# Lecture 10 : Rigid Body III.

- Rotation about a moving axis.
- Rolling without slipping.
- Rolling friction
- Work & Power

So far we know:

① Translation motion of CM : $\sum \vec{F}_{ext} = m\,\vec{a}_{cm}$

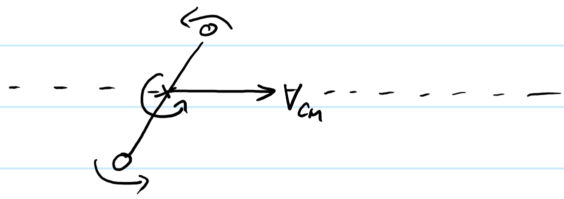
② Rotation of rigid body about a fixed axis : $\sum \tau_{ext_z} = I_z \alpha_z$

Combine both types of motion : a rigid body rotating about a moving axis.

e.g. Rolling wheel.

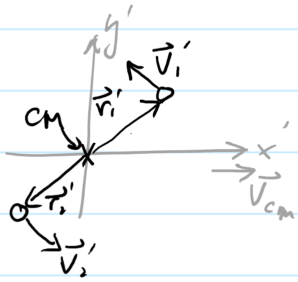
$\boxed{\text{Translation motion of CM + Rotational motion about a axis through its CM.}}$

Energy.




In CM-frame

Pure rotation




choose

$$\vec{0} = \vec{r}_{cm}' = \frac{m_1\vec{r}_1' + m_2\vec{r}_2'}{m_{tot}}$$

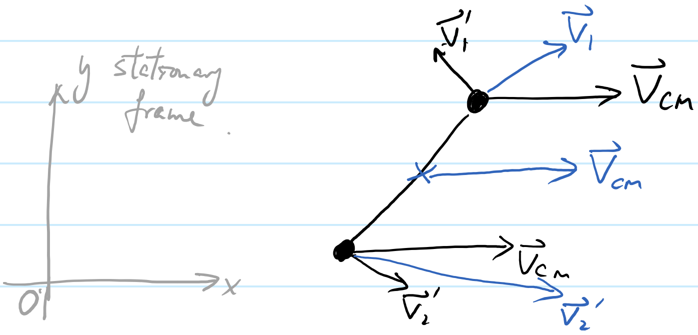
$$\frac{d}{dt}: \vec{0} = \vec{V}_{cm}' = \frac{m_1\vec{v}_1' + m_2\vec{v}_2'}{m_{tot}}.$$

velocity of cm in CM frame

Add $\vec{V}_{cm}$ to all points on the object.

i.e. add a translational motion.

$\vec{v}_i$ : velocity measured by a stationary reference frame




In a stationary frame,

$$K = \sum_i \frac{1}{2} m_i |\vec{v}_i|^2$$

$$= \frac{1}{2}\sum m_i (\vec{v}_i \cdot \vec{v}_i)$$

$$= \frac{1}{2}\sum m_i (\vec{v}_i' + \vec{V}_{cm}) \cdot (\vec{v}_i' + \vec{V}_{cm})$$

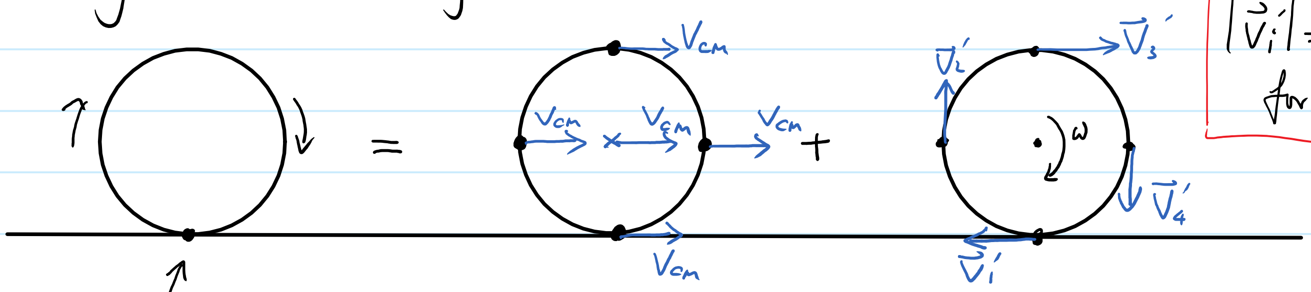
$$= \frac{1}{2}\sum m_i |\vec{v}_i'|^2 + \frac{1}{2}\sum m_i |\vec{V}_{cm}|^2 + \frac{1}{2}\sum m_i (2\vec{V}_{cm}\cdot\vec{v}_i')$$

$$= \frac{1}{2}\sum m_i (r_i\omega)^2 + \frac{1}{2} m_{tot} V_{cm}^2 + \vec{V}_{cm}\cdot\underbrace{\sum m_i\vec{v}_i'}_{\vec{V}_{cm}' = \vec{0}}$$

