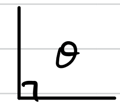
MATH1012

Trigonometric functions

1. The radian of an angle:



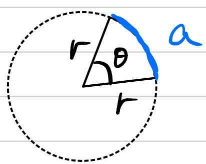
$\theta = 90^\circ = \frac{\pi}{2}$ rad.

In general $1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ$.

$2 \text{ rad} = 2 \times \frac{180^\circ}{\pi} = \frac{360^\circ}{\pi} \approx 114.6^\circ$.

$\pi \text{ rad} = 180^\circ$

$2\pi \text{ rad} = 360^\circ$.



↑

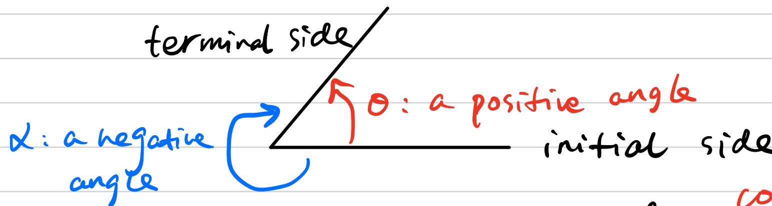
a circle with radius r.

a: the length of the arc spanned by θ.

$$\frac{\theta}{2\pi} = \frac{a}{\text{circumference}} = \frac{a}{2\pi r}.$$

$\rightarrow a = \theta \cdot r$.

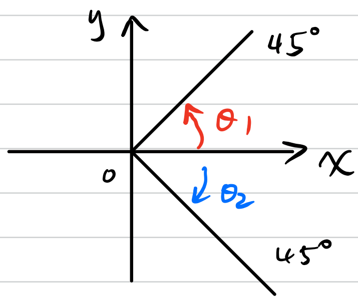
2. The sign of an angle.

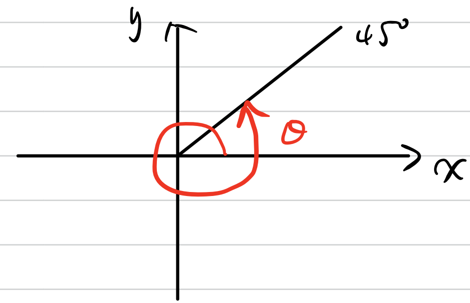


a positive angle : initial side $\xrightarrow{\text{counterclockwise}}$ terminal side.

a negative angle : initial side $\xrightarrow{\text{clockwise}}$ terminal side.

Example :





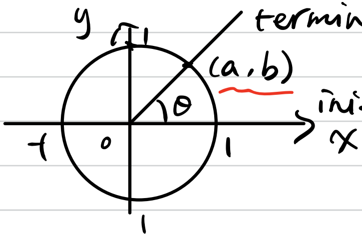
$\theta_1 = 45^\circ = \frac{\pi}{4}$ rad

$\theta_2 = -45^\circ = -\frac{\pi}{4}$ rad.

$\theta = \frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$ rad

3. The trigonometric functions

1). $\theta \in (-\infty, +\infty)$.

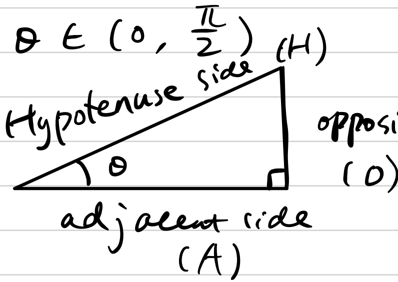


(a, b) is on the unit circle.

$a =$ a function of $\theta \xrightarrow{\text{denoted by}} \cos\theta$

$b =$ a function of $\theta = \sin\theta$

2) $\theta \in (0, \frac{\pi}{2})$



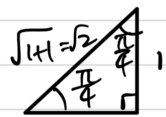
$\sin\theta = \frac{O}{H}$ $\cos\theta = \frac{A}{H}$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$.

Some special values of $\sin\theta$, $\cos\theta$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

① $\theta = 0$. $(a,b) = (1,0)$. $\cos 0 = 1$. $\sin 0 = 0$. $\tan 0 = 0$

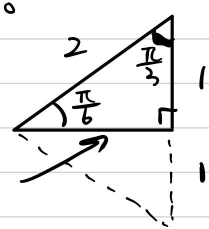
② $\theta = \frac{\pi}{4} = 45^\circ$.



$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = 1$

③ $\theta = \frac{\pi}{6} = 30^\circ$



$\sqrt{2^2-1} = \sqrt{3}$

→ a half of an equilateral triangle

$\sin\frac{\pi}{6} = \frac{1}{2}$. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

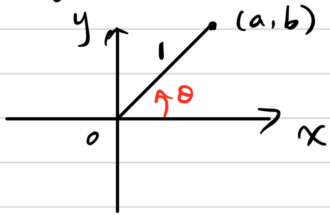
$\theta = \frac{\pi}{3}$. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. $\cos\frac{\pi}{3} = \frac{1}{2}$. $\tan\frac{\pi}{3} = \sqrt{3}$.

