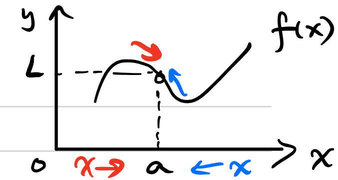
Review : Calculation of Limits.



(1). $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$.

It is allowed that $f(a) \neq L$ or $f(a)$ is not defined.

If $\lim\limits_{x\to a} f(x) = f(a)$, then f is continuous at $x=a$.

(2) Laws of Limits : $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist $\Rightarrow$ $\lim\limits_{x\to a} f(x) \pm g(x)$. $\lim\limits_{x\to a} f(x)\cdot g(x)$. $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$.

(3). Squeeze theorem : $\left.\begin{array}{l} h(x) \le f(x) \le g(x) \text{ when } x \text{ is near } a \\ \lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = L \end{array}\right\} \Rightarrow \lim\limits_{x\to a} f(x) = L.$

Example: $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$ because $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ and $\lim\limits_{x\to 0} -x^2 = \lim\limits_{x\to 0} x^2 = 0$.

(4). Algebraic tricks $\Rightarrow \frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$.

Example : $\lim\limits_{x\to+\infty} \sqrt{x+2} - \sqrt{x} = \lim\limits_{x\to+\infty} \frac{\sqrt{x+2}-\sqrt{x}}{1} = \lim\limits_{x\to+\infty} \frac{(\sqrt{x+2}-\sqrt{x})(\sqrt{x+2}+\sqrt{x})}{1\cdot(\sqrt{x+2}+\sqrt{x})} = \lim\limits_{x\to+\infty} \frac{\cancel{x}+2-\cancel{x}}{\sqrt{x+2}+\sqrt{x}} = \lim\limits_{x\to+\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}} = 0$

(5) L'Hopital's Rule : $\left.\begin{array}{l} \lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \\ \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists or equals } \infty \end{array}\right\} \Rightarrow \lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}.$

$\Rightarrow \frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\cdot\infty, 1^\infty, 0^0, \infty^0.$

Example : $\lim\limits_{x\to 0} \frac{1-\cos x}{\frac{1}{2}x^2} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}} \lim\limits_{x\to 0} \frac{(1-\cos x)'}{(\frac{1}{2}x^2)'} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$

$\frac{0}{0}$ type: $1-\cos x \to 0$ as $x\to 0$
$\frac{1}{2}x^2 \to 0$ as $x \to 0$

Review : Calculation of $f'(x) = \frac{dy}{dx}$

(1) use the definition : $\boxed{\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

(2) use rules of differentiation : $f', g' \Rightarrow (f\pm g)', (f\cdot g)', (\frac{f}{g})', (c\cdot f)'$.

(3). use the chain rule : If $y = f(u)$, $u = g(x)$, then $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$

Example : $\frac{d}{dx} \tan(\underbrace{2x^2+1}_{}) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{d}{du} \tan u \cdot \frac{du}{dx} = \frac{1}{\cos^2 u} \cdot 4x = \frac{4x}{\cos^2(2x^2+1)}$

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(4). use implicit differentiation if y is implicitly defined by an equation $F(x,y)=0$.

$F(x,y)=0 \Rightarrow \frac{d}{dx}F(x,y) = \frac{d}{dx}(0) = 0 \Rightarrow$ solve $\frac{d}{dx}F(x,y)=0$ for $\frac{dy}{dx}$.

(5) use logarithmic differentiation if $y = \frac{h_1(x) \cdots h_n(x)}{g_1(x) \cdots g_m(x)}$ or $y = h(x)^{g(x)}$

Example : $y = x^x \xRightarrow[\text{logarithm}]{\text{take}} \ln y = x \ln x \Rightarrow \frac{d}{dx} \underline{\ln y} = \frac{d}{dx}(x \ln x)$.

$\xRightarrow[\text{rule}]{\text{chain}} \underline{\frac{y'}{y}} = \ln x + x \cdot \frac{1}{x} \Rightarrow y' = y \cdot (\ln x + 1) = x^x(\ln x + 1)$.

product rule: $(f\cdot g)' = f'\cdot g + f\cdot g'$.

