Math1014 Calculus II

Week 7-8: Brief Review and Some Practice Problems

Polar Coordinates, Partial Fractions, Numerical Integration

- Get use to using polar coordinates (r, θ) to describe points in the plane, which are related to the rectangular coordinates by
$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$
- When dealing with derivative problems of polar curves $r = r(\theta)$, Chain Rule is useful: $\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$.
- When dealing with area or arc length problems in polar coordinates, i.e., when using
$$\text{area } = \frac{1}{2}\int_a^b r^2 d\theta, \quad \text{arc length } = \int_a^b \sqrt{r^2 + [r'(\theta)]^2} d\theta$$
be careful with the appropriate choice $[a, b]$ of the range of the "polar angle".
- The method of partial fractions is just about breaking up a rational function $f(x)$ into sum of terms like $\frac{A}{(ax+b)^2}$, or $\frac{Ax+B}{a^2(x+b)^2+c^2}$, whose indefinite integrals could be found by standard integration techniques, say by substitution $u = ax + b$, or $x + b = \frac{c}{a}\tan\theta$.
- Numerical integration:
  - how to use rectangles, trapeziums, or quadratic polynomials to approximate integrals;
  - how to use the *error bounds* of the numerical integration methods;

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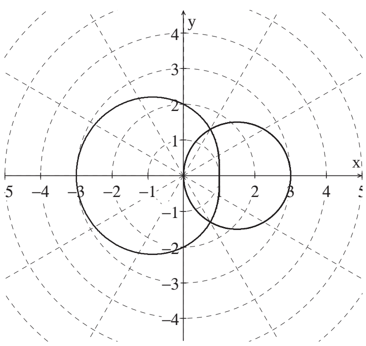
1. Find the area of the region that lies inside the first curve and outside the second curve given by the following polar equations. (Try sketching the curves first.)

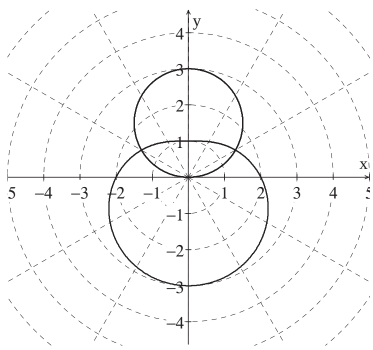
   (i) $r = 3\cos\theta$, $r = 2 - \cos\theta$ (ii) $r = 3\sin\theta$, $r = 2 - \sin\theta$.

   (i) When the two curves intersect, $3\cos\theta = 2 - \cos\theta$, i.e., $\cos\theta = \frac{1}{2}$. $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$.
$$\text{area } = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [(3\cos\theta)^2 - (2-\cos\theta)^2] d\theta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (8\cos^2\theta + 4\cos\theta - 4) d\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos 2\theta + 4\cos\theta) d\theta = \Big[\sin 2\theta + 2\sin\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = 3\sqrt{3}$$
   (ii) Similar to (i). (Actually a rotation of (i).)
$$\text{area } = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3\sin\theta)^2 - (2-\sin\theta)^2\right] d\theta = 3\sqrt{3}$$

(i)



(ii)



2. Find the slope of the tangent line to the polar curve at the point with angular coordinate $\theta = \frac{\pi}{3}$, and also the length of the polar curve.

   (i) $r = e^{2\theta}$, $0 \le \theta \le \pi$ (ii) $r = \cos^2\frac{\theta}{2}$.

(i) Slope at the point with angular coordinate $\theta = \frac{\pi}{3}$ is

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\Big|_{\theta=\frac{\pi}{3}} = \frac{\frac{de^{2\theta}\sin\theta}{d\theta}}{\frac{de^{2\theta}\cos\theta}{d\theta}}\Big|_{\theta=\frac{\pi}{3}} = \frac{2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta}{2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta}\Big|_{\theta=\frac{\pi}{3}} = \frac{2\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}}{2\cdot\frac{1}{2}-\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}+1}{2-\sqrt{3}}$$

The arc length is

$$L = \int_0^{\pi}\sqrt{(r)^2 + (\frac{dr}{d\theta})^2}\,d\theta = \int_0^{\pi}\sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}d\theta = \int_0^{\pi}\sqrt{5e^{4\theta}}d\theta$$

$$= \int_0^{\pi}\sqrt{5}e^{2\theta}d\theta = \frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^{\pi} = \frac{\sqrt{5}}{2}(e^{2\pi}-1)$$

(ii) The underlying curve is run through one round for $0 \le \theta \le 2\pi$.

