Infinite Sequences and Infinite Series

An infinite repeating decimal can be interpreted in two ways:

$0.9999\ldots$

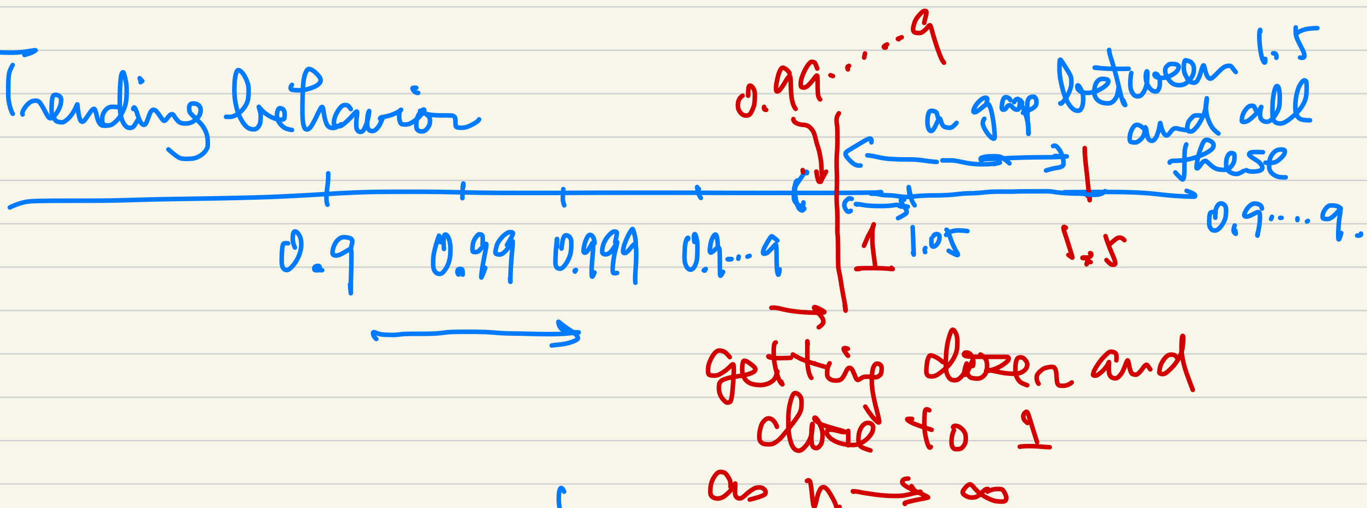
① limit of an infinite sequence

$$\lim_{n\to\infty} 0.\underbrace{99\ldots9}_{n \text{ decimal places.}} = 1$$

② As the sum of an infinite series

$0.9 + 0.09 + 0.009 + \cdots = 1.$

Trending behavior



① $$\boxed{\lim_{n\to\infty} a_n = L}$$ Meaning of this?

Limit of an Infinite Sequence

An infinite sequence is just an <u>ordered</u> list of infinite many numbers.

$a_1, a_2, a_3, \ldots, a_n, \ldots$, or $\{a_n\}_{n=1}^{\infty}$

$1, 2, 3, 4, 5, 6, \ldots$

$2, 3, 1, 5, 4, 6, \ldots$

different sequence!

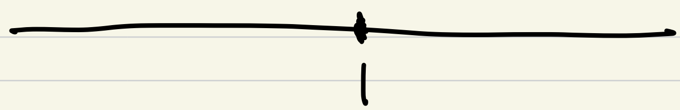
(2) $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ Sum of infinitely many numbers.

|| In summation notation

$$\sum_{n=1}^{\infty} a_n = S$$

What does this mean?

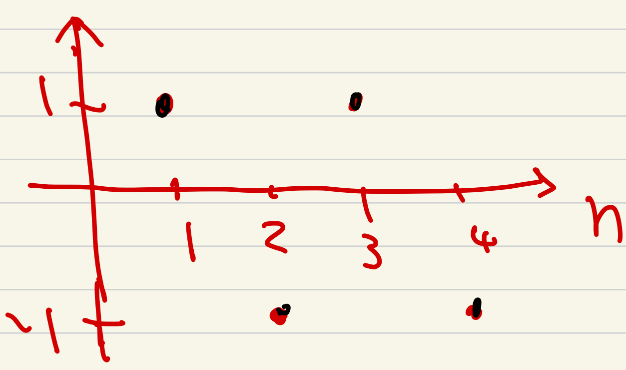
Example.



