FINA 1303

FOUNDATIONS OF INTEREST RATES Part I

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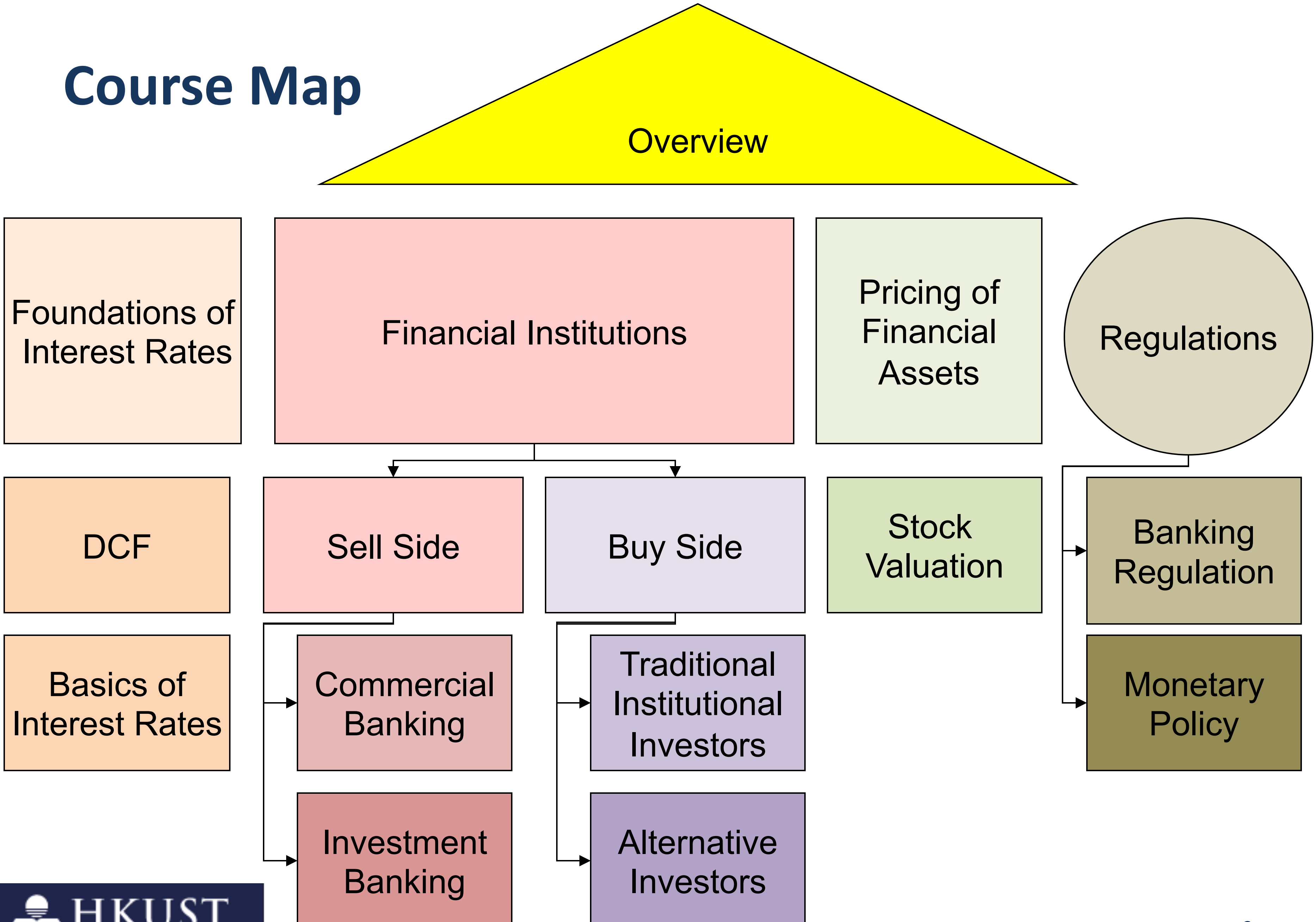
Course Map


Overview
Foundations of Interest Rates
Financial Institutions
Pricing of Financial Assets
Regulations
DCF
Sell Side
Buy Side
Stock Valuation
Banking Regulation
Basics of Interest Rates
Commercial Banking
Traditional Institutional Investors
Monetary Policy
Investment Banking
Alternative Investors

Overview

Foundations of Interest Rates

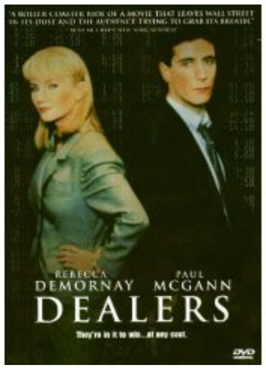
DCF

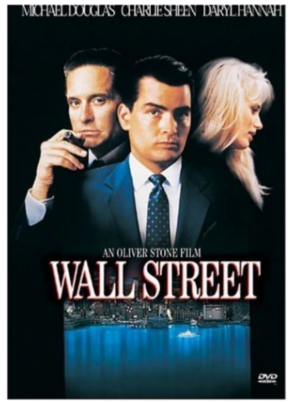
Basics of Interest Rates

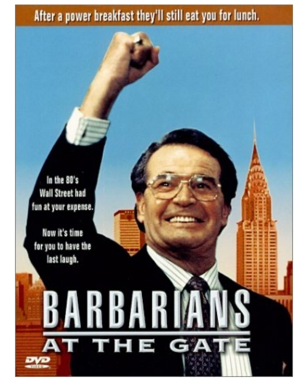
Students will establish an understanding of

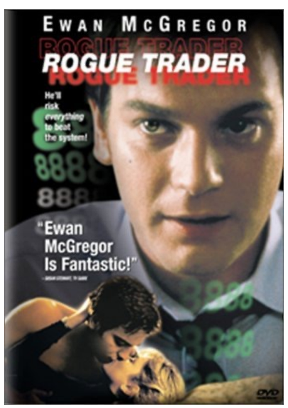
1. Time and the value of payments
2. Present value versus future value
3. DCF
4. Basic Formulas

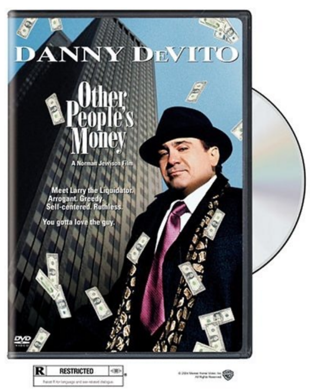
Get a Glimpse: Movies

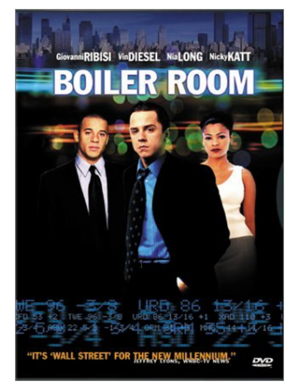




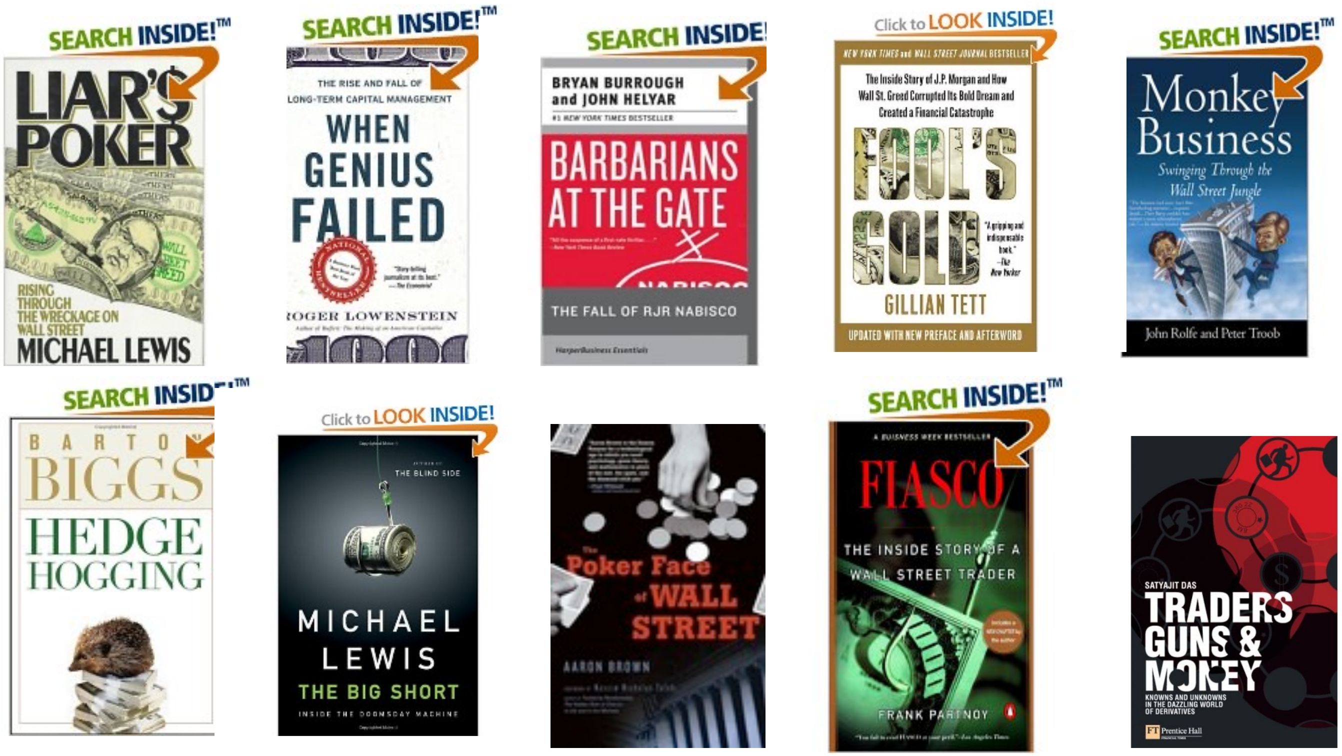








Get a Glimpse: Books



Get a Glimpse: Newsletters and Blogs

- General:
  - Matt Levine “Money Stuff”: https://www.bloomberg.com/opinion/authors/ARbTQlRLRjE/matthew-s-levine
  - Marc Rubinstein “Net Interest”: https://netinterest.substack.com/
  - Barry Ritholtz “The Big Picture”: https://ritholtz.com/
  - John Authers: https://www.bloomberg.com/opinion/authors/AT2bBytfUHQ/john-authers
- Corporate Finance:
  - Vernimmen: http://www.vernimmen.com/ (check out the newsletter)
  - Damodaran: http://aswathdamodaran.blogspot.com/

FINA 1303 – **Basics of financial calculations**

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Agenda

- Part 1: Time Value of Money
  - Present Value
  - Future Value
- Part 2: Valuing a Stream of Cash Flows
- Part 3: Applications
  - IRR
  - Mortgages

Interest in History & Islamic Finance

- Throughout history, money lenders have been unpopular because they charge interest
- Shakespeare's famous play "The Merchant of Venice" is an illustration of this
  - Christianity banned the charging of interest (usury) for centuries, and only Jews were involved in providing credit
- Today, Islamic Finance prohibits the payment and charging of interest (*riba*) which is contrary to Islamic laws (*sharia*). Yet, the Islamic Finance market is growing fast including in Hong Kong, where the HKSAR Government has issued two *sukuks* (Islamic bonds)

Important things to know before we start!

