Gravitation II

PHYS1112

Lecture 13

Intended Learning Outcomes

- After this lecture you will learn:
  1) Kepler's laws of planetary motion
  2) gravitation effect of a spherical mass distribution is the same as a point mass
  3) the apparent weight due to rotation of the earth
  4) the idea of a black hole starting from the concept of escape speed

Kepler's Laws of Planetary Motion

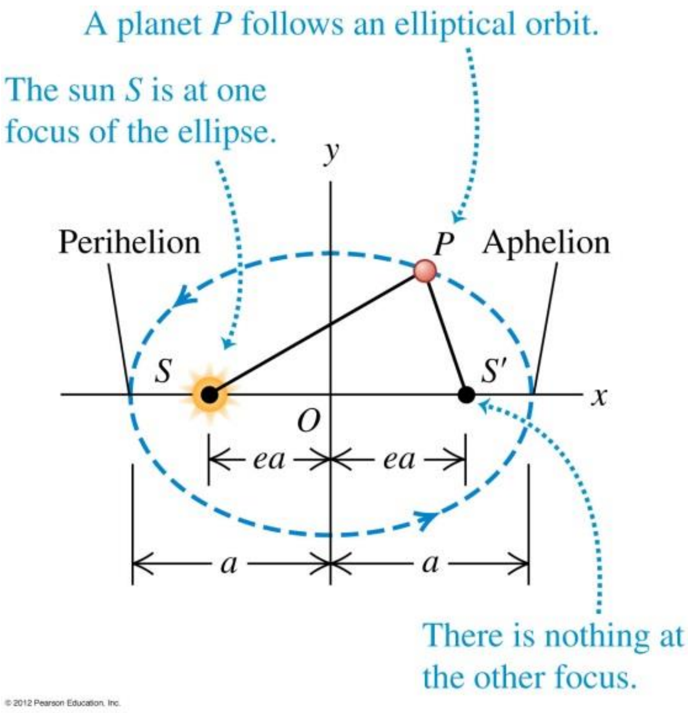
Purely phenomenological $\longleftrightarrow$ first principles

– Kepler didn't know why

Later derived by Newton using his laws of motion and gravitation

– significance: heavenly objects obey the same physical laws as terrestrial objects, don't need, e.g., Greek myths!

First Law: Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.



An **ellipse** is defined by the locus of a point P such that $|PS'| + |SP| = \text{constant}$

S and S' are the two **focuses** of the ellipse

Semi-major axis a (⚠ a length, not an axis)

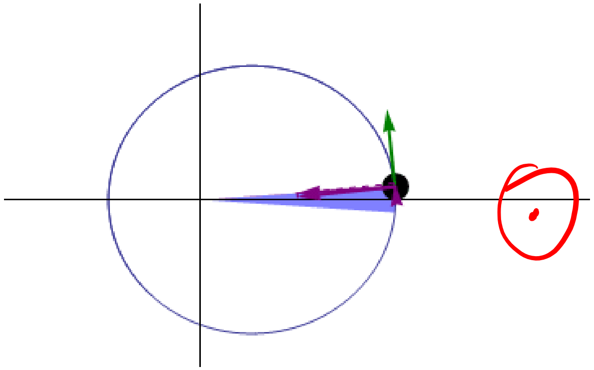
Eccentricity e ($e = 0$ for circle, $0 < e < 1$ for ellipse)

Aphelion – furthest $[(1+e)a]$ point from sun

Perihelion – closest $[(1-e)a]$ point to sun

Note: aphelion distance + perihelion distance = $2a$

Second Law: A line from the sun to a given planet sweeps out equal areas in equal times. See http://en.wikipedia.org/wiki/File:Kepler-second-law.gif



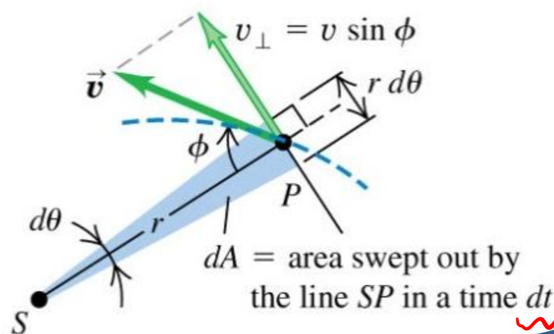
$$dA \approx \text{area of blue triangle} = \tfrac{1}{2}(r\,d\theta)r$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$v_\perp = v \sin\phi = r\frac{d\theta}{dt}$$

$$\therefore \quad \frac{dA}{dt} = \frac{1}{2} r v \sin\phi = \frac{1}{2m}|\vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}| = \frac{L}{2m}$$

(b)

