COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Fall Semester – Tutorial 6

Question 1: Ramesh can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will be late. The probability that he is late when he rides his bicycle is only 5%. Ramesh arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

(a) Suppose the boss assumes that there is a 1/3 chance that Ramesh takes each of the three ways he can get to work. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem under this assumption?

(b) Suppose the boss knows that Ramesh drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem using this information?

Solution : Let L be the event that Ramesh is late, and let B, C, and O (standing for "omnibus") be the events that he went by bicycle, car, and bus, respectively. We are told that $p(L|B) = 0.05, p(L|C) = 0.50$, and $p(L|O) = 0.20$. We are asked to find $p(C|L)$.

(a) We are to assume here that $p(B) = p(C) = p(O) = 1/3$. Then by the generalized version of Bayes' theorem,

$$\begin{aligned} p(C|L) &= \frac{p(L|C)p(C)}{p(L|B)p(B) + p(L|C)p(C) + p(L|O)p(O)} \\ &= \frac{(0.50)(1/3)}{(0.05)(1/3) + (0.50)(1/3) + (0.20)(1/3)} = \frac{2}{3} \end{aligned}$$

(b) Now we are to assume here that $p(B) = 0.60, p(C) = 0.30, p(O) = 0.10$. Then by the generalized version of Bayes' theorem,

$$\begin{aligned} p(C|L) &= \frac{p(L|C)p(C)}{p(L|B)p(B) + p(L|C)p(C) + p(L|O)p(O)} \\ &= \frac{(0.50)(0.30)}{(0.05)(0.60) + (0.50)(0.30) + (0.20)(0.10)} = \frac{3}{4} \end{aligned}$$

Question 2: Consider rolling two fair dice and let D_1 and D_2 be the results respectively. Define $X = \max\{D_1, D_2\}$. In other words, X is the maximum of D_1 and D_2. Answers with summation are accepted.

(a) What is the distribution function of X?

(b) What is $E(X)$?

(c) What is $V(X)$?

Solution : (a) $P(X = k) = \frac{1+2\cdot(k-1)}{36}$ for $k \in \{1, 2, 3, 4, 5, 6\}$.

(b) $\sum_{k=1}^{6} k \frac{1+2\cdot(k-1)}{36}$.

(c) $\sum_{k=1}^{6} (k - E(X))^2 \frac{1+2\cdot(k-1)}{36}$.

Question 3: Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

(a) exactly three boys?

(b) at least one boy?

(c) at least one girl?

(d) all children of the same sex?

Solution : These questions are applications of the binomial distribution. Let $p = 0.51$ and $n = 5$ for this problem.

(a) We are asked for the probability that $k = 3$. By the Theorem the answer is $C(5, 3)0.51^3 0.49^2 \approx 0.32$.

(b) There will be at least one boy if there are not all girls. The probability of all girls is 0.49^5, so the answer is $1 - 0.49^5 \approx 0.972$.

(c) This is just like part (b): The probability of all boys is 0.51^5, so the answer is $1 - 0.51^5 \approx 0.965$.

(d) There are two ways this can happen. The answer is clearly $0.51^5 + 0.49^5 \approx 0.063$.

Question 4: A marketing representative from a company randomly approaches people on a random street in Hong Kong until he finds a person who is a customer of his company. Let p, the probability that he succeeds in finding such a person, equal 0.2. And let X denote the number of people he approaches until he finds his first success. What is the probability that the marketing representative

(a) must approach 4 people before he finds one customer of his company?

(b) must approach more than 4 people before he finds one customer of his company?

Solution : This problem is modeled using the geometric distribution.

(a) $p(X = 4) = (1 - p)^3 p = 0.8^3 \times 0.2 = 0.1024$.