- In this section, unless otherwise indicated, interest rates are **yearly** (annual) interest rates (in the markets we use the term “per annum” or “p.a.”)
- Financial institutions are generally required to indicate the interest rates they charge on an *annual percentage rate* (**APR**) basis so that consumers can compare prices
- The way we calculate interest payments differs depending on the financial instrument we use. You need to verify the applicable conditions in the contract!
- In particular, debt contracts of less than one year and money market securities use simpler formulas (we will discuss this later)

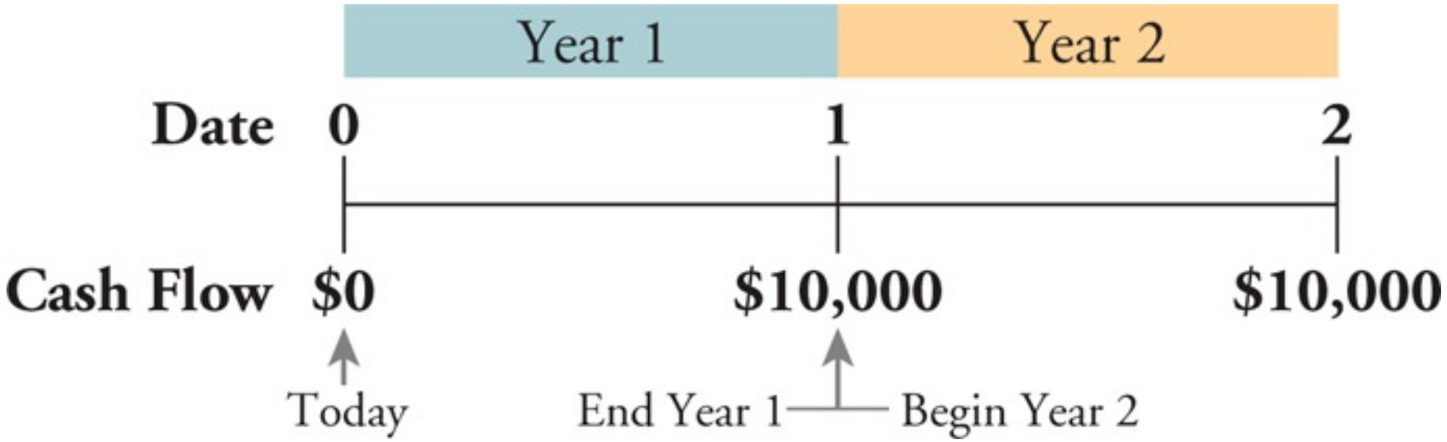
Part 1: Time Value of Money

Valuing Monetary Payments now and in the Future: Introducing DCF ("Discounted Cash Flows")

- How do we compare the return on different financial instruments?
  - Different debt instruments have different streams of cash payments to the holder (known as **Cash Flows**), with different **timing**.
- How and why is the promise to pay $X on T1 more or less valuable than the promise to pay $Y on T2?
- To find out, we use **DCF calculations**

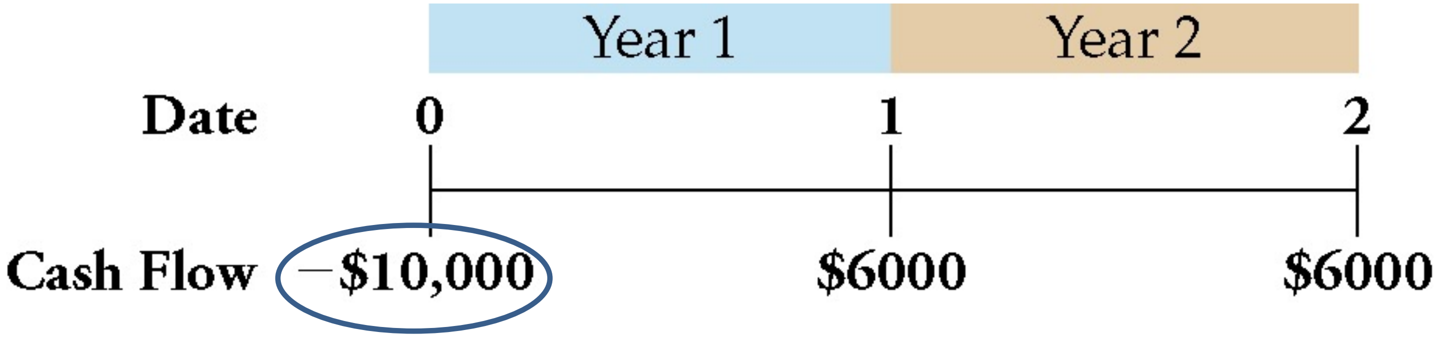
Timelines

- Constructing a Timeline
  - Identifying Dates on a Timeline
    - Date 0 is today, the beginning of the first year
    - Date 1 is the end of the first year



Timelines

- Distinguishing Cash Inflows from Outflows (negative signs indicate outflows)



- Representing Various Time Periods
  - Indicate the label :“Year” or “Month” or …..

DCF Introduction

- The concept of discounted value is based on the notion (assumption) that a dollar of cash flow paid to you some time from now is less valuable to you than a dollar paid to you today: "a dollar is worth more today than tomorrow" or the **time value of money**
  - Why? This notion has been GENERALLY true because you could invest the dollar in a savings account that earns interest
  - BUT in an era of NEGATIVE interest rates this is no longer true everywhere
- We will learn basic pricing of financial instruments **under the assumption of positive interest rates**. Negative interest rates change significantly our allocation of resources and distort financial markets.

Class Discussion

- What do you think will happen to interest rates and why?

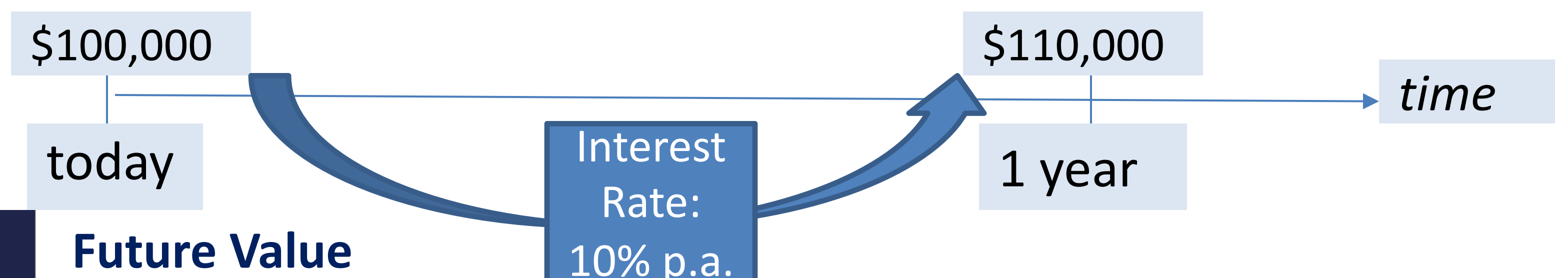
The Time Value of Money and Interest Rates

- The Interest Rate: Converting Cash Across Time
  - Present Value PV
    - The value of a cost or benefit computed in terms of cash **today**
  - Future Value FV
    - The value of a cash flow that is moved forward **in time**

Future Value and Interest

- **Future value** is the value on a later date of an investment today.
  - $100,000 invested today at 10% interest gives $110,000 in a year.

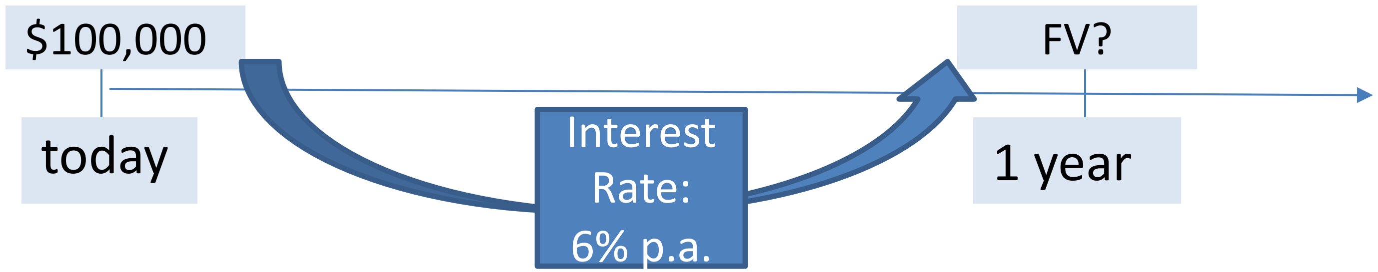
    => the **future value** of $100,000 today at 10% interest is $110,000 **one year from now**.
  - The $100,000 *yields* $10,000, which is why interest rates are sometimes called **yield** or yield to maturity.



Future Value

Future Value example (single period)

- Calculate the value of $100,000 Investment in 1 year at a rate of 6% (p.a.)

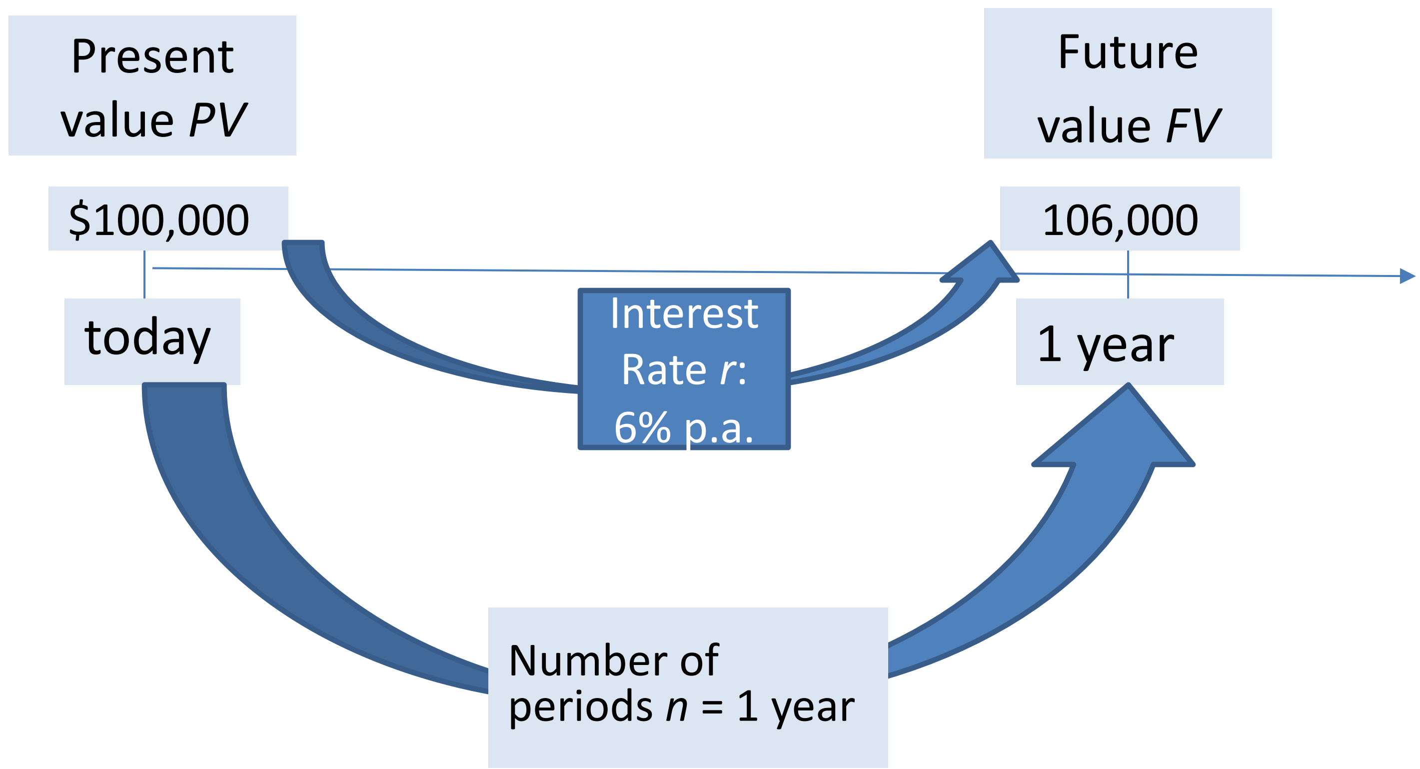


Future Value example (single period)

