§1.7 linear independence

Ex: $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

Solutions: unique ?

Solution: $\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix} \xrightarrow[③-3①]{②-2①} \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{pmatrix}$

$\xrightarrow{③+2②} \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{②}{-3}} \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{①-4②} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \end{cases}$$

If we take $x_3 = 1$, we get a solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

In particular, we can write

$$\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$$

Thus, span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = span $\{\vec{v}_1, \vec{v}_2\}$ is a plane

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Def: A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = 0$ has only the trivial solution. The set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, \cdots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}.$$

* Linear Independence of Matrix Column vectors

$A = [\vec{a}_1, \cdots, \vec{a}_n]$ a matrix

$$A\vec{x} = 0 \overset{\text{equivalent}}{\iff} x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0}$$

The column vectors of a matrix A are linearly independent if and only if $A\vec{x} = \vec{0}$ has only the trivial solution.

* Sets of one or Two vectors

iff

1) $\{\vec{v}\}$ is linearly independent if and only if $\vec{v} \neq \vec{0}$.
2) $\{\vec{v}_1, \vec{v}_2\}$ linearly independent if and only if neither of the vectors is a multiple of the other

$\iff$ $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

Proof: 1) If there exists $c_1 \neq 0$ such that

$$c_1\vec{v} = 0.$$

Then $\vec{v} = 0$.

so $\{\vec{v}\}$ is linearly dependent if $\vec{v} = 0$.

If $\vec{v} = \vec{0}$, then $c_1 \vec{0} = \vec{0}$ has many nontrivial solution. so $\{\vec{0}\}$ is linearly dependent.
Therefore, $\{\vec{v}\}$ is linearly dependent $\Leftrightarrow \vec{v} = \vec{0}$
i.e. $\{\vec{v}\}$ is linearly independent iff $\vec{v} \neq \vec{0}$

2) If $\vec{v}_1, \vec{v}_2$ are linearly dependent, then
$\exists$ exists c_1, c_2 not all zero such that
$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$.
Without loss of generality, assume $c_1 \neq 0$,
then $c_1 \vec{v}_1 = -c_2 \vec{v}_2$
$\Rightarrow \vec{v}_1 = -\frac{c_2}{c_1} \vec{v}_2$
i.e., $\vec{v}_1$ is a multiple of $\vec{v}_2$
On the other hand, if one vector of $\{\vec{v}_1, \vec{v}_2\}$ is a multiple of another vector, say, $\vec{v}_1 = c\vec{v}_2$, then we have $\vec{v}_1 - c\vec{v}_2 = \vec{0}$, thus $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent.

Ex: Determine if the following two vectors are linearly independent.

(a) $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ (b) $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

(c) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution: (a) It is easy to see $2\vec{v}_1 = \vec{v}_2$,
so $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent.

(b) Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 3 & 6 & 0 \\ 2 & 2 & 0 \end{pmatrix} \xrightarrow{②-\frac{2}{3}①} \begin{pmatrix} 3 & 6 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{①}{3}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow[\frac{②}{-2}]{①+②} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

So $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ are linearly independent.

(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

So $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are linearly dependent

* Characterization of linearly Dependent Sets

$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}_{p\geq 2}$ is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

Proof: If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ are linearly dependent, there exist $c_1, \cdots, c_p$ not all zero, $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$.

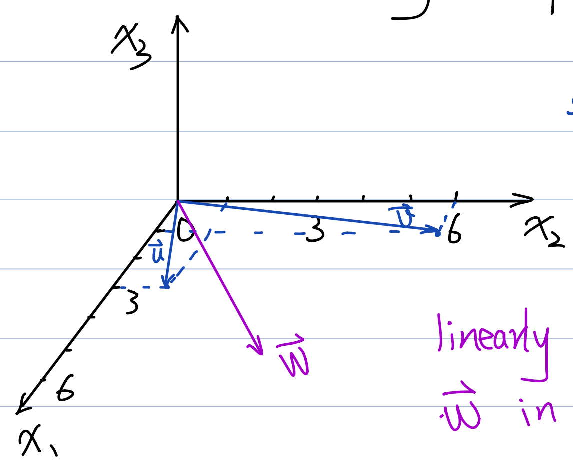
Assume $c_1 \neq 0$, then

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \cdots - \frac{c_p}{c_1}\vec{v}_p .$$

Conversely, if $\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$, then
$\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_p\vec{v}_p = \vec{0}$.

