DYNAMICS OF RIGID BODIES II

Intended Learning Outcomes – after this lecture you will learn:
1. how to calculate the moment of inertia of simple symmetric rigid bodies
2. the parallel axis theorem to find the moment of inertia about different rotation axis
3. vector product
4. torque, and the Newton's second law in rotational dynamics

Textbook Reference: Ch 9.5, 10.1, 10.2

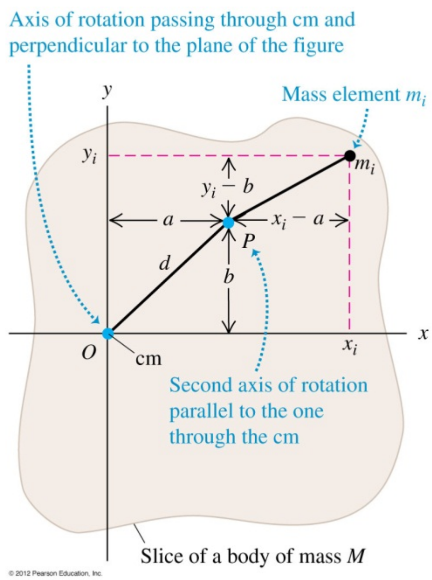
Parallel axis theorem

I_{cm}: moment of inertia about an axis through its CM

I_p: moment of inertia about another axis ∥ to the original one and at ⊥ distance d

$$\boxed{I_p = I_{\text{cm}} + Md^2}$$

Proof: take CM as the origin, rotation axis as the z axis. A point mass m_i in the solid has coordinates (x_i, y_i, z_i).



⚠ *not* $\sum m_i(x_i^2 + y_i^2 + z_i^2)$

$$I_{\text{cm}} = \sum m_i \overbrace{(x_i^2 + y_i^2)}^{\text{square of } \perp \text{ distance of } m_i \text{ to rotation axis}}$$

$$I_p = \sum m_i[(x_i - a)^2 + (y_i - b)^2]$$

$$= \underbrace{\sum m_i(x_i^2 + y_i^2)}_{I_{\text{cm}}} - 2a\underbrace{\sum m_i x_i}_{Mx_{\text{cm}} = 0} - 2b\underbrace{\sum m_i y_i}_{My_{\text{cm}} = 0}$$

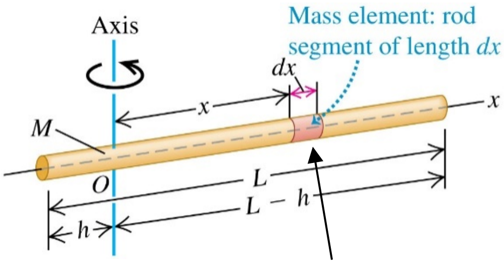
$$+(a^2 + b^2)\underbrace{\sum m_i}_{M}$$

Question:
A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Does it have a larger moment of inertia
(i) for an axis through the thicker end of the rod and perpendicular to the length of the rod, or
(ii) for an axis through the thinner end of the rod and perpendicular to the length of the rod?

Answer: see inverted text on P. 316 of textbook

Significance of the parallel axis theorem: need formula for I_{cm} only.

Example A thin rod with uniform linear density $\rho = M/L$



⚠ Before calculating moment of inertia, must specify rotation axis

$$I = \sum m_i r_i^2 \longrightarrow \int r^2\, dm$$

$\perp$ distance of m_i to rotation axis

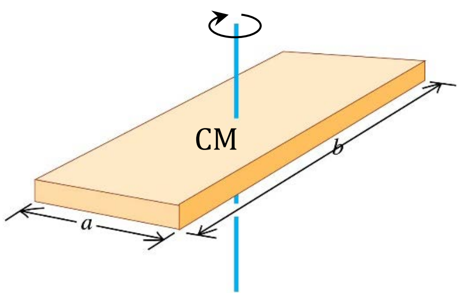
Choose a convenient mass element dm – a segment of length dx at a $\perp$ distance x from the axis, and mass $dm = \rho dx$

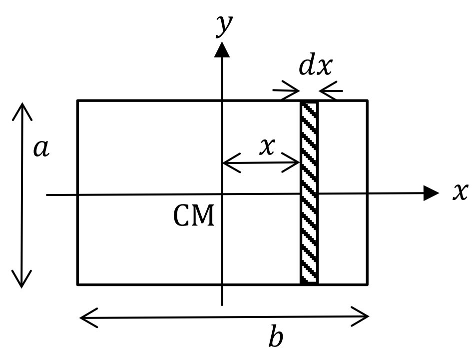
$$I_O = \int_{-h}^{L-h} x^2\,(\rho dx) = \frac{\rho}{3}[(L-h)^3 + h^3] = \frac{M}{3}(L^2 - 3Lh + 3h^2)$$

⚠ Put $h = L/2$, we get $I_{\text{cm}} = ML^2/12$.

