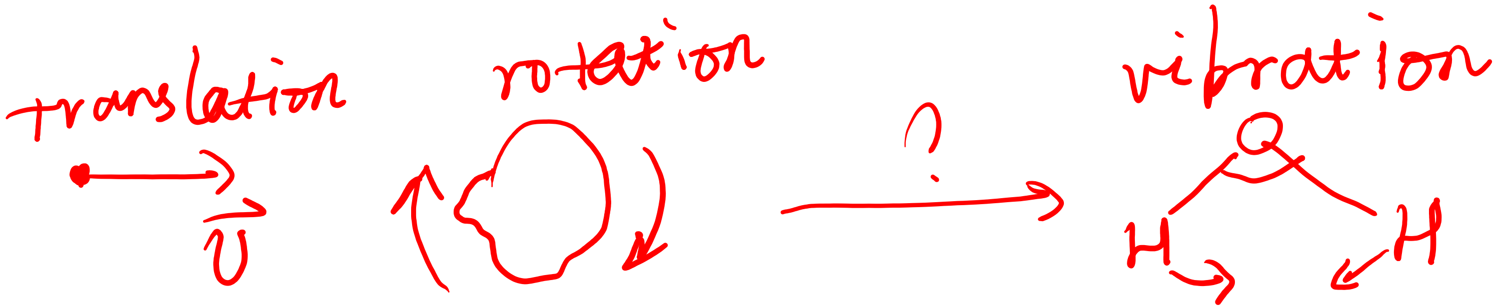


PERIODIC MOTION I

PHYS1112

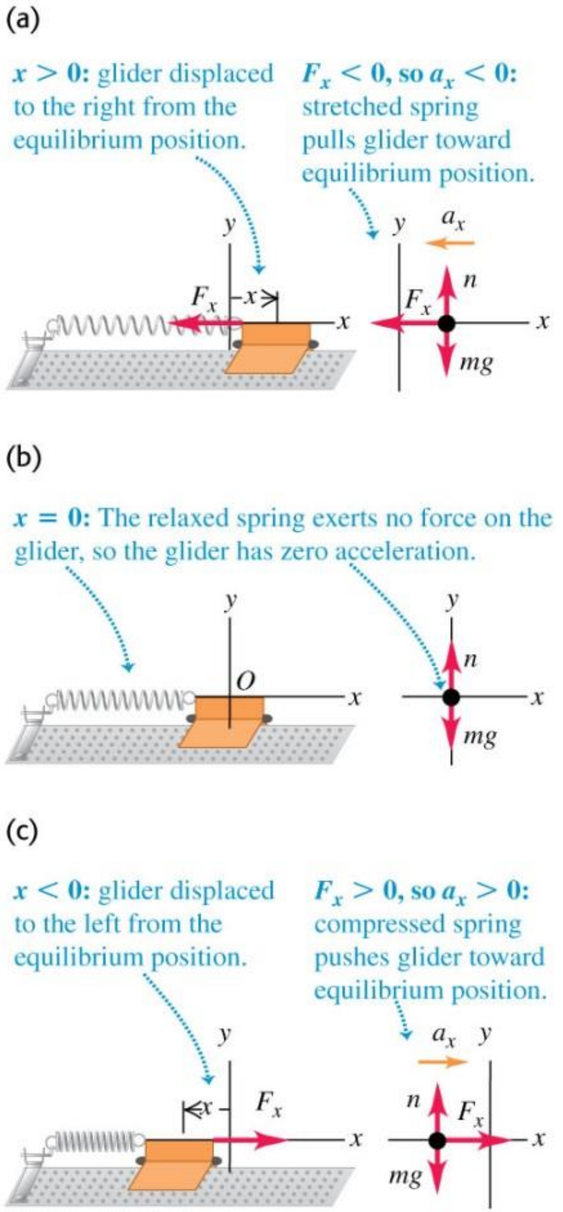
Lecture 14

Intended Learning Outcomes

- After this lecture you will learn:
  1) definition of simple harmonic motion
  2) relation between uniform circular motion and simple harmonic motion
  3) description of simple harmonic motion in terms of phasor diagram
  4) kinetic, potential, and total energy in simple harmonic motion

Simple Harmonic Motion (SHM)

Simplest example: a spring and mass system



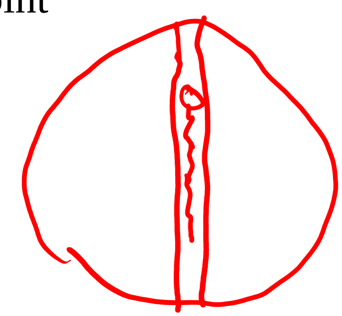
Hooke's law: $F_x = -kx$

$\vec{F} = -k\vec{x}$

restoring force

displacement (+/-) from equilibrium point

Newton's law



$$a_x = \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

ordinary

ODE

a *differential equation* of the form

$$\ddot{x} = -\alpha x,\ \alpha > 0,$$

$\alpha = \frac{k}{m}$

called **simple harmonic motion (SHM)**

A system executing simple harmonic motion is called a **harmonic oscillator**

ODE : $\frac{d^2x}{dt^2} = -\alpha x$ $\quad \alpha = \frac{k}{m}$

$$\boxed{\ddot{x} = -x}$$

$$x = A\cos\omega t$$

$$-\cancel{A}\omega^2\cancel{\cos\omega t} = -\alpha\cancel{A\cos\omega t}$$

$$\omega^2 = \alpha = \frac{k}{m} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\begin{cases} v = \dot{x} = -A\omega\sin\omega t \\ a = \dot{v} = -A\omega^2\cos\omega t \\ \quad = -\frac{k}{m}x \end{cases}$$

