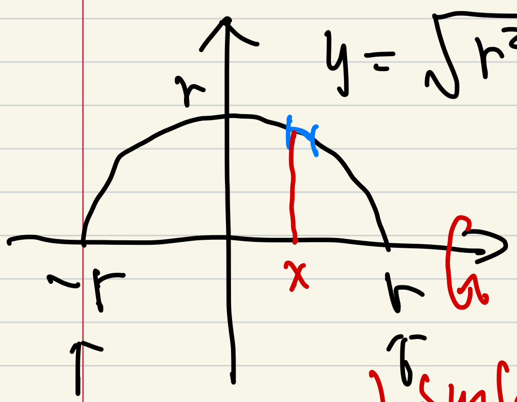
Area of a Surface of Revolution.



$\longleftrightarrow$ arc length

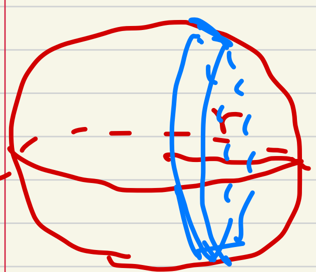
$$= \int_{-r}^{r} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \pi r$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+\left(\frac{dx}{dy}\right)}\, dy$$

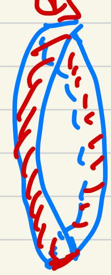
area $= 4\pi r^2$

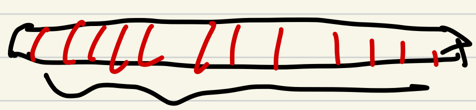


sphere

$$\boxed{\text{surface area} = \int_{-r}^{r} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy}$$

thin band area $\approx 2\pi y\, ds$ ← summing by integration





$$\frac{dy}{dx} = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}\cdot(-2x)$$

$$= \frac{-x}{\sqrt{r^2-x^2}}$$

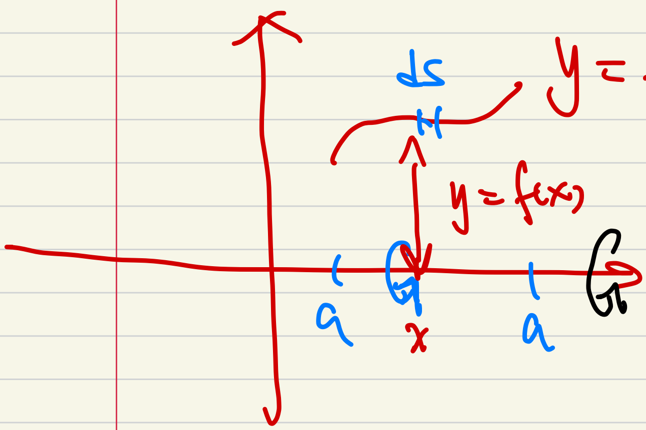
area of

$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\, dx$$

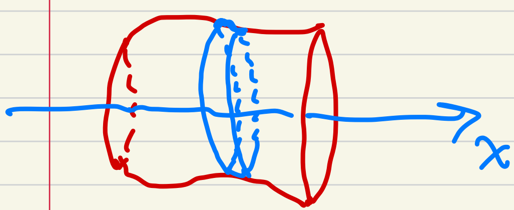
$$= \int_{-r}^{r} 2\pi\sqrt{r^2\cancel{-x^2}\cancel{+x^2}}\, dx$$

$$= 2\pi r\cdot[x]_{-r}^{r} = 4\pi r^2$$

In general,



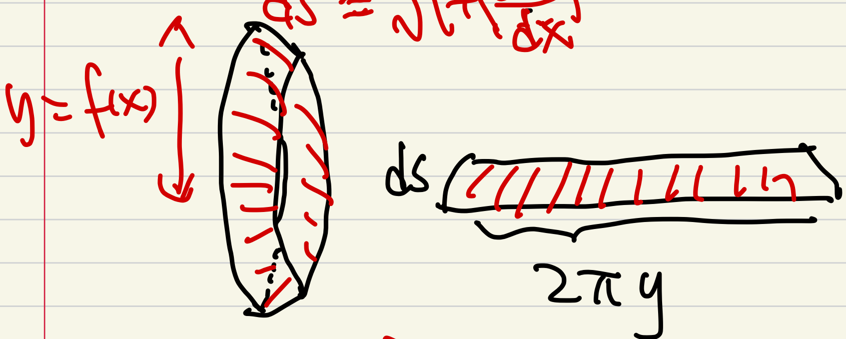
Rotate the graph about the x-axis to generate a surface of revolution



