MATH 2111: Tutorial 13

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Review

- The Gram-Schmidt Process
- Least-Squares Problem
- Applications to Linear Models
- Diagonalization of Symmetric Matrices

Example 1

Let W be a subspace of $\mathbb{R}^n$, and let $\{\mathbf{u_1}, \mathbf{u_2}\}$ be an orthonormal basis for W. So for each $\mathbf{y} \in \mathbb{R}^n$, we have,

$$\text{proj}_W(\mathbf{y}) = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2.$$

Based on this formula proving the following:
(1) $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.
(2) The kernel of proj_W is $W^\perp$.
(Hint: show that $\mathbf{y}$ is in the kernel of proj_W if and only if $\mathbf{y}$ is in $W^\perp$)
(3) What is proj_W^2? (Hint: if $\mathbf{y}$ is in W, $\text{proj}_W(\mathbf{y}) = \mathbf{y}$)

Example 2

Find a QR factorization of the matrix.

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

Example 3

State whether each of the following statement is true or false. (If it is true, give a brief justification; if it is false, give a counterexample.)

(1) Let U be an orthogonal matrix. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^n$, then $\{U\mathbf{v}_1, U\mathbf{v}_2, U\mathbf{v}_3\}$ is also an orthogonal set.

(2) Let U and W be subspaces of $\mathbb{R}^n$, and $U \subseteq W$. Then $U^\perp \subseteq W^\perp$.

(3) If U is a square matrix with orthonormal columns, then the rows of U are also orthonormal.

(4) Suppose $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are vectors in $\mathbb{R}^n$. If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

Example 4

Given data $(x_1, y_1), \ldots, (x_n, y_n)$, for a least-squares problem $y = \hat{\beta}_0 + \hat{\beta}_1 x$. And $(\hat{\beta}_0, \hat{\beta}_1)$ is the least-squares solution to

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

From linear algebra perspective, prove the formula for regression coefficients $\hat{\beta}_0, \hat{\beta}_1$ from statistics:

$$\hat{\beta}_1 = \frac{\mathrm{SS}_{xy}}{\mathrm{SS}_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where,

$$\mathrm{SS}_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}, \quad \mathrm{SS}_{xx} = \Sigma xx - \frac{(\Sigma x)^2}{n}, \quad \bar{y} = \frac{(\Sigma y)}{n}, \quad \bar{x} = \frac{(\Sigma x)}{n}$$

Example 4 - Continued

(1) By considering the normal equations, find a matrix M such that $M\left[\begin{array}{c}\hat{\beta}_0\\ \hat{\beta}_1\end{array}\right]=\left[\begin{array}{c}\Sigma y\\ \Sigma xy\end{array}\right]$. (The entries of M will depend on n, x and y.)

(2) Assume that $x_1, \ldots, x_n$ are not all the same, explain why this indicates $\hat{\beta}_0$ and $\hat{\beta}_1$ are unique.

(3) The uniqueness of $\hat{\beta}_0$ and $\hat{\beta}_1$ implies that the matrix M is invertible. By inverting M, show that $\hat{\beta}_1$ has the formula given above, then show that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$.

Example 5

Let A be the symmetric matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 & 5 \\ 2 & 1 & 0 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \\ 2 & 1 & 0 \\ 0 & -2 & 5 \end{bmatrix}^{-1}$$

Find an orthogonal matrix Q such that

$$A = Q \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} Q^{-1}.$$

Or explain why this Q doesn't exist.