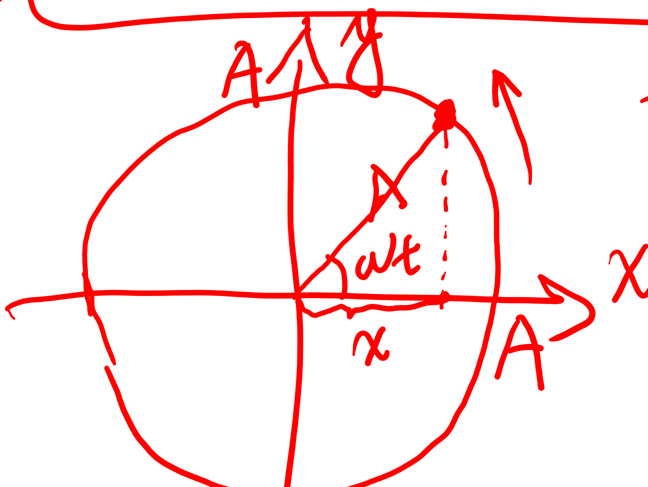
ω : angular speed
frequency

$$\frac{d^2 f(t)}{dt^2} - f(t) : \begin{cases} \sin(t) \\ \cos(t) \\ e^{it} \quad e^{-it} \end{cases}$$

$$\frac{d^2 e^{-t}}{dt^2} = +e^{-t} \qquad \boxed{i^2 = -1}$$

$$\begin{cases} e^{it} = \cos t + i\sin t \\ e^{-it} = \cos t - i\sin t \end{cases}$$

$$\sin t = \cos\left(t - \frac{\pi}{2}\right)$$



$$T = \frac{2\pi}{\omega}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$\omega = 2\pi f$$

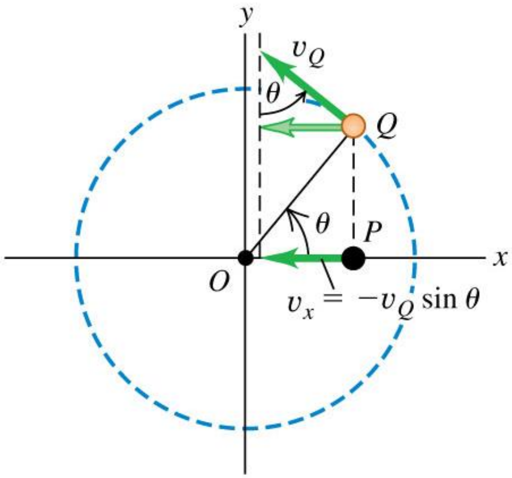
How to solve the differential equation? Consider a particle Q executing uniform circular motion with angular speed ω and radius A. P is its projection along x axis.

position of P: $x = A\cos\theta$

velocity of P: $v_x = -v_Q \sin\theta$

acceleration of P:
$$a_x = -a_Q\cos\theta$$
$$= -(\omega^2 A)\cos\theta$$
$$= -\omega^2 x \qquad c.f.\quad a = -(k/m)x$$

Conclusion: a harmonic oscillator is the same as a particle in uniform circular motion with $\omega = \sqrt{k/m}$ projected along the x direction

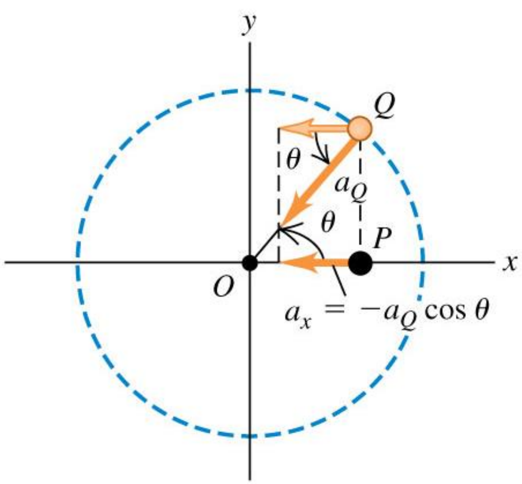


frequency f = number of cycles per unit time
$$= \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

period T = time for one complete cycle
$$= \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

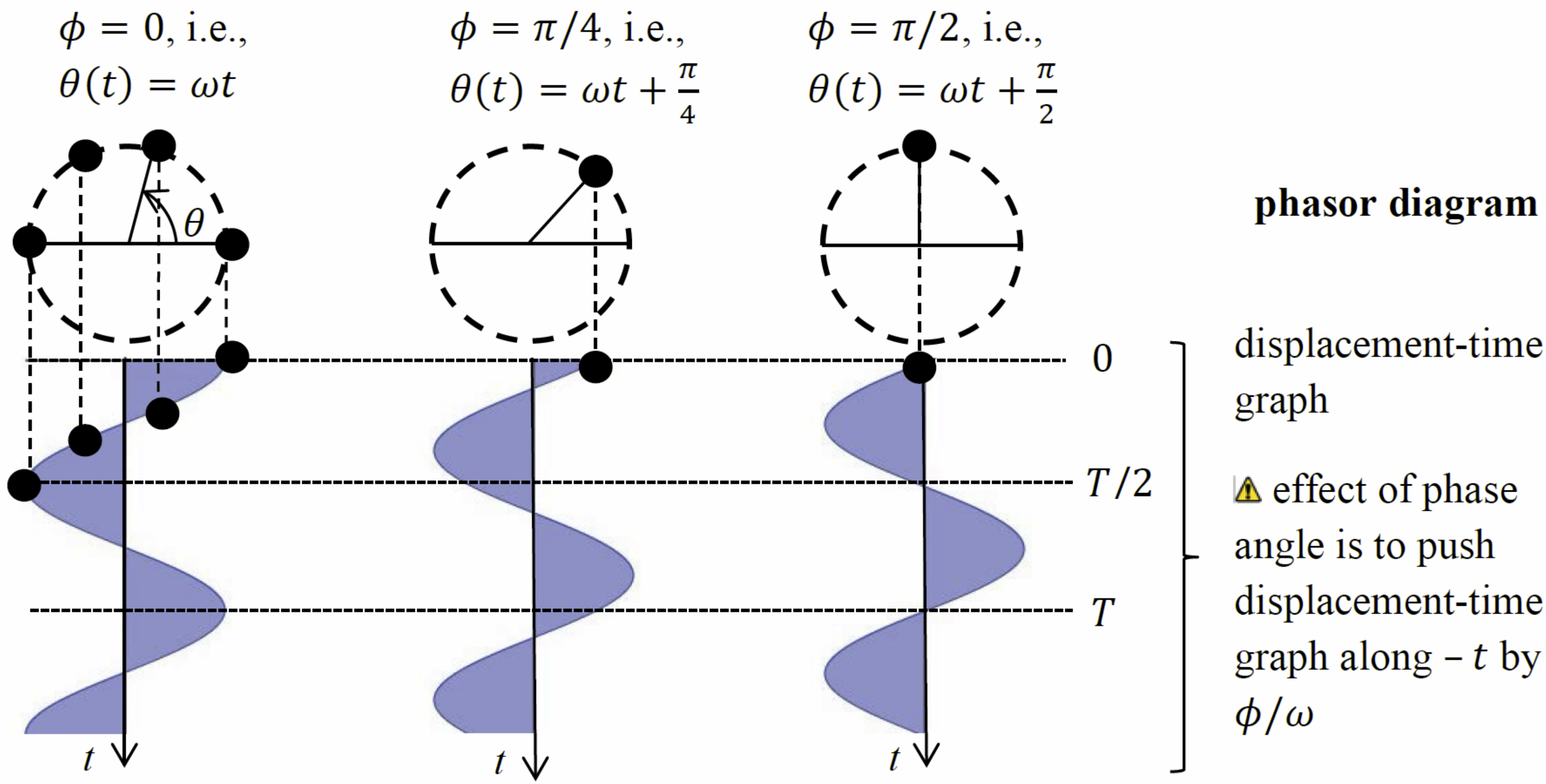
angular frequency ω = angle (in radian)per unit time
$$= 2\pi f$$