Slope at the point with angular coordinate $\theta = \frac{\pi}{3}$ is

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\Big|_{\theta=\frac{\pi}{3}} = \frac{\frac{d\cos^2\frac{\theta}{2}\sin\theta}{d\theta}}{\frac{d\cos^2\frac{\theta}{2}\cos\theta}{d\theta}}\Big|_{\theta=\frac{\pi}{3}} = \frac{-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\theta + \cos^2\frac{\theta}{2}\cos\theta}{-\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\theta - \cos^2\frac{\theta}{2}\sin\theta}\Big|_{\theta=\frac{\pi}{3}}$$

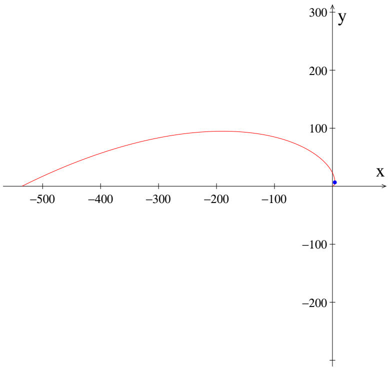
$$= \frac{-\frac{1}{2}\sin\theta\sin\theta + \frac{1}{2}(1+\cos\theta)\cos\theta}{-\frac{1}{2}\sin\theta\cos\theta - \frac{1}{2}(1+\cos\theta)\sin\theta}\Big|_{\theta=\frac{\pi}{3}} = \frac{-\frac{3}{4}+\frac{3}{4}}{\frac{-\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}} = 0$$

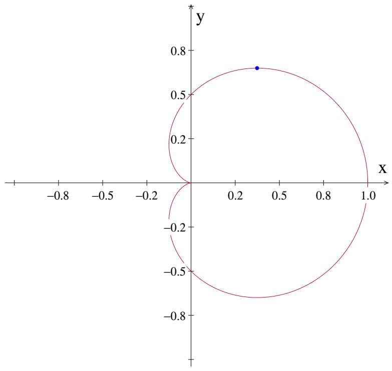
The arc length is

$$L = \int_0^{2\pi}\sqrt{\cos^4\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\,,d\theta = \int_0^{2\pi}\sqrt{\cos^2\frac{\theta}{2}}\,d\theta = \int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta$$

$$= 2\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 2\Big[2\sin\frac{\theta}{2}\Big]_0^{\pi} = 4$$

(Note that $\cos\frac{\theta}{2} \le 0$ for $\pi \le \theta \le 2\pi$.)





3. Evaluate the following integrals.

(i) $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}dx$ (ii) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}dx$ (iii) $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx$,

(iv) $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$ (v) $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}dx$ (vi) $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}dx$

(vii) $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x}dx$ (viii) $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx$ (vi) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x-\cos x}dx$

Solution

(i) $$\int_0^1 \frac{x-1}{x^2+3x+2}dx = \int_0^1 \frac{x-1}{(x+2)(x+1)}dx = \int_0^1 \Big[\frac{3}{x+2} - \frac{2}{x+1}\Big]dx$$

$$= \Big[3\ln|x+2| - 2\ln|x+1|\Big]_0^1 = 3\ln 3 - 5\ln 2$$

(ii) $$\int \frac{x^2+2x-1}{x^3-x}dx = \int \frac{x^2+2x-1}{x(x-1)(x+1)}dx = \int \Big[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\Big]dx$$

Note that

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1))(x+1)}$$
$$x^2+2x-1 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Putting in $x=0$, we have $A=1$. Putting in $x=1$, we have $B=1$. Putting in $x=-1$, we have $C=-1$. Thus

$$\int \frac{x^2+2x-1}{x^3-x}dx = \int \Big[\frac{1}{x}+\frac{1}{x-1}+\frac{-1}{x+1}\Big]dx = \ln x + \ln|x-1| - \ln|x+1| + C$$

