Math1014 Calculus II
Brief Answers to Some Practice Problems on Definite Integrals: Net Changes, Areas. Volumes

1. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year, and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \le t \le 10$. What does this area represent?

$$\text{“area”} = \int_0^{10} (b(t) - d(t))dt = \int_0^{10} (2200e^{0.024t} - 1460e^{0.018t})dt$$
$$= \Big[\frac{2200}{0.024}e^{0.024t} - \frac{1460}{0.018}e^{0.018t}\Big]_0^{10} = \frac{275000}{3}e^{6/25} - \frac{730000}{9}e^{9/50} - \frac{95000}{9}$$

The area represents the net change in the population in the 10 year period.

2. If the amount of capital that a company has at time t is $f(t)$, then the derivative $f'(t)$ is called the *net investment flow*. Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in year). Find the *increase in capital* (also called the *capital formation*) from the fourth year to the eighth year.

capital formation during the period $= f(8) - f(4) = \displaystyle\int_4^8 \sqrt{t}dt = \Big[\frac{2}{3}t^{3/2}\Big]_4^8 = \frac{16}{3}(2\sqrt{2} - 1)$ (million dollars)

3. Evaluate the integral $\displaystyle\int_0^4 \left|\sqrt{x+2} - x\right| dx$ and interpret it as the area of a region. Sketch the region.

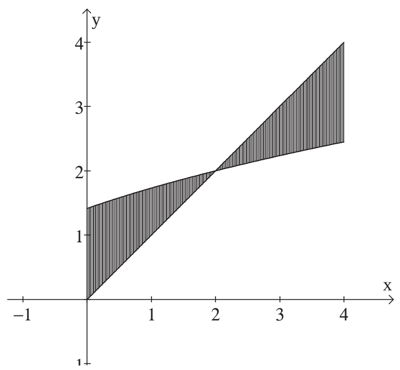
Note that $\sqrt{x+2} - x = \begin{cases} > 0 & \text{if } 0 \le x < 2 \\ = 0 & \text{if } x = 2 \\ < 0 & \text{if } 2 < x \le 4 \end{cases}$.

$$\int_0^4 \left|\sqrt{x+2} - x\right| dx$$
$$= \int_0^2 (\sqrt{x+2} - x)\, dx + \int_0^2 (-\sqrt{x+2} + x)\, dx$$
$$= \Big[\tfrac{2}{3}(x+2)^{3/2} - \tfrac{1}{2}x^2\Big]_0^2 + \Big[-\tfrac{2}{3}(x+2)^{3/2} + \tfrac{1}{2}x^2\Big]_0^2$$
$$= -4\sqrt{6} + \tfrac{44}{3} - \tfrac{4}{3}\sqrt{2}$$

= area between the curves given by

$y = \sqrt{x+2}$ and $y = x, 0 \le x \le 4$

Or as area under the graph of the non-negative function $|\sqrt{x+2} - 2|$ over the interval $[0, 4]$.



4. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

(i) $4x + y^2 = 12$, $x = y$.

(ii) $y = 3x^2$, $y = 8x^2$, $4x + y = 4$, $x \ge 0$.

(i) Intersection of the curves: $4x + y^2 = 12$, $x = y$

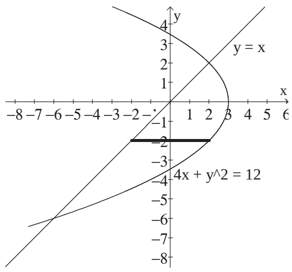
Putting $x = y$ into the first equation, we have

$4y + y^2 = 12 \iff (y+6)(y-2) = 0$

Intersection points: $(-6,-6), (2,2)$

By considering horizontal thin approximating rectangles:

$$\begin{aligned} \text{area} &= \int_{-6}^{2} \Big((3 - \frac{y^2}{4}) - y \Big) dy \\ &= \Big[3y - \frac{y^3}{12} - \frac{y^2}{2} \Big]_{-6}^{2} \\ &= \frac{64}{3}. \end{aligned}$$



$$\begin{cases} x = 3 - \frac{y^2}{4} \\ x = y \end{cases}$$

Or by considering vertical thin approximating rectangles:

$$\text{area} = \int_{-6}^{2} (x + \sqrt{12 - 4x}) dx + 2 \int_{2}^{3} \sqrt{12 - 4x} dx$$

$$= \ldots\ldots = \frac{64}{3}$$

(Draw some vertical thin approximating rectangles.)

