Functions

1.1 Functions and their Graphs

Definition 1.1 *A function f is a rule assigning a number to each of the numbers. The number assigned to the number x via the rule f is usually denoted by $f(x)$.*

Remark 1.2 *The objective of our course "Calculus" is to study functions.*

Example 1.3 *Let f be the rule of assignment so that for each number x, its square x^2 is assigned. The handy notation to define this function is to write $f(x) = x^2$ for all x.*

Example 1.4 *Similarly,*

1. $g(x) = |x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \end{cases}$.
2. $h(x) = 1 + x^4$ *for all* x.

are example of functions. However,

1. $p(x) = \frac{1}{x}$
2. $q(x) = \sqrt{x}$

are somewhat different because they do not assign a number to EVERY choice of a number.

Definition 1.5 *The domain of a function f is the collection of numbers x such that a number is assigned to x via the rule f (we also say that $f(x)$ is defined).*

Example 1.6 *The domain of the function $p(x) = \frac{1}{x}$ is the collection of all numbers except 0.*

Example 1.7 *The domain of the function $q(x) = \sqrt{x}$ is the collection of all non-negative numbers.*

Remark 1.8 *With the concept of domain, an amended definition of a function is that it is a rule which assigns a number to each choice of a number in its domain.*

Definition 1.9 *We have following standard collection of numbers:*
$[a, b]$ *is the collection of those* x *such that* $a \leq x \leq b$.
(a, b) *is the collection of those* x *such that* $a < x < b$.
$[a, b)$ *is the collection of those* x *such that* $a \leq x < b$.
$(a, b]$ *is the collection of those* x *such that* $a < x \leq b$.
$[a, +\infty)$ *is the collection of those* x *such that* $a \leq x$.
$(-\infty, b)$ *is the collection of those* x *such that* $x < b$, *etc.*

Example 1.10 *If* $p(x) = 1/x$ *for all* $x \neq 0$, *the domain of* p *is* $(-\infty, 0) \cup (0, +\infty)$. *Here the symbol* $\cup$ *means to be the union of collections.*

Example 1.11 *If* $q(x) = \sqrt{x}$ *for all* $x \geq 0$, *the domain of* q *is* $[0, +\infty)$.

Definition 1.12 *Two coordinate-axes in the plane are two straight lines perpendicular to each other. One of them is usually called the "x-axis", another line is usually called the "y-axis". For every point* P *in the plane, the first coordinate (or x-coordinate) of* P *is the perpendicular distance from* P *to the y-axis. The second coordinate (or y-coordinate) of* P *is the perpendicular distance from* P *to the x-axis. If* a, b *are the first and second coordinate of* P *respectively,* (a, b) *are called the coordinates of* P.

Definition 1.13 *Let* f *be a function. The graph of* f *is the collection of points in the plane collecting all the points* (x, y) *such that* $y = f(x)$.

Remark 1.14 *The graph of a function* f *carries important information about* f. *One of our main objective is to study a function via its graph.*

Example 1.15 *Show that the circle with center* O *(the origin) and radius* 1 *is not the graph of any function.*

proof:
Suppose the contrary, the given circle is the graph of a function f. Since $(0, 1)$ belongs to the given circle (the distance from $(0, 1)$ to the origin is 1), $f(0) = 1$. On the other hand, $(0, -1)$ belongs to the given circle also (why?), $f(0) = -1$. Combining the two equations yields $1 = -1$ which is nonsense. Therefore, the circle cannot be the graph of any function.

Remark 1.16 *A plane curve is the graph of a certain function if and only if it cuts every vertical line at at most one point.*

1.2 Combining Functions, Shifting and Scaling Graphs

Definition 1.17 *Let* f, g *be functions. Define the following new functions:*

- $(f + g)(x) = f(x) + g(x)$ *for all* x.
- $(f - g)(x) = f(x) - g(x)$ *for all* x.
- $(fg)(x) = f(x)g(x)$ *for all* x.

- $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ *whenever* $g(x) \neq 0$.
- $f \circ g(x) = f(g(x))$.