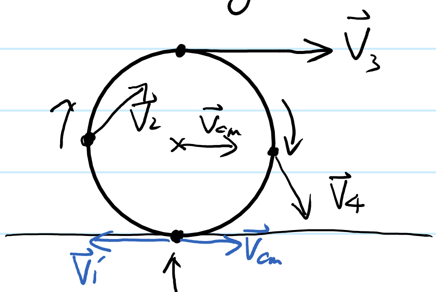
$$\boxed{K = \underbrace{\frac{1}{2} m_{tot} V_{cm}^2}_{K_{tran}} + \underbrace{\frac{1}{2} I\omega^2}_{K_{rot.cm}}}$$

# Rolling without slipping

$|\vec{V}_i'| = R\omega$ for all $i$.




stationary      translation + rotation about CM.




$\vec{V}_1 = \vec{V}_{cm} + \vec{V}_1'$

Velocity of each point relative to the ground

$$\vec{V}_i = \vec{V}_i' + \vec{V}_{cm}$$

Velocity relative to CM + velocity of CM

Require: $\vec{V}_1 = \vec{0}$      stationary point.

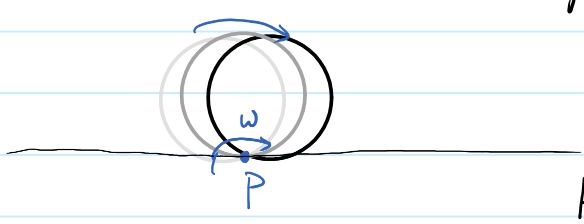
$\Rightarrow \vec{V}_1' = -\vec{V}_{cm}$

$$V_{cm} = V_1' = R\omega$$

Kinematic constrain for rolling without slipping.

## Alternative view on rolling.

Rolling = swinging about the point of contact "P" but the contact point changes continuously.



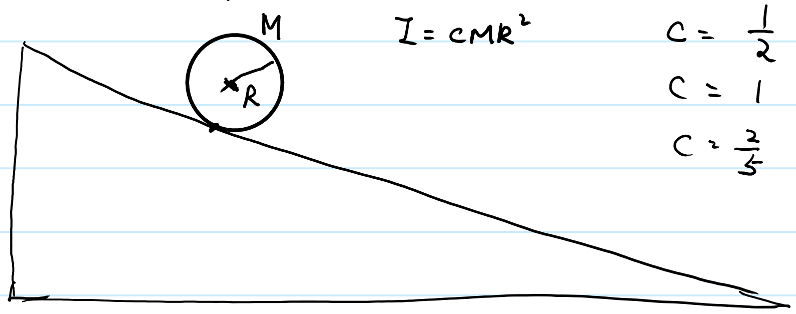

$K = \frac{1}{2} I_P \omega^2$ , $I_P = I_{cm} + MR^2$

$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MR^2 \cdot \omega^2$

$= \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M V_{cm}^2$    $\because R\omega = V_{cm}$

same as previous page

Rolling down inclined plane.




$I = cMR^2$

$c = \frac{1}{2}$ cylinder

$c = 1$ ring

$c = \frac{2}{5}$ sphere

contact point does not move $\Rightarrow$ static friction $\Rightarrow W_{friction} = 0$

$$E_i = E_f$$

$$Mgh = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 \qquad R\omega = V_{cm}$$

$$Mgh = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \frac{cMR^2}{R^2} V_{cm}^2$$

$$V_{cm} = \sqrt{\frac{2gh}{1+c}}$$

depends only on type of rolling object. c.

not M or R.

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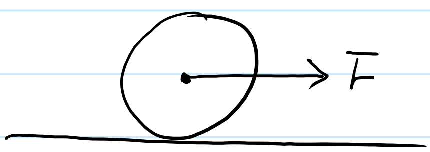
## Torque Approach.

