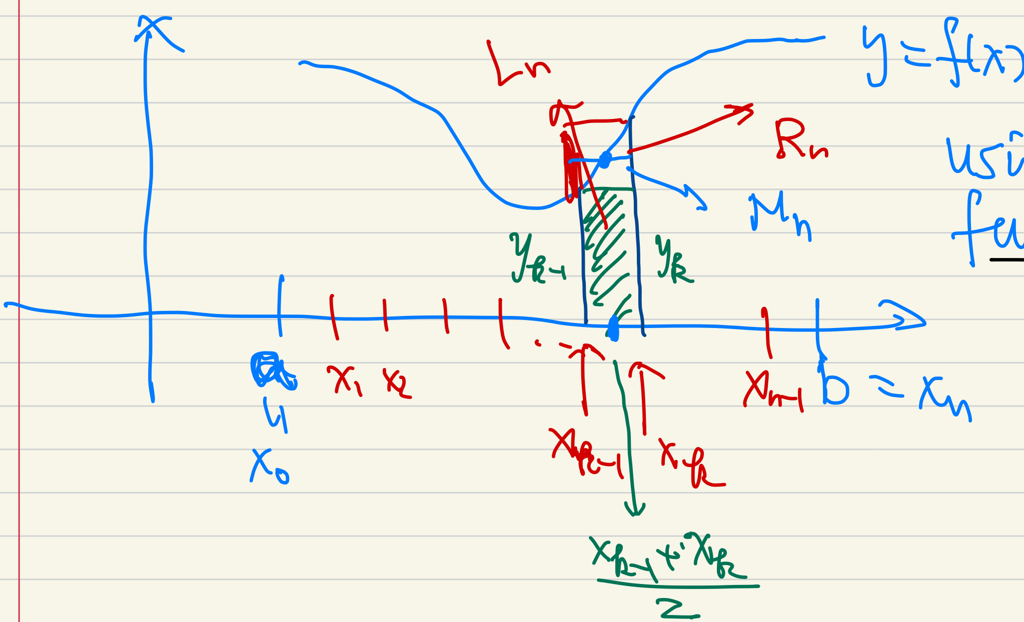
Numerical Integration: $\int_a^b f(x)\,dx$

$$L_n = \frac{b-a}{n}\left[y_0 + y_1 + y_2 + \cdots + y_{n-1}\right]$$

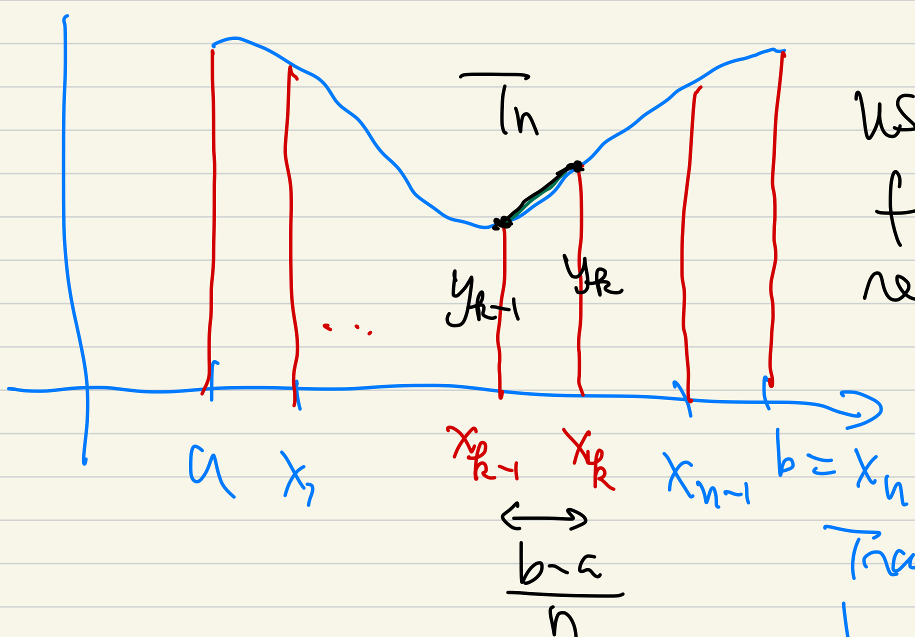
$$R_n = \frac{b-a}{n}\left[y_1 + y_2 + \cdots + y_n\right]$$

