§6.2 Orthogonal Sets

Def: A set of vectors $\{\vec{u}_1, \cdots, \vec{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$.

Example: Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set, where

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{pmatrix}$$

Solution:

$$\vec{u}_1 \cdot \vec{u}_2 = 3 \cdot (-1) + 1 \cdot 2 + 1 \cdot 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 3 \cdot (-\tfrac{1}{2}) + 1 \cdot (-2) + 1 \cdot \tfrac{7}{2} = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = -1 \cdot (-\tfrac{1}{2}) + 2 \cdot (-2) + 1 \cdot \tfrac{7}{2} = 0$$

Each pair of distinct vectors is orthogonal, and so $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

Thm: If $S = \{\vec{u}_1, \cdots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof: If $\vec{0} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$

$$0 = \vec{0} \cdot \vec{u}_1 = (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p) \cdot \vec{u}_1$$

$$= c_1\vec{u}_1 \cdot \vec{u}_1 + \cdots + c_p\vec{u}_p \cdot \vec{u}_1$$

$$= c_1 \|\vec{u}_1\|^2 \Rightarrow c_1 = 0$$

Similarly, we get $c_2 = 0, \cdots, c_p = 0$

Def: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Thm: Let $\{\vec{u}_1, \cdots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each y in W, the weights in the linear combination

$$\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$$

are given by

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}, \quad j = 1, \cdots, p$$

Proof: The orthogonality of $\{\vec{u}_1, \cdots, \vec{u}_p\}$ shows that

$$\vec{y}\cdot\vec{u}_1 = (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p)\cdot\vec{u}_1 = c_1\,\vec{u}_1\cdot\vec{u}_1$$

Since $\vec{u}_1\cdot\vec{u}_1 \neq 0$, $c_1 = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$

Similarly, we get $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$, $j = 2, \cdots, p$

Ex: $\left\{\vec{u}_1 = \begin{pmatrix}3\\1\\1\end{pmatrix}, \vec{u}_2 = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \vec{u}_3 = \begin{pmatrix}-\frac{1}{2}\\-2\\7/2\end{pmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express the vector $\vec{y} = \begin{pmatrix}6\\1\\-8\end{pmatrix}$ as a linear combination of the vectors in S.

Solution: $\vec{y}\cdot\vec{u}_1 = 11$, $\quad \vec{y}\cdot\vec{u}_2 = -12$, $\quad \vec{y}\cdot\vec{u}_3 = -33$

$\vec{u}_1\cdot\vec{u}_1 = 11$, $\quad \vec{u}_2\cdot\vec{u}_2 = 6$, $\quad \vec{u}_3\cdot\vec{u}_3 = 33/2$

Then $\vec{y} = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\cdot\vec{u}_2 + \dfrac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\cdot\vec{u}_3$

$$= \frac{11}{11}\vec{u}_1 + \frac{-12}{6}\vec{u}_2 + \frac{-33}{33/2}\vec{u}_3$$

$$= \vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3$$

Remark: If the basis were not orthogonal, it would be necessary to solve a system of linear equations in order to find the weights.

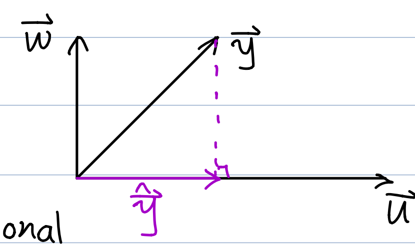
* An Orthogonal Projection

Let $\vec{u} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$

Write $\vec{y} = \hat{\vec{y}} + \vec{w}$ such that $\hat{\vec{y}} = c\vec{u}$ (parallel to $\vec{u}$)

and $\vec{w}$ is perpendicular to $\vec{u}$.

$\hat{\vec{y}}$ is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$; $\vec{w}$ is called the component of $\vec{y}$ orthogonal to $\vec{u}$.



$L \triangleq \text{span}\{\vec{u}\}$: line parallel to $\vec{u}$.