Example 1.18 *Let* $f(x) = 1 + x$, $g(x) = x^2$ *for all* x. *Evaluate* $f \circ g$ *and* $g \circ f$.

solution:
For every number x,

$$f \circ g(x) = f(g(x)) = 1 + g(x) = 1 + x^2,$$

$$g \circ f(x) = g(f(x)) = (f(x))^2 = (1 + x)^2 = 1 + 2x + x^2.$$

Remark 1.19 *In general,* $f \circ g \neq g \circ f$.

Example 1.20 *Let* $f(x) = 1/(x+1)$ *and* $g(x) = 1/x$ *whenever they are defined. Find the domain of* $f \circ g$.

solution:
Since g is not defined at 0 only, and f is not defined at -1 only while $g(-1) = -1$, therefore $f \circ g(x) = f(g(x))$ is not defined at 0 and -1. The domain of $f \circ g$ is thus $(-\infty, -1) \cup (-1, 0) \cup (0, +\infty)$.

Proposition 1.21 *Let* f *be a function and* a *be a number.*

1. *If* $g(x) = f(x + a)$ *for all* x, *the graph of* g *is obtained from that of* f *by shifting to the left by* a *units.*
2. *If* $g(x) = f(x) + a$ *for all* x, *the graph of* g *is obtained from that of* f *by shifting up by* a *units.*
3. *If* $g(x) = -f(x)$ *for all* x, *the graph of* g *is obtained from that of* f *by flipping over the* x*-axis.*
4. *If* $a > 0$ *and* $g(x) = af(x)$ *for all* x, *the graph of* g *is obtained from that of* f *by scaling by a factor of* a *vertically.*
5. *If* $g(x) = f(-x)$ *for all* x, *the graph of* g *is obtained from that of* f *by flipping over the* y*-axis.*
6. *If* $a > 0$ *and* $g(x) = f(\frac{x}{a})$ *for all* x, *the graph of* g *is obtained from that of* f *by scaling by a factor of* a *horizontally.*

Example 1.22 *A polynomial is a function of the following type*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

for all x. *Where* n *is a non-negative integer and* a_0,..., a_n *are certain numbers. If* $a_n \neq 0$, n *is called the degree of* p. a_n *is called the leading coefficient of* p *and* a_0 *is called the constant term of* p.

Example 1.23 *A rational function is a quotient of two polynomials.*

1.3 Exponential and Logarithm Functions

Definition 1.24 *Let $a > 0$. The function $f(x) = a^x$ for all x is called the exponential function with base a.*

Proposition 1.25 *If $a, b > 0$ and x is a number,*

1. $a^x a^y = a^{x+y}$.
2. $\frac{a^x}{a^y} = a^{x-y}$.
3. $(a^x)^y = a^{xy}$.
4. $a^x b^x = (ab)^x$.
5. $a^{-x} = \frac{1}{a^x}$.

Proposition 1.26 $a^0 = 1$ *for all* $a > 0$.

Remark 1.27 0^0 *is not defined.*

Example 1.28 (carbon dating) *A tree stops absorbing ^{14}C at its death. After that, the ^{14}C in its body decay (it is radioactive decay in this case) such that its concentration is halved after every 5730 years. Suppose that the concentration of ^{14}C in the tree is one unit initially. Evaluate the concentration of ^{14}C after* 10000 *years.*

solution:
Initially, the concentration of ^{14}C in the tree is 1 unit.
After 5730 years, the concentration in the tree would be $\frac{1}{2}$ unit.
After 2×5730 years, the concentration of ^{14}C in the tree would be $\frac{1}{2} \times \frac{1}{2} = (\frac{1}{2})^2$ units.
After 3×5730 years, the concentration of ^{14}C in the tree would be $(\frac{1}{2})^2 \times \frac{1}{2} = (\frac{1}{2})^3$ units.
In general, the concentration of ^{14}C in the tree would be $(\frac{1}{2})^{t/5730}$ unit after t years. Therefore, the concentration of ^{14}C in the tree after 10000 years is

$$(\frac{1}{2})^{10000/5730} \text{ unit.}$$

Remark 1.29 *In the previous example, we saw that the concentration of ^{14}C in a dead object follows an exponential function. In the example, $\frac{1}{2}$ was the base of the exponential function.* 5730 *years is usually called the half-life of ^{14}C.*

