COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Fall Semester – Tutorial 2

Question 1: Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet", where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

(a) No one in the class has chatted with Bob.

(b) Not everyone in your class has an Internet connection.

(c) Everyone except one student in your class has an Internet connection.

(d) There are two students in your class who have not chatted with each other over the Internet.

Solution : (a) $\neg\exists x C(x, \mathrm{Bob})$.

(b) $\neg\forall x I(x)$.

(c) $\exists x \forall y (x \neq y \leftrightarrow I(y))$.

(d) $\exists x \exists y (x \neq y \wedge \neg C(x, y))$

Question 2: Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\forall x \exists y \forall z\ T(x, y, z)$

(b) $\forall x \exists y\ P(x, y) \vee \forall x \exists y\ Q(x, y)$

(c) $\forall x \exists y\ (P(x, y) \wedge \exists z\ R(x, y, z))$

(d) $\forall x \exists y\ (P(x, y) \rightarrow Q(x, y))$

Solution : As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

(a)

$$\begin{aligned} \neg\forall x \exists y \forall z\ T(x, y, z) &\equiv \exists x \neg\exists y \forall z\ T(x, y, z) \\ &\equiv \exists x \forall y \neg\forall z\ T(x, y, z) \\ &\equiv \exists x \forall y \exists z\ \neg T(x, y, z) \end{aligned}$$

(b)

$$\begin{aligned} \neg(\forall x \exists y\ P(x, y) \vee \forall x \exists y\ Q(x, y)) &\equiv \neg\forall x \exists y\ P(x, y) \wedge \neg\forall x \exists y\ Q(x, y) \\ &\equiv \exists x \forall y\ \neg P(x, y) \wedge \exists x \forall y\ \neg Q(x, y) \end{aligned}$$

(c)

$$\begin{aligned}
\neg\forall x\exists y\ (P(x,y) \wedge \exists z\ R(x,y,z)) &\equiv \exists x\forall y\ \neg(P(x,y) \wedge \exists z\ R(x,y,z)) \\
&\equiv \exists x\forall y\ (\neg P(x,y) \vee \neg\exists z\ R(x,y,z)) \\
&\equiv \exists x\forall y\ (\neg P(x,y) \vee \forall z\ \neg R(x,y,z))
\end{aligned}$$

(d)

$$\begin{aligned}
\neg\forall x\exists y\ (P(x,y) \rightarrow Q(x,y)) &\equiv \exists x\forall y\ \neg(P(x,y) \rightarrow Q(x,y)) \\
&\equiv \exists x\forall y\ (P(x,y) \wedge \neg Q(x,y))
\end{aligned}$$

Question 3: In this question, p, q, r and s are logic propositions. In each part, a set of premises and a conclusion are given. You are asked to derive the conclusion from the premises using the rules of inference that you have learned in class. Show all steps and explain each line of your derivations.

(a) Premises:
1. $\neg p$
2. $\neg r$
3. $\neg(p \vee q) \rightarrow r$

Conclusion: q

(b) Premises:
1. $\neg q$
2. $p \rightarrow q$
3. $\neg p \rightarrow (r \wedge s)$

Conclusion: r

Solution : (a)

1. $\neg(p \vee q) \rightarrow r$	Premise
2. $\neg r$	Premise
3. $p \vee q$	Modus tollens (1,2)
4. $\neg p$	Premise
5. q	Disjunctive syllogism (3,4)

(b)

1. $p \rightarrow q$	Premise
2. $\neg q$	Premise
3. $\neg p$	Modus tollens (1,2)
4. $\neg p \rightarrow (r \wedge s)$	Premise
5. $r \wedge s$	Modus Ponens (3,4)
6. r	Simplification (5)

Question 4: Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

Solution : We can set this up in two-column format.

Step	Reason
1. $\forall x(P(x) \wedge R(x))$	Premise
2. $P(a) \wedge R(a)$	Universal instantiation using (1)
3. $P(a)$	Simplification using (2)
4. $\forall x(P(x) \to (Q(x) \wedge S(x)))$	Premise
5. $Q(a) \wedge S(a)$	Universal modus ponens using (3) and (4)
6. $S(a)$	Simplification using (5)
7. $R(a)$	Simplification using (2)
8. $R(a) \wedge S(a)$	Conjunction using (7) and (6)
9. $\forall x(R(x) \wedge S(x))$	Universal generalization using (5)

Question 5: Identify the error or errors in this argument that supposedly shows that if $\exists x\ P(x) \wedge \exists x\ Q(x)$ is true then $\exists x\ (P(x) \wedge Q(x))$ is true.

1.	$\exists x\ P(x) \wedge \exists x\ Q(x)$	Premise
2.	$\exists x\ P(x)$	Simplification from (1)
3.	$P(c)$ for some c	Existential instantiation from (2)
4.	$\exists x\ Q(x)$	Simplification from (1)
5.	$Q(c)$ for some c	Existential instantiation from (4)
6.	$P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7.	$\exists x\ (P(x) \wedge Q(x))$	Existential generalization

Solution : The error occurs in step (5), because we cannot assume, as is being done here, that the c that makes P true is the same as the c that makes Q true.

Question 6: Use a direct proof to show that if $m+n$ and $n+p$ are even integers, where m, n, and p are integers, then $m+p$ is even.

Solution : Suppose that $m+n$ is even. Then $m+n=2s$ for some integer s. Suppose that $n+p$ is even. Then $n+p=2t$ for some integer t. If we add these (this step is inspired by the fact that we want to look at $m+p$), we get $m+p+2n=2s+2t$. Subtracting $2n$ from both sides and factoring, we have $m+p=2s+2t-2n=2(s+t-n)$. Since we have written $m+p$ as 2 times an integer, we conclude that $m+p$ is even, as desired.

Question 7: Recall that an integer n is odd if and only if it can be written $n=2k+1$ for some integer k. Use a direct proof to show that every odd integer can be written as the difference of two squares.

Solution : Assuming n is odd, we can write $n=2k+1$ for some integer k. Then we can write $(k+1)^2-k^2=k^2+2k+1-k^2=2k+1=n$