COMP 2711 Discrete Mathematical Tools for Computer Science
2020 Spring Semester – Final Exam (Part 2)

Note 1: For all non-proof questions, showing your steps is not necessary unless required otherwise. However, in case your answer is wrong, showing your steps may earn you some partial credits.

Note 2: You can express your answers using binomial coefficients, factorials, and D_n (derangement number). However, you should not have summation $\sum$ in your final answers. For example, $\binom{10}{3} D_9 + 4!$ and $1! + 2! + 3! + 4!$ are valid, but $\sum_{i=0}^{n} \binom{n}{i}$ or $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ is not. The latter has to be simplified to 2^n.

Question 1: (8 pts) Prove that $\binom{n}{1} + 2^2 \cdot \binom{n}{2} + ... + n^2 \cdot \binom{n}{n} = n(n+1) \cdot 2^{n-2}$, where n is a positive integer. [*Hint:* $i^2 = i(i-1) + i$.]

Solution:

$$\binom{n}{1} + 2^2 \cdot \binom{n}{2} + ... + n^2 \cdot \binom{n}{n}$$

$$= \sum_{i=1}^{n} i^2 \cdot \binom{n}{i} = \sum_{i=1}^{n} i(i-1) \cdot \binom{n}{i} + \sum_{i=1}^{n} i \cdot \binom{n}{i}$$

We have proved $\sum_{i=1}^{n} i \cdot \binom{n}{i} = n \cdot 2^{n-1}$ in tutorial 9,

$$\sum_{i=1}^{n} i \cdot \binom{n}{i} = \sum_{i=1}^{n} i \cdot \frac{n!}{i!(n-i)!}$$

$$= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!}$$

$$= \sum_{i=1}^{n} n \cdot \frac{(n-1)!}{(i-1)!((n-1)-(i-1))!}$$

$$= n \cdot \sum_{i=0}^{n-1} \binom{n-1}{i}$$

$$= n \cdot 2^{n-1}$$

And we use the same method to calculate $\sum_{i=1}^{n} i(i-1) \cdot \binom{n}{i}$,

$$\sum_{i=1}^{n} i(i-1) \cdot \binom{n}{i} = \sum_{i=1}^{n} i(i-1) \cdot \frac{n!}{i!(n-i)!}$$

$$= \sum_{i=1}^{n} \frac{n!}{(i-2)!(n-i)!}$$

$$= \sum_{i=1}^{n} n(n-1) \cdot \frac{(n-2)!}{(i-2)!((n-2)-(i-2))!}$$

$$= n(n-1) \cdot 2^{n-2}$$

Combine the two terms together, we have,

$$\begin{aligned}
&\binom{n}{1} + 2^2 \cdot \binom{n}{2} + ... + n^2 \cdot \binom{n}{n} \\
=&\sum_{i=1}^{n} i(i-1) \cdot \binom{n}{i} + \sum_{i=1}^{n} i \cdot \binom{n}{i} \\
=&n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2} \\
=&n(n+1) \cdot 2^{n-2}
\end{aligned}$$

Question 2: (5 pts) Three seminars were held by CSE department last week. 70% of students attended the seminar on Monday, 80% of students attended the seminar on Wednesday, and 90% of students attended the seminar on Friday. Prove that at least 40% of students attended all three seminars.

Solution: U - Set of all students;
A - Set of students having attended the seminar on Monday, $|A| = 70\%|U|$;
B - Set of students having attended the seminar on Wednesday, $|B| = 80\%|U|$;
C - Set of students having attended the seminar on Friday, $|C| = 90\%|U|$.

$$\begin{aligned}
|A \cap B \cap C| &= |U| - |\overline{A} \cup \overline{B} \cup \overline{C}| \\
&\geq |U| - (|\overline{A}| + |\overline{B}| + |\overline{C}|) \\
&= |U| - (30\%|U| + 20\%|U| + 10\%|U|) \\
&= 40\%|U|
\end{aligned}$$

The inequality uses the fact that $|X \cup Y \cup Z| \leq |X| + |Y| + |Z|$.

Question 3: (5 pts) Let A and B be two events. It is known that $p(A) = 0.7, p(B) = 0.6, p(A \cap B) = 0.5$, what is $p(B|A \cup \overline{B})$?

