COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Spring Semester – Final Exam (Part 1)

Note 1: For all non-proof questions, showing your steps is not necessary unless required otherwise. However, in case your answer is wrong, showing your steps may earn you some partial credits.
Note 2: You can express your answers using binomial coefficients, factorials, and D_n (derangement number). However, you should not have summation $\sum$ in your final answers. For example, $\binom{10}{3}D_9+4!$ and $1!+2!+3!+4!$ are valid, but $\sum_{i=0}^{n}\binom{n}{i}$ or $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}$ is not. The latter has to be simplified to 2^n.

Question 1: (10 pts) Toss a fair coin $2n$ times. You earn \$1 for each heads, and lose \$1 for each tails. Calculate the probabilities of the following events:

(a) (3 pts) Your net return is 0.

(b) (3 pts) Your net return is -2.

(c) (4 pts) Your net return is a multiple of 4. Please justify of your answer.

(d) (Bonus problem) Your final net return is 0, and your net return is never negative during the whole game. [Hint: consider the connection between (a) and (b).] Please justify your answer.

Solution:

(a) You must get exactly n heads, so the probability is $\binom{2n}{n}/2^{2n}$.

(b) You must get exactly $n-1$ heads, so the probability is $\binom{2n}{n-1}/2^{2n}$.

(c) If n is an odd number, you will need an odd number of heads, and there are $\binom{2n}{1}+\binom{2n}{3}+\cdots+\binom{2n}{2n-1}$ ways to do it. If n is an even number, you will need an even number of heads, and there are $\binom{2n}{0}+\binom{2n}{2}+\cdots+\binom{2n}{2n}$ ways. By the corollary on slide 11, both are equal to $2^{2n}/2$. So the probability is $1/2$.

(d) We will count the number of ways (d) can happen, which is $\binom{2n}{n}-y$, where y is the number of ways such that the final net return is 0 and the net return becomes negative at some point. We will show that $y=\binom{2n}{n-1}$, i.e., it is the same as the number of ways where the net return is -2. We prove so by showing a bijection f. Let $x_1,\ldots,x_{2n}$ be the results of the coin flips where $x_i=1$ represents a heads and $x_i=-1$ represents a tails. For any such sequence where the net return is negative at some point, we consider the first time it becomes negative, say k, then we have $x_1+\cdots+x_k=-1$. Define $f(x_1,\ldots,x_{2n})=(x_1,\ldots,x_k,-x_{k+1},\ldots,-x_{2n})$. Because $x_1+\cdots+x_{2n}=0$, we have $x_{k+1}+\cdots+x_{2n}=1$, and thus $x_1+\cdots+x_k-x_{k+1}-\cdots-x_{2n}=-2$, i.e., $f(x_1,\ldots,x_{2n})$ is a sequence with net return -2. It can be verified that f is a bijection. So the answer is $(\binom{2n}{n}-\binom{2n}{n-1})/2^{2n}$.

Question 2: (8 pts) At a gathering of 30 people, there are 20 people who all know each other, while the other 10 people don't know anyone. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?

Solution: One way to solve this problem is as follows. Each one of the ten people has to shake hands with all the 20 other people they don't know. So $10 \cdot 20 = 200$. From there, we calculate how many handshakes occurred between the people who don't know each other. This is simply counting how many ways to choose two people to shake hands, or $\binom{10}{2} = 45$. Thus the answer is $200 + 45 = 245$.

Another way to consider this problem is as follows. We can focus on how many handshakes the 10 people get.

The 1st person gets 29 handshakes.

2nd get 28

...

And the 10th receives 20 handshakes. We can write this as the sum of an arithmetic sequence.

$$\frac{10(20+29)}{2} = 5 \cdot 49 = 245.$$

Therefore the answer is 245.

Question 3: (10 pts) Consider a web system for online shopping that will generate an ID for each order. The order ID is a sequence of 7 characters where each character could be either one of the 26 upper case English letters (A to Z) or one of the 10 digits (0 to 9).

(a) (2 pts) How many possible order IDs are there?

(b) (2 pts) How many possible order IDs are there if no two characters are the same? (E.g. A034NXD is valid, A034NAD is invalid)

(c) (3 pts) How many possible order IDs are there if the leftmost character must be a digit and no two consecutive characters are the same? (E.g. 3ABA3C6 is valid. C3ARX7B and 3ARRX7B are invalid)

(d) (3 pts) How many possible order IDs are there if characters must alternative between a letter and a digit? (E.g. A3R4B7Q and 4V7X3V7 are valid, B7XB4Q3 is invalid)

Solution:

(a) 36^7

(b) $P(36, 7)$ or $7!\binom{36}{7}$

(c) $10 \cdot 35^6$. The leftmost character has 10 choices, then the second leftmost character has 35 choices because it cannot be the same as the previous character, and so on.