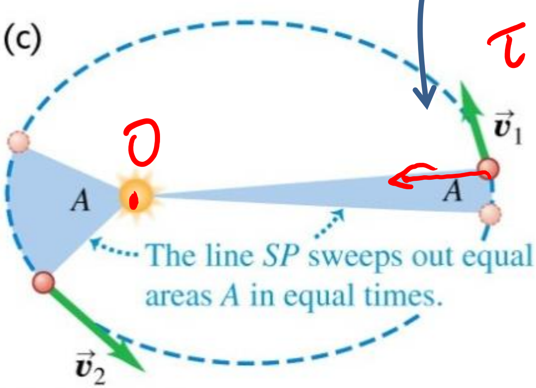


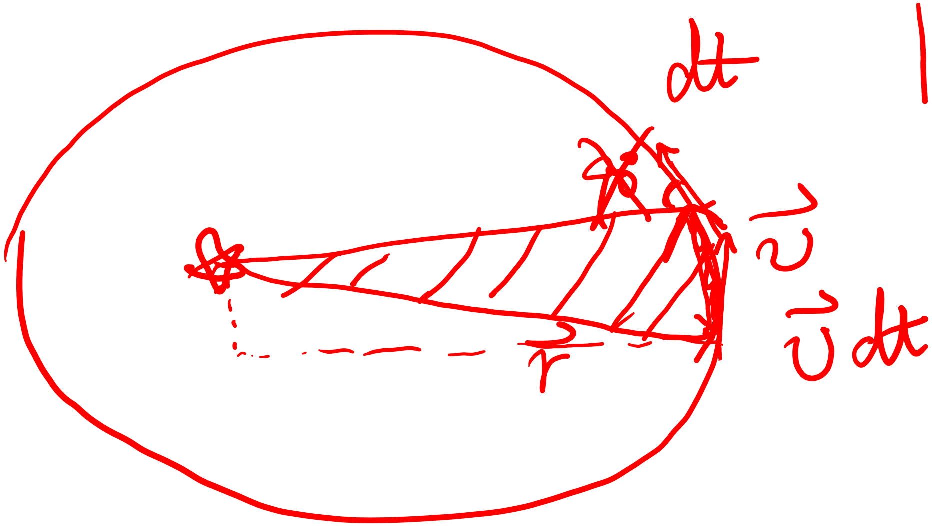
i.e., Kepler's second law ⇔ conservation of angular momentum

$\vec\tau = \vec r \times \vec F$ = 0

- ⚠ Angular momentum is conserved because gravitational force (a central force) produces no torque
- ⚠ Another consequence of conservation of angular momentum – orbit lies in a plane

(c)





$$m\vec{v} = \vec{p}$$

$$|d\vec{A}| = \frac{|\vec{r} \times m\vec{v}\,dt|}{m \cdot 2}$$

$$= \frac{|\vec{r} \times \vec{p}|}{2m}\,dt \qquad |\vec{r} \times \vec{p}| = |\vec{L}|$$

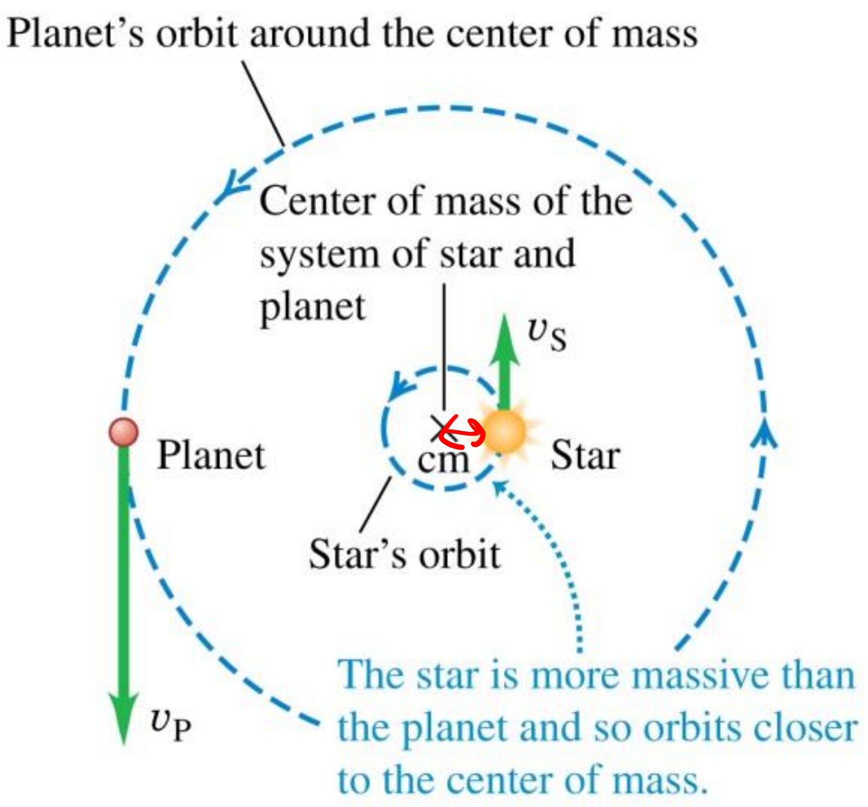
$$|d\vec{A}| = \frac{|\vec{L}|}{2m}\,dt$$

