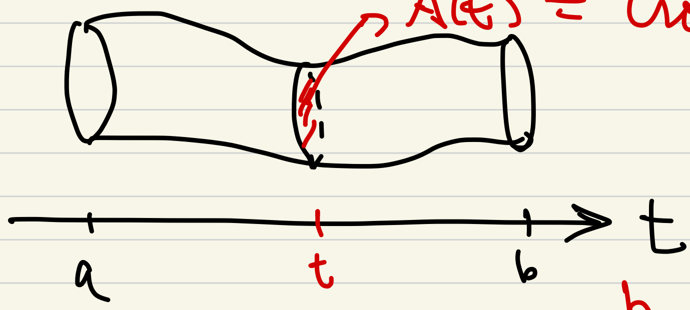
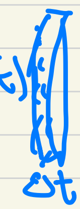
# Volume by the Method of Slicing

$A(t)$ = cross section area



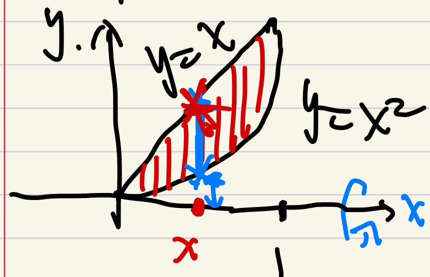


$A(t)$ ≈ volume $A(t)\Delta t$

$$\text{volume} = \int_a^b A(t)\,dt$$

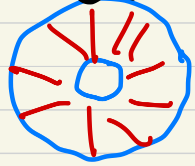
Riemann Sum!

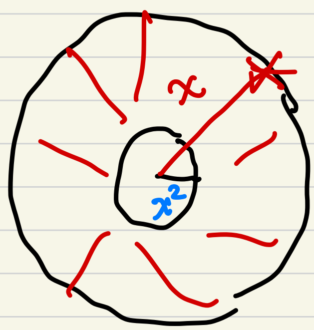
**Example**:



If the area is rotated about the x-axis, the volume of the solid of revolution is given by

$$\text{volume} = \int_0^1 \text{area}\;dx$$





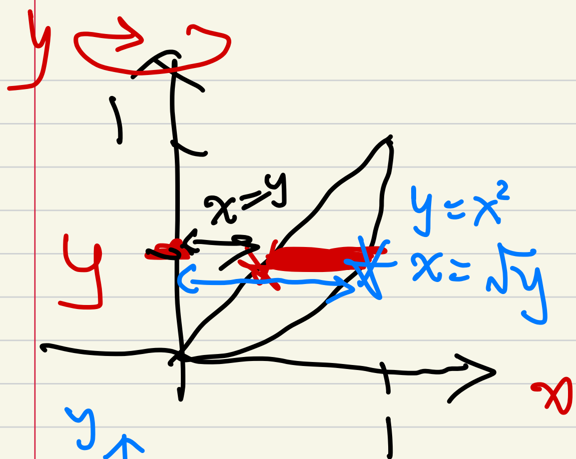
$$= \int_0^1 \left[\pi x^2 - \pi (x^2)^2\right] dx$$

area of outer circle − area of the inner circle

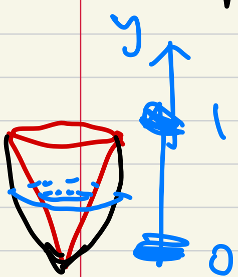
$$= \pi \int_0^1 (x^2 - x^4)\,dx$$

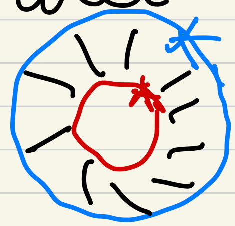
$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= \frac{2}{15}\pi \quad \text{(cubic units).}$$



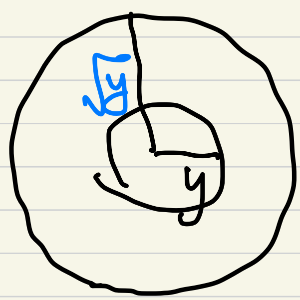

Rotate the area about the y-axis to generate a solid of revolution.