$$M_n = \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

$$T_n = \frac{b-a}{2n}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right]$$



using constant function to replace the original f on $[x_{k-1}, x_k]$

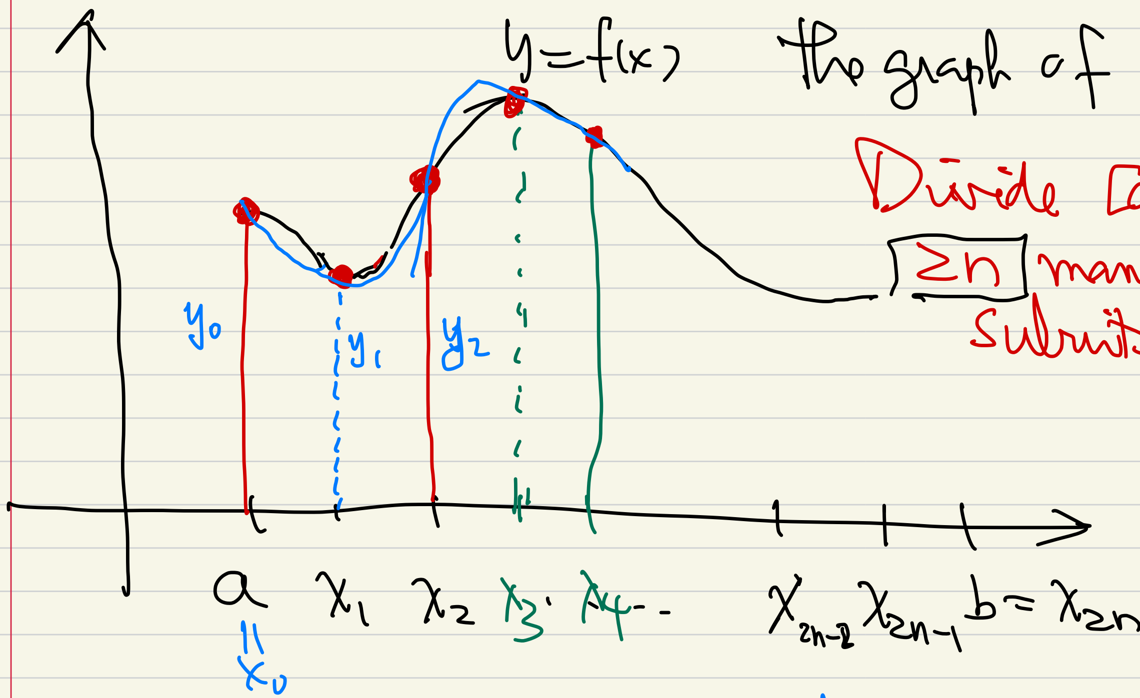


using a linear function to replace f on $[x_{k-1}, x_k]$

Trapezoidal area

$$\frac{1}{2}\cdot\frac{b-a}{n}\left(y_{k-1} + y_k\right)$$

Simpson's Rule $\Leftrightarrow$ using quadratic functions to approximate the graph of f



Divide $[a,b]$ into $\boxed{2n}$ many subintervals of equal length $= \dfrac{b-a}{2n} = x_1 - x_0$

There is a quadratic function $P_1(x)$ over $[x_0, x_2]$ such that

$$P_1(x_0) = y_0, \quad P_1(x_1) = y_1, \quad P_1(x_2) = y_2$$

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} P_1(x)\,dx \overset{?}{=} \frac{x_2 - x_0}{6}\left[y_0 + 4y_1 + y_2\right]$$

$$\frac{2 \cdot \frac{b-a}{2n}}{6}$$

do the same for all subintervals $[x_{2n-2}, x_{2n}]$

$$\int_{x_2}^{x_4} f(x)\,dx \approx \int_{x_2}^{x_4} P_2(x)\,dx = \frac{x_4 - x_2}{6}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots \qquad \vdots \qquad \downarrow \qquad \vdots$$