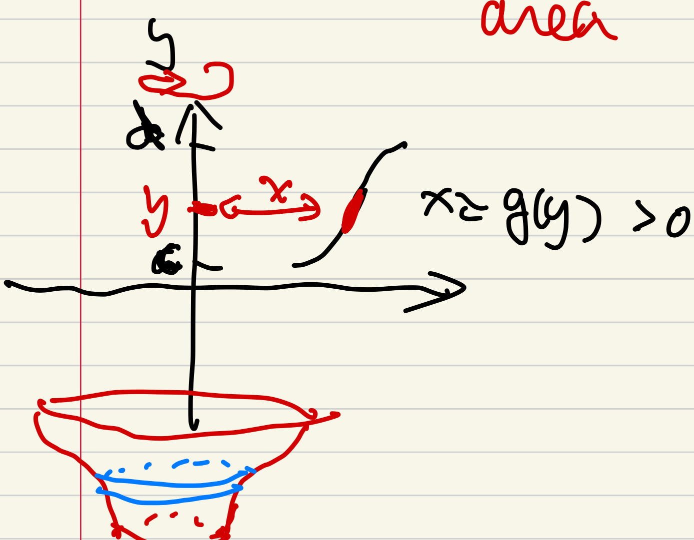
$$\text{area} = \int_a^b 2\pi y \, ds$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

$$ds \approx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



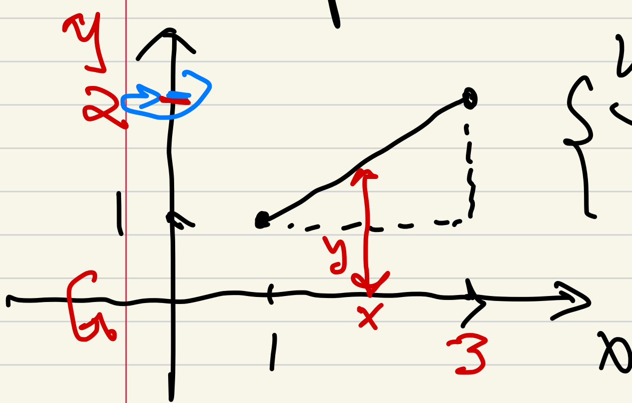
thin band area $\approx 2\pi y \, ds$



$$\text{area} = \int_c^d 2\pi x \, ds$$

$$= \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy$$

Example



$$\begin{cases} y = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2} \\ \frac{dy}{dx} = \frac{1}{2} \end{cases}$$

$$\begin{cases} x = 2y-1 \\ \frac{dx}{dy} = 2 \end{cases}$$

area of the surface of revolution about the x-axis $= \int_1^3 2\pi y \, ds$

$$= \int_1^3 \cancel{2}\pi\left(\frac{1}{\cancel{2}}x + \frac{1}{\cancel{2}}\right) \overset{\frac{\sqrt{3}}{2}}{\boxed{\sqrt{1+\left(\frac{1}{2}\right)^2}}} \, dx$$

$$= \frac{\sqrt{3}}{2}\pi \int_1^3 (x+1)\, dx$$

$$= \frac{\sqrt{3}}{2}\pi \left[\frac{x^2}{2} + x\right]_1^3$$

$$= \frac{\sqrt{3}}{2}\pi \left[\underbrace{\frac{9}{2}}_{4} + 3 - \underbrace{\frac{1}{2}}_{+2} - 1\right]$$

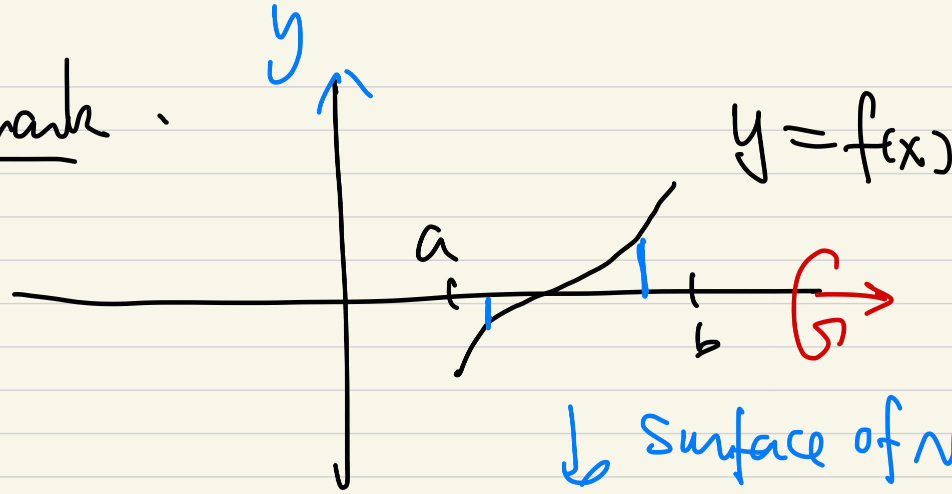
$= 3\sqrt{3}\pi$ (sq units)

area of surface of revolution about the y-axis $= \int_1^2 2\pi x \, ds = \int_1^2 2\pi(2y-1)\sqrt{1+4}\, dy$