(b) $p(X > 4) = 1 - p(X \leq 4) = 1 - \sum_{k=1}^{4}(1-p)^{k-1}p = 1 - \sum_{k=1}^{4} 0.8^{k-1}0.2 = 0.4096$

Question 5: A gift for Tom is randomly hidden at one of n locations. Everytime Tom randomly checks a location from all the n locations, i.e. the same location may be checked more than once before Tom finds his gift. Let X be the number of locations that Tom checks until he finds the gift. Answers with summation are accepted.

(a) What is the distribution function of X?

(b) What is $E(X)$?

(c) What is $V(X)$?

Solution : (a) $P(X = i) = \frac{1}{n}(\frac{n-1}{n})^{i-1}$ for all $i \in \{1, 2, \ldots, n\}$.

(b) $\frac{1}{1/n} = n$.

(c) $\sum_{i=1}^{\infty}(i - n)^2 \frac{1}{n}(\frac{n-1}{n})^{i-1}$

Question 6: Consider a game where one repeatedly throws a fair die until the total number of dots in all the throws reaches or exceeds 3. This means that if one gets 3, 4, 5, or 6 at the first throw, one stops right away. If one gets 1 or 2 at the first throw, however, one must continue.

(a) Describe the sample space for the game by listing all the possible outcomes. Here are two examples: The outcome "12" means that one gets 1 at the first throw and 2 at the second throw, and then the game is over. The outcome "113" means that one gets 1 at the first throw, 1 at the second throw and 3 at the third throw, and then the game is over.

(b) Give the probability for each possible outcome.

(c) Let X be the number of times one gets an odd number of dots during the game. What are the possible values of X? For the two outcomes "12" and "113", the values of X are 1 and 3 respectively.

(d) For each possible value x of X, describe the event "$X = x$". Recall that an event is a *a subset of the sample space*. To answer this question you need to give the subset of the sample space corresponding to each value of x.

(e) For all possible values x, give the probability that "$X = x$".

(f) What is $E(X)$?

(g) What is $V(X)$?

Solution :

(a) All possible outcomes are: {"3", "4", "5", "6", "12", "13", "14", "15", "16", "21", "22", "23", "24", "25", "26", "111", "112", "113", "114", "115", "116"}, i.e., there are 21 possible outcomes.

(b) For {"3", "4", "5", "6"}, the probability of each of these outcomes is 1/6.

For {"12", "13", "14", "15", "16", "21", "22", "23", "24", "25", "26"}, the probability of each of these outcomes is $(1/6)^2$.

For {"111", "112", "113", "114", "115", "116"}, the probability of each of these outcomes is $(1/6)^3$.

(c) The possible values are 0, 1, 2, 3.

(d) The event "$X = 0$" includes the following subsets: {"22", "24", "26", "4", "6"}.

The event "$X = 1$" includes the following subsets: {"3", "5", "12", "14", "16", "21", "23", "25"}.

The event "$X = 2$" includes the following subsets: {"112", "114", "116", "13", "15"}.

The event "$X = 3$" includes the following subsets: {"111", "113", "115"}.

(e) $p(X = 0) = 2 \cdot \frac{1}{6} + 3 \cdot (\frac{1}{6})^2 = \frac{5}{12}$;

$p(X = 1) = 2 \cdot \frac{1}{6} + 6 \cdot (\frac{1}{6})^2 = \frac{1}{2}$;

$p(X = 2) = 2 \cdot (\frac{1}{6})^2 + 3 \cdot (\frac{1}{6})^3 = \frac{5}{72}$;

$p(X = 3) = 3 \cdot (\frac{1}{6})^3 = \frac{1}{72}$.

(f) $E(X) = 0 \cdot \frac{5}{12} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{72} + 3 \cdot \frac{1}{72} = \frac{49}{72}$

(g) First, $E(X^2) = 0^2 \cdot \frac{5}{12} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{5}{72} + 3^2 \cdot \frac{1}{72} = \frac{65}{72}$;

Thus, $V(X) = E(X^2) - E(X)^2 = \frac{65}{72} - (\frac{49}{72})^2 = \frac{2279}{5184}$.