⚠ Check the parallel axis theorem $I_O = I_{\text{cm}} + M(\qquad\qquad)^2$

Example A rectangular plate





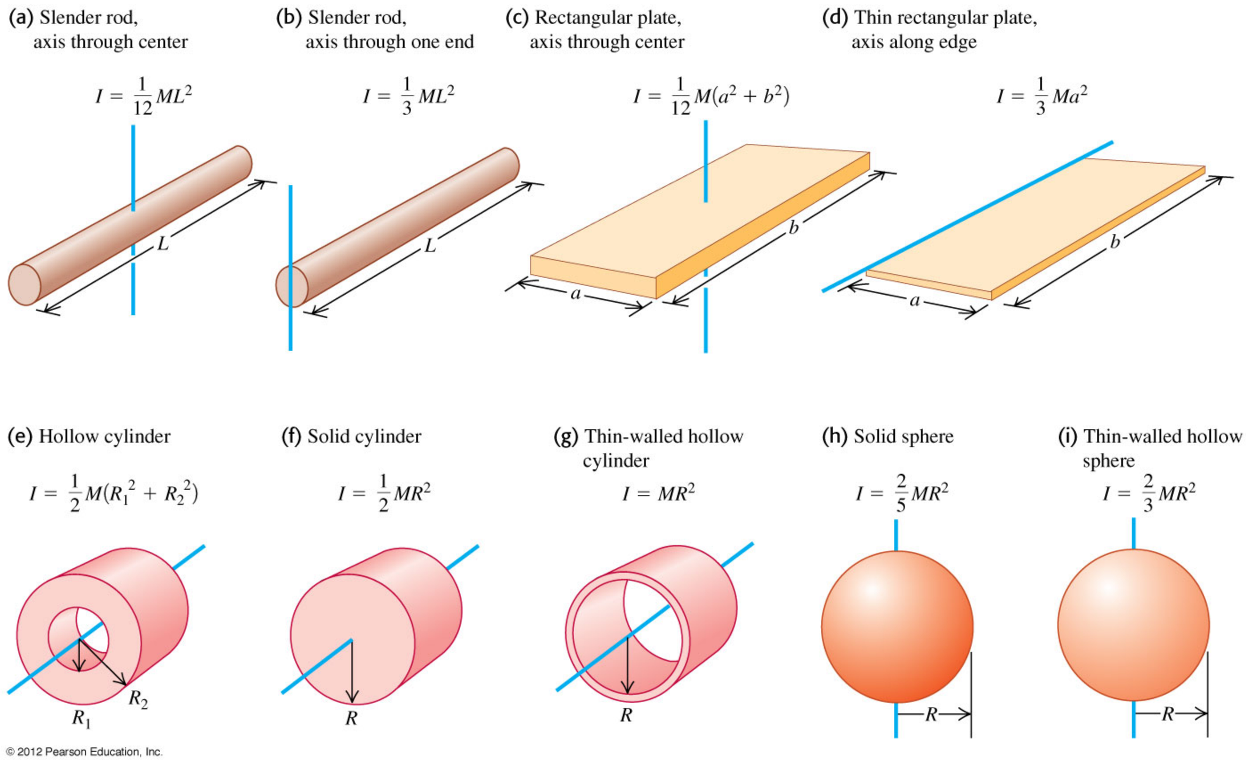
Choose the mass element dm to be a rod at $\perp$ distance x from the axis. *Why? Because you know its moment of inertia!*

$$dI = \underbrace{\frac{(dm)a^2}{12}}_{\text{about CM of the rod, not of the plate}} + \underbrace{(dm)x^2}_{\text{parallel axis theorem}}$$

Since $dm = \left(\frac{M}{b}\right) dx$

$$I = \int dI = \frac{M}{b}\int_{-b/2}^{b/2}\left[\frac{a^2}{12} + x^2\right] dx = \frac{1}{12}M(a^2 + b^2)$$

