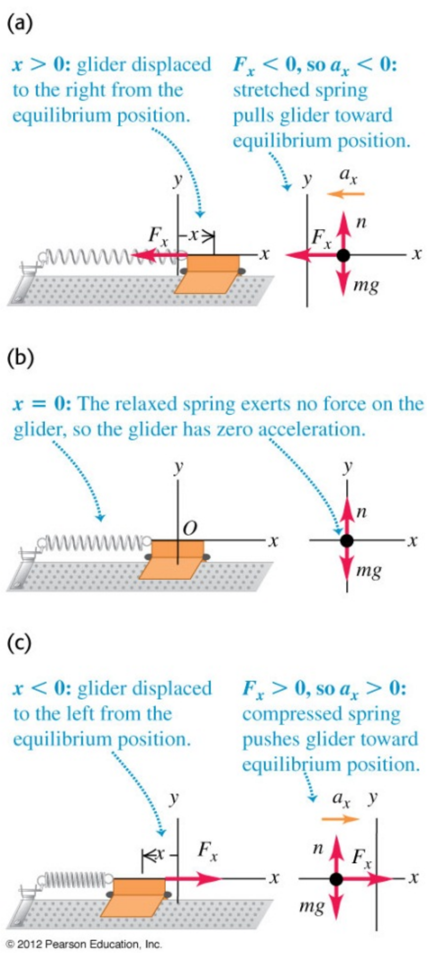
PERIODIC MOTION I

Intended Learning Outcomes – after this lecture you will learn:
1. definition of simple harmonic motion
2. relation between uniform circular motion and simple harmonic motion
3. description of simple harmonic motion in terms of phasor diagram
4. kinetic, potential, and total energy in simple harmonic motion

Textbook Reference: Ch 14.1 – 14.3

Simple Harmonic Motion (SHM)
Simplest example: a spring and mass system



Hooke's law: $F_x = -kx$

restoring force

displacement (+/-) from equilibrium point

Newton's law $\quad a_x = \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$

a *differential equation* of the form $\ddot{x} = -\alpha x$, $\alpha > 0$, called **simple harmonic motion (SHM)**

A system executing simple harmonic motion is called a **harmonic oscillator**

How to solve the differential equation? Consider a particle Q executing uniform circular motion with angular speed ω and radius A. P is its projection along x axis.

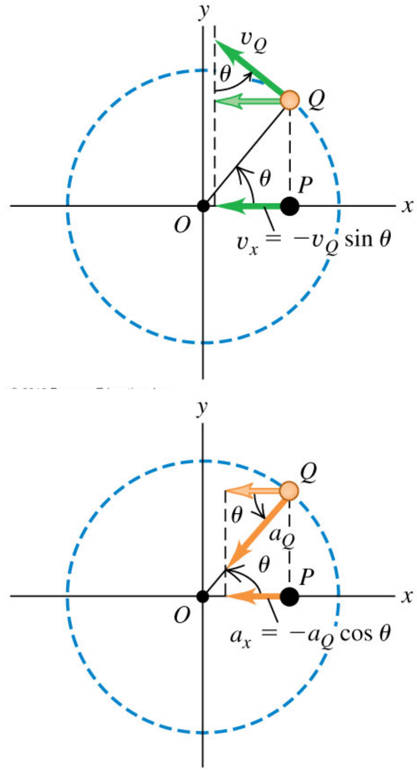
position of P: $x = A\cos\theta$

velocity of P: $v_x = -v_Q \sin\theta$

acceleration of P: $a_x = -a_Q\cos\theta$

$$= -(\omega^2 A)\cos\theta$$

$$= -\omega^2 x \quad c.f. \quad a = -(k/m)x$$



Conclusion: a harmonic oscillator is the same as a particle in uniform circular motion with $\omega = \sqrt{k/m}$ projected along the x direction

