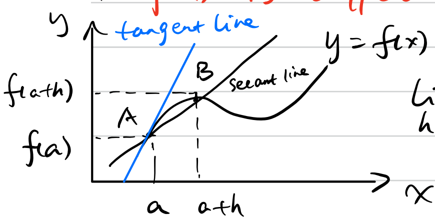
Review:

1. $f(x)$ is differentiable at $x=a \Leftrightarrow \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists.



$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \overset{\text{(if exists)}}{=} \text{slope of the tangent line of } f \text{ at } x=a$$

$$= \text{rate of change of } f \text{ at } x=a.$$

$f(x)$ is differentiable on $I \Leftrightarrow f(x)$ is differentiable at each point in I.

2. f is differentiable at $x=a \underset{\not\Leftarrow}{\Rightarrow} f(x)$ is continuous at $x=a$. $\left(\lim\limits_{x\to a} f(x) = f(a)\right)$

3. The derivative of $f(x)$: $f'(x) \overset{\text{defined by}}{=} \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ (if exists).

$f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ (if exists). $f'(a)$ is the derivative of f at $x=a$.

$f(x)$ is differentiable at $x=a \Leftrightarrow f'(a)$ exists.

Some other notations for $f'(x)$:

$y = f(x)$.

$$f'(x) = y' = \frac{d}{dx} y = \frac{d}{dx} f = Dy = Df \xRightarrow{\text{represent}} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\frac{d}{dx}$, D : differentiation operations.

(we can also write $\frac{dy}{dx}$ and $\frac{df}{dx}$).

$$f'(a) = y' \Big|_{x=a} = \frac{dy}{dx} \Big|_{x=a} = \frac{df}{dx} \Big|_{x=a} = Df \Big|_{x=a}.$$

Example: $y = f(x) = \sin x$.

We can use $(\sin x)'$, $\frac{d}{dx} \sin x$, $\frac{d \sin x}{dx}$, $D \sin x$ to represent the derivative of $\sin x$.

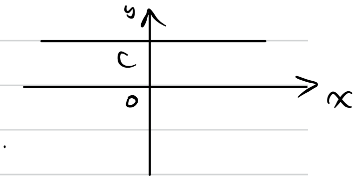
Notice: $\underbrace{(\sin x)' \Big|_{x=a}}_{\text{the derivative of } \sin x \text{ at } x=a} \neq \underbrace{(\sin a)'}_{\text{the derivative of the constant number } \sin a}$.

Computation of derivatives

also called "compute the derivative by first principle".

1. compute the derivative by definition : $\frac{dy}{dx} = f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(1) $f(x) = C$. (C is a constant). $f'(x) = 0$.



$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{C-C}{h} = \lim_{h\to 0} 0 = 0.$$

(2). $f(x) = x^n$, (n is a positive integer). $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$.

$f'(x) = n \cdot x^{n-1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{(\cancel{x+h-x})\left((x+h)^{n-1} + (x+h)^{n-2}\cdot x + \cdots + x^{n-1}\right)}{\cancel{h}}$$

$$= \lim_{h\to 0} (x+h)^{n-1} + (x+h)^{n-2}\cdot x + \cdots + x^{n-1} = x^{n-1} + x^{n-2}\cdot x + \cdots + x^{n-1} = n\cdot x^{n-1}.$$

If $f(x) = x$, then $f'(x) = 1$.

If $f(x) = x^2$, then $f'(x) = 2x$.

If $f(x) = x^3$, then $f'(x) = 3\cdot x^2$.

(3) $f(x) = \sqrt{x}$. $\quad f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h \quad (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x + \cancel{h} - x}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(4). $f(x) = \sin x$. $\quad f'(x) = \cos x$. Recall the addition formula: $\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \frac{\cosh - 1}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h}.$$

Recall that $\displaystyle \lim_{h\to 0} \frac{\sin h}{h} = 1$. $\quad \displaystyle \lim_{h\to 0} \frac{1 - \cos h}{\frac{1}{2} h^2} = 1$ $\quad \displaystyle \Rightarrow \lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{1 - \cos h}{\frac{1}{2} \cdot h^2} \cdot \left(-\frac{1}{2} h\right)$

$$= \lim_{h\to 0} \frac{1 - \cos h}{\frac{1}{2} h^2} \times \lim_{h\to 0} \left(-\frac{1}{2} h\right)$$

$$= 1 \times 0 = 0.$$

$$f'(x) = \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

(5). $f(x) = \cos x$. $\quad f'(x) = -\sin x$ $\quad$ (using the similer method).

means → $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$ exist.

2. Rules of differentiation

Suppose that f and g are differentiable.

① $(f\pm g)'(x) = f'(x) \pm g'(x)$ ② $(c\cdot f)'(x) = c\cdot f'(x)$. ($c$ is a constant).

