§5.3 Diagonalization

Example: If $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$, then

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 \\ 0 & 3^3 \end{pmatrix}$$

In general,

$$D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix} \quad \text{for } k \geq 1$$

If $A = PDP^{-1}$, P invertible, D diagonal, then
$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$
In general, $A^k = PD^kP^{-1}$

Example: Let $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$. Find a formula for A^k,

given that $A = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution: It is easy to see $P^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

In general,

$$A^k = PD^kP^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{pmatrix}$$

Def: A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix. That is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Thm: (The Diagonalization Theorem)

An $n\times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.

$A = PDP^{-1}$ with D a diagonal matrix iff the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Proof: $P = (\vec{v}_1, \cdots, \vec{v}_n)$, $\vec{v}_i$ column vectors.

Then $AP = A(\vec{v}_1, \cdots, \vec{v}_n) = (A\vec{v}_1, \cdots, A\vec{v}_n)$

If A is diagonalizable, then

$$AP = PD = (\vec{v}_1, \cdots, \vec{v}_n)\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$= (\lambda_1\vec{v}_1, \cdots, \lambda_n\vec{v}_n)$$

Thus, we get

$$A\vec{v}_1 = \lambda_1\vec{v}_1, \ A\vec{v}_2 = \lambda_2\vec{v}_2, \ \cdots, \ A\vec{v}_n = \lambda_n\vec{v}_n$$

which implies $\vec{v}_1, \cdots, \vec{v}_n$ are eigenvectors.

Since P is invertible, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent.

Conversely, if $A\vec{v}_1 = \lambda_1\vec{v}_1, \cdots, A\vec{v}_n = \lambda_n\vec{v}_n$ with $\{\vec{v}_1, \cdots, \vec{v}_n\}$ linearly independent, then

$AP = PD$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent, then P is invertible, thus $A = PDP^{-1}$.

* Diagonalizing Matrices

Example: Diagonalize the following matrix, if possible.

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Step 1: Find eigenvalues

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & -2-\lambda & -2-\lambda \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(1-\lambda)\begin{vmatrix} 1 & 1 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= (\lambda-1)(\lambda+2)[1-\lambda-3]$$

$$= -(\lambda-1)(\lambda+2)^2$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -2$.

Step 2: Find linearly independent eigenvectors of A

For eigenvalue $\lambda_1 = 1$, $(A-\lambda_1 I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{pmatrix}\vec{x} = 0$$

$$\begin{pmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{pmatrix} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

Solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

We choose $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as an eigenvector

corresponding to 1.

For eigenvalue $\lambda_2 = -2$, solve $(A - \lambda_2 I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 1+2 & 3 & 3 \\ -3 & -5+2 & -3 \\ 3 & 3 & 1+2 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

Solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

We take $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as eigenvector

corresponding to -2.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent.

Step 3. Construct P

$$P = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det P = 1\cdot(-1)^{3+1}\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} + 1\cdot(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= 1 + 1 - 1 = 1$$

Step 4. Construct D

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Example: Diagonalize the following matrix, if possible.

$$A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

Solution: $$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & 3 & 1-\lambda \\ -4 & -6-\lambda & -3 \\ 2-\lambda & 4 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 1 & \frac{1}{3} - \frac{\lambda}{3} \\ 4 & 6+\lambda & 3 \\ 2-\lambda & 4 & 3 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & \frac{1}{3} - \frac{\lambda}{3} \\ 0 & \lambda+2 & \frac{5}{3} + \frac{4\lambda}{3} \\ 0 & \lambda+2 & 3 - \frac{1}{3}(1-\lambda)(2-\lambda) \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} \lambda+2 & \frac{1}{3}(5+4\lambda) \\ \lambda+2 & 3 - \frac{1}{3}(1-\lambda)(2-\lambda) \end{vmatrix}$$

$$= 3 \cdot (\lambda+2) \begin{vmatrix} 1 & \frac{1}{3}(5+4\lambda) \\ 1 & \frac{7}{3} + \lambda - \frac{\lambda^2}{3} \end{vmatrix}$$

$$= 3(\lambda+2)\left[\frac{7}{3} + \lambda - \frac{\lambda^2}{3} - \frac{1}{3}(5+4\lambda)\right]$$

$$= 3(\lambda+2)\left[\quad + \lambda - \frac{4}{3}\lambda - \frac{\lambda^2}{3}\right]$$

$$= 3(\lambda+2)\left[\frac{2}{3} - \frac{1}{3}\lambda - \frac{\lambda^2}{3}\right]$$

$$= (\lambda+2)(2 - \lambda - \lambda^2)$$

$$= -(\lambda+2)(\lambda-1)(\lambda+2)$$

$$= -(\lambda-1)(\lambda+2)^2$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$.

