Math1014 Calculus II
Brief Summary and Basic Problems on Vectors in the Plane and Space

- Algebraic and geometric operations on position vectors ("coordinate vectors"), e.g., parallelogram law for vector addition (triangle law, and also for free vectors too), and stretching vectors by scalar multiplication.

Vector Addition	Scalar Multiplication
$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle a_1 + a_2, b_1 + b_2\rangle$	$k\langle a, b\rangle = \langle ka, kb\rangle$
$\langle a_1, b_1, c_1\rangle + \langle a_2, b_2, c_2\rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2\rangle$	$k\langle a, b, c\rangle = \langle ka, kb, kc\rangle$
$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) + (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$	$k(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$
$= (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$	$= ka\mathbf{i} + kb\mathbf{j} + kc\mathbf{k}$

- Length of a vector: $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$ for 2D-vectors, and $|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$ for 3D-vectors.
- **Dot Product And Cross Product**

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \qquad \mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \qquad \mathrm{Comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

where $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and θ is the angle between these two vectors.

- 2× determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

  3 × 3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

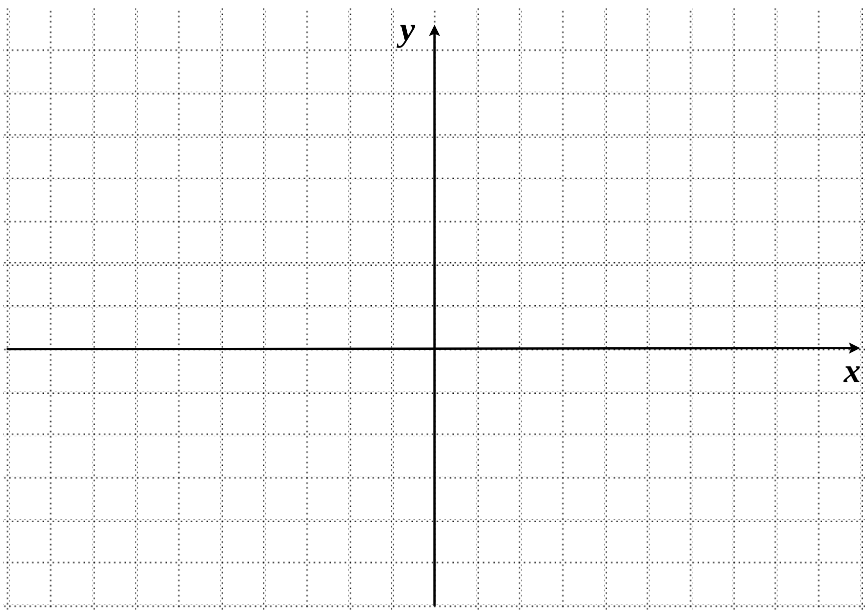
---

1. Let $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -1, 3\rangle$, $\mathbf{a} = \langle -3, -3\rangle$.

   (a) Draw arrows with initial point at the origin to represent these vectors in the plane.

   (b) Draw arrows to represent the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} + \mathbf{v} - \frac{1}{3}\mathbf{a}$ without algebraic calculation.

   (c) Find and draw a unit vector in the opposite direction of $\mathbf{a}$.



(d) Calculate $\mathbf{u}+\mathbf{v}-\frac{1}{3}\mathbf{a}$, then write it as a linear combination of the standard basis vector $\mathbf{i}$, $\mathbf{j}$.

(e) Find constants α and β such that $\mathbf{a}=\alpha\mathbf{u}+\beta\mathbf{v}$.

(f) Find the cosine of the angles between these vectors by using dot product.

(g) Find the projection of $\mathbf{u}$ on $\mathbf{a}$.

2. Let $\mathbf{a}=\langle -2,1,-1\rangle$, $\mathbf{b}=\langle -1,0,2\rangle$, $\mathbf{c}=\langle -3,1,0\rangle$.

   (a) Find a unit vector in the same direction as $\mathbf{b}$.

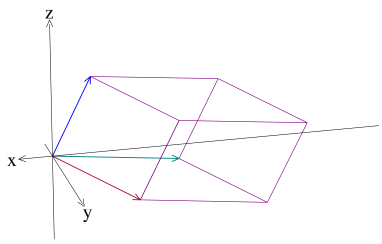
   (b) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

   (c) Find a vector perpendicular (orthogonal) to both $\mathbf{a}$ and $\mathbf{b}$.

   (d) Find the projection of $\mathbf{c}$ on the direction orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

   (e) Find the area of the triangle whose vertices are given by these three vectors.

   (f) Find the volume of the parallelopiped generated by these three vectors.



3. Given three points $P:(-2,1,3)$, $Q:(-1,0,2)$, $R:(-3,1,0)$ in space.

   (a) Use cross product to find a vector perpendicular to the plane generated by $\langle -1,0,2\rangle$ and $\langle -3,1,0\rangle$ (i.e., the plane containing the two arrows).

   (b) Use suitable orthogonal projection to find the distance from the point P to the plane in (a).

   (c) Find the orthogonal projection of the vector $\langle -2,1,3\rangle$ on the plane in (a).