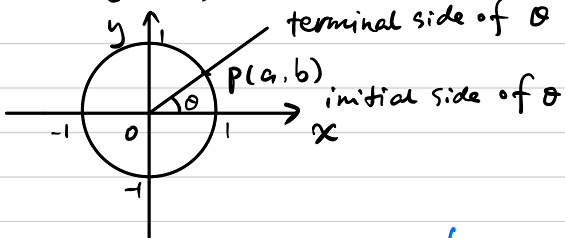
Review

1. The trigonometric functions.

$\theta \in (-\infty, +\infty)$.



$P(a, b)$ is on the unit circle.

$a = \cos\theta$.

$b = \sin\theta$

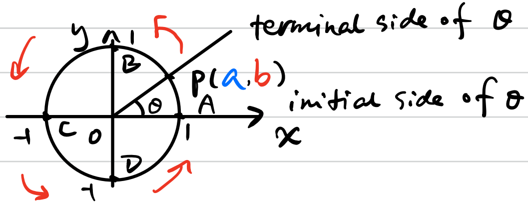
$\tan\theta = \frac{\sin\theta}{\cos\theta}$. $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$.

$\sin(\theta + 2\pi) = \sin\theta$, $\cos(\theta + 2\pi) = \cos\theta$ $\tan(\theta + \pi) = \tan\theta$.

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$.

$\cos\theta$: even function. $\sin\theta$, $\tan\theta$: odd function.

2. Graphs of $y=\sin\theta$ and $y=\cos\theta$.



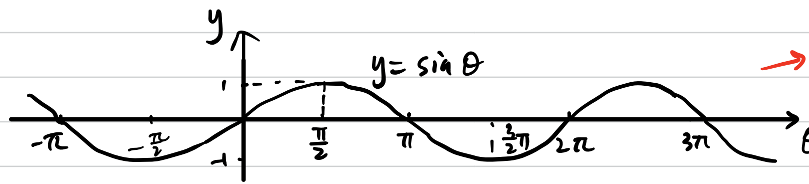
$\theta:\ 0 \to \frac{\pi}{2} \to \pi \to \frac{3}{2}\pi \to 2\pi$

$P:\ A \to B \to C \to D \to A$

$b:\ 0 \to 1 \to 0 \to -1 \to 0$

$a:\ 1 \to 0 \to -1 \to 0 \to 1$

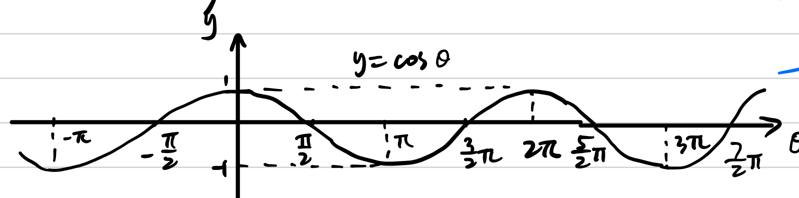
1) $y=\sin\theta = b$



→ odd function
period $=2\pi$

domain: $(-\infty,+\infty)$. range: $[-1,1]$. $\sin k\pi=0$ for any integer k.

2) $y=\cos\theta = a$

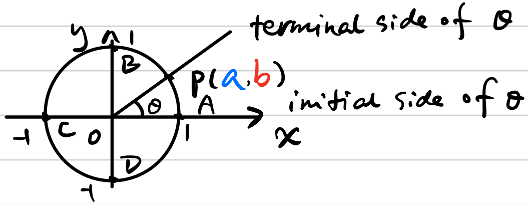


→ even function.
period $=2\pi$.

domain: $(-\infty,+\infty)$. range: $[-1,1]$. $\cos(k\pi+\frac{\pi}{2})=0$ for any integer k.

