§2.2 The inverse of a matrix

Def: An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \text{ and } AC = I_n.$$

In this case, C is an inverse of A and denoted by A^{-1}.

Remark: The inverse of A is unique.

Proof: If B is another inverse of A, then

$$AB = BA = I, \quad AC = CA = I.$$

Hence, $B = B \cdot I = B \cdot AC = (BA)C = C$

Def: If a matrix is not invertible, it is called singular matrix.

Example: If $A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$ and $C = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$, then

$$A \cdot C = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times (-7) + 5 \times 3 & 2 \times (-5) + 5 \times 2 \\ -3 \times (-7) + (-7) \times 3 & -3 \times (-5) + (-7) \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $ad-bc=0$, then A is not invertible.
$ad-bc$ is called the determinant of A, denoted by $\det(A)$.

Proof: $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Example: Find the inverse of $A=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution: Since $\det A = 3\cdot 6 - 5\cdot 4 = -2 \neq 0$, A is invertible.

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix}.$$

Thm: If A is an invertible $n\times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$.

Proof: If we replace $\vec{x}$ by $A^{-1}\vec{b}$, then

$$A\vec{x} = A\cdot A^{-1}\vec{b} = (AA^{-1})\vec{b} = \vec{b}.$$

So $A^{-1}\vec{b}$ is a solution.

Now we prove the uniqueness. Let $\vec{u}$ be any solution for $A\vec{x}=\vec{b}$.

Multiplying both sides by A^{-1} and obtain

$$A^{-1}Au = A^{-1}\vec{b}, \quad I\vec{u} = A^{-1}\vec{b} \text{ and } u = A^{-1}\vec{b}$$

Example: Use the inverse of the matrix to solve the system

$$3x_1 + 4x_2 = 3$$
$$5x_1 + 6x_2 = 7$$

Solution: $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Thm: (a) If A is invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverse of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

(c) If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof: (a) $A^{-1}A = AA^{-1} = I_n$, so $(A^{-1})^{-1} = A$.

(b) $(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_nA^{-1}) = AA^{-1} = I_n$

$$(B^{-1}A^{-1})\cdot(AB) = B^{-1}[A^{-1}(AB)] = B^{-1}(A^{-1}A)B$$
$$= B^{-1}(I_nB) = B^{-1}B = I_n$$

(c) $AA^{-1} = I_n \Rightarrow (A^{-1}A)^T = I_n \Rightarrow A^T (A^{-1})^T = I_n$

Thus $(A^T)^{-1} = (A^{-1})^T$.

Elementary matrices

Example: Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Compute E_1A, E_2A and E_3A, and describe how these products can be obtained by elementary row operations on A.

Solution: $$E_1A = \begin{pmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{pmatrix}$$

$$E_2A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$E_3A = \begin{pmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{pmatrix}$$

* If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m
* Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example: Find the inverse of $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$

Solution: $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$.

Thm: An $n \times n$ matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transform I_n into A^{-1}.

Proof: A is invertible iff $A\vec{x} = \vec{b}$ has the unique solution for each $\vec{b}$ in $\mathbb{R}^n$ $\Leftrightarrow$ every column is a pivot column (or every row contains a pivot) $\Leftrightarrow$ $A \sim I_n$ (row operations to echelon form).

Conversely, if $A \sim I_n$, each row operation corresponds to multiplying A by an elementary matrix, so $E_p \cdots E_1 A = I_n$, E_i: elementary matrix.

so $A = (E_p \cdots E_1)^{-1} I_n = E_1^{-1} \cdots E_p^{-1}$.

Thus $A^{-1} = (E_1^{-1} \cdots E_p^{-1})^{-1} = (E_p^{-1})^{-1} \cdots (E_1^{-1})^{-1}$

$= E_p \cdots E_1$.

* An algorithm for finding A^{-1}

Row operation reduce the matrix $[A \;\; I_n]$, If A is row equivalent to I_n, then $[A \;\; I_n]$ is row equivalent to $[I_n, A^{-1}]$. Otherwise, A doesn't have an inverse.

Proof: $E_p \cdots E_1 [A \;\; I_n] = (E_p \cdots E_1 A \quad E_p \cdots E_1 I_n)$

$= (I_n, A^{-1})$

Ex: $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$, find A^{-1}.

Solution: $\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix}$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$

Exercise: Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}, \quad \text{if it exists.}$$

Solution: $(A \quad I_n) = \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow[\text{③}-5\text{①}]{\text{②}+\text{①}} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{③}{3}} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{③-6②} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right)$$

So the inverse of A doesn't exist.

Ex: If A is an invertible matrix, prove that $5A$ is an invertible matrix.

Proof: Since A is invertible, there exists a matrix C such that $AC = I = CA$.

Let $D = \frac{1}{5}C$. Then

$$(5A)D = 5A \cdot \frac{1}{5}C = AC = I.$$

$$D(5A) = \frac{1}{5}C \cdot (5A) = CA = I$$

Thus $\frac{1}{5}C$ is the inverse of $5A$.

Ex: Let $A = \begin{pmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}$. Find the third column of A^{-1} without computing the other columns.

Solution: Since the algorithm $[A \quad I] \sim [I \quad A^{-1}]$ gives the inverse of A. Thus $[A \quad e_3] \sim [I \quad b_3]$ will give the third column of A^{-1}.

By performing row operations, we have

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Hence, the third column of A^{-1} is $\begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$.