HKUST

MATH 2111 Matrix Algebra and Applications

Solutions to Sample Fall 2022-2023 Midterm Test

Name:________________________

Student ID:________________________

Lecture Section:________________________

Directions:

- DO NOT open the exam until instructed to do so.
- This is a closed book examination. No calculators nor formula sheet is allowed to use in this examination.
- Please switch all mobile phones to silent mode. And all electronic communication devices (laptops, tablets, smart watches, etc.) must be kept away from your body.
- Please write your name, ID number, and lecture section in the space provided above.
- When instructed to open the exam, please check that you have **6** pages (excluding this cover page) of **5** questions.
- **Answer all questions**. Show an appropriate amount of work for each problem. If you do not show enough work, you will get only partial credit.
- You may write on the backside of the pages, but if you use the backside, clearly indicate that you have done so.

Please read the following statement and sign your signature.

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature :

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Question No.	Marks	Out of
Qn. 1		32
Qn. 2		18
Qn. 3		20
Qn. 4		15
Qn. 5		15
Total Marks		100

Qn. 1 (32 marks) Choose a correct option for each question. No justification is required. Each correct answer is worth 4 marks (no deduction for wrong answers).

Write down your answers into the boxes provided at the bottom of next page.

(1) Let A be a $p \times q$ matrix and suppose that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^p$. Then the number of pivot positions in A must be

(A) at least equal to p.

(B) less than p

(C) at least equal to q

(D) less than q

Solution: (B) As $A\mathbf{x} = \mathbf{b}$ is in consistent for some $\mathbf{b}$, A cannot have pivot position in every row. Since A has p rows, the number of pivot positions in A must be less than p.

(2) Let $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be linearly independent set. Then which of the following sets is NOT linearly independent?

(A) $\{2\mathbf{u}, 2\mathbf{v}, 2\mathbf{w}\}$

(B) $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$

(C) $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\}$

(D) None of the above

Solution: (C). It is easy to see that $(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) + (\mathbf{w} - \mathbf{u}) = \mathbf{0}$, so the three vectors are linearly dependent. Both (A), (B) are linearly independent sets.

(3) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a linear transformation such that $T(\mathbf{e}_1) = \mathbf{e}_1 + \mathbf{e}_2, T(\mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3$, and $T(\mathbf{e}_3) = \mathbf{e}_3 + \mathbf{e}_1$. Then T is:

(A) both one-to-one and onto.

(B) one-to-one but not onto

(C) not one-to-one but onto

(D) neither one-to-one nor onto.

solution: (D). The standard matrix T is given by $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Its echelon form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Not every column has a pivot position, so it is not one-to-one. In the echelon form, not every row has pivot position, so it is not onto.

(4) Let $A, B.Q$ be general $n \times n$ matrices. Which of the following conditions will imply that A is row-equivalent to B?

(A) $AB = I_n$

(B) $A = QB$

(C) $A = BQ$

(D) None of the above

Solution: (A). By the invertible matrix theorem, both A and B are invertible and hence they are row-equivalent to I_n. Then A is row-equivalent to B by the sequence of EROs that bring A to I_n and then the sequence of EROs that bring I_n back to B.

As Q might be not invertible, $A = QB$ cannot guarantee that A is row-equivalent to B. Even in the case of invertible Q, BQ corresponds to column operations on B instead.

(5) Let A, B be $n \times n$ invertible matrices. Which of the followings is NOT correct?

(A) $\det(AB) = det(BA)$.

(B) $\det(AB^{-1}) = (det(BA^{-1}))^{-1}$

(C) $\det(ABAB) = \det(AB)^2$

(D) $\det(A - B) = -\det(B - A)$

(E) None of the above

Solution: (D). The correct formula for (D) should be $\det(A - B) = (-1)^n \det(B - A)$. The others are correct by the product formula of determinants: $\det(AB) = (\det A)(\det B)$.

(6) Which of the following statement is NOT true?

(A) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

(B) The columns of any 4×5 matrix are linearly dependent.

(C) The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(D) Two vectors are linearly dependent if and only if they lie on a line through the origin.

Solution: (A)

(7) Find the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1)$ and $(6, 4)$.