Volume $= \int_0^1$ area  ← a slice at y $\,dy$

$$= \int_0^1 \left[\pi(\sqrt{y})^2 - \pi(y)^2\right] dy$$

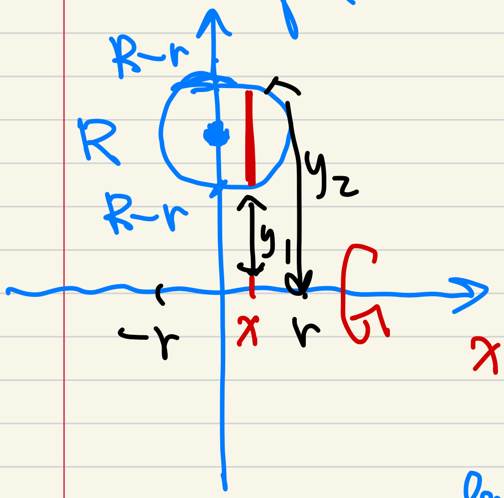
$$= \pi\int_0^1 (y - y^2)\, dy$$



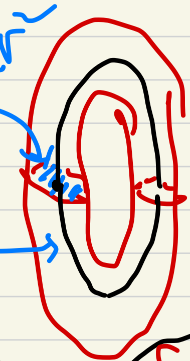

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}\pi$$

# Example: Volume of a Torus




Rotate the circle about the x-axis:




$$x^2+(y-R)^2=r^2$$

$$y=R\pm\sqrt{r^2-x^2}$$

$$y_1=R-\sqrt{r^2-x^2}$$

$$y_2=R+\sqrt{r^2-x^2}$$

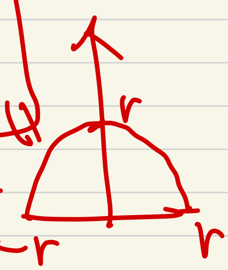
$0<r<R$

volume $= \int_{-r}^{r}$  $dx$

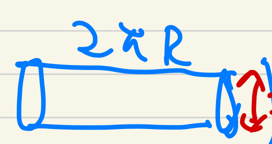

$$= \int_{-r}^{r}\left[\pi\left(R+\sqrt{r^2-x^2}\right)^2-\pi\left(R-\sqrt{r^2-x^2}\right)^2\right]dx$$

$$= \pi\int_{-r}^{r}4R\sqrt{r^2-x^2}\,dx$$

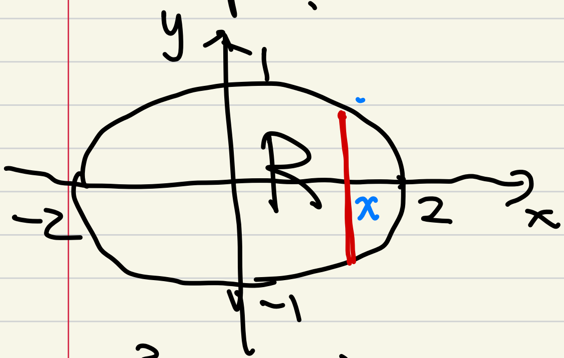
$$= 4\pi R\boxed{\int_{-r}^{r}\sqrt{r^2-x^2}\,d}$$

area of 


$$= 4\pi R\cdot\boxed{\frac{\pi r^2}{2}}$$

$$= 2\pi R\cdot\pi r^2 = \text{vol}\Big($$  $$\Big)$$

## Example.

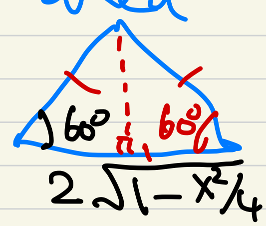



Suppose a solid with base R has all it cross sections along the x-axis as equilateral triangles. Find the volume of this solid.

$$\frac{x^2}{4} + y^2 = 1$$

$$y = \pm\sqrt{1-\frac{x^2}{4}}$$


$$\text{volume} = \int_{-2}^{2} \text{area} \; dx$$




$$= \int_{-2}^{2} \frac{1}{\cancel{2}} \cdot \underbrace{\cancel{2}\sqrt{1-x^2/4}}_{\substack{\text{base} \\ \text{length of} \\ \text{the } \Delta}} \cdot \underbrace{2\sqrt{1-x^2/4} \cdot \sin 60^\circ}_{\text{height}} \, dx$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

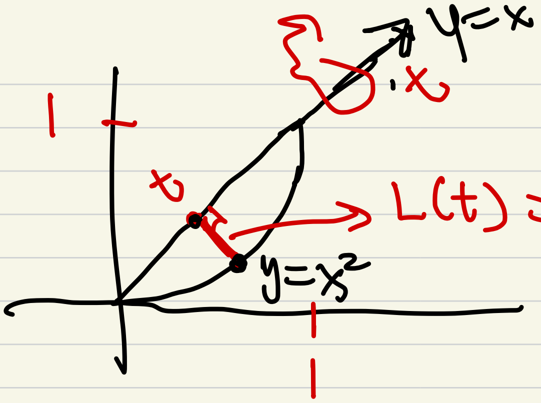
$$= \int_{-2}^{2} \sqrt{3}\left(1-\frac{x^2}{4}\right) dx$$

$$= \frac{\sqrt{3}}{4}\int_{-2}^{2} (4-x^2)\, dx$$

$$= \frac{\sqrt{3}}{4} \cdot \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{8\sqrt{3}}{3}$$

