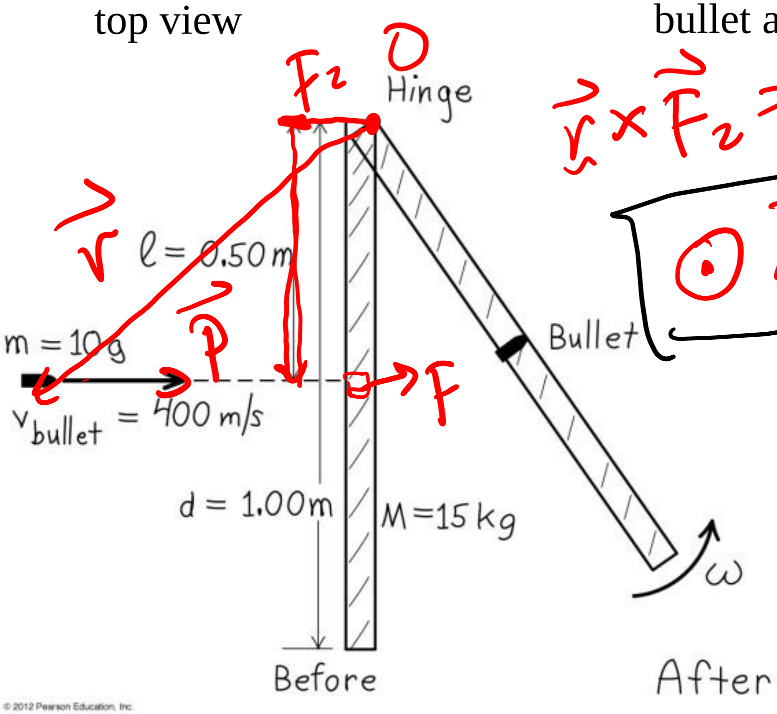
A bullet hits a door in a perpendicular direction, embeds in it and swings it open.

During the collision:

Linear momentum is not conserved because ______________________________

Angular momentum along the rotation axis is conserved because ______________

top view



$\vec{r} \times \vec{F}_2 = 0$

$\boxed{\odot\ \vec{L}}$

initial angular momentum of bullet about hinge

$$mvl = \underbrace{\left(\frac{Md^2}{3}\right)}_{\text{moment of inertia of door about hinge}}\omega + \overset{I\omega}{\underbrace{\left(ml^2\right)}_{\text{moment of inertia of bullet about hinge after embedded in door}}}\omega$$

$$\Rightarrow \qquad \omega = \frac{mvl}{\frac{1}{3}Md^2 + ml^2}$$

"$\vec{L}_z = I\vec{\omega}$"

Question: the hinge is not a symmetry axis! Why is the angular momentum $I\omega$?

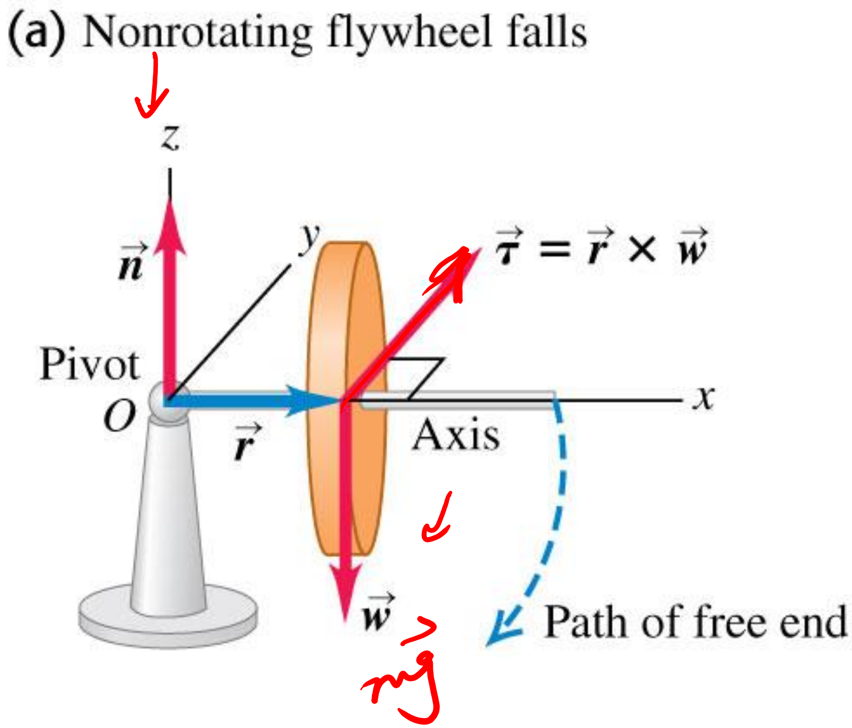
Question: If the polar ice caps were to completely melt due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (increase / decrease / remain the same).

