Work and Kinetic Energy

PHYS1112

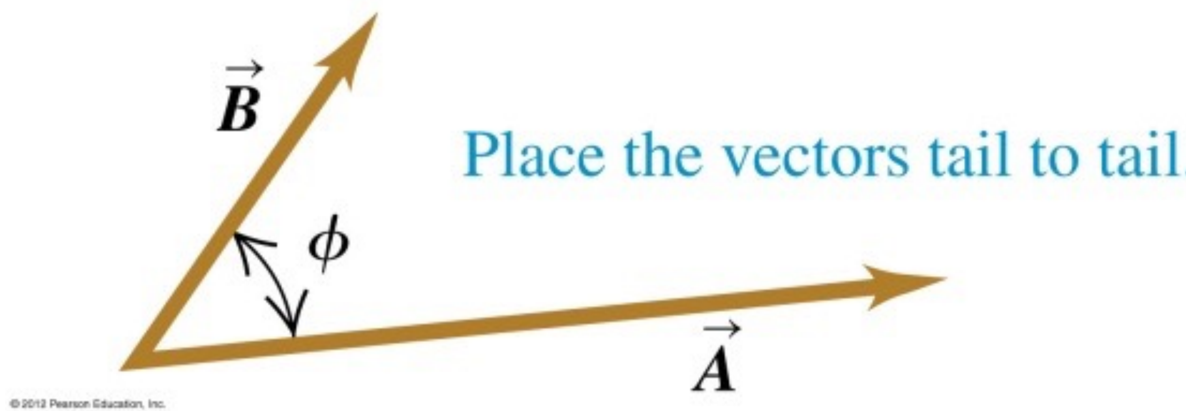
Lecture 4

Intended Learning Outcomes

- After this lecture you will learn:
  1) scalar product of vectors
  2) the meanings of +ve and –ve work done.
  3) the Hooke's law as an example of a variable force.
  4) the work-energy theorem in the general case.

Scalar Product

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB\cos\phi$$



Special cases:

(i) if $\overrightarrow{\boldsymbol{A}} \parallel \overrightarrow{\boldsymbol{B}}$, $\overrightarrow{\boldsymbol{A}} \cdot \overrightarrow{\boldsymbol{B}} = AB$, in particular, $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$

(ii) if $\overrightarrow{\boldsymbol{A}} \perp \overrightarrow{\boldsymbol{B}}$, $\overrightarrow{\boldsymbol{A}} \cdot \overrightarrow{\boldsymbol{B}} = 0$, in particular, $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$

In analytical form, $\boxed{\overrightarrow{\boldsymbol{A}} \cdot \overrightarrow{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z}$

Q1.4

Consider the two vectors

$$\vec{\mathrm{A}} = 3\hat{i} + 4\hat{j}$$

$$\vec{\mathrm{B}} = -8\hat{i} + 6\hat{j}$$

What is the dot product $\vec{A} \bullet \vec{B}$?

A. zero

B. 14

C. 48

D. 50

E. none of these

Consider the two vectors

$$\vec{A} = 3\hat{i} + 4\hat{j}$$

$$\vec{B} = -8\hat{i} + 6\hat{j}$$

What is the dot product $\vec{A} \bullet \vec{B}$?