④ $\theta = \frac{\pi}{2}$. $(a,b) = (0,1)$. $\sin\frac{\pi}{2} = 1$. $\cos\frac{\pi}{2} = 0$.

$\tan\frac{\pi}{2}$ is not defined.

Recall $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, $\cot\theta = \frac{1}{\tan\theta}$

4. Trigonometric identities

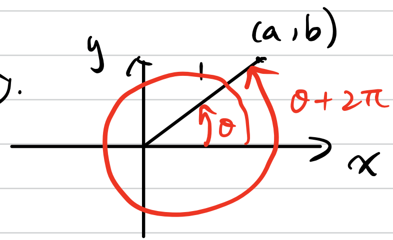
① 

(a, b) is on the unit circle $\rightarrow a^2+b^2=1$.

$\sin\theta = b$, $\cos\theta = a$.

$\rightarrow \sin^2\theta + \cos^2\theta = 1.$ $\xrightarrow{\text{divide by } \cos^2\theta} \tan^2\theta + 1 = \sec^2\theta$.

$\xrightarrow{\text{divide by } \sin^2\theta} 1 + \cot^2\theta = \csc^2\theta$.

②. 

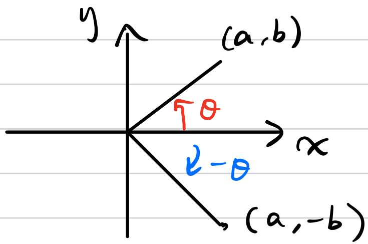
$\sin\theta = b$. $\cos\theta = a$. $\tan\theta = \frac{b}{a}$.

$\sin(\theta+2\pi) = b$ $\cos(\theta+2\pi) = a$ $\tan(\theta+2\pi) = \frac{b}{a}$

$\rightarrow$ $\boxed{\begin{aligned}\sin(\theta+2\pi) &= \sin\theta \\ \cos(\theta+2\pi) &= \cos\theta\end{aligned}}$ for any $\theta \in (-\infty, +\infty)$.

$\tan(\theta+2\pi) = \tan\theta$.

③



$\sin\theta = b.$ $\cos\theta = a.$ $\tan\theta = \frac{b}{a}.$

$\sin(-\theta) = -b.$ $\cos(-\theta) = a.$ $\tan(-\theta) = -\frac{b}{a}.$

$\rightarrow$ $\sin(-\theta) = -\sin\theta.$ for any $\theta \in (-\infty, +\infty).$

$\cos(-\theta) = \cos\theta.$

$\tan(-\theta) = -\tan\theta$

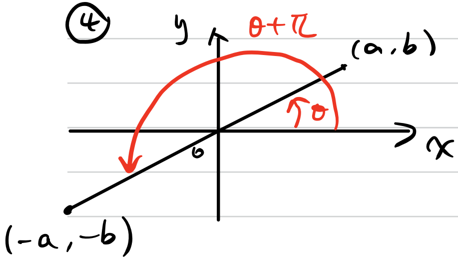
$\rightarrow$ $y=\sin\theta$ and $y=\tan\theta$ are odd functions.

$y=\cos\theta$ is an even function.

Recall: 1). f is odd if and only if $f(-x) = -f(x)$ for all x

2). f is even if and only if $f(-x) = f(x)$ for all x.

4



$\sin\theta = b \qquad \cos\theta = a \qquad \tan\theta = \frac{b}{a}.$

$\sin(\theta+\pi) = -b. \qquad \cos(\theta+\pi) = -a \qquad \tan(\theta+\pi) = \frac{-b}{-a} = \frac{b}{a}.$

$\rightarrow \quad \sin(\theta+\pi) = -\sin\theta$ for any $\theta \in (-\infty, +\infty)$.

$\cos(\theta+\pi) = -\cos\theta$

$\boxed{\tan(\theta+\pi) = \tan\theta.}$

$f(x+p) = f(x)$ for all x.

(p is a positive constant)

Notice: 1. $y=\sin\theta$, $y=\cos\theta$, $y=\tan\theta$ are all periodic functions.

2.

	$y=\sin\theta$	$y=\cos\theta$	$y=\tan\theta$
period	2π	2π	π

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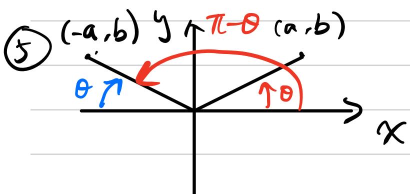


$\sin\theta = b \qquad \cos\theta = a \qquad \tan\theta = \frac{b}{a}.$

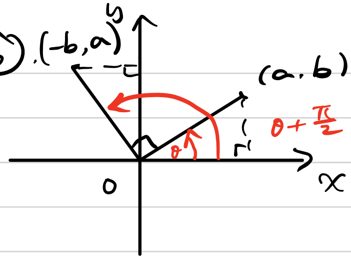
$\sin(\pi-\theta) = b \qquad \cos(\pi-\theta) = -a \qquad \tan(\pi-\theta) = -\frac{b}{a}.$

$\rightarrow \quad \sin(\pi-\theta) = \sin\theta$ for any $\theta \in (-\infty, +\infty)$.