- the cost of the investment is:
  - ($100,000 today) + ($100,000 x 6% p.a.) = $106,000 in one year
  - We can also express it as:
  - ($100,000 today) $\times$ (1.06 $ in one year/$ today) = $106,000 in one year
- $106,000 is the **opportunity cost** of spending $100,000 today
  - The firm gives up the $106,000 it would have had in one year if it had left the money in the bank
  - Alternatively, by borrowing the $100,000 from the same bank, the firm would owe $106,000 in one year.

Future Value example (single period)

- Calculate the value of $100,000 Investment in 1 year at a rate of 6% (p.a.)



Future Value and Interest (single period)

- Notice that
  - If the interest rate **r** is **10%**, the future value **FV** in one year is:
    - $100,000 + ($100,000 x 0.10) = $110,000
  - If the interest rate **r** is **6%**, the future value **FV** in one year is:
    - $100,000 + ($100,000 x 0.06) = $106,000
- **The higher the interest rate, the higher the future value**.
- $\boxed{FV = PV + PV \times r = PV \times (1 + r)}$

Future Value and Compound Interest

- The higher the interest rate, the higher the future value.
- The higher the amount invested, the higher the future value.
- Most financial instruments are not so simple, so what happens when time to repayment varies?
- When using one-year interest rates to compute the value repaid **more** than one year from now, we must consider compound interest.

Compound interest is the interest on the interest.

Future Value and Compound Interest

- What if you leave your $100 in the bank for two years at 6% yearly interest rate?
  - Yearly: Calculated and credited at the end of each year!
- The future value is:

$$\$100 + (\$100 \times 0.06) + (\$100 \times 0.06) + (\$6 \times 0.06) = \$112.36$$

$$\$100 \times 1.06 \times 1.06 = \$100 \times (1.06)^2$$

- In general

$$\boxed{FV_n = PV \times (1 + r)^n}$$

Future Value and Compound Interest

Today		**Year 1**		**Year 2**
$100	Principal	$100	Principal	$100
			Interests from Y2 on Principal	$100 x 6% = $6
	Interests from Y1 on Principal	$100 x 6% = $6	Interests from Y1	$6
			Interests on Interests	$6 x 6% = $0.26

$100 Year 0 Principal + 6% => $106 Year 1 Principal + 6% => $ 112.36

Future Value and Compound Interest

Future Value of $100 at 6.00% annual interest

Nbr Years	Calculations	Future Value
1	$\$100 \times (1+6\%)$	$ 106.00
2	$\$100 \times (1+6\%)^2$	$ 112.36
3	$\$100 \times (1+6\%)^3$	$ 119.10
4	$\$100 \times (1+6\%)^4$	$ 126.25
5	$\$100 \times (1+6\%)^5$	$ 133.82
6	$\$100 \times (1+6\%)^6$	$ 141.85
7	$\$100 \times (1+6\%)^7$	$ 150.36
8	$\$100 \times (1+6\%)^8$	$ 159.38
9	$\$100 \times (1+6\%)^9$	$ 168.95
10	$\$100 \times (1+6\%)^{10}$	$ 179.08

Pop Quiz!

- The time value of money: “a dollar is worth _____today than tomorrow”
- The higher the interest rate, the ______ the future value.
- The higher the amount invested, the ______ the future value.
- Present Value = The value of a cost or benefit computed in terms of cash _____
- Compound interest is _________________

Compounding Frequency

- If the *annual* interest rate is 6%, what is the equivalent *monthly* rate if interests are calculated and credited on a monthly basis?
- Assume j is a one-month interest rate and n is the number of months, then a one year (n = 12) deposit of \$100 will have a future value of: \$100 x $(1 + j)^{12}$.

Compounding Frequency

- In one year the future value is \$100 x 1.06 so we can find j:

$$(1 + j)^{12} = 1.06$$

$$(1 + j) = (1.06)^{1/12} = 1.0049$$

$$j = \mathbf{0.49\%} = 49\text{bp}$$

- A **basis point** **(bp)** is one one-hundredth of a percentage point, 0.01 percent.

Changing Compounding Frequency

- The frequency of the compounding (calculating and crediting interest) changes substantially the future value

- What if compounding is:
  - 12% annual,
  - 6% semi-annual,
  - 3% quarterly,
  - 1% monthly?

Changing Compounding Frequency

Final Value	Annual	Semi-Annual	Quarterly	Monthly
Months	12%	6%	3%	1%
0	$100.00	$100.00	$100.00	$100.00
1				$ 101.00
2				$ 102.01
3			$ 103.00	$ 103.03
4				$ 104.06
5				$ 105.10
6		$ 106.00	$ 106.09	$ 106.15
7				$ 107.21
8				$ 108.29
9			$ 109.27	$ 109.37
10				$ 110.46
11				$ 111.57
12	$ 112.00	$ 112.36	$ 112.55	$ 112.68
Difference		$ 0.36	$ 0.55	$ 0.68

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Compounding

Changing Compounding Frequency

Final Value	Annual	Semi-Annual	Quarterly	Monthly
Months	24%	12%	6%	2%
0	$100.00	$100.00	$100.00	$100.00
1				$ 102.00
2				$ 104.04
3			$ 106.00	$ 106.12
4				$ 108.24
5				$ 110.41
6		$ 112.00	$ 112.36	$ 112.62
7				$ 114.87
8				$ 117.17
9			$ 119.10	$ 119.51
10				$ 121.90
11				$ 124.34
12	$ 124.00	$ 125.44	$ 126.25	$ 126.82
Difference		$ 1.44	$ 2.25	$ 2.82

With higher interest rates, the difference from a higher compounding frequency is even larger

Present Value

- Financial instruments promise future cash payments. Valuing those payments under the same or consistent assumptions enables us to compare several alternatives.
- **Present value** is the value **today** of a payment to be made in **the future**.
- Present value is the equivalent amount to invest today in order to be equal to a specific amount on a given future date.

Present Value for 1 Period

- Knowing the Future Value, we can find easily the Present Value:

$$FV = PV \times (1+r), \text{ so}$$

$$\boxed{PV = \frac{FV}{(1+r)}}$$

- This is just the future value calculation inverted.

Present Value for *n* Periods

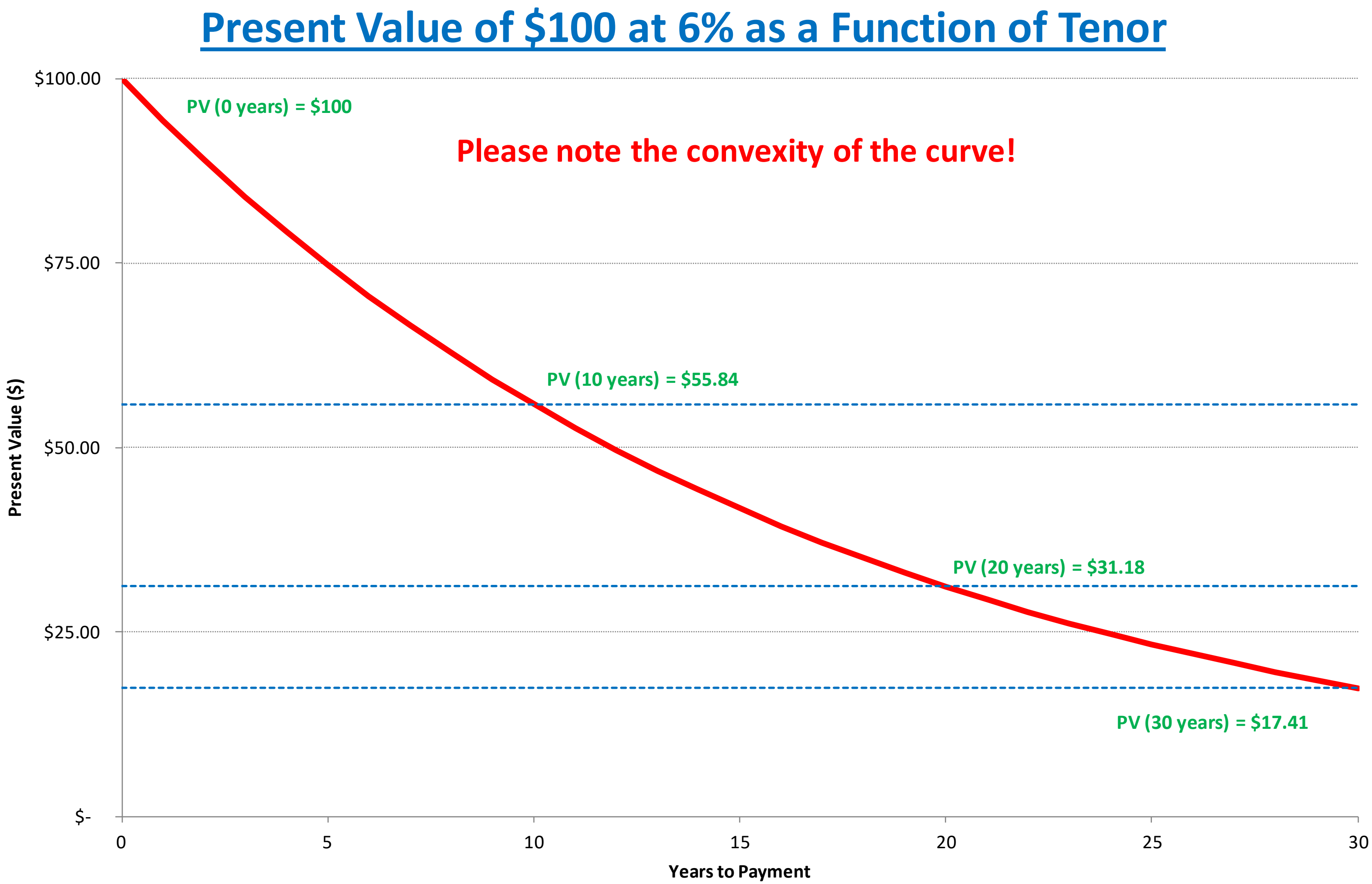
- We can generalize the process as we did for future value.
- Present Value of one payment credited ***n*** years in the future, but calculated at the end of each year:

$$\boxed{\text{PV} = \frac{FV}{(1+r)^n}}$$

Present Value: THE Basis for DCF

- The present value is higher:
  - The higher future value of the payment, FVn.
  - The shorter time period until payment, ***n***.
  - The lower the interest rate, *r*.
- **Present value is the single most important relationship in our study of financial instruments. It is THE basis for Discounted Cash Flows analysis.**