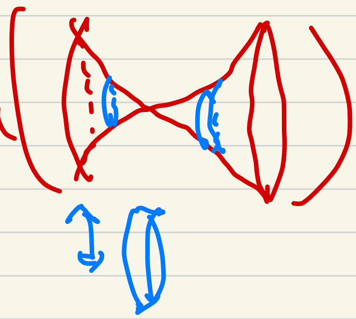
$$= 2\pi\sqrt{5}\left[y^2 - y\right]_1^2 = 4\sqrt{5}\pi$$

Remark.

(1)

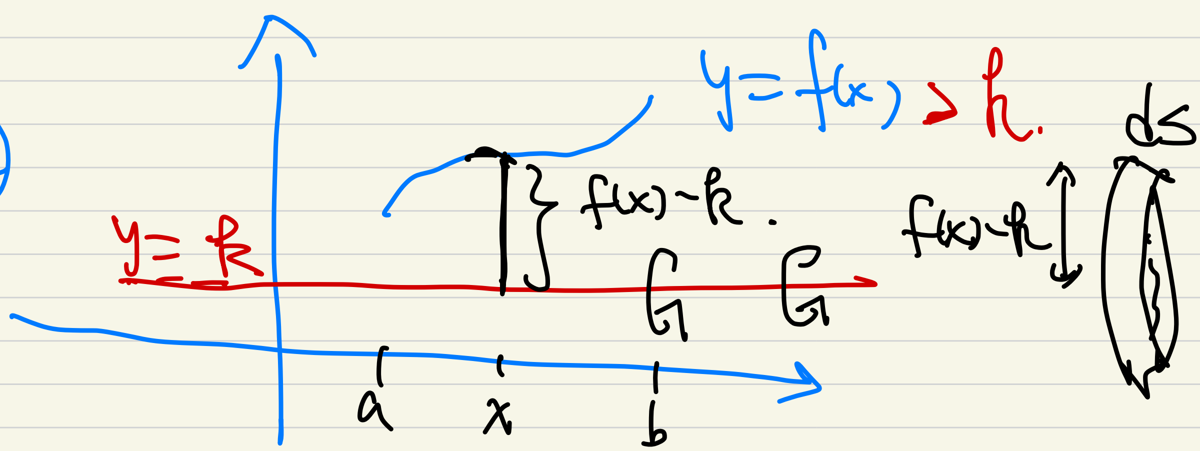


↳ surface of revolution.

area 

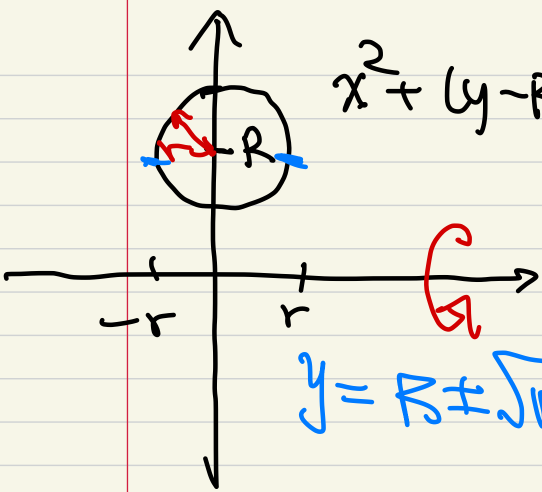
$$= \int_a^b 2\pi\,|f(x)|\,\sqrt{1+(y')^2}\,dx$$

(2)



Area of the surface of revolution about $y = k$

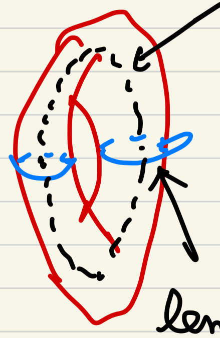
$$= \int_a^b 2\pi\,[f(x)-k]\,\sqrt{1+[f'(x)]^2}\,ds$$



$x^2+(y-R)^2=r^2$

$y = R \pm \sqrt{r^2-x^2}$

$0<r<R$



Find the area of this torus.

$$\frac{dy}{dx} = \mp \frac{x}{\sqrt{r^2-x^2}}, \quad \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2-x^2}$$

$$\text{area} = \int_{-r}^{r} 2\pi\Big(\underbrace{R+\sqrt{r^2-x^2}}_{\text{from upper semi-circle}}\Big)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$+\int_{-r}^{r} 2\pi\Big(\underbrace{R-\sqrt{r^2-x^2}}_{\text{from lower semi-circle}}\Big)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