A. zero

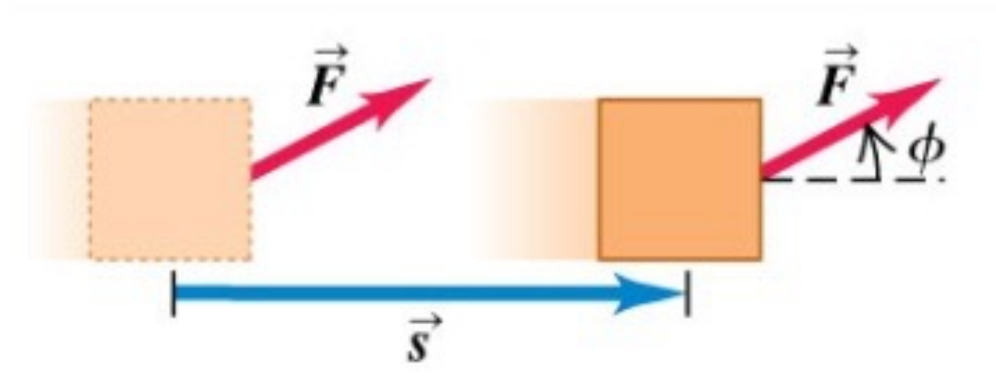
B. 14

C. 48

D. 50

E. none of these

From high school,

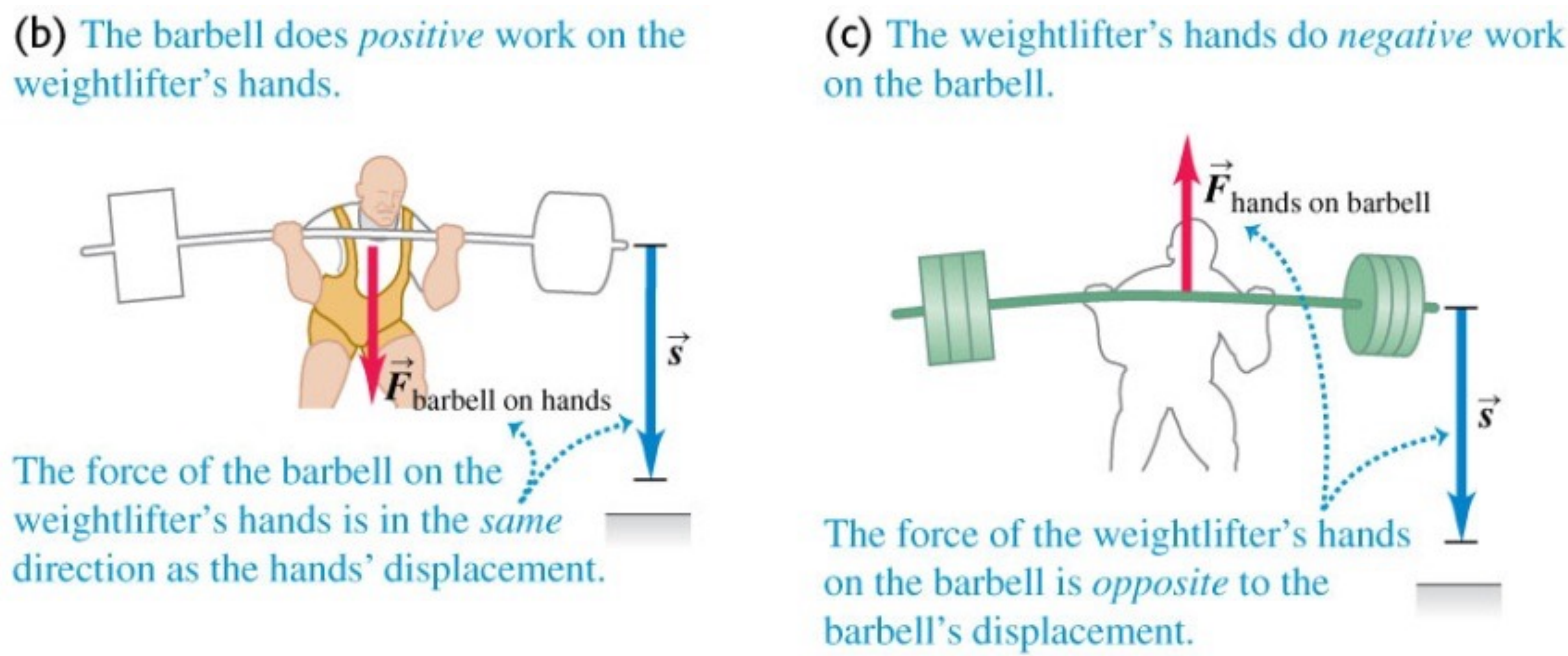


work done $W = Fs\cos\phi$ SI unit: joule 1 J = 1 Nm

$\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}}$, see how useful vector notation is!!

In general, $W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}} = F_x s_x + F_y s_y + F_z s_z$

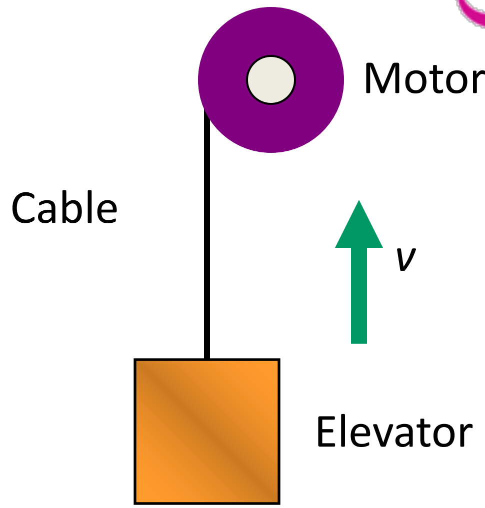
W must refer to the work done by a **specific force** on a **body**, otherwise you may be confused by the sign as illustrated below:



Action and reaction – a body does work on a second body, the second body does an equal and opposite amount of work on the first.