③ $(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ ④ $\underline{\left(\frac{f}{g}\right)'(x)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$

proof of ①: $(f+g)'(x) = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} = \lim_{h\to 0}\underbrace{\frac{f(x+h)-f(x)}{h}}_{\sim} + \underline{\underline{\frac{g(x+h)-g(x)}{h}}} = \underset{\sim}{f'} + \underline{\underline{g'}}$.

proof of ②. $(c\cdot f)'(x) = \lim_{h\to 0}\frac{c\cdot f(x+h) - c\cdot f(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\cdot f'$.

proof of ③ $(f\cdot g)'(x) = \lim_{h\to 0}\frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\,(g(x+h)-g(x))}{h} + \lim_{h\to 0}\frac{g(x)\,(f(x+h)-f(x))}{h}$$

$$= \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f(x)g'(x) + g(x)f'(x)$$

proof of ④. $\left(\frac{f}{g}\right)'(x)$. We consider $f(x)$ as $g(x)\cdot\frac{f(x)}{g(x)}$. Rule ③ gives $f' = g'\cdot\frac{f}{g} + g\cdot\underset{\sim}{\left(\frac{f}{g}\right)'}$

$$\Rightarrow \left(\frac{f}{g}\right)' = \frac{f' - g'\cdot\frac{f}{g}}{g} = \frac{f'\cdot g - g'\cdot f}{g^2}.$$

Example 1: $f(x) = \frac{1}{x^n} = x^{-n}$. ($n$ is a positive integer).

$$f'(x) = \frac{(1)' \cdot x^n - 1 \cdot (x^n)'}{(x^n)^2} = \frac{-n \cdot x^{n-1}}{x^{2n}} = (-n) \cdot x^{-n-1}.$$

In general, $f(x) = x^n$, (n is an integer) $\Rightarrow f'(x) = n \cdot x^{n-1}$.

Example 2: $f(x) = \tan x = \frac{\sin x}{\cos x}$. $f'(x) = \frac{1}{(\cos x)^2}$.

Recall $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

$$f'(x) = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2} = \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{(\cos x)^2} = \frac{1}{(\cos x)^2}.$$

Recall $\cos^2 x + \sin^2 x = 1$. for any x.

3. The Chain rule → compute the derivative of composite functions.

Suppose g is differentiable at x and f is differentiable at $g(x)$.

Then the composite function $f \circ g = f(g(x))$ is differentiable at x,

and $$\underbrace{(f \circ g)'(x)}_{\text{rate of change of } f(g(x)) \text{ at } x.} = \underbrace{f'(g(x))}_{\text{rate of change of } f \text{ at } g(x).} \times \underbrace{g'(x)}_{\text{rate of change of } g \text{ at } x.}$$

Another statement: If $y = f(u)$, and $u = g(x)$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Notice: $y = f(u) = f(g(x)) = (f \circ g)(x)$ $\quad \frac{dy}{dx} = (f \circ g)'(x) = (f(g(x)))'$

$\frac{dy}{du} = f'(u) = f'(g(x))$. $\quad \frac{du}{dx} = g'(x)$.

the derivative of $f(g(x))$ at x. the derivative of f at $g(x)$.

means ↑ ↑ means

The Chain rule: $\underbrace{\big(f(g(x))\big)'} = \underbrace{f'(g(x))} \cdot g'(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}. \qquad y=f(u),\ u=g(x).$$

Example 1: $y=(2x^2+3x)^3 = f(g(x))$ $\qquad y=f(u)=u^3.$ $\qquad u=g(x)=2x^2+3x.$

$f'(u)=3u^2$ $\qquad g'(x)=4x+3.$

① : $\big(f(g(x))\big)' = f'(g(x))\cdot g'(x) = 3\cdot g^2(x)\cdot g'(x) = 3(2x^2+3x)^2\cdot(4x+3).$

② : $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2\cdot(4x+3) = 3(2x^2+3x)^2\cdot(4x+3).$

Example 2. $y=f(x)=x^{\frac{p}{q}}$. (p and q are two integers).

Our aim is to compute $\frac{dy}{dx}$. Let $h(x)=y^q=x^p$

Notice that $h(x)=x^p \Rightarrow \frac{dh}{dx} = \frac{d(x^p)}{dx} = p\cdot x^{p-1}.$

By the chain rule, we have $\frac{dh}{dx} = \frac{dh}{dy}\cdot\frac{dy}{dx} = q\cdot y^{q-1}\cdot\frac{dy}{dx}.$

$\Rightarrow q\cdot y^{q-1}\cdot\frac{dy}{dx} = p\cdot x^{p-1}$

$\Rightarrow \frac{dy}{dx} = \frac{p}{q}\,\frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}}$

$= \frac{p}{q}\cdot x^{\frac{p}{q}-1}.$

In general, $f(x)=x^n$ (n is a rational number) $\Rightarrow f'(x)=n\cdot x^{n-1}.$

In general, $f(x)=(g(x))^n$ (n is a rational number) $\Rightarrow f'(x)=n\cdot(g(x))^{n-1}\cdot g'(x).$

(consider $f(u)=u^n$, $u=g(x)$.).