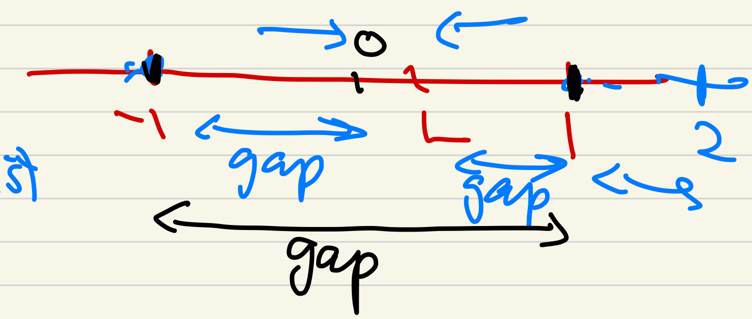
(1) 1, 1, 1, 1, ...

$$\lim_{n\to\infty} 1 = 1$$

(2) $1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$



nth term of the infinite sequence.



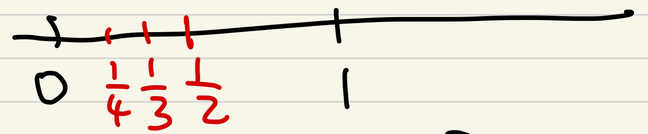
The limit doesnot exist

"$\lim_{n\to\infty} (-1)^{n+1}$"

cannot get as closed as you want to any number.

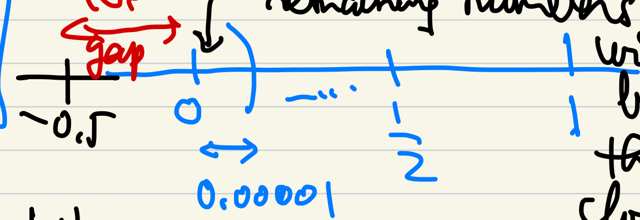
③ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots \xrightarrow{\text{as } n\to\infty} 0$

getting closer and closer to 0



$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0}$$

as $n > 10^5$, all the remaining numbers will be that closed to 0.



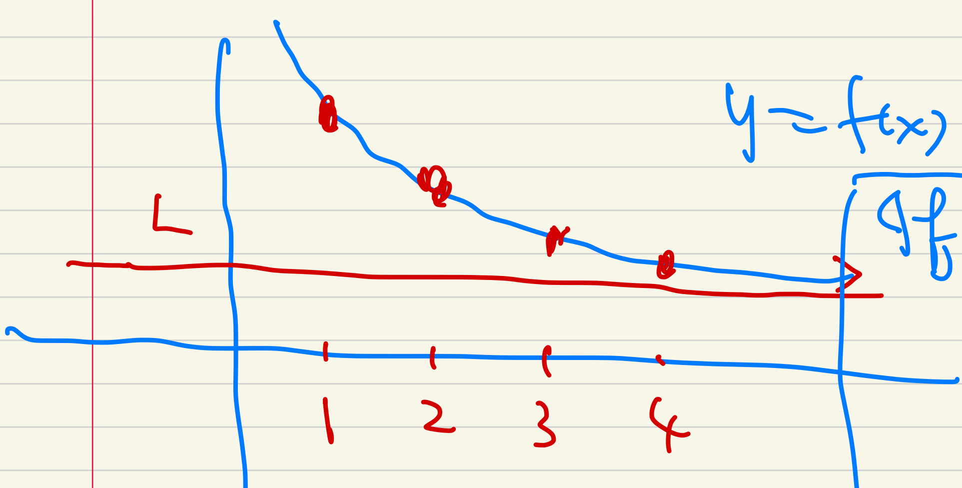
Compare it with

$$\lim_{x\to 0} \frac{1}{x} = 0$$

When n is large enough, we have

$$\frac{1}{n} < 0.00001$$

e.g. $\boxed{n > 10^5}$



If $\lim_{x\to\infty} f(x) = L$

$\Downarrow$

$\lim_{n\to\infty} f(n) = L$.

$\lim_{n\to\infty} a_n$

① The limit exists as a finite number

② The limit does not exist as a finite number.

- ∞, e.g. $1, 2, 3, 4, \dots$
- $-\infty$, e.g. $-1, -2, -3, \dots$
- Oscillating without a limit.

Example.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \text{does not exist} & \text{if otherwise} \end{cases}$$

$r>1$: ∞

$r<-1$: oscillating without a limit

2^n : $2, 2^2, 2^3, 2^4, \dots \longrightarrow \infty$

$(-2)^n$: $-2, 2^2, -2^3, 2^4, \dots$, oscillating between large numbers.