Q6.1

An elevator is being *lifted* at a constant speed by a steel cable attached to an electric motor. Which statement is correct?



A. The cable does positive work on the elevator, and the elevator does positive work on the cable.

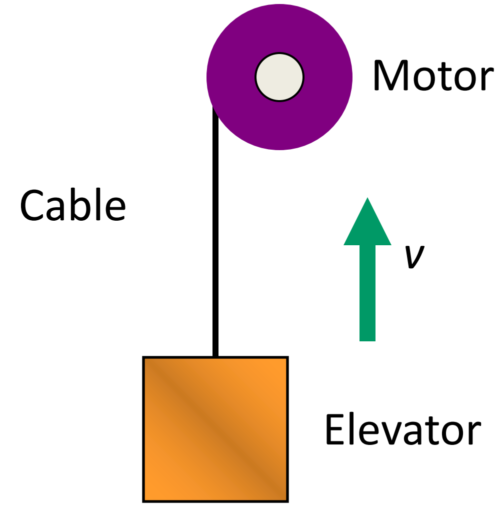
B. The cable does positive work on the elevator, and the elevator does negative work on the cable.

C. The cable does negative work on the elevator, and the elevator does positive work on the cable.

D. The cable does negative work on the elevator, and the elevator does negative work on the cable.

A6.1

An elevator is being *lifted* at a constant speed by a steel cable attached to an electric motor. Which statement is correct?



A. The cable does positive work on the elevator, and the elevator does positive work on the cable.

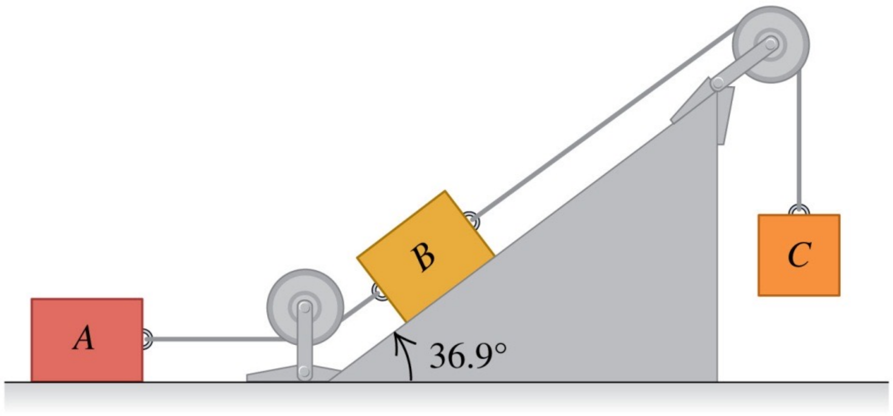
B. The cable does positive work on the elevator, and the elevator does negative work on the cable.

C. The cable does negative work on the elevator, and the elevator does positive work on the cable.

D. The cable does negative work on the elevator, and the elevator does negative work on the cable.

Q6.8

Three blocks are connected as shown. The ropes and pulleys are of negligible mass. When released, block *C* moves downward, block *B* moves up the ramp, and block *A* moves to the right.



After each block has moved a distance *d*, the force of gravity has done

A. positive work on *A*, *B*, and *C.*

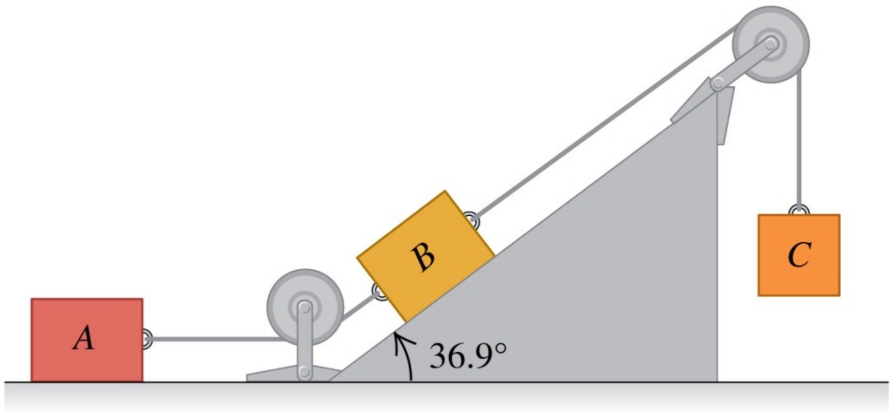
B. zero work on *A,* positive work on *B*, and negative work on *C.*

C. zero work on *A,* negative work on *B*, and positive work on *C.*

D. none of these

A6.8

Three blocks are connected as shown. The ropes and pulleys are of negligible mass. When released, block *C* moves downward, block *B* moves up the ramp, and block *A* moves to the right.