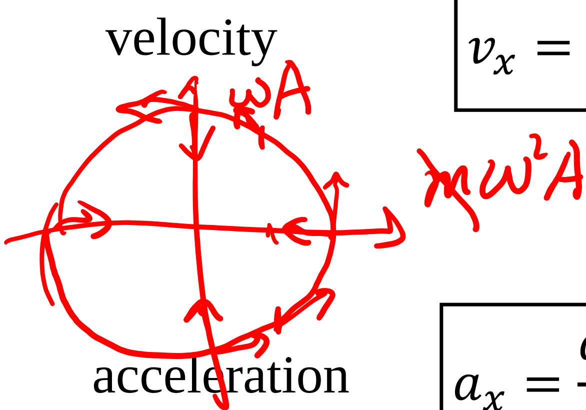
speed



General solution: $x = A\cos\theta(t) = A\cos(\omega t + \phi)$, where the **phase angle** $\phi = \theta(0)$

A is the **amplitude** (maximum displacement) of the oscillation





$$\boxed{v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) = \omega A \cos\left(\omega t + \phi + \tfrac{\pi}{2}\right)}$$

$$v_{max} = \omega A$$

$$\boxed{a_x = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi) = \omega^2 A \cos(\omega t + \phi + \pi)}$$

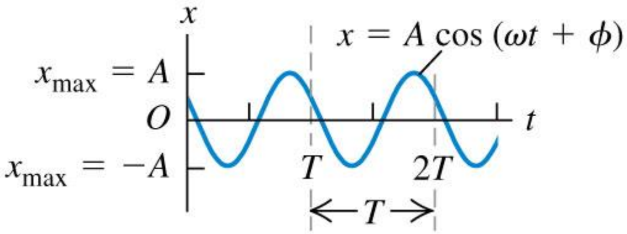
$$\omega = \sqrt{\frac{k}{m}} \qquad a_{max} = \omega^2 A$$

How to find A and ϕ? If given initial condition $x(0) = x_0$, $v(0) = v_{0x}$

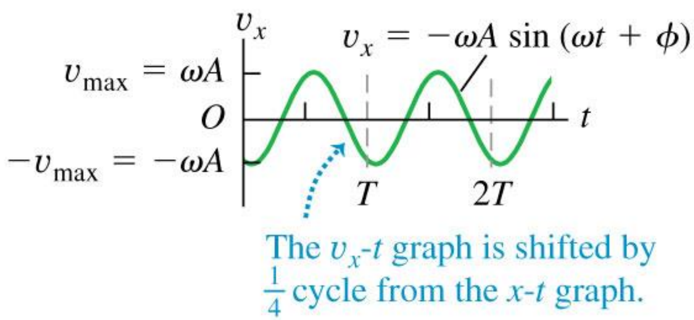
$$\frac{v_{0x}}{x_0} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega \tan\phi \implies \boxed{\phi = \begin{cases} \tan^{-1}\left(-\dfrac{v_{0x}}{\omega x_0}\right), & \text{if } x_0 > 0 \\ \tan^{-1}\left(-\dfrac{v_{0x}}{\omega x_0}\right) + \pi, & \text{if } x_0 < 0 \end{cases}}$$

$$x_0^2 + \frac{v_{0x}^2}{\omega^2} = A^2\left(\cos^2\phi + \sin^2\phi\right) = A^2 \implies \boxed{A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}}}$$

