Limits

2.1 Limit of a Function

Definition 2.1 *Let f be a function and c be a number. If there is number L such that $f(x)$ approaches L as x approaches c, we say that L is the limit of $f(x)$ as x tends to c. Symbolically we write*

$$\lim_{x \to c} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to c.$$

Example 2.2 *Evaluate*

$$\lim_{x \to 2} x^2.$$

solution:
As x approaches the number 2, x^2 approaches $2^2 = 4$. So,

$$\lim_{x \to 2} x^2 = 4.$$

Example 2.3 *Evaluate*

$$\lim_{x \to 0} \frac{1}{x}.$$

solution:
As x gets closer and closer to 0, $\frac{1}{x}$ becomes as large as one wish. In other words, $\frac{1}{x}$ does not approach to any limit as x approaches 0.

Example 2.4 *Evaluate*

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

solution:
Observe that

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1$$

for all $x \neq 1$. We are asking for the behavior of $\frac{x^2 - 1}{x - 1}$ as x approaches 1 but at the same time not equal to 1. Hence it is irrelevant if

$$\frac{x^2 - 1}{x - 1} = x + 1$$

when $x = 1$. As long as we know that it holds for x close to 1, we apply the previous equation to conclude that

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} x + 1 = 2.$$

Remark 2.5 *It does not make any sense to evaluate* $\frac{x^2 - 1}{x - 1}$ *at the number* 1. *However, the symbol*

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

does make sense and it turned out to be 2.

Example 2.6 *Let*

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

Evaluate

$$\lim_{x \to 0} f(x).$$

solution:
As x is close to 0, $f(x)$ can sometimes be 0 and sometimes be 1. In other words, $f(x)$ does not approach any number (neither 0 nor 1) as x approaches 0. We say that the limit

$$\lim_{x \to 0} f(x)$$

does not exist.

Theorem 2.7 *Let f and g be functions and c be a number such that both* $\lim_{x \to c} f(x)$ *and* $\lim_{x \to c} g(x)$ *exist. Then,*

1. $\lim_{x \to c} (f + g)(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
2. $\lim_{x \to c} (f - g)(x) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.
3. $\lim_{x \to c} (fg)(x) = (\lim_{x \to c} f(x)) \times (\lim_{x \to c} g(x))$.
4. $\lim_{x \to c} \frac{f}{g}(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ *provided that* $\lim_{x \to c} g(x) \neq 0$.
5. $\lim_{x \to c} (af)(x) = a \lim_{x \to c} f(x)$ *whenever a is a number.*
6. $\lim_{x \to c} (f^k)(x) = (\lim_{x \to c} f(x))^k$ *whenever* $(\lim_{x \to c} f(x))^k$ *is defined.*

proof:
It is way beyond the scope of our course. One is enough to know that there are difficult proofs behind them.

Example 2.8 *Evaluate* $\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1}$.

solution:
First of all,

$$\begin{aligned} & \frac{\sqrt{x+3}-2}{x-1} \\ = & \frac{\sqrt{x+3}-2}{x-1} \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \\ = & \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \\ = & \frac{1}{\sqrt{x+3}+2} \end{aligned}$$

for $x \neq 1$. Now,

$$\begin{aligned} & \lim_{x \to 1} x+3 \text{ exists and is } 4 \\ & \lim_{x \to 1} \sqrt{x+3} \text{ exists and is } 2 \\ & \lim_{x \to 1} \sqrt{x+3}+2 \text{ exists and is } 4 \\ & \lim_{x \to 1} \frac{1}{\sqrt{x+3}+2} \text{ exists and is } \frac{1}{4} \end{aligned}$$

Each step follows by a certain part of the previous theorem. Hence,

$$\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1} = \frac{1}{4}.$$

Theorem 2.9 (Sandwich) *Let c be a number and f, g, h be functions so that $f(x) \leq g(x) \leq h(x)$ for all x near c. If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then,*

$$\lim_{x \to c} g(x) = L.$$

proof:
Again it is too difficult for beginners.

Example 2.10 *Evaluate* $\lim_{x \to 0} \sin x$.

solution:
Let P be the point on the unit circle whose inclination is x radians. Now, $\sin x$ is the perpendicular (and hence shortest) distance from P to the x-axis. x is the length of a curve (indeed an arc) connecting P to the x-axis. $-|x| < \sin x < |x|$ for x near 0. Moreover,

$$\lim_{x \to 0} |x| = \lim_{x \to 0} -|x| = 0.$$

By the sandwich theorem, $\lim_{x \to 0} \sin x = 0$.