$(-1)^n$: $1, -1, 1, -1, \dots$

$\frac{1}{2^n}$: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \longrightarrow 0.$

① $r = (1+a)^n$, $a>0$, $r>1$

$$= 1+na+\frac{n(n-1)}{2}a^2+\dots+a^n$$

$$> na \longrightarrow \infty \text{ as } n\to\infty$$

② $0<r<1$, $0<r=\left(\frac{1}{1+a}\right)^n = \frac{1}{1+na+\dots} < \frac{1}{na}$ $\to 0$ as $n\to\infty$

Example.

$$a_n = \frac{n^2+1}{2n^2+n+3}$$

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+n+3} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\left(2+\frac{1}{n}+\frac{3}{n^2}\right)}$$

$\frac{1}{n^2} \to 0$, $\frac{1}{n} \to 0$, $\frac{3}{n^2} \to 0$

$$= \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2+x+3} = \frac{1}{2}$$

Properties of Limits of Infinite Sequences.

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$ (if these limits exist)

2. $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

4. If $c_n \leq a_n \leq b_n$, then $\lim_{n\to\infty} c_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

"Squeeze Theorem"

Example.

$$\lim_{n\to\infty} \frac{n^2}{e^n}$$

$$= \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{L'Hôpital's Rule}}{=} \lim_{x\to\infty} \frac{2x}{e^x}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Example.

$$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2}$$

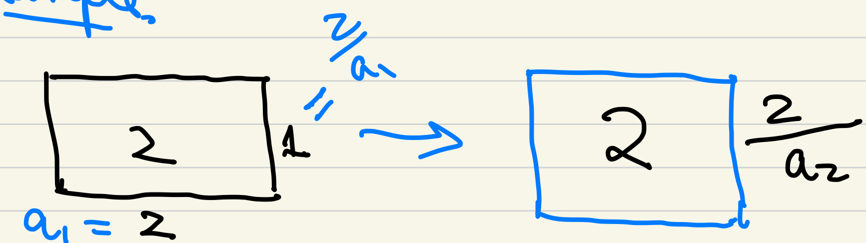
$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1-\frac{2}{x}}\cdot\frac{2}{\cancel{x^2}}}{-1/\cancel{x^2}}}$$

$$= \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$$

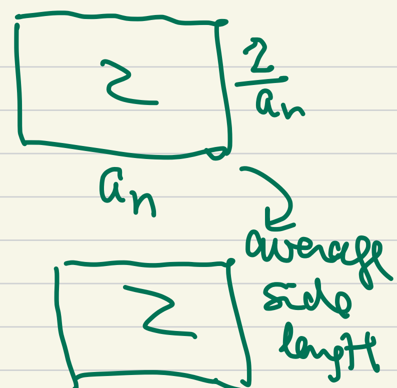
$$= \lim_{x\to\infty} e^{x\ln\left(1-\frac{2}{x}\right)} = \lim_{x\to\infty} e^{\boxed{\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}}}}$$

$\frac{0}{0}$-type

Example

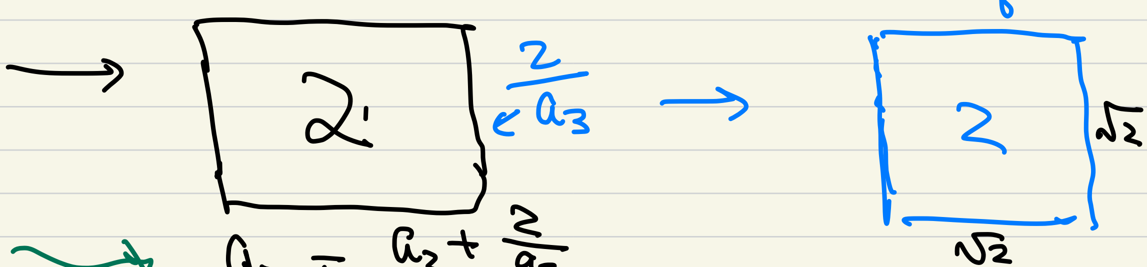


$a_2 = \frac{2+1}{2}$ = average of 2 and 1



$a_{n+1} = \frac{a_n + 2/a_n}{2}$

a square



$$a_3 = \frac{a_2 + \frac{2}{a_2}}{2}$$

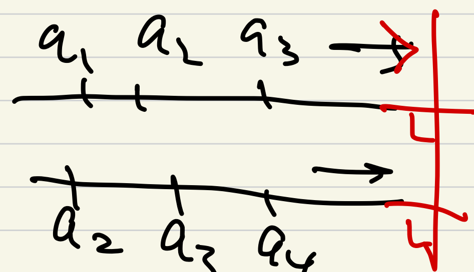
A recursive relation defining the infinite sequence.

$a_1 = 2,\ a_2 = \frac{3}{2}, \ldots$

$$\boxed{a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}} \rightarrow \sqrt{2}$$

If we let $\lim_{n\to\infty} a_n = L$.

Then $\lim_{n\to\infty} a_{n+1} = L$



Therefore

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + \frac{2}{a_n}}{2}$$

$$L = \frac{L + \frac{2}{L}}{2}$$

$$2L = L + \frac{2}{L} \iff L^2 = 2,$$

$$L = \sqrt{2}$$