(ii) Intersection of the curves:

$y = 3x^2$, $y = 8x^2$, $4x + y = 4$, $x \geq 0$

Putting $y = 4 - 4x$ into the first equation $y = 3x^2$:

$4 - 4x = 3x^2 \iff (3x-2)(x+2) = 0$

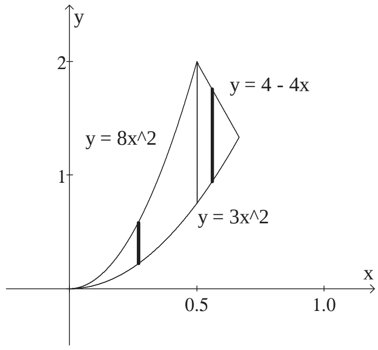
Intersection points: $(\frac{2}{3}, \frac{4}{3})$.

Putting $y = 4 - 4x$ into the second equation $y = 8x^2$:

$4 - 4x = 8x^2 \iff (8x-4)(x+1) = 0$

Intersection points: $(\frac{1}{2}, 2)$.

$$\begin{aligned} \text{area} &= \int_{0}^{\frac{1}{2}} (8x^2 - 3x^2) dx + \int_{\frac{1}{2}}^{\frac{2}{3}} (4 - 4x - 3x^2) dx \\ &= \Big[\frac{5}{3} x^3 \Big]_{0}^{\frac{1}{2}} + \Big[\, 4x - 2x^2 - x^3 \Big]_{\frac{1}{2}}^{\frac{2}{3}} = \frac{17}{54} \end{aligned}$$



Or by considering horizontal thin approximating rectangles:

$$\text{area} = \int_{0}^{\frac{4}{3}} (\sqrt{\frac{y}{3}} - \sqrt{\frac{y}{8}}) dy + \int_{\frac{4}{3}}^{2} (\frac{4-y}{4} - \sqrt{\frac{y}{8}}) dy$$

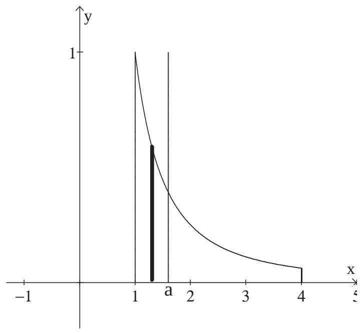
$$= \ldots\ldots = \frac{17}{54}\ .$$

(Draw some horizontal thin approximating rectangles.)

5. Consider the area under the curve $y = \dfrac{1}{x^2}$, $1 \leq x \leq 4$.

   (i) Find the number a such that $x = a$ bisects the given area.

   (ii) Find the number b such that $y = b$ bisects the given area.



(i) $\displaystyle\int_{1}^{a} \frac{1}{x^2} dx = \frac{1}{2} \int_{1}^{4} \frac{1}{x^2} dx \iff \Big[-\frac{1}{x} \Big]_{1}^{a} = \frac{1}{2} \Big[-\frac{1}{x} \Big]_{1}^{4}$

Hence $-\frac{1}{a} + 1 = \frac{1}{2}(-\frac{1}{4} + 1) = \frac{3}{8}$; i.e., $a = \dfrac{8}{5}$.

(ii) Note that $\frac{1}{16} < b < 1$, since $\frac{1}{16}(4-1) < \frac{3}{8}$.

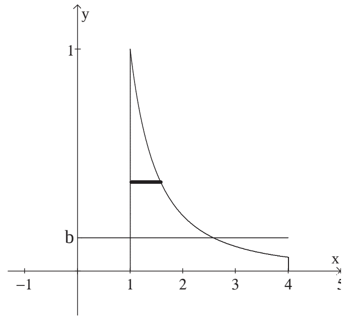
$$\int_b^1 \left(\frac{1}{\sqrt{y}} - 1\right)dy = \frac{3}{8}$$

$$\Big[2\sqrt{y} - y\Big]_b^1 = \frac{3}{8}$$

$$b - 2\sqrt{b} + \frac{5}{8} = 0 \Longleftrightarrow \sqrt{b} = \frac{2 \pm \sqrt{4 - \frac{5}{2}}}{2}$$

Checking: $\sqrt{b} = \frac{2+\sqrt{4-\frac{5}{2}}}{2} > \frac{2}{2} > 1$.
i.e., $\sqrt{b} = \frac{2-\sqrt{4-\frac{5}{2}}}{2} = 1 - \frac{\sqrt{6}}{4}$, and hence

$$b = (1 - \frac{\sqrt{6}}{4})^2 = \frac{11}{8} - \frac{\sqrt{6}}{2}$$



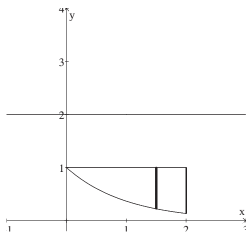
6. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the sold and a typical disk or washer.