Remark 1.30 *The realistic use of carbon dating is usually the other way round. We measure the concentration of ^{14}C and deduce how old is that piece of object.*

Definition 1.31 *A function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.*

Remark 1.32 *A function f is one-to-one if the graph of f intersects any choice of a horizontal line at at most one point.*

Definition 1.33 *Let f be a one-to-one function. Its inverse is a function g defined as follows: If $f(a) = b$, then $g(b) = a$.*

Remark 1.34 *The graph of the inverse of a function f is obtained by reflecting the graph of f across the line through origin and with slope 1.*

Example 1.35 *Let*

$$f(x) = \frac{x-1}{x+1}.$$

Compute the inverse function of f.

solution:
Suppose that a, b are numbers such that $f(a) = b$. That is,

$$b = \frac{a-1}{a+1}.$$

Rearrange the equation, we have

$$\begin{array}{rcl} b(a+1) &=& a-1 \\ ba+b &=& a-1 \\ a-ab &=& 1+b \\ a(1-b) &=& 1+b \\ a &=& \frac{1+b}{1-b} \end{array}$$

Thus,

$$g(b) = a = \frac{1+b}{1-b}$$

for every number b.

Definition 1.36 *Let f be a function. The range of f is the collection of numbers of the form $f(x)$ for some x.*

Remark 1.37 *Let f be a one-to-one function with inverse g. Then the domain of g is the range of f*

Definition 1.38 *The inverse function of the exponential function with base $a > 0$ is called the logarithm function with base a. The logarithm function with base e is also called the natural log. The symbol for the logarithm function with base a is $\log_a$. That is,*

$$y = \log_a x \text{ means that } x = a^y.$$

Example 1.39 *Evaluate* $\log_2 4$.

solution:
Since $2^2 = 4$, $\log_2 4 = 2$.

Example 1.40 *Evaluate* $\log_3 1$.

solution:
Since $3^0 = 1$, $\log_3 1 = 0$.

Example 1.41 *Evaluate* $2^{\log_2 3}$.

solution:
$\log_2 3$ stands for a number so that 2 to such a power is 3. So, $2^{\log_2 3} = 3$.

Remark 1.42 $\log_a x$ *is NOT defined when* $x \leq 0$. *In other words, the domain of* $\log_a$ *is* $(0, +\infty)$.

Proposition 1.43 *For any numbers* x, y, a, $b > 0$ *and for any number* k,

1. $\log_a xy = \log_a x + \log_a y$.
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$.
3. $\log_a x^k = k \log_a x$.
4. $\log_a x = \frac{\log_b x}{\log_b a}$

proof:
Let $\log_a x = u$, $\log_a y = v$,
proof of 1): Since

$$a^{u+v} = a^u a^v = xy,$$

we have

$$\log_a xy = u + v = \log_a x + \log_a y.$$

proof of 2):

$$\log_a x = \log_a(\frac{x}{y} y) = \log_a \frac{x}{y} + \log_a y.$$

proof of 3): Since

$$a^{ku} = (a^u)^k = x^k,$$

we have

$$\log_a x^k = ku = k \log_a x.$$

proof of 4): Since

$$u \log_b a = \log_b a^u = \log_b x,$$

we have

$$\log_a x = u = \frac{\log_b x}{\log_b a}.$$

Remark 1.44 *Every positive number can be the base of an exponential (or log) function. However, one particular choice turns out to be most convenient. This is the exponential function with base*

$$e \approx 2.718281828459045.$$

The logarithm function with base e is usually written as log *or* ln *(both of them are used in webwork). We will come back to the reason of such a choice later.*

Example 1.45 *The concentration of* ^{14}C *in a piece of wood is found to be* 0.7 *unit (the concentration of* ^{14}C *in the nature is regarded as 1 unit.). How long ago the piece of wood was cut from a tree?*

solution:
Let this piece of wood be cut from a tree t years ago. Then,

$$(\frac{1}{2})^{t/5730} = 0.7.$$

Thus,

$$t/5730 = \log_{\frac{1}{2}} 0.7 = \frac{\ln 0.7}{\ln 0.5} \approx 0.5146.$$

This piece of wood was cut $5730 \times 0.5146 \approx 2949$ years ago.