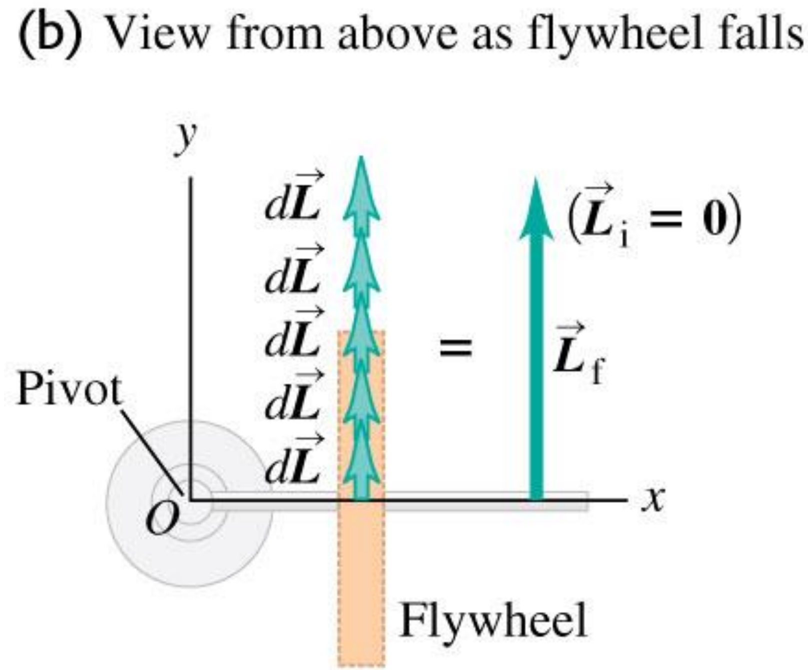
$\uparrow I \, \omega \downarrow$

$\uparrow T = \frac{2\pi}{\downarrow \omega}$

Gyroscopes and Precession

Case 1: when the flywheel is not spinning – it falls down





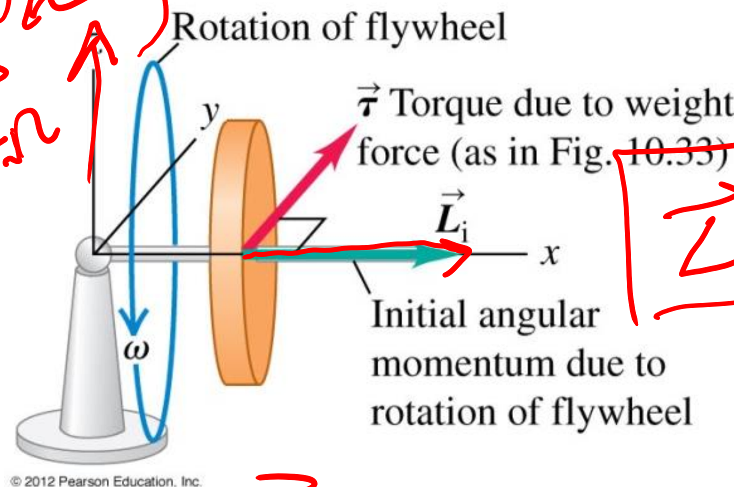
torque $\vec{\tau}$ due to weight of the flywheel $\vec{w}$ causes it to fall in the x-z plane