frequency f = number of cycles per unit time

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

period T = time for one complete cycle

$$\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

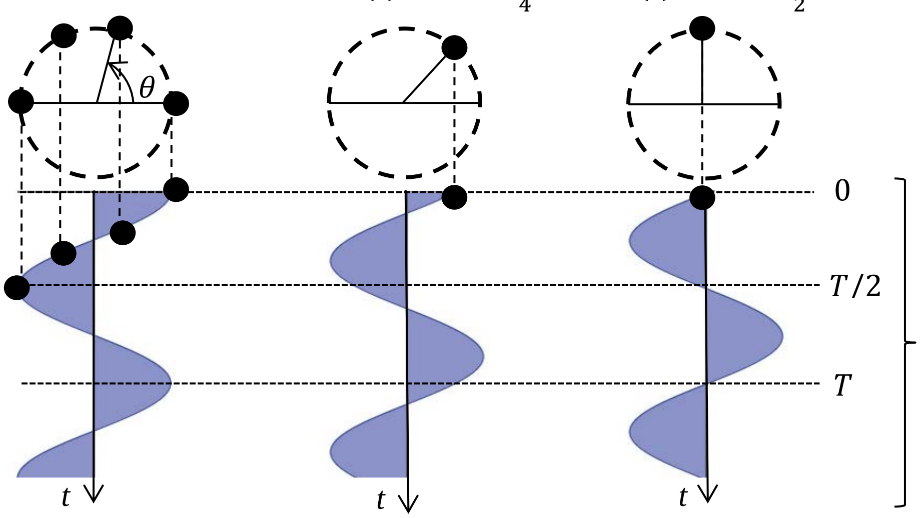
angular frequency ω = angle (in radian) per unit time

$$\boxed{\omega = 2\pi f}$$

General solution: $x = A\cos\theta(t) = A\cos(\omega t + \phi)$, where the **phase angle** $\phi = \theta(0)$

A is the **amplitude** (maximum displacement) of the oscillation

$\phi = 0$, i.e., $\theta(t) = \omega t$ | $\phi = \pi/4$, i.e., $\theta(t) = \omega t + \frac{\pi}{4}$ | $\phi = \pi/2$, i.e., $\theta(t) = \omega t + \frac{\pi}{2}$



phasor diagram

displacement-time graph

⚠ effect of phase angle is to push displacement-time graph along $-t$ by ϕ/ω

velocity $\boxed{v_x = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) = \omega A\cos\left(\omega t+\phi+\frac{\pi}{2}\right)}$, $\quad v_{max} = \omega A$

acceleration $\boxed{a_x = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t+\phi) = \omega^2 A\cos(\omega t+\phi+\pi)}$, $\quad a_{max} = \omega^2 A$

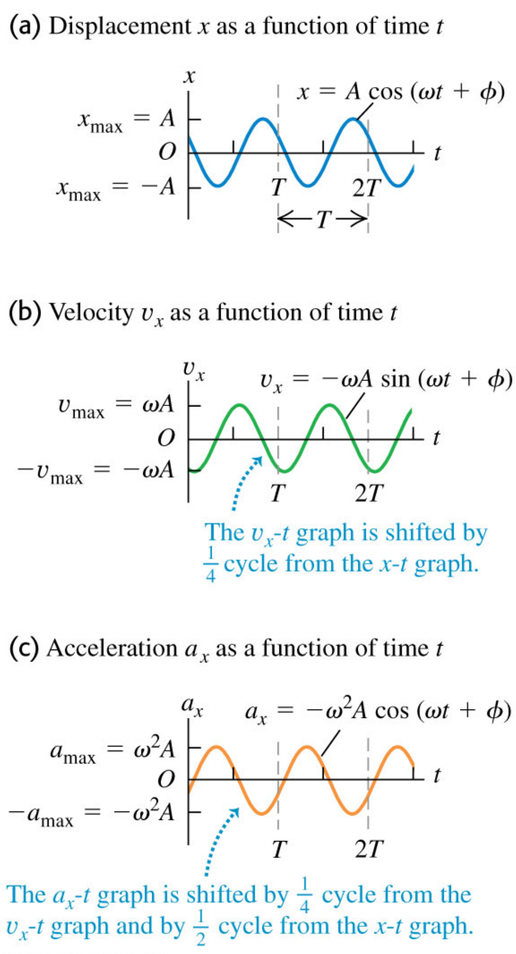
⚠ see Appendix I about changing from sine to cosine function.

How to find A and ϕ? If given initial condition $x(0) = x_0$, $v(0) = v_{0x}$

$$\frac{v_{0x}}{x_0} = \frac{-\omega A\sin\phi}{A\cos\phi} = -\omega\tan\phi \quad\Longrightarrow\quad \boxed{\phi = \begin{cases} \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right), & \text{if } x_0 > 0 \\ \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right)+\pi, & \text{if } x_0 < 0 \end{cases}}$$