Example: Find $\boxed{\frac{dy}{dx}}$ if $F(x,y) = xy^2 - \sin(xy+1) = 0$

Step 1: Differentiate both sides with respect to x:

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}(0) = 0.$$

Step 2: Express $\frac{d}{dx}F(x,y)$ in terms of x, y and $\frac{dy}{dx}$:

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}(xy^2) - \frac{d}{dx}\sin(xy+1)$$

Note: $\frac{d}{dx}(x\cdot y^2) \overset{\text{product rule}}{=} \left(\frac{d}{dx}x\right)\cdot y^2 + \underline{\left(\frac{d}{dx}y^2\right)}\cdot x \overset{\text{chain rule}}{=} y^2 + \underline{2y\cdot\frac{dy}{dx}}\cdot x.$

$$\frac{d}{dx}\sin(\underbrace{xy+1}_{u}) \overset{\text{chain rule}}{=} \frac{d}{du}\sin u \cdot \frac{du}{dx} = \cos u \cdot \underline{\frac{d}{dx}(xy+1)}$$

$$= \cos u \cdot \underline{\left[\left(\frac{d}{dx}x\right)\cdot y + \left(\frac{d}{dx}y\right)\cdot x\right]} \quad \text{product rule}$$

$$= \cos(xy+1)\cdot\left(y + x\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d}{dx}F(x,y) = y^2 + 2xy\cdot\boxed{\frac{dy}{dx}} - \cos(xy+1)\cdot\left(y + x\cdot\boxed{\frac{dy}{dx}}\right) \overset{\text{step 1}}{=} 0.$$

Step 3: solve $\frac{d}{dx}F(x,y) = 0$ for $\frac{dy}{dx}$

$$\left[2xy - \cos(xy+1)\cdot x\right]\cdot\boxed{\frac{dy}{dx}} = y\cdot\cos(xy+1) - y^2 \Longrightarrow \boxed{\frac{dy}{dx}} = \frac{y\cdot\cos(xy+1) - y^2}{2xy - x\cdot\cos(xy+1)}.$$

Review some notable derivatives and antiderivatives:

$$\frac{d}{dx} x^p = p \cdot x^{p-1} \quad \text{for any constant } p.$$

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}.$$

$$\frac{d}{dx} a^x = \ln a \cdot a^x$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x} \quad (\text{when } x>0)$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad (\text{when } x>0)$$

$$\int x^p \, dx = \frac{1}{p+1} x^{p+1} + C \quad \text{for } p \neq -1.$$

$$\int \sin x \, dx = -\cos x + C. \qquad \int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln\left|\frac{1}{\cos x}\right| + C.$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C$$

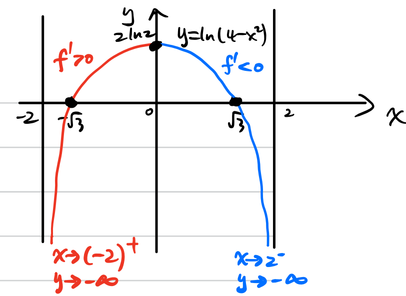
$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \begin{cases} \ln x + C. & (\text{when } x>0) \\ \ln(-x) + C & (\text{when } x<0) \end{cases}$$

$$= \ln|x| + C.$$

How to sketch the graph of $y=f(x)$?

Example 1: $y=f(x)=\ln(4-x^2)$.



Step 1: Find the domain. $\{x \mid 4-x^2>0\} = (-2,2)$.

Step 2: Find vertical / horizontal asymptote

Recall: $x=a$ is a vertical asymptote if $\lim_{x\to a^\pm} f(x) = \pm\infty$.

$y=L$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = L$.

Note: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \ln(4-x^2) = -\infty$, $\lim_{x\to(-2)^+} f(x) = \lim_{x\to(-2)^+} \ln(4-x^2) = -\infty$

$\Rightarrow$ $x=-2$ and $x=2$ are two vertical asymptotes of y.

Note: domain of y is $(-2,2)$

$\Rightarrow$ $y=f(x)$ does not have horizontal asymptotes.

