Error Bounds for M_n, T_n, S_{2n}

$$E_{M_n} = \int_a^b f(x)\,dx - M_n$$
$$T_n = \int_a^b f(x)\,dx - T_n$$
$$S_{2n} = \int_a^b f(x)\,dx - S_{2n}$$

Some Theoretical Error Bounds

① $|E_{T_n}| \le \dfrac{K(b-a)^3}{12n^2}$

② $|E_{M_n}| \le \dfrac{K(b-a)^3}{24n^2}$

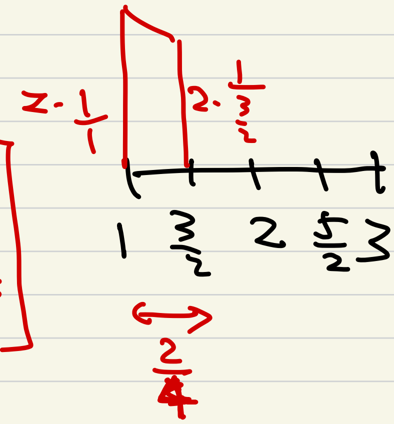
if $|f''(x)| \le K$ for some constant K for all $a \le x \le b$

③ $|S_{2n}| \le \dfrac{K(b-a)^5}{180(2n)^4}$ if $|f''''(x)| \le K$ for some constant K and for all $a \le x \le b$.

$2n$ = number of subintervals

Example $\displaystyle\int_1^3 \frac{1}{x}\,dx = \ln x \Big|_1^3 = \ln 3$

Suppose we use 4 subintervals to estimate the integral by T_4, M_4, S_4.



$$T_4 = \frac{1}{2}\,\underline{\frac{3-1}{4}}\left[\frac{1}{1} + 2\cdot\frac{1}{\frac{3}{2}} + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{5/2} + \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{1}{3}\right]$$

$$|E_{T_4}| \le \frac{2\cdot(3-1)^3}{12\cdot 4^2} = \frac{1}{12}$$

$$|E_{T_4}| \le \frac{1}{12}$$

$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$ decreasing

$$\boxed{\frac{2}{3^3} \le f''(x) \le \frac{2}{1^3}}$$

$1 \le x \le 3$

$|f''(x)| \le 2$ on $[1,3]$

Similarly

$$|E_{M_4}| \le \frac{1}{24}$$

Simpson's Rule

$$S_4 = \frac{K(b-a)^5}{180(2n)^4} = \frac{24\cdot 2^5}{180\cdot 4^4}$$

$$= \frac{2^3\cdot 3\cdot 2^5}{180\cdot 2^8} = \frac{1}{60}$$

$f'''(x) = -\frac{6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$

$|f^{(4)}(x)| \le \frac{24}{1^5}$ for all $x \in [1,3]$

$|E_{M_n}|$, $|E_{T_n}|$, $|E_{S_{2n}}|$?

What if we want to assure that these errors are less than 10^{-4}?

In general,

$$|E_{T_n}| \le \frac{2(2)^3}{12 \cdot n^2}$$

$$|E_{M_n}| \le \frac{2 \cdot 2^3}{24 \cdot n^2}$$

$$|E_{S_{2n}}| \le \frac{24 \cdot 2^5}{180(2n)^4}$$

How large an n we should choose?

Consider:

$$\frac{4}{3n^2} = \frac{2(2)^3}{12 \cdot n^2} < 10^{-4}$$

$$\frac{4}{6n^2} = \frac{2 \cdot 2^3}{24 \cdot n^2} < 10^{-4}$$

$$\frac{24 \cdot 2^5}{180(2n)^4} < 10^{-4}$$

$$\frac{4}{15} \frac{\cancel{16}}{\cancel{60}} = \frac{\cancel{2^8} \cdot \cancel{3}}{\cancel{180} \cdot \cancel{2^4} \cdot n^4}$$

T_n : $n^2 > \frac{4}{3} \cdot 10^4 = 115.5$

$n = 116$

M_n : $n^2 > \frac{2}{3} \cdot 10^4 = 81.7$

$n = 82$.

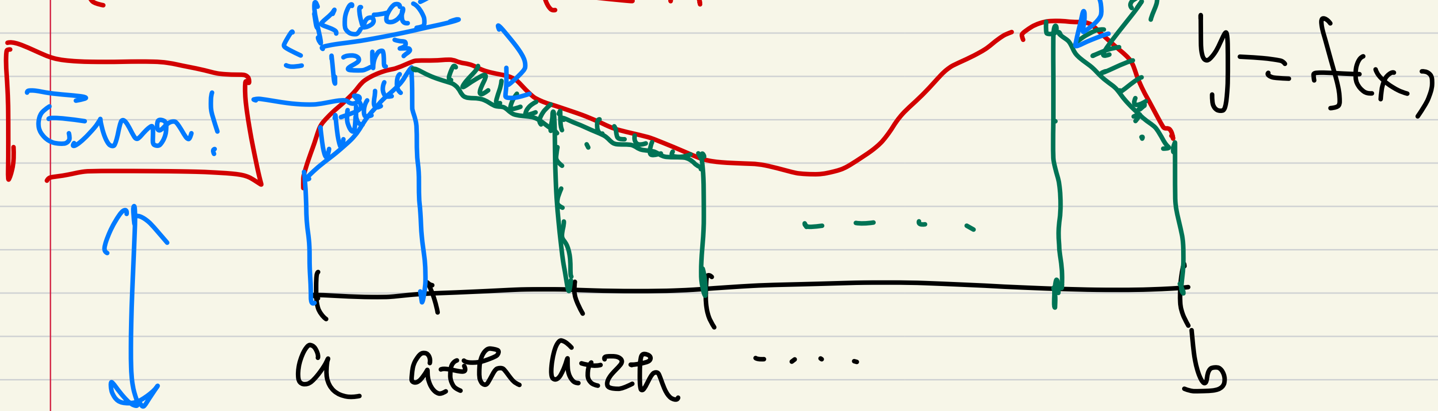
S_{2n} : $n^4 > \frac{4}{15} \cdot 10^4 \approx 9.5$

