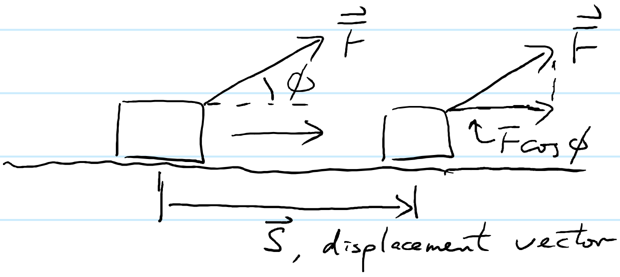
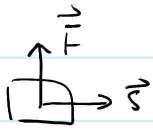
Lecture 4. Work and Kinetic Energy

Work (energy transfer to an object due to a force)

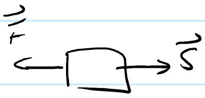
Work done by a force



$$W = F \cdot s \cdot \cos\phi$$

for $\phi = 90°$,  $\Rightarrow W = 0$ $\because$ No changes in (kinetic) energy $\therefore W = 0$

for $\phi = 0°$,  $\Rightarrow W = F \cdot s$, $a > 0$ $\because$ speed increases $\therefore W > 0$

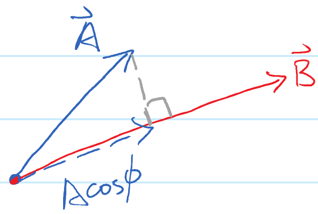
for $\phi = 180°$,  $\Rightarrow W = -F \cdot s$, $a < 0$ $\because$ speed decreases $\therefore W < 0$

S.I. unit of work : 1 Joule = 1 J = 1 N·m.

Dot Product

we write: $W = \vec{F} \cdot \vec{S} = |\vec{F}| \cdot |\vec{s}| \cdot \cos\phi$

$$\vec{A} \cdot \vec{B} = A\,B\cos\phi = \vec{B} \cdot \vec{A}$$



if $\phi = 0$, $\vec{A} \cdot \vec{B} = A \cdot B$
if $\phi = 90°$, $\vec{A} \cdot \vec{B} = 0$

$$\Rightarrow \begin{cases} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{cases}$$

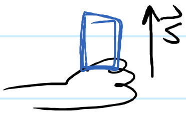
$$\vec{A} = A_x\hat{i} + A_y\hat{j}, \quad A_x = \vec{A} \cdot \hat{i}, \quad A_y = \vec{A} \cdot \hat{j}.$$

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j}) \cdot (B_x\hat{i} + B_y\hat{j}) = A_x B_x \overset{1}{\cancel{\hat{i} \cdot \hat{i}}} + A_x B_y \overset{0}{\cancel{\hat{i} \cdot \hat{j}}} + A_y B_x \overset{0}{\cancel{\hat{j} \cdot \hat{i}}} + A_y B_y \overset{1}{\cancel{\hat{j} \cdot \hat{j}}}$$

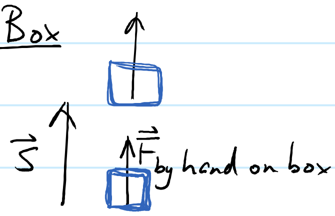
$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y}$$

$$W = \vec{F}\cdot\vec{s} = F_x s_x + F_y s_y + F_z s_z$$ <u>scalar product.</u>

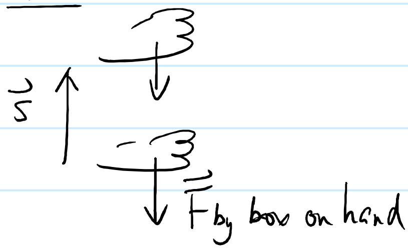
e.g. <u>Hand lifts a box</u>



<u>Box</u>



<u>Hand</u>



$$W_{\text{on box}} = \vec{F}_{\text{by hand on box}}\cdot\vec{s} > 0 \quad (\text{same direction})$$

$$W_{\text{on hand}} = \vec{F}_{\text{by box on hand}}\cdot\vec{s} < 0 \quad (\text{opp. direction})$$