(iii) Note that $\dfrac{x^2-5x+16}{(2x+1)(x-2)^2} = \dfrac{A}{2x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$; i.e;

$$x^2-5x+16 = A(x-2)^2+B(2x+1)(x-2)+C(2x+1)$$

Putting in $x=-\frac{1}{2}$, we have $A=3$. Putting in $x=2$, we have $C=2$. Putting in $x=0$, we have

$$16 = 4A-2B+C = 12-2B+2, \quad \text{i.e., } B=-1$$

Hence

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx \int \frac{3}{2x+1}dx + \int \frac{-1}{x-2}dx + \int \frac{2}{(x-2)^2} = \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

(iv) $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2+1}$

$$x^2-2x-1 = A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2$$

Putting in $x=1$, we have $-2=2B$, i.e., $B=-1$. Putting $B=-1$ back to the above equation, we have

$$2x^2-2x = 2x(x-1) = A(x-1)(x^2+1)+(Cx+D)(x-1)^2 \iff 2x = A(x^2+1)+(Cx+D)(x-1)$$

Putting in $x=1$, we have $A=1$, and hence $(Cx+D)(x-1) = -x^2+2x-1 = (-x+1)(x-1)$; i.e. $C=-1$, $D=1$. Consequently, we have

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int \frac{1}{x-1}dx + \int \frac{-1}{(x-1)^2}dx + \int \frac{-x+1}{x^2+1}dx$$
$$= \ln|x-1| + \frac{1}{x-1} - \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx = \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

(v) $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \dfrac{Ax+B}{(x+1)^2+1}+\dfrac{Cx+D}{[(x+1)^2+1]^2}$

$$x^3+2x^2+3x-2 = [Ax+B][(x+1)^2+1]+Cx+D$$

By comparing the coefficients of the x^3 term on both sides of the equation, we have $A=1$.

By comparing the x^2 term on both sides of the equation, we have $2=2A+B$; i.e. $B=0$. Hence

$$x^3+2x^2+3x-2 = x(x^2+2x+2)+Cx+D \iff x-2 = Cx+D$$

i.e., $C=1$, $D=-2$.

$$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}dx = \int \frac{x}{(x+1)^2+1}dx + \int \frac{x-2}{[(x+1)^2+1]^2}dx$$
$$= \int \frac{x+1}{(x+1)^2+1}dx - \int \frac{1}{(x+1)^2+1} + \int \frac{x+1}{[(x+1)^2+1]^2}dx - \int \frac{3}{[(x+1)^2+1]^2}dx$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - \tan^{-1}(x+1) - \frac{1}{2[(x+1)^2+1]} - \int \frac{3\sec^2\theta}{\sec^4\theta}d\theta \qquad \text{(by letting } x+1=\tan\theta\text{)}$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - \tan^{-1}(x+1) - \frac{1}{2[(x+1)^2+1]} - \int 3\cos^2\theta$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - \tan^{-1}(x+1) - \frac{1}{2[(x+1)^2+1]} - \int \frac{3}{2}(1+\cos 2\theta)d\theta$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - \tan^{-1}(x+1) - \frac{1}{2[(x+1)^2+1]} - \frac{3}{2}\theta - \frac{3}{4}\sin 2\theta + C$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - \tan^{-1}(x+1) - \frac{1}{2[(x+1)^2+1]} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3}{2}\;\frac{x+1}{(x+1)^2+1} + C$$

$$= \frac{1}{2}\ln|(x+1)^2+1| - \frac{3x+4}{2[(x+1)^2+1]} - \frac{5}{2}\tan^{-1}(x+1) + C$$

(vi) Let $u = x^{1/3}$ such that $du = \frac{1}{3}x^{-2/3}dx$, i.e., $3u^2 = dx$. Note also that $u = 0$ when $x = 0$, and $u = 1$ when $x = 1$. Hence

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}dx = \int_0^1 \frac{3u^2}{1+u}du = \int_0^1 \Big[3u - 3 + \frac{3}{1+u}\Big]du$$
$$= \Big[\frac{3}{2}u^2 - 3u + 3\ln|u+1|\Big]_0^1 = 3\ln 2 - \frac{3}{2}$$

