MATH2111 Tutorial 11

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1 Diagonalization

1. **Definition**. A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix. i.e. If $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

2. **Theorem (The Diagonalization Theorem)**.
   (a) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
   (b) $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

3. **Procedures to Diagonalize a Matrix** A.
   (a) Find all the eigenvalues and the corresponding eigenvectors of A.
   (b) Construct D from the eigenvalues in step (a) to fill all the diagonal entries in D.
   (c) Construct P from the corresponding eigenvectors in step (a) to form the columns of P.

4. **Theorem**. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

5. **Theorem**. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.
   (a) For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
   (b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if
      i. the characteristic polynomial factors completely into linear factors and
      ii. the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.
   (c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

2 Exercises

1. Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix. Given λ and ρ are two distinct eigenvalues of A. Show that eigenspaces of λ and ρ are orthogonal. Namely, for any vectors $\mathbf{x}_1 \in \mathcal{E}_\rho(A), \mathbf{x}_2 \in \mathcal{E}_\lambda(A)$, it has $\mathbf{x}_1^\top \mathbf{x}_2 = 0$.

2. Given $A \in \mathbb{R}^{n \times n}$ and its characteristic function $f(\lambda) = \lambda^2(\lambda + 1)(\lambda - 1)(3 - \lambda)^{n-4}$.
(1) Write down eigenvalues and their multiplicities.
(2) What is characteristics function of matrix $A + 2I$?

3. Suppose $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(1) Find out characteristics function of A.

(2) Determine whether A is diagonalizable.

4. Diagonalize the following matrix, if possible,

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

5. Determine range of α such that the following matrix is similar to some real diagonal matrix,

$$A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}.$$