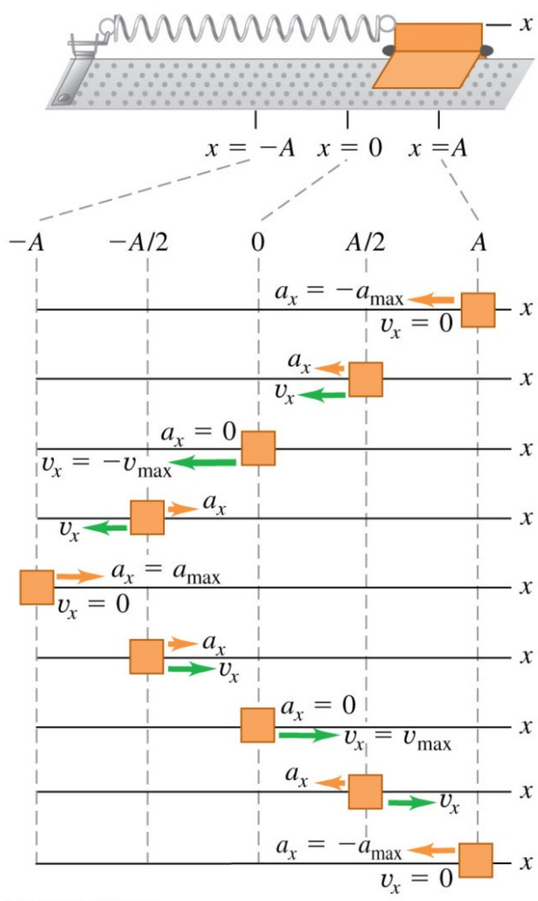
⚠ see Appendix II

$$x_0^2 + \frac{v_{0x}^2}{\omega^2} = A^2(\cos^2\phi + \sin^2\phi) = A^2 \quad\Longrightarrow\quad \boxed{A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}}}$$



⚠ x pushed towards $-t$ by $\frac{\pi}{2}$, i.e., $\frac{T}{4}$

⚠ x pushed towards $-t$ by π, i.e., $\frac{T}{2}$



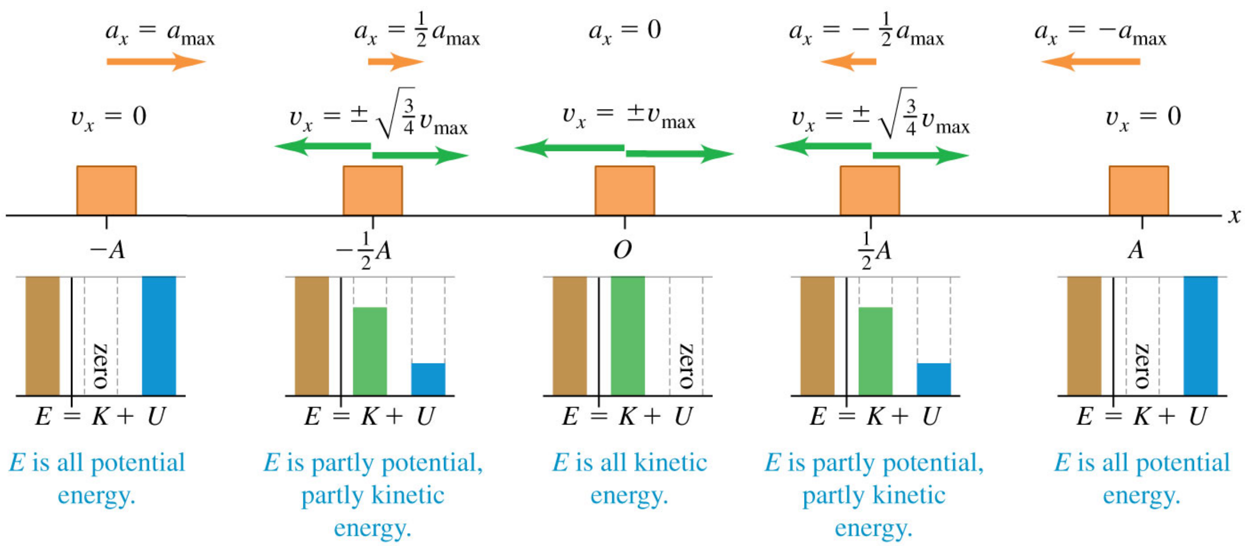
Question
Suppose the glider in the above diagram is moved to x = 0.10 m and is released from rest at t = 0, then A = ____ m and ϕ = ____ .
Suppose instead the glider in the above diagram at t = 0 is at x = 0.10 m and is moving to the right, then A is (> / < / =) 0.10 m and ϕ is (> / < / =) 0.
Answer: see inverted text on P. 466