3) $y = \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{b}{a}$

Domain $= \{\theta \mid \cos\theta \neq 0\} = \{\theta \mid \theta \neq k\pi + \frac{\pi}{2}$ for any integer $k\}$



① when $\theta \in [0, \frac{\pi}{2})$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

θ approaches $\frac{\pi}{2}$

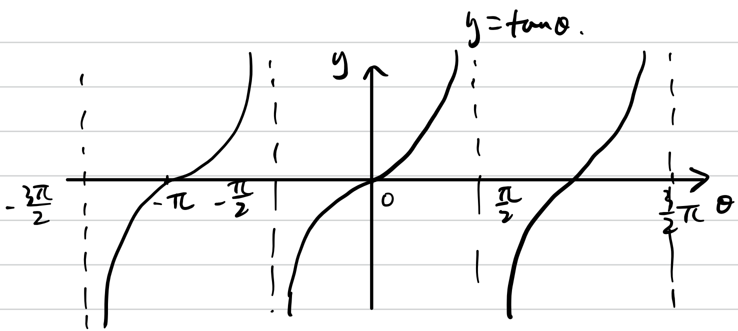
$\tan\theta$ approaches $+\infty$.

② when $\theta \in (-\frac{\pi}{2}, 0]$

θ	0	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$
$\tan\theta$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$

θ approaches $-\frac{\pi}{2}$

$\tan\theta$ approaches $-\infty$



→ odd function.

period $= \pi$.

Notice: $\tan k\pi = 0$ for any integer k.

3. Some useful formulas.

(1) Addition formulas:

① $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ → Express $\sin(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$.

② $\cos(\alpha+\beta) = \sin\left(\frac{\pi}{2} - (\alpha+\beta)\right)$. Recall $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$= \sin\left(\left(\frac{\pi}{2} - \alpha\right) + (-\beta)\right)$

$= \sin\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) + \cos\left(\frac{\pi}{2} - \alpha\right)\sin(-\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

divided by $\cos\alpha \cdot \cos\beta$.

③ $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$

(2). Subtraction formulas (substitute $-\beta$ for β in addition formulas)

① $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

② $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

③ $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$.

(3) Double-angle formulas (substitute β for α in addition formulas).

① $\sin 2\alpha = \sin\alpha\cdot\cos\alpha + \cos\alpha\cdot\sin\alpha = 2\sin\alpha\cdot\cos\alpha.$ Recall $\cos^2\theta+\sin^2\theta=1$

② $\cos 2\alpha = \cos\alpha\cos\alpha - \sin\alpha\cdot\sin\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$
$= 1 - 2\sin^2\alpha$

③ $\tan 2\alpha = \dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\cdot\tan\alpha} = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Example 1: Compute $\tan\frac{5\pi}{12}$.

$\frac{5\pi}{12} = 75^\circ = 45^\circ + 30^\circ$. Recall $\tan 45^\circ = 1$. $\tan 30^\circ = \frac{1}{\sqrt{3}}$.

$\tan 75^\circ = \tan(45^\circ+30^\circ) \overset{\text{using (1).③}}{=} \dfrac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ\cdot\tan 30^\circ} = \dfrac{1+\frac{1}{\sqrt{3}}}{1-1\times\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}.$

Example 2: Compute $\cos\frac{\pi}{12}$.

$\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \underset{\text{using (3)②}}{=} 2\cdot\cos^2\frac{\pi}{12} - 1. \rightarrow \cos^2\frac{\pi}{12} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}+1\right) = \frac{\sqrt{3}+2}{4}. \rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$

4. The inverse trigonometric functions.

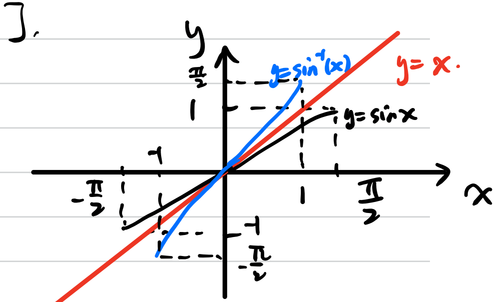
Recall : (1) The inverse functions are defined for one-to-one functions.

(2) For a one-to-one function f: y is uniquely given by x.

x is uniquely given by y.

