Integration

5.1 Antiderivatives

Definition 5.1 *If f and F are functions such that $F' = f$, F is called an antiderivative of f.*

Remark 5.2 *If F is an antiderivative of f, so is $F + C$ for any constant function C. Moreover, these are the only antiderivatives of f (a corollary of mean value theorem).*

Definition 5.3 *If F is an antiderivative (or indefinite integral) of f, we write*

$$\int f(x)dx = F(x) + C.$$

Remark 5.4 *We get formula right away from the formula for derivatives. For instance,*

- $\int x^k dx = \frac{x^{k+1}}{k+1} + C$ *if* $k \neq -1$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \csc^2 x dx = -\cot x + C$
- $\int \tan x \sec x dx = \sec x + C$
- $\int \cot x \csc x dx = -\csc x + C$
- $\int e^x dx = e^x + C$
- $\int \dfrac{1}{x} dx = \ln x + C$

Lemma 5.5 *If f and g are functions and a is a number,*

1. $\int (f + g)(x)dx = \int f(x)dx + \int g(x)dx$
2. $\int (af)(x)dx = a \int f(x)dx$

proof of 1) Suppose that $F' = f$ and $G' = g$. Then, $(F+G)' = F'+G' = f+g$. Hence, an antiderivative of $f+g$ is

$$\int (f+g)(x)dx = F+G = \int f(x)dx + \int g(x)dx.$$

Example 5.6 *An antiderivative of*

$$f(x) = x^2 + \frac{2}{x} + 3\sin x$$

is

$$F(x) = \frac{1}{3}x^3 + 2\ln x - 3\cos x + C.$$

Example 5.7 *Evaluate* $\int \tan^2 x dx$.

solution:

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C.$$

5.2 Definite Integrals

Definition 5.8 *If* a_1,...,a_n *are numbers,*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_{n-1} + a_n.$$

Lemma 5.9 *If* a_1,...,a_n,b_1,...,b_n,c *are numbers. Then,*

1. $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$
2. $\sum_{i=1}^n ca_i = c\sum_{i=1}^n a_i$

proof of 1):

$$\begin{aligned}
&\textstyle\sum_{i=1}^n (a_i + b_i) \\
&= (a_1 + b_1) + (a_2 + b_2) + ...(a_n + b_n) \\
&= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\
&= \textstyle\sum_{i=1}^n a_i + \sum_{i=1}^n b_i
\end{aligned}$$

Remark 5.10 *For every summation* $\sum_{i=1}^n a_i$, *we have*

$$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}.$$

Example 5.11 *Evaluate* $\sum_{i=1}^{100} i$.

solution:

$$\begin{aligned}
&\textstyle\sum_{i=1}^{100} i\\
&= \textstyle\sum_{i=1}^{100} \dfrac{i((i+1)-(i-1))}{2}\\
&= \tfrac{1}{2}\textstyle\sum_{i=1}^{100}((i+1)i - i(i-1))\\
&= \tfrac{1}{2}[\textstyle\sum_{i=1}^{100}(i+1)i - \sum_{i=1}^{100} i(i-1)]\\
&= \tfrac{1}{2}[\textstyle\sum_{i=1}^{100}(i+1)i - \sum_{i=0}^{99}(i+1)i]\\
&= \tfrac{1}{2}[\textstyle\sum_{i=1}^{99}(i+1)i + 101\times 100 - 1\times 0 - \sum_{i=1}^{99}(i+1)i]\\
&= \tfrac{1}{2}(101\times 100 - 1\times 0)\\
&= 5050.
\end{aligned}$$

Example 5.12 *Evaluate* $\sum_{i=1}^{n} i^2$.

solution:

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} i^2\\
&= \textstyle\sum_{i=1}^{n} \dfrac{(i+2)(i+1)i-(i+1)i(i-1)}{3} - i\\
&= \tfrac{1}{3}[\textstyle\sum_{i=1}^{n}(i+2)(i+1)i - \sum_{i=1}^{n}(i+1)i(i-1)] - \sum_{i=1}^{n} i\\
&= \tfrac{1}{3}[\textstyle\sum_{i=1}^{n}(i+2)(i+1)i - \sum_{i=0}^{n-1}(i+2)(i+1)i] - \tfrac{1}{2}n(n+1)\\
&= \tfrac{1}{3}[(\textstyle\sum_{i=1}^{n-1}(i+2)(i+1)i) + (n+2)(n+1)n - (2)(1)(0) - \sum_{i=1}^{n-1}(i+2)(i+1)i] - \tfrac{1}{2}n(n+1)\\
&= \tfrac{1}{3}(n+2)(n+1)n - \tfrac{1}{2}n(n+1)\\
&= \tfrac{1}{6}(2n+1)n(n+1).
\end{aligned}$$

Definition 5.13 *Let f be a function and a, b be numbers. The definite integral of f from a to b is defined to be the limit of*

$$\sum_{i=1}^{N} f(c_i)(a_i - a_{i-1}) \quad \textit{(which is called a Riemann sum)}$$

as all of the $a_i - a_{i-1}$ tend to zero. Here a_i are numbers satisfying

$$a = a_0 < a_1 < a_2 < \ldots < a_N = b.$$

c_i is a number chosen such that $a_{i-1} \leq c_i \leq a_1$.

Remark 5.14 *The symbol for the definite integral of f from a to b is*

$$\int_a^b f(x)dx.$$

Remark 5.15 *If f is a function such that $f(x) \geq 0$ for all $a \leq x \leq b$. R is the region consisting of points (x, y) satisfying $a \leq x \leq b$ and $0 \leq y \leq f(x)$. Then, the area of R is $\int_a^b f(x)dx$.*

Example 5.16 *For each $a > 0$, evaluate*

$$\int_0^a x^2 dx.$$

solution:
For every positive integer n, we subdivide the interval $[0, a]$ into n parts. Then we evaluate the sum giving the definite integral $\int_0^a x^2 dx$, with the square function evaluated at the left end point of each of the subintervals, which is

$$\begin{aligned} & \textstyle\sum_{i=1}^n [(i-1)\frac{a}{n}]^2[\frac{a}{n}] \\ & = \frac{a^3}{n^3}\textstyle\sum_{i=1}^n (i-1)^2 \\ & = \frac{a^3 n(n-1)(2n-1)}{6n^3} \end{aligned}$$

which tends to $a^3/3$ as n tends to infinity. Consequently,

$$\int_0^a x^2 dx = \frac{a^3}{3}.$$

Lemma 5.17 *If f and g are functions and a, b, c are numbers,*

1. $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b (cf)(x)dx = c\int_a^b f(x)dx$

proof: omitted. But it is a direct consequence of the corresponding formula for summations.

Lemma 5.18 *If f is a function and $a \le b \le c$, then,*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

proof: omitted.

Lemma 5.19 *If f is a function and a is a number, $\int_a^a f(x)dx = 0$.*

proof: omitted.

Remark 5.20 *If $a > b$, the symbol $\int_a^b f(x)dx$ is interpreted as $-\int_b^a f(x)dx$.*

Lemma 5.21 *If f is a function and a, b are numbers,*

$$\int_a^b f(x)dx = \int_a^b f(y)dy.$$

Lemma 5.22 *If $f(x) \ge 0$ for all $a \le x \le b$, then $\int_a^b f(x)dx \ge 0$.*

proof: This integral is interpreted as the area of a region.