(a) Displacement x as a function of time t

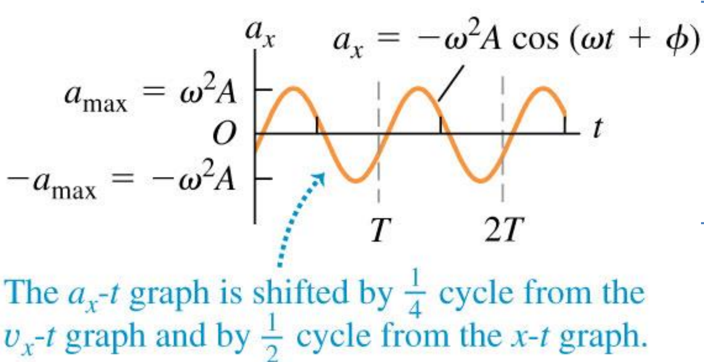


(b) Velocity v_x as a function of time t

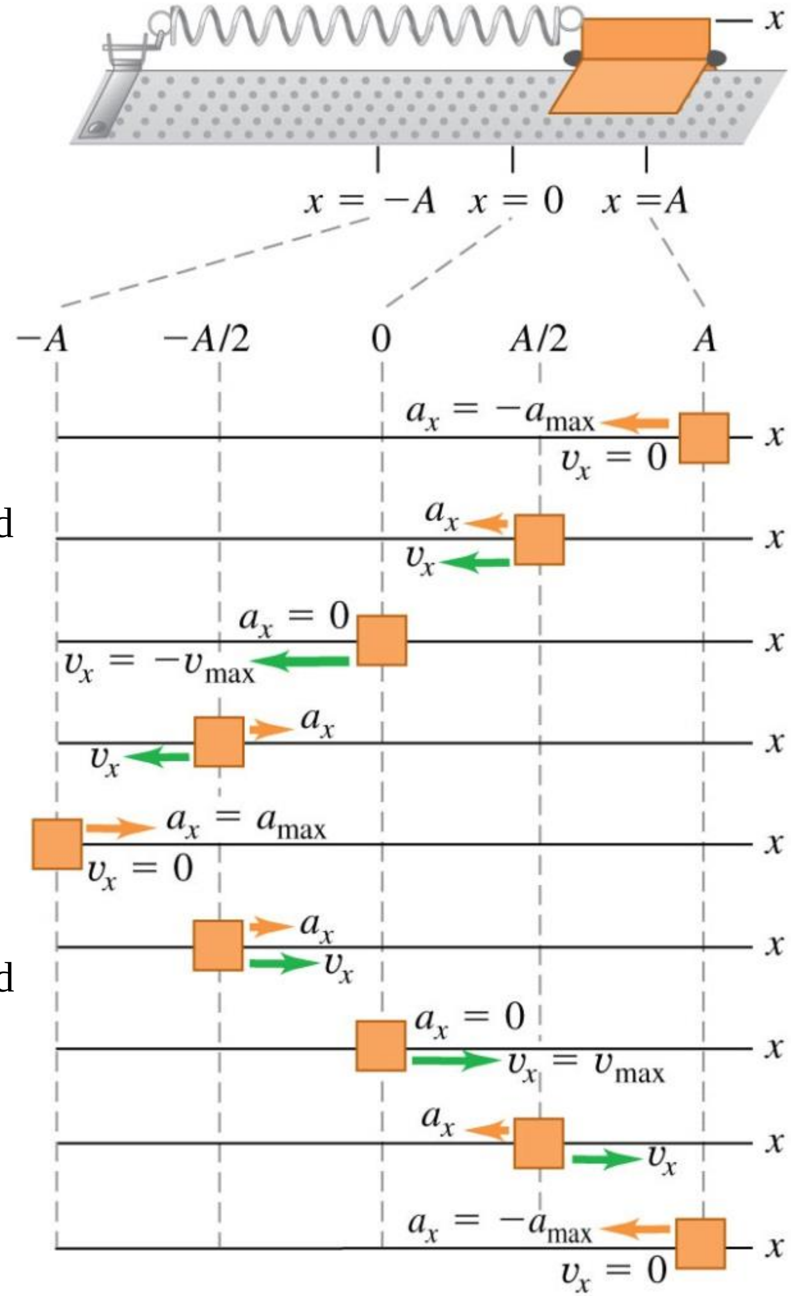


x pushed towards $-t$ by $\frac{\pi}{2}$, i.e., $\frac{T}{4}$

(c) Acceleration a_x as a function of time t



x pushed towards $-t$ by π, i.e., $\frac{T}{2}$



Question

Suppose the glider in the above diagram is moved to x = 0.10 m and is released from rest at t = 0, then A = 0.10 m and ϕ = 0 .

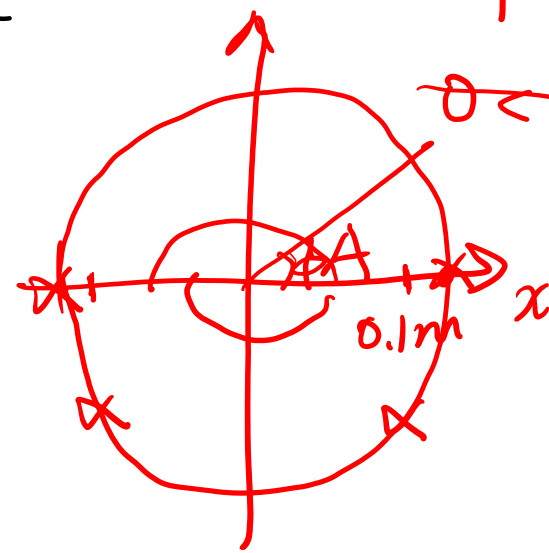
$v = 0$

Suppose instead the glider in the above diagram at t = 0 is at x = 0.10 m and is moving to the right, then A is (> / < / =) 0.10 m and ϕ is (> / < / =) 0.

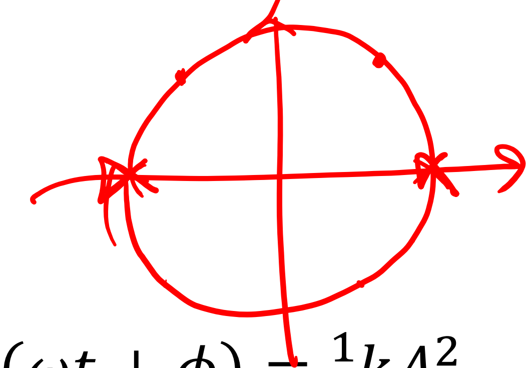
" $-\pi < \phi \leq \pi$ "

$>$ $<$

~~$0 < \phi \leq 2\pi$~~



Energy in Simple Harmonic Motion



$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2$$
$$= \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) + \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) = \underline{\underline{\tfrac{1}{2}kA^2}}$$