(vii) Let $u = \sin x$ such that $du = \cos x dx$. Then

$$\int \frac{\cos x}{\sin^2 x + \sin x}dx = \int \frac{1}{u^2+u}du = \int \frac{1}{u}du - \int \frac{1}{u+1}du = \ln|u| - \ln|u+1| + C = \ln|\sin x| - \ln|1+\sin x| + C$$

(viii) Let $u = e^x$ such that $du = e^x dx$. Then

$$\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx = \int \frac{1}{(u-2)(u^2+1)}du = \int \Big[\frac{\frac{1}{5}}{u-2} - \frac{\frac{1}{5}u + \frac{2}{5}}{u^2+1}\Big]du$$
$$= \frac{1}{5}\ln|u-2| - \frac{1}{10}\ln|u^2+1| - \frac{2}{5}\tan^{-1}u + C$$
$$= \frac{1}{5}\ln|e^x-2| - \frac{1}{10}\ln|e^{2x}+1| - \frac{2}{5}\tan^{-1}e^x + C$$

(vi) Let $u = \tan\frac{x}{2}$. Then $du = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}(u^2+1)dx$; i.e., $dx = \frac{2du}{u^2+1}$. Moreover,

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1 = \frac{2}{u^2+1} - 1 = \frac{-u^2+1}{u^2+1}$$
$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\frac{1}{\sec^2\frac{x}{2}} = \frac{2u}{u^2+1}$$

Hence

$$\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x - \cos x}dx = \int_{\tan\frac{\pi}{6}}^{\tan\frac{\pi}{4}} \frac{2}{1 + \frac{2u}{u^2+1} - \frac{-u^2+1}{u^2+1}}\frac{1}{u^2+1}du$$
$$= \int_{\tan\frac{\pi}{6}}^{\tan\frac{\pi}{4}} \frac{1}{u^2+u}du = \int_{\tan\frac{\pi}{6}}^{\tan\frac{\pi}{4}} [\frac{1}{u} - \frac{1}{u+1}]du$$
$$= \Big[\ln|u| - \ln|u+1|\Big]_{1/\sqrt{3}}^{1} = -\ln 2 + \ln(1+\sqrt{3})$$

4. Use (a) the Trapezoidal rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified valued of n. (Round your answers to six decimal places.)

(i) $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}dx, \quad n = 4$ (ii) $\displaystyle\int_0^1 \sqrt{z}e^{-z}dz, \quad n = 4$

(i) (a) Trapezoidal Rule:

$$T_{10} = \frac{1}{2}\cdot\frac{2}{10}\Big[\frac{1}{\sqrt{1+0^3}} + \frac{2}{\sqrt{1+0.2^3}} + \frac{2}{\sqrt{1+0.4^3}} + \cdots + \frac{2}{\sqrt{1+1.8^3}} + \frac{1}{\sqrt{1+2^3}}\Big] \approx 1.401435$$

(b) Midpoint Rule:

$$M_{10} = \frac{2}{10}\Big[\frac{1}{\sqrt{1+0.1^3}} + \frac{1}{\sqrt{1+0.3^3}} + \frac{1}{\sqrt{1+0.5^3}} + \cdots + \frac{1}{\sqrt{1+1.7^3}} + \frac{1}{\sqrt{1+1.9^3}}\Big] \approx 1.402558$$

(c) Simpson's Rule:

$$S_{10} = \frac{2}{3\cdot 10}\left[\begin{array}{l} \frac{1}{\sqrt{1+0^3}} + \frac{4}{\sqrt{1+0.2^3}} + \frac{2}{\sqrt{1+0.4^3}} + \frac{4}{\sqrt{1+0.6^3}} + \frac{2}{\sqrt{1+0.8^3}} \\ + \frac{4}{\sqrt{1+1.0^3}} + \frac{2}{\sqrt{1+1.2^3}} + \frac{4}{\sqrt{1+1.4^3}} + \frac{2}{\sqrt{1+1.6^3}} + \\ + \frac{4}{\sqrt{1+1.8^3}} + \frac{1}{\sqrt{1+1.9^3}} \end{array}\right] \approx 1.402206$$

(ii) (a) Trapezoidal Rule:

$$T_{10} = \frac{1}{2}\cdot\frac{1}{10}\Big[\sqrt{0}e^{-0} + 2\sqrt{0.1}e^{-0.1} + 2\sqrt{0.2}e^{-0.2} + \cdots + 2\sqrt{0.9}e^{-0.9} + \sqrt{1}e^{-1}\Big] \approx 0.372299$$

