Review some notable derivatives:

$f'(x) \overset{\text{defined by}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (if exists).

$= $ rate of change of f at x.

$\frac{d}{dx} x^n = n \cdot x^{n-1}$. (n is a real number).

$\frac{d}{dx} \sin x = \cos x$. $\quad \frac{d}{dx} \cos x = -\sin x$. $\quad \frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$.

$\frac{d}{dx} a^x = \ln a \cdot a^x$. $\quad \frac{d}{dx} e^x = e^x$.

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$. $\quad \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$. $\quad \frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$.

$\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}$. $\quad \frac{d}{dx} \ln x = \frac{1}{x}$.

We can combine these derivatives with the chain rule.

For example: Suppose $y = f(x)$. Then $\frac{d}{dx} \ln f(x) = \frac{d}{dx} \ln y = \frac{d \ln y}{dy} \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}$

Example: If $y = x^2+1$, then $\frac{d}{dx} \ln(x^2+1) = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x^2+1} \frac{d(x^2+1)}{dx} = \frac{2x}{x^2+1}$

If $y = \ln x$. then $\frac{d}{dx} \ln(\ln x) = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{d \ln x}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$.

Logarithmic Differentiation

A trick to simplify the calculation of $\frac{dy}{dx}$.

If $y = \frac{h_1(x)\cdots h_m(x)}{g_1(x)\cdots g_n(x)}$ or $h(x)^{g(x)}$, to calculate $\frac{dy}{dx}$ we take logarithms of both sides.

(e.g. $y = \frac{(3x+1)^4 \sqrt{x^2+1}\cdot \cos x \cdot e^x}{(4x-3)^5\cdot(x+1)^3\cdot(\sin x)^2}$, $y=(1+x)(1+x^2)(1+x^3)(1+x^4)$, $y = x^{\sqrt{x}}$).

Example: Calculate $\frac{dy}{dx}$ if $y = f(x) = x^{\sqrt{x}}$.

Step 1: Take natural logarithms of both sides of $y=f(x)$.

$$\ln y = \ln\left(x^{\sqrt{x}}\right) \xrightarrow{\text{use laws of logarithms to simplify } \ln f(x).} = \sqrt{x}\ln x.$$

Step 2: Differentiate both sides with respect to x.

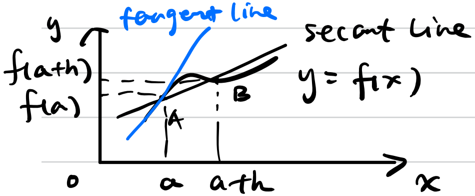
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sqrt{x}\ln x).$$

Notice: By the chain rule, $\frac{d}{dx}\ln y = \frac{d\ln y}{dy}\cdot\frac{dy}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \left(\frac{d}{dx}\sqrt{x}\right)\cdot \ln x + \sqrt{x}\cdot\left(\frac{d}{dx}\ln x\right) = \frac{1}{2\sqrt{x}}\cdot \ln x + \sqrt{x}\cdot\frac{1}{x} = \frac{2+\ln x}{2\sqrt{x}}.$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\frac{2+\ln x}{2\sqrt{x}} = x^{\sqrt{x}}\cdot\frac{2+\ln x}{2\sqrt{x}}.$$

The expression for the tangent line



Recall: As $h \to 0$, secant line AB $\longrightarrow$ tangent line.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ = the slope of the tangent line at $(a, f(a))$.

= rate of change of f at $x = a$

The point-slope form of the tangent line at $(a, f(a))$: $y = \underline{\underline{f'(a)}}(x-a) + f(a)$.

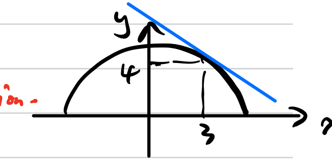
Example: Suppose that $y = f(x) = 3^x + 2x^2 + 1$.
Find the tangent line of $f(x)$ at $(1, 6)$.

Our aim is to calculate $f'(1)$.

$$f'(x) = (3^x)' + (2x^2)' + (1)' = \ln 3 \cdot 3^x + 4x \Rightarrow f'(1) = 3\ln 3 + 4.$$

Tangent line at $(1, 6)$: $y = f'(1)(x-1) + 6 = (3\ln 3 + 4)(x-1) + 6$.

Example: Suppose $F(x,y) = x^2+y^2-25=0$ $(y>0)$. $\to$ y is defined implicitly
Find the tangent line of y at $(3,4)$.



To calculate $\left.\frac{dy}{dx}\right|_{x=3}$, we use the implicit differentiation.