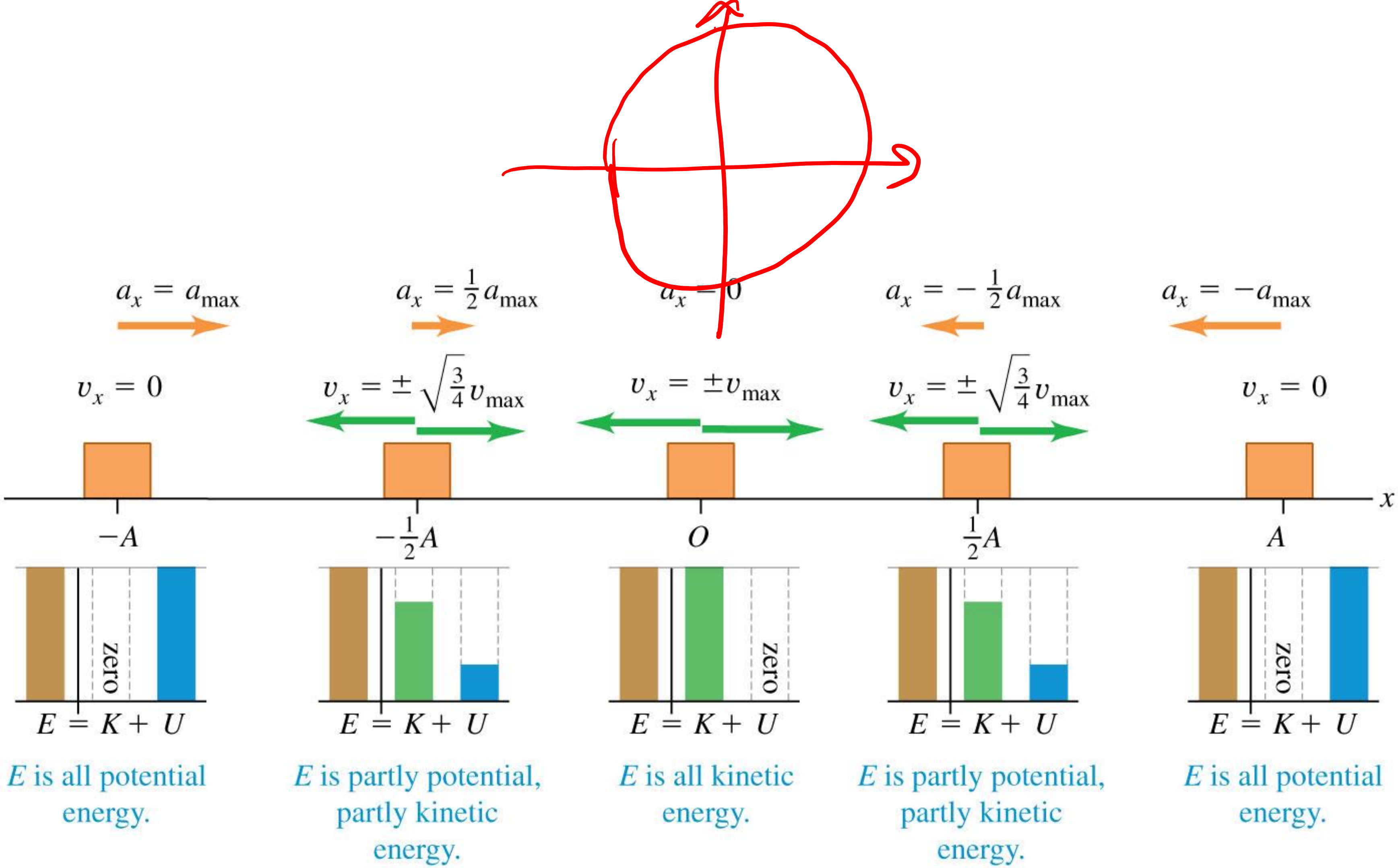
$\frac{k}{m}$

Conservation of mechanical energy!

To find velocity:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \quad \Rightarrow$$

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$



$a_x = a_{max}$
$a_x = \frac{1}{2}a_{max}$
$a_x = 0$
$a_x = -\frac{1}{2}a_{max}$
$a_x = -a_{max}$
$v_x = 0$
$v_x = \pm\sqrt{\frac{3}{4}}v_{max}$
$v_x = \pm v_{max}$
$v_x = \pm\sqrt{\frac{3}{4}}v_{max}$
$v_x = 0$
x
$-A$
$-\frac{1}{2}A$
O
$\frac{1}{2}A$
A
zero
zero
zero
$E = K + U$
$E = K + U$
$E = K + U$
$E = K + U$
$E = K + U$
E is all potential energy.
E is partly potential, partly kinetic energy.
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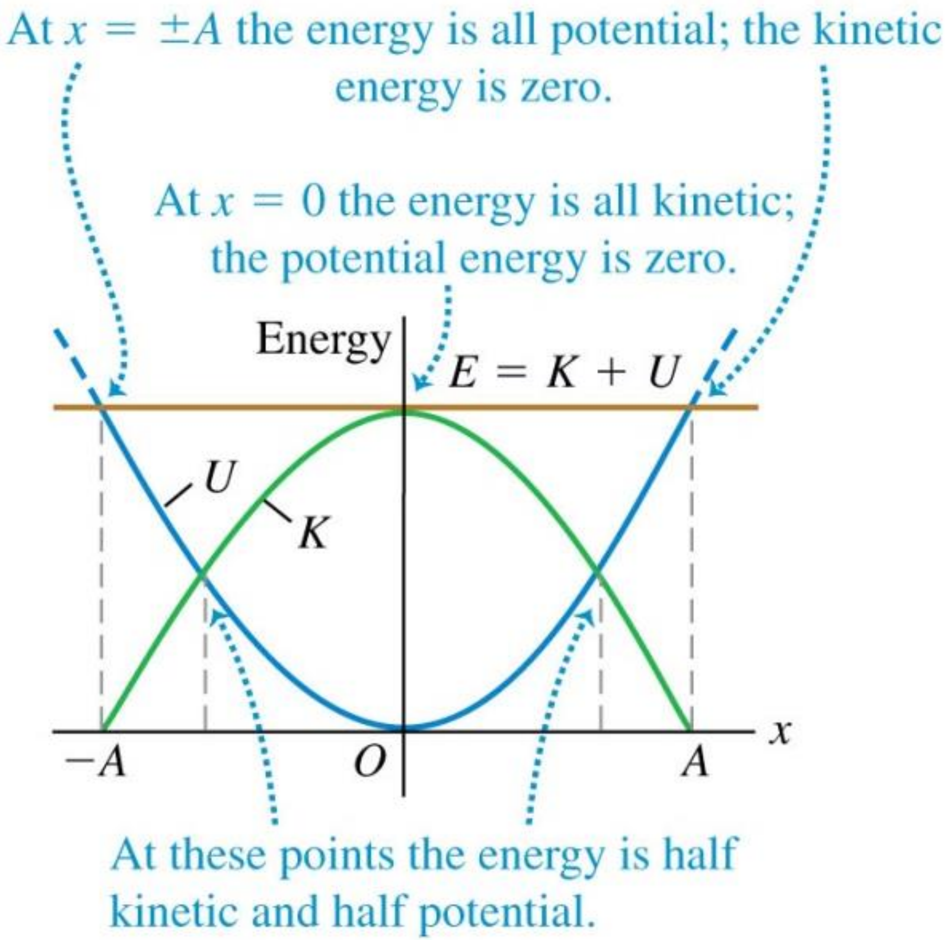


