MATH2111 Tutorial 4

QUAN Xueyang (xquan@connect.ust.hk)
T1A: Tue (18:00-18:50) 2465
T1B: Thu (18:00-18:50) 6591

1 Linear Independence

1. **Definition (Linear Independence):**
   (a) An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation
   $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$
   has only the trivial solution.
   (b) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that
   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$
2. **Theorem (Linear Independence of Columns of Matrix):**
   The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has **only** the trivial solution.
3. **Theorem (Characterization of Linearly Dependent Sets):**
   An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.
4. **Theorem (Conditions For Linear Dependence):**
   (a) If a set contains more vectors than the entries in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
   (b) If a set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

2 Transformations

1. **Definition (Transformation):**
   A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
   The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T.
   The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
   For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the range of T.

3 Exercises

1(a). Express the general solutions of the following non-homogeneous systems in terms of the given particular solutions.

$$\begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 5 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 4 \\ -x_1 - x_2 + x_3 - x_5 = -1 \end{cases}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the above linear system.

1(b). Denote the coefficient matrix as $\boldsymbol{A}$. Use as many columns of $\boldsymbol{A}$ as possible to construct a matrix $\boldsymbol{B}$ with the property that the equation $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution. (Solve $\boldsymbol{Bx} = \mathbf{0}$ to verify your work.)

2. Find conditions on p and q such that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ p \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 9 \\ q \end{bmatrix} \right\}$$

is linearly independent.

3. Consider matrix $\boldsymbol{A}$,

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \end{bmatrix},$$

Find a vector which is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and also in Span $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$, or explain why such a vector cannot exist.

(Given $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $\boldsymbol{Ax} = \mathbf{0}$.)

4. State whether each of the following statement is true or false. (If it is true, give a brief justification; if it is false, give a counterexample.)

(a) If $\boldsymbol{A}\begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \mathbf{0}$, then $\boldsymbol{A}\boldsymbol{e}_4$ is a linear combination of the first three columns of $\boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a 4×3 matrix with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, and suppose $\boldsymbol{b}$ is a vector in $\mathbb{R}^4$ such that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}\}$ is linearly dependent. Then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

5. Consider

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ 0 \\ 3x_1 - x_2 \end{bmatrix}$$

(a) What is the domain of F?

(b) Find the image of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ under F.