Math1014 Calculus II, Spring 2016
Midterm Exam Solution

Part I: MC Questions.

	1	2	3	4	5	6	7	8	9	10	11
Makeup	*	A	C	E	A	A	C	E	B	D	C

1. What is the color version of your examination paper? Make sure your ID number has also been marked correctly in the I.D. No. Box in the MC answer form. **If you do not do both correctly, you lose the points of this question.**

   (a) Green (b) Orange (c) White (d) Yellow (e) None of the previous

2. The curves intersect each other when

$$(5y-2y^2)-(y^2-y)=6y-3y^2=3y(2-y)=0$$

The area enclosed is

$$\int_0^2[(5y-2y^2)-(y^2-y)]dy=\int_0^2(6y-3y^2)dy=\Big[3y^2-y^3\Big]_0^2=4$$

3. Let $u=\cos(2x)$ such that $du=-2\sin(2x)dx$.

$$\int_0^{\frac{\pi}{4}}30\sin(2x)\cos^2(4x)\,dx=\int_0^{\frac{\pi}{4}}30\sin(2x)(2\cos^2(2x)-1)^2dx$$

$$=\int_1^0-15(2u^2-1)^2du=-15\Big[\frac{4}{5}u^5-\frac{4}{3}u^3+u\Big]_1^0=7$$

4.

$$\int_{-4}^4x^2\sqrt{16-x^2}\,dx\overset{u=4\sin x}{=}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}16\sin^2x\cdot4\cos x\cdot4\cos xdx$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}16^2\sin^2u\cos^2udu=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}64\sin^22udu=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}32(1-\cos4u)du=32\pi$$

5.

$$\int_0^2\frac{x+3}{(x+1)(x+2)}dx=\int_0^2\Big(\frac{2}{x+1}-\frac{1}{x+2}\Big)dx$$

$$=\Big[2\ln|x+2|-\ln|x+1|\Big]_0^2=2\ln3-\ln2$$

6. The average value is
$$\frac{1}{32}\int_0^{32}\pi x^2 dy = \frac{1}{32}\int_0^{32}\pi\sqrt{\frac{y}{2}}dy = \frac{\pi}{32\sqrt{2}}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^{32} = \frac{8\pi}{3}$$

7. The volume is
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}\sin\frac{\pi}{3}(4\cos x - 2\sin 2x)^2dx = \sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos^2 x - 4\sin 2x\cos x + \sin^2 2x)dx$$
$$= \sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2 + 2\cos 2x - 4\sin 2x\cos x + \frac{1-\cos 4x}{2})dx = \frac{5\sqrt{3}\pi}{2}$$
(Note that $4\sin 2x\cos x$ is an odd function.)

8.
$$\int_0^1 f'(x)\ln(f(x))dx = \int_0^1 \ln f(x)df(x)$$
$$= \Big[f(x)\ln f(x)\Big]_0^1 - \int_0^1 f'(x)dx$$
$$= f(1)\ln f(1) - f(0)\ln f(0) - f(1) + f(0) = 6\ln 2 - 2$$

9. The was a typo in the question. The interval should have been $0 \le x \le 4$, not $0 \le y \le 4$.
$$y' = \frac{1}{\sqrt{2x}}$$
and the surface area is
$$\int_0^4 2\pi y\sqrt{1+(y')^2}dx = \int_0^4 2\pi\sqrt{2x}\sqrt{1+\frac{1}{2x}}\,dx$$
$$= 2\pi\int_0^4(2x+1)^{1/2}dx = \frac{2\pi}{3}(2x+1)^{3/2}\Big|_0^4 = \frac{52\pi}{3}$$
If you want to stick to the interval $0 \le y \le 4$, the surface area is given by the integral $\displaystyle\int_0^3 2\pi y\sqrt{1+y^2}dy$.