$n = 10$

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \ ?$$

$\frac{K(b-a)^3}{12n^3}$

error!



$$\left| \int_a^{a+h} f(x)\,dx - \boxed{\frac{1}{2} h [f(a) + f(a+h)]} \right| < ?$$

? not

$$\int_a^{a+h} f(x)\, d\left(x - \left(a + \frac{h}{2}\right)\right)$$

$$= f(x)\left(x - \left(a + \frac{h}{2}\right)\right)\Big|_a^{a+h} - \int_a^{a+h} \left(x - a - \frac{h}{2}\right) f'(x)\,dx$$

$$= f(a+h)\cdot\frac{h}{2} + f(a)\frac{h}{2} - \int_a^{a+h} \left(x - a - \frac{h}{2}\right) f'(x)\,dx$$

$$\boxed{\int_a^{a+h} f(x)\,dx - \frac{h}{2}[f(a) + f(a+h)]} = -\int_a^{a+h} \left(x - a - \frac{h}{2}\right) f'(x)\,dx$$

error of the approximation on the 1st subinterval

$$\boxed{\int_a^{a+h} f(x)\,dx - \frac{h}{2}\left[f(a)+f(a+h)\right]}$$

$$= -\int_a^{a+h}\left(x-a-\frac{h}{2}\right) f'(x)\,dx$$

$$= \int_a^{a+h} f'(x)\, \underbrace{d\left(\frac{(x-a)^2}{2}-\frac{h}{2}(x-a)\right)}_{\left(x-a-\frac{h}{2}\right)dx}$$

this term is zero at $x=a$ or $x=a+h$

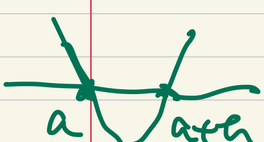
$$= \underline{f'(x)\left(\frac{(x-a)^2}{2}-\frac{h}{2}(x-a)\right)\Bigg|_a^{a+h}}$$

$= 0$

$$-\int_a^{a+h}\left(\frac{(x-a)^2}{2}-\frac{h}{2}(x-a)\right) f''(x)\,dx$$

$$\left|\text{Error on } 1^{st} \text{ subinterval}\right| \le \int_a^{a+h} |f''(x)|\left|\frac{(x-a)^2}{2}-\frac{h}{2}(x-a)\right| dx$$

$$\le K\int_a^{a+h}\left[\frac{h}{2}(x-a)-\frac{(x-a)^2}{2}\right] dx$$



$\frac{(x-a)^2}{2}-\frac{h}{2}(x-a)$

$$= K\left[\frac{h}{2}\frac{(x-a)^2}{2}-\frac{(x-a)^3}{6}\right]_a^{a+h}$$

$$= K\left[\frac{h^3}{4}-\frac{h^3}{6}\right] = \frac{Kh^3}{12}$$

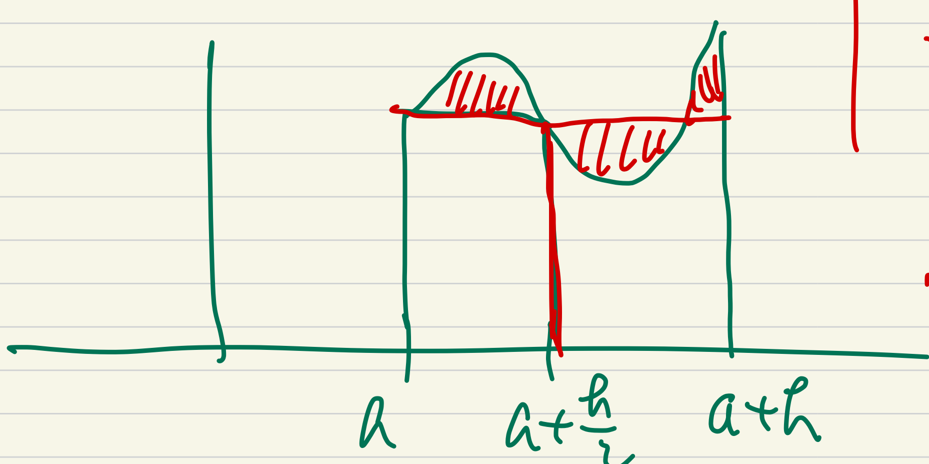
$h = \frac{b-a}{n}$,

$$= K\left(\frac{b-a}{n}\right)^3 = \frac{K(b-a)^3}{12n^3}$$

$$|E_{T_n}| \le \underbrace{\frac{K(b-a)^3}{12n^3} + \frac{K(b-a)^3}{12n^3} + \cdots + \frac{K(b-a)^3}{12n^3}}_{n \text{ terms}}$$

$$= \frac{K(b-a)^3}{12\,n^2}.$$

$$|E_{M_n}| \le \frac{K(b-a)^3}{24\,n^2}$$



$$\left| \int_a^{a+\frac{h}{2}} f(x)\,dx - \frac{h}{2}\cdot f\!\left(a+\frac{h}{2}\right) \right| \le \frac{K(b-a)^3}{24\,n^3}$$

$$h$$

$$\frac{h}{2} = \left(\frac{b-a}{2n}\right)$$