§5.1 Eigenvectors and Eigenvalues

Ex: Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Then

$$A\vec{v} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times 2 + (-2)\times 1 \\ 1\times 2 + 0\times 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\vec{v}$$

$$A\vec{u} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times(-1) + (-2)\times 1 \\ 1\times(-1) + 0\times 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$\vec{v}$ is an eigenvector corresponding to the eigenvalue 2 of A.

Def: An eigenvector of an $n\times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ.

λ is called an eigenvalue of A

$\vec{x}$ is called an eigenvector corresponding to λ.

Ex: Let $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$, and $\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

Solution:

$$A\vec{u} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} 1\times 6 + 6\times(-5) \\ 5\times 6 + 2\times(-5) \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix}$$

$$= -4\cdot\begin{pmatrix} 6 \\ -5 \end{pmatrix} = -4\vec{u}$$

$$A\vec{v} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \neq \lambda\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Thus $\vec{u}$ is an eigenvector corresponding to an eigenvalue (-4). But $\vec{v}$ is not an eigenvector of A, because $A\vec{v}$ is not a multiple of $\vec{v}$.

Ex: Show that 7 is an eigenvalue of matrix $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$, and find the corresponding eigenvectors.

Solution: The scalar 7 is an eigenvalue of A if and only if the equation

$$A\vec{x} = 7\vec{x}$$

has a nontrivial solution.

$$A\vec{x} = 7\vec{x} \Leftrightarrow (A - 7I)\vec{x} = 0$$

$$[A-7I, \vec{0}] = \begin{pmatrix} 1-7 & 6 & 0 \\ 5 & 2-7 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 = 0$, $x_1 = x_2$

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to 7.

Ex: Let $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$. An eigenvalue of A is 2.

Find a basis for the corresponding eigenspace.

Solution:

$$A - 2I = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2x_1 - x_2 + 6x_3 = 0$$

$$x_1 = \tfrac{1}{2}x_2 - 3x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} \tfrac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Hence the basis of the eigenspace is

$$\begin{pmatrix} \tfrac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

Thm: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & \ddots & & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$.

Then $A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & \ddots & & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$

$$= \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ & \ddots & & \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{pmatrix}$$

Hence the scalar λ is an eigenvalue.

$\Leftrightarrow$ $(A-\lambda I)\vec{x} = \vec{0}$ has a nontrivial solution.

$\Leftrightarrow$ $(A-\lambda I)\vec{x} = \vec{0}$ has a free variable

$\Leftrightarrow$ at least one of the entries on the diagonal of $A-\lambda I$ is zero. (# of pivots of $A - \lambda I \leq n$)

$\Leftrightarrow$ $\lambda = a_{11}$ or $\lambda = a_{22}$ or $\cdots$ $\lambda = a_{nn}$ #

Remark: For A is lower triangular, we have the same conclusion.

Ex: Let $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{pmatrix}$.

The eigenvalues of A are 3, 0, 2.

The eigenvalues of B are 4, 1, 4.

Exercise: If $\vec{x}$ is an eigenvector of A corresponding to λ, what is $A^3\vec{x}$?

Solution: Since $A\vec{x} = \lambda\vec{x}$,

$$A^3\vec{x} = A^2(A\vec{x}) = A^2(\lambda\vec{x}) = \lambda A^2\vec{x}$$
$$= \lambda A(A\vec{x}) = \lambda A(\lambda\vec{x}) = \lambda^2 A\vec{x} = \lambda^3\vec{x}.$$

Thm: If $\vec{v}_1, \cdots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n\times n$ matrix A, then the set $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent.

Proof: Suppose $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly dependent. Since $\vec{v}_1 \neq 0$, one of the vectors in the set is a linear combination of the preceding vectors.

Let p be the least index such that $\vec{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors.

Then there exist scalars $c_1, \cdots, c_p$ such that

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1}. \quad \cdots\cdots \quad (1)$$

Mupliplying both sides of (1) by A, we get

$$c_1 A\vec{v}_1 + \cdots + c_p A\vec{v}_p = A\vec{v}_{p+1}$$
$$c_1\lambda_1\vec{v}_1 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1} \quad \cdots (2)$$

(2) $-\lambda_{p-1} *$ (1), we have

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}$$

Since $\vec{v}_1, \cdots, \vec{v}_p$ are linearly independent,

$$c_1(\lambda_1 - \lambda_{p+1}) = 0, \quad \cdots, \quad c_p(\lambda_p - \lambda_{p+1}) = 0$$

But $\lambda_1 - \lambda_{p+1} \neq 0$, $\cdots$, $\lambda_p - \lambda_{p+1} \neq 0$ since the eigenvalues are distinct.

Hence $c_1 = 0$, $c_2 = 0$, $\cdots$, $c_p = 0$

Substitute it into (1), we get $\vec{v}_{p+1} = \vec{0}$ which is imposible.

Hence $\{\vec{v}_1, \cdots, \vec{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.

Exercise: Suppose that $\vec{b}_1$ and $\vec{b}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively, and suppose that $\vec{b}_3$ and $\vec{b}_4$ are linearly independent eigenvectors corresponding to a third distinct eigenvalue λ_3. Does it necessarily follow that $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ is a linearly independent set?

Solution: Yes, $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ is a linearly independent set.

Suppose $c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$.

Since
$$
\begin{aligned}
A(c_3\vec{b}_3 + c_4\vec{b}_4) &= c_3A\vec{b}_3 + c_4A\vec{b}_4 \\
&= c_3\lambda_3\vec{b}_3 + c_4\lambda_3\vec{b}_4 \\
&= \lambda_3(c_3\vec{b}_3 + c_4\vec{b}_4).
\end{aligned}
$$

So $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$ or $c_3\vec{b}_3 + c_4\vec{b}_4$ is an eigenvector of A corresponding to eigenvalue λ_3.

If $c_3\vec{b}_3 + c_4\vec{b}_4$ is an eigenvector for λ_3, then using the Theorem above, we know $\{\vec{b}_1, \vec{b}_2, c_3\vec{b}_3 + c_4\vec{b}_4\}$ would be a linearly independent set which implies $c_1 = 0$, $c_2 = 0$ and $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$, which is a contradiction.

So we must have $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$.

implying $c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{0}$

Since $\vec{b}_1, \vec{b}_2$ are linearly independent, $c_1 = c_2 = 0$.

Moreover, $\{\vec{b}_3, \vec{b}_4\}$ is a linearly independent set so $c_3 = c_4 = 0$.

Therefore, $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ is a linearly independent set.

§5.2 The characteristic equation

Ex: Find the eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

Solution: Find all scalars λ such that

$(A - \lambda I)\vec{x} = \vec{0}$ has a non-trivial solution

$\Leftrightarrow \det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3\times 3$$

$$= \lambda^2 + 4\lambda - 21 = (\lambda - 3)(\lambda + 7)$$

The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -7$.

Def: The characteristic equation: A : $n \times n$ matrix

$\det(A - \lambda I) = 0$ is called the characteristic equation of A.

$\det(A - \lambda I)$ is called the characteristic polynomial of A, a polynomial of degree n in the variable λ.

Thm: λ is an eigenvalue of A iff λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.