MATH 2111: Tutorial 5 Linear Transformations and Matrix Operations

T1A&T1B QUAN Xueyang
T1C&T2A SHEN Yinan
T2B&T2C ZHANG Fa

Department of Mathematics, HKUST

Review

- Null Space
- Column Space
- Kernel
- Range
- Basis

Determine whether the following is a subspace or not.

(1) $\{(1+a,\ b,\ a+b)\,|a, b \in \mathbb{R}\}$,

(2) $\{(1+a, b,\ 1+a+b)\,|a,\ b \in \mathbb{R}\}$,

(3) $\{(a, 3b, a+2b, 2b-a)\,|a, b \in \mathbb{R}\}$

(1). not a subspace. since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is not in that.

(2). is a subspace,

Denote the set as B

① take $a=-1,\ b=0$ get $\begin{pmatrix}0\\0\\0\end{pmatrix} \in B$

② if $\begin{pmatrix}x\\y\\z\end{pmatrix} \in B$ the we know, exists $a_0, b_0 \in \mathbb{R}$ such that

$$x = a_0+1,\quad y = b_0,\quad z = a_0+1+b_0$$

for any scalar $c \in \mathbb{R}$,

$$cx = ca_0 + c = (ca_0+c-1)+1$$
$$cy = cb_0$$
$$cz = ca_0+c+cb_0 = (ca_0+c-1)+1+cb_0.$$

take $a = ca_0+c-1$, $b = cb_0$, we know

$$c\begin{pmatrix}x\\y\\z\end{pmatrix} \in B$$

③ if $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} \in B$, $\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \in B$, then we have,

$$\begin{pmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{pmatrix} \in B$$

(3).

Denote the set as C.

$$\begin{pmatrix}a\\3b\\a+2b\\2b-a\end{pmatrix} = a\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} + b\begin{pmatrix}0\\3\\2\\2\end{pmatrix}.$$

hence, $C = \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\3\\2\\2\end{pmatrix}\right\}$. is a subspace of $\mathbb{R}^4$

Example Two

Null Space

Determine the null space of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}, \tag{1}$$

if $col(A)$ is subspace of $\mathbb{R}^k$, what is k?

① $Ax = 0.$

$$\begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix} \begin{matrix} ① \\ ② \end{matrix} \xrightarrow{② - 2\cdot ①} \begin{pmatrix} 1 & 3 & 2 & 8 \\ 0 & 1 & -2 & -13 \end{pmatrix} \begin{matrix} ①' \\ ②' \end{matrix}$$

$$\xrightarrow{①' - 3\cdot ②'} \begin{pmatrix} 1 & 0 & 8 & 47 \\ 0 & 1 & -2 & -13 \end{pmatrix}$$

that is, $\begin{cases} x_1 = -8x_3 - 47x_4 \\ x_2 = 2x_3 + 13x_4 \end{cases}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -8 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -47 \\ 13 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Null}(A) = \text{span}\left\{ \begin{pmatrix} -8 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -47 \\ 13 \\ 0 \\ 1 \end{pmatrix} \right\}$$

② If col(A) is space of $\mathbb{R}^k$, then $k = 2$, What is the base of the range for the above given matrix?

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}, \tag{2}$$

pivot positions are in the first two columns,

then, $\text{range}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$

(1) Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$?

(2) $S_1 = \{1, x, x^2\}$ is a basis of $\mathbb{P}_2$. Is $S_2 = \{1, x+1, (x+1)^2\}$ also a basis of $\mathbb{P}_2$?

(1) no, since $\begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$ doesn't have pivot positions in each row.

(2). yes, ① for any $a+bx+cx^2 \in \mathbb{P}_2$, it is a linear combinition of S_1

on the other hand, $x^2 = (x+1)^2 - 2(x+1) + 1$

$x = (x+1) - 1$

so, cx^2+bx+a

$= c\cdot[(x+1)^2-2(x+1)+1] + b[(x+1)-1] + a\cdot 1$

$= c\cdot(x+1)^2 + (b-2c)(x+1) + (c-b+a)\cdot 1$

$\Rightarrow cx^2+bx+a$ is also a linear combinition of S_2.

②. show S_2 is linearly independent,

if exists $u_1\ u_2\ u_3 \in \mathbb{R}$ such that

$u_1(x+1)^2 + u_2(x+1) + u_3\cdot 1 = 0$,

it is

$u_1x^2 + (2u_1+u_2)x + (u_1+u_2+u_3) = 0$,

it infers (because S_1 is linearly independent),

$$\begin{cases} u_1 = 0 \\ 2u_1+u_2 = 0 \\ u_1+u_2+u_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = 0 \\ u_2 = 0 \\ u_3 = 0 \end{cases}$$

$\Rightarrow S_2$ is linearly independent,

(1) Is $\left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ linearly independent?

(2) Suppose non-zero vectors $v_1, v_2, \cdots, v_n$ are orthogonal to each other, namely, $v_i^\top v_j = 0$ holds for any $i \neq j$, $i, j = 1, \cdots, n$. Prove $v_1, v_2, \cdots, v_n$ are linearly independent.

(1) yes,

(2), if there exist $a_1, \dots, a_n \in \mathbb{R}$ such that

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0,$$

then, multiply with v_1^T, it's $a_1 v_1^T v_1 + a_2 \underbrace{v_1^T v_2}_{\text{zero}} + \dots + a_n \underbrace{v_1^T v_n}_{\text{zero,}} = 0$

$\Rightarrow \quad a_1 \underbrace{v_1^T v_1}_{\text{non-zero}} = 0$

$\Rightarrow \quad a_1 = 0$

similary, get $a_2 = 0, \dots, a_n = 0$

$\Rightarrow \quad v_1, \dots, v_n$ are linearly independent,