$\vec{L}$ increases as flywheel falls

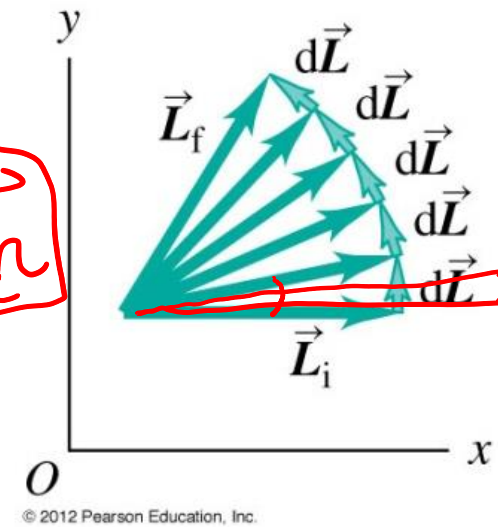
Case 2: when flywheel spinning with initial angular moment $\vec{L}_i$ – it **precesses**

Ω

$\vec{L}_\Omega$



$\boxed{\vec{L} = \vec{L}_i + \vec{L}_\Omega}$



$d\vec{L}$

$|\vec{\omega}| >> |\vec{\Omega}|$

$|\vec{\omega}| \sim |\vec{\Omega}|$

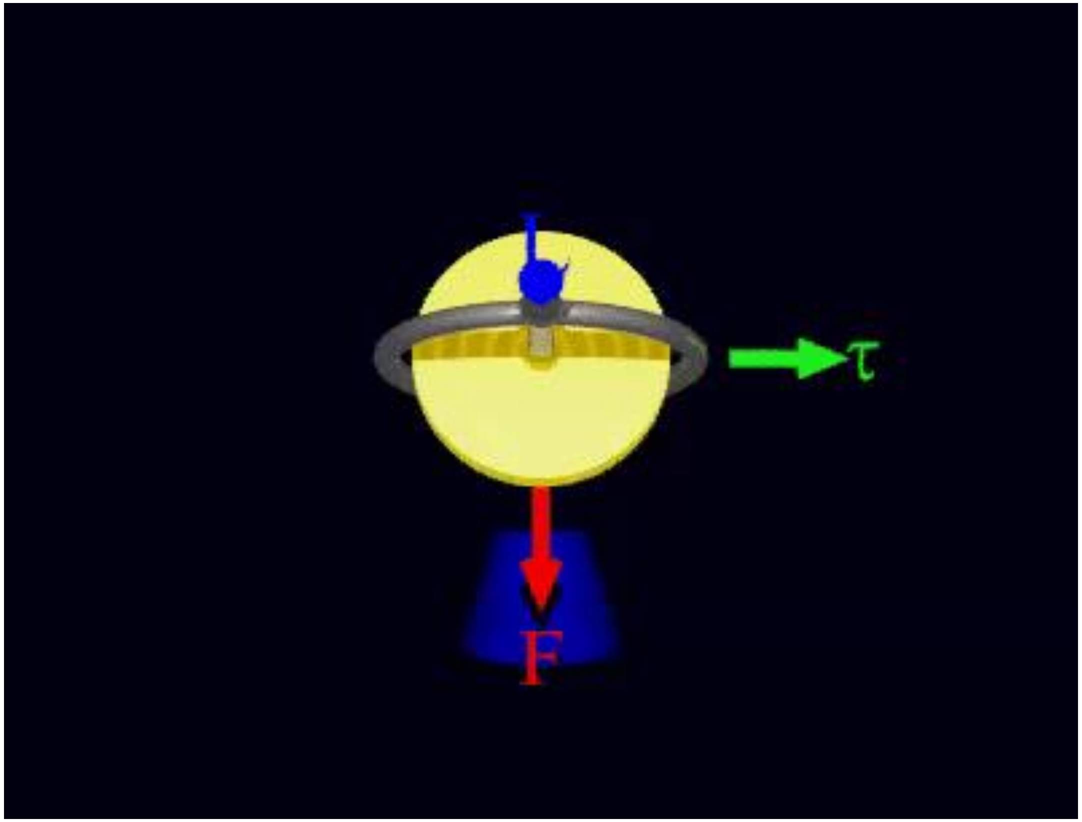
$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Since $\vec{L} \perp d\vec{L}$, flywheel axis execute circular motion called **precession**, $|\vec{L}|$ remains constant