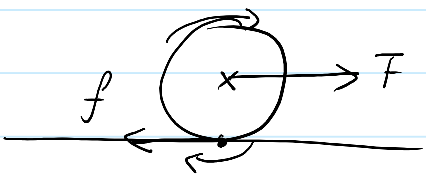



no friction $\Rightarrow$ no torque about cm

$\Rightarrow$ no angular acceleration

$\Rightarrow$ only sliding occurs.

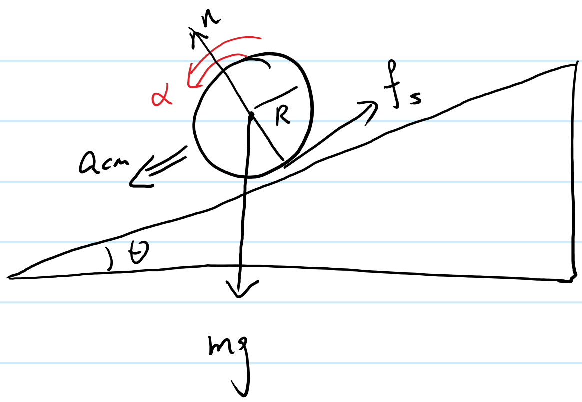
To make it rolls,




If it is not slipping $\Rightarrow$ contact point not sliding

$\Rightarrow$ static friction.

$f_s$.




A cylinder rolling down without slipping

given m, R.,
solve for $a_{cm}$ & α.

Translation motion of CM governed by Newton's 2nd Law.

$$\Sigma \vec{F}_{ext} = m\,\vec{a}_{cm}.$$

Rotational motion about CM governed by

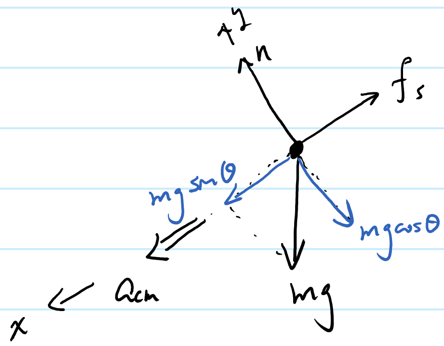
$$\Sigma \tau_{ext_{cm}} = I_{cm}\,\alpha$$

In general, $\vec{a}_{cm}$ and α are independent of each other.

But for rolling without slipping, $\underline{v_{cm} = R\omega} \Rightarrow \underline{a_{cm} = R\alpha}$.

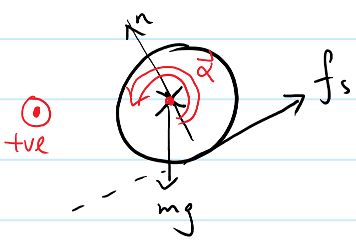
We have

<u>For Translation motion</u>




$$y: \quad n - mg\cos\theta = 0 \quad \text{——— ①}$$
$$x: \quad mg\sin\theta - f_s = m\,a_{cm} \quad \text{——— ②}$$

<u>For rotational motion about CM</u>




only $f_s$ produce a torque about cm.

$$\tau_{cm} = I_{cm}\,\alpha$$
$$+f_s \cdot R = +I_{cm}\,\alpha \quad \text{——— ③}$$

Kinematic constrain: $a_{cm} = R\alpha$ —— ④

$$\Rightarrow \begin{cases} n = mg\cos\theta & \text{—— ①} \\ mg\sin\theta - f_s = m a_{cm} & \text{—— ②} \\ f_s R = I_{cm}\alpha & \text{—— ③} \\ a_{cm} = R\alpha & \text{—— ④} \end{cases}$$

$\Rightarrow$ eliminating $f_s$ and $\alpha$, ② ③ & ④ gives..

$$\Rightarrow mg\sin\theta - I_{cm}\frac{a_{cm}}{R^2} = m a_{cm}$$

$$a_{cm} = \frac{g\sin\theta}{1 + \frac{I_{cm}}{mR^2}}$$

$$\Rightarrow a_{cm} = \frac{2}{3} g\sin\theta \qquad \because I_{cm} = \frac{1}{2}mR^2 \text{ for cylinder.}$$

$$\alpha = \frac{a_{cm}}{R} = \frac{2}{3}\frac{g\sin\theta}{R}$$

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Summary. To solve dynamical problem with both translational and rotational motions, we can use.