Table 9.2 Moments of Inertia of Various Bodies



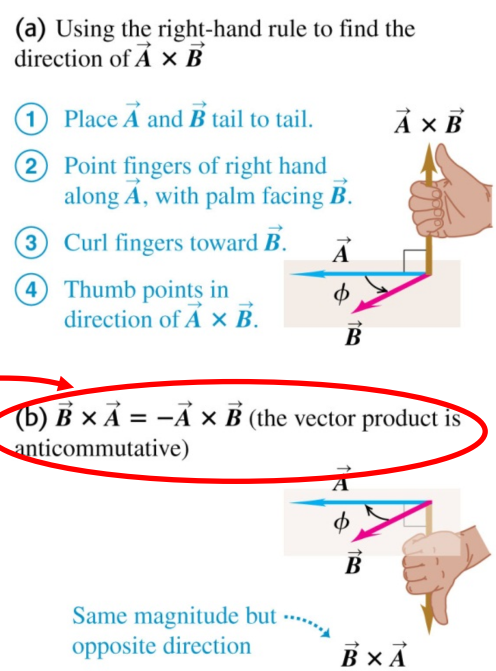
Vector (cross) product:

$\vec{C} = \vec{A} \times \vec{B}$

Magnitude: $C = AB \sin\phi$

direction determined by *Right Hand Rule*

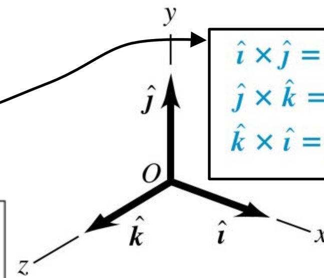
Important!



Special cases:

(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = 0$, in particular, $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB$, in particular,



$\hat{\imath} \times \hat{\jmath} = \hat{k}$
$\hat{\jmath} \times \hat{k} = \hat{\imath}$
$\hat{k} \times \hat{\imath} = \hat{\jmath}$

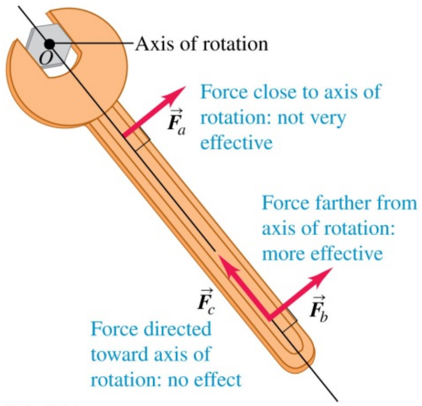
In analytical form (no need to memorize this)

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$$
$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

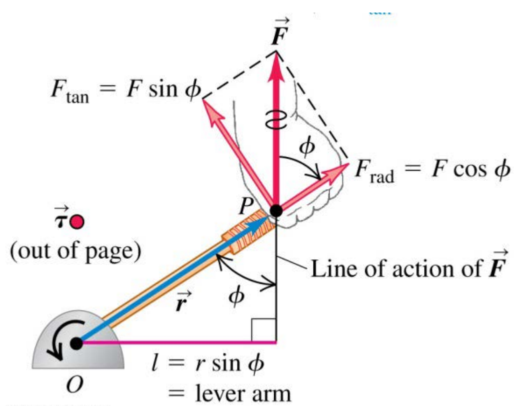
don't worry if you have not learnt determinants in high school

Torque

Besides magnitude and direction, the **line of action** of a force is important because it produces rotation effect.



$\vec{\boldsymbol{F}}_a$ and $\vec{\boldsymbol{F}}_b$ have the same magnitudes and directions, but different line of action: they produce different physical effects – which force would you apply if you were to tighten/loosen the screw?



Define **torque** about a point O as a vector

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}}$$

⚠ $\vec{\boldsymbol{\tau}}$ is $\perp$ to both $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$

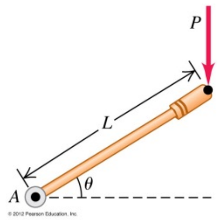
Magnitude: $\tau = r\underbrace{(F\sin\phi)}_{\text{component of } \vec{\boldsymbol{F}} \perp \text{ to } \vec{\boldsymbol{r}}} = \underbrace{(r\sin\phi)}_{\perp \text{ distance from } O \text{ to line of actions of } \vec{\boldsymbol{F}}}F$

Direction gives the sense of rotation about O through the right-hand-rule.

