MATH 2111: Tutorial 9
Coordinate Systems, Dimension and Rank

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- The unique representation theorem, coordinate mapping
- The dimension of a vector space, the basis theorem
- The rank of a matrix, the rank theorem
- The invertible matrix theorem

Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

1. $c_1 b_1 + c_2 b_2 = x$

$$\begin{cases} c_1 + 5c_2 = 4 \\ -2c_1 - 6c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -6 \\ c_2 = 2 \end{cases} \quad [x]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

2. $[b_1\ b_2\ b_3][x_B] = x$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right] \xrightarrow{3-3\,1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 2 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{3-2\,2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 0 \\ c_3 = 5 \end{cases}$$

$$[x]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

$$\mathcal{B} = \{b_1, b_2, b_3\} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$$

1. Show that the set $\mathcal{B}$ is a basis of $\mathbb{R}^3$

2. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.

3. Write the equation that relates x in $\mathbb{R}^3$ to $[x]_{\mathcal{B}}$

1. $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix} \xrightarrow[\substack{② + 3① \\ ③ + 4①}]{① \leftrightarrow ②} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 2 \\ 0 & -5 & -1 \end{bmatrix}$

$\xrightarrow{③ + \frac{5}{2}②} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

b_1, b_2, b_3 are linearly independent, then $\mathcal{B}$ is a basis of $\mathbb{R}^3$

2. $P_{\mathcal{B}} = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$

3. $x = P_{\mathcal{B}} [x]_{\mathcal{B}}$

For each subspace, find a basis, and state the dimension

$$\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$$

$$\left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

1. $\begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} t$

basis : $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$, dimension : 2

2. $\begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} a + \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} b + \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} c$

$\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

basis : $\left\{ \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}$, dimension : 2

Determine the dimensions of Nul A and Col A for the matrices

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Two pivots in column 1, 3, then dim $\mathrm{Col}A = 2$
Two free variables x_2, x_4, then $\dim \mathrm{Nul}A = 2$

2. Three pivots in column 1, 3, 4, then dim $\mathrm{Col}A = 3$
Three free variables x_2, x_5, x_6, then $\dim \mathrm{Nul}A = 3$

Assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since A, B row equivalent, $\operatorname{rank} A = \operatorname{rank} B = 3$

According to Rank Theorem, $\dim \operatorname{Null} A = 6 - \operatorname{rank} A = 3$

The pivots of B are in column $1, 2, 4$, row $1, 2, 3$

Then for A, $\operatorname{Col} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$

$\operatorname{Row} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 7 \\ 9 \\ -9 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \\ -2 \end{bmatrix}^T \right\}$

The reduced echelon form of A is $C = \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$AX = 0 \Leftrightarrow CX = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Then $\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$