Step 3: Find x-intercepts and y-intercept. $\begin{cases} \text{To find } x\text{-intercepts we solve } f(x)=0 \text{ for } x. \\ y\text{-intercept}: (0, f(0)). \end{cases}$

Note: solve $f(x)=0 \Leftrightarrow$ solve $4-x^2=1$. $\Rightarrow f(x)=0$ has two roots: $-\sqrt{3}$ and $\sqrt{3}$. $\Rightarrow$ x-intercepts: $(-\sqrt{3},0)$ and $(\sqrt{3},0)$

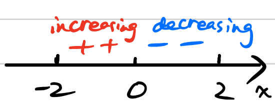
Note: $f(0)=\ln 4 = 2\ln 2$ $\Rightarrow$ y-intercept: $(0, 2\ln 2)$,

Step 4: Find the intervals where f is increasing and decreasing. (calculate the sign of $f'(x)$)

$$f'(x) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{(4-x^2)'}{4-x^2} = \frac{-2x}{4-x^2}$$

$$\Rightarrow f'(x) = \begin{cases} >0 & \text{when } x\in(-2,0) \\ =0 & \text{when } x=0 \\ <0 & \text{when } x\in(0,2). \end{cases}$$

sign of $f'(x)$:



$f(0)=2\ln 2$ is a local maximum.

Review: Related rates problem.

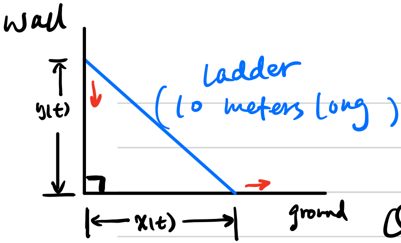
In a related rates problem, we have at least two different rates of change: $\frac{d}{dt}x(t)$ and $\frac{d}{dt}y(t)$, and they are closely related to each other.

Question: Given $\frac{d}{dt}x(t)$, how to calculate $\frac{d}{dt}y(t)$?

To calculate $\frac{d}{dt}y(t)$ from $\frac{d}{dt}x(t)$ we take 2 steps:

step 1: Find a relation between $x(t)$ and $y(t)$. (e.g. $y(t) = x^2(t)$)

Step 2: Differentiation both sides of this equation with respect to t and express $\frac{d}{dt}y(t)$ in term of $\frac{d}{dt}x(t)$.



Suppose that the bottom of the ladder slides away from the wall at a rate of $\underset{\text{“}\frac{d}{dt}x(t)\text{”}}{\underline{2}}$ meters per second.

Question: How fast is the top of this ladder sliding down the wall after 3 seconds?

Answer: $\frac{3}{2}$ meters per second.

$x(t)$: distance from the bottom of this ladder to the wall at time t.

$y(t)$: distance from the top of this ladder to the ground at time t.

Given that $\frac{d}{dt}x(t) = 2$, our aim is to find $\frac{d}{dt}y(t)$ when $t=3$.

Step 1: Find a relation between $x(t)$ and $y(t)$.

Notice: $x^2(t) + y^2(t) = 10^2 = 100$ (Pythagorean Theorem)

Step 2: Differentiate both sides with respect to t.

$$\frac{d}{dx}\left(x^2(t)+y^2(t)\right) = \frac{d}{dx}(100) = 0.$$

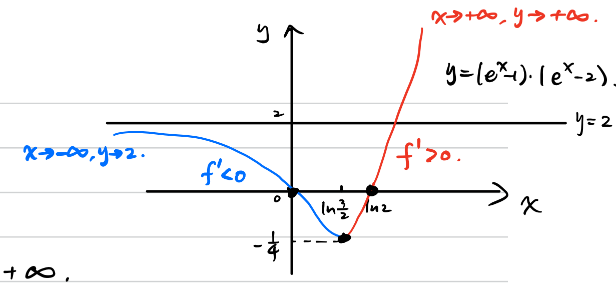
$$\Rightarrow 2x(t)\cdot\frac{d\,x(t)}{dt} + 2y(t)\frac{dy(t)}{dt} = 0. \Rightarrow \frac{d}{dt}y(t) = -\frac{x(t)}{y(t)}\cdot\underbrace{\frac{d}{dt}x(t)}_{\text{“}2\text{”}} = -2\cdot\frac{x(t)}{y(t)}.$$

Notice: When $t=3$, we have $x(t) = 3\times 2 = 6$ and $y(t) = \sqrt{10^2-6^2} = \sqrt{64} = 8$.