Solution:

$$p(B|A \cup \overline{B}) = \frac{p(B \cap (A \cup \overline{B}))}{p(A \cup \overline{B})}$$

Since $B \cap (A \cup \overline{B}) = (B \cap A) \cup (B \cap \overline{B}) = (B \cap A) \cup \varnothing = B \cap A$,

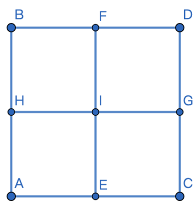
$$p(B \cap (A \cup \overline{B})) = p(B \cap A) = 0.5$$

$$p(A \cup \overline{B}) = 1 - p(\overline{A} \cap B) = 1 - (p(B) - p(A \cap B)) = 0.9$$

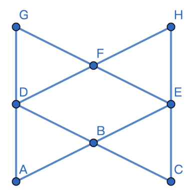
The final result is,

$$p(B|A \cup \overline{B}) = \frac{p(B \cap (A \cup \overline{B}))}{p(A \cup \overline{B})} = \frac{5}{9}$$

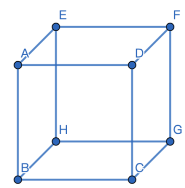
Question 4: (8 pts) For each of the following graphs, determine whether it has an Euler cycle. If not, insert the least number of edges so that it has an Euler cycle. (You can either draw a picture or write down which edges should be inserted.)



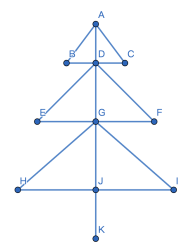
(a) Figure 1



(b) Figure 2



(c) Figure 3



(d) Figure 4

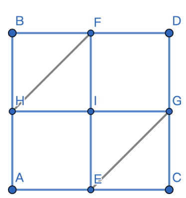
Solution:

(a) It is not an Euler circuit, since E, H, G, F have odd degree.

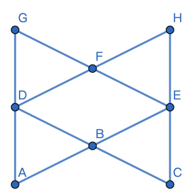
(b) It is an Euler circuit, since all vertices have even degree.

(c) It is not an Euler circuit, since all vertices have odd degree.

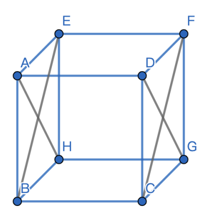
(d) It is not an Euler circuit, since A, K have odd degree.



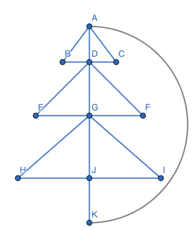
(a) Figure 1



(b) Figure 2



(c) Figure 3



(d) Figure 4

Grading scheme: 2, 2, 2, 2

Question 5: (11 pts) There is a random undirected graph G with n vertices, where an edge exists between every pair of vertices with probability p independently. Let X be the random variable denoting the number of edges in the graph.

(a) (2 pts) Suppose we remove a vertex u and all edges incident to u, and get a subgraph G'. Given G' is connected, what is the probability that G is also connected?

(b) (4 pts) What is the expectation and variance of X?

(c) (2 pts) If G has exactly k connected components, what is the minimum possible value of X?

(d) (3 pts) What is the probability that G with $n = 4$ vertices is connected and X takes its minimal possible value?

Solution: (a) G is also connected if and only if u connects with at least one vertex in G'. Since we know u links to any other vertex with probability p, the probability that it has no edge with other vertices is, $(1-p)^{n-1}$. So the probability that G is connected is $1-(1-p)^{n-1}$.

(b) X_e is the indicator random variable that indicates the event, pair of vertices $e = (u, v)$ has an edge. X_e is a Bernoulli trial, thus $E(X_e) = p$, $V(X_e) = p(1-p)$.

There are $\binom{n}{2}$ pairs of vertices in total. So that $E(X) = \binom{n}{2}p$ and $V(X) = \binom{n}{2}p(1-p)$.

(c) Suppose that the k connected components are $C_1, C_2, ..., C_k$. Since C_i is a connected component with v_i vertices, the minimum number of edges to connect them is $v_i - 1$, which is a tree.

The minimum of X is equal to all components reach minimal the same time.

$$\min(X) = \min_{C_i} \sum_{i=1}^{k} \min(C_i) = \min_{C_i} \sum_{i=1}^{k} (v_i - 1) = n - k$$

(d) G is connected and X is minimal if and only if G is a tree. The probability of having 3 edges in G is $\binom{6}{3}p^3(1-p)^3$. However, if the 3 edges form a triangle, then the last vertex will be disconnected, which happens with probability $\binom{4}{3}p^3(1-p)^3$. So, the final probability is

$$\binom{6}{3}p^3(1-p)^3 - \binom{4}{3}p^3(1-p)^3 = 16p^3(1-p)^3$$

Question 6: (10 pts) Consider the permutations of the 10 digits (0–9). We denote a permutation as A, whose elements are $A[0], A[1], \ldots, A[9]$. Calculate the number of permutations that satisfy each of the following conditions:

(a) (2 pts) There is exactly one i such that $A[i] = i$.