$U = \frac{1}{2}kx^2$

both U and K are quadratic (i.e., parabolic), and they add up to a constant

$$E = \tfrac{1}{2}kA^2$$

Question

To double the total energy of a spring and mass system oscillating in SHM, one should increase the amplitude by a factor of $\underline{\sqrt{2}}$. As a result of this amplitude change, the frequency of the oscillator will (be larger / be smaller / have no change).

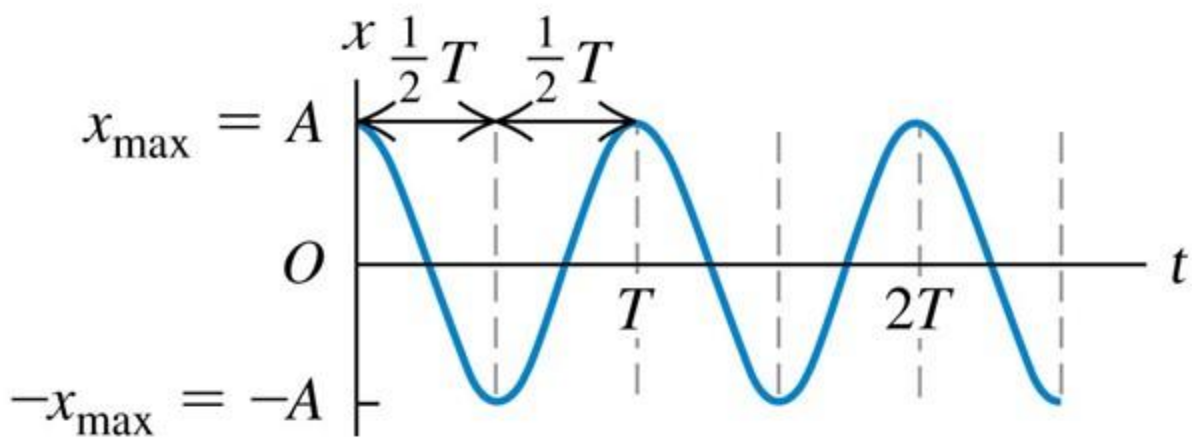
$$E = \underline{\frac{1}{2} k A^2}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$f = \underline{\frac{\omega}{2\pi}}$$

Q14.6

This is an *x-t* graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *potential energy* of the spring the greatest?



A. $t = T/8$

B. $t = T/4$

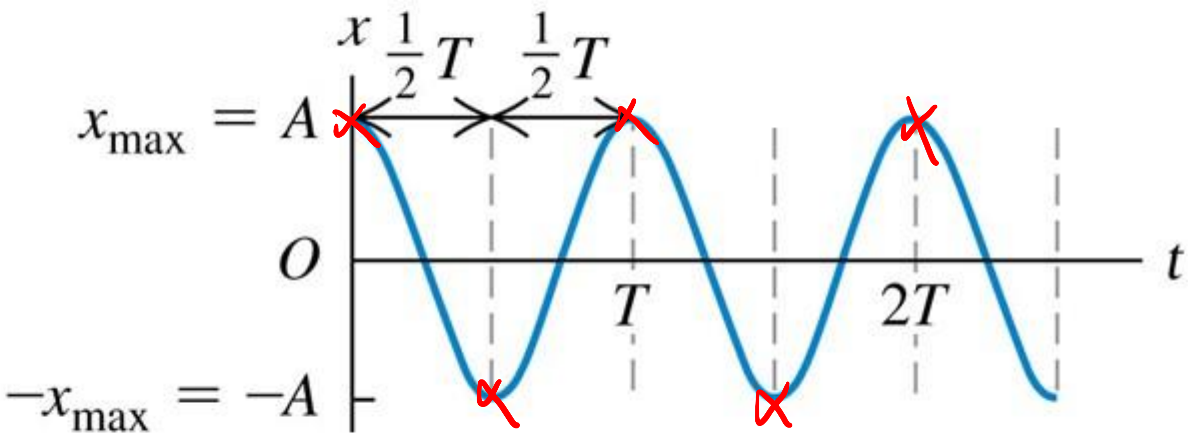
C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest potential energy.

A14.6

This is an *x*-*t* graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *potential energy* of the spring the greatest?