Remark 2.11 *The limit $\lim_{x \to c} \sin x$ is simply $\sin c$. A similar result holds for other trigonometric functions, whenever it is defined at c.*

2.2 One-sided Limits

Definition 2.12 *Let f be a function and c be a number. If $f(x)$ approaches a number L when x is bigger (smaller) than but close to the number c, we say that $f(x)$ approaches (tends to) L as x approaches c from the right (left). Symbolically we write*

$$\lim_{x \to c^+} f(x) = L \quad (\lim_{x \to c^-} f(x) = L).$$

Theorem 2.13 *Let f be a function and c be a number. $\lim_{x \to c} f(x)$ exists if and only if both*

$$\lim_{x \to c^+} f(x) \text{ and } \lim_{x \to c^-} f(x)$$

exist and $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$. In this case all the three limits mentioned are the same.

proof:
Not as difficult as the previous ones but we will still omit it.

Example 2.14 *Let*

$$f(x) = \begin{cases} x & \text{when } x \geq 0 \\ 1 - x^2 & \text{when } x < 0 \end{cases}$$

Determine if the limit

$$\lim_{x \to 0} f(x)$$

exists.

solution:
Since $f(x) = x$ whenever $x > 0$,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$$

On the other hand $f(x) = 1 - x^2$ whenever $x < 0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 - x^2 = 1.$$

We see that $\lim_{x \to 0^+} f(x)$ is different from $\lim_{x \to 0^-} f(x)$. The limit $\lim_{x \to 0} f(x)$ does not exist by the previous theorem.

Example 2.15 *Determine if the limit*

$$\lim_{x \to 0} \sin |x|$$

exists.

solution:
Since $|x| = x$ whenever $x > 0$, $\lim_{x\to 0^+} \sin|x| = \lim_{x\to 0^+} \sin x = 0$.
On the other hand $|x| = -x$ whenever $x < 0$, $\lim_{x\to 0^-} \sin|x| = \lim_{x\to 0^-} \sin(-x) = 0$.
We see that both $\lim_{x\to 0^+} \sin|x|$ and $\lim_{x\to 0^-} \sin|x|$ exist and they are the same. The limit $\lim_{x\to 0} \sin|x|$ exists and is 0.

Remark 2.16 *The results concerning sums and products of limits and the sandwich theorem hold for one-sided limits.*

Theorem 2.17

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

proof:
First of all, suppose that $x > 0$. Let O be the center of the unit circle. X is the point on the unit circle whose inclination is x. A is the intersection of the positive x-axis and the unit circle. The line OX extends to meet the tangent of the unit circle at A in the point Y. The foot of perpendicular from X and Y to the x-axis are X', Y' respectively. Then,

$$\text{area of } \Delta OXX' \leq \text{area of the sector } XOA \leq \text{area of } \Delta OYY'.$$

Therefore,

$$\frac{1}{2}\sin x \cos x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x.$$

Rearranging yields (remember that $x > 0$)

$$\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.$$

Now, both $\lim_{x\to 0^+} \cos x$ and $\lim_{x\to 0^+} \frac{1}{\cos x}$ exist and they are both 1. By the sandwich theorem, $\lim_{x\to 0^+} \frac{x}{\sin x} = 1$. By a similar reasoning, $\lim_{x\to 0^-} \frac{x}{\sin x} = 1$. Therefore, $\lim_{x\to 0} \frac{x}{\sin x}$ exists and is 1.