$$\frac{|d\vec{A}|}{dt} = \frac{|\vec{L}|}{2m}$$

Third Law: The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}}$$

Center of mass



Both sun and planet orbit around their center of mass

Mass of sun ~ 750 times the total mass of planets → sun effectively at rest

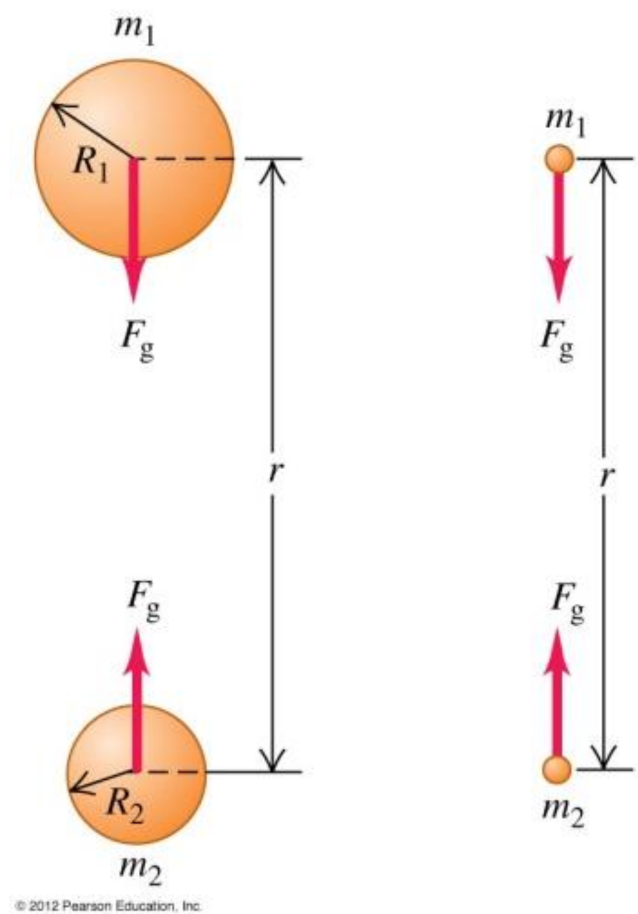
A **binary star** consists of two stars orbiting about their common CM, one called primary (the brighter one) and the other secondary. Can detect the secondary based on wobbling of the primary around their CM

Spherical Mass Distribution

Means density $\rho(r)$ depends on distance from the center only, not on the direction. Can be a shell or solid.

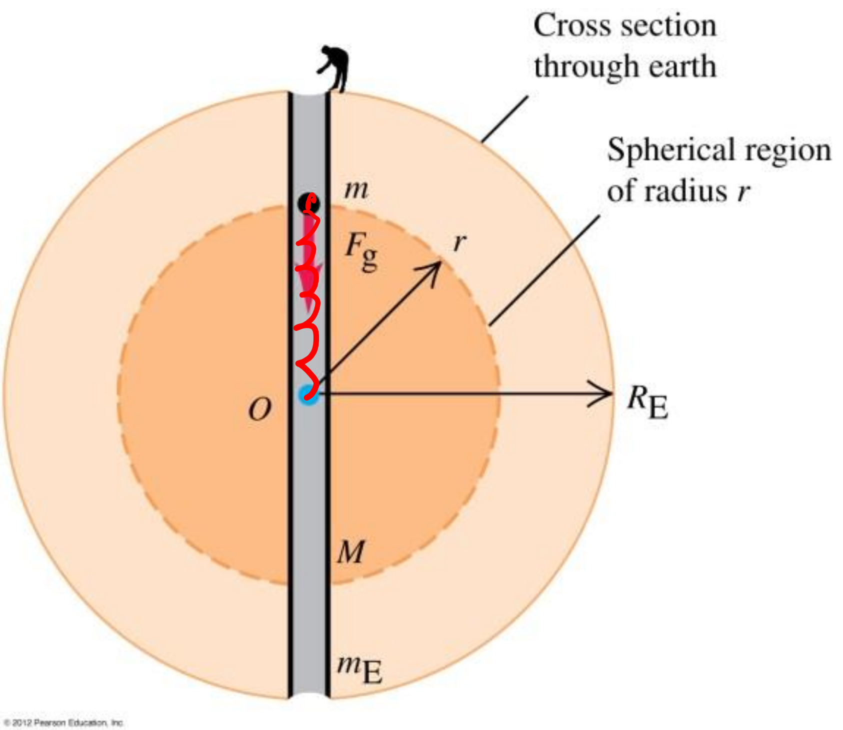
Major results: (see textbook for proofs)

1. The gravitational effect *outside* a spherical mass distribution is the same as if all the mass is concentrated at the center of the sphere.



2. The gravitational effect *inside* a spherical mass distribution is the same as if all the mass *interior* to that point is concentrated at the center of the sphere.