(A)-28 (B) 28 (C) -6 (D) 6 (E) None of the above

Solution: (B)

(8) Let A, B, C be $n \times n$ matrices. Which of the following formula is NOT correct?

(A) $A(BC) = (AB)C$

(B) $A(B + C) = AB + AC$

(C) $(A + B)^T = A^T + B^T$

(D) $(AB)^T = A^T B^T$

(E) None of the above

Solution: (D) The correct formula should be $(AB)^T = B^T A^T$

qn:	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
ans:	B	C	D	A	D	A	B	D

Qn. 2 Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{a}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Set $S = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$, $T = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5\}$.

(a) (8 marks) Find the reduced row-echelon form of the matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{b}]$.

(b) (5 marks marks) Are S, T linearly independent sets? Why or why not?

(c) (5 marks) Is $\mathbf{b}$ a linear combination of vectors in S? or in T? Why or why not?

Solution:

(a) We apply the standard row reduction algorithm:

$$\begin{bmatrix} 1&1&0&0&1&0\\ 1&0&0&1&0&1\\ 0&0&1&1&1&0\\ 0&1&1&0&0&1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1&1&0&0&1&0\\ 0&-1&0&1&-1&1\\ 0&0&1&1&1&0\\ 0&1&1&0&0&1 \end{bmatrix} \xrightarrow{R_2+R_4} \begin{bmatrix} 1&1&0&0&1&0\\ 0&-1&0&1&-1&1\\ 0&0&1&1&1&0\\ 0&0&1&1&-1&2 \end{bmatrix}$$

$$\xrightarrow{-R_3+R_4} \begin{bmatrix} 1&1&0&0&1&0\\ 0&-1&0&1&-1&1\\ 0&0&1&1&1&0\\ 0&0&0&0&-2&2 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_4]{-R_2} \begin{bmatrix} 1&1&0&0&1&0\\ 0&1&0&-1&1&-1\\ 0&0&1&1&1&0\\ 0&0&0&0&1&-1 \end{bmatrix} \xrightarrow[-R_4+R_1]{\substack{R_4+R_3\\ -R_4+R_2}} \begin{bmatrix} 1&1&0&0&0&1\\ 0&1&0&-1&0&0\\ 0&0&1&1&0&1\\ 0&0&0&0&1&-1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1} \begin{bmatrix} 1&0&0&1&0&1\\ 0&1&0&-1&0&0\\ 0&0&1&1&0&1\\ 0&0&0&0&1&-1 \end{bmatrix}$$

(b) Based on the computations in (a), we see that the matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ formed by vectors in S will contain 3 pivot positions and the last column contain no pivot position; while the matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_5]$ formed by vectors in T will contain 4 pivot positions occupying all the 4 columns. Hence S is linearly dependent and T is linearly independent.

(c) Again based on the computations in (a), the augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4 \mid \mathbf{b}]$ represents an inconsistent system. So $\mathbf{b}$ is not a linear combination of vectors in S, but $\mathbf{b}$ is a linear combination of vectors in T.

Qn 3

(a) (10 marks) Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

(b) (10 marks) Solve the following system:

$$\begin{cases} x_1 + 2x_4 = 3a \\ x_2 + 2x_3 = 3b \\ 2x_2 + x_3 = 3c \\ 2x_1 + x_4 = 3d \end{cases}$$

Solution: (a) We perform row operations to the combined matrix $[A \mid I_4]$:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_2+R_3]{-2R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_4]{-\frac{1}{3}R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} \end{array}\right] \xrightarrow[-2R_4+R_1]{-2R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & -\frac{1}{3} \end{array}\right]$$

Hence A is invertible and

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

(b) The given system can be written as a matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3a \\ 3b \\ 3c \\ 3d \end{bmatrix}.$$