1.4 Trigonometric Functions

Definition 1.46 *The functions* cos *and* sin *are defined as follows. For each number* x*,* $\cos x$*,* $\sin x$ *are respectively the first and second coordinate of the point on the unit circle whose inclination is* x *radians.*

Definition 1.47 *For each number* x*,*

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Proposition 1.48

$$\begin{array}{ccccc}
\sin 0 = 0 & \sin \frac{\pi}{6} = \frac{1}{2} & \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} & \sin \frac{\pi}{2} = 1 \\
\cos 0 = 1 & \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} & \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos \frac{\pi}{3} = \frac{1}{2} & \cos \frac{\pi}{2} = 0 \\
\tan 0 = 0 & \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} & \tan \frac{\pi}{4} = 1 & \tan \frac{\pi}{3} = \sqrt{3} &
\end{array}$$

Proposition 1.49 *For every number* x*,*

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x.$$

Definition 1.50 *A function* f *is even if* $f(-x) = f(x)$ *for all* x*. We say that* f *is odd if* $f(-x) = -f(x)$ *for all* x*.*

Corollary 1.51 sin *is an odd function while* cos *is an even function.*

Definition 1.52 *A function f is periodic if there is a number T such that $f(x+T) = f(x)$ for all x. The smallest such number T is called the period of f.*

Proposition 1.53 $\sin$ *and* $\cos$ *are periodic functions with period 2π.*

Proposition 1.54 *For every number x, $\sin x = \cos(\frac{\pi}{2} - x)$, $\cos x = \sin(\frac{\pi}{2} - x)$.*

Theorem 1.55 $\sin^2 x + \cos^2 x = 1$ *for all x.*

proof:
For every number x, $(\sin x, \cos x)$ is a point on the unit circle with center $(0,0)$. Hence, the distance from $(\sin x, \cos x)$ to $(0,0)$ is 1. i.e. $\sin^2 x + \cos^2 x = 1$.

Corollary 1.56 *For every number x,*

$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x.$$

Theorem 1.57 *For every number x, y*

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Corollary 1.58 *For every number x,*

- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

Example 1.59 *Evaluate* $\cos \dfrac{\pi}{12}$.

solution:
From the double angle formula for cosine,

$$\frac{\sqrt{3}}{2} = \cos \frac{\pi}{6} = 2\cos^2 \frac{\pi}{12} - 1.$$

Solving yields $\cos \dfrac{\pi}{12} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

Proposition 1.60 *The function* sin *is one-to-one when its domain is restricted to* $[-\frac{\pi}{2}, \frac{\pi}{2}]$. *The inverse of such function is denoted by* $\sin^{-1}$ *or* arcsin.

Proposition 1.61 *The function* cos *is one-to-one when its domain is restricted to* $[0, \pi]$. *The inverse of such function is denoted by* $\cos^{-1}$ *or* arccos.

Proposition 1.62 *The function* tan *is one-to-one when its domain is restricted to* $(-\frac{\pi}{2}, \frac{\pi}{2})$. *The inverse of such function is denoted by* $\tan^{-1}$ *or* arctan.

Proposition 1.63 *The domain of* $\sin^{-1}$ *and* $\cos^{-1}$ *are both* $[-1, 1]$.

proof: The range of both sin and cos is $[-1, 1]$.

Example 1.64 *Evaluate* $\cos^{-1} \frac{-1}{2}$.

solution:
Since $\cos \frac{2\pi}{3} = \frac{-1}{2}$ and $0 \leq \frac{2\pi}{3} \leq \pi$, we conclude that $\cos^{-1} \frac{-1}{2} = \frac{2\pi}{3}$.