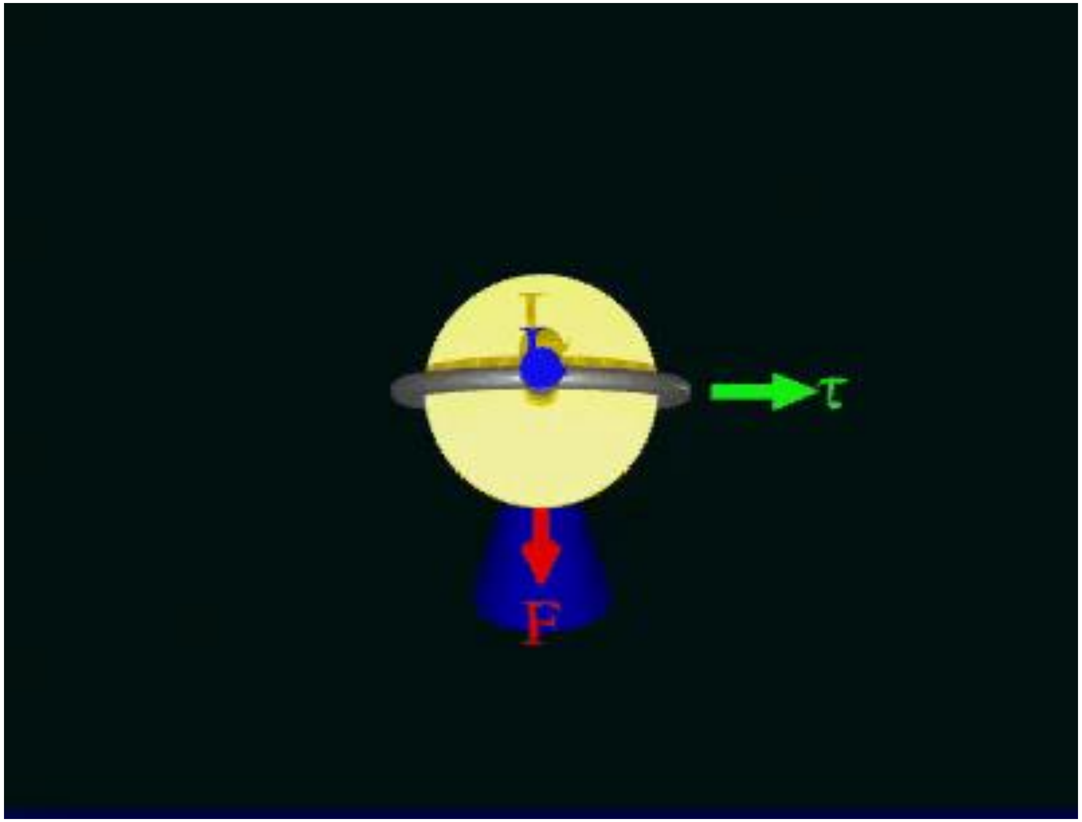
$d\vec{L} = \vec{\tau}\,dt$

⚠ faster spinning ω → slower precession Ω

animation of the vectors $\vec{\boldsymbol{w}}$, $\vec{\boldsymbol{\tau}}$, and $\vec{\boldsymbol{L}}$ at
http://phys23p.sl.psu.edu/phys_anim/mech/gyro_s1_p.avi