(b) (1 pts) There are exactly nine i's such that $A[i] = i$.

(c) (1 pts) $A[0] = 1, A[1] = 0$.

(d) (1 pts) $A[4] = 4, A[5] = 5$.

(e) (5 pts) Satisfying any of the above conditions.

Solution: (a) D_n is the number of derangements of a set with n elements,

$$D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - ... + (-1)^n \frac{1}{n!}]$$

Exactly 1 digit appears in its index position means that, we randomly select 1 digit and put it in its index position and dearrange the remaining 9 digits. So the answer is,

$$\binom{10}{1} D_9 = 10 \cdot D_9$$

(b) If 9 digits appear in their corresponding index positions, the remaining one digit must also appear in its index position, so the answer is 0.

(c) Fix the first two digits, and the remaining 8 digits are randomly permuted, so the answer is 8!.

(d) Fix the middle two digits, and the remaining 8 digits are randomly permuted, so the answer is 8!.

(e) We let A, B, C, D denote the sets of permutations that satisfy the above conditions (a), (b), (c), (d), respectively. Obviously, $B = \varnothing$, and we want to calculate $|A \cup B \cup C \cup D| = |A \cup C \cup D|$.

Considering set $A \cap C$, fixed the first two digits, and pick 1 digit from $\{2, 3, ..., 9\}$ to put in the corresponding index position and then dearrange the remaining 7 digits. Thus,

$$|A \cap C| = 8 \cdot D_7$$

Considering set $C \cap D$, fix 4 digits and randomly permute the remaining, thus,

$$|C \cap D| = 6!$$

Considering set $A \cap D$, middle two index is 4 and 5, so $A \cap D = \varnothing$. And $A \cap C \cap D = C \cap \varnothing = \varnothing$. So,

$$|A \cap D| = |A \cap C \cap D| = 0$$

By using inclusion-exclusion theorem,

$$\begin{aligned} |A \cup C \cup D| &= |A| + |C| + |D| - |A \cap C| - |A \cap D| - |C \cap D| + |A \cap C \cap D| \\ &= 10 \cdot D_9 + 2 \cdot 8! - 8 \cdot D_7 - 6! \end{aligned}$$

Bonus: (10 pts) During a game show, two players are competing for a prize according to the following rule. They take turns to answer a randomly chosen question, and if the answer is wrong, the player will be eliminated. The player who remains last will be the winner. Suppose each question can be answered correctly with probability 1/2 by any player.

(a) (2 pts) Is this a fair game? If not, who has an advantage?

(b) (4 pts) Compute the probability that the two players win, respectively.

(c) (4 pts) Redo (b) with 3 players.

Solution: (a) Not fair, the second player has an advantage.

(b) Suppose player 1 has a winning probability of p_1, and player 2 has winning probability p_2. We have

$$p_1 + p_2 = 1. \tag{1}$$

Meanwhile

$$\begin{aligned} p_1 =& p(\text{player 1 answers first question correctly}) \\ &\cdot p(\text{player 1 wins} \mid \text{player 1 answers first question correctly}) \\ =& \frac{1}{2} \cdot p(\text{player 1 wins} \mid \text{player 1 answers first question correctly}). \end{aligned}$$

Note that $p(\text{player 1 wins} \mid \text{player 1 answers first question correctly}) = p_2$ as under the condition "player 1 answers first question correctly", player 1 takes the role of player 2. So

$$p_1 = \frac{1}{2} \cdot p_2. \tag{2}$$

Solving (1) and (2), we have $p_1 = \frac{1}{3}, p_2 = \frac{2}{3}$.

(c) Let the probabilities of the 3 players winning be q_1, q_2, q_3. With similar reasoning as above, we have

$$\begin{aligned} q_1 + q_2 + q_3 &= 1, \\ q_1 &= \frac{1}{2} q_3. \end{aligned}$$

For the third equation, consider q_2:

$$\begin{aligned} q_2 =& p(\text{player 1 answers first question correctly}) \\ &\cdot p(\text{player 2 wins} \mid \text{player 1 answers first question correctly}) \\ &+ p(\text{player 1 answers first question wrongly}) \\ &\cdot p(\text{player 2 wins} \mid \text{player 1 answers first question wrongly}). \\ =& \frac{1}{2} q_1 + \frac{1}{2} \cdot \frac{1}{3}. \end{aligned}$$

Solving the 3 equations, we obtain $q_1 = \frac{5}{21}, q_2 = \frac{6}{21}, q_3 = \frac{10}{21}$.