Example: when passing through a tunnel through the earth (assume constant density ρ), only the spherical region of radius r contributes to the gravitational force at r



$F = kr$

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left[\left(\frac{4}{3}\pi r^3\right)\rho\right] = \frac{Gm_E m}{R_E^3} r$$

$$\rho = \frac{m_E}{\frac{4}{3}\pi R_E^3}$$

$\propto r$, not $1/r^2$

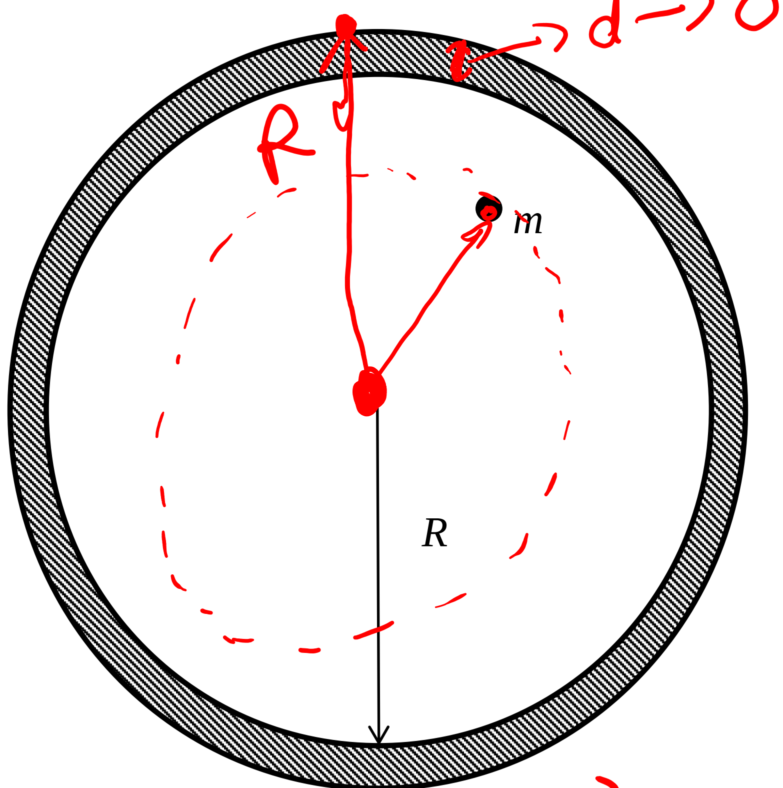
$$U = -\frac{GMm}{R}$$

3. The gravitational PE of a mass *m* *anywhere inside* a hollow sphere is a constant

$$U = -\frac{GMm}{R}$$

Gravitational force

$$\boxed{\vec{F}_g = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = \underline{0}}$$



$$W = \int \underbrace{\vec{F}}_{=0} \cdot d\vec{l} = 0 = -\Delta U$$

$$= 0$$

Question

If the earth were a hollow sphere and were not rotating, would it be possible to stand and walk on its inner surface?

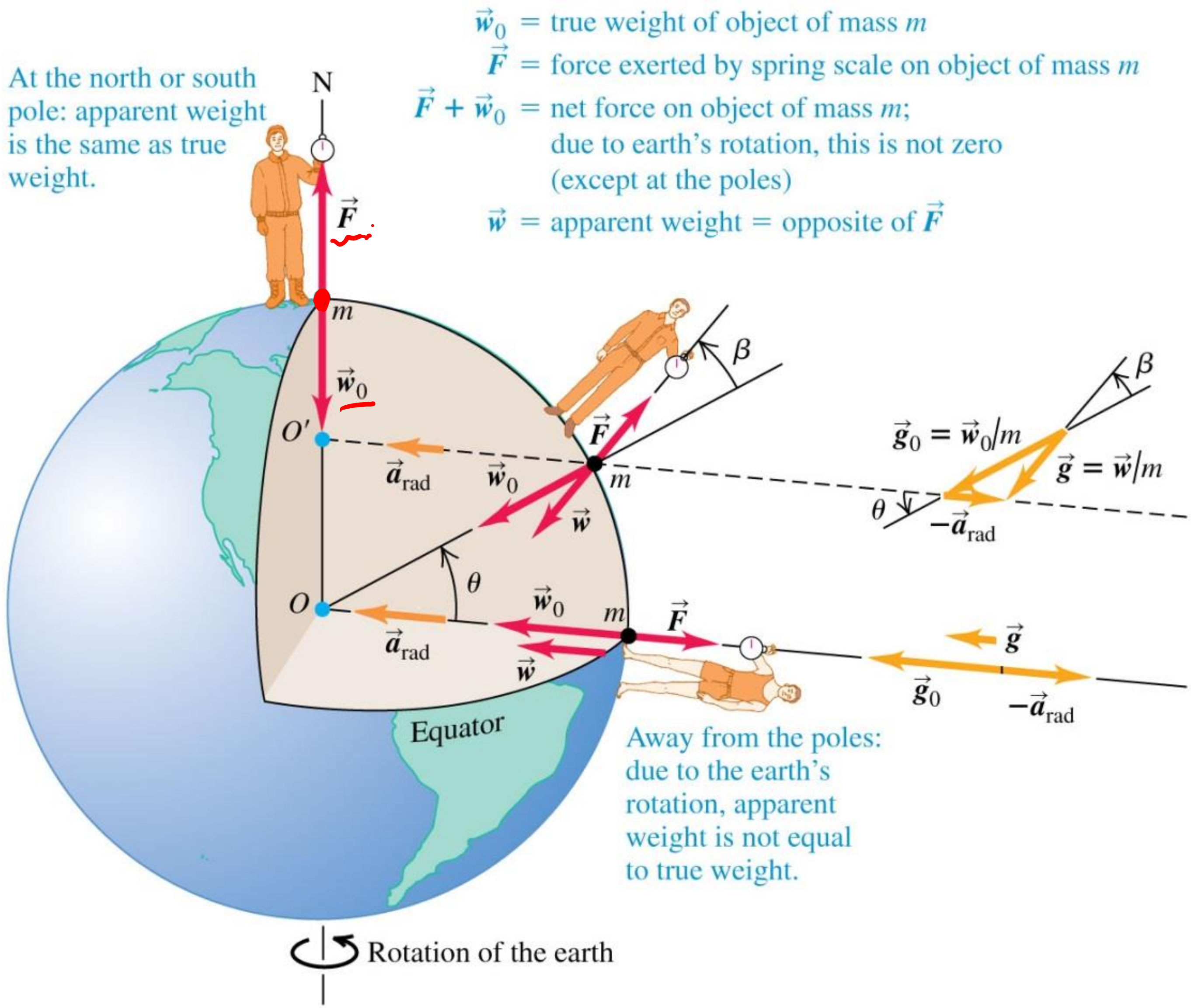
Apparent Weight and the Earth's Rotation