Step 1: $\frac{d}{dx}F(x,y) = \frac{d}{dx}(0) = 0$.

Step 2: $\frac{d}{dx}F(x,y) = \frac{d}{dx}(x^2+y^2-25) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 - \frac{d}{dx}25$.

$$= 2x + 2y\cdot\frac{dy}{dx}$$

Step 3: Solve $\frac{d}{dx}F(x,y)=0$ for $\frac{dy}{dx}$ $\quad 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$.

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = -\frac{3}{4}.$$

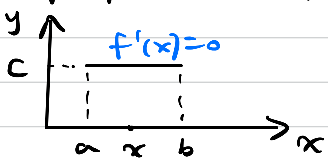
Tangent line at $(3,4)$ $\quad y = \left.\frac{dy}{dx}\right|_{x=3}(x-3)+4 = -\frac{3}{4}(x-3)+4$.

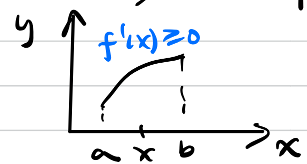
Two basic facts about $f'(x)$

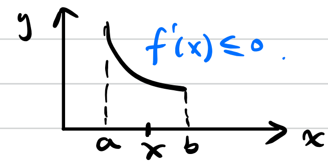
Recall : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ = rate of change of y at x.

$\Rightarrow$ the value of f does not change

(1). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).



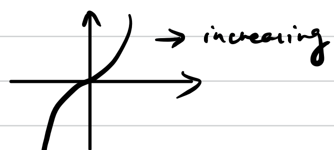




(2). ① $f'(x) \geq 0$ for all $x \in (a, b)$ $\Leftrightarrow$ f is increasing on (a, b).

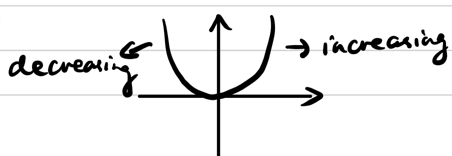
② $f'(x) \leq 0$ for all $x \in (a, b)$ $\Leftrightarrow$ f is decreasing on (a, b).

Example : $y = x^3$



$y' = 3x^2 \geq 0$

Example : $y = x^2$



$y' = 2x$. $\begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$.

Higher-order derivaties

a function of x.

The derivative of $y = f(x)$: $f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ → rate of change of f

called the second derivative of y

Similarly, we can define the derivative of $f'(x)$: $f''(x) = \frac{d}{dx} f'(x)$. → rate of change of f'

Alternative notations: $f''(x) = f^{(2)}(x) = y'' = y^{(2)} = \frac{d^2y}{dx^2}$.

In general, we can define the n-th deriuatie of $y = f(x)$:

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d}{dx} f^{(n-1)}(x)$$

Example 1: $y = x^4 + x^3 + x^2 + x + 1$. Recall $(x^n)' = n \cdot x^{n-1}$

$y' = 4x^3 + 3x^2 + 2x + 1$. → 1st derivative of y

$y'' = \frac{d}{dx} y' = 12x^2 + 6x + 2$ → 2nd derivative of y.

$y^{(3)} = \frac{d}{dx} y^{(2)} = 24x + 6$ → 3rd derivatie of y.

$y^{(4)} = \frac{d}{dx} y^{(3)} = 24$. → 4th derivatie of y.

$y^{(5)} = \frac{d}{dx} y^{(4)} = 0$ → 5th derivative of y.

Example 2. $y = \sin x$. Recall $(\sin x)' = \cos x$. $(\cos x)' = -\sin x$.

$$y' = \frac{d}{dx} \sin x = \cos x.$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \cos x = -\sin x.$$

$$y^{(3)} = \frac{d}{dx} y'' = \frac{d}{dx} (-\sin x) = -\frac{d}{dx} \sin x = -\cos x.$$

$$y^{(4)} = \frac{d}{dx} y^{(3)} = \frac{d}{dx} (-\cos x) = -\frac{d}{dx} \cos x = \sin x.$$

Example 3: $y = e^x$. Recall $(e^x)' = e^x$.

$$y' = \frac{d}{dx} y = \frac{d}{dx} e^x = e^x.$$

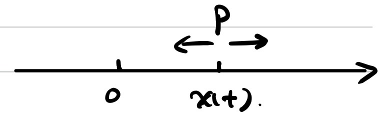
$$y'' = \frac{d}{dx} y' = \frac{d}{dx} e^x = e^x$$

$$\vdots$$

$$\Rightarrow \frac{d^n y}{dx^n} = e^x \quad \text{for each } n \geq 1.$$

Rates of change in the velocity problem

Let P be a moving particle along a line

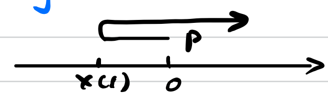


Let $x(t)$ be the position function of P with respect to time t.