Example 2.18 *Evaluate* $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$.

solution:
First of all

$$\frac{\sin 2x}{\sin 3x} = \tfrac{2}{3}\frac{\sin 2x}{2x}\frac{3x}{\sin 3x}.$$

Now, $\lim_{x\to 0} \frac{\sin 2x}{2x} = 1$, $\lim_{x\to 0} \frac{3x}{\sin 3x} = 1$. Thus, the limit of the product $\lim_{x\to 0} \frac{2}{3}\frac{\sin 2x}{2x}\frac{3x}{\sin 3x}$ exists and is $\frac{2}{3} \times 1 \times 1 = \frac{2}{3}$. Finally

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \frac{2}{3}.$$

Example 2.19 *Evaluate* $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

solution:
Since
$$\begin{aligned} &\frac{1-\cos x}{x^2} \\ &= \frac{1-\cos 2(\frac{x}{2})}{x^2} \\ &= \tfrac{1}{2}\frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2} \end{aligned}$$
and
$$\lim_{x\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} = 1.$$
So,
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

2.3 Limits at Infinity

Definition 2.20 *Let f be a function. If $f(x)$ approaches a number L when x is sufficiently large, we say that $f(x)$ approaches L as x tends to positive infinity, or we write symbolically as*
$$\lim_{x\to +\infty} f(x) = L.$$

Remark 2.21 *The symbol $\lim_{x\to -\infty} f(x) = L$ is defined in a similar way.*

Remark 2.22 *Results concerning sums and products of limits and the sandwich theorem hold for limits at infinity.*

Example 2.23 *Evaluate* $\lim_{x\to +\infty} \dfrac{1}{x}$.

solution:
Since $\dfrac{1}{x}$ is as small as we wish provided that $\dfrac{1}{x}$ is sufficiently large,
$$\lim_{x\to +\infty} \frac{1}{x} = 0.$$

Example 2.24 *Evaluate*
$$\lim_{x\to +\infty} \frac{3x^4+2x^2+1}{x^4+2x^2+3}.$$

solution:

$$\begin{aligned}&\frac{3x^4+2x^2+1}{x^4+2x^2+3}\\&=\frac{3+2\frac{1}{x^2}+\frac{1}{x^4}}{1+2\frac{1}{x^2}+3\frac{1}{x^4}}\end{aligned}$$

Now, both $\dfrac{1}{x^2}$ and $\dfrac{1}{x^4}$ tend to 0 as x tends to positive infinity. The denominator $3+2\dfrac{1}{x^2}+\dfrac{1}{x^4}$ tends to 3 as x tends to positive infinity. Similarly, the denominator $1+2\dfrac{1}{x^2}+3\dfrac{1}{x^4}$ tends to 1 as x tends to positive infinity. Therefore, the quotient tends to $\frac{3}{1}=3$ as x tends to positive infinity. Symbolically,

$$\lim_{x\to+\infty}\frac{3x^4+2x^2+1}{x^4+2x^2+3}=3.$$

Example 2.25 *Evaluate*

$$\lim_{x\to+\infty}\frac{\cos x}{x}.$$

solution:
Since $-1\leq\cos x\leq 1$,

$$-\frac{1}{x}\leq\frac{\cos x}{x}\leq\frac{1}{x}\text{ for }x>0.$$

Now,

$$\lim_{x\to+\infty}\frac{1}{x}=\lim_{x\to+\infty}-\frac{1}{x}=0.$$

By the sandwich theorem,

$$\lim_{x\to+\infty}\frac{\cos x}{x}=0.$$

Definition 2.26 *If either $\lim_{x\to+\infty}f(x)=L$ or $\lim_{x\to-\infty}f(x)=L$, we say that the graph of f has a horizontal asymptote $y=L$ (that is, the graph of the constant function L).*

2.4 Infinite Limits

Definition 2.27 *Let f be a function and c be a number. If $f(x)$ is as large as we wish when x is sufficiently close to c, we say that $f(x)$ tends to positive infinity as x approaches c. Symbolically,*

$$\lim_{x\to c}f(x)=+\infty.$$

Remark 2.28 *One-sided infinite limits* $\lim_{x\to c^+} f(x) = +\infty$ *and* $\lim_{x\to c^-} f(x) = +\infty$ *are defined in a similar way, except that "x approaches c" is modified into "x approaches and is bigger (smaller) than c".*

The symbol $\lim_{x\to c} f(x) = -\infty$ *means that* $\lim_{x\to c} -f(x) = +\infty$. *The one-sided analogues are defined similarly.*

We may also have the limits $\lim_{x\to+\infty} f(x) = +\infty$ *or other similar symbols. The spirit is there already and we do not formulate them one by one here.*