$\hat{\vec{y}}$ is called the orthogonal projection of $\vec{y}$ onto L, denoted by $\text{proj}_L \vec{y}$

$$\vec{y} = \hat{\vec{y}} + \vec{w}$$

$$\vec{u}\cdot\vec{y} = \vec{u}\cdot\hat{\vec{y}} + \vec{u}\cdot\vec{w}$$

$$= \vec{u}\cdot\hat{\vec{y}}$$

$$= \vec{u}\cdot c\vec{u}$$

So $\quad c = \dfrac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}$

Thus $\quad \hat{\vec{y}} = \dfrac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}\vec{u}, \qquad \vec{w} = \vec{y} - \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$

Ex: Let $\vec{y} = \begin{pmatrix}7\\6\end{pmatrix}, \quad \vec{u} = \begin{pmatrix}4\\2\end{pmatrix}$.

1) Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$.

2) Write $\vec{y}$ as the sum of two orthogonal vectors, one in $L = \text{span}\{\vec{u}\}$ and one orthogonal to $\vec{u}$

3) Find the distance from $\vec{y}$ to L.

Solution: 1) $\text{Proj}_L \vec{y} = \hat{\vec{y}} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$

$$= \frac{7\cdot 4 + 2\cdot 6}{4^2+2^2}\begin{pmatrix}4\\2\end{pmatrix}$$

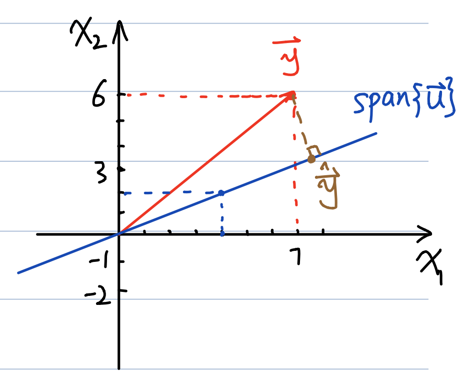
$$= \begin{pmatrix}8\\4\end{pmatrix}$$

2) $\vec{w} = \vec{y} - \hat{\vec{y}} = \begin{pmatrix}7\\6\end{pmatrix} - \begin{pmatrix}8\\4\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

Then $\vec{y} = \hat{\vec{y}} + \vec{w}$

$$= \begin{pmatrix}8\\4\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix}$$

3) $\|\vec{y} - \hat{\vec{y}}\| = \sqrt{(-1)^2+2^2} = \sqrt{5}$



* Orthonormal Sets

Def: $\{\vec{u}_1, \cdots, \vec{u}_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors, i.e. $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$ and $\vec{u}_i \cdot \vec{u}_i = 1$.

If W is a subspace spanned by an orthonormal set $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p\}$, then $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_p\}$ is an orthonormal basis for W.

Ex: $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

Ex: Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -\frac{1}{\sqrt{66}} \\ -\frac{4}{\sqrt{66}} \\ \frac{7}{\sqrt{66}} \end{pmatrix}$$

Solution: $\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{\sqrt{11}} \cdot \left(-\frac{1}{\sqrt{6}}\right) + \frac{1}{\sqrt{11}} \cdot \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{11}} \cdot \frac{1}{\sqrt{6}} = 0$

$$\vec{v}_2 \cdot \vec{v}_3 = \left(-\frac{1}{\sqrt{6}}\right) \cdot \left(-\frac{1}{\sqrt{66}}\right) + \frac{2}{\sqrt{6}} \cdot \left(-\frac{4}{\sqrt{66}}\right) + \frac{1}{\sqrt{6}} \cdot \frac{7}{\sqrt{66}} = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{3}{\sqrt{11}} \cdot \left(-\frac{1}{\sqrt{66}}\right) + \frac{1}{\sqrt{11}} \cdot \left(-\frac{4}{\sqrt{66}}\right) + \frac{1}{\sqrt{11}} \cdot \frac{7}{\sqrt{66}} = 0$$

$$\|\vec{v}_1\| = \sqrt{\left(\frac{3}{\sqrt{11}}\right)^2 + \left(\frac{1}{\sqrt{11}}\right)^2 + \left(\frac{1}{\sqrt{11}}\right)^2} = 1, \quad \|\vec{v}_2\| = \sqrt{\left(-\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2} = 1$$

$$\|\vec{v}_3\| = \sqrt{\left(-\frac{1}{\sqrt{66}}\right)^2 + \left(-\frac{4}{\sqrt{66}}\right)^2 + \left(\frac{7}{\sqrt{66}}\right)^2} = 1$$