But $\underline{W_{\text{on box}}} = \vec{F}_{\text{by hand on box}}\cdot\vec{s} = -\vec{F}_{\text{by box on hand}}\cdot\vec{s} = -\underline{W_{\text{on hand}}}$

action - reaction pair.
(Newton's 3rd Law)

Energy transfer to the box +ve = Energy transfer away from the hand −ve

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Multiple Forces: $W_{tot} = \sum_i \vec{F}_i\cdot\vec{s} = \sum_i W_i$

Recall: <u>Work-Energy Theorem</u> $K = \frac{1}{2}mv^2$

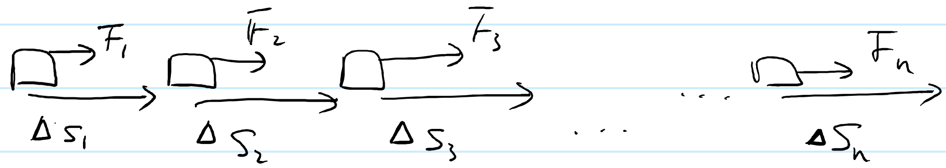
$$\Delta K = W_{tot} \Rightarrow W_{tot} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Constant force $\Rightarrow$ constant a $\Rightarrow$ $2as = v_f^2 - v_i^2$

$$\Rightarrow \underset{\sim}{ma}\cdot s = \frac{1}{2}m(v_f^2 - v_i^2)$$

$$W_{tot} = \sum F_i \cdot s = \frac{1}{2}m(v_f^2 - v_i^2) \quad ✓$$

Non-constant force



$$W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \dots + \Delta W_n$$

$\Rightarrow$ Consider $F = F(x)$ function of position.

$$\boxed{W = \sum_i \Delta W_i = \sum_i F_i \cdot \Delta S_i \longrightarrow \int_{x_i}^{x_f} F(x) \cdot dx}$$

in 3D
$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x \cdot dx + \int_{y_i}^{y_f} F_y \cdot dy + \int_{z_i}^{z_f} F_z \cdot dz$$

where $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Proof: work-energy theorem with non-constant force. (1D)

$$W_{tot} = \int_{x_i}^{x_f} \Sigma F_{i_x} \, dx$$

$$= \int_{x_i}^{x_f} m a_x \cdot dx \qquad \because a_x = \frac{dv_x}{dt} = \frac{dx}{dt} \cdot \frac{dv_x}{dx} = v_x \frac{dv_x}{dx}$$

$$= m \int_{x_i}^{x_f} v_x \frac{dv_x}{dx} \, dx$$

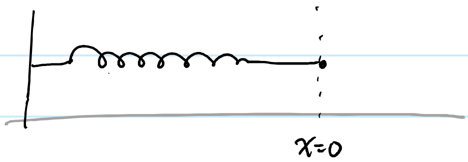
$$= m \int_{v_i}^{v_f} v_x \, dv_x$$

$$= m \left(\frac{1}{2} v_x^2 \right) \Big|_{v_i}^{v_f}$$

$$W_{tot} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \Delta K.$$

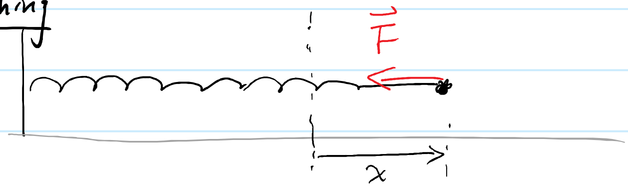
Hooke's Law.