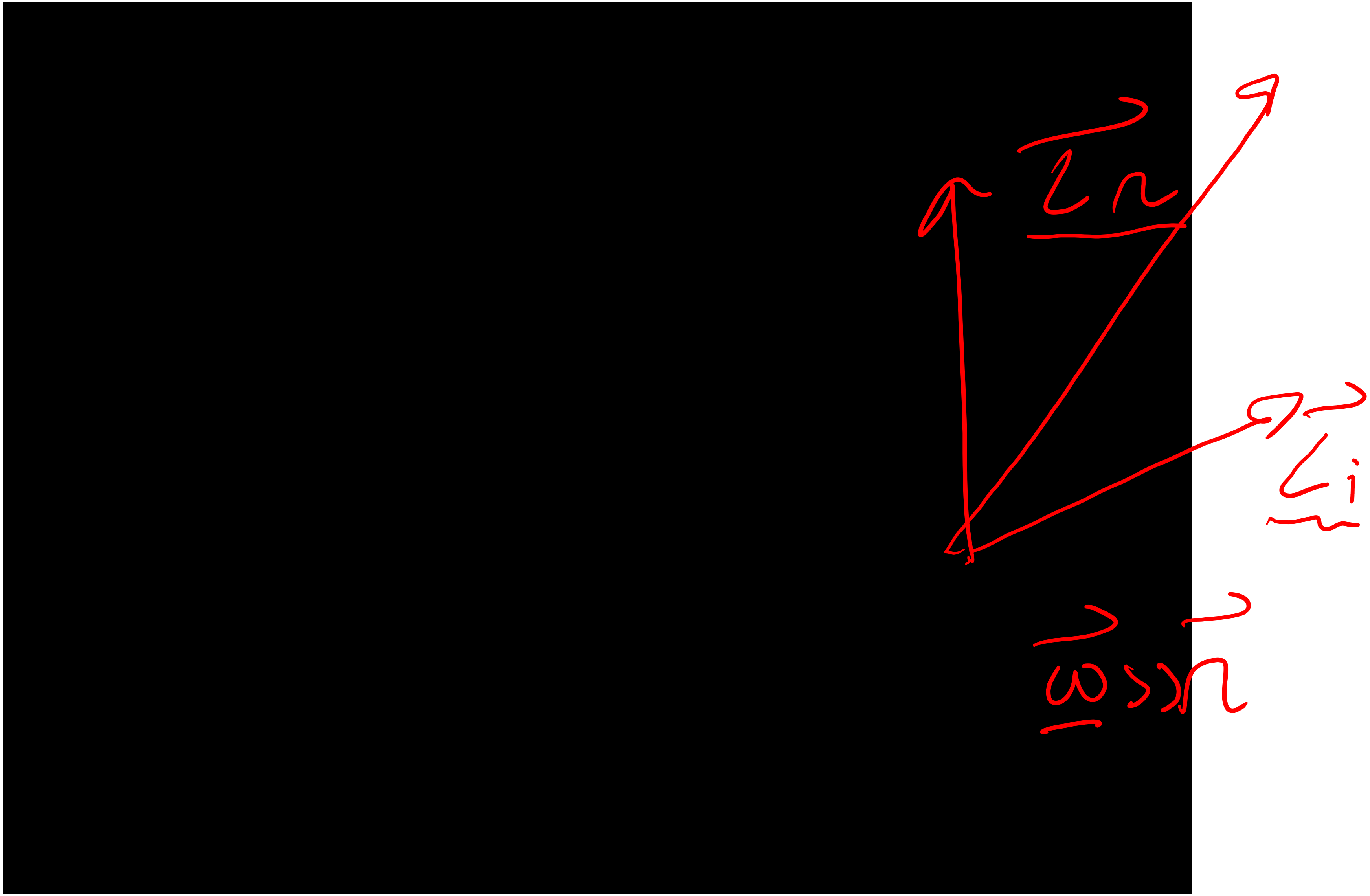


If $\omega \gg \Omega$, can ignore angular momentum due to precession. Otherwise there is *nutation* of the flywheel axis – it wobbles up and down



