Math2001 Answer to Homework 7

Exercise 3.23

When $k = 1$, $m(n1) = mn = (mn)1$.

By induction, suppose associativity holds for k, then consider the case $k+1$. There is $m(n(k+1)) = m(nk+n) = m(nk) + mn = (mn)k + mn = mn(k+1)$. Therefore, associativity holds for all $k \in \mathbb{N}$.

Exercise 3.24

$m_1 + n_2 = n_1 + m_2$ and $k_1 + l_2 = l_1 + k_2$ imply $(m_1 + n_1)(k_1 + l_2) = (m_1 + n_1)(l_1 + k_2)$ and $(m_1 + n_2)(k_2 + l_2) = (m_2 + n_1)(k_2 + l_2)$. They furtherly imply that $[m_1k_1 + n_1l_1, m_1l_1 + n_1k_1] = [m_1k_2 + n_1l_2, m_1l_2 + n_1k_2]$ and $[m_1k_2 + n_1l_2, m_1l_2 + n_1k_2] = [m_2k_2 + n_2l_2, m_2l_2 + n_2k_2]$. By transitivity of equivalence relationship, there is $[m_1k_1 + n_1l_1, m_1l_1 + n_1k_1] = [m_2k_2 + n_2l_2, m_2l_2 + n_2k_2]$.

Exercise 3.25

(2) *Distributivity:* Suppose $a = [m, n]$, $b = [k, l]$, $c = [s, t]$.

$(a + b)c = [m + k, n + l][s, t] = [(m + k)s + (n + l)t, (m + k)t + (n + l)s]$, and $ac + bc = [m, n][s, t] + [k, l][s, t] = [ms + nt, mt + ns] + [ks + lt, kt + ls] = [ms + nt + ks + lt, mt + ns + kt + ls]$. By distributivity in natural numbers, there is $[(m + k)s + (n + l)t, (m + k)t + (n + l)s] = [ms + nt + ks + lt, mt + ns + kt + ls]$, thus $(a + b)c = ac + bc$.

$a(b+c) = [m, n][k+s, l+t] = [m(k+s)+n(l+t), m(l+t)+n(k+s)]$, and $ab+ac = [m, n][k, l] + [m, n][s, t] = [mk + nl, ms + nt] + [ms + nt, mt + ns] = [mk + nl + ms + nt, ms + nt + mt + ns]$. By distributivity in natural numbers, $[m(k + s) + n(l + t), m(l + t) + n(k + s)] = [mk + nl + ms + nt, ms + nt + mt + ns]$, thus $a(b + c) = ab + ac$.

(3) *Associativity:* Suppose $a = [m, n]$, $b = [k, l]$, $c = [s, t]$.

$a(bc) = [m, n][ks + lt, kt + ls] = [m(ks + lt) + n(kt + ls), m(kt + ls) + n(ks + lt)] = [m(ks) + m(lt) + n(kt) + n(ls), m(kt) + m(ls) + n(ks) + n(lt)]$.

$(ab)c = [mk + nl, ml + nk][s, t] = [(mk + nl)s + (ml + nk)t, (mk + nl)t + (ml + nk)s] = [(mk)s + (nl)s + (ml)t + (nk)t, (mk)t + (nl)t + (ml)s + (nk)s]$. By associativity in natural numbers, there is $a(bc) = (ab)c$.

(4) *Commutativity:* Suppose $a = [m, n]$, $b = [k, l]$.

$ab = [mk + nl, ml + nk]$ and $ba = [km + ln + lm + kn]$. By commutativity in $\mathbb{N}$, $ab = ba$.

(5) *One:* It suffices to show that $a1 = a$. Suppose $a = [m, n]$ and $1 = [l + 1, l]$. $a1 = [m, n][l + 1, l] = [m(l + 1) + nl, ml + n(l + 1)] = [ml + nl + m, ml + nl + n]$. Since $(ml + nl + n) + m = (ml + nl + m) + n$, there is $a1 = a$.

Exercise 3.26

Since $-c > 0$, according to Prop 3.4.3.(8) $a > b \iff (-c)a > (-c)b$. Meanwhile, $(-c)a > (-c)b \iff -c(a - b) > 0 \iff c(b - a) > 0 \iff ac < bc$. Therefore, $a > b \iff ac < bc$.

Exercise 3.31

By Prop 3.5.2.(5), there exists $-t$ for t such that $(-t) + t = 0$. Hence $r = (r + t) + (-t) = (s + t) + (-t) = s + (t + (-t)) = s$.

By Prop 3.5.2.(8), there exists reciprocal t^{-1} for t. Hence $r = rtt^{-1} = stt^{-1} = s(tt^{-1}) = s$.

Exercise 3.32

If $r = 0 = \frac{0}{1}$, suppose $s = \frac{a}{b}$, then $rs = \frac{0}{1} \cdot \frac{a}{b} = \frac{0}{b} = 0$. Likewise for the case $s = 0$.

If $rs = 0$, suppose $r = \frac{a}{b}$ and $s = \frac{c}{d}$, then $ac = 0$ implies $a = 0$ or $c = 0$. Thus $r = 0$ or $s = 0$.

Exercise 3.34

Firstly, we show that $|rs| = |r||s|$. If $rs > 0$, then either r and s are both positive or they are both negative. In the first case, we have $|r||s| = rs = |rs|$. In the second case, there is $|r||s| = (-r)(-s) = (-1)^2 rs = rs = |rs|$. Besides, if $rs < 0$, then one of them if negative and another one is positive. Thus $|r||s| = -rs = |rs|$.

Since $|r+s|$ and $|r|+|s|$ are both positive, it is equivalent to show that $|r+s|^2 \leqslant (|r|+|s|)^2$. $(|r| + |s|)^2 - |r + s|^2 = (r^2 + s^2 + 2|r||s|) - (r^2 + s^2 + 2rs) = 2(|rs| - rs) \geqslant 0$. Hence $|r + s|^2 \leqslant (|r| + |s|)^2$ holds.

$|r| < s \iff -s < r < s$: Suppose $|r| < s$ holds. Then we have $s > |r| \geqslant 0$. If $r > 0$, $-s < 0 < r < s$. If $r < 0$, then $-r < s$. Multiply -1 on both sides there is $-s < r$. It can be concluded that $-s < r < 0 < s$.

Suppose $-s < r < s$ holds. If $r > 0$, then $|r| = r < s$. If r¡0, then $|r| = -r < s$ by $-s < r$.

Exercise 3.35

If $r > s$, then $\max\{r, s\} = r$, $\min\{r, s\} = s$. Thus $\max\{r, s\} + \min\{r, s\} = r + s$ and $\max\{r, s\} - \min\{r, s\} = r - s$. If $r < s$, then $\max\{r, s\} = s$, $\min\{r, s\} = r$. Thus $\max\{r, s\} + \min\{r, s\} = s + r = r + s$ and $\max\{r, s\} - \min\{r, s\} = s - r = -(r - s)$.

Therefore, $\max\{r, s\} + \min\{r, s\} = r + s$, $\max\{r, s\} - \min\{r, s\} = |r - s|$.