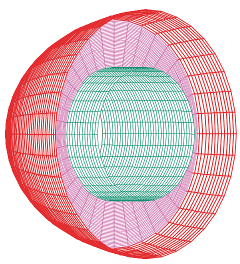
(i) $y = e^{-x}$, $y = 1$, $x = 2$; about $y = 2$.

(ii) $y = x$, $y = \sqrt{x}$; about $x = 2$.

(i) Cross-sections perpendicular to the x-axis are washers of outer radius $2 - e^{-x}$ and inner radius 1. By integrating the cross-section area function over an x-interval, we have

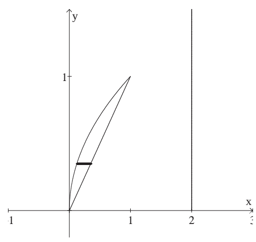
$$\begin{aligned} \text{volume} &= \int_0^2 [\pi(2 - e^{-x})^2 - \pi(1)^2]dx \\ &= \pi \int_0^2 (3 - 4e^{-x} + e^{-2x})dx \\ &= \pi \Big[3x + 4e^{-x} - \tfrac{1}{2}e^{-2x}\Big]_0^2 \\ &= \left(\frac{5}{2} + 4e^{-2} - \tfrac{1}{2}e^{-4}\right)\pi \end{aligned}$$

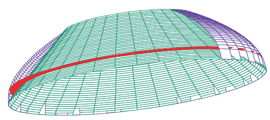




(ii) Cross-sections perpendicular to the y-axis are washers of outer radius $2 - y^2$ and inner radius $2 - y$. By integrating the cross-section area function over a y-interval, we have

$$\begin{aligned} \text{volume} &= = \int_0^1 [\pi(2 - y^2)^2 - \pi(2 - y)^2]dy \\ &= \pi \int_0^1 (4y - 5y^2 + y^4)dy \\ &= \pi \Big[2y^2 - \tfrac{5}{3}y^3 + \tfrac{1}{5}y^5\Big]_0^1 \\ &= \frac{8\pi}{15} \end{aligned}$$





7. Find the volume of the described solid S.

(i) The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

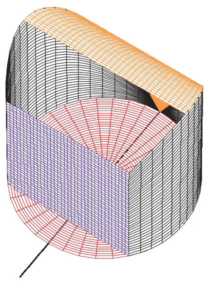
(ii) The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.

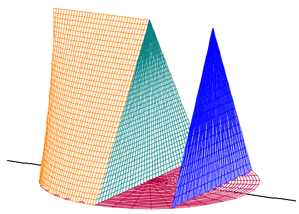
(iii) The common region S of two cylinders with the same radius r, if the axes of the cylinders intersect at right angles.

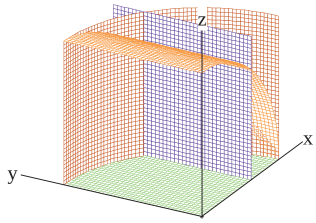
(i) volume $= 2\int_0^r \left[2\sqrt{r^2-x^2}\right]^2 dx = \dfrac{16}{3}r^3$

(ii) volume $= 2\int_0^r h\sqrt{r^2-x^2}\,dx = \dfrac{1}{2}\pi r^2 h$

(iii) Same as (i), since cross-section areas are the same as those of (i) .







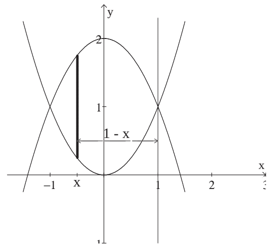
8. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

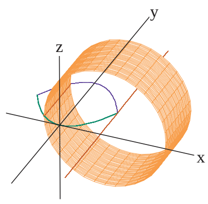
(i) $y = x^2$, $y = 2 - x^2$; about $x = 1$.

(ii) $y = x^2$, $x = y^2$; about $y = -1$.

(i)

$$\int_{-1}^{1} 2\pi(1-x)(2-2x^2)dx = \frac{16}{3}\pi$$





(ii)

$$\int_{0}^{1} 2\pi(y+1)(\sqrt{y}-y^2)dy = \frac{29}{30}\pi$$