MATH2111 Tutorial 6

T1A&T1B QUAN Xueyang
T1C&T2A SHEN Yinan
T2B&T2C ZHANG Fa

1 The Inverse of a Matrix

1. **Definition (Inverse of a Matrix)**
   An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that
   $$CA = I_n \text{ and } AC = I_n$$
   where I_n denote the $n \times n$ identity matrix. In this case, C is the inverse of A, and is denoted by A^{-1}. A matrix that is not invertible is called a singular matrix, and an invertible matrix is called a nonsingular matrix.

2. **Theorem**. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
   If $ad - bc \neq 0$, then A is invertible and
   $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
   If $ad - bc = 0$, then A is not invertible.
   Note: The quantity $ad - bc$ is called the determinant of A, and we write
   $$\det A = ad - bc$$

3. **Theorem**. If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

4. **Theorem**.
   (a) If A is an invertible matrix, then A^{-1} is invertible and
   $$\left(A^{-1}\right)^{-1} = A$$
   (b) If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. i.e.
   $$(AB)^{-1} = B^{-1}A^{-1}$$
   (c) If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. i.e.
   $$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

5. **Theorem**. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

6. **Algorithm for Finding** A^{-1}
(a) Row reduce the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
(b) If A is row equivalent to I, then $\left[\begin{array}{c|c} A & I \end{array}\right]$ is row equivalent to $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$. Otherwise, A does not have an inverse.

2 Characterizations of Invetible Matrices

1. **Theorem (The Invertible Matrix Theorem)**
Let A be a square $n \times n$ matrix. Then the following statements are equivalent.
(a) A is an invertible matrix.
(b) A is row equivalent to the $n \times n$ identity matrix.
(c) A has n pivot positions.
(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A form a linearly independent set.
(f) The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
(j) There is an $n \times n$ matrix C such that $CA = I$.
(k) There is an $n \times n$ matrix D such that $AD = I$.
(l) A^T is an invertible matrix.

2. **Definition**. A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that
$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n, \text{ and } T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

3. **Theorem**. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the definition.

3 Determinants

3.1 Calculate Determinants

1. **Notation**.
(1) A_{ij} is the submatrix got from matrix A by deleting the ith row and jth column of A.
(2) $C_{ij} = (-1)^{i+j} \det A_{ij}$ is called the (i, j)-**cofactor** of A.

2. **Definition**.
For $n \geq 2$, the determinant of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is given by
$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

3. **Theorem**.
The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\begin{aligned} \det A &= (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots + (-1)^{i+n} a_{in} \det A_{in} \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{(expansion across row } i\text{)} \end{aligned}$$

$$\begin{aligned} \det A &= (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \cdots + (-1)^{n+j} a_{nj} \det A_{nj} \\ &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{(expansion down column } j\text{)} \end{aligned}$$

Note: Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.2 Properties of Determinants

1. **Theorem (Row Operations)**. Let A be a square matrix.
(a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
(b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.
(c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

2. **Theorem**. A square matrix A is invertible if and only if $\det A \neq 0$.

3. **Corollary**. $\det A = 0$ if the rows of A are linearly dependent.

4. **Theorem**. If A is an $n \times n$ matrix, then $\det A^T = \det A$.

5. **Theorem**. If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

4 Exercises

1. Find the inverses of the matrices below, if they exist.

(a) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

Q1.

① $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2+3R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3+3R_1 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$

$\xrightarrow[\frac{1}{2}R_3 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$

$\xrightarrow[R_1+2R_3 \to R_1]{R_2+2R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right] = [I \mid A^{-1}]$

② $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1 \to R_3]{R_2-4R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3-2R_2 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$

A^{-1} doesn't exist.

2. Suppose T and U are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(U\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Is it true that $U(T\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$? Why or why not?

Q2. Let A, B be standard matrices of T, U.

Then standard matrix of $T(U\vec{x})$ is AB.

$\because$ $T(U\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

$\therefore$ we have $AB = I_n$

For $A, B \in \mathbb{R}^{n\times n}$, by invertible matrix theorem,

A, B are invertible, $B = A^{-1}$, $BA = I$

And BA is standard matrix of $UT(\vec{x})$,

so we have $UT(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

3. Compute the determinants by cofactor expansions.

(a) $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

Q3. (a)

$$\det(A) = \begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = (-3) \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix}$$

$$= (-3)\cdot(2) \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= (-3)\cdot(2)\cdot\left(4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right)$$

$$= -6 \cdot \left[4\left(2\times 2-(-3)\times(-1)\right) - 5\left(3\times 2-(-1)\times(-5)\right)\right]$$

$$= -6(4\times 1 - 5\times 1) = 6$$

(b)

$$\det(B) = \begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = (-2) \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix}$$

$$= (-2)(-1) \begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 2\left[3 \times \begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right]$$

$$= 2\cdot\left[3\times(-4\times 2 - 3\times 1) + 2\times\left(2\times 1 - 4\times(-4)\right)\right] = 6$$

4. Find the determinant by row reduction to echelon form.

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$$

Q4.

$$\det(A) = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix} \overset{R_3+2R_1 \to R_3}{\underset{R_4-R_1 \to R_4}{=\!=\!=}} \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 2 & -5 & 2 & -1 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$$

$$\overset{R_4-R_2 \to R_4}{\underset{R_5-R_2 \to R_5}{=\!=\!=}} \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$$

$$\overset{R_3 \Leftrightarrow R_4}{=\!=\!=} (-1) \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$$

$$\underset{R_5-\frac{7}{3}R_4 \to R_5}{=\!=\!=} (-1) \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1) \cdot 1 \cdot 2 \cdot (-1) \cdot 3 \cdot 1 = 6$$

5. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

Q5. $\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - bc$

$= 1 + ad + a + d - bc$

$\det(A) + \det(B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1 + ad - bc$

Thus $\det(A+B) = \det(A) + \det(B) \iff 1 + ad + a + d - bc = 1 + ad - bc$

$\iff a + d = 0$