ChatBox

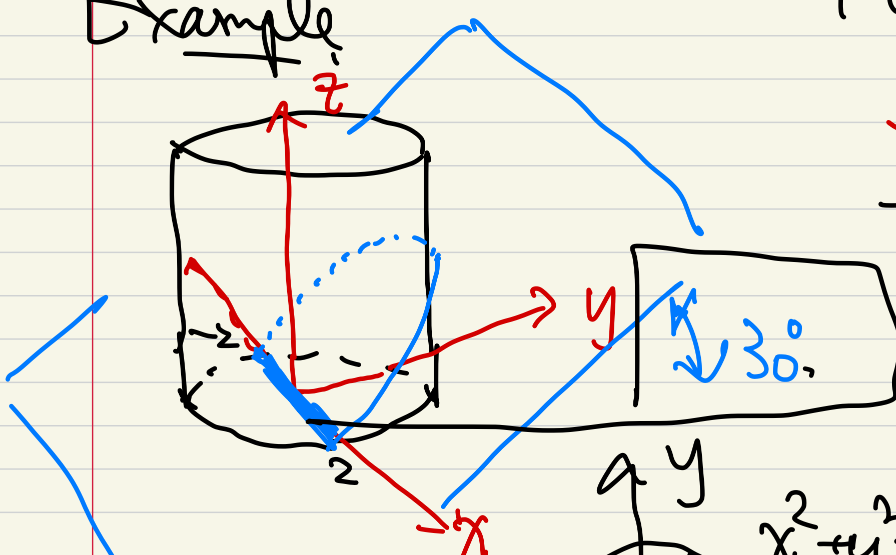
**Question.**

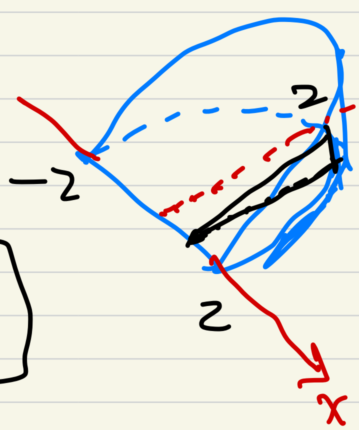



$$\mathrm{vol}\left(\quad\right) = \int_0^{\sqrt{2}} \pi \left[L(t)\right]^2 \, dt$$

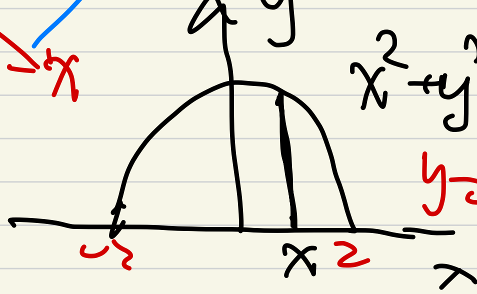
**Example**

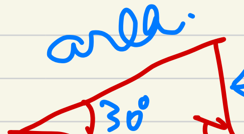
Find the volume of a wedge!




