COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Fall Semester – Tutorial 3

Question 1: Prove these statements by contrapositive.

(a) Suppose $x, y \in \mathbb{R}$. Show that if $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.

(b) For any integer a, b, $a + b \geq 15$ implies $a \geq 8$ or $b \geq 8$.

(c) Assume A, B, C are three sets. Prove $A \cap B \subseteq C \implies (A - C) \cap B = \emptyset$

Solution : (a) We will prove by contrapositive.
So we assume $y > x$. $y > x \implies y - x > 0$.
As $(x^2 + y^2)$ is a positive value we multiply it by both sides.

$(y - x)(x^2 + y^2) > 0(x^2 + y^2) \implies (y - x)(x^2 + y^2) > 0$.
$(y - x)(x^2 + y^2) > 0 \implies y^3 + yx^2 > x^3 + xy^2$
Which is the negation of our assumption.

(b) We will prove by contrapositive. We assume $a < 8$ and $b < 8$ which is $a \leq 7$ and $b \leq 7$.
So $a + b \leq 14$ and this is the negation of $a + b \geq 15$

(c) We will prove by contrapositive. Let's assume $(A - C) \cap B$ is not empty. So there exists an element x such that: $x \in (A - C) \cap B$.

$x \in (A - C)$ and $x \in B$. ($x \in A$ and $x \notin C$) and $x \in B$. This implies $x \in (A \cap B)$ and $x \notin C$ which means $(A \cap B) \nsubseteq C$. And this is the negation of our assumption.

Question 2: Determine whether each of these functions is a bijection from **R** to **R**.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^3$

(d) $f(x) = (x + 1)/(x + 2)$

Solution : Definitions:
one-to-one: $\forall x \forall y \; [(f(x) = f(y)) \rightarrow (x = y)]$
onto: $\forall y \exists x \; f(x) = y$

(a) To show that the function is one-to-one, note that if $2x + 1 = 2x' + 1$, then $x = x'$. To show that the function is onto, note that $2((y - 1)/2) + 1 = y$, so every number is in the range.

(b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written as $[1, \infty)$), not all of $\mathbf{R}$. (It is not injective either.)

(c) This function is a bijection, its inverse function is $f(y) = y^{1/3}$ (obtained by solving $y = x^3$ for x).

(d) This function is not a bijection. It is easy to see that it is not injective because there is no real number x such that $x + 1/x + 2 = 1$.

Question 3: Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) the even integers

(b) the real numbers between 0 and $\frac{1}{2}$

(c) the integers that are multiples of 7

Solution : (a) The even integers are countably infinite. We can list the set of even integers in the order $0, 2, -2, 4, -4, 6, -6, \ldots$, and pair them with the positive integers listed in their natural order. Thus $1 \leftrightarrow 0, 2 \leftrightarrow 2, 3 \leftrightarrow -2, 4 \leftrightarrow 4$, and so on. There is no need to give a formula for this correspondence—the discussion given is quite sufficient; but it is not hard to see that we are pairing the positive integer n with the even integer $f(n)$, where $f(n) = n$ if n is even and $f(n) = 1 - n$ if n is odd.

(b) The proof that the set of real numbers between 0 and 1 is not countable (in the lecture notes) can easily be modified to show that the set of real numbers between 0 and $\frac{1}{2}$ is not countable. We need to let the digit d_i be something like 2 if $d_{ii} \neq 2$ and 3 otherwise. The number thus constructed will be a real number between 0 and $\frac{1}{2}$ that is not in the list.

(C) This set is countably infinite, exactly as in part (b); the only difference is that there we are looking at the multiples of 2 and here we are looking at the multiples of 7. The correspondence is given by pairing the positive integer n with $7n/2$ if n is even and $-7(n-1)/2$ if n is odd: $0, 7, -7, 14, -14, 21, -21, \ldots$.

Question 4: Show that the set of irrational numbers is an uncountable set.

Solution : We know that the set of rational numbers is countable. If the set of irrational numbers were also countable, then the union of these two sets would also be countable. But their union, the set of real numbers, is known to be uncountable. This contradiction tells us that the set of irrational numbers is not countable.

Question 5: Answer these questions.

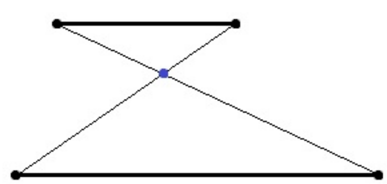
(a) Is it true that every uncountable set has the same cardinality?

(b) Suppose you have two line segments X, Y of different lengths. the set of the points on each segment (both endings are included) is uncountable. we call these sets S_X, S_Y accordingly. Prove that the $|S_X| = |S_Y|$.

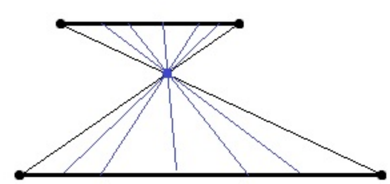
Solution : (a) No! As an example $\mathbb{R}$ and $P(\mathbb{R})$ are both uncountable. But $|\mathbb{R}| \neq |p(\mathbb{R})|$.

(b) Solution 1: This question is equivalent to this question. We want to prove the cardinality of the real numbers in range $[a, b]$ is the same as the cardinality of the real numbers in range $[c, d]$.
Basically we want to create a one-to-one mapping between the real numbers in $[a, b]$ to $[c, d]$.
$f(x) = [(x - a) \cdot \frac{d-c}{b-a}] + c$. This function is clearly a bijection between $[a, b]$ and $[c, d]$. (why?)

Solution 2: Here is a more visible(!) solution. By putting these two line segments in parallel. Connect the opposite endings as shown in picture a. Notice that after finding the collision point of these two new lines we can find the mapping of each point from one line segment by connecting it to the collision point and continuing it to hit the other line segment as shown in picture b. this mapping is clearly a bijection. (why?)



a)



b)