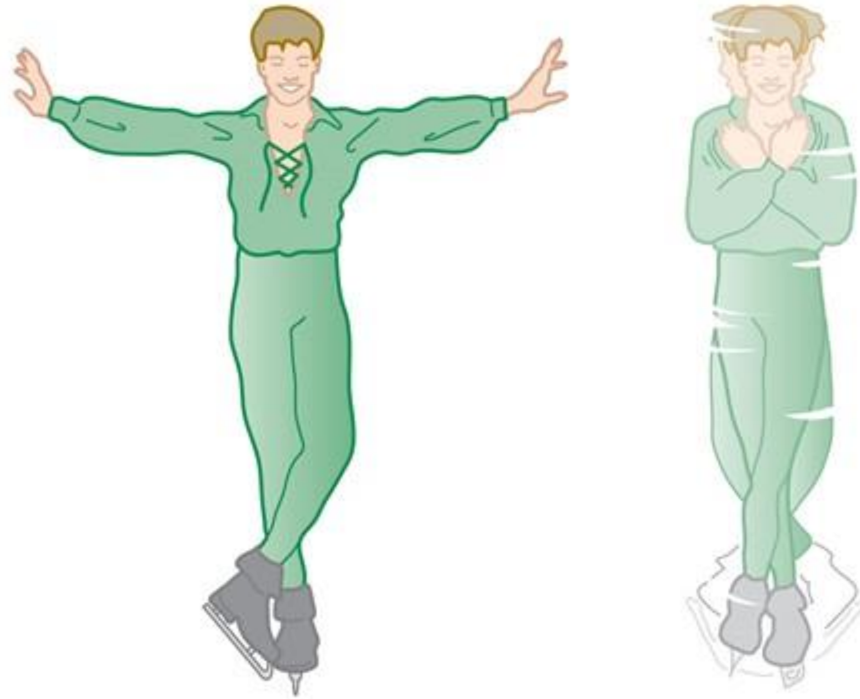
$\underline{\omega} \sim \underline{\Omega}$

A formal gyroscope



Q10.12

A spinning figure skater pulls his arms in as he rotates on the ice. As he pulls his arms in, what happens to his angular momentum L and kinetic energy K?



A. L and K both increase.

B. L stays the same; K increases.

C. L increases; K stays the same.

D. L and K both stay the same.

E. None of the above.

A10.12

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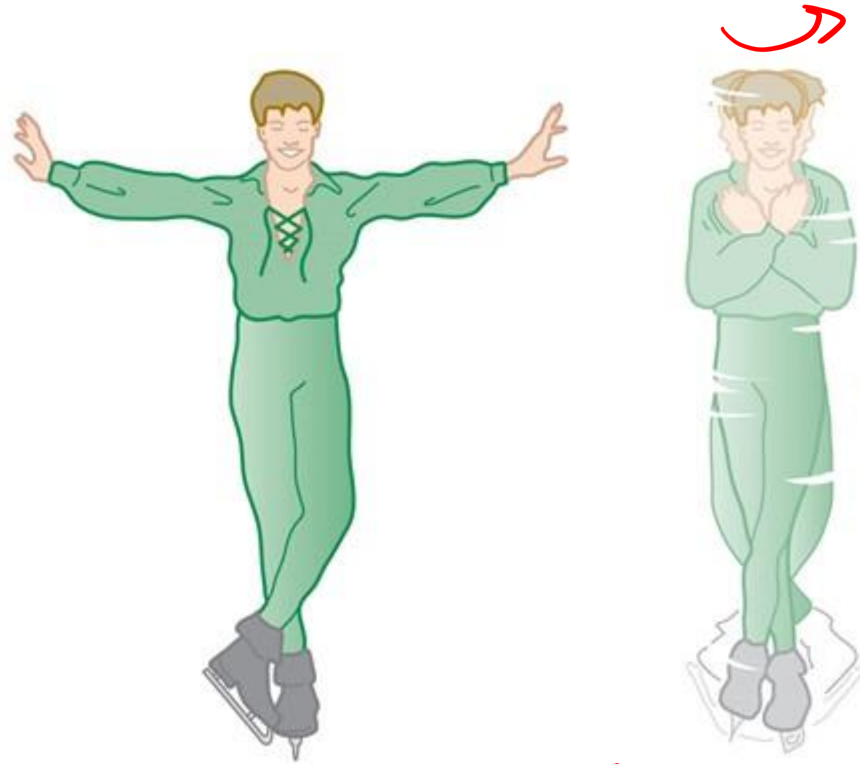


$\vec{L} = \downarrow I \, \vec{\omega} \uparrow$

A. L and K both increase.

✔ B. L stays the same; K increases.