Present Value as a Function of Tenor



Present Value of $100 Payment

	PV		PV		PV	
Rate/Tenor (Yr)	1	Difference	5	Difference	10	Difference
1.00%	$99.01		$95.15		$90.53	
2.00%	$98.04	$ (0.97)	$90.57	$ (4.57)	$82.03	$ (8.49)
3.00%	$97.09		$86.26		$74.41	
4.00%	$96.15		$82.19		$67.56	
5.00%	$95.24		$78.35		$61.39	
6.00%	$94.34	$ (0.90)	$74.73	$ (3.63)	$55.84	$ (5.55)
7.00%	$93.46		$71.30		$50.83	
8.00%	$92.59		$68.06		$46.32	
9.00%	$91.74		$64.99		$42.24	
10.00%	$90.91	$ (0.83)	$62.09	$ (2.90)	$38.55	$ (3.69)
11.00%	$90.09		$59.35		$35.22	
12.00%	$89.29		$56.74		$32.20	
13.00%	$88.50		$54.28		$29.46	
14.00%	$87.72		$51.94		$26.97	
15.00%	$86.96	$ (0.76)	$49.72	$ (2.22)	$24.72	$ (2.26)

- Higher interest rates are associated with lower PV no matter the timing of the payment.
- At any interest rate, a longer tenor reduces PV.
- For 1% interest rate variation, the difference in PV varies with rate and tenor: Convexity

Example 1: Comparing Revenues at Different Points in Time

Problem:

- The launch of Sony's PlayStation 3 was delayed until November 2006, giving Microsoft's Xbox 360 a full year on the market without competition. Sony did not repeat this mistake in 2013 when the PS4 launched at the same time as the Xbox One.
- It is November 2005 and you are the marketing manager for the PlayStation. You estimate that if the PlayStation 3 were ready to be launched immediately, you could sell **$2 billion** worth of the console in its first year.
- However, if your launch is delayed a year, you believe that Microsoft's head start will reduce your first-year sales by **20%** to **$1.6 billion**.
- If the interest rate is **8%**, what is the cost of a delay in terms of dollars in 2005?

Example 1: Comparing Revenues at Different Points in Time

Solution:

Plan:

- Revenues if released today: $2 billion
- Revenue if delayed: $1.6 billion
- Interest rate: 8%
- We need to compute the revenues if the launch is delayed and compare them to the revenues from launching today.
- However, in order to make a fair comparison, we need to convert the future revenues of the PlayStation if delayed into an equivalent present value of those revenues today.

Example 1: Comparing Revenues at Different Points in Time

Execute:

- If the launch is delayed to 2006, revenues will drop by 20% of $2 billion, or $400 million, to $1.6 billion.
- To compare this amount to revenues of $2 billion if launched in 2005, we must convert it using the interest rate of 8% (this is calculating the present value of the $1.6Bn):

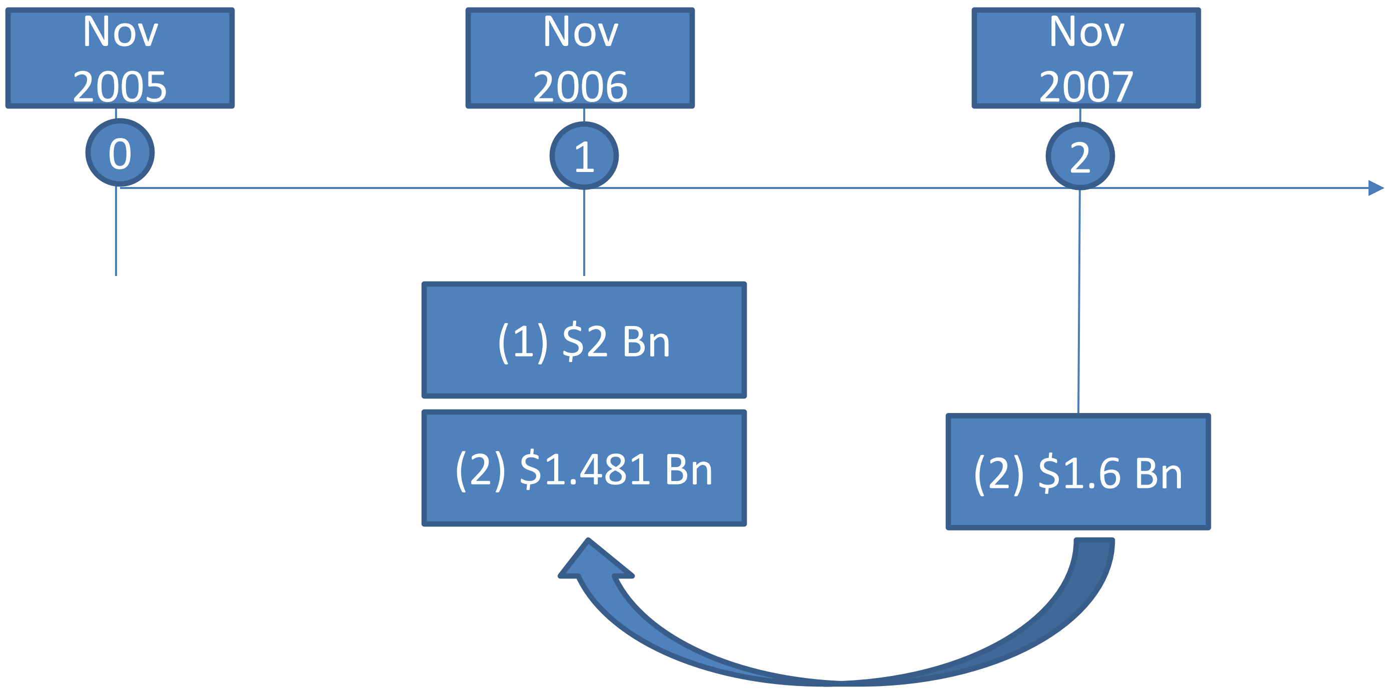
  $1.6 billion/1.08 = $1.481 billion in 2005
- Therefore, the cost of a delay of one year is

  $2 billion - $1.481 billion = **$0.519 billion ($519 million).**

$$\text{PV} = \frac{FV}{(1+r)^n}$$

In other words, we multiply the future value ($1.6Bn) by the discount factor (1/1+r) to get the present value

Example 1: Comparing Revenues at Different Points in Time



Example 1: Comparing Revenues at Different Points in Time

Evaluate:

- Delaying the project for one year was equivalent to giving up $519 million in cash.
- In this example, we focused only on the effect on the first year's revenues. However, delaying the launch delays the entire revenue stream by one year, so the total cost would be calculated in the same way by summing the cost of delay for each year of revenues.

Your Turn!

Problem:

- The launch of Sony's PlayStation 3 was delayed until November 2006, giving Microsoft's Xbox 360 a full year on the market without competition.
- It is November 2005 and you are the marketing manager for the PlayStation. You estimate that if the PlayStation 3 were ready to be launched immediately, you could sell **$3 billion** worth of the console in its first year.
- However, if your launch is delayed a year, you believe that Microsoft's head start will reduce your first-year sales by **35%**.
- If the interest rate is **6%**, what is the cost of a delay in terms of dollars in 2005?

Your Turn! (PRS, please)

- The cost of delay is
  - $519 million
  - $1.160 billion
  - $2 billion
  - I don't care I don't like video games

Solution to Example: Comparing Revenues at Different Points in Time

Solution:

Plan:

- Revenues if released today: \$3 billion,
- Revenue decrease if delayed: 35% ,
- Interest rate: 6%
- We need to compute the revenues if the launch is delayed and compare them to the revenues from launching today.
- However, in order to make a fair comparison, we need to convert the future revenues of the PlayStation if delayed into an equivalent present value of those revenues today.

Solution to Example: Comparing Revenues at Different Points in Time

Execute:

- If the launch is delayed to 2006, revenues will drop by 35% of \$3 billion, or \$1.05 billion, to \$1.95 billion.
- To compare this amount to revenues of \$3 billion if launched in 2005, we must convert it using the interest rate of 6%:

  \$1.95 billion/1.06 = \$1.840 billion in 2005

$$\text{PV} = \frac{FV}{(1+r)^n}$$

- Therefore, the cost of a delay of one year is

  \$3 billion - \$1.840 billion = \$1.160 billion

Solution to Example: Comparing Revenues at Different Points in Time

Evaluate:

- Delaying the project for one year was equivalent to giving up $1.16 billion in cash.
- In this example, we focused only on the effect on the first year's revenues.
- However, delaying the launch delays the entire revenue stream by one year, so the total cost would be calculated in the same way by summing the cost of delay for each year of revenues.

The Three Rules of Valuing Cash Flows

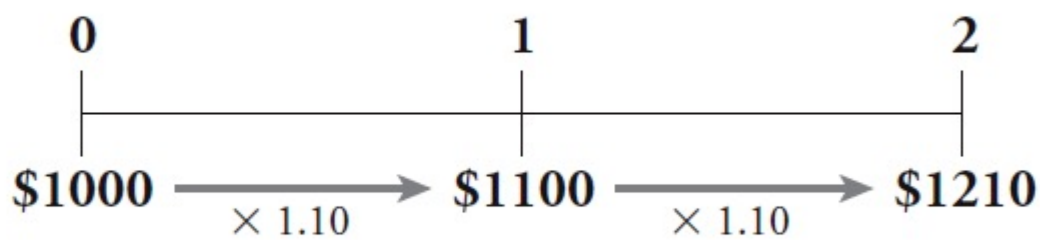
Rule	Formula
1: Only values at the same point in time can be compared or combined.	None
2: To calculate a cash flow's future value, we must compound it.	Future value of a cash flow: $FV_n = C \times (1 + r)^n$
3: To calculate the present value of a future cash flow, we must discount it.	Present value of a cash flow: $PV = C \div (1 + r)^n = \dfrac{C}{(1 + r)^n}$

Valuing Cash Flows at Different Points in Time

- **Rule 1**: Comparing and Combining Values
  - It is only possible to compare or combine values **at the same point in time**

Valuing Cash Flows at Different Points in Time

- **Rule 2**: Compounding
  - To calculate a cash flow's **future value**, you must **compound** it



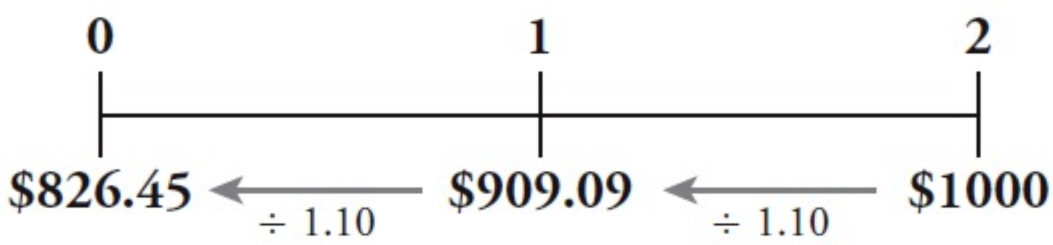
  - **Compound Interest = The effect of earning "interest on interest"**

Future Value of a Cash Flow

$$FV_n = C \times \underbrace{(1+r) \times (1+r) \times \cdots \times (1+r)}_{n \text{ times}} = C \times (1+r)^n \qquad (3.1)$$

Valuing Cash Flows at Different Points in Time

- **Rule 3**: Discounting
  - To calculate the value of a future cash flow at an earlier point in time, we must **discount** it.

