MATH2111 Tutorial 8

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1 Null Spaces and Column Spaces

1. **Definition (Null Space)**. The null space of an $m \times n$ matrix A, written as $\mathrm{Nul}\, A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\mathrm{Nul}\ A = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

2. **Theorem**. The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

3. **Definition (Column Space)**. The column space of an $m \times n$ matrix A, written as $\mathrm{Col}\, A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\mathrm{Col}\ A = \mathrm{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

4. **Theorem**. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

2 Kernel and Range

1. **Definition (Linear Transformation)**. A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that for all $\mathbf{u}, \mathbf{v}$ in V and all scalars c,
(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(b) $T(c\mathbf{u}) = cT(\mathbf{u})$

2. **Definition (Kernel and Range)**. For a linear transformation $T : V \longrightarrow W$,
(a) the kernel of T is defined as

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

(b) the range (image) of T is defined as

$$\mathrm{range}\, T = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

3. **Theorem**. Let $T : V \longrightarrow W$ be any linear transformation.
(a) ker T, range T are both vector subspaces (of V, W respectively)
(b) T is injective(one-to-one) iff $\ker T = \{\mathbf{0}\}$
(c) T is surjective(onto) iff range $T = W$

3 Basis

1. **Theorem**. An indexed set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ of two or more vectors, with $\boldsymbol{v}_1 \neq 0$, is linearly dependent if and only if some $\boldsymbol{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$.

2. **Definition (Basis)**. Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if
(a) B is a linearly independent set, and
(b) the subspace spanned by B coincides with H. that is,
$$H = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

3. **Fact**. $\{\mathbf{v}_1, \cdots \mathbf{v}_p\}$ is a basis for $\mathbb{R}^n$ if and only if:
(1) $p = n$ (i.e. the set has exactly n vectors), and
(2) $\det \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & | & | \end{bmatrix} \neq 0$.

4. **Theorem (The Spanning Set Theorem)**. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
(a) If one of the vectors in S, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
(b) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

5. **Theorem (casting-out algorithm)**. The pivot columns of a matrix A form a basis for Col A.

4 Exercises

1. Determine whether the following is a subspace or not.
(1) $\{(1 + a, b, a + b) \mid a, b \in \mathbb{R}\}$,
(2) $\{(1 + a, b, 1 + a + b) \mid a, b \in \mathbb{R}\}$,
(3) $\{(a, 3b, a + 2b, 2b - a) \mid a, b \in \mathbb{R}\}$

2. Determine the null space of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}$$

if $\mathrm{col}(A)$ is subspace of $\mathbb{R}^k$, what is k?

3. What is the base of the range for the above given matrix?

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}$$

4. (1) Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$?

(2) $S_1 = \{1, x, x^2\}$ is a basis of $\mathbb{P}_2$. Is $S_2 = \{1, x+1, (x+1)^2\}$ also a basis of $\mathbb{P}_2$?

5. (1) Is $\left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ linearly independent?

(2) Suppose nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are orthogonal to each other, namely, $\mathbf{v}_i^\top \mathbf{v}_j = 0$ holds for any $i \neq j$, $i, j = 1, \cdots, n$. Prove $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.