$\cos(\pi-\theta) = -\cos\theta$

$\tan(\pi-\theta) = -\tan\theta$

⑥



$\sin\theta = b.$ $\cos\theta = a.$ $\tan\theta = \frac{b}{a}.$

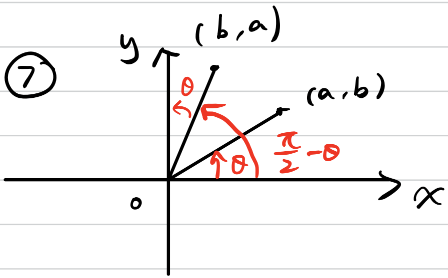
$\sin(\theta+\frac{\pi}{2}) = a$ $\cos(\theta+\frac{\pi}{2}) = -b.$ $\tan(\theta+\frac{\pi}{2}) = -\frac{a}{b}.$

$\rightarrow$ $\sin(\theta+\frac{\pi}{2}) = \cos\theta$ for any $\theta \in (-\infty, +\infty).$

$\cos(\theta+\frac{\pi}{2}) = -\sin\theta$

$\tan(\theta+\frac{\pi}{2}) = -\frac{1}{\tan\theta}$

⑦



$\sin\theta = b.$ $\cos\theta = a.$ $\tan\theta = \frac{b}{a}.$

$\sin(\frac{\pi}{2}-\theta) = a$ $\cos(\frac{\pi}{2}-\theta) = b.$ $\tan(\frac{\pi}{2}-\theta) = \frac{a}{b}.$

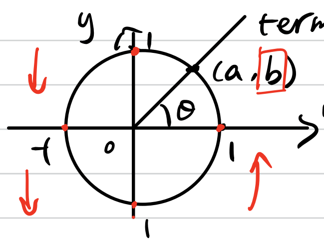
$\rightarrow$ $\sin(\frac{\pi}{2}-\theta) = \cos\theta.$ for any $\theta \in (-\infty, +\infty).$

$\cos(\frac{\pi}{2}-\theta) = \sin\theta.$

$\tan(\frac{\pi}{2}-\theta) = \frac{1}{\tan\theta}$

5. Graphs of the trigonometric functions.

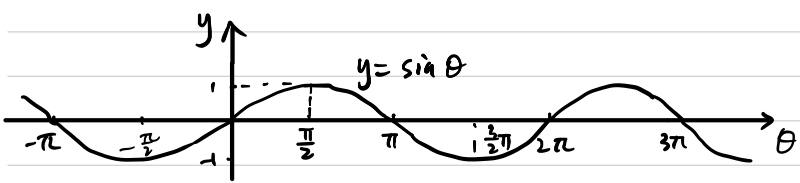
1) $y = \sin\theta = b$.



$\theta: \ 0 \to \frac{\pi}{2} \to \pi \to \frac{3\pi}{2} \to 2\pi \to \cdots$

$b: \ \underbrace{0 \to 1 \to 0 \to -1 \to 0}_{\text{This process keeps repeating.}} \to \cdots$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0.



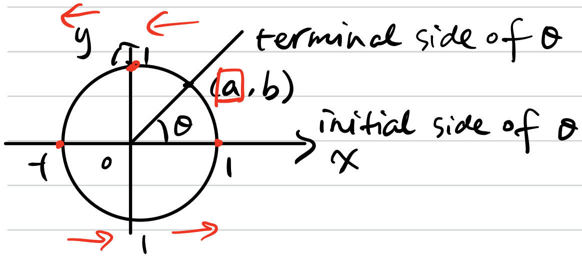
→ odd function.

period $= 2\pi$.

Notice: $\sin k\pi = 0$ $\quad \sin\left(2k\pi + \frac{\pi}{2}\right) = 1$. $\quad \sin\left(2k\pi - \frac{\pi}{2}\right) = -1$ for any integer k

Domain : $(-\infty, +\infty)$. $\qquad$ Range : $[-1, 1]$

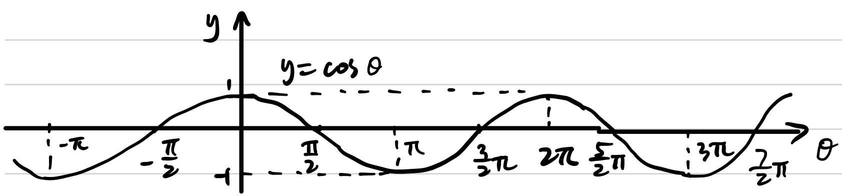
2) $y = \cos\theta = a$



$$\theta: 0 \to \frac{\pi}{2} \to \pi \to \frac{3\pi}{2} \to 2\pi \to \cdots$$

$$\underbrace{a: 1 \to 0 \to -1 \to 0 \to 1.}_{\text{This process keeps repeating.}} \to \cdots$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1



→ even function

period $= 2\pi$.

Notice: $\cos(k\pi + \frac{\pi}{2}) = 0$ $\quad \cos 2k\pi = 1.$ $\quad \cos(2k\pi + \pi) = -1$ for any integer k.

Domain: $(-\infty, +\infty)$. Range: $[-1, 1]$.