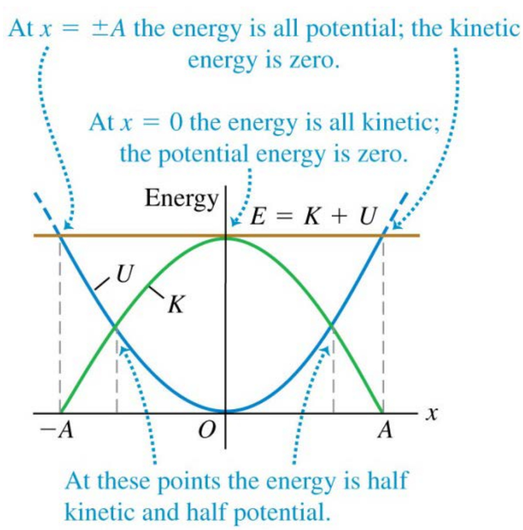
Energy in Simple Harmonic Motion

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) + \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) = \tfrac{1}{2}kA^2$$

Conservation of energy! To find velocity:

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \quad \Rightarrow \quad v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$



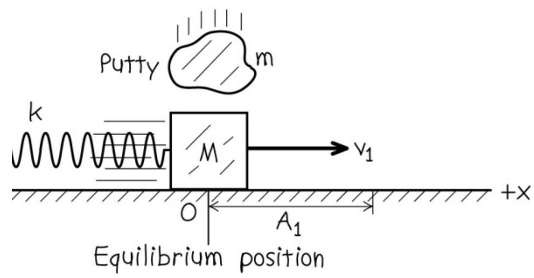


both U and K are quadratic (i.e., parabolic), and they add up to a constant $E = \frac{1}{2}kA^2$

Question
To double the total energy of a spring and mass system oscillating in SHM, one should increase the amplitude by a factor of ___. As a result of this amplitude change, the frequency of the oscillator will (be larger / be smaller / have no change).
Answer: see inverted text on P. 469 of textbook

Example 14.5 P. 469 Energy and momentum in SHM

Given: an oscillator with amplitude A_1
When it is at *x* = 0, a putty of mass *m* hits, and then stays on the block after collision



During the collision:
y component of momentum (is / is not) conserved
x component of momentum (is / is not) conserved

New velocity at *x* = 0:

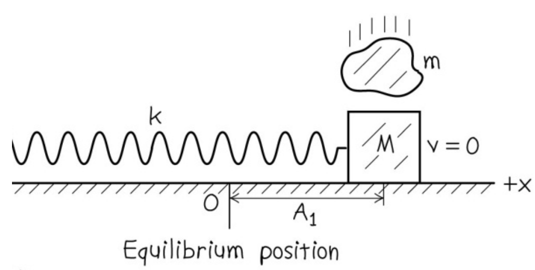
$$Mv_1 + 0 = Mv_2 + mv_2 \quad \Rightarrow \quad v_2 = \frac{M}{M+m} v_1$$

New amplitude:

$$\underbrace{\tfrac{1}{2}kA_2^2}_{E \text{ in terms of amplitude after collision}} = \underbrace{\tfrac{1}{2}(M+m)v_2^2}_{K \text{ right after collision}} = \left(\frac{M}{M+m}\right)\tfrac{1}{2}Mv_1^2 = \left(\frac{M}{M+m}\right)\tfrac{1}{2}kA_1^2 \quad \Rightarrow \quad A_2 = A_1\sqrt{\frac{M}{M+m}}$$

Total energy of the oscillator (increase / decrease). Where does the energy go?

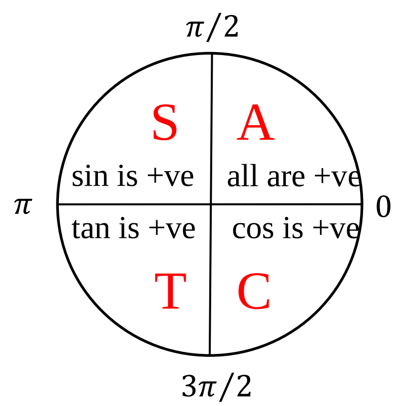
Suppose the putty hits when the block is at $x = A_1$
No change in horizontal velocity (why?)
No change in *K* (why?)
Does the total energy of the oscillator change? Why?
Is the energy of the system (oscillator + putty) conserved? Why?