$$\frac{b-a}{6n}$$

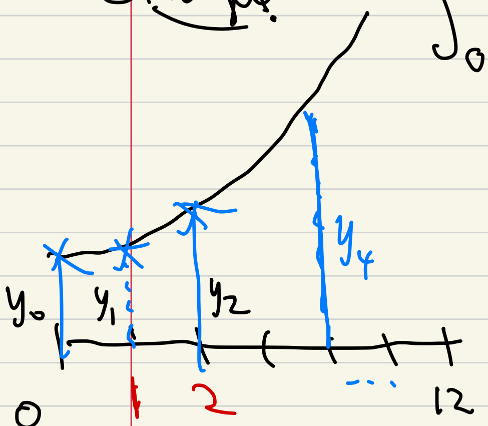
$$\int_a^b f(x)\,dx \approx S_{\underline{\underline{2n}}} = \frac{b-a}{6n}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{2n-1} + y_{2n}\right]$$

Simpson's Rule

= a weighted sum of T_n and M_n

$$S_{2n} \approx \frac{b-a}{6n}\Big[y_0+4y_1+2y_2+4y_3+2y_4+\cdots + 2y_{2n-2}+4y_{2n-1}+y_{2n}\Big]$$

Example $\displaystyle\int_0^{12} e^{x^2}\,dx \approx S_{12}$



$$S_{12} = \frac{12-0}{6\cdot 6}\Big[y_0+4y_1+2y_2+4y_3+2y_4+\cdots + 4y_{11}+y_{12}\Big]$$

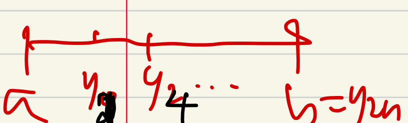
$S_{12} = S_{2\cdot 6}$, $\quad 2n$, $\quad 6\cdot n$

$$\approx 2\Big[e^0+4e^{1^2}+2e^{2^2}+4e^{3^2}+2e^{4^2}+4e^{5^2}+2e^{6^2}+\cdots+2e^{10^2}+4e^{11^2}+e^{12^2}\Big]$$

Remark:

$$S_{2n} = \frac{b-a}{2n}\Big[\tfrac13 y_0 + \tfrac43 y_1 + \cancel{\tfrac13 y_2} + \tfrac23 y_2 + \tfrac43 y_3 + \cancel{\tfrac13 y_4} + \tfrac23 y_4 + \tfrac43 y_5 + \cancel{\tfrac13 y_6} + \cdots + \tfrac23 y_{2n-2} + \tfrac43 y_{2n-1} + \boxed{\tfrac13 y_n}\Big]$$

$\frac{b-a}{n}\cdot\frac23\left[y_1+y_3+\cdots y_{2n-1}\right]$



$$\boxed{S_{2n} = \frac13 T_n + \frac23 M_n}$$

corrected, Feb 26.

Why?

$$\int_a^{a+2h} (\underbrace{Ax^2+Bx+C}_{=p(x)})\,dx$$

$u = \cancel{x}-a-h$

$\cancel{x}=a \Rightarrow u=-h$

$x=a+2h \Rightarrow u=h$

$$= \frac{2h}{6}\left[p(a)+4p(a+h)+p(a+2h)\right]$$

$$= (\underset{2h}{\text{interval length}})\cdot\left[\frac{1}{6}y_0+\frac{4}{6}y_1+\frac{1}{6}y_2\right]$$

$$\int_{-h}^{h} \boxed{p(u+a+h)}\,du \qquad = p(u+a+h)$$

$$= \int_{-h}^{h}(\overbrace{\alpha u^2+\beta u+\gamma}^{q(u)})\,du \overset{?}{=} \frac{2h}{6}\left[q(-h)+4q(0)+q(h)\right]$$