Notation: ⊙ out of the plane ⊗ into the plane

SI unit for torque: N·m (just like work done)

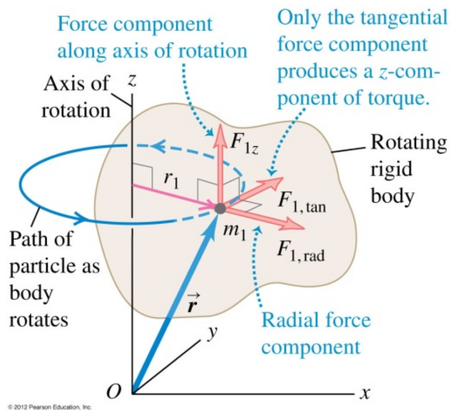
Question



A force P is applied to one end of a lever of length L. The magnitude of the torque of this force about point A is ($PL\sin\theta$ / $PL\cos\theta$ / $PL\tan\theta$)

Answer: see inverted text on P. 333 of textbook

Suppose a rigid body is rotating about a fixed axis which we arbitrarily call the z axis. m_1 is a small part of the total mass.



$F_{1,\text{rad}}$, $F_{1,\tan}$, and $F_{1,z}$ are the 3 components of the total force acting on m_1

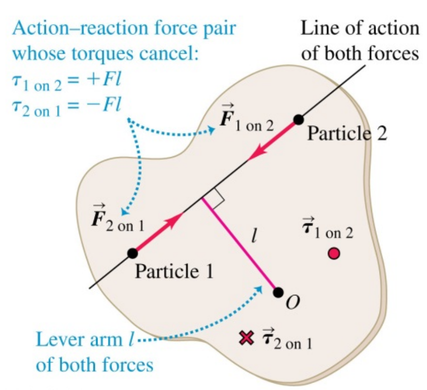
Only $F_{1,\tan}$ produces the desired rotation, $F_{1,\text{rad}}$ and $F_{1,z}$ produce some other effects which are irrelevant to the rotation about the z axis.

$$F_{1,\tan} = m_1 a_{1,\tan} = m_1(r_1\alpha_z)$$

$$\underbrace{F_{1,\tan} r_1}_{\text{torque on } m_1 \text{ about } z,\ \tau_{1z}} = m_1 r_1^2 \alpha_z$$

Sum over all mass in the body, since they all have the same α_z

$$\sum \tau_{iz} = \left(\sum m_r r_i^2\right)\alpha_z = I\alpha_z$$



⚠ Need to consider torque due to external forces only. Internal forces (action and reaction pairs) produce equal and opposite torques which have no net rotational effect.

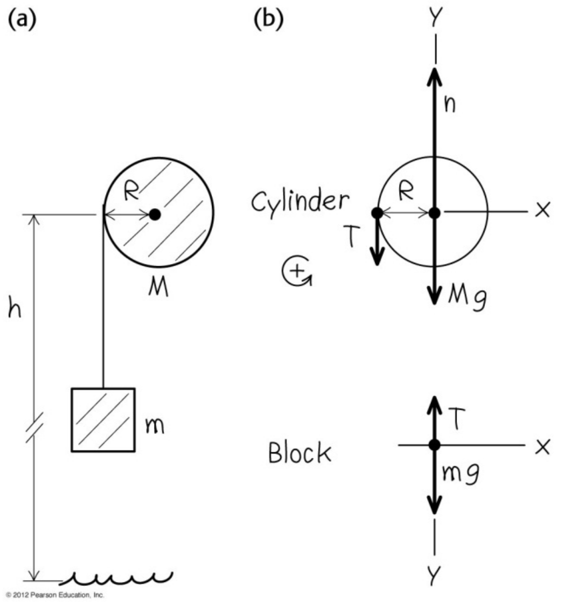
Conclusion: for rigid body rotation about a fixed axis,

$$\boxed{\sum \tau_{\text{ext}} = I\alpha}$$

c.f. Newton's second law $\sum \vec{\boldsymbol{F}}_{\text{ext}} = M\vec{\boldsymbol{a}}$

Example 10.3 P. 335 An unwinding cable

Pulley rotates about a fixed axis. What is the acceleration a of the block?