After each block has moved a distance *d*, the force of gravity has done

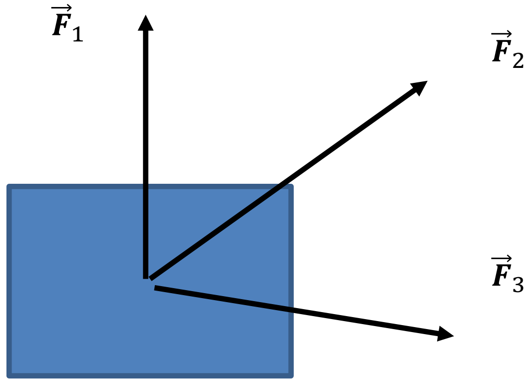
A. positive work on *A*, *B*, and *C.*

B. zero work on *A,* positive work on *B*, and negative work on *C.*

C. zero work on *A,* negative work on *B*, and positive work on *C.*

D. none of these

Workdone by multiple forces:



$$W = \left(\sum \vec{\boldsymbol{F}}\right) \cdot \vec{s} = \sum \vec{\boldsymbol{F}} \cdot \vec{s}$$

work done by
resultant force

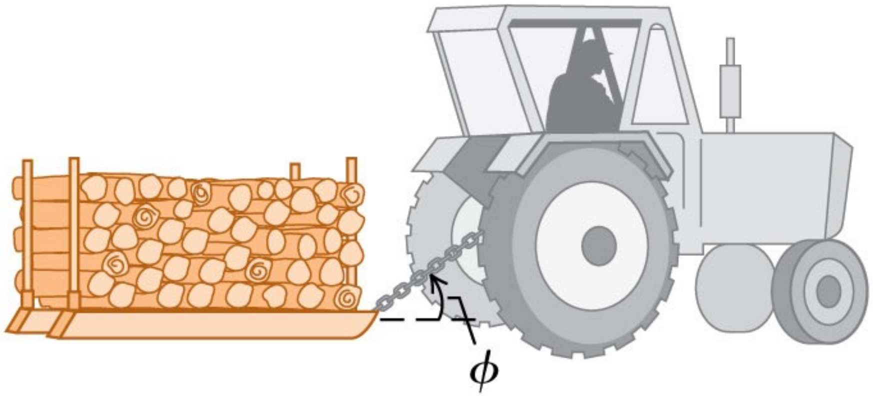
sum of work done
by individual forces

Question

- An electron moving in a straight line with a constant speed of 8×10^7 m/s. You are told that it has electric, magnetic, and gravitational forces acting on it. During a 1 m displacement, the total work done on the electron is
  1) +ve
  2) –ve
  3) Zero
  4) not enough information given to decide?

Q6.4

A tractor driving at a constant speed pulls a sled loaded with firewood. There is friction between the sled and the road.



The total work done on the sled after it has moved a distance *d* is

A. positive.

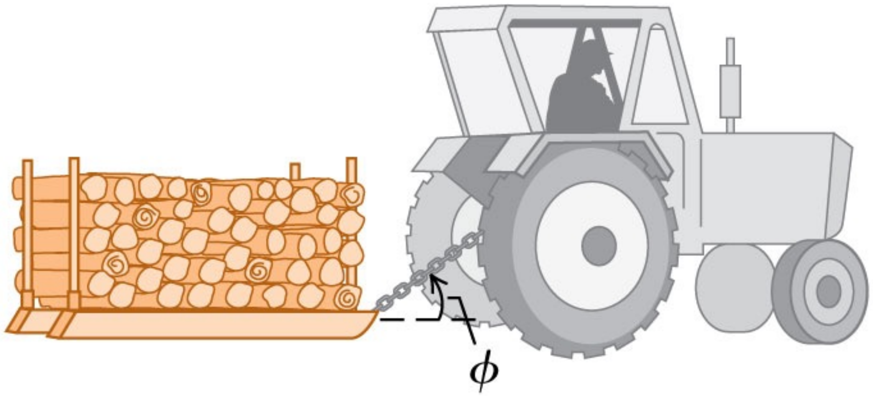
B. negative.