(b) Midpoint Rule:

$$M_{10} = \frac{1}{10}\Big[\sqrt{0.05}e^{-0.05} + \sqrt{0.15}e^{-0.15} + \sqrt{0.25}e^{-0.25} + \cdots + \sqrt{0.95}e^{-0.95}\Big] \approx 0.380894$$

(c) Simpson's Rule:

$$S_{10} = \frac{1}{3\cdot 10}\left[\begin{array}{l} \sqrt{0}e^{-0} + 4\sqrt{0.1}e^{-0.1} + 2\sqrt{0.2}e^{-0.2} + 4\sqrt{0.3}e^{-0.3} + 2\sqrt{0.4}e^{-0.4} \\ +4\sqrt{0.5}e^{-0.5} + 2\sqrt{0.6}e^{-0.6} + 4\sqrt{0.7}e^{-0.7} + 2\sqrt{0.8}e^{-0.8} \\ +4\sqrt{0.9}e^{-0.9} + \sqrt{1}e^{-1} \end{array}\right] \approx 0.376330$$

5. Find the approximation T_{10} and M_{10} for $\int_1^2 e^{1/x}dx$, and then estimate the errors in the approximations. How large do we have to choose n so that the approximation T_n and M_n to the integral are accurate to within 0.0001?

$$T_{10} = \frac{1}{20}[e^1 + 2e^{1/1.1} + 2e^{1/1.2} + \cdots + 2e^{1/1.9} + e^2] \approx 2.021976$$
$$M_{10} = \frac{1}{10}[e^{1/1.05} + e^{1/1.15} + e^{1/1.25} + \cdots + e^{1/1.95}] \approx 2.019102$$

Since $f(x) = e^{1/x}$, $f'(x) = -x^{-2}e^{1/x}$, $f''(x) = \frac{1+2x}{x^4}e^{1/x}$, we have $|f''(x)| \leq (1 + 2(2))e = 5e$ for $1 \leq x \leq 2$. Using the error bound for the trapezoidal rule, and respectively for the midpoint rule, we have the estimations

$$E_T \leq \frac{5e}{12 \cdot 10^2} \approx 0.011326, \qquad E_M \leq \frac{5e}{24 \cdot 10^2} \approx 0.005663$$

To make sure that $E_T \leq 0.0001$, we may pick n satisfying $\frac{5e}{12n^2} \leq 0.0001$, i.e., $n^2 \geq \frac{50000e}{12} \approx 11326.1743$. For example, pick $n = 107$.

To make sure that $E_M \leq 0.0001$, we may pick n satisfying $\frac{5e}{24n^2} \leq 0.0001$, i.e., $n^2 \geq \frac{50000e}{24} = 5663.087$. For example, pick $n = 76$.

6. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}dx$ is accurate to within 0.00001?

$f(x) = e^{x^2}$, $f'(x) = 2xe^{x^2}$, $\quad f''(x) = 2e^{x^2} + 4x^2e^{x^2} \quad f'''(x) = 12xe^{x^2} + 8x^3e^{x^2}$,

and $f''''(x) = (12 + 48x^2 + 16x^3)e^{x^2}$, hence $|f''''(x)| \leq (12 + 48 + 16)e = 76e$ for $0 \leq x \leq 1$.

To get $E_S \leq 0.00001$, just pick an n satisfying $\frac{76e \cdot 1^5}{180n^4} \leq 0.00001$, i.e.,

$$n^4 \geq \frac{7600000e}{180} \approx 114771.8994$$

For example, pick $n = 20$. (Even number of subintervals for the Simpson's Rule.)

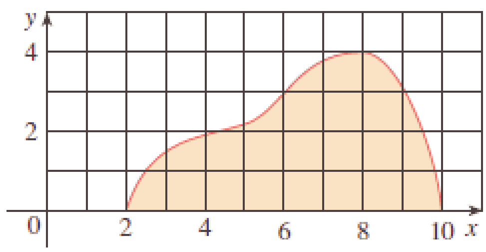
7. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's Rule to estimate the distance the runner covered during those 5 seconds.