1). inverse function of $y=f(x)=\sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. → one-to-one.

$x \xrightarrow{f} \sin x$.		$\sin x \xrightarrow{g} x$	
$-\frac{\pi}{2}$	$\to -1$	-1	$\to g(-1) = -\frac{\pi}{2}$.
$-\frac{\pi}{6}$	$\to -\frac{1}{2}$	$-\frac{1}{2}$	$\to g(-\frac{1}{2}) = -\frac{\pi}{6}$.
$\frac{\pi}{6}$	$\to \frac{1}{2}$	$\frac{1}{2}$	$\to g(\frac{1}{2}) = \frac{\pi}{6}$
$\frac{\pi}{2}$	$\to 1$	1	$\to g(1) = \frac{\pi}{2}$.



denoted by

$g(x) = \arcsin x$ or $\sin^{-1} x$.

$y = \sin^{-1} x \iff \sin y = x$.

domain of g = range of f = $[-1, 1]$.

range of g = domain of f = $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

2). inverse function of $y=\cos x$ $x\in[0,\pi]$. → one-to-one.

$x \xrightarrow{f} \cos x$.

$0 \to 1$

$\frac{\pi}{3} \to \frac{1}{2}$

$\frac{2\pi}{3} \to -\frac{1}{2}$

$\pi \to -1$

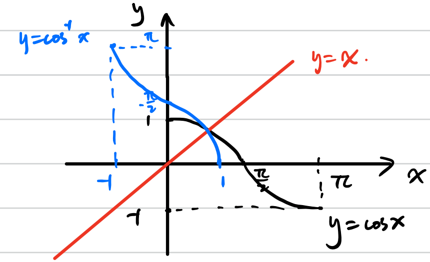
$\cos x \xrightarrow{g} x$

$1 \to 0$

$\frac{1}{2} \to \frac{\pi}{3}$.

$-\frac{1}{2} \to \frac{2\pi}{3}$.

$-1 \to \pi$.



$g(x) = \arccos x$ or $\cos^{-1}x$.

$y=\cos^{-1}(x) \Leftrightarrow \cos y = x$.

domain of g = range of f = $[-1,1]$.

range of g = domain of f = $[0,\pi]$.

3). inverse function of $y=\tan x$. → one-to-one $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$x \xrightarrow{f} \tan x$.

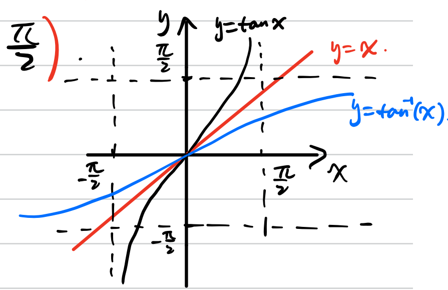
$-\frac{\pi}{4} \to -1$

$\frac{\pi}{4} \to 1$

$\tan x \xrightarrow{g} x$

$-1 \to -\frac{\pi}{4}$

$1 \to \frac{\pi}{4}$.



$g(x) = \arctan x$ or $\tan^{-1}x$.

$y=\tan^{-1}x \Leftrightarrow \tan y = x$.

domain of g = range of f = $(-\infty,+\infty)$.

range of g = domain of f = $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

4). To conclude :

		domain	range
$f(x) = \sin x$. $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.	$\to g(x) = \sin^{-1} x$.	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$.
$f(x) = \cos x$. $x \in [0, \pi]$	$\to g(x) = \cos^{-1} x$.	$[-1, 1]$.	$[0, \pi]$.
$f(x) = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$\to g(x) = \tan^{-1} x$	$(-\infty, +\infty)$.	$(-\frac{\pi}{2}, \frac{\pi}{2})$.

Example: Compute the domain and range of $y = \sin^{-1}(3x+1)$.

$3x+1 \in$ domain of $y = \sin^{-1} x$. $\to 3x+1 \in [-1, 1]$.

$\to x \in [-\frac{2}{3}, 0]$.

domain : $[-\frac{2}{3}, 0]$. range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.