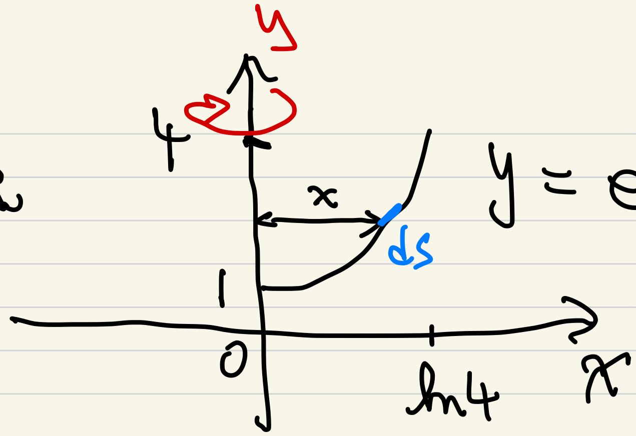
$$= 4\pi R\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}\,dx \qquad \begin{aligned} x &= r\sin\theta \\ dx &= r\cos\theta\, d\theta \end{aligned}$$

$$= 4\pi R\int_{-\pi/2}^{\pi/2}\frac{r}{\cancel{r\cos\theta}}\,\cancel{r\cos\theta}\,d\theta$$

$$= 4\pi R r\cdot\pi$$

$$= (2\pi R)(2\pi r)$$

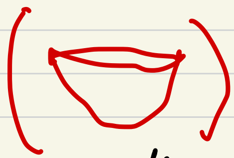
Example



$y = e^x, \quad x = \ln y$

$\frac{dy}{dx} = e^x$

$\frac{dx}{dy} = \frac{1}{y}$

① area()

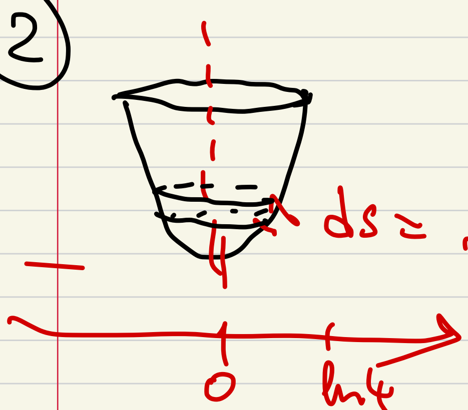
$= \int_1^4 2\pi x \, ds$

Easy to write the integral

$= \int_1^4 2\pi \ln y \sqrt{1 + \frac{1}{y^2}} \, dy$

Hard to integrate !!
(Try Wolfram Alpha!).

②

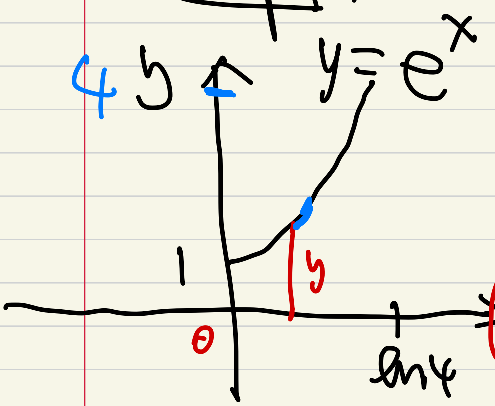


$ds = \sqrt{1 + (y')^2} \, dx$

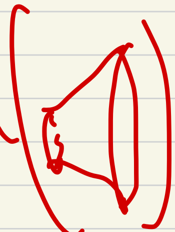
$\text{area} = \int_0^{\ln 4} 2\pi x \sqrt{1 + (e^x)^2} \, dx$

$\left(\frac{dy}{dx}\right)^2$

Example.



$\frac{dy}{dx} = e^x$

area() $= \int_0^{\ln 4} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$x = \ln y$

$$= \int_0^{\ln 4} 2\pi e^x \sqrt{1+e^{2x}}\,dx$$

Let $u = e^x$
$du = e^x dx$

$$\to = \int_1^4 2\pi \sqrt{1+u^2}\,du$$

Let $u = \tan\theta$

$$= \cancel{2}\pi \left[\frac{1}{\cancel{2}} u\sqrt{1+u^2} + \frac{1}{\cancel{2}} \ln\left|u+\sqrt{1+u^2}\right|\right]_1^4$$

$$= \pi\left[4\sqrt{17} + \ln|4+\sqrt{17}| - \sqrt{2} - \ln(1+\sqrt{2})\right]$$

Exercise: Think about

"area $= \int_1^4 2\pi y \sqrt{1+\left(\frac{d \ln y}{dy}\right)^2}\,dy$"