Math1014 Calculus II
Brief Review: Infinite Sequences and Series

- Computing limits of simple infinite sequences $\lim\limits_{n\to\infty} f(n)$ is very often the same as computing the limit of function values $\lim\limits_{x\to\infty} f(x)$ as long as $f(x)$ is well-defined for all sufficiently large real numbers. E.g., can use similar limit laws, L'Hôpital Rule, Squeeze Theorem, etc. For sequence defined by recurrence relation, just take the limits of both sides of the relation, as long as you know the limit of the sequence exists.
- The meaning of the sum of the *infinite series* $\sum\limits_{n=1}^{\infty} a_n$ is just the limit of the corresponding **partial sum sequence** of the series:

$$\sum_{n=1}^{\infty} a_n \overset{means}{=} \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \lim_{n\to\infty} s_n \quad \text{whenever the limit exists}$$

  The problems to look at are mainly the convergence (when the limit of partial sum sequence above exists) or divergence (when the limit of partial sum sequence above does not exist) of infinite series:

$$\sum_{n=1}^{\infty} a_n \quad \longleftrightarrow \quad \begin{cases} \text{convergent} \begin{cases} \text{absolutely convergent: i.e., convergence of } \sum\limits_{n=1}^{\infty} |a_n| \\ \text{conditionally convergent: i.e., convergence of } \sum\limits_{n=1}^{\infty} a_n \text{ but not } \sum\limits_{n=1}^{\infty} |a_n| \end{cases} \\ \text{divergent} \end{cases}$$

- Important basic infinite series:
  geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ for any $-1 < r < 1$, and divergent otherwise.
  ***p*-series** $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for all $p > 1$, and diverges otherwise.
- Tools to solve convergence/divergence problems of infinite series: **divergence test**, **integral test**, **comparison test** and **limit comparison test**, **ratio test**, **root test**, alternating series test, etc. (Roughly speaking, the main trick is to see what $f(n)$ in $\sum\limits_{n=1}^{\infty} f(n)$ looks like for sufficiently large n (say, for $n \geq N$), which indicates that the series is essentially behaving like certain well-known series, or similar to certain improper integral.)
- Finding the domain of a power series $c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$, i.e., the **interval of convergence**, and the respective **radius of convergence** of the power series, is a matter of using the ratio/root test.
- We can integrate and differentiate power seriesto get new ones without changing the radius of convergence.
- Taylor or Maclaurin series of an infinitely differentiable function:

$$f(x) \longrightarrow f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n + \cdots$$

  (Maclurin series is the special case when $a = 0$.)

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1. Find the limit $\lim\limits_{n\to\infty} a_n$ of the sequence whenever it exists.

   (i) $a_n = \sqrt{\dfrac{n+1}{9n+1}}$ (ii) $a_n = \cos(2/n)$ (iii) $a_n = \tan^{-1}(2n)$

   (iv) $a_n = \dfrac{\ln n}{\ln(2n)}$ (v) $a_n = \dfrac{(-3)^n}{n!}$ (vi) $a_n = \sqrt[n]{3^n + 5^n}$

2. Assuming the existence of the limit, find $\lim_{n\to\infty} a_n$.

   (i) $a_1 = 2,\ a_{n+1} = \dfrac{1}{3 - a_n}$ (ii) $a_1 = 1,\ a_{n+1} = 1 + \dfrac{1}{1 + a_n}$

3. Determine if the infinite series is convergent or divergent.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$ (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} [2(0.1)^n) + (0.2)^n]$

   (iv) $\displaystyle\sum_{k=1}^{\infty} (\cos 1)^k$ (v) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ (vi) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

4. Work out the partial sums of the series, and then determine if the infinite series is convergent or divergent.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

5. Find the values of x for which the geometric series converges. Find the sum of the series for those values of x.

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ (ii) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$

6. Use the integral test to determine whether the series is convergent or divergent.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

7. Find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ is convergent.

8. Determine if the series is convergent or divergent by comparing it with certain geometric or p-series. (Comparison or Limit Comparison Test: i.e., does the general term of the series look like ar^n or n^{-p} for large n?).

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

   (iv) $\displaystyle\sum_{k=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ (v) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (vi) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

9. Test the alternating series for convergence or divergence:

   (i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

10. Approximate the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$ to four decimal places.

11. Determine whether the series converges absolutely or not by the Ratio or Root Test.

    (i) $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$ (ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ (iii) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^n$

    (vi) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$

12. Find the radius of convergence and interval of convergence of the power series.

    (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (ii) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$ (vi) $\displaystyle\sum_{n=2}^{\infty} \frac{(3x-2)^n}{n3^n}$

    (Apply Ratio/Root Test, and check the convergence/divergence at the interval endpoints.)

13. Find the power series representation for $\dfrac{1}{(1+x)^2}$ and $\dfrac{1}{(1+x)^3}$ by differentiating some known power series.

14. Show that the function $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $f'(x) = f(x)$. Hence show that $f(x) = e^x$ by considering $F(x) = f(x)e^{-x}$.

15. Use the power series for $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, $-1 \le x \le 1$, to show

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} \qquad \text{(What } x \text{ should you put into the power series?)}$$

16. Find the Maclaurin series of the function, and find the radius of convergence of the power series obtained.

    (i) $f(x) = \cosh x$ (ii) $f(x) = (1-x)^{2/3}$