$\vec{w}_0$ true weight (due to earth's gravitational attraction)

If earth not rotating, body in equilibrium,

$$\vec{F} = -\vec{w}_0$$

(this is true at the north/south poles of the rotating earth)



At equator

$$w_0 - F = \frac{mv^2}{R_E}$$

$W_0 > F$

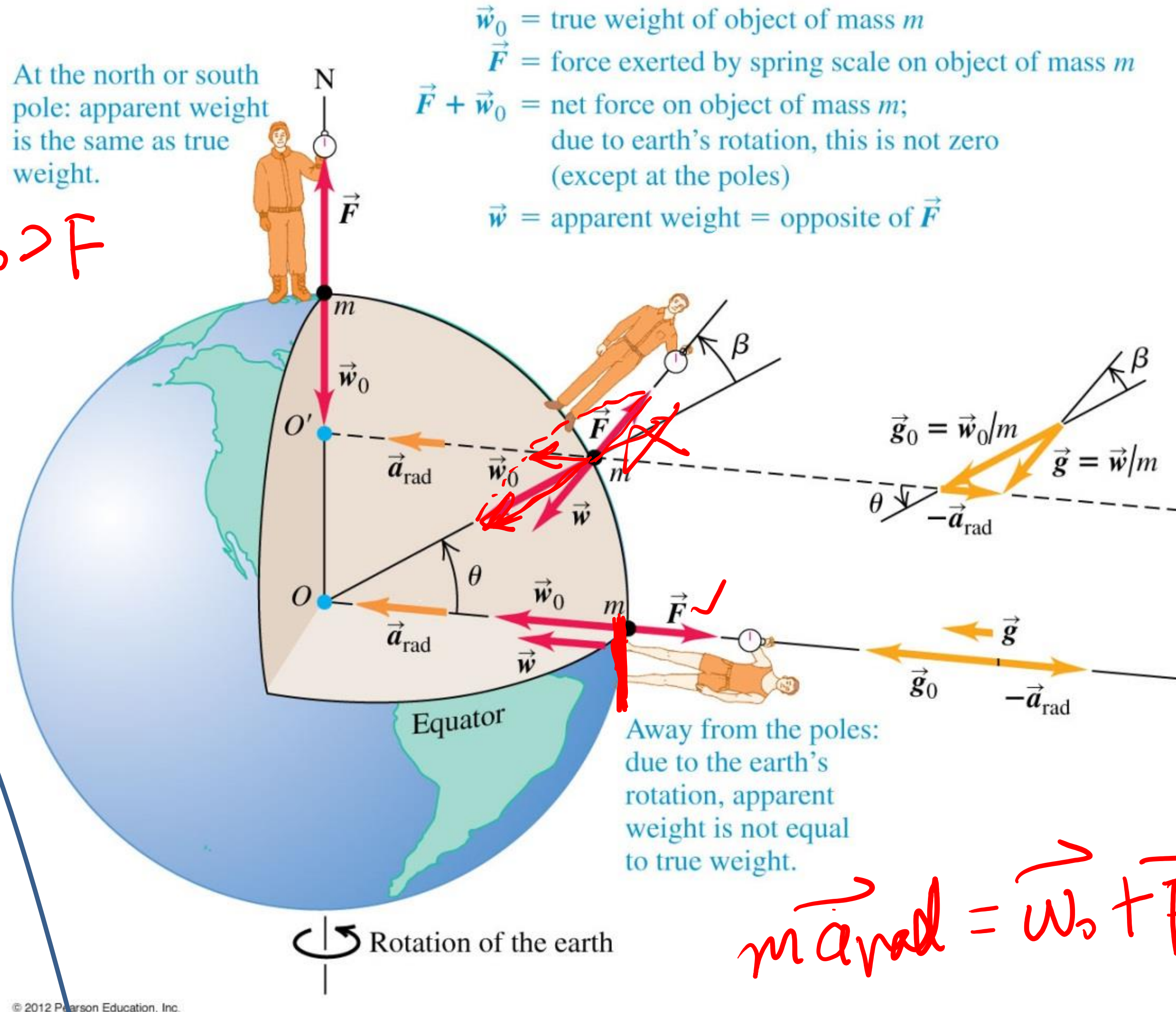
Apparent weight

$$w = F$$

$$= \frac{Gm_E}{R_E^2}m - \frac{mv^2}{R_E}$$

$$= mg$$

acceleration due to gravity of a rotating earth



$\vec{m a_{rad}} = \vec{w_0} + \vec{F}$

$$g = g_0 - \frac{v^2}{R_E}$$

$\sim 0.0339\ \mathrm{m/s^2}$

g_0, acceleration due to gravity of a non-rotating earth

Question

A planet has the same mass and radius as the earth, but rotates 10 times faster. The difference between the acceleration of gravity at its equator and poles is (0.00339 / 0.0339 / 0.339 / 3.39) m/s^2.

$$\frac{v^2}{R_E} \rightarrow 10^2$$