A. $t = T/8$

B. $t = T/4$

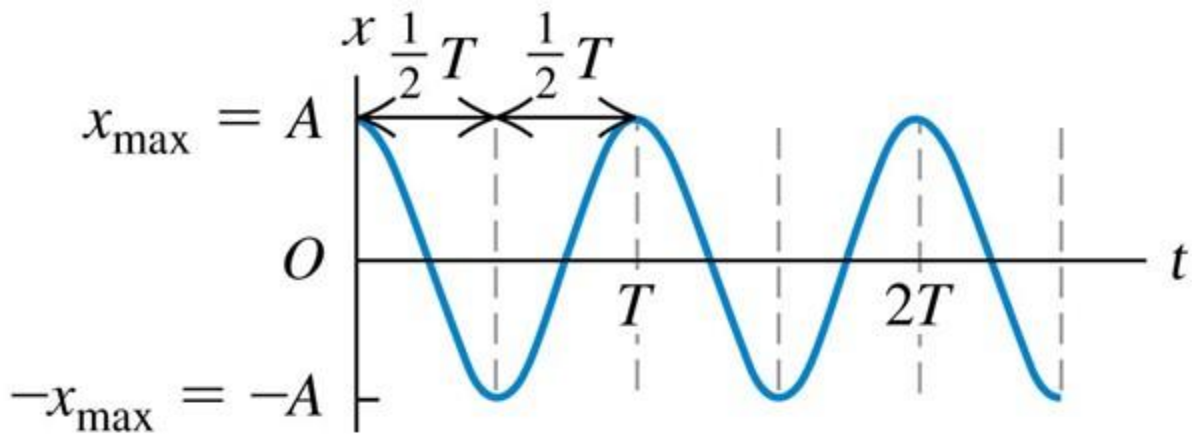
C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest potential energy.

Q14.7

This is an x-t graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *kinetic energy* of the object the greatest?



A. $t = T/8$

B. $t = T/4$

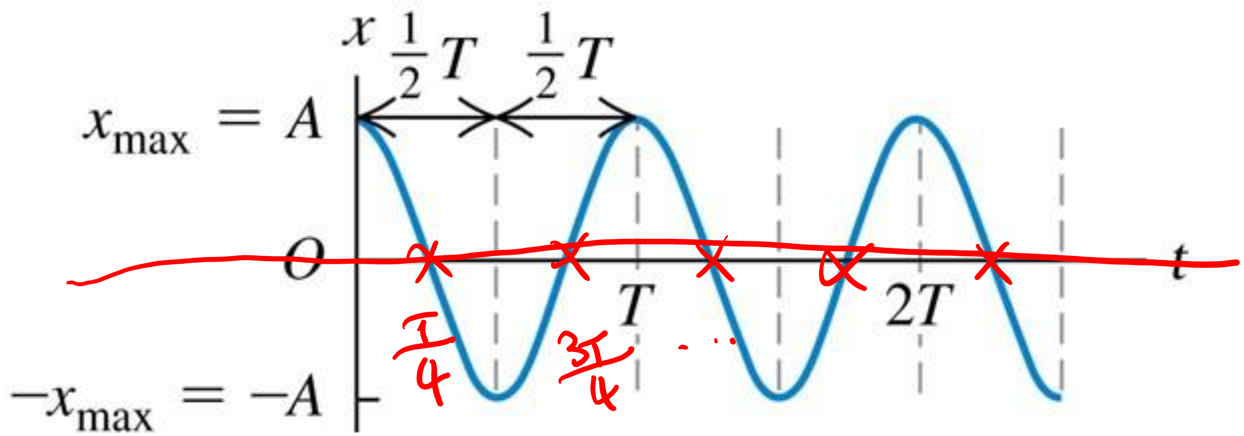
C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest kinetic energy.

A14.7

This is an x-t graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *kinetic energy* of the object the greatest?



A. $t = T/8$

✔ B. $t = T/4$

C. $t = 3T/8$

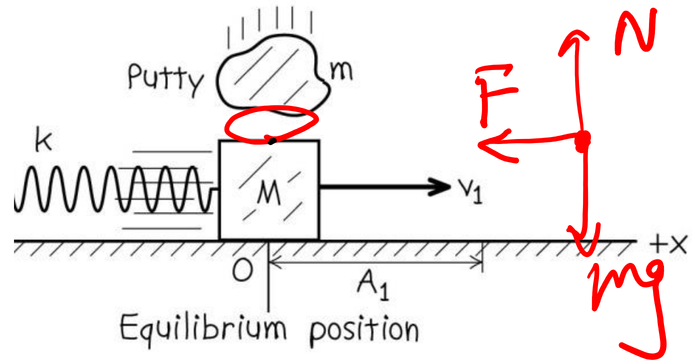
D. $t = T/2$

E. Two of the above are tied for greatest kinetic energy.

Example Energy and momentum in SHM

$x = 0$ equilibrium

Given: an oscillator with amplitude A_1
When it is at $x = 0$, a putty of mass m hits, and then stays on the block after collision



During the collision:
y component of momentum (is / is not) conserved
x component of momentum (is / is not) conserved

Oscillator: M -> M+m

New velocity at $x = 0$:

$x = 0$

$\frac{1}{2} M v_1^2$

$$M v_1 + 0 = M v_2 + m v_2 \quad \Rightarrow \quad v_2 = \frac{M}{M+m} v_1$$

$\frac{1}{2}$

New amplitude:

< 1

$$\cancel{\tfrac{1}{2}} k A_2^2 = \tfrac{1}{2}(M+m) v_2^2 = \left(\frac{M}{M+m}\right) \tfrac{1}{2} M v_1^2 = \boxed{\left(\frac{M}{M+m}\right)} \cancel{\tfrac{1}{2}} k A_1^2$$

E in terms of amplitude after collision

K right after collision

$$\Rightarrow \quad A_2 = A_1 \sqrt{\frac{M}{M+m}}$$

Total energy of the oscillator (increase/decrease). Where does the energy go?