Corollary 5.23 *If f and g are functions such that $f(x) \le g(x)$ for $a < x < b$. Then,*

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

Definition 5.24 *If f is a function and $a < b$ are numbers. The average (or mean value) of f over $[a, b]$ is*

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 5.25 (mean value) *If f is a continuous function on $[a, b]$, then there exists c in $[a, b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

proof:
Since f is continuous on $[a, b]$, f attains absolute maximum and minimum on $[a, b]$. We may let α, β be in $[a, b]$ such that

$$m = f(\alpha) \leq f(x) \leq f(\beta) = M \quad \text{for all } x \text{ in } [a, b].$$

Now the continuous function $f - \frac{1}{b-a}\int_a^b f(x)dx$ evaluates at α, β to be negative and positive respectively. By the intermediate value theorem, this function evaluate at a certain c in $[a, b]$ to be zero.

Theorem 5.26 (Fundamental Theorem of Calculus, 1st version) *If f is a continuous function and a is a number. Define*

$$F(x) = \int_a^x f(y)dy \text{ for all } x.$$

Then, $F'(x) = f(x)$ for all x.

proof:
For every $h \neq 0$,

$$\begin{aligned}
&\tfrac{F(x+h)-F(x)}{h} \\
&= \tfrac{1}{h}[\textstyle\int_a^{x+h} f(y)dy - \int_a^x f(y)dy] \\
&= \tfrac{1}{h}[\textstyle\int_x^{x+h} f(y)dy] \\
&= \text{average of } f \text{ over } [x, x+h] \\
&= f(c) \quad \text{for a certain } c \text{ in } [x, x+h]
\end{aligned}$$

by mean value theorem for integrals. Thus, its limit is $f(x)$ as h tends to 0.

Corollary 5.27 (Fundamental Theorem of Calculus, 2nd version) *If f is a function. a and b are numbers. F is an antiderivative of f. Then,*

$$\int_a^b f(x)dx = F(b) - F(a).$$

proof:
Define $G(x) = \int_a^x f(y)dy$ for all x. Then G is an antiderivative of f by the fundamental theorem of calculus. By hypothesis, F is an antiderivative of f also. Thus, $G = F + C$ for some constant function C. Now, $0 = G(a) = F(a) + C$ so that $C = -F(a)$. Moreover, $G(b) = F(b) + C = F(b) - F(a)$. Therefore,

$$\int_a^b f(y)dy = G(b) = F(b) - F(a).$$

Example 5.28 *Evaluate the derivative of* $f(x) = \int_0^{x^2} e^{y^2} dy$.

solution:
Let

$$g(x) = \int_0^x e^{y^2} dy$$

and

$$h(x) = x^2$$

for all x. Then, $f(x) = g(h(x))$. Now, by the fundamental theorem of calculus,

$$g'(x) = e^{x^2}$$

for all x. Now, we apply the chain rule and get,

$$f'(x) = g'(h(x))h'(x) = e^{h(x)^2} h'(x) = 2xe^{x^4}$$

for all x.

Example 5.29 *Evaluate the area of the region bounded by the x-axis, the line defined by* $x = 1$ *and the graph of* $f(x) = \sqrt{x}$.

The area of the given region is

$$\int_0^1 \sqrt{x} dx.$$

Now, $F(x) = \frac{2}{3}\sqrt{x}^3$ is an antiderivative of $f(x) = \sqrt{x}$. Hence,

$$\int_0^1 \sqrt{x} dx = F(1) - F(0) = \frac{2}{3}.$$

In another usual notation,

$$\int_0^1 \sqrt{x} dx = \frac{2}{3}\sqrt{x}^3|_0^1 = \frac{2}{3}\sqrt{1}^3 - \frac{2}{3}\sqrt{0}^3 = \frac{2}{3}.$$

Example 5.30 *Evaluate*

$$\int_0^{2\pi} |\sin x| dx.$$

solution:

$\int_0^{2\pi} |sinx| dx$
$= \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx$
$= \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x dx$
(as $|\sin x| = \sin x$ for $0 \le x \le \pi$ and $|\sin x| = -\sin x$ for $\pi \le x \le 2\pi$)
$= -\cos x|_0^{\pi} + \cos x|_{\pi}^{2\pi}$
$= 4.$

