COMP 2711 Discrete Math Tools for Computer Science

2022 Fall Semester - Homework 5

Question 1: Analyze the worst-case time complexity of the following algorithm for finding the first term of a sequence of integers equal to some previous term.

procedure find duplicate($a_1, a_2, \ldots, a_n$: integers)
location := 0 {no match found yet}
$i := 2$
while $i \leq n$ and *location* = 0
    $j := 1$
    while $j < i$ and *location* = 0
      if $a_i = a_j$ **then** *location* := i
      else $j := j + 1$
    $i := i + 1$
return *location*
{location is the subscript of the first value that repeats a previous value in the sequence and is 0 if there is no such value}

Question 2: A computer system considers a string of digits to be a valid code word if and only if it contains an even number of 0's. For instance, 1203045 is valid, whereas 780900 is not. Let V_n be the number of valid code words of length n.

It is clear that $V_1 = 9$ because "1", "2", ..., "9" are valid code words of length 1, while "0" is an invalid code word of length 1.

For $n > 1$, find a recurrence for V_n by determining how it is related to V_{n-1}. Solve the recurrence to get a closed-form formula for V_n.

Question 3: Use induction to prove that, for any integer $n \geq 1$,

$$5^n + 2 \cdot 11^n \text{ is divisible by 3.}$$

Question 4: You are given a real number a and a positive integer n that is a power of 2, i.e. $n = 2^k$ for some integer $k \geq 0$.

(a) Devise a recursive algorithm to find a^n. Your algorithm should use as few multiplications as possible.

(b) Give a recurrence equation of the number of multiplications used in your algorithm in (a).

(c) Solve your recurrence equation in (b).

Note that in this question, we assume

(i) b^m uses $m - 1$ multiplications for any real number b and positive integer m.

(ii) b/m is not counted as a multiplication if m is a positive integer.

(iii) In the computation of $(f(n))^m$, $f(n)$ is evaluated only once. But, in the computation of $f(n) \cdot f(n)$, $f(n)$ is evaluated twice.