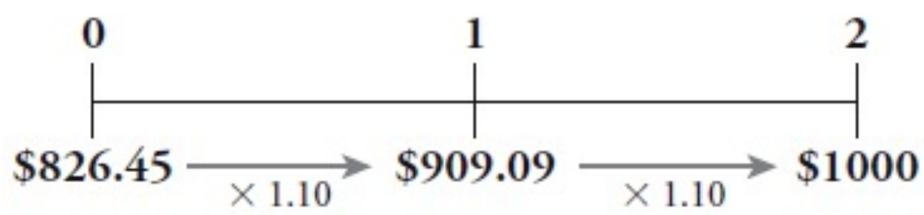


Present Value of a Cash Flow

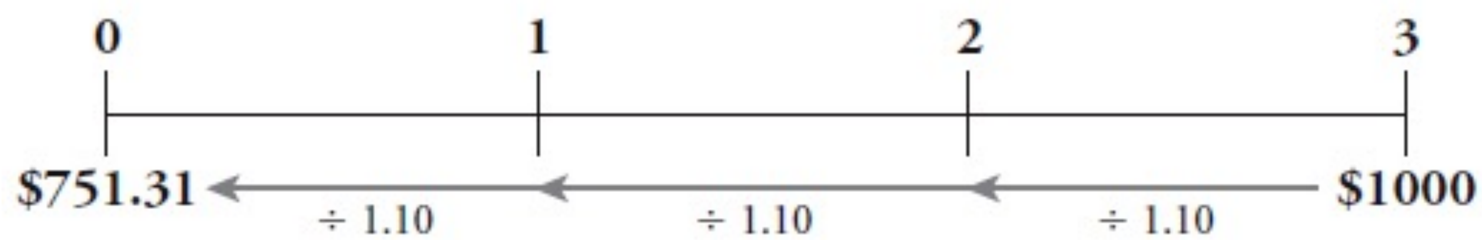
$$PV = C \div (1 + r)^n = \frac{C}{(1 + r)^n} \tag{3.2}$$

Valuing Cash Flows at Different Points in Time

- Rule 3: Discounting & Compounding **- Example**
  - If \$826.45 is invested today for **two** years at 10% interest, the **future value** will be \$1,000-



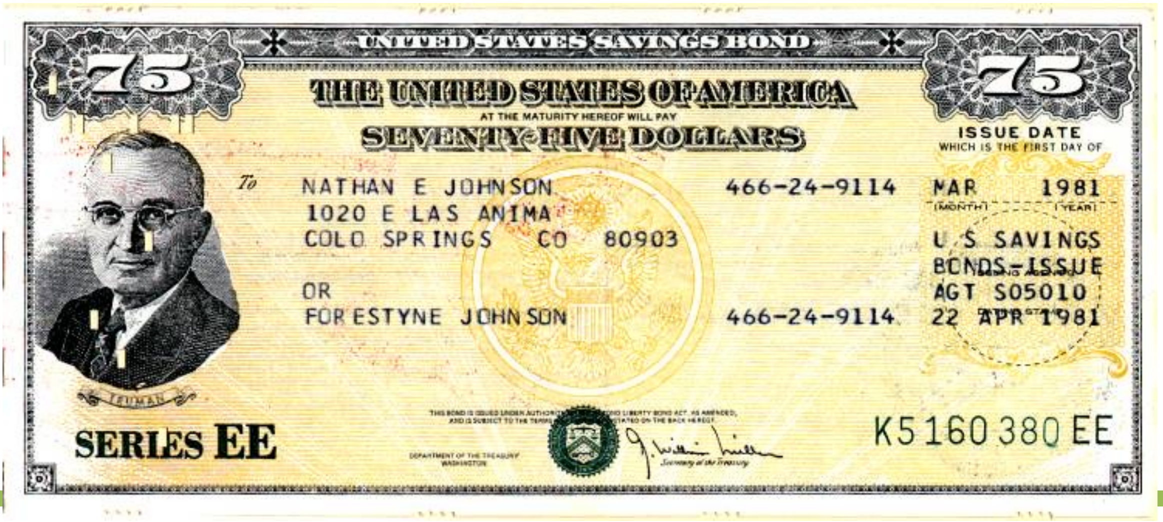
  - If \$1,000- were **three** years away, the **present value**, if the interest rate is 10%, would be \$751.31



Example: Present Value of a Single Future Cash Flow

Problem:

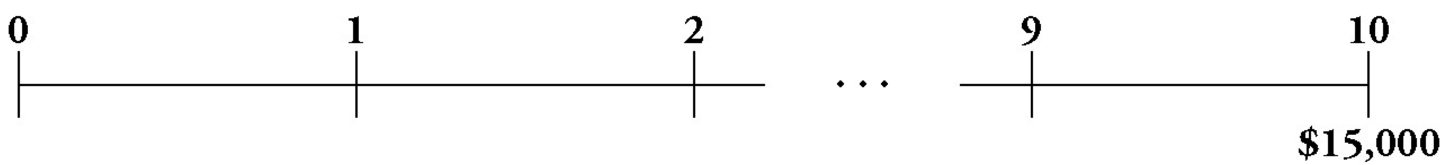
- You are considering investing in a savings bond that will pay **$15,000** in **10 years**.
- If the competitive market interest rate is fixed at **6%** per year, **what is the bond worth today?**



Example: Present Value of a Single Future Cash Flow

Solution:

- First set up your timeline. The cash flows for this bond are represented by the following timeline:



- Thus, the bond is worth $15,000 in 10 years. To determine the value today, we compute the present value using the PV equation and our interest rate of 6%.

Present Value of a Cash Flow

$$PV = C \div (1 + r)^n = \frac{C}{(1 + r)^n}$$

Where C is the single cash flow – in this case, it's also the Future Value

Example: Present Value of a Single Future Cash Flow

Execute: $$PV = \frac{15,000}{1.06^{10}} = \$8,375.92 \text{ today}$$

Using the financial calculator we can input the relevant variables

	N	I/Y	PV	PMT	FV
Given:	10	6		0	15,000
Solve for:			**-8,375.92**		
Excel Formula: =PV(RATE,NPER, PMT, FV) = PV(0.06,10,0,15000)					

Example: Present Value of a Single Future Cash Flow

Evaluate:

- The bond is worth much less today than its final payoff because of the time value of money.

Food for thought:
What if interest rates were lower?

Your Turn!

Problem:

- XYZ Company expects to receive a cash flow of **$2 million** in **five** years.
- If the competitive market interest rate is fixed at **4%** per year, how much can they borrow **today** in order to be able to repay the loan in its entirety with that cash flow?

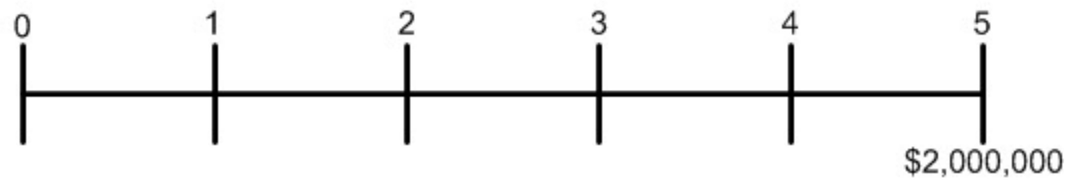
Your Turn! (PRS, please)

- We can borrow
  - $1.5 million
  - $1.6 million
  - $2 million
- Borrowing is bad, it leads to trouble!

Solution to Example: Present Value of a Single Future Cash Flow

Solution:

- First set up your timeline. The cash flows for the loan are represented by the following timeline:



- Thus, XYZ Company will be able to repay the loan with its expected $2 million cash flow in five years. To determine the value today, we compute the present value using our interest rate of 4%.

Present Value of a Cash Flow

$$PV = C \div (1 + r)^n = \frac{C}{(1 + r)^n}$$

Where C is the single cash flow – in this case, it's also the Future Value

Solution to Example: Present Value of a Single Future Cash Flow

Execute:

$$PV = \frac{\$2,000,000}{(1.04)^5} = \$1,643,854.21$$

Using the financial calculator we can input the relevant variables

	N	I/Y	PV	PMT	FV
Given:	5	4		0	2,000,000
Solve for:			**-1,643,854.21**		
Excel Formula: =PV(RATE,NPER, PMT, FV) = PV(0.04,5,0,2000000)					

Solution to Example: Present Value of a Single Future Cash Flow

Evaluate:

- The loan is much less than the $2 million the company will pay back because of the time value of money.

Part 2: Valuing a Stream of Cash Flows

Valuing a Stream of Cash Flows

Applying the Rules of Valuing Cash Flows

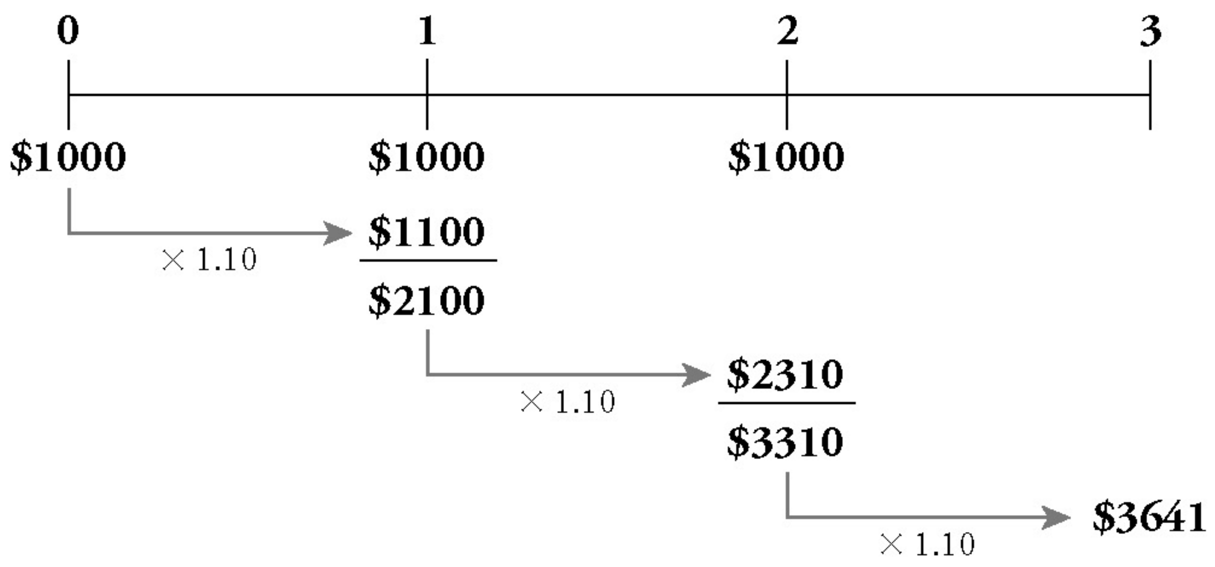
- Suppose we plan to save $1,000 today, and $1,000 at the end of each of the next two years
- If we earn a fixed 10% interest rate on our savings, how much will we have three years from today (future value)?