C. zero.

D. not enough information given to decide

A6.4

A tractor driving at a constant speed pulls a sled loaded with firewood. There is friction between the sled and the road.



The total work done on the sled after it has moved a distance *d* is

A. positive.

B. negative.

C. zero.

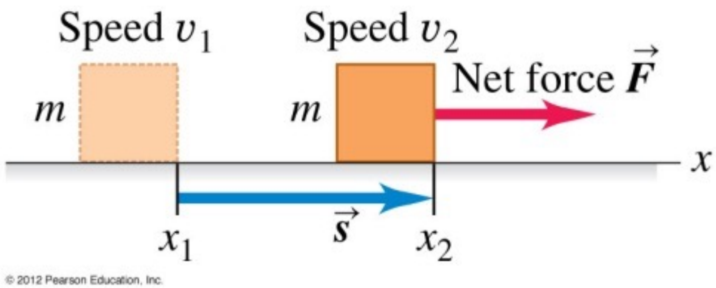
D. not enough information given to decide

Also from high school:

•Definition of **kinetic energy**, $K = \frac{1}{2}mv^2$

•**Work-energy theorem**

Work done by the net external force = change in KE of the particle



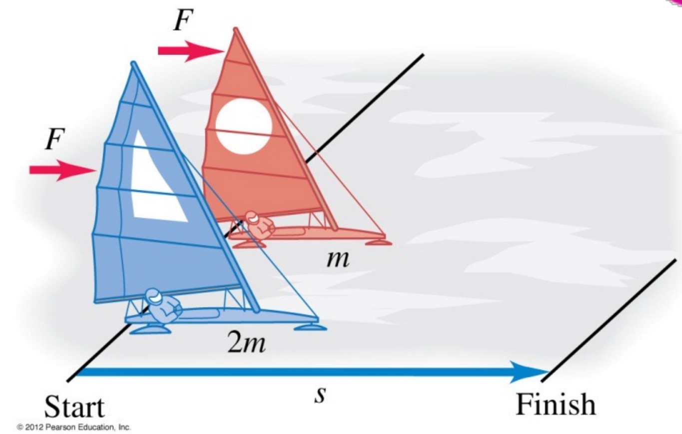
⚠ When accelerating a particle, work done by an external force $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 > 0$.

⚠ When decelerating a particle, $W < 0$.

The above results are easy to prove if you consider 1D motion under a constant external force. You have done it high school. If you have forgotten, see Textbook.

Q6.3

Two iceboats (one of mass *m*, one of mass 2*m*) hold a race on a frictionless, horizontal, frozen lake. Both iceboats start at rest, and the wind exerts the same constant force on both iceboats.



Which iceboat crosses the finish line with more kinetic energy (KE)?

A. The iceboat of mass *m*: it has twice as much KE as the other.

B. The iceboat of mass *m*: it has 4 times as much KE as the other.

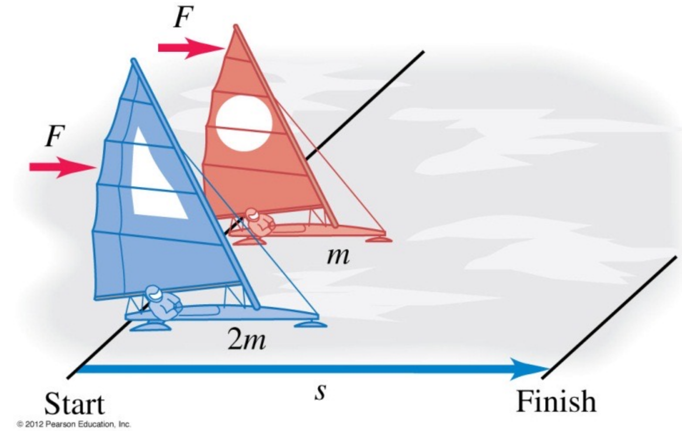
C. The iceboat of mass 2*m*: it has twice as much KE as the other.

D. The iceboat of mass 2*m*: it has 4 times as much KE as the other.

E. They both cross the finish line with the same kinetic energy.

A6.3

Two iceboats (one of mass *m*, one of mass 2*m*) hold a race on a frictionless, horizontal, frozen lake. Both iceboats start at rest, and the wind exerts the same constant force on both iceboats.