- Restoring force e.g. tension of a spring.



natural position / equilibrium position.

stretching

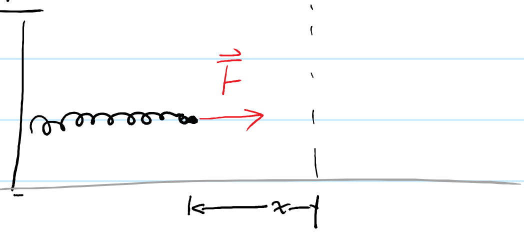


Force by the spring $(\vec{F} = F_x \hat{i})$

$F_x = -kx.$ $(x>0)$

to the left

Compress



but $x<0$

$F_x = k|x| = -kx$

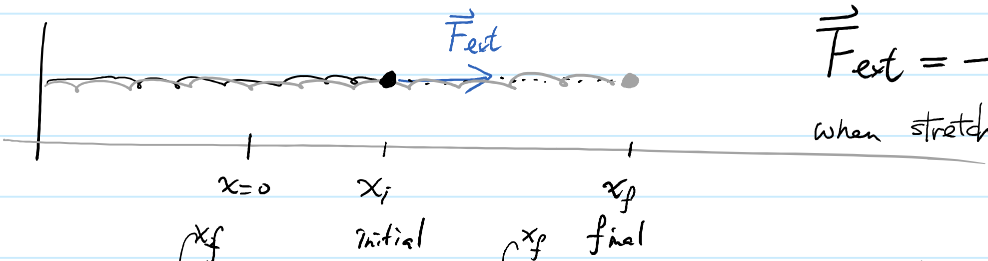
to the right.

$$\vec{F}_{spring} = -k\,x\,\hat{i}$$

x: displacement from the natural position.

k: force constant / spring constant (Force per unit Length.)

Work by an extenal force on a spring (at equilibrium. $\vec{a}=\vec{0}$)



$\vec{F}_{ext} = -\vec{F}_{spring}$

when stretching slowly.

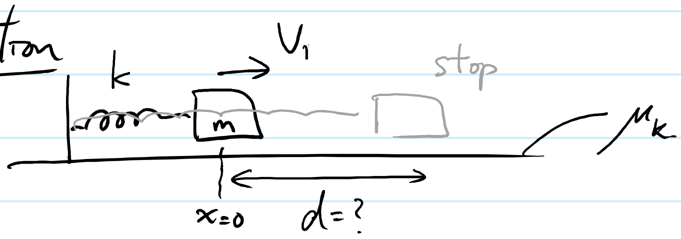
$$W_{\text{on spring}} = \int_{x_i}^{x_f} \vec{F}_{ext} \cdot dx\,\hat{i} = -\int_{x_i}^{x_f} \vec{F}_{spring} \cdot dx\,\hat{i} = k\int_{x_i}^{x_f} x\,dx$$

$$= \underline{\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2}$$

stretching: $x_f > x_i$, $W>0$ work done on spring.

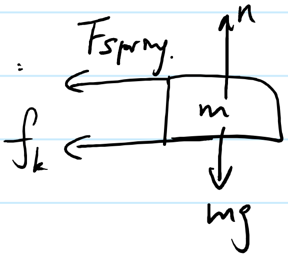
releasing: $x_f < x_i$, $W<0$, work done by spring.

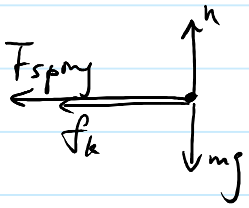
Application



$$\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m V_i^2 = 0 - \frac{1}{2} m V_1^2$$