Therefore, when $t=3$, $\frac{d}{dt}y(t) = -2\cdot\frac{x(t)}{y(t)} = -2\cdot\frac{6}{8} = \underline{-\frac{3}{2}}$ meters per second.

$\frac{d}{dt}y(t)$ is negative because this ladder is sliding down and $y(t)$ is decreasing.

More examples for sketching the graph of $y=f(x)$

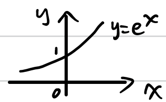


Example 1: $y=f(x)=(e^x-1)\cdot(e^x-2)=e^{2x}-3\cdot e^x+2.$

Step 1: Find the domain. $(-\infty,+\infty)$.

Step 2: Find vertical / horizontal asymptote

Recall :



$\lim_{x\to-\infty} e^x = 0.$ $\lim_{x\to+\infty} e^x=+\infty.$

Note : $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(e^x-1)\cdot\lim_{x\to-\infty}(e^x-2)=(-1)\times(-2)=2 \Rightarrow y=2$ is a horizontal asymptote.

Note : $f(x)$ does not have vertical asymptotes because $\lim_{x\to a} f(x)=(e^a-1)\cdot(e^a-2)$ is always a finite number.
$x\to a \to$ a finite number

Step 3 : Find x-intercepts and y-intercept.

Note : solve $f(x)=0 \Rightarrow e^x-1=0$ or $e^x-2=0 \Rightarrow x=0$ or $\ln 2$. $\Rightarrow$ x-intercepts : $(0,0)$ and $(\ln 2, 0)$.

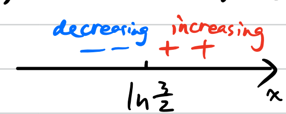
Note : $f(0)=0 \Rightarrow$ y-intercept : $(0,0)$.

Step 4 : Find the intervals where f is increasing and decreasing.

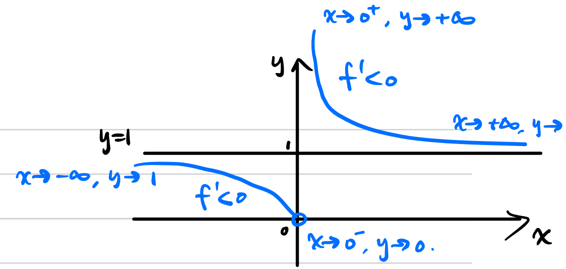
$$f'(x)=(e^x-2)\cdot\frac{d}{dx}(e^x-1)+(e^x-1)\cdot\frac{d}{dx}(e^x-2)=(e^x-2)\cdot e^x+(e^x-1)\cdot e^x=2\left(e^x-\tfrac{3}{2}\right)\cdot e^x.$$

$$\Rightarrow f'(x)\begin{cases} <0 & \text{when } x<\ln\frac{3}{2} \\ =0 & \text{when } x=\ln\frac{3}{2} \\ >0 & \text{when } x>\ln\frac{3}{2}. \end{cases}$$

sign of $f'(x)$:



$f(\ln\frac{3}{2})=-\frac{1}{4}$ is a local minimum.

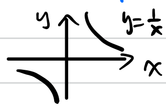


Example 2 : $y=f(x)=e^{\frac{1}{x}}$

Step 1 : Find the domain. $\{x \mid x\neq 0\} = (-\infty,0)\cup(0,+\infty)$

Step 2 : Find vertical / horizontal asymptote

Recall :



$\lim\limits_{x\to-\infty}\frac{1}{x}=\lim\limits_{x\to+\infty}\frac{1}{x}=0$, $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$, $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$.

Note : $\left.\begin{array}{l}\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}e^{\frac{1}{x}}=e^{\lim\limits_{x\to-\infty}\frac{1}{x}}=e^{0}=1\\ \lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}e^{\frac{1}{x}}=e^{\lim\limits_{x\to+\infty}\frac{1}{x}}=e^{0}=1\end{array}\right\}\Rightarrow y=1$ is a horizontal asymptote.