Example 5.31 *Evaluate*

$$\int_0^4 (|x^2 - 1| - x|x - 3|) dx.$$

solution:

$\int_0^4 (|x^2 - 1| - x|x - 3|) dx$
$= \int_0^1 (|x^2 - 1| - x|x - 3|) dx + \int_1^3 (|x^2 - 1| - x|x - 3|) dx + \int_3^4 (|x^2 - 1| - x|x - 3|) dx$
$= \int_0^1 [-x^2 + 1 + x(x - 3)] dx + \int_1^3 [x^2 - 1 + x(x - 3)] dx + \int_3^4 [x^2 - 1 - x(x - 3)] dx$
$= [-\frac{3}{2}x^2 + x]|_0^1 + [\frac{2}{3}x^3 - \frac{3}{2}x^2 - x]|_1^3 + [\frac{3}{2}x^2 - x]|_3^4$
$= -7/6.$

Example 5.32 *Evaluate*

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3.$$

solution:
Let $f(x) = x^3$ for all x. We study the Riemann sum of f over $[0, 1]$ by partitioning $[0, 1]$ into n parts uniformly, and by evaluating f at the right endpoint of each subinterval, which is

$$\sum_{i=1}^{n} (\frac{i}{n})^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^{n} i^3.$$

Hence its limit as n tends to infinity exists and is the definite integral $\int_0^1 x^3 dx = 1/4$.

Example 5.33 *Evaluate*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n + i}.$$

solution:
Let $f(x) = 1/x$ for $x > 0$. For each positive integer n, partition the interval $[1, 2]$ by the points $1 + \dfrac{i}{n}$ where i runs from 0 to n. Consider the Riemann sum

of f over $[1,2]$ corresponding to this partition and by evaluating f at the right end point of each subinterval, it becomes

$$\sum_{i=1}^{n} \frac{1}{1+i/n} \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n+i}.$$

Its limit as n tends to ∞ is thus

$$\int_1^2 \frac{dx}{x} = \ln x|_1^2 = \ln 2.$$

5.3 Integration by Substitution

Theorem 5.34 *If f and g are functions, $f(g(x))$ is an antiderivative of $f'(g(x))g'(x)$. That is,*

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

proof: nothing different from the ordinary chain rule.

Example 5.35 *Evaluate*

$$\int 2x(1+x^2)^8 dx.$$

solution:
Let $g(x) = 1 + x^2$ and $h(x) = x^8$. We are asking for an antiderivative of $g'(x)h(g(x))$. Now if $f(x) = \frac{1}{9}x^9$, f is an antiderivative of h. In other words, we are asking for an antiderivative of $g'(x)f'(g(x))$. According to the version of chain rule above, an antiderivative of it is $f(g(x)) = \frac{1}{9}(1+x^2)^8$. Thus,

$$\int 2x(1+x^2)^8 dx = \frac{1}{9}(1+x^2)^9 + C.$$

Remark 5.36 *The previous procedure can easily be memorized by putting $u = x^2 + 1$, $du = 2xdx$. Then,*

$$\int 2x(1+x^2)^8 dx = \int u^8 du = \frac{1}{9}u^9 + C = \frac{1}{9}(1+x^2)^9 + C.$$

Example 5.37 *Evaluate*

$$\int e^{4x} dx.$$

Let $u = 4x$ so that $du = 4dx$. Then,

$$\begin{aligned} &\int e^{4x} dx \\ &= \int e^u \tfrac{1}{4} du \\ &= \tfrac{1}{4} e^u + C \\ &= \tfrac{1}{4} e^{4x} + C. \end{aligned}$$

Example 5.38 *Evaluate*
$$\int e^{\sin x} \cos x dx.$$

solution:
Let $u = \sin x$ so that $du = \cos x dx$. Then,

$$\begin{aligned}
&\int e^{\sin x} \cos x dx \\
&= \int e^u du \\
&= e^u + C \\
&= e^{\sin x} + C.
\end{aligned}$$