$t\,(s)$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$v\,(m/s)$	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.76	10.81	10.81

$$\text{distance } = \int_0^5 v(t)dt$$

$$\approx \frac{5}{3 \cdot 10}\Big[0{+}4(4.67){+}2(7.34){+}4(8.86){+}2(9.73){+}4(10.22){+}2(10.51){+}4(10.67){+}2(10.76){+}4(10.81){+}10.81\Big] \approx 44.735 \ m$$

8. If the region shown in the figure is rotated about the y-axis to form a solid, use Simpson's Rule with $n = 8$ to estimate the volume of the solid.



Using the cylindrical shell method, we have by the Simpson's Rule that

$$\text{volume } = \int_2^{10} 2\pi x f(x)dx$$

$$\approx \frac{10-2}{3(8)} 2\pi \big[2\cdot f(2) + 4\cdot 3\cdot f(3) + 2\cdot 4\cdot f(4) + 4\cdot 5\cdot f(5) + 2\cdot 6\cdot f(6) + 4\cdot 7\cdot f(7) + 2\cdot 8\cdot f(8) + 4\cdot 9\cdot f(9) + 10\cdot f(10)\big]$$

$$\approx \frac{2\pi}{3}\big[2\cdot 0 + 4(3)(1.5) + 2(4)(2) + 4(5)(2.2) + 2(6)(3) + 4(7)(3.8) + 2(8)(4) + 4(9)(3) + 10(0)\big] \approx 821.8406$$

9. If f is a positive function and $f''(x) < 0$ for $a \le x \le b$, show that

$$T_n < \int_a^b f(x)dx < M_n$$

f is concave downward since $f''(x) < 0$, hence the trapezoid over any subinterval for T_n is under the graph of the positive function f. Thus

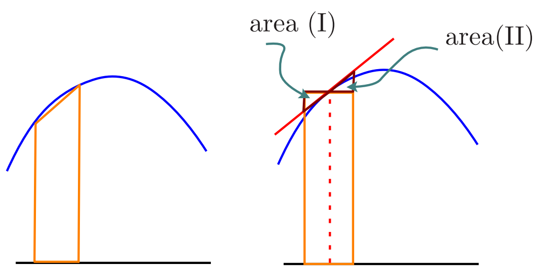
$$T_n = \begin{array}{c}\text{sum of trapezoidal areas}\\ \text{over subintervals}\end{array} \le \int_a^b f(x)dx\ .$$

On the other hand, the tangent line to the graph of f at the point $\left(\frac{a_{k-1}+a_k}{2}, f\left(\frac{a_{k-1}+a_k}{2}\right)\right)$ is above the graph. Hence if $h = \frac{b-a}{n}$, on the k-th subinterval $a_{k-1} = a + (k-1)h \le x \le a_k = a + kh$,

$$h \cdot f\left(\frac{a_{k-1}+a_k}{2}\right) - \int_{a_{k-1}}^{a_k} f(x)dx \ge \text{area } (I) - \text{area } (II) = 0$$

since I and II are congruent triangles. (In fact, it is easy to see that $h \cdot f\left(\frac{a_{k-1}+a_k}{2}\right)$ = area of the trapezoid under the tangent line over the subinterval.)

Hence $M_n \ge \displaystyle\int_a^b f(x)dx$.



Remark To show rigorously that the graph is squeezed between the chord and the tangent line as shown above, consider the increasing/decreasing properties of the functions $F(x) = f(x) - f(a_{k-1}) - \frac{f_{a_k} - f_{a_{k-1}}}{h}(x - a_{k-1})$ and $G(x) = f(x) - f(c) - f'(c)(x-c)$, where $c = (a_{k-1}+a_k)/2$.

(Or, another approach is to apply integration by parts to

$$\int_{a_{k-1}}^{a_k} f(x)d(x-a-h/2), \text{ and also } \int_{a_k-h/2}^{a_k} f(x)d(x-a_k-h/2) \text{ and } \int_{a_{k-1}+h/2}^{a_k} f(x)d(x-a_{k-1})$$

10. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\displaystyle\int_a^b f(x)dx$.

$|f''''(x)| = 0 \le 0$ if f is a polynomial of degree 3, hence the error bound for the Simpson's Rule says that the erorr $E_S = 0$.