$$= \left[\frac{\alpha u^3}{3}+\frac{\beta u^2}{2}+\gamma u\right]_{-h}^{h}$$

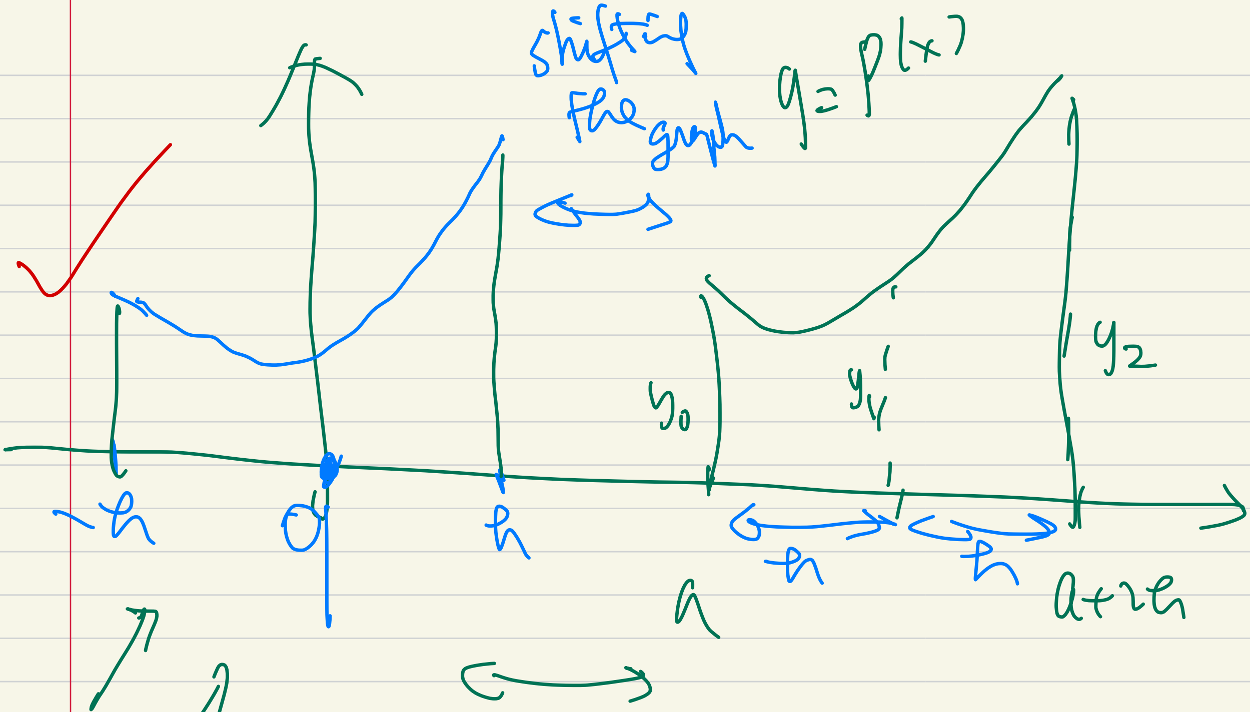
$$= \frac{2\alpha h^3}{3} + 2\gamma h$$

$$= \frac{h}{3}\left[2\alpha h^2+6\gamma\right] \overset{\text{o.k.}}{=} \frac{h}{3}\left[2\alpha h^2 + 6\gamma\right]$$

$$= \frac{h}{3}\cdot\left[\alpha h^2 - \cancel{\beta h} + \gamma + 4\gamma + \alpha h^2 + \cancel{\beta h} + \gamma\right]$$

$$= \frac{h}{3}\left[q(-h)+4q(0)+q(h)\right]$$

$$= \frac{h}{3}\left[p(a)+4p(a+h)+p(a+2h)\right]$$



same area.

$$Ax^2+Bx+C = f(x)$$

$$\int_{-h}^{h} f(x)\,dx$$

easier to compute and check that

$$= \frac{2h}{6}\left[f(-h) + 4f(0) + f(h)\right]$$

$$y_0 + 4y_1 + y_2$$

$$\int_a^{a+2h} (Ax^2+Bx+C)\,dx$$

= ✓

a little bit messy to compute.

$$= \frac{2h}{6}\left[y_0 + 4y_1 + y_2\right].$$