Thm: An $m \times n$ matrix U has orthonormal columns iff $U^T U = I$.

proof: $U = (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)$, $U^T = \begin{pmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix}$

$\vec{u}_i^T$ is a row vector of U^T.

$$U^T U = \begin{pmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_n^T \end{pmatrix} (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)$$

$$= \begin{pmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \cdots & \vec{u}_1^T\vec{u}_n \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \cdots & \vec{u}_2^T\vec{u}_n \\ \vdots & & \ddots & \\ \vec{u}_n^T\vec{u}_1 & \vec{u}_n^T\vec{u}_2 & \cdots & \vec{u}_n^T\vec{u}_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Thm: Let U be an $m \times n$ matrix with orthonormal columns, and let $\vec{x}$ and $\vec{y}$ be in $\mathbb{R}^n$. Then

a) $\|U\vec{x}\| = \|\vec{x}\|$

b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

c) $(U\vec{x}) \cdot (U\vec{y}) = 0$ if and only if $\vec{x} \cdot \vec{y} = 0$.

Remark: Properties a) and c) say that the linear mapping $\vec{x} \mapsto U\vec{x}$ preserves lengths and orthogonality.

Example: Let $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix}$. Notice that

U has orthonormal columns and

$$U^TU = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

verify that $\|U\vec{x}\| = \|\vec{x}\|$.

Proof:

$$U\vec{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\|U\vec{x}\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|\vec{x}\| = \sqrt{2+9} = \sqrt{11}$$

Exercise: Let U be an $n\times n$ matrix with orthonormal columns. Show that $\det U = \pm 1$.

Proof: Since $U^TU = I$,

$$|U^TU| = |U^T|\cdot|U| = |U|\cdot|U| = |U|^2$$

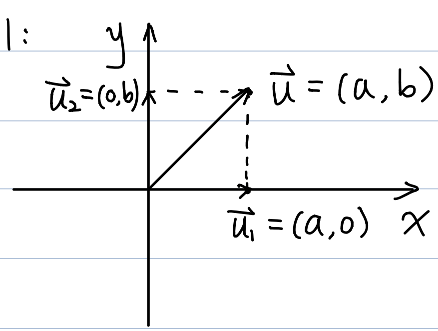
$|I| = 1$, we have

$$|U|^2 = 1$$

That is, $|U| = \pm 1$.

§6.3 Orthogonal Projection

Ex 1:

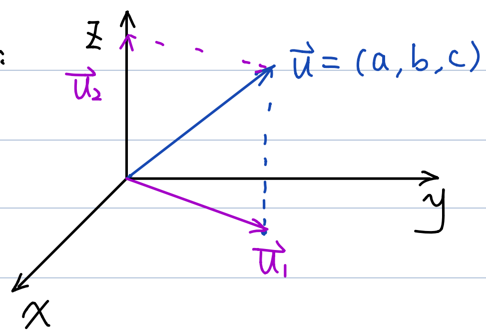


$\vec{u} = \vec{u}_1 + \vec{u}_2$

Let W — x-axis

Then $\vec{u}_1 \in W$ and $\vec{u}_2 \in W^{\perp}$

Ex 2:



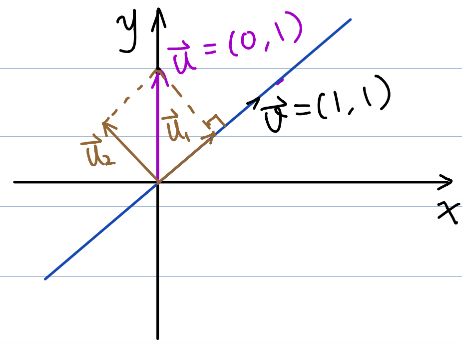
$W = xy\text{-plane}$

$\vec{u}_1 \in W, \quad \vec{u}_2 \in W^{\perp}$

$\vec{u} = \vec{u}_1 + \vec{u}_2$

$(a,b,c) = (a,b,0) + (0,0,c)$

Ex 3:



$W = \text{span}\{\vec{v}\}$

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} + \left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right)$$

$\vec{u}_1 \in W, \quad \vec{u}_2 \perp W$

$$\vec{u} = (0,1) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

In general, find a decomposition of $\vec{u}$ in $\mathbb{R}^n$ as $\vec{u} = \vec{u}_1 + \vec{u}_2$, $\vec{u}_1$ is in a subspace W and $\vec{u}_2$ is in $W^{\perp}$.

$\vec{u}_1$ is called the orthogonal projection of $\vec{u}$ onto the subspace W.

Thm: (The Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{y} = \hat{\vec{y}} + \vec{z}$$

where $\hat{\vec{y}}$ is in W and $\vec{z}$ is in $W^{\perp}$. In fact, if $\{\vec{u}_1, \cdots, \vec{u}_p\}$ is any orthogonal basis of W, then

$$\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

and $\vec{z} = \vec{y} - \hat{\vec{y}}$.

$\hat{\vec{y}}$ is called orthogonal projection of $\vec{y}$ onto W, denoted by $\text{proj}_W \vec{y}$

Proof: Let $\{\vec{u}_1, \cdots, \vec{u}_p\}$ be any orthogonal basis for W.

$$\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p \in W$$

Let $\vec{z} = \vec{y} - \hat{\vec{y}}$. Then

$$\vec{z}\cdot\vec{u}_1 = (\vec{y} - \hat{\vec{y}})\cdot\vec{u}_1$$

$$= \vec{y}\cdot\vec{u}_1 - \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1\cdot\vec{u}_1 - \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p\cdot\vec{u}_1$$

$$= \vec{y}\cdot\vec{u}_1 - \vec{y}\cdot\vec{u}_1 = 0$$

Thus $\vec{z}$ is orthogonal to $\vec{u}_1$.

Similarly, $\vec{z}\cdot\vec{u}_2 = 0$, $\cdots$, $\vec{z}\cdot\vec{u}_p = 0$

To prove the uniqueness.

Suppose $\vec{y}$ has two orthogonal decompositions:

$$\vec{y} = \hat{\vec{y}}_1 + \hat{\vec{z}}_1 = \hat{\vec{y}}_2 + \hat{\vec{z}}_2, \text{ where } \hat{\vec{y}}_1, \hat{\vec{y}}_2 \in W$$

and $\hat{\vec{z}}_1$ and $\hat{\vec{z}}_2$ are in $W^\perp$.

Then $\hat{\vec{y}}_1 - \hat{\vec{y}}_2 = \hat{\vec{z}}_2 - \hat{\vec{z}}_1$

So $(\hat{\vec{y}}_1 - \hat{\vec{y}}_2)\cdot(\hat{\vec{y}}_1 - \hat{\vec{y}}_2) = (\hat{\vec{y}}_1 - \hat{\vec{y}}_2)(\hat{\vec{z}}_2 - \hat{\vec{z}}_1) = 0$

since $\hat{\vec{y}}_1 - \hat{\vec{y}}_2 \in W$ and $\hat{\vec{z}}_1 - \hat{\vec{z}}_2 \in W^\perp$.

Hence $\|\hat{\vec{y}}_1 - \hat{\vec{y}}_2\| = 0 \Rightarrow \hat{\vec{y}}_1 = \hat{\vec{y}}_2$

and $\hat{\vec{z}}_1 = \hat{\vec{z}}_2$.

Example: Let $\vec{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. observe that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$. Write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W.

Solution: The orthogonal projection of $\vec{y}$ onto W is

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{9}{30} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{3}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{pmatrix}$$

Also, $\vec{y} - \hat{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{pmatrix}$

Hence $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{pmatrix}$