MATH2111 Tutorial 8

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1 Null Spaces and Column Spaces

1. **Definition (Null Space)**. The null space of an $m \times n$ matrix A, written as $\operatorname{Nul} A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\operatorname{Nul} A = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

2. **Theorem**. The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

3. **Definition (Column Space)**. The column space of an $m \times n$ matrix A, written as $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

4. **Theorem**. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

2 Kernel and Range

1. **Definition (Linear Transformation)**. A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that for all $\mathbf{u}, \mathbf{v}$ in V and all scalars c,
(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(b) $T(c\mathbf{u}) = cT(\mathbf{u})$

2. **Definition (Kernel and Range)**. For a linear transformation $T : V \to W$,
(a) the kernel of T is defined as

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

(b) the range (image) of T is defined as

$$\operatorname{range} T = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

3. **Theorem**. Let $T : V \to W$ be any linear transformation.
(a) $\ker T$, $\operatorname{range} T$ are both vector subspaces (of V, W respectively)
(b) T is injective(one-to-one) iff $\ker T = \{\mathbf{0}\}$
(c) T is surjective(onto) iff $\operatorname{range} T = W$

3 Basis

1. **Theorem**. An indexed set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ of two or more vectors, with $\boldsymbol{v}_1 \neq 0$, is linearly dependent if and only if some $\boldsymbol{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$.

2. **Definition (Basis)**. Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if
(a) B is a linearly independent set, and
(b) the subspace spanned by B coincides with H. that is,
$$H = \text{Span}\left\{\mathbf{b}_1, \ldots, \mathbf{b}_p\right\}$$

3. **Fact**. $\{\mathbf{v}_1, \cdots \mathbf{v}_p\}$ is a basis for $\mathbb{R}^n$ if and only if:
(1) $p = n$ (i.e. the set has exactly n vectors), and
(2) $\det \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & | & | \end{bmatrix} \neq 0$.

4. **Theorem (The Spanning Set Theorem)**. Let $S = \left\{\mathbf{v}_1, \ldots, \mathbf{v}_p\right\}$ be a set in V, and let $H = \text{Span}\left\{\mathbf{v}_1, \ldots, \mathbf{v}_p\right\}$.
(a) If one of the vectors in S, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
(b) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

5. **Theorem (casting-out algorithm)**. The pivot columns of a matrix A form a basis for Col A.

4 Exercises

1. Determine whether the following is a subspace or not.

(1) $\{(1+a, b, a+b) \mid a, b \in \mathbb{R}\}$, $= S$

(2) $\{(1+a, b, 1+a+b) \mid a, b \in \mathbb{R}\}$, $= Q$

(3) $\{(a, 3b, a+2b, 2b-a) \mid a, b \in \mathbb{R}\}$ $= W$

(1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is NOT in S, thus not a subspace

(2) is a subspace.

Denote the set as Q,

① take $a=-1$, $b=0$,

$$\begin{bmatrix} 1+a \\ b \\ 1+a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in Q$$

② if $\vec{q_1}, \vec{q_2} \in Q$.

$$\vec{q_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1+a_1 \\ b_1 \\ 1+a_1+b_1 \end{bmatrix}, \quad \vec{q_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1+a_2 \\ b_2 \\ 1+a_2+b_2 \end{bmatrix} \qquad a_i, b_i \in \mathbb{R}.$$

then $$\vec{q_3} = \vec{q_1} + \vec{q_2} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix} = \begin{bmatrix} 1+a_1+1+a_2 \\ b_1+b_2 \\ 1+a_1+b_1+1+a_2+b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(a_1+a_2+1) \\ b_1+b_2 \\ 1+(a_1+a_2+1)+(b_1+b_2) \end{bmatrix} = \begin{bmatrix} 1+a_3 \\ b_3 \\ 1+a_3+b_3 \end{bmatrix} \qquad a_3, b_3 \in \mathbb{R}$$

$\therefore \vec{q_3} \in Q$

$\therefore Q$ is closed under addition.

③ for $c \in \mathbb{R}$, if

$$\vec{q}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1+a_1 \\ b_1 \\ 1+a_1+b_1 \end{bmatrix} \in Q$$

$$\vec{q}_4 = c\vec{q}_1 = c\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = c\begin{bmatrix} 1+a_1 \\ b_1 \\ 1+a_1+b_1 \end{bmatrix} = \begin{bmatrix} c+ca_1 \\ cb_1 \\ c+ca_1+cb_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(c+ca_1-1) \\ cb_1 \\ 1+(c+ca_1-1)+cb_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+a_4 \\ b_4 \\ 1+a_4+b_4 \end{bmatrix} \qquad a_4, b_4 \in \mathbb{R}.$$

$\therefore \vec{q}_4 \in Q$

$\therefore$ Q is closed under scalar multiplication.