To find W_{tot}:





$$f_k = \mu_k m g$$

$$W_{tot} = W_f + W_{by\ spring}$$

$$W_{by\ spring} = -W_{on\ spring}$$

or $$= \int F_{spring} \cdot dx$$

$$= \int_{x_i}^{x_f} (-kx)\, dx$$

$$W_f = \vec{f_k} \cdot \vec{d} = -\mu_k m g d$$

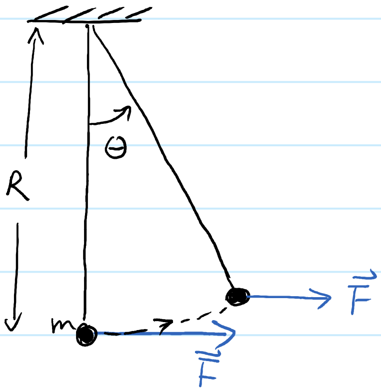
$$W_{by\ spring} = -\left(\frac{1}{2} k x_f^2 - \frac{1}{2} k x_i^2\right) = -\frac{1}{2} k d^2 - 0$$

$W_{tot} = \Delta K$ gives.

$$-\frac{1}{2} k d^2 - \mu_k m g d = -\frac{1}{2} m V_1^2$$

$$d = -\frac{\mu_k m g}{k} \pm \sqrt{\left(\frac{\mu_k m g}{k}\right)^2 + \frac{m V_1^2}{k}}$$

Example in 2D. Swing



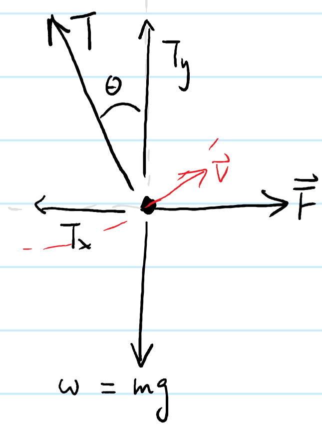
Situation: apply a horizontal force $\vec{F}$ to push the swing up from $\theta = 0$ to $\theta = \theta_0$.

Find the work done by each force acting on the object ●

Assumption: $\vec{F}$ is applied such that the object move up slowly at a constant speed (at equilibrium any time)

$$\Rightarrow \sum \vec{F} = \vec{0} \quad \Rightarrow \quad \Delta K = 0$$

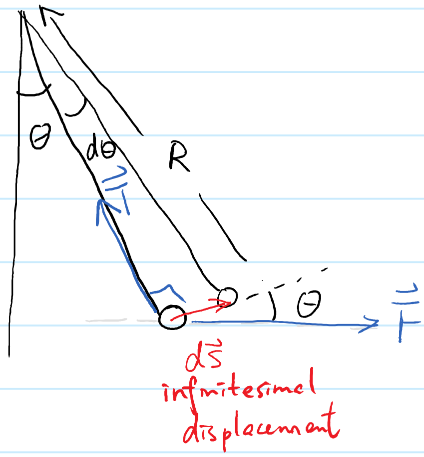
At any angle θ,



$\vec{a} = \vec{0}$

$$\left.\begin{aligned} x:&\quad F = T_x = T\sin\theta \\ y:&\quad W = T_y = T\cos\theta \end{aligned}\right\} \quad \frac{F}{W} = \tan\theta$$

$$\text{or} \quad F = W\tan\theta = mg\tan\theta$$

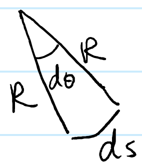


$$W_T = \int \vec{T}\cdot d\vec{s} = 0$$

$$\because \vec{T} \perp d\vec{s}$$

$$W_F = \int dW_F$$

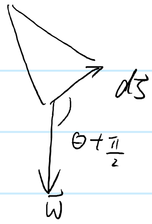
$$\begin{aligned} dW_F &= \vec{F}\cdot d\vec{s} \\ &= F\cdot ds\cdot \cos\theta \\ &= mg\tan\theta \cdot R\,d\theta\cos\theta \\ &= mgR\sin\theta\,d\theta \end{aligned}$$



$$|d\vec{s}| = R\,d\theta$$

$$W_F = \int_0^{\theta_0} mgR\sin\theta\,d\theta = mgR\,(1-\cos\theta_0)$$

$$W_w = \int_0^{\theta_0} mg\, R\, d\theta\, \underbrace{\cos\left(\theta + \frac{\pi}{2}\right)}_{-\sin\theta}$$



$$= -mgR \int_0^{\theta_0} \sin\theta\, d\theta$$

$$= -mgR \left(-\cos\theta\right)\Big|_0^{\theta_0} = mgR\left(\cos\theta_0 - 1\right)$$

check: $W_{tot} = W_T + W_F + W_W = \Delta K$

$\Delta K = 0 \quad \because \vec{a} \sim \vec{0}$ throughout the motion.

