Example

$1+\tan^2\theta = \sec^2\theta, \quad \sec^2\theta - 1 = \tan^2\theta$

$9x^2-1$

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^3\sqrt{9x^2-1}}$$

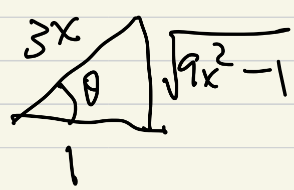
Let $x = \frac{1}{3}\sec\theta$

$\frac{dx}{d\theta} = \frac{1}{3}\sec\theta\tan\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\frac{1}{27}\sec^{\cancel{3}\,2}\theta\,\cancel{\sqrt{\tan^2\theta}}}$$

$$\boxed{\begin{aligned} &9x^2-1 \\ &= \sec^2\theta - 1 = \tan^2\theta \end{aligned}}$$

$$= 9\int_{\pi/4}^{\pi/3} \cos^2\theta\, d\theta$$

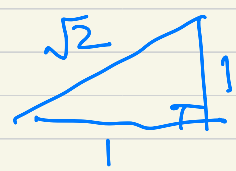


$x = \sqrt{2}/3$

$\frac{\sqrt{2}}{\cancel{3}} = \frac{1}{\cancel{3}}\sec\theta$

$$= \frac{9}{2}\int_{\pi/4}^{\pi/3} (1+\cos 2\theta)\, d\theta$$

$x = \frac{\sqrt{2}}{3}$

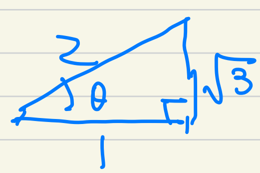


$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$x = 2/3$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/3}$$



$\frac{2}{\cancel{3}} = \frac{1}{\cancel{3}}\sec\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3$

$$= \frac{9}{2}\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \frac{\pi}{4} - \frac{1}{2}\right]$$

"Variations"

Let $x+1=\sqrt{6}\sin\theta$

$\dfrac{dx}{d\theta}=\sqrt{6}\cos\theta$

$$\int \frac{x^2}{\sqrt{5-2x-x^2}}\,dx$$

Completing the Square

$+5-(x^2+2x+1)$

$\parallel$

$6-(x+1)^2$

$\updownarrow$

$\boxed{6-6\sin^2\theta}$

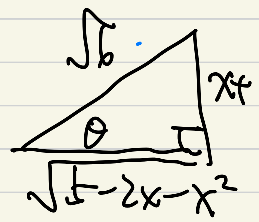
$$=\int \frac{x^2}{\sqrt{6-(x+1)^2}}\,dx$$

$$=\int \frac{(\sqrt{6}\sin\theta-1)^2}{\cancel{\sqrt{6(1-\sin^2\theta)}}}\,\cancel{\sqrt{6}\cos\theta}\,d\theta$$

$x=\sqrt{6}\sin\theta-1$

$$=\int\left(6\sin^2\theta-2\sqrt{6}\sin\theta+1\right)d\theta$$

$\hookrightarrow 2\sin^2\theta=1-\cos2\theta$



$$=\int\left(3(1-\cos2\theta)-2\sqrt{6}\sin\theta+1\right)d\theta$$

$2\sin\theta\cos\theta$

$$=4\theta-\frac{3\,\boxed{\sin2\theta}}{2}+2\sqrt{6}\cos\theta+C$$

$$=4\sin^{-1}\frac{x+1}{\sqrt{6}}-\cancel{3}\,\frac{(x+1)\sqrt{5-2x-x^2}}{\cancel{6}\ 2}+2\cancel{\sqrt{6}}\,\frac{\sqrt{5-2x-x^2}}{\cancel{\sqrt{6}}}+C$$

Completing the Square

$A \neq 0$,

$$Ax^2 + Bx + C$$

$$= A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}\right) + C - \frac{B^2}{4A}$$

$$= A\left(x + \frac{B}{2A}\right)^2 + \left(C - \frac{B^2}{4A}\right).$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

Integration By Parts $\leftrightarrow$ Product Rule in integration form!

$$[u(x)v(x)]' = \underline{\underline{u(x)v'(x)}} + v(x)u'(x)$$

$$u(x)v'(x) = [u(x)v(x)]' - v(x)u'(x)$$

$\downarrow$ antidifferentiation! $\downarrow$

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx}$$

Example

$$\int \underbrace{x}_{u}\,\underbrace{e^{-2x}}_{v'}\,dx$$

Let $u = x \rightarrow u' = 1$

$v' = \underline{\underline{e^{-2x}}} \rightarrow v = \frac{e^{-2x}}{-2}$

$$= \underbrace{x \cdot \frac{e^{-2x}}{-2}}_{uv} - \int \underbrace{\frac{e^{-2x}}{-2}}_{v} \cdot \underbrace{1}_{u'}\,dx$$

$$= -\frac{\cancel{x}}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx \longleftarrow \text{easier!}$$

$$= -\frac{\cancel{x}}{2}e^{-2x} + \frac{1}{2} \cdot \frac{e^{-2\cancel{x}}}{-2} + C$$

added after lecture

Example. $\displaystyle\int \underbrace{x}_{u} \underbrace{\cos x}_{v'}\, dx$

$u = x,\ u' = 1$

$v' = \cos x,\ v = \sin x$

$$= \underbrace{x \sin x}_{uv} - \overbrace{\int \underbrace{\sin x}_{v} \cdot \underbrace{1}_{u'} \cdot dx}^{\text{easier!}}$$

$$= x \sin x + \cos x + C.$$

What if we try: $\begin{cases} u = \cos x \Longleftrightarrow u' = -\sin x \\ v' = x\ ,\ v = \frac{x^2}{2} \end{cases}$

$$\int x \cos x\, dx = \frac{x^2}{2} \cos x - \int \frac{x^2}{2}(-\sin x)\, dx$$

$$= \frac{x^2}{2}\cos x + \frac{1}{2}\underbrace{\int x^2 \sin x\, dx}_{\text{Harder one?}}$$