cross sections are right triangles



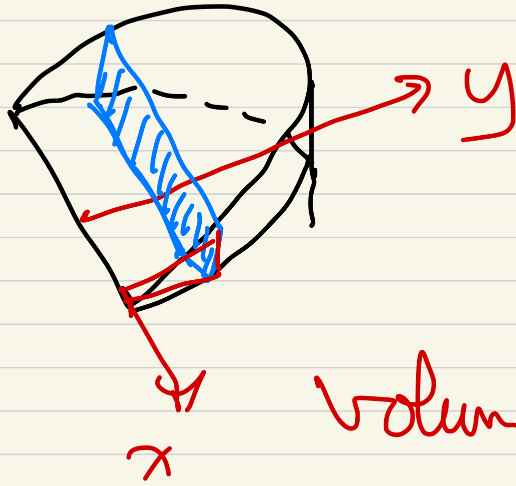

volume $= \int_{-2}^{2}$  $dx$

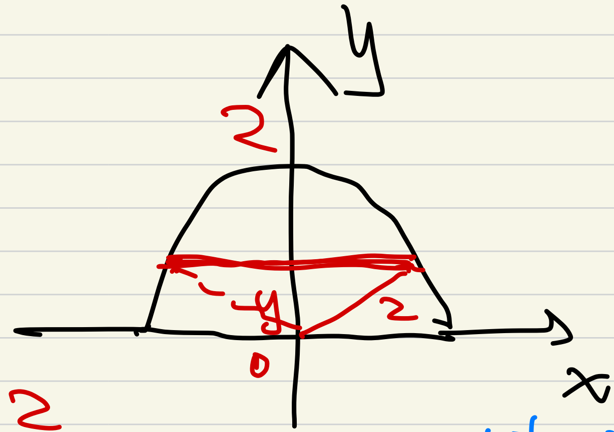


$\tan 30° = \frac{\sqrt{3}}{3}$

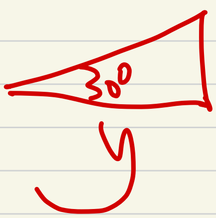
$$= \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2} \cdot \boxed{\sqrt{4-x^2} \cdot \tan 30°} \, dx$$

$$= \frac{\sqrt{3}}{6} \int_{-2}^{2} (4-x^2)\, dx = \frac{\sqrt{3}}{6}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$




volume $= \displaystyle\int_0^2 \underbrace{\boxed{\phantom{xxxxxxxx}}}_{2\sqrt{2^2-y^2}} \leftarrow y\tan 30^\circ \; dy$




$$= \int_0^2 2\sqrt{4-y^2} \cdot y \cdot \frac{\sqrt{3}}{3}\, dy$$

$$= \frac{\sqrt{3}}{3} \int_0^2 2y\,(4-y^2)^{1/2}\, dy$$

$$= \frac{\sqrt{3}}{3} \left[ -\frac{2}{3}(4-y^2)^{3/2} \right]_0^2$$

$$= \frac{2\sqrt{3}}{9} \cdot 4^{3/2} = \frac{16\sqrt{3}}{9}$$

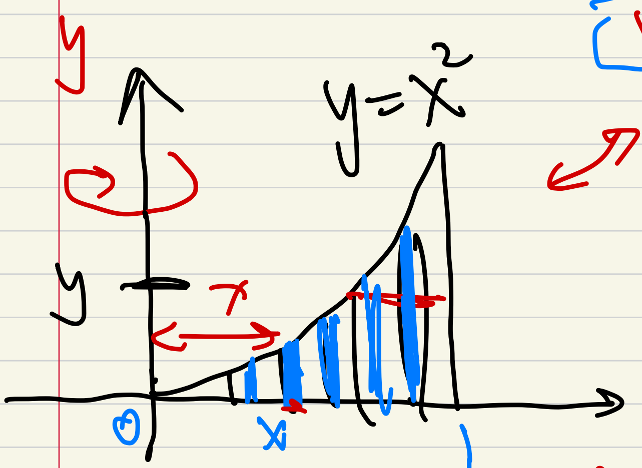
# Method of Cylindrical

(Another way to decompose a solid of revolution)

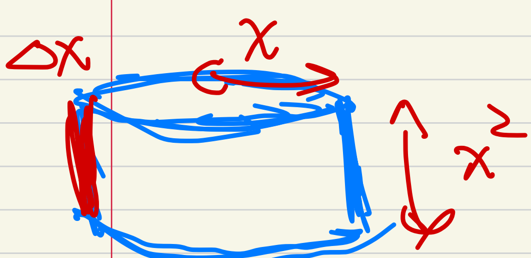
By slicing $\pi - \int_0^{y} \pi y\,dy$

$= \pi - \frac{\pi}{2}$

$= \frac{\pi}{2}$

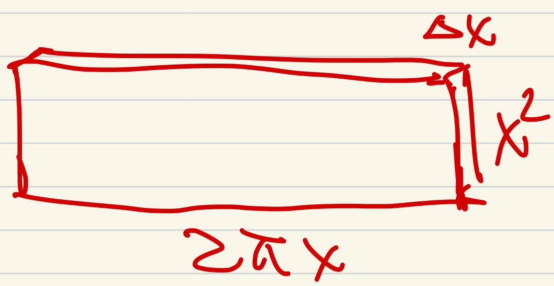



Volume of the solid of revolution = ?




≈

thin volume




$2\pi x \cdot x^2 \cdot \Delta x$

↓ summing as an integral

$$\text{volume} = \int_0^1 2\pi x \cdot x^2\,dx$$

$$= 2\pi \int_0^1 x^3\,dx$$

$$= \frac{\pi}{2}$$