We can do this in several ways

- First, take the deposit at date 0 and move it forward to date 1
- Combine those two amounts and move the combined total forward to date 2

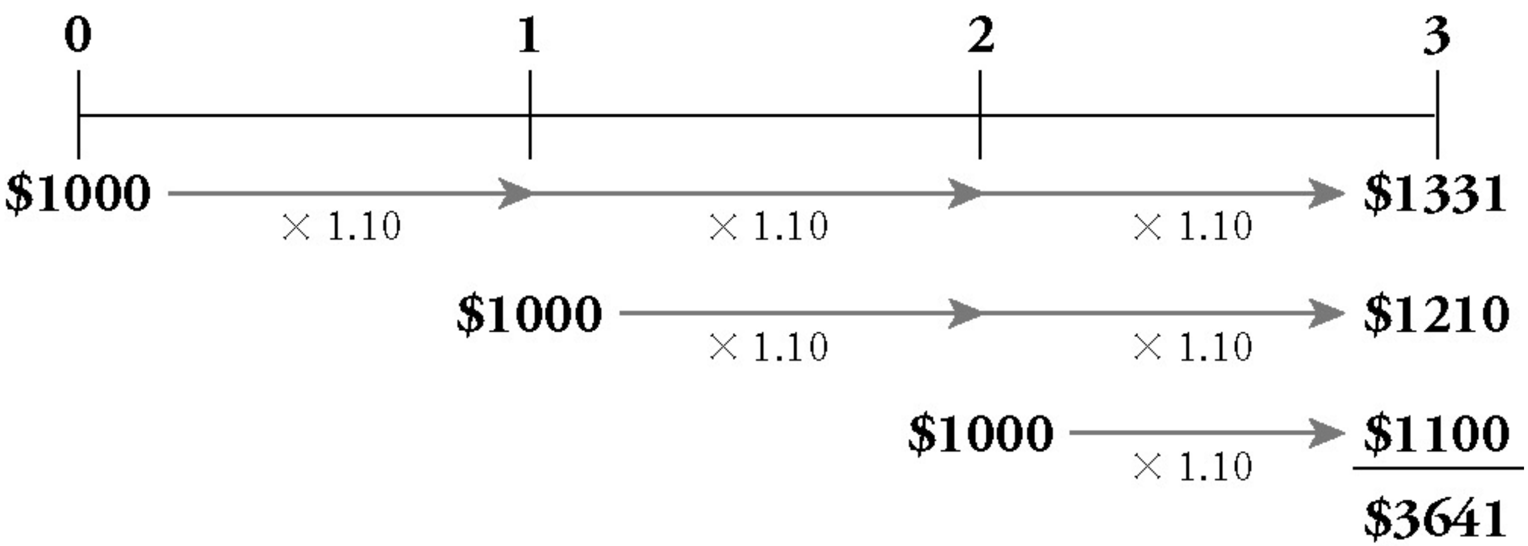
Valuing a Stream of Cash Flows

- Continuing in the same fashion, we can solve the problem as follows:



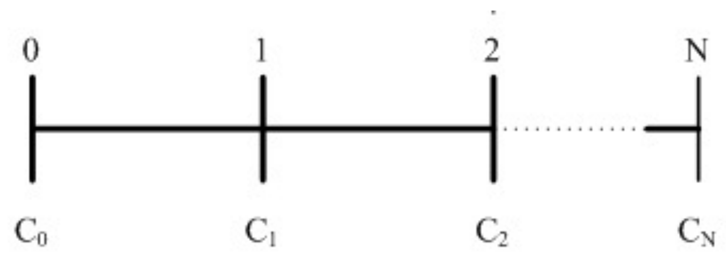
Valuing a Stream of Cash Flows

- Another approach is to compute the **future value** in year 3 of **each cash flow** separately
- Once all amounts are in year 3 dollars, combine them



Valuing a Stream of Cash Flows

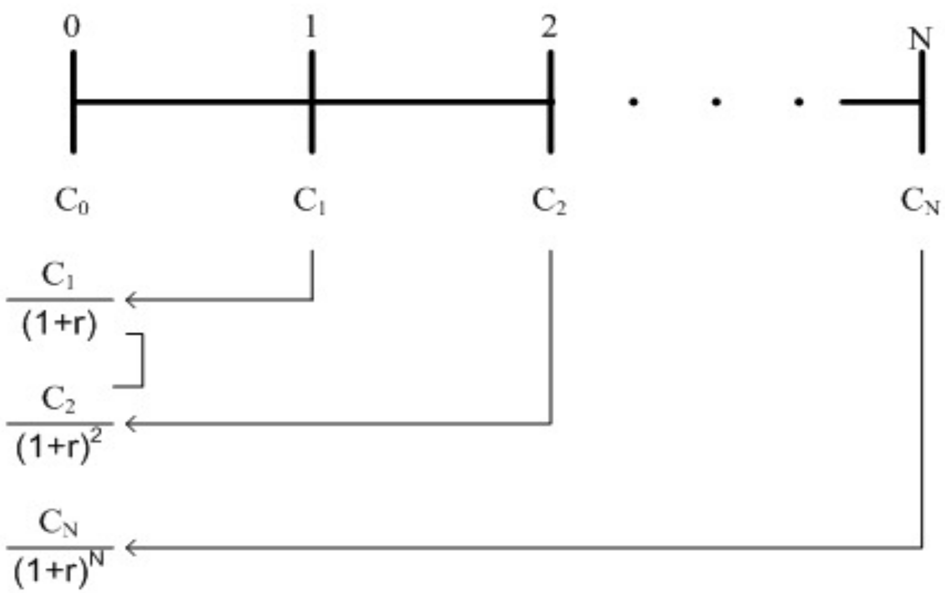
- Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, and so on, up to C_N at date N



- We compute the **present value** of this cash flow stream in two steps

Valuing a Stream of Cash Flows

- First, compute the present value of each cash flow
- Then combine the present values: the PV of a stream of cash flows is the sum of the present values of each cash flow



Valuing a Stream of Cash Flows

- **The present value of a cash flow stream is the sum of the present values of each cash flow**

$$PV = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N}$$

Your Turn!

Problem:

- You have just graduated and need money to buy a laptop
- Your aunt will lend you the money so long as you agree to pay her back **within six years**.
- You offer to pay her the rate of interest that she would otherwise get by putting her money in a savings account.
- Based on your earnings and living expenses, you think you will be able to pay her **$70** next year, **$85** in each of the next two years, and then **$90** each year for years 4 through 6.
- If your aunt would otherwise earn **0.5%** per year on her savings, **how much can you borrow from her?**

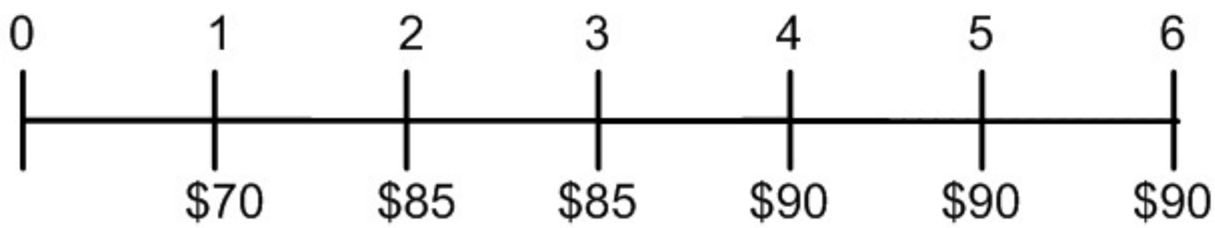
Your Turn! (PRS, please)

- We can borrow
  - $600.7
  - $500.9
  - $400.5
- Borrowing is bad, it leads to trouble!

Solution : Present Value of a Stream of Cash Flows

Plan:

- The cash flows you can promise your aunt are as follows:



- She should be willing to give you an amount equal to these payments in **present value** terms.

Solution : Present Value of a Stream of Cash Flows (cont'd)

Plan:

- We will:
  - Solve the problem using the equation

$$PV = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_N}{(1+r)^N}$$

  - the present value of a series of cash flows is the sum of the present values of each of the cash flows; so we calculate the present value of each cash flow and add up all the present values
  - Verify our answer by calculating the **future value** of this amount.

Solution : Present Value of a Stream of Cash Flows (cont'd)

Execute:

- We can calculate the PV as follows:

$$PV = \frac{70}{1.005} + \frac{85}{1.005^2} + \frac{85}{1.005^3} + \frac{90}{1.005^4} + \frac{90}{1.005^5} + \frac{90}{1.005^6}$$
$$= \$69.65 + \$84.16 + \$83.74 + \$88.22 + \$87.78 + \$87.35$$
$$\boxed{= \$500.90}$$

Solution : Present Value of a Stream of Cash Flows (cont'd)

Part 2

- Now, suppose that your aunt gives you the money, and then deposits your payments in the bank each year.
- How much will she have six years from now?