10.
$$r(\theta) = 3 + \sin 2\theta - \cos^2\theta, \qquad \frac{dr}{d\theta} = 2\cos 2\theta + 2\sin\theta\cos\theta$$
$$r(\frac{\pi}{4}) = 3 + 1 - \frac{1}{2} = \frac{7}{2}, \qquad r'(\frac{\pi}{4}) = 1$$
$$\left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{(\frac{7}{2}+1)\frac{\sqrt{2}}{2}}{(-\frac{7}{2}+1)\frac{\sqrt{2}}{2}} = -\frac{9}{5}$$

11.
$$\frac{dr}{d\theta} = \frac{1}{2}\frac{2\sec 2\theta\tan 2\theta}{\sqrt{\sec 2\theta}}$$
$$\int_0^{\frac{\pi}{6}}\sqrt{r^2+(r')^2}\,d\theta = \int_0^{\frac{\pi}{6}}\sqrt{\sec 2\theta + \sec 2\theta\tan^2 2\theta}\,d\theta = \int_0^{\frac{\pi}{6}}\sqrt{\sec^3 2\theta}\,d\theta$$

Part II: Long Questions

12. (a) The volume is given by the integral

$$V = \int_0^3 2\pi x[\ln(12 - x^2) - \ln 3]dx$$

(b) The volume is

$$V = \int_0^3 \pi[\ln(12 - x^2) - \ln 3)]dx^2 = \pi x^2[\ln(12 - x^2) - \ln 3)]\Big|_0^3 + \pi \int_0^3 x^2 \cdot \frac{2x}{12 - x^2}dx$$

$$= \int_0^3 \pi\Big[-2x + \frac{24}{x}12 - x^2\Big]dx = \Big[-x^2 - 12\ln|12 - x^2|\Big]_0^3 =$$

$$= \pi(12\ln 12 - 12\ln 3 - 9)$$

13. The work required is

$$W = \int_0^8 \rho g \pi x^2(y + 5)dy = \rho g \pi \int_0^8 \frac{1}{4} y^{2/3}(y + 5)dy$$

$$= \frac{9800}{4}\pi\Big[\frac{3}{8}y^{\frac{8}{3}} + 3y^{5/3}\Big]_0^8 = 470400\pi \quad (J)$$

14. (a) Since $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^2\sin^2\theta = r^2\cos^2\theta \cdot \frac{3 - r\cos\theta}{1 + r\cos\theta}$$

$$\sin^2\theta + r\cos\theta\sin^2\theta = 3\cos^2\theta - r\cos^3\theta$$

$$r\cos\theta(\sin^2\theta + \cos^2\theta) = 3\cos^2\theta - \sin^2\theta$$

$$r = (3\cos^2\theta - \sin^2\theta)\sec\theta$$

(b) Noting that $r = 0$ when $\theta = \frac{\pi}{3}$ and by the symmetry of the curve, the area of the loop is

$$A = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}(3\cos^2\theta - \sin^2\theta)^2\sec^2\theta d\theta = \int_0^{\frac{\pi}{3}} (4\cos^2\theta - 1)^2\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{3}} (16\cos^2\theta - 8 + \sec^2\theta)d\theta = 8\int_0^{\frac{\pi}{3}} \cos 2\theta d\theta + \Big[\tan\theta\Big]_0^{\frac{\pi}{3}}$$

$$= \Big[4\sin 2\theta\Big]_0^{\frac{\pi}{3}} + \sqrt{3} = 3\sqrt{3}$$

(c) The two curves meets at an angle θ when

$$(3\cos^2\theta - \sin^2\theta)\sec\theta = 2\cos\theta$$

$$3\cos^2\theta - \sin^2\theta = 2\cos^2\theta$$

$$\cos^2\theta = \sin^2\theta$$

As the two curves meets each other at $\theta = \pm\frac{\pi}{4}$, the required area is

$$A = \int_0^{\frac{\pi}{4}} (3\cos^2\theta - \sin^2\theta)^2\sec^2\theta d\theta - \int_0^{\frac{\pi}{4}} (2\cos\theta)^2\theta$$

$$= \int_0^{\frac{\pi}{4}} (4\cos^2\theta - 1)^2 \sec^2\theta d\theta - \int_0^{\frac{\pi}{4}} 4\cos^2\theta\theta$$

$$= \int_0^{\frac{\pi}{4}} (16\cos^2\theta - 8 + \sec^2\theta) d\theta - \int_0^{\frac{\pi}{4}} 4\cos^2\theta d\theta$$

$$= 6\int_0^{\frac{\pi}{4}} (1 + \cos 2\theta) d\theta + \Big[-8\theta + \tan\theta \Big]_0^{\frac{\pi}{4}}$$

$$= 6\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{4}} - 2\pi + 1 = 4 - \frac{\pi}{2}$$