Therefore, Q is a subspace of $\mathbb{R}^3$.

(3) Denote the set as W,

$$\begin{bmatrix} a \\ 3b \\ a+2b \\ 2b-a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix} \qquad a, b \in \mathbb{R}$$

$\therefore$ $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^4$ by theorem.

2. Determine the null space of the following matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}$$

if $\mathrm{col}(A)$ is subspace of $\mathbb{R}^k$, what is k?

① $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \to R_2]{} \begin{bmatrix} 1 & 3 & 2 & 8 \\ 0 & 1 & -2 & -13 \end{bmatrix}$$

$$\xrightarrow[R_1 - 3R_2 \to R_1]{} \begin{bmatrix} 1 & 0 & 8 & 47 \\ 0 & 1 & -2 & -13 \end{bmatrix}$$

$$\therefore \begin{cases} x_1 = -8x_3 - 47x_4 \\ x_2 = 2x_3 + 13x_4 \end{cases}$$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -47 \\ 13 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

$$\therefore \mathrm{Null}(A) = \mathrm{span}\left\{ \begin{bmatrix} -8 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -47 \\ 13 \\ 0 \\ 1 \end{bmatrix} \right\}$$

② if $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^k$,

then $k = 2$.

3. What is the base of the range for the above given matrix?

$$A = \begin{pmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 2 & 3 \end{pmatrix}$$

The pivot positions are in the first 2 columns,

then, range$(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$.

4. (1) Is $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\right\}$ basis for $\mathbb{R}^3$?

(2) $S_1 = \{1, x, x^2\}$ is a basis of $\mathbb{P}_2$. Is $S_2 = \{1, x+1, (x+1)^2\}$ also a basis of $\mathbb{P}_2$?

(1) No. Since $\begin{bmatrix}1 & 2\\2 & 2\\1 & 1\end{bmatrix}$ doesn't have pivot in every row.

(2) Yes.

① Since $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.

$\therefore\ \forall\ v \in \mathbb{P}_2,\quad v = a + bx + cx^2$ for $a, b, c \in \mathbb{R}$

(a linear combination of S_1)

Also, $x^2 = (x+1)^2 - 2(x+1) + 1$

$x = (x+1) - 1$

So, $cx^2 + bx + a$

$= c\left[(x+1)^2 - 2(x+1) + 1\right] + b\left[(x+1) - 1\right] + a \cdot 1$

$= c(x+1)^2 + (b - 2c)(x+1) + (a - b + c) \cdot 1$

(a linear combination of S_2)

② Show S_2 is linearly independent.

if there exists $u_1, u_2, u_3 \in \mathbb{R}$ s.t.

$u_1(1+x)^2 + u_2(1+x) + u_3 \cdot 1 = 0$

then, arrange as:

$u_1 x^2 + (2u_1 + u_2)x + u_1 + u_2 + u_3 = 0$

Since S_1 is a basis, which means,

S_1 is a linearly independent set,

$$\begin{cases} u_1 = 0 \\ 2u_1 + u_2 = 0 \\ u_1 + u_2 + u_3 = 0 \end{cases} \Longrightarrow \begin{cases} u_1 = 0 \\ u_2 = 0 \\ u_3 = 0 \end{cases}$$

Thus, S_2 is linearly independent.

5. (1) Is $\left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ linearly independent?

(2) Suppose nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are orthogonal to each other, namely, $\mathbf{v}_i^\top \mathbf{v}_j = 0$ holds for any $i \neq j$, $i, j = 1, \cdots, n$. Prove $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.

(1) Yes.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2+R_1 \to R_2]{} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ REF}$$

(2) if there exist $a_1, \dots, a_n \in \mathbb{R}$ s.t.

$$a_1\vec{v_1} + a_2\vec{v_2} + \dots + a_n\vec{v_n} = \vec{0} \qquad (*)$$

multiply $\vec{v_1}^T$ on left for both sides

then, $a_1 \underbrace{\vec{v_1}^T\vec{v_1}}_{\text{non-zero}} + a_2 \underbrace{\vec{v_1}^T\vec{v_2}}_{\text{zero}} + \dots + a_n \underbrace{\vec{v_1}^T\vec{v_n}}_{\text{zero}} = \vec{v_1}^T\vec{0} = 0$

$\Rightarrow \quad a_1 \underbrace{\vec{v_1}^T\vec{v_1}}_{\text{non zero}} = 0$

Thus, $a_1 = 0$.

Similarly, $a_2 = 0$, $a_3 = 0$, $\dots$, $a_n = 0$

Therefore, $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ is linearly independent.