For the cylinder

$$\underbrace{TR}_{\text{torque due to } T} = \underbrace{\left(\tfrac{1}{2}MR^2\right)}_{\text{moment of inertia of cylinder}} \underbrace{\left(\frac{a}{R}\right)}_{\text{angular acceleration}}$$

i.e.

$$T = \tfrac{1}{2}Ma$$

For the block

$$mg - T = ma$$

Therefore

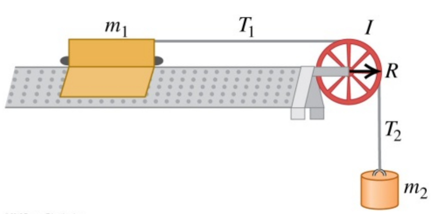
$$a = \frac{g}{1 + M/2m}$$

Suppose the block is initially at rest at height h. At the moment it hits the floor:

$$v^2 = 0 + 2\left(\frac{g}{1 + M/2m}\right)h \quad \Longrightarrow \quad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

c.f. lecture 8 in which we get the same result using energy conservation.

Question



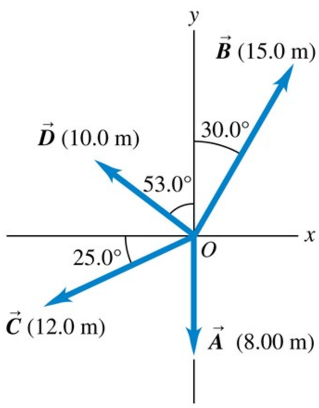
Mass m_1 slides on a frictionless track. The pulley has moment of inertia I about its rotation axis, and the string does not slip nor stretch. When the hanging mass m_2 is released, arrange the forces T_1, T_2, and m_2g in increasing order of magnitude.

Answer: see inverted text on P. 336 of textbook

Clicker Question:

Q1.14

Consider the vectors shown. What is the cross product $\vec{A} \times \vec{C}$?



A. $(96.0\ \text{m}^2)\sin 25.0°\hat{\boldsymbol{k}}$

B. $(96.0\ \text{m}^2)\cos 25.0°\hat{\boldsymbol{k}}$

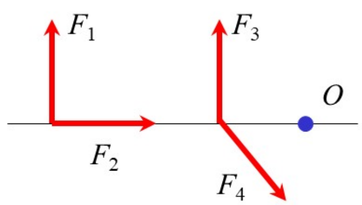
C. $-(96.0\ \text{m}^2)\sin 25.0°\hat{\boldsymbol{k}}$

D. $-(96.0\ \text{m}^2)\cos 25.0°\hat{\boldsymbol{k}}$

E. none of these

Q10.2

Which of the four forces shown here produces a torque about O that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

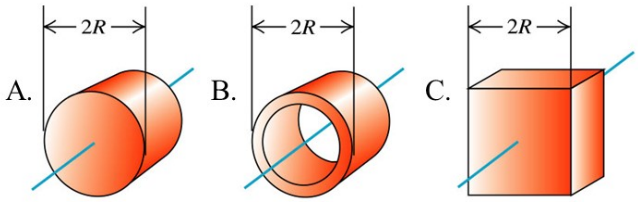
C. F_3

D. F_4

E. more than one of these

Q-RT9.2

Objects A, B, and C all have the same mass, all have the same outer dimension, and are all uniform. Each object is rotating about an axis through its center (shown in blue). All three objects have the same rotational kinetic energy.

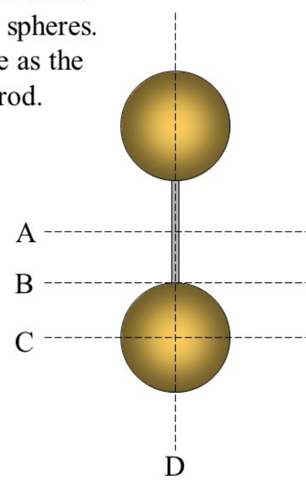


Rank these objects in order of their *angular speed* of rotation, from fastest to slowest.

Q-RT9.3

Two identical uniform solid spheres are attached by a solid uniform thin rod. The rod lies on a line connecting the centers of mass of the two spheres. Axes A, B, C, and D are in the same plane as the centers of mass of the spheres and of the rod.

For the combined object of two spheres plus rod, **rank** the object's *moments of inertia* about the four axes, from largest to smallest.



Ans: Q1.14) D, Q10.2) D, Q-RT9.2) ACB, Q-RT9.3) CBAD