For $\lambda_1 = 1$, Solve $(A - \lambda_1 I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 2-1 & 4 & 3 \\ -4 & -6-1 & -3 \\ 3 & 3 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & -9 & -9 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

choose eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For eigenvalue $\lambda_2 = -2$, solve $(A - \lambda_2 I)\vec{x} = \vec{0}$.

$$\begin{pmatrix} 2+2 & 4 & 3 & 0 \\ -4 & -6+2 & -3 & 0 \\ 3 & 3 & 1+2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 = 0 \\ \quad -x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 \\ x_3 = 0 \end{cases}$$

Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Choose eigenvector $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

There are no other eigenvalues, and every eigenvector of A is a multiple of either $\vec{v}_1$ or $\vec{v}_2$. Hence it is impossible to construct a basis of $\mathbb{R}^3$ using eigenvectors of A.
So A is not diagonalizable.

Thm: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof: Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be eigenvectors corresponding to n distinct eigenvalues of a matrix A. Then $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent. Hence A is diagonalizable.

Example: Determine if the following matrix is diagonalizable

$$A = \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$$

Solution: Since the matrix is triangular, its eigenvalues are $5, 0, -2$.
Since A is 3×3 matrix with three distinct eigenvalues, A is diagonalizable.

* Matrices Whose Eigenvalues Are Not Distinct

Thm: Let A be an $n\times n$ matrix whose distinct eigenvalues are $\lambda_1, \cdots, \lambda_p$.

a) For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b) The matrix A is diagonalizable iff the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c) If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \cdots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example: Diagonalize the following matrix, if possible.

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{pmatrix}$$

Solution: Since A is a triangular matrix, the eigenvalues are 5 and -3, each with multiplicity 2.

For $\lambda_1 = 5$, solve $(A - \lambda_1 I)\vec{x} = \vec{0}$, we can get the basis $\vec{v}_1 = \begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -16 \\ 4 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = -3$, solve $(A - \lambda_2 I)\vec{x} = 0$, we can get the basis $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent. So the matrix $P = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4]$ is invertible, and $A = PDP^{-1}$, where

$$P = \begin{pmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Example: Show that if A is both diagonalizable and invertible, then so is A^{-1}.

Proof: If A is diagonalizable, there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Since A is invertible,

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = PD^{-1} P^{-1}$$

which implies A^{-1} is diagonalizable.

Exercise: Show that if A has n linearly independent eigenvectors, then so does A^T.

Proof: If A has n linearly independent eigenvectors, there exists invertible matrix P and diagonal matrix D such that

$$A = PDP^{-1}.$$

Then $A^T = (PDP^{-1})^T$

$$= (P^T)^{-1} D P^T$$

So A^T is diagonalizable which implies A^T has n linearly independent eigenvectors.

Example: $A = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}$. Find the range of a where A has no real eigenvalues.

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & a \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - a$$

$$= \lambda^2 - 2\lambda + 1 - a$$

$\lambda^2 - 2\lambda + (1-a) = 0$ has no real roots iff

$(-2)^2 - 4 \cdot (1-a) < 0 \iff a < 0.$

§5.6. Discrete Dynamical Systems

Example: Denote the owl /aul/ and wood rat populations at time k by $\vec{x}_k = \begin{pmatrix} O_k \\ R_k \end{pmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$O_{k+1} = 0.5\,O_k + 0.4\,R_k$$
$$R_{k+1} = -0.104\,O_k + 1.1\,R_k.$$

Determine the evolution of this system.

Solution: Let $A = \begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}$.

Then $\vec{x}_{k+1} = A\vec{x}_k$.

It is easy to find the eigenvalues for A is $\lambda_1 = 1.02$, $\lambda_2 = 0.58$. The corresponding eigenvector are

$$\vec{v}_1 = \begin{pmatrix} 10 \\ 13 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Obviously, $\vec{v}_1, \vec{v}_2$ form a basis in $\mathbb{R}^2$.

For any initial $\vec{x}_0$, there exists c_1 and c_2 such that

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

Hence
$$\begin{aligned}\vec{x}_k &= A^k\vec{x}_0 \\ &= A^k(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1A^k\vec{v}_1 + c_2A^k\vec{v}_2\end{aligned}$$

$$= c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

$$= c_1 (1.02)^k \begin{pmatrix} 10 \\ 13 \end{pmatrix} + c_2 (0.58)^k \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

As $k \to \infty$, $\vec{x}_k \to c_1 (1.02)^k \begin{pmatrix} 10 \\ 13 \end{pmatrix}$.

Hence for large k,

$$\vec{x}_{k+1} \approx c_1 (1.02)^{k+1} \begin{pmatrix} 10 \\ 13 \end{pmatrix}$$

$$= 1.02\, c_1 (1.02)^k \begin{pmatrix} 10 \\ 13 \end{pmatrix}$$

$$\approx 1.02\, \vec{x}_k$$

This approximation says that eventually both entries of $\vec{x}_k$ (the numbers of owls and rats) grow by a factor of almost 1.02 each month, a 2% monthly growth rate.