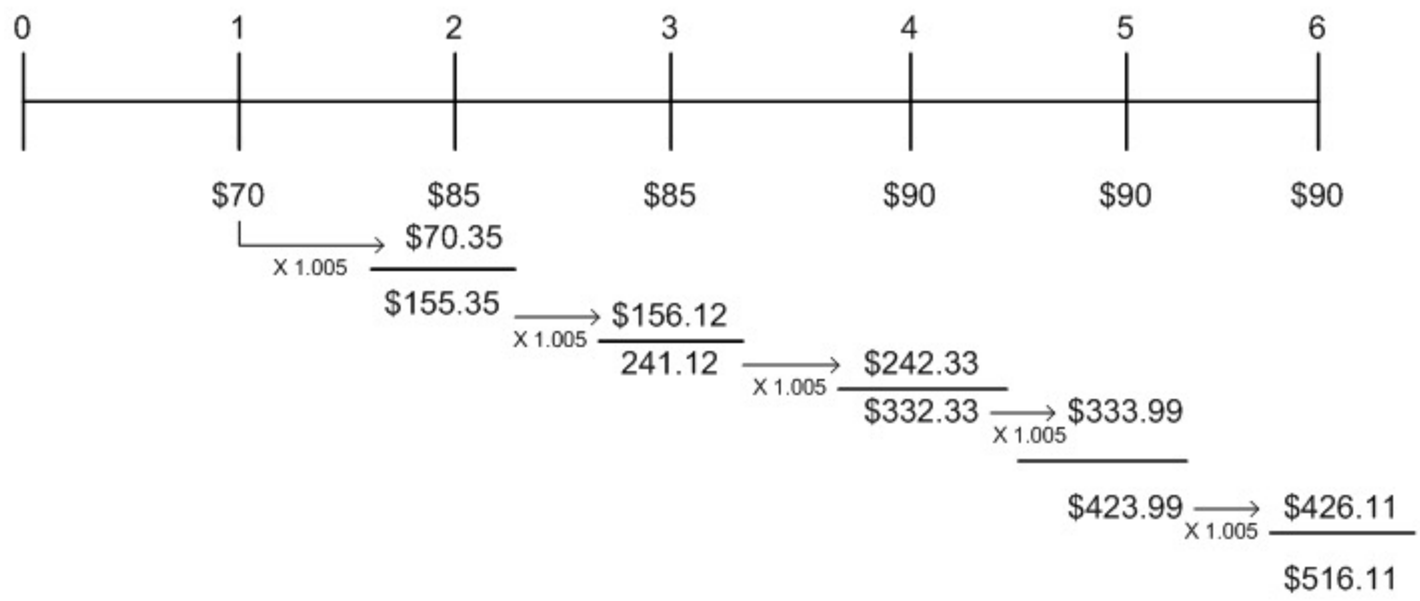
Your Turn! (PRS, please)

- She will have
  - $600
  - $516
  - $475

Solution : Present Value of a Stream of Cash Flows (cont'd)

Execute:

- We need to compute the **future value** of the yearly deposits.
- One way is to compute the bank balance each year.



Solution : Present Value of a Stream of Cash Flows (cont'd)

Execute:

- To verify our answer, suppose your aunt kept her **$500.90** in the bank today earning 0.5% interest.
- In six years she would have:

$$FV = \$500.90 \times (1.005)^6 = \mathbf{\$516.11} \text{ in 6 years}$$

Your Turn!

Problem:

- We plan to save $1,000 today and at the end of each of the next **two** years.
- At a fixed **6%** interest rate, **how much will we have in the bank three years from today?**

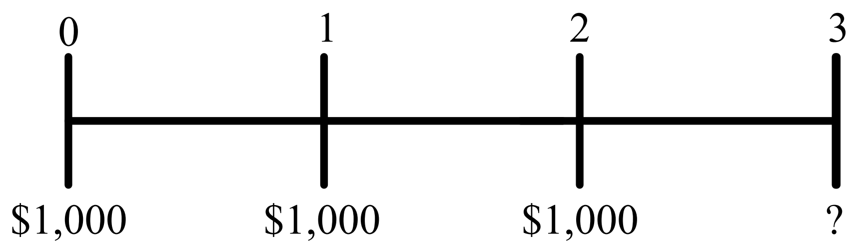
Your Turn! (PRS, please)

- We will have
  - $3,505.5
  - $2,833.4
  - $3,374.6

Solution : Computing the Future Value (cont'd)

Plan:

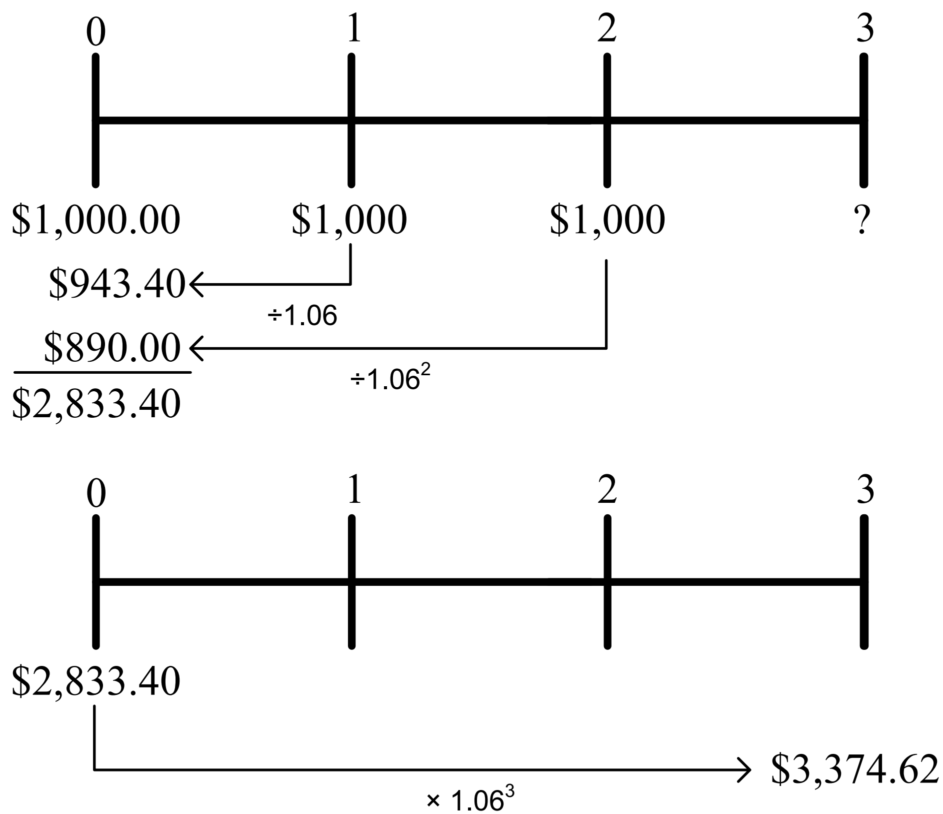
- We'll start with the timeline for this savings plan:



  - First we'll compute the present value of the cash flows.
  - Then we'll compute its value three years later (its future value).

Solution : Computing the Future Value (cont'd)

Execute:



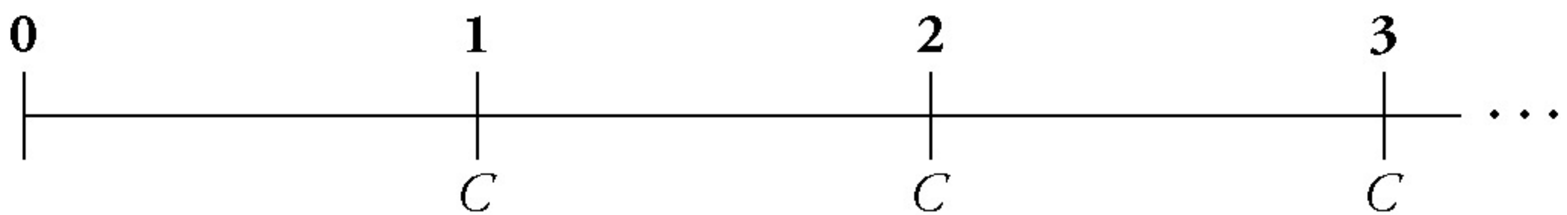
This is how we've done it so far – this works no matter the cash flows

Perpetuities

Examples?

Perpetuities (also called consols)

- A perpetuity is a stream of **equal** cash flows that occur at regular intervals and last **forever**
- Here is the timeline for a perpetuity:



- the first cash flow does not occur immediately; it arrives at the **end of the first period**

Perpetuities

- Using the formula for present value, the present value of a perpetuity with payment C and interest rate r is given by:

$$\boxed{PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots\ldots}$$

- Notice that **all the cash flows are the same**
- Also, the first cash flow starts at time 1 (not 0)

Perpetuities/Consols

- Present Value of a Perpetuity

$$PV(C \text{ in Perpetuity}) = \frac{C}{r}$$

- This formula **only** works when ***r*** is **positive**!

Example of consol

Perpetuities

Your Turn!

Problem:

- You just won the lottery, and you want to endow a professorship at your *alma mater*.
- You are willing to donate **$4 million** of your winnings for this purpose.
- If the university earns **5%** per year on its investments, and the professor will be receiving her first payment **in one year**, how much will the endowment pay her each year?

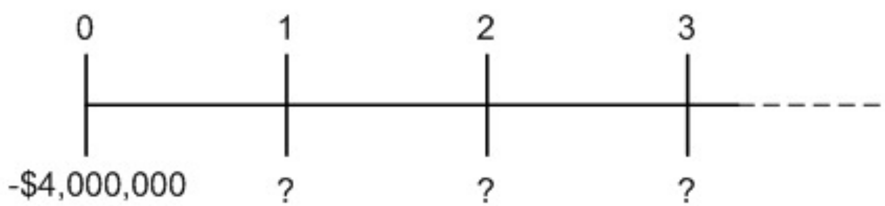
Your Turn! (PRS, please)

- The endowment will pay
  - $200,000
  - $283,334
  - $337,456

Solution : Endowing a Perpetuity

Plan:

- The timeline of the cash flows you want to provide is:



- This is a standard perpetuity. The amount she can withdraw each year while keeping the principal intact is the cash flow ***C*** when solving the equation:

$$PV(C \text{ in Perpetuity}) = \frac{C}{r}$$

Solution : Endowing a Perpetuity (cont'd)

Execute:

- From the formula for a perpetuity,

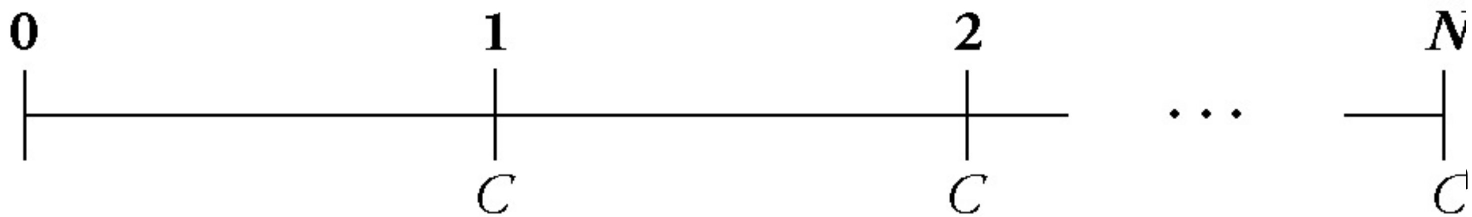
$$\boxed{PV = \frac{C}{r}, so\ C = PV\ X\ r}$$

$$C = \$4,000,000\ X\ .05$$
$$= \$200,000$$

Annuities

Examples?

- Annuities (also called fixed payment loans)
  - An **annuity** is a stream of *N* **equal** cash flows *C* paid at regular intervals



  - ds

    after some fixed number of payments (*N*)

Annuities

Present Value of An Annuity

- Note that, just as with the perpetuity, we assume the first payment takes place one period from today (time 1)

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots\ldots \frac{C}{(1+r)^N}$$

$$PV(\text{Annuity of } C \text{ for } N \text{ Periods with Interest Rate } r) = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

Annuities

Future Value of an Annuity

$$
\begin{aligned}
FV(\text{Annuity}) &= PV \times (1 + r)^N \\
&= \frac{C}{r}\left(1 - \frac{1}{(1 + r)^N}\right) \times (1 + r)^N \\
&= C \times \frac{1}{r}((1 + r)^N - 1)
\end{aligned}
$$

Your Turn!