(d) $10^4 \cdot 26^3 + 10^3 \cdot 26^4$. It can either be four digits and three letters, or three digits and four letters. The positions of digits and alphabets are fixed in either case.

Question 4: (8 pts) Use induction to prove that, for any integer $n \geq 1$,

$$5^n + 2 \cdot 11^n \text{ is divisible by } 3.$$

Solution: **Base case:** When $n = 1$, we have $5^1 + 2 \cdot 11^1 = 27$, which is divisible by 3. So, the statement is true for $n = 1$.

Induction hypothesis: Now let $n > 1$. Assume the statement is true for $n - 1$, i.e. $5^{n-1} + 2 \cdot 11^{n-1}$ is divisible by 3.

Induction step: Consider the case of n:

$$\begin{aligned}
5^n + 2 \cdot 11^n &= 5^{n-1} + 4 \cdot 5^{n-1} + 2 \cdot 11^{n-1} + 20 \cdot 11^{n-1} \\
&= 5^{n-1} + 2 \cdot 11^{n-1} + 4(5^{n-1} + 5 \cdot 11^{n-1}) \\
&= 5^{n-1} + 2 \cdot 11^{n-1} + 4(5^{n-1} + 2 \cdot 11^{n-1} + 3 \cdot 11^{n-1}) \\
&= 3y + 4(3y + 3 \cdot 11^{n-1}) \\
&= 3(5y + 4 \cdot 11^{n-1})
\end{aligned}$$

which is divisible by 3.

Question 5: (12 pts) Five couples (i.e., 10 people) sit down at random in a row of 10 seats.

(a) (3 pts) Two of the 5 couples are Peter/Mary and John/Helen. Since Peter and John are friends, they want to sit next to each other. What is the probability that Peter and Mary sit together, John and Helen sit together, and Peter and John sit together?

(b) (3 pts) What is the probability that every couple sits together?

(c) (6 pts) What is the probability that no couple sits together? For this question, you may use the summation sign $\sum$ to express your answer.

Solution: (a) There are totally 10! ways to seat the 5 couples.

Since each of these two couples sits together, and Peter and John sit next to each other, we treat the two couples as one single unit involving the four persons in these two couples. We can randomly permute the 7 units in 7! different ways. For each permutation, there are 2 ways to seat the two couples (i.e., (Mary, Peter, John, Helen) and (Helen, John, Peter, Mary)). Therefore, the probability that each of these two couples sits together, and Peter and John sit next to each other is

$$\frac{7! \cdot 2}{10!} = \frac{1}{360} \approx 0.0027778$$

(b) There are totally 10! ways to seat the 6 couples.

If a couple sits together, we treat it as one single unit. There are 5! ways of permuting 5 single units. For each permutation, there are 2^5 ways to seat the 5 bound couples. Therefore, the probability that every couple sits together is

$$\frac{5! \cdot 2^5}{10!} = \frac{1}{945} \approx 0.0010582$$

(c) There are totally 10! ways to seat the 5 couples.

If a couple sits together, we treat it as one single unit. Thus, for $k(1 \leq k \leq 5)$ specified couples to sit together, we can randomly permute the $(10-k)$ units in $(10-k)!$ different ways. For each permutation, there are 2^k ways to seat the k bound couples. Therefore, the probability that k specified couples end up sitting together (regardless of whether the other $5-k$ couples sit together or not) is

$$\frac{(10-k)! \cdot 2^k}{10!}$$

Let E_i denote the event that the i-th couple sits together. The probability that at least one couple sits together can be computed using the inclusion-exclusion principle as

$$P(\cup_{i=1}^{5} E_i) = \sum_{k=1}^{5} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k \\ 1 \leq i_1 < i_2 < \ldots < i_k \leq 5}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

Thus, we have

$$P(\cup_{i=1}^{5} E_i) = \sum_{k=1}^{5} (-1)^{k+1} \binom{5}{k} \frac{(10-k)! \cdot 2^k}{10!}$$

Thus, the probability that no couple sits together is

$$\begin{aligned} 1 - P(\cup_{i=1}^{5} E_i) &= 1 - \sum_{k=1}^{5} (-1)^{k+1} \binom{5}{k} \frac{(10-k)! \cdot 2^k}{10!} \\ &= \sum_{k=0}^{5} (-1)^{k} \binom{5}{k} \frac{(10-k)! \cdot 2^k}{10!} \end{aligned}$$