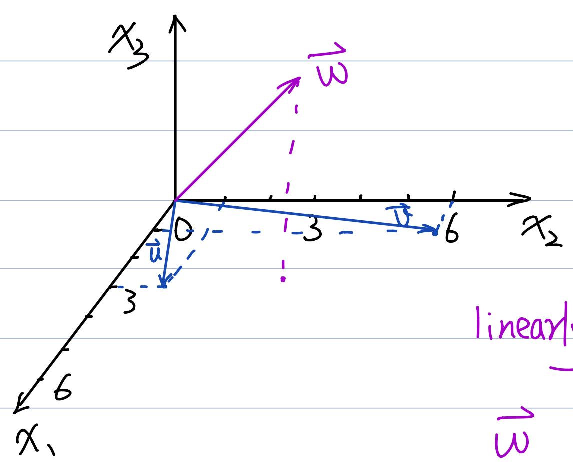
Ex: Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$. Describe the set spanned by $\vec{u}$ and $\vec{v}$, and explain why a vector $\vec{w}$ is in $\mathrm{Span}\{\vec{u}, \vec{v}\}$ if and only if $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

Solution:



span$\{\vec{u}, \vec{v}\} = x_1x_2$-plane

linearly dependent,
$\vec{w}$ in span$\{\vec{u}, \vec{v}\}$



linearly independent

$\vec{w}$ not in span$\{\vec{u}, \vec{v}\}$

Theorem: If $\{\vec{v}_1, \cdots, \vec{v}_p\}_{p\geq 2}$, $\vec{v}_1, \cdots, \vec{v}_p$ are vectors in $\mathbb{R}^n$.
If $p > n$, then $\{\vec{v}_1, \cdots, \vec{v}_p\}$ must be linearly dependent

Proof: $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ $(*)$

$(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p, \vec{0})$ is $n\times(p+1)$ matrix

number of variables p

number of equations: n

Since $p>n$, there are more variables than equations, so there must be free variables. So the equation $(*)$ must have non-trivial solution.

i.e. $\{\vec{v}_1, \cdots, \vec{v}_p\}$ must be linearly dependent.

Theorem: If a set $S=\{\vec{v}_1, \cdots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof: Assume $\vec{v}_1 = \vec{0}$ then

$$1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \cdots + 0\cdot\vec{v}_p = \vec{0}.$$

which implies S is linearly dependent.

§1.8 Introduction to linear Transformations

Ex:

$$\underbrace{\begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 5 \\ 8 \end{pmatrix}}_{\vec{b}}$$

$$\underbrace{\begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 4 \\ -1 \\ 3 \end{pmatrix}}_{\vec{u}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{0}}$$

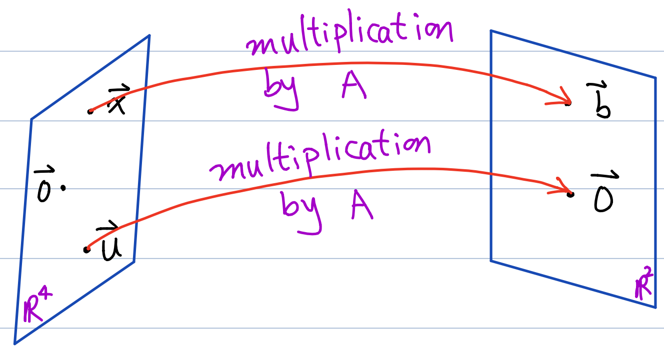


Figure: Transforming vectors via matrix multiplication

Def: A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ is a rule that assigns to each vector in $\mathbb{R}^n$ to a vector $T(\vec{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$: domain of T

$\mathbb{R}^m$: codomain of T

For $\vec{x} \in \mathbb{R}^n$, $T(\vec{x})$ is called the image of $\vec{x}$. The set of all images $T(\vec{x})$ is called the range of T.

A: matrix $m \times n$, $\vec{x} \in \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{x}) \triangleq A\vec{x}$.

Ex: Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

and define a tranformation $T: \mathbb{R}^2 \mapsto \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$, so that

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{pmatrix}$$

a) Find $T(\vec{u})$, the image of $\vec{u}$ under the transformation T.

b) Find an $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\vec{b}$.

c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?

d) Determine if $\vec{c}$ is in the range of the transformation T.

Solution: a) $T(\vec{u}) = A\vec{u} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

b) Solve $T(\vec{x}) = \vec{b}$

$$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{pmatrix} \xrightarrow[③+①]{②-3①} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{pmatrix}$$

$$\xrightarrow[\frac{③}{4}]{\frac{②}{14}} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 1 & -0.5 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{①+3②} \begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 1.5 \\ x_2 = -0.5 \end{cases}$$

c) Since the augmented matrix has no free variables, so the solution is unique.

(d)
$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{pmatrix}$$

Since the last column is a pivot column, the equation $A\vec{x} = \vec{c}$ has no solution, i.e., $\vec{c}$ is not in the range of T.