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Math1014 Midterm Exam, Spring 2014

Part I: Multiple Choice Questions

Brief MC Solution

1.

$$\text{“net change”} \;=\; \int_0^{20} \frac{t^2(30-t)}{100}dt = \frac{1}{100}\Big[10t^3 - \frac{t^4}{4}\Big]_0^{20} = 400$$

2. Find the intersection points of the curves first by solving $2y^2 - (4y^4 - 2y^2) = 4y^2(1-y^2) = 0$; i.e., $(0,0)$, $(2,1)$.

$$\int_0^1 [2y^2 - (4y^4 - 2y^2)]dy = \int_0^1 (4y^2 - 4y^4)dy = \Big[\frac{4y^3}{3} - \frac{4y^5}{5}\Big]_0^1 = \frac{8}{15}$$

3. The area of an equilateral triangle with base length $2x$ is given by $\frac{1}{2}(2x)(2x\sin\frac{\pi}{3})$. By the slicing method,

$$\text{volume} \;=\; 2\int_0^2 \frac{1}{2}(2x)^2 \sin\frac{\pi}{3}dy = 2\sqrt{3}\int_0^2 (1-y^2)dy = \frac{8\sqrt{3}}{3}$$

4. Let $u = \sec 2x$, such that $du = 2\sec 2x \tan 2x dx$. Note that when $x = 0$, $u = \sec 0 = 1$; $x = \frac{\pi}{8}$, $u = \sec\frac{\pi}{4} = \sqrt{2}$.

$$\int_0^{\frac{\pi}{8}} 4\tan^3(2x)\sec^2(2x)\,dx = \int_1^{\sqrt{2}} 2(u^2-1)udu = \frac{1}{2}$$

5.

$$\int_0^{\frac{\pi}{3}} 3\cos^2(3x)\cos(6x)\,dx = \frac{3}{2}\int_0^{\frac{\pi}{3}} (1+\cos(6x))\cos 6x dx$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{3}} (\cos 6x + \frac{1+\cos 12x}{2})dx = \frac{3}{2}\Big[\frac{\sin 6x}{6} + \frac{x}{2} + \frac{\sin 12x}{24}\Big]_0^{\frac{\pi}{3}} = \frac{\pi}{4}$$

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6. Let $u = e^x$ such that $du = e^x dx = \frac{dx}{u}$. Note that when $x = 0$, $u = 1$; $x = \ln 3$, $u = 3$. By the substitution and the method of partial fractions,

$$\int_0^{\ln 3} \frac{e^x - 3}{(e^x + 1)(e^x + 3)}\,dx = \int_1^3 \frac{u - 3}{(u + 1)(u + 3)u}\,du$$

$$= \int_1^3 \Big(\frac{2}{u+1} - \frac{1}{u+3} - \frac{1}{u}\Big)\,du = \left[\ln\left|\frac{(u+1)^2}{u(u+3)}\right|\right]_1^3 = 3\ln 2 - 2\ln 3$$

7.

$$\int_{-1}^{2} \frac{1}{\sqrt{x^2 + 2x + 5}}\,dx = \int_{-1}^{2} \frac{1}{\sqrt{(x+1)^2 + 4}}\,dx \overset{x+1=2\tan\theta}{=} \int_0^{\tan^{-1}\frac{3}{2}} \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta|\Big|_0^{\tan^{-1}\frac{3}{2}} = \ln\frac{\sqrt{13} + 3}{2}$$

8.

$$\text{arc length} \;=\; \int_0^1 \sqrt{1 + (y')^2}dx$$

$$= \int_0^1 \sqrt{1 + (\sqrt{e^x - 1})^2}dx = \int_0^1 e^{x/2}dx = 2\sqrt{e} - 2$$

9. Approximate the volume integral $\displaystyle\int_0^8 \pi[y_{top}^2 - y_{down}^2]dx$ by Simpson's Method on 4 subintervals:

$$\frac{2}{3}\pi\Big[(3^2 - 2^2) + 4(3^2 - 1^2) + 2(3^2 - 3^2) + 4(2^2 - 0^2) + (1^2 - 1^2)\Big] = \frac{106\pi}{3}$$

10.

$$\frac{dy}{dx}\bigg|_{\theta=\frac{\pi}{4}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\bigg|_{\theta=\frac{\pi}{4}} = \frac{\frac{d}{d\theta}(\theta\sin^2\theta)}{\frac{d}{d\theta}(\theta\sin\theta\cos\theta)}\bigg|_{\theta=\frac{\pi}{4}}$$

$$= \frac{\sin^2\theta + 2\theta\sin\theta\cos\theta}{\sin\theta\cos\theta + \theta\cos^2\theta - \theta\sin^2\theta}\bigg|_{\theta=\frac{\pi}{4}} = \frac{2+\pi}{\pi}$$