$$W_{tot} = W_T + W_F + W_W$$
$$= 0 + mgR\left(1 - \cos\theta_0\right) + mgR\left(\cos\theta_0 - 1\right) = 0$$

$$\Rightarrow \quad W_{tot} = \Delta K \quad \checkmark$$

<u>Power</u> $\quad \dfrac{\text{Work}}{\text{time}} = \dfrac{\Delta W}{\Delta t}$

$P = \dfrac{dW}{dt}$ $\qquad$ unit: 1 Watt = 1 J/s , 1 horse power = 746 W

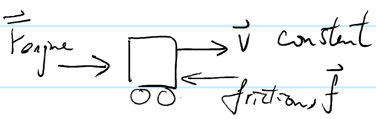
$$P = \frac{dW}{dt} = \frac{d}{dt}\int \vec{F} \cdot d\vec{s} = \frac{d}{dt}\int \vec{F} \cdot \frac{d\vec{s}}{dt}\, dt = \frac{d}{dt}\int \vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot \vec{v}$$

$$\boxed{P = \vec{F} \cdot \vec{v}}$$

<u>example</u>. car moving at constant speed v
given power of the car engine.

find frictional force acting on the car.

<u>simple model</u>.

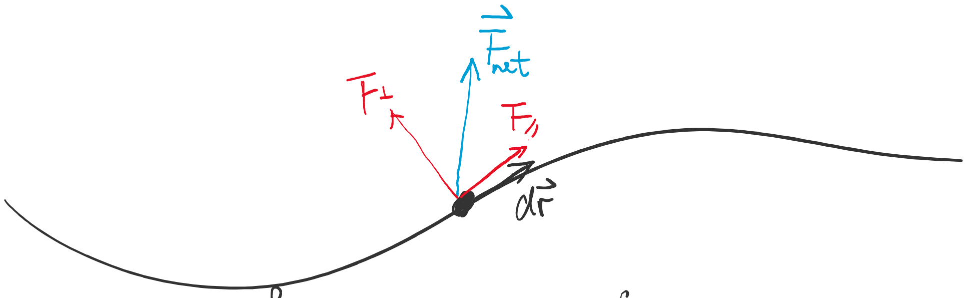


$\vec{F}_{engine} - \vec{f} = 0 \quad \because \vec{a} = \vec{0}$

$$P = \vec{F}_{engine} \cdot \vec{v} = F_{engine}\, v = f \cdot v$$

$$\Rightarrow \quad f = \frac{P}{v}$$

Alternative proof on work-energy theorem.



$$W_{tot} = \int_i^f \vec{F}_{net} \cdot d\vec{r} = \int_i^f F_\parallel \cdot dr$$

$\because F_\parallel = m a_\parallel$

$a_\parallel = \frac{dv}{dt}$

$v = v_\parallel = \frac{d|\vec{r}|}{dt} = \text{speed}$

$$= \int m a_\parallel \cdot dr$$

$$= \int m \frac{dv}{dt} \cdot dr \qquad \because dr = \frac{dr}{dt} \cdot dt = v\,dt$$

$$= \int m \frac{dv}{dt} v\,dt$$

$$= \int_{v_i}^{v_f} m\, v\,dv$$

$$= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

$$W_{tot} = K_f - K_i \quad \text{where } K = \frac{1}{2} m v^2$$