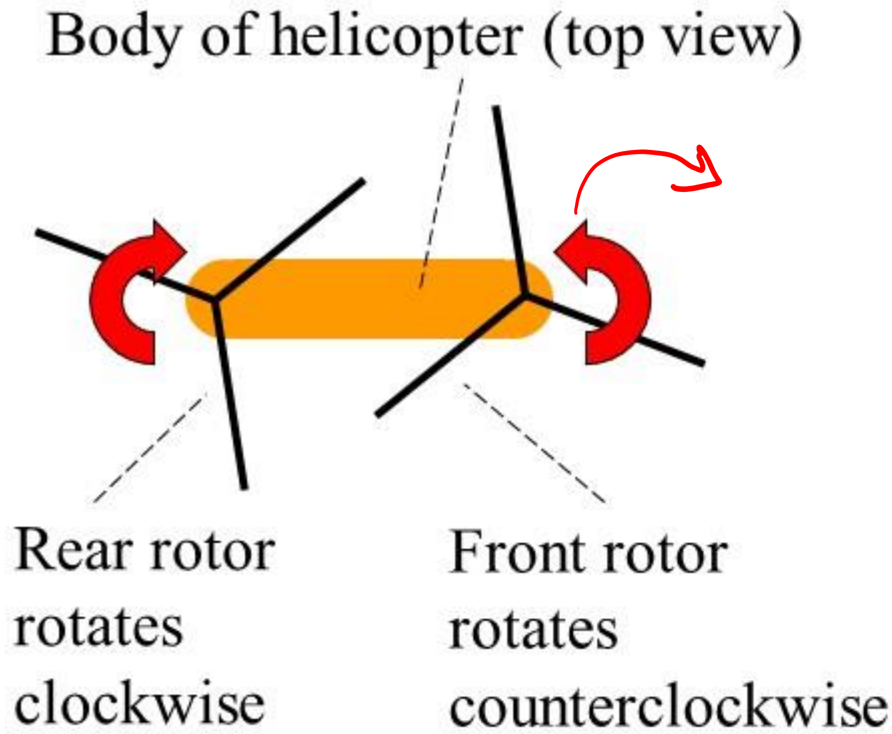
C. L increases; K stays the same.

D. L and K both stay the same.

E. None of the above.

Q10.13

Some helicopters have two large rotors that rotate in *opposite* directions as shown. If instead they *both* rotated in the clockwise direction as seen from above, what would happen to the body of the helicopter if the pilot increased the rotation speed of both rotors?



A. The body would rotate clockwise.

B. The body would rotate counterclockwise.

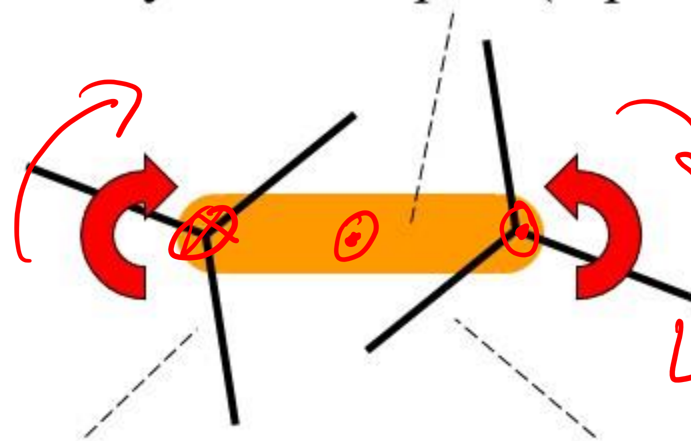
C. Nothing—there would be no effect on the body.

D. The answer depends on how fast the rotors rotate.

A10.13

$\vec{F}$

Some helicopters have two large rotors that rotate in *opposite* directions as shown. If instead they *both* rotated in the clockwise direction as seen from above, what would happen to the body of the helicopter if the pilot increased the rotation speed of both rotors?



$\vec{L}$

L_{body}

$\vec{L} = I\vec{\omega}$

A. The body would rotate clockwise.

B. The body would rotate counterclockwise.

C. Nothing—there would be no effect on the body.

D. The answer depends on how fast the rotors rotate.