Black Holes

Recall *escape speed* from a star of mass *M* and radius *R*

$$v = \sqrt{\frac{2GM}{R}}$$

What if a star collapses, keeping the same *M* but *R* decreases? v increases.

When *R* small enough (reaches a critical value R_S), $v \to c$, no light can escape?

$$c = \sqrt{\frac{2GM}{R_S}} \quad \rightarrow \quad \boxed{R_S = \frac{2GM}{c^2}}$$ **Schwarzschild radius**

$\boxed{E = mc^2}$

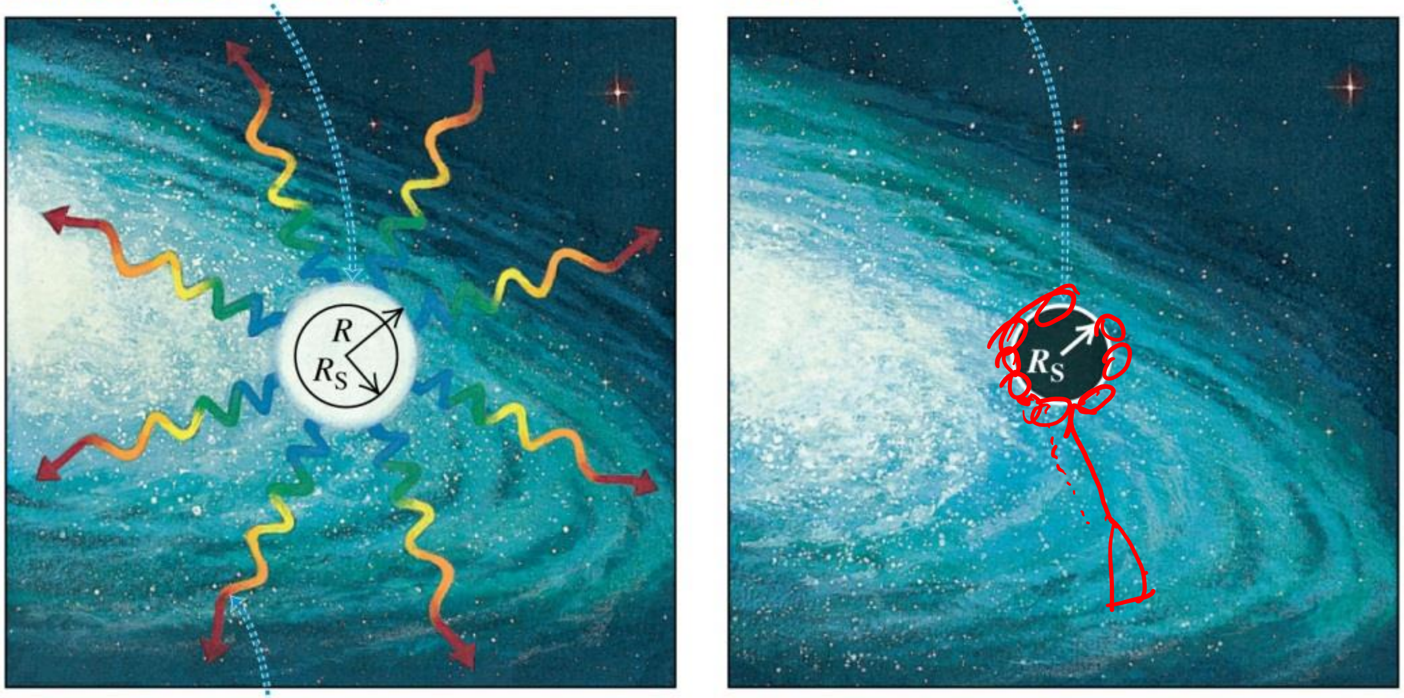
Problems: 1) KE of light (photon) is not $\frac{1}{2}mc^2$

2) gravitational PE near black hole is not $-GMm/r$

Schwarzschild (1916) derived exactly the same critical radius using *General Relativity* (a relativistic theory of gravitation)

(a) When the radius R of a body is greater than the Schwarzschild radius R_S, light can escape from the surface of the body.