Appendix I Summary of trigonometrical relations

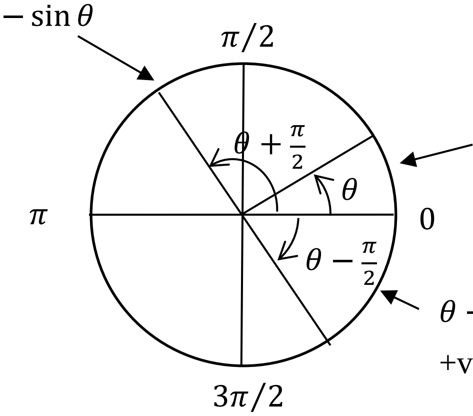
In this Chapter we have used the relations $\boxed{\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp \sin\theta}$

If two angles ϕ_1 and ϕ_2 differ by $\frac{\pi}{2}$, then sin and cos interchanged: $|\sin\phi_1| = |\cos\phi_2|$, the sign is determined by the following rule for trigonometric function in different quadrants:



$\theta + \frac{\pi}{2}$ in second quadrant, cos is -ve, therefore

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$$



θ is in first quadrant, sin is +ve

$\theta - \frac{\pi}{2}$ in fourth quadrant, cos is +ve, therefore

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$

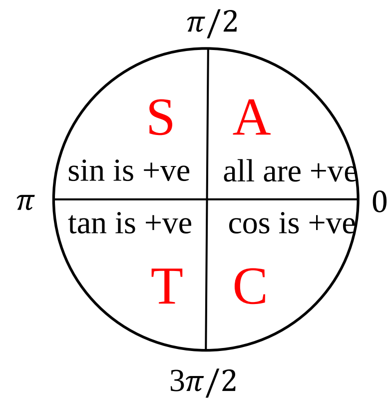
Likewise, $\boxed{\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta}$

For students with more advanced mathematics background only. Others may ignore this part.

Appendix II

The formula $\phi = \tan^{-1}(-v_{0x}/\omega x_0)$ does not always give the correct answer. One needs to determine ϕ in the correct quadrant through the conditions

$$\sin\phi = -v_{0x}/\omega A$$
$$\cos\phi = x_0/A$$



But you can easily convince yourself that the general formula

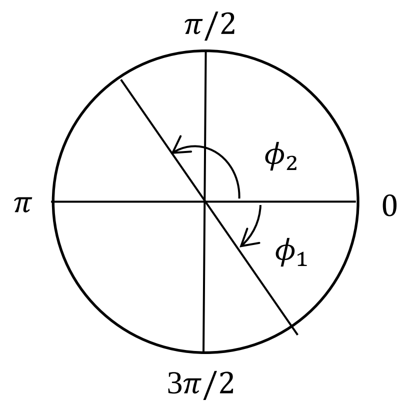
$$\text{is } \phi = \begin{cases} \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right), & \text{if } x_0 > 0 \\ \tan^{-1}\left(-\frac{v_{0x}}{\omega x_0}\right) + \pi, & \text{if } x_0 < 0 \end{cases},$$

irrespective of whether v_{0x} is positive or negative, as illustrated in the following example:

Example

Given $v_{0x} = 0.40$ m/s, $x_0 = 0.015$ m, $\omega = 20$ rad/s, then

$$\phi_1 = \tan^{-1}\left(-\frac{0.40\ \text{m/s}}{(20\ \text{rad/s})(0.015\ \text{m})}\right) = -0.93\ \text{rad}$$



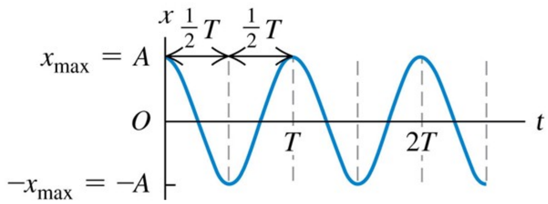
But if $v_{0x} = -0.40$ m/s, $x_0 = -0.015$ m, then $\sin\phi_2 > 0$ and $\cos\phi_2 < 0$, i.e., ϕ_2 in the second quadrant (between $\pi/2$ and π) , and the correct phase angle is

$$\phi_2 = \pi - 0.93\ \text{rad} = 2.21\ \text{rad}$$

Clicker questions

Q14.6

This is an *x-t* graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *potential energy* of the spring the greatest?



A. $t = T/8$

B. $t = T/4$

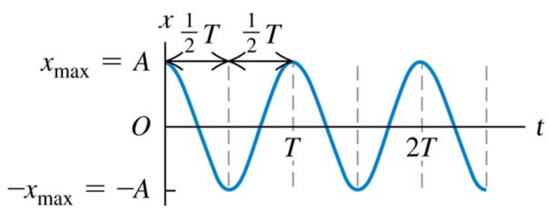
C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest potential energy.

Q14.7

This is an *x-t* graph for an object connected to a spring and moving in simple harmonic motion. At which of the following times is the *kinetic energy* of the object the greatest?



A. $t = T/8$

B. $t = T/4$

C. $t = 3T/8$

D. $t = T/2$

E. Two of the above are tied for greatest kinetic energy.

Ans: Q14.6) D, Q14.7) B