Example 2.29 *Evaluate*

$$\lim_{x\to 0^-} \frac{1}{x}.$$

solution:
As x approaches 0, the magnitude of $\frac{1}{x}$ is as large as one wish. If x is restricted to be a negative number and approaches 0 at the same time, $\frac{1}{x}$ is a negative quantity as large as one wish. Therefore,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Remark 2.30 *Although we had the symbols like* $\lim_{x\to c} f(x) = +\infty$ *to mean something as defined above. We still consider the limit* $\lim_{x\to c} f(x)$ *fail to exist under such circumstances.*

Definition 2.31 *Let* f *be a function and* c *be a number. If either* $\lim_{x\to c^-} f(x)$ *or* $\lim_{x\to c^+} f(x)$ *is* $+\infty$ *or* $-\infty$, *we say that the graph of* f *has a vertical asymptote* $x = c$ *(that is, the vertical line collecting the points whose first coordinate is* c*).*

Example 2.32 *Find the vertical asymptotes of the function* f *if*

$$f(x) = \frac{x}{x^2-1} \text{ for all } x \neq 1 \text{ or } -1.$$

solution:
For all $c \neq 1$ or -1, $\lim_{x\to c} x^2 - 1 = c^2 - 1$ exists and is not zero. Therefore, the limit

$$\lim_{x\to c} \frac{x}{x^2-1}$$

exists and the graph of f does not have a vertical asymptote $x = c$. On the other hand,

$$\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty$$

and

$$\lim_{x\to -1^-} \frac{x}{x^2-1} = -\infty.$$

Thus, the graph of f has vertical asymptotes $x = 1$ and $x = -1$.

2.5 Continuity

Definition 2.33 *Let f be a function and c be a number. f is continuous at c if*

$$\lim_{x \to c} f(x) = f(c).$$

We also say that f is continuous if f is continuous at all numbers.

Example 2.34
1. *All polynomials are continuous.*
2. *All rational functions are continuous except at the zeros of their denominators.*
3. *All trigonometric functions are continuous, except at the points where they are not defined.*
4. *The exponential function is continuous.*
5. *The logarithm function is continuous at all positive numbers.*

Example 2.35
1. *The sum of two continuous functions is continuous.*
2. *The difference between two continuous functions is continuous.*
3. *The product of two continuous functions is continuous.*
4. *The quotient of two continuous functions is continuous, except at the zeros of the latter function.*
5. *The composition of two continuous functions is continuous.*
6. *The inverse of a one-to-one continuous function is continuous.*
7. *The power of a continuous function is continuous, whenever it is defined.*

Remark 2.36 *Intuitively, a function is continuous means that we are able to draw its graph so that the pencil never have to leave the paper.*

Theorem 2.37 *Let f, g be functions so that f is continuous. Then,*

$$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$$

proof:
difficult

Theorem 2.38 *Let f be continuous on $[a, b]$. Then there is a number c such that $a \leq c \leq b$ such that $f(x) \leq f(c)$ for all x.*

proof: difficult

Theorem 2.39 (Intermediate Value) *Let f be a continuous function so that $f(a) > 0$ and $f(b) < 0$. Then, there is a c between a and b such that $f(c) = 0$.*

proof:
difficult

Example 2.40 *Two guys are moving along a line segment so that they are at different endpoints initially and ultimately reach the opposite endpoints. Prove that these guys meet somewhere in the line segment.*

proof:
Let the initial location of a first guy be O, and call another guy the second guy, who is at a distance a meters away from O on the line segment initially. Denote the distance of the first and second guy from O after t seconds by $f(t)$, $g(t)$ respectively, and define $h = f - g$. Then if the first guy reaches the other endpoint after T seconds,

$$h(0) = -a < 0, \text{ and } h(T) = a - g(T) \geq 0.$$

Note that both f and g are continuous, so is $h = f - g$. We apply intermediate value theorem to yield a number c such that $0 < c \leq T$ and $h(c) = 0$. In other words, $f(c) = g(c)$ so that these guys meet after c seconds.

Example 2.41 *Show that the equation*

$$2^x = 4x^2$$

has a root.

solution:
Let $f(x) = 2^x - 4x^2$ for all x. Then f is continuous and

$$f(0) = 1 > 0, \quad f(1) = -2 < 0.$$

By intermediate value theorem, there exists a number c such that $0 < c < 1$ and $f(c) = 0$. That is, $2^c = 4c^2$, so that the given equation has a root.