(b) If all the mass of the body lies inside radius R_S, the body is a black hole: No light can escape from it.



Gravity acting on the escaping light "red shifts" it to longer wavelengths.

Surface of the sphere with radius R_S surrounding a black hole is called the **event horizon**: light inside cannot escape ∴ cannot know what happens inside a black hole

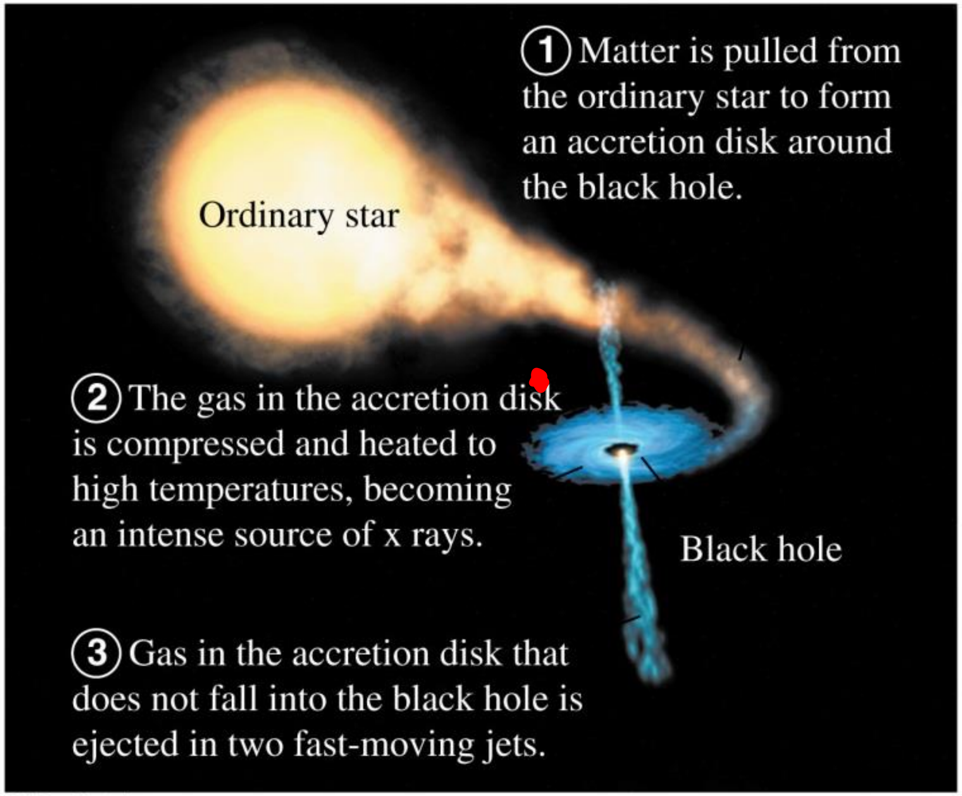
We know a black hole's

•mass – through its gravitational force on others

•electric charge – through its electric force on other charged bodies

•angular momentum – through its rotating gravitational field that drags space and everything in that space around it

Can’t see light from a black hole, how to detect it?



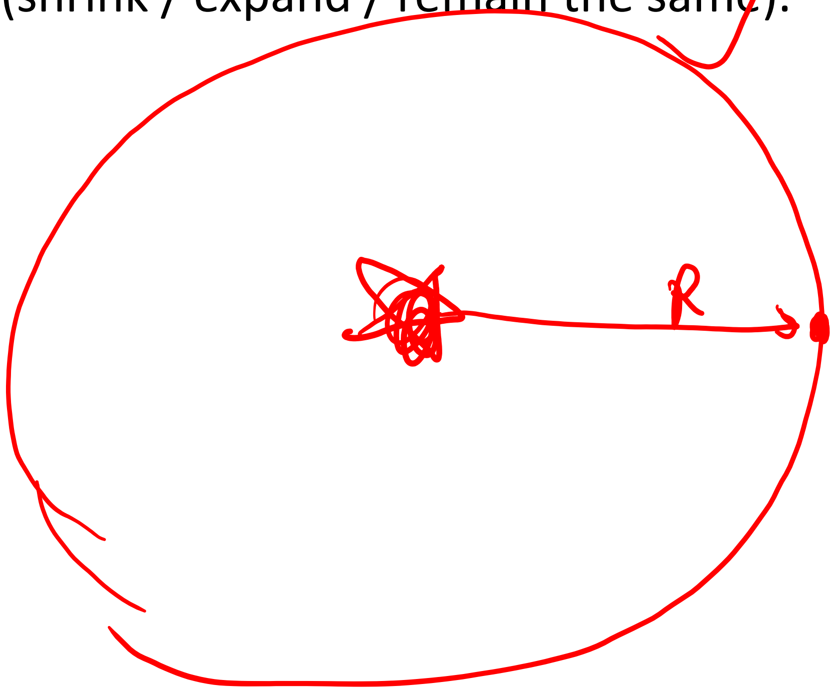
In a binary star system (ordinary star + black hole), look for x-ray source

Demonstration: gravity well

Or study orbits of surrounding stars in other cases

Question

If the sun collapses to form a black hole, the orbit of the earth would (shrink / expand / remain the same).