Suppose the putty hits when the block is at $x = A_1$ (one end)

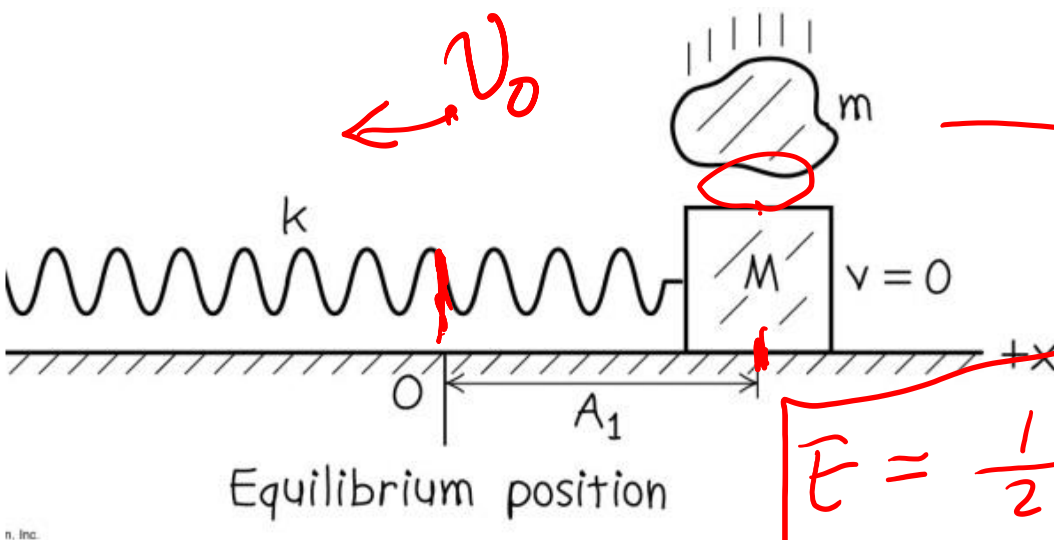
No change in horizontal velocity at $x = A_1$(why?)

No change in K (why?)

Does the energy of the oscillator change? Why?

(Oscillator: M -> M+m)

$M \rightarrow M+m$



$$\boxed{E = \frac{1}{2} k A_1^2}$$

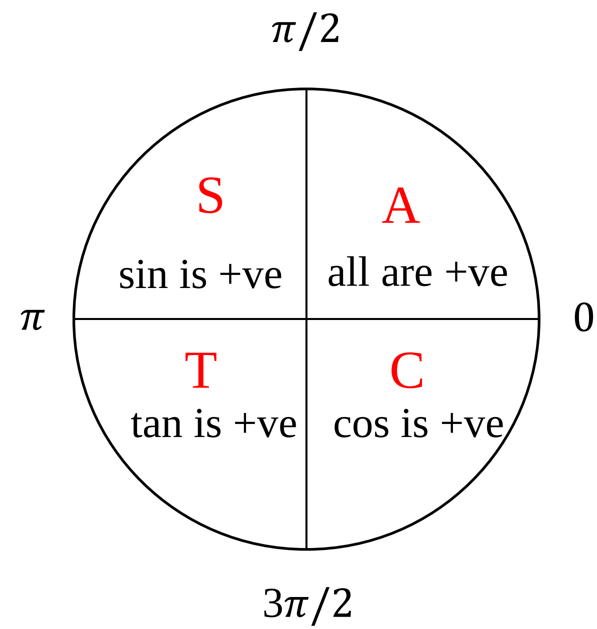
$$= \frac{1}{2}(M+m) v_0^2$$

For advanced students only. Others may ignore this part

Appendix

The formula $\phi = \tan^{-1}(-v_{0x}/\omega x_0)$ does not always give the correct answer. One needs to determine ϕ in the correct quadrant through the conditions

$$\sin\phi = -v_{0x}/\omega A$$
$$\cos\phi = x_0/A$$



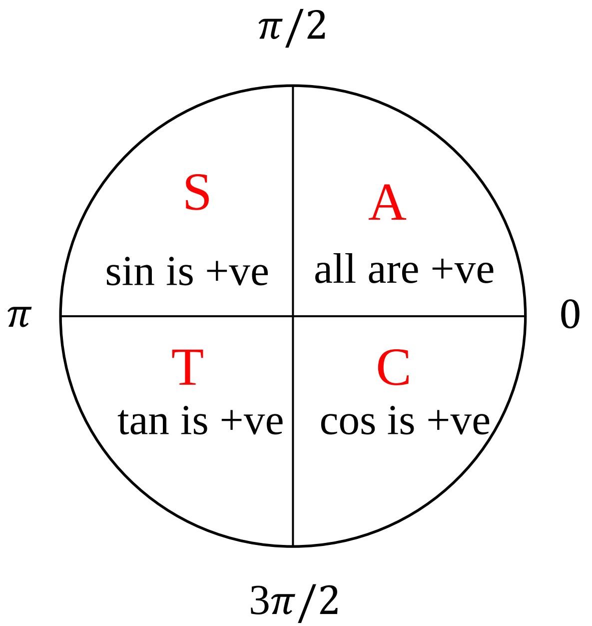
But you can easily convince yourself that the general formula is $\phi = \begin{cases} \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right), & \text{if } x_0 > 0 \\ \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right) + \pi, & \text{if } x_0 < 0 \end{cases}$,

irrespective of whether $v_{0x}$is positive or negative,as illustrated in the following example:

Example

Given $v_{0x} = 0.40\ \mathrm{m/s}$, $x_0 = 0.015\ \mathrm{m}$, $\omega = 20\ \mathrm{rad/s}$, then

$$\phi_1 = \tan^{-1}\left(-\frac{0.40\ \mathrm{m/s}}{(20\ \mathrm{rad/s})(0.015\ \mathrm{m})}\right) = -0.93\ \mathrm{rad}$$



But if $v_{0x} = -0.40\ \mathrm{m/s}$, $x_0 = -0.015\ \mathrm{m}$, then $\sin\phi_2 > 0$ and $\cos\phi_2 < 0$, i.e., ϕ_2 in the second quadrant, and the correct phase angle is

$$\phi_2 = \pi - 0.93\ \mathrm{rad} = 2.21\ \mathrm{rad}$$