Example

$$\int \underbrace{x^3}_{v'} \underbrace{\ln x}_{u} \, dx \qquad u = \ln x, \quad u' = \frac{1}{x}$$
$$v' = x^3, \quad v = \frac{x^4}{4}$$

$$= \underbrace{\frac{x^4}{4} \ln x}_{uv} - \int \underbrace{\frac{x^{\cancel{4}\,3}}{4}}_{v} \cdot \underbrace{\frac{1}{\cancel{x}}}_{u'} \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C.$$

Example

$$\int_0^1 \underbrace{(x^2 + x)}_{u} \underbrace{e^{2x}}_{v'} \, dx \qquad \boxed{u' = 2x+1} \qquad \boxed{v = \frac{e^{2x}}{2}}$$

$$= \underbrace{(x^2+x)}_{u} \underbrace{\frac{e^{2x}}{2}}_{v} \Big|_0^1 - \boxed{\int_0^1 \frac{e^{2x}}{2} \cdot (2x+1) \, dx}$$

simpler!

$$= \frac{\cancel{2} e^2}{\cancel{2}} - \frac{1}{2} \int \underbrace{(2x+1)}_{u} \underbrace{e^{2x}}_{v'} \, dx \qquad u = 2x+1, \quad u' = 2$$

$$= e^2 - \frac{1}{2} \left[(2x+1) \frac{e^{2x}}{2} \Big|_0^1 - \int \underbrace{\frac{e^{2x}}{\cancel{2}}}_{v} \cdot \underbrace{\cancel{2}}_{u'} \, dx \right]$$

$$= e^2 - \frac{1}{2} \left[\frac{3e^2 - 1}{2} - \frac{e^2 - 1}{2} \right] \qquad \left[\frac{e^{2x}}{2} \right]_0^1$$

Exercise $\int p(x) e^{kx} dx$ $\quad p(x) =$ a polynomial

$$= \frac{p(x) e^{kx}}{k} - \frac{1}{k}\int p'(x) e^{kx} dx$$

$$\int u v' dx = uv - \int v u' dx$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\int u\, dv = uv - \int v\, du$$

Example $\int x^2 \cos x\, dx = \int x^2 d\sin x$

$$= x^2 \sin x - \int \sin x\, dx^2$$

$$= x^2 \sin x - 2\int x \sin x\, dx$$

$$= x^2 \sin x + 2\int x\, d\cos x$$

$$= x^2 \sin x + 2x\cos x - 2\int \cos x\, dx$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + C.$$

$$\int \sin^{4}\theta \, d\theta \begin{cases} \to \text{double angle formula} \\ \to \text{integration by parts !} \end{cases}$$

n

$$\int \sin^{3}\theta \sin\theta \, d\theta = \int \sin^{3}\theta \cdot d(-\cos\theta)$$

$n-1$ $\quad$ $n-1$

$$= -\sin^{3}\theta\cos\theta + \int \cos\theta \, d\sin^{3}\theta$$

$$= -\sin^{3}\theta\cos\theta + \int \cos\theta \cdot 3\sin^{2}\theta\cos\theta \, d\theta$$

$$= -\sin^{3}\theta\cos\theta + 3\int \sin^{2}\theta \underbrace{\cos^{2}\theta}_{(1-\sin^{2}\theta)} \, d\theta$$

$$\int \sin^{4}\theta \, d\theta = -\sin^{3}\theta\cos\theta + 3\boxed{\int \sin^{2}\theta \, d\theta} \text{ easier}$$

$$-3\boxed{\int \sin^{4}\theta \, d\theta} \to \text{the same integral}$$

$$\cancel{4}\int \sin^{4}\theta \, d\theta = \frac{-\sin^{3}\theta\cos\theta}{4} + \boxed{\frac{3}{4}\int \sin^{2}\theta \, d\theta}$$

$$\boxed{\int \sin^{n}\theta \, d\theta = \frac{-\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\boxed{\int \sin^{n-2}\theta \, d\theta}}$$

Reduction Formula !

$$\int \sin^2\theta \, d\theta = -\frac{\sin\theta\cos\theta}{2} + \frac{1}{2}\int 1 \, d\theta$$

$$= -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C$$

$$\int_0^{\pi/2} \sin^6\theta \, d\theta = \underbrace{\left. -\frac{\sin^5\theta\cos\theta}{6} \right|_0^{\pi/2}}_{0} + \underline{\underline{\frac{5}{6}\int_0^{\pi/2} \sin^4\theta \, d\theta}}$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 1 \, d\theta$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

Exercise:

$$\int \cos^n\theta \, d\theta = \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta \, d\theta.$$

$$n = 2, 3, 4, 5, \ldots$$