The velocity of P at time t : $V(t) = x'(t) = \lim\limits_{h \to 0} \boxed{\frac{x(t+h) - x(t)}{h}}$ → average velocity of P from t to $t+h$.

means (in a positive direction)

$V(t) > 0 \Rightarrow P$ is moving forward $V(t) < 0 \Rightarrow P$ is moving backward. $V(t) = 0 \Rightarrow P$ is at rest.



Example: $x(t) = t^2 - 2t$. $V(t) = 2t - 2$.

At time $t = 0$, P is moving backward at a rate of $V(0) = -2$.

When $0 \le t < 1$, P is moving backward.

At time $t = 1$, P is at rest.

When $t > 1$, P is moving forward.

$a(t) = V'(t) = x''(t) = 2$.

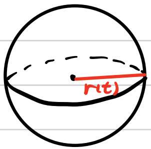
The acceleration of P at time t : $a(t) = V'(t) = \lim\limits_{h \to 0} \frac{V(t+h) - V(t)}{h}$ → rate of change of the velocity at t.

$= x''(t)$.

$V(t)$ increases at a rate of $L\ m^3/s$. $\Rightarrow$ $\frac{d}{dt}V(t)$.
$\Uparrow$ means

Related rates problems

1. a spherical air balloon



Air is pumped into this balloon at a rate of L m^3/s.

Question: How fast is the surface area increasing when the radius is equal to r meters. Answer: $\frac{2L}{r}\ m^2/s$.

$r(t)$: radius $\quad$ $S(t)$: surface area. $\quad$ $V(t)$: volume $\quad$ t: time.

Given that $\frac{d}{dt}V(t)=L$, our aim is to find $\frac{d}{dt}S(t)\Big|_{r(t)=r}$.

To find $\frac{d}{dt}S(t)$ from $\frac{d}{dt}V(t)$, we take the following 2 steps:

Step 1: Find a relation between $S(t)$ and $V(t)$.

Notice: $V(t)=\frac{4\pi}{3}\cdot r^3(t)$, $\quad S(t)=4\pi r^2(t)$.

$$r(t)=\left(\frac{S(t)}{4\pi}\right)^{\frac{1}{2}} \Rightarrow V(t)=\frac{4\pi}{3}\left(\frac{S(t)}{4\pi}\right)^{\frac{3}{2}}=\frac{1}{6\sqrt{\pi}}\left(S(t)\right)^{\frac{3}{2}}.$$

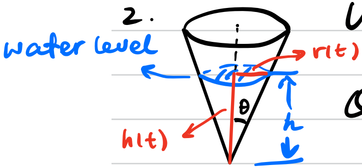
Step 2: Differentiate both sides with respect to t.

$$\frac{d}{dt}V(t)=\frac{d}{dt}\left(\frac{1}{6\sqrt{\pi}}S(t)^{\frac{3}{2}}\right)\overset{\text{chain rule}}{=\!=}\frac{1}{6\sqrt{\pi}}\cdot\frac{3}{2}S(t)^{\frac{1}{2}}\cdot\frac{dS(t)}{dt}.$$

$$\Rightarrow \frac{d}{dt}S(t)=\frac{4\sqrt{\pi}}{\sqrt{S(t)}}\cdot\frac{d}{dt}V(t). \Rightarrow \frac{d}{dt}S(t)\Big|_{r(t)=r}=\frac{4\sqrt{\pi}}{\sqrt{4\pi r^2}}\cdot L=\frac{2L}{r}\ m^2/s.$$

an inverted conical tank

2.



$V(t)$ increases at a rate of L m^3/s.

means $\Uparrow$ $\qquad \frac{d}{dt}V(t)$

Water is flowing into this tank at a rate of L m^3/s.

Question: How fast is the water level rising when the height of water in the tank is h meters.

$r(t)$: radius of the water level, $h(t)$: height of water $V(t)$: volume of water in the tank.

Given that $\frac{d}{dt}V(t) = L$, our aim is to find $\left.\frac{d}{dt}h(t)\right|_{h(t)=h}$.

Step 1: Find a relation between $h(t)$ and $V(t)$.

Step 2: Differentiate both sides with respect to t.