§4.6 Rank

* The Row Space

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \overset{\Delta}{=} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{pmatrix}$$

Row Space of A (denoted by Row A) $= \text{Span}\{\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m\}$ is a subspace of $\mathbb{R}^n$.

Thm: If two matrices A and B are row equivalent, then their row spaces are the same.
If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

* The Rank Theorem

Def: The rank of A is the dimension of the column space of A. $\text{Rank} A = \dim \text{Col} A$

Theorem: The rank Theorem $A: m \times n$

1) $\dim \text{Col} A = \dim \text{Row} A$
$\qquad \parallel$
$\quad$ # of pivots in A

2) $\text{rank} A + \dim \text{Nul} A = n$

Proof: Since the pivot columns of a matrix A form a basis for Col A, so

dim Col A = # of pivots in A = dim Row A

$$\underbrace{\#\text{ of pivots}}_{\substack{\shortparallel \\ \text{rank}A}} + \underbrace{\#\text{ of free variables}}_{\substack{\shortparallel \\ \dim \text{Nul}A}} = n$$

Ex: 1) If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?

2) Could a 6×9 matrix have a two-dimensional Null space?

Solution: (1) $\text{rank}\,A = 9-2 = 7$

(2) If $\dim \text{Nul}\,A = 2$, then

$\text{rank}A = 9-2 = 7 = \dim \text{Row}\,A \le 6$. Contradiction

So a 6×9 matrix can't have a two-dimensional null space.

Thm: The invertible Matrix Theorem.

Let A be an $n\times n$ matrix. Then the following statements are equivalent.

(a) A is an invertible matrix

(b) A is row equivalent to the $n\times n$ identity matrix.

(c) A has n pivot positions

(d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.

(g) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.

(h) The columns span $\mathbb{R}^n$.

(i) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is an invertible matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$

(n) $\operatorname{Col} A = \mathbb{R}^n$

(o) $\dim \operatorname{Col} A = n$

(p) $\operatorname{rank} A = n$

(q) $\operatorname{Nul} A = \{0\}$

(r) $\dim \operatorname{Nul} A = 0$

(s) $\det(A) \neq 0$

Ex: The matrices below are row equivalent.

$$A = \begin{pmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1) Find $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$

2) Find bases for $\operatorname{Col} A$ and $\operatorname{Row} A$

3) What is the next step to perform to find a basis for Nul A?

4) How many pivot columns are in a row echelon form of A^T?

Solution: 1) $\text{rank}A = 2$

$\dim \text{Nul} = 5 - 2 = 3$

2) The pivot columns of A are the first two columns. So a basis for Col A is

$$\{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{pmatrix} 2 \\ 1 \\ -7 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 8 \\ -5 \end{pmatrix} \right\}$$

The nonzero rows of B form a basis for Row A, namely,

$\{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$.

3) For nul A, the next step is to perform row operations on B to obtain the reduced echelon form of A.

4). Since $\text{Col}A^T = \text{Row}A$, $\text{Rank}A^T = \text{rank}A$. So A^T has two pivot positions.