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Part II: Long Qestions

11. By the "Cylindrical Shell Mehtod", the volume is

$$V = \int_0^{\sqrt{2\pi}} 2\pi x \cdot x^2 \sin\frac{x^2}{4}\, dx$$

Let $u = \frac{x^2}{4}$ such that $du = \frac{x}{2}dx$. Using "Integration by Parts", we have

$$V = \int_0^{\frac{\pi}{2}} 16\pi u \sin u\, du = -16\pi u \cos u\Big|_0^{\frac{\pi}{2}} + 16\pi \int_0^{\frac{\pi}{2}} = 16\pi \sin u\Big|_0^{\frac{\pi}{2}} = 16\pi$$

Alternatively,

$$V = \int_0^{\sqrt{2\pi}} -4\pi x^2\, d\cos\frac{x^2}{4} = -4\pi x^2 \cos\frac{x^2}{4}\Big|_0^{\sqrt{2\pi}} + 8\pi \int_0^{\sqrt{2\pi}} x\cos\frac{x^2}{4}dx$$

$$= \left[16\pi \sin\frac{x^2}{4}\right]_0^{\sqrt{2\pi}} = 16\pi$$

12. (a) Note that $y(1+x^2) = 2x^2$, i.e., $x^2 = \dfrac{y}{2-y}$, where $0 \le y \le \frac{8}{5}$.

$$\text{volume } = \int_0^1 \pi x^2 dy = \pi \int_0^1 \frac{\pi y}{2-y}dy \qquad (\text{“Disc Method”})$$

or

$$\text{volume } = \int_0^1 2\pi x\Big[1 - \frac{2x^2}{1+x^2}\Big]dx \qquad (\text{“Cylindrical Shell Method”})$$

(b)

$$\text{work } = \int_0^1 9.8 \cdot 100 \cdot \pi \cdot \frac{y}{2-y}\Big(\frac{8}{5} - y\Big)\, dy$$

or

$$\text{work } = \int_0^1 9800\pi x^2 \Big[\frac{8}{5} - \frac{2x^2}{1+x^2}\Big] d\left(\frac{2x^2}{1+x^2}\right)$$

(c) Using

$$\frac{dy}{dx} = \frac{4x}{(1+x^2)^2} \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{\sqrt{y(2-y)^3}}$$

$$\text{area } = \int_0^2 2\pi x \sqrt{1 + \frac{16x^2}{(1+x^2)^4}}\, dx$$

or

$$\text{area } = \int_0^{\frac{8}{5}} 2\pi \sqrt{\frac{y}{2-y}} \sqrt{1 + \frac{1}{y(2-y)^3}}\, dy$$

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13. (a)

	$\theta=\pi+\frac{\pi}{3}$	$\theta=-\frac{\pi}{4}$
x coordinate	$-(1+\frac{\sqrt{3}}{2})\frac{1}{2}$	$(1+\frac{\sqrt{2}}{2})\frac{\sqrt{2}}{2}$
y coordinate	$-(1+\frac{\sqrt{3}}{2})\frac{\sqrt{3}}{2}$	$-(1+\frac{\sqrt{2}}{2})\frac{\sqrt{2}}{2}$

(b) An obvious intersection point is $(0,0)$.
Solve $1-\sin\theta=-3\sin\theta$, i.e. $\sin\theta=-\frac{1}{2}$ to find one angular coordinate: $\theta=-\frac{\pi}{6}$.
The other intersection points are: $\big(\pm(1+\sin\frac{\pi}{6})\cos\frac{\pi}{6}, -(1+\sin\frac{\pi}{6})\sin\frac{\pi}{6}\big)$.

(c) Since the curves are symmetric with respect to the y-axis,

$$\text{area } = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{2}(1-\sin\theta)^2\,d\theta - 2\int_{-\frac{\pi}{6}}^{0}\frac{1}{2}(-3\sin\theta)^2\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}(1-2\sin\theta+\sin^2\theta)\,d\theta - 9\int_{-\frac{\pi}{6}}^{0}\sin^2\theta\,d\theta$$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}(1-2\sin\theta+\frac{1-\cos 2\theta}{2})\,d\theta - 9\int_{-\frac{\pi}{6}}^{0}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \Big[\frac{3}{2}\theta+2\cos\theta-\frac{\sin 2\theta}{4}\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} - 9\Big[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\Big]_{-\frac{\pi}{6}}^{0} = \frac{\pi}{4}$$