Example 5.39 *Evaluate the area of the region bounded by the graph of $f(x) = e^{\sin x} \cos x$, the x-axis, the lines $x = 0$ and $x = \pi/2$.*

solution:
Note that $f(x) \geq 0$ for $0 \leq x \leq \pi/2$. Hence, the area of the given region is

$$\int_0^{\pi/2} e^{\sin x} \cos x dx.$$

We saw that $g(x) = e^{\sin x}$ is an antiderivative of $f(x) = e^{\sin x} \cos x$. By the fundamental theorem of calculus,

$$\int_0^{\pi/2} e^{\sin x} \cos x dx = e^{\sin x}|_0^{\pi/2} = e^1 - e^0 = e - 1$$

which is the area of the given region.

Remark 5.40 *In the previous two examples, we had $e^{\sin x}$ to be an antiderivative of $e^{\sin x} \cos x$. Then we evaluated the antiderivative at $\pi/2$ and 0, the difference was the definite integral we want. However, an antiderivative of $e^{\sin x} \cos x$ was known to be e^u also, where $u = \sin x$. Evaluating $e^{\sin x}$ at $\pi/2$ is the same as to evaluate e^u at 1. Evaluating $e^{\sin x}$ at 0 is the same as to evaluate e^u at 0. Therefore,*
$$\int_0^{\pi/2} e^{\sin x} \cos x dx = \int_0^1 e^u du.$$

Example 5.41 *Evaluate the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2+1}}$, the x-axis, the lines $x = 0$ and $x = 1$.*

solution:
Note that $f(x) \geq 0$ for $0 \leq x \leq 1$. Hence the area of the given region is

$$\int_0^1 \frac{xdx}{\sqrt{x^2 + 1}}.$$

Let $u = x^2 + 1$ so that $du = 2xdx$. Moreover, $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. Thus,

$$\int_0^1 \frac{xdx}{\sqrt{x^2+1}}$$
$$= \int_1^2 \frac{du}{2\sqrt{u}}$$
$$= \sqrt{u}|_1^2$$
$$= \sqrt{2} - 1$$

which is the area of the given region.

Example 5.42 *Evaluate*

$$\int_1^2 \sqrt{\frac{x-1}{x^5}}dx$$

solution:
Let $u = 1 - 1/x$. Then $du = dx/x^2$. Moreover, $u = 0$ when $x = 1$ and $u = 1/2$ when $x = 2$

$$\int_1^2 \sqrt{\frac{x-1}{x^5}}dx$$
$$= \int_1^2 \sqrt{1 - \frac{1}{x}}\frac{dx}{x^2}$$
$$= \int_0^{1/2} \sqrt{u}du$$
$$= \tfrac{2}{3}\sqrt{u}^3|_0^{1/2}$$
$$= 1/3\sqrt{2}.$$

Example 5.43 *Evaluate the area of the region bounded by the graph of* $f(x) = \dfrac{1}{x\ln x}$, *the x-axis, the lines* $x = 1/e^2$ *and* $1/e$.

solution:
Note that $f(x) \leq 0$ for $1/e^2 \leq x \leq 1/e$. The area of the given region is

$$-\int_{1/e^2}^{1/e} \frac{dx}{x\ln x}$$

Let $u = -\ln x$. So, $du = -\dfrac{dx}{x}$; $u = 2$ when $x = 1/e^2$ and $u = 1$ when $x = 1/e$. Thus,

$$-\int_{1/e^2}^{1/e} \frac{dx}{x\ln x}$$
$$= -\int_2^1 \frac{du}{u}$$
$$= -\ln u|_2^1$$
$$= \ln 2$$

which is the area of the given region.