Problem:

- Your parents have made you an offer you can't refuse.
- They're planning to give you part of your inheritance early (!).
- They've given you a choice.
  - Option a: They'll pay you **\$10,000** per year for each of the next **seven** years (beginning **today,** last payment at the end of the 7^{th} year)
  - Option b: they'll give you their BMW Convertible, which you can sell for **\$61,000** (guaranteed) **today**.
- If you can earn **7%** annually on your investments, which should you choose?

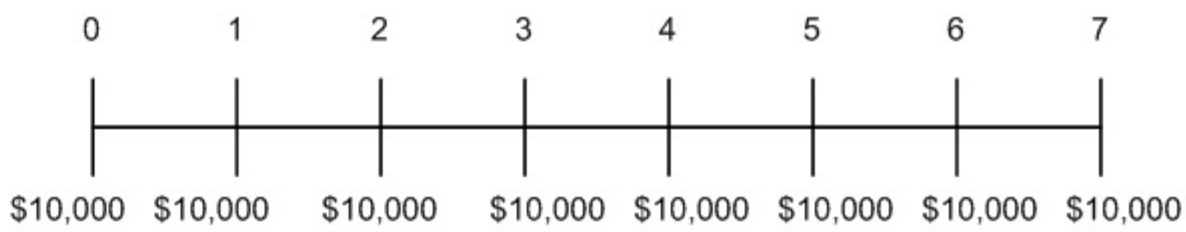
Your Turn! (PRS, please)

- We should choose
  - A
  - B

Solution : Present Value of an Annuity

Plan:

- Option (a) provides $10,000 paid over time. To evaluate it correctly, we must convert it to a **present value**. Here is the timeline:



- The $10,000 at date 0 is already stated in present value terms, but we need to compute the present value of the remaining payments.
- Fortunately, this case looks like a 7-year annuity of $10,000 per year, so we can use the annuity formula.

Solution : Present Value of an Annuity (cont'd)

Execute:

$$\boxed{PV(\text{Annuity of } C \text{ for } N \text{ Periods with Interest Rate } r) = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right)}$$

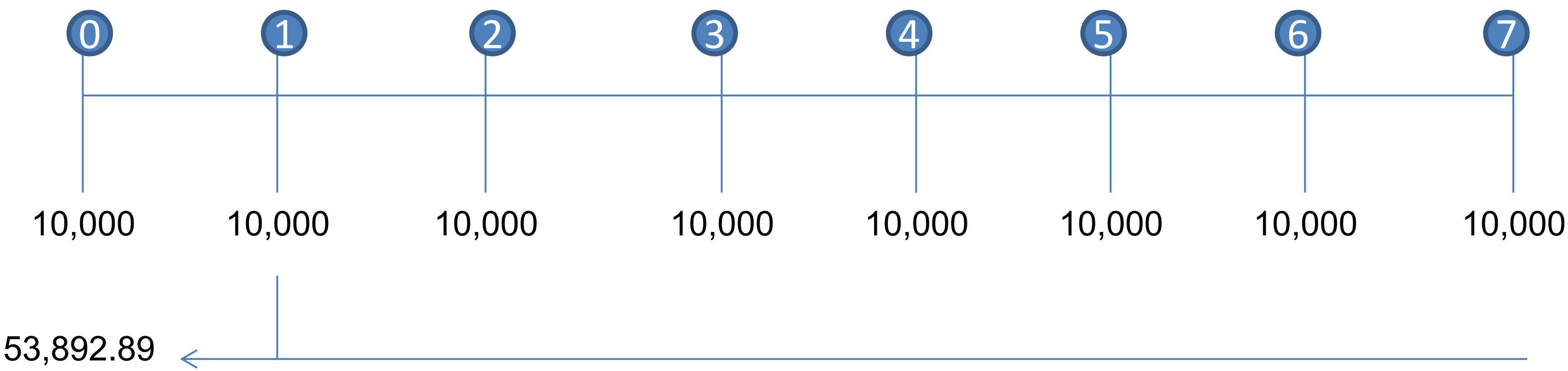
$$PV = 10{,}000 \times \frac{1}{0.07} \times \left(1 - \frac{1}{1.07^7}\right)$$

- *PV* = 53,892.89

Solution : Present Value of an Annuity (cont'd)

Execute (cont'd):

- Thus, the total present value of the cash flows is $10,000 + 53,892.89 = **$63,892.89** which is **more** than the value of the car. In timeline form:



63,892.89

Solution : Present Value of an Annuity (cont'd)

Execute (cont'd):

- Financial calculators or Excel can handle annuities easily—just enter the cash flow in the annuity as the PMT:

	N	I/Y	PV	PMT	FV
Given:	7	7		10000	0
Solve for:			**-53,892.89**		
Excel Formula: =PV(RATE,NPER, PMT, FV) = PV(0.07,7,10000,0)					

Your Turn!

Problem:

- Adam is **25** years old, and he has decided it is time to plan seriously for his retirement.
- He will save **$10,000** in a retirement account at the end of each year until he is **45**.
- At that time, he will **stop** paying into the account, though he does not plan to retire until he is **65**.
- If the account earns **10%** per year, **how much will Adam have saved at age 65?**

Your Turn! (PRS, please)

- Adam will have saved
  - $572,750
  - $1,545,250
  - $3,853,715

Solution : Retirement Savings Plan Annuity

Solution:

Plan:

- As always, we begin with a timeline. In this case, it is helpful to keep track of both the dates and Adam's age:

Solution : Retirement Savings Plan Annuity

Execute:

$$FV(at\ age\ 45) = \$10,000 \times \left(1.10^{20} - 1\right) \text{ X } 1/0.10 = \$572,750$$

Using Financial calculators or Excel:

	N	I/Y	PV	PMT	FV
Given:	20	10.0	0	-10,000	
Solve for:					$572,750
Excel Formula: =FV(RATE,NPER, PMT, PV) = FV(0.10,20,-10000,0)					

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Solution : Retirement Savings Plan Annuity

Execute:

$$\begin{aligned} FV(at\ age\ 65) \\ &= \$572,750 \times (1.10)^{20} \\ &= \$3,853,175 \end{aligned}$$

Using Financial calculators or Excel:

	N	I/Y	PV	PMT	FV
Given:	20	10.0	-$572,750	0	
Solve for:					$3,853,175
Excel Formula: =FV(RATE,NPER, PMT, PV) = FV(0.10,20,0,-572750)					

Part 3: Applications

Application 1: Your Credit Card in Hong Kong

- Check out your credit card statement
- What does it show at the end of the statement?
- Why do you think the financial regulators are forcing the credit card issuers to include this information?

Credit Cards “Easy Money”: Pay Fast!

- Credit cards are useful, but sometimes too easy.

Number of months to pay off a HK$20,000 credit card debt

	Monthly Paiements		
Annual Rate	HK$500	HK$600	HK$700
10.00%	**48.4**	**38.9**	**32.6**
12.50%	**51.1**	**40.6**	**33.7**
15.00%	**54.3**	**42.5**	**35.0**
17.50%	**58.0**	**44.6**	**36.4**
20.00%	**62.4**	**47.0**	**37.9**

Applying PV : Example - calculating the Internal Rate of Return

- Imagine that you run a toy manufacturing company and that you are considering purchasing a new machine.
  - Machine costs $4 million and can produce 50,000 toys per year.
  - You sell the toys for $10, generating $500,000 in revenue per year.
  - Assume that the machine is the only input, you have certainty about the revenue, there is no maintenance and the machine has a 10 year lifespan.

Internal Rate of Return: Example

- If you borrow $4 million, is the revenue enough to make the payments?
- We need to calculate the internal rate of return (IRR) of this investment:
  - It is **the interest rate that equates the present value from the cash flow of an investment with its cost.**

Internal Rate of Return: Example

- Balance the cost of the machine against the revenue.
  - $4 million today versus $500,000 a year for ten years.
- At the internal rate of return, the cost of the machine is equal to the present value of all the yearly revenues.
  - Solve for ***i*** - the internal rate of return. (NB: we can use the notation *i* or *r*)

Internal Rate of Return: Example

- Solving for *i*,

$$\$4{,}000{,}000 = \frac{\$500{,}000}{(1+i)^1} + \frac{\$500{,}000}{(1+i)^2} + \frac{\$500{,}000}{(1+i)^3} + \ldots\ldots + \frac{\$500{,}000}{(1+i)^{10}}$$

- *i* = 4.28%
- As long your borrowing cost is less than 4.28%, then you should buy the machine.
- **Knowing the cash flows (and dates), this IRR calculation is a way to identify the break even funding cost**.

Key Formulas: Fixed Cash Flows + Residual Value

- Often there are cases with fixed cash flows over the tenor of the transaction and one final **but different** amount on maturity date leading to this:

$$DCF = \sum_{k=1}^{k=n} \frac{C}{(1+i)^k} + \frac{P}{(1+i)^n}$$

Annex

Footer TBD

Basic Math Time!

- Dealing with Cash Flows, Amounts and Dates, an Excel spreadsheet (or the HP12C) with the Excel functions FV, PV, PMT, Rate, NPRER, XNPV, XIRR,…is of course a solution but relatively straightforward cases could be and should be dealt with some analytical formulas in order to understand how the different variables impact the output.
- Please keep in mind that 25/30 years ago, many derivative markets were open-outcry and market markers had NO access real time to a computer when asked to quote derivatives!
- For those having doubts about Geometric Sequences and Sums of consecutive terms, you might want to look at the Annex!

Basic Math Refresher

- Adding 5% interest p.a. (per annum) compounding for several years to $100 is like multiplying by 1.05 for each year, so the basic instrument to be fluent with is the **Geometric Sequence** as well as the Sum of ***n*** consecutive terms of a geometric sequence!

Basic Math Refresher

- A geometric sequence is defined as such:

$$\begin{cases} u_1 \\ u_n = u_{n-1} . q \end{cases}$$

- Which is equivalent to the general term:

$$u_{n=}u_1 . q^{n-1} , \forall n \geq 1$$

- So once the first term and the ratio are defined everything there is to know about it is easily accessible

Basic Math Refresher

- One useful aspect to know is the **Sum of *n* consecutive terms of a geometric sequence**
- One solution is to learn by heart the simple formula; another solution lowering the risk level is to know **how** to find again this formula quickly !

Basic Math Refresher

- Here is the way:

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$q.S_n = q.u_1 + q.u_2 + q.u_3 + \cdots + q.u_n$$

$$q.S_n = u_2 + u_3 + u_4 \ldots + u_{n+1}$$

$$q.S_n - S_n = u_2 + u_3 + u_4 \ldots + u_{n+1} - u_1 - u_2 - u_3 - \cdots - u_n$$

$$q.S_n - S_n = u_{n+1} - u_1$$

$$S_n.(q-1) = u_1.q^n - u_1 = u_1.(q^n - 1)$$

$$\boxed{S_n = u_1 \frac{q^n - 1}{q - 1}}$$

Basic Math Refresher

- Worth noticing is the case where ***n*** is $\propto$
- AND

$$\boxed{0 < q < 1}$$

Only works with positive interest rates

- In that case

$$S_\infty = u_1 \frac{0-1}{q-1}$$

- And

$$\boxed{S_\infty = \frac{u_1}{1-q}}$$

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Annex