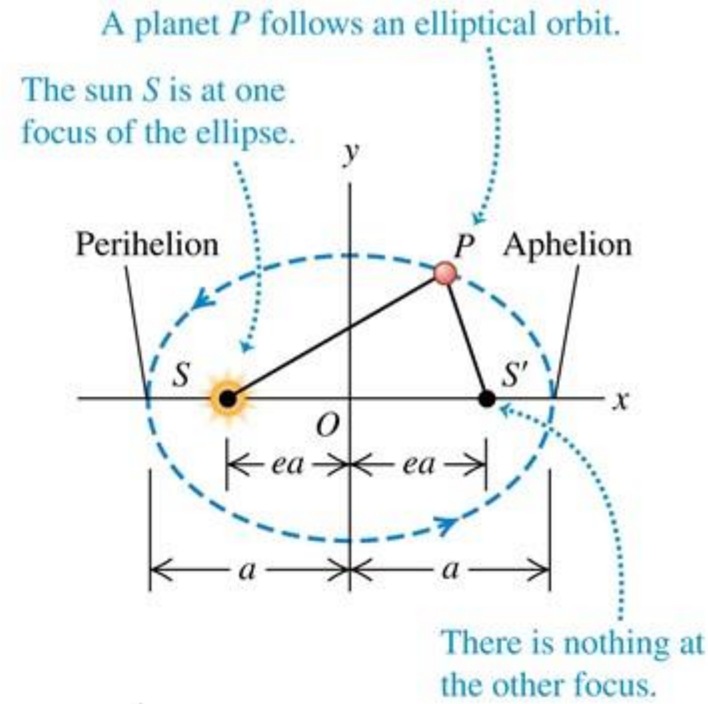


$$F = G\frac{Mm}{R^2} \quad ✓$$

$$= m\frac{v^2}{R}$$

Q13.12

A planet (P) is moving around the sun (S) in an elliptical orbit. As the planet moves from aphelion to perihelion, the planet's angular momentum



A. increases at all times.

B. decreases at all times.

C. decreases during part of the motion and increases during the other part.

D. increases, decreases, or remains the same during various parts of the motion.

E. remains the same at all points between aphelion and perihelion.

A13.12

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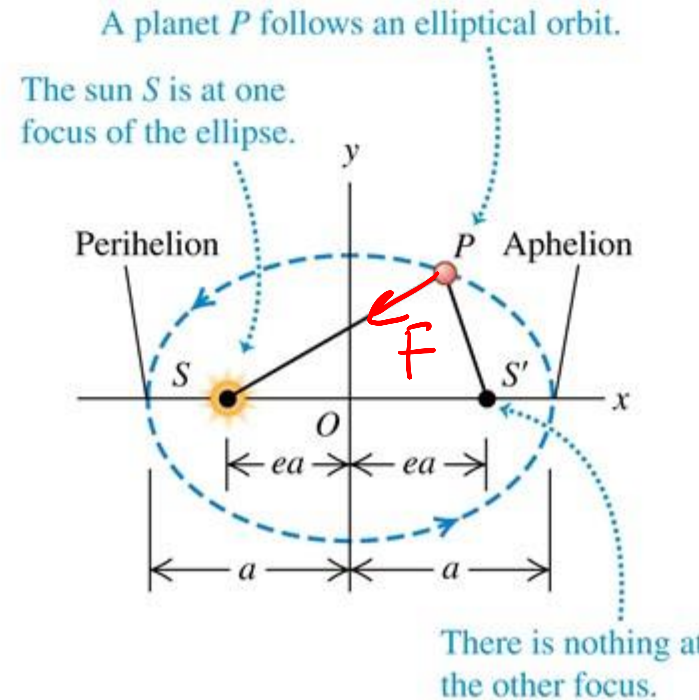

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Q13.8

Star X has twice the mass of the sun. One of star X's planets moves in a circular orbit around star X. This orbit has the same radius as the earth's orbit around the sun. The orbital *speed* of this planet of star X

A. is faster than the earth's orbital speed.

B. is the same as the earth's orbital speed.

C. is slower than the earth's orbital speed.

D. depends on the mass of the planet.

E. depends on the mass and radius of the planet.

A13.8

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$$G\frac{M\cancel{m}}{R^2} = \cancel{m}\frac{v^2}{\cancel{R}}$$

A. is faster than the earth's orbital speed.

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C. is slower than the earth's orbital speed.

D. depends on the mass of the planet.

E. depends on the mass and radius of the planet.

$$v = \sqrt{\frac{GM^{\times 2}}{R}}$$

$$\sqrt{2}$$