Remark 5.44 *Let $f(x) = 1/x$ for $x > 0$, then $\int_1^x \frac{dy}{y} = \ln x$ for $x > 0$ is an antiderivative of f. Now if $g(x) = 1/x$ for $x < 0$, then*

$$\int_{-1}^{x} \frac{dy}{y} = \int_{1}^{-x} \frac{dz}{z} = \ln(-x)$$

is an antiderivative of g. Combine these results and we see that $\ln|x|$ is an antiderivative of $h(x) = 1/x$ for $x \neq 0$. In other words,

$$\int \frac{dx}{x} = \ln|x| + C$$

when both sides are regarded as functions defined on the whole real line except 0.

Example 5.45 *Evaluate*

$$\int_{3\pi/4}^{\pi} \tan x dx.$$

solution:
Let $u = \cos x$. So, $du = -\sin x dx$; $u = -1/\sqrt{2}$ when $x = 3\pi/4$ and $u = -1$ when $x = \pi$. Thus,

$$\begin{aligned}
&\textstyle\int_{3\pi/4}^{\pi} \tan x dx \\
&= \textstyle\int_{3\pi/4}^{\pi} \dfrac{\sin x dx}{\cos x} \\
&= \textstyle\int_{-1/\sqrt{2}}^{-1} \dfrac{-du}{u} \\
&= -\ln|u||_{-1/\sqrt{2}}^{-1} \\
&= -\dfrac{\ln 2}{2}.
\end{aligned}$$

5.4 Even and Odd Functions

Definition 5.46 *A function f is an even (odd) function if*

$$f(-x) = f(x) \ \ (-f(-x))$$

for all x.

Lemma 5.47 *If f is an odd function and a is a number,*

$$\int_{-a}^{a} f(x)dx = 0.$$

proof:
First of all,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$

We will evaluate the first term on RHS. To do this, let $u = -x$ so that $du = -dx$; $u = a$ when $x = -a$ and $u = 0$ when $x = 0$. Thus,

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = -\int_{0}^{a} f(u)du \text{ (since } f \text{ is odd)} = -\int_{0}^{a} f(x)dx$$

Finally,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

Lemma 5.48 *If f is an even function and a is a number,*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

solution:
First of all, we evaluate $\int_{-a}^{0} f(x)dx$. Let $u = -x$ so that $du = -dx$; $u = a$ when $x = -a$ and $u = 0$ when $x = 0$. Thus,

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_{0}^{a} f(u)du \text{ (since } f \text{ is even)} = \int_{0}^{a} f(x)dx$$

Finally,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Example 5.49 *Evaluate*

$$\int_{-1}^{1} \frac{1 + x + x^2}{1 + x^2} dx.$$

solution:

$$\int_{-1}^{1} \frac{1 + x + x^2}{1 + x^2} dx = \int_{-1}^{1} \frac{x}{1 + x^2} dx + \int_{-1}^{1} \frac{1 + x^2}{1 + x^2} dx = \int_{-1}^{1} \frac{x}{1 + x^2} dx + \int_{-1}^{1} dx.$$

Now, $\frac{x}{1+x^2}$ is an odd function so that the first term on the RHS is zero. Obviously, $\int_{-1}^{1} dx = 2$. Therefore,

$$\int_{-1}^{1} \frac{1 + x + x^2}{1 + x^2} dx = 2.$$

Example 5.50 *Evaluate*

$$\int_{-2}^{2} (2x + 3)\sqrt{4 - x^2} dx.$$

solution:
First of all,

$$\int_{-2}^{2}(2x+3)\sqrt{4-x^2}dx = 2\int_{-2}^{2}x\sqrt{4-x^2}dx + 3\int_{-2}^{2}\sqrt{4-x^2}dx.$$

Since $x\sqrt{4-x^2}$ is an odd function. Its integral from -2 to 2 is zero. On the other hand,

$\int_{-2}^{2}\sqrt{4-x^2}dx$
$=$ area of the region under the graph of $\sqrt{4-x^2}$ from -2 to 2
$=$ area of the semi disc with radius 2
$= 2\pi$.

Thus,

$$\int_{-2}^{2}(2x+3)\sqrt{4-x^2}dx = 6\pi.$$