Note : $\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}e^{\frac{1}{x}}=e^{\lim\limits_{x\to 0^+}\frac{1}{x}}=+\infty \Rightarrow x=0$ is a vertical asymptote.

Step 3 : Find x-intercepts and y-intercept.

Note : $f(x)=e^{\frac{1}{x}}>0$ for any $x\neq 0$ $\Rightarrow$ y does not have x-intercepts.

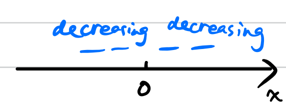
Note : $f(0)$ is not defined $\Rightarrow$ y does not have y-intercepts.

Step 4 : Find the intervals where f is increasing and decreasing.

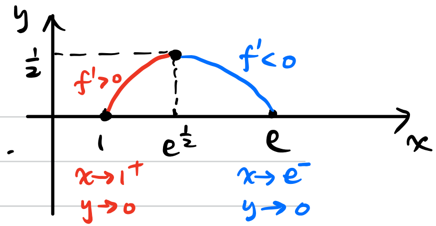
$$f'(x)=\frac{d}{dx}e^{\underbrace{\frac{1}{x}}_{}=u}\ \underset{\text{rule}}{\overset{\text{chain}}{=\!=}}\ \frac{d}{du}e^{u}\cdot\frac{du}{dx}=e^{u}\cdot\left(-\frac{1}{x^2}\right)=-e^{\frac{1}{x}}\cdot\frac{1}{x^2}.$$

$$\Rightarrow f'(x)\begin{cases}<0 & \text{when } x<0\\ <0 & \text{when } x>0\end{cases}$$

sign of $f'(x)$:



Example 3: $y = f(x) = \sqrt{-(\ln x - 1)\cdot \ln x}$



Step 1: Find the domain. $\{x \mid x > 0 \text{ and } \ln x \cdot (\ln x - 1) \le 0\} = [1, e]$.

solve $\ln x \cdot (\ln x - 1) \le 0 \Rightarrow$ solve $0 \le \ln x \le 1 \Rightarrow 1 \le x \le e$.

Step 2: Find vertical / horizontal asymptote

Note: For a finite number $a \in [1, e]$, $\lim_{x \to a} f(x) = f(a)$ is always a finite number.

$\Rightarrow f(x)$ does not have vertical asymptotes

Note: domain is $[1, e]$ $\Rightarrow f(x)$ does not have horizontal asymptotes

Step 3: Find x-intercepts and y-intercept.

Note: solve $f(x) = 0 \Leftrightarrow$ solve $(\ln x - 1)\cdot \ln x = 0 \Rightarrow f(x) = 0$ has 2 roots: 1 and e

$\Rightarrow$ x-intercepts: $(1, 0)$ and $(e, 0)$

Note: $f(0)$ is not defined $\Rightarrow f(x)$ does not have y-intercept.

Step 4: Find the intervals where f is increasing and decreasing.

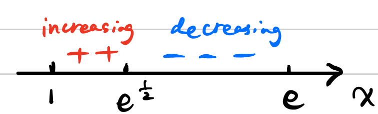
$$f'(x) = \frac{d}{dx}\sqrt{\underbrace{-(\ln x - 1)\ln x}_{=u}} = \frac{d}{du}\sqrt{u}\cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot \frac{d}{dx}\left(-(\ln x - 1)\ln x\right)$$

$$= -\frac{1}{2\sqrt{u}}\cdot\left(\ln x \cdot \frac{d}{dx}(\ln x - 1) + (\ln x - 1)\cdot \frac{d}{dx}\ln x\right) = -\frac{1}{2\sqrt{-(\ln x - 1)\cdot \ln x}}\cdot \frac{1}{x}\cdot (2\ln x - 1)$$

Note: $2\sqrt{-(\ln x - 1)\cdot \ln x}$ and x are positive when $x \in [1, e]$.

$$\Rightarrow f'(x) \begin{cases} > 0 & x \in [1, e^{\frac{1}{2}}) \\ = 0 & x = e^{\frac{1}{2}} \\ < 0 & x \in (e^{\frac{1}{2}}, e] \end{cases}$$

sign of $f'(x)$:



$f(e^{\frac{1}{2}}) = \frac{1}{2}$ is a local maximum.