$$\begin{cases} \text{Newton's 2nd Law for the trans. motion of its cm.} \\ + \\ \text{Newton's 2nd Law for rotation for the rotational motion about its cm.} \end{cases}$$

i.e. $\boxed{\sum \vec{F}_{ext} = m a_{cm}}$ & $\boxed{\sum \tau_{cm} = I_{cm}\alpha}$

For $\boxed{\text{Rolling without slipping,}}$ we add the

kinematic constrain: $V_{cm} = R\omega$ , $a_{cm} = R\alpha$

## Rolling without slipping

- contact point does not slide.
- friction on the wheel is static friction.

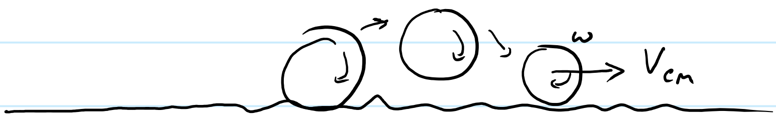
$\Rightarrow$ $W_{friction} = 0$ static friction does not do work.

$\Rightarrow$ No energy loss!

$\Rightarrow$ Roll forever if no other resistances.

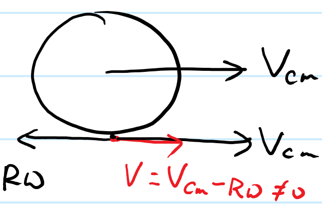
But, in reality, rolling without slipping is rare.

The surface of the road is not perfectly flat.



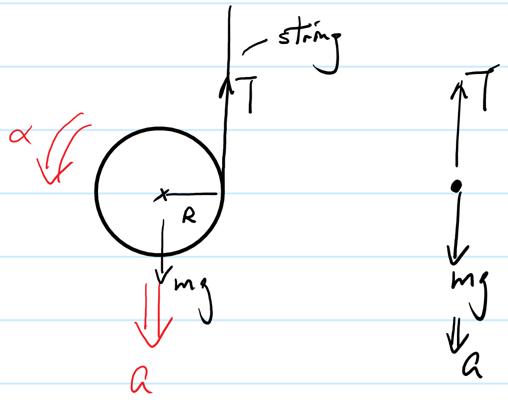

If the wheel hits a bump, the wheel could jump up.
When it touches the ground again, its $V_{cm}$ and $\omega$ may not satisfy $V_{cm} = R\omega$ anymore.

If $V_{cm} \neq R\omega$, the point of the wheel contacting the ground will slide.




The sliding motion will introduce a kinetic friction which leads to energy loss.

## Yo-yo : vertical rolling without slipping along a string.




assume the yo-yo is a solid cylinder for simplicity.

$$I_{cm} = \frac{1}{2} m R^2$$

Translational motion : $\Sigma F = mg - T = ma$ —— ①

Rotational motion about CM : $\Sigma \tau_{cm} = T \cdot R = I_{cm} \alpha$ —— ②

Kinematic constrain : $\alpha = a/R$ —— ③

$$\Rightarrow \quad mg - \frac{I_{cm}\alpha}{R} = ma$$

$$\Rightarrow \quad mg - \frac{I_{cm} a}{R^2} = ma$$

$$\Rightarrow \quad \begin{cases} a = \frac{2}{3} g \\ T = \frac{1}{2} ma = \frac{1}{3} mg \end{cases}$$

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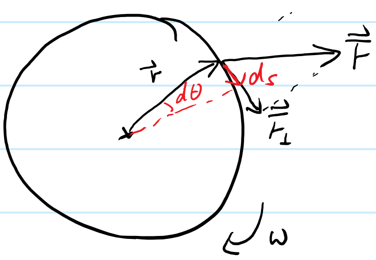
## Work and Power




$$dW = \vec{F} \cdot d\vec{s}$$
$$= F_\perp \cdot ds$$
$$= F_\perp \cdot r\, d\theta$$
$$= \tau\, d\theta$$
$$\Rightarrow \boxed{W = \int \tau\, d\theta}$$

$$\boxed{P = \vec{\tau} \cdot \vec{\omega}}$$

c.f. $W = \int \vec{F} \cdot d\vec{s}$ & $P = \vec{F} \cdot \vec{v}$