So the solution will be given by $\mathbf{x} = A^{-1}\mathbf{b}$, namely

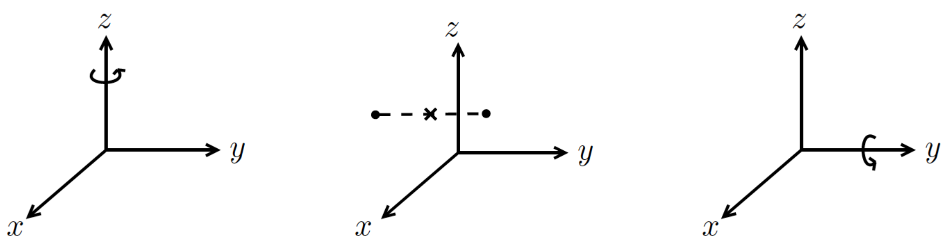
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3a \\ 3b \\ 3c \\ 3d \end{bmatrix} = \begin{bmatrix} -a + 2d \\ -b + 2c \\ 2b - c \\ 2a - d \end{bmatrix}$$

Qn 4 Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the linear transformation obtained by performing the following 3 operations in sequence:

(1) Rotation about the positive z-axis by 90^0.

(2) Reflection about the xz-plane.

(3) Rotation about the positive y-axis by 90^0.



(1) (2) (3)

(a) (10 marks) Find the standard matrix A of T.

(b) (5 marks) Find the image of the vector $[1 \quad 2 \quad 3]^T$ after the above sequence of operations.

Solution: We check of the sequence of operations on the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}0\\-1\\0\end{bmatrix} \xrightarrow{(3)} \begin{bmatrix}0\\-1\\0\end{bmatrix} = T(\mathbf{e}_1)$$

$$\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}-1\\0\\0\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}-1\\0\\0\end{bmatrix} \xrightarrow{(3)} \begin{bmatrix}0\\0\\1\end{bmatrix} = T(\mathbf{e}_2)$$

$$\mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{(1)} \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{(2)} \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{(3)} \begin{bmatrix}1\\0\\0\end{bmatrix} = T(\mathbf{e}_3)$$

So the standard matrix of T is given by

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)] = \begin{bmatrix}0 & 0 & 1\\-1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

(b) Using the standard matrix A of T, we have $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in $\mathbb{R}^3$ and hence:

$$T\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix}$$

Qn. 5 Let A denote the following matrix:

$$A = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix}$$

(a) (7 marks)Assume $a = 1, b = 2, c = 3, d = 4$. Evaluate $\det A$ in this special case.

(b) (4 marks) Note that the sum of each column in A is $(a+b+c+d)$. Use suitable row operations to show that $\det A$ contains a factor $(a + b + c + d)$.

(c) (4 marks) Show that $\det A$ also contains a factor $(a - b + c - d)$.

Solution: (a)Use row operations to simplify the determinant first.

$$\begin{vmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{vmatrix} \xlongequal[-4R_1+R_4]{\substack{-2R_1+R_2 \\ -3R_1+R_3}} \begin{vmatrix} 1 & 4 & 3 & 2 \\ 0 & -7 & -2 & -1 \\ 0 & -10 & -8 & -2 \\ 0 & -13 & -10 & -7 \end{vmatrix} \xlongequal[expansion]{cofactor} \begin{vmatrix} -7 & -2 & -1 \\ -10 & -8 & -2 \\ -13 & -10 & -7 \end{vmatrix} \xlongequal[-7R_1+R_3]{-2R_1+R_2} \begin{vmatrix} -7 & -2 & -1 \\ 4 & -4 & 0 \\ 36 & 4 & 0 \end{vmatrix}$$

$$\xlongequal[expansion]{cofactor} (-1) \begin{vmatrix} 4 & -4 \\ 36 & 4 \end{vmatrix} = (-1)(16 + 144) = -160$$

(b) By using EROs $r_2 + r_1, r_3 + r_1, r_4 + r_1$, we get:

$$\det(A) = \begin{vmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} a+b+c+d & a+b+c+d & a+b+c+d & a+b+c+d \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix}$$

$$= (a+b+c+d) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix}$$

So $\det A$ contains a factor $(a + b + c + d)$.

(c) Use suitable row operations to produce the factor $(a - b + c - d)$:

$$\begin{vmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix} \xlongequal[-R_4+R_1]{\substack{-R_2+R_1 \\ R_3+R_1}} \begin{vmatrix} a-b+c-d & d-a+b-c & c-d+a-b & b-c+d-a \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix}$$

$$= (a-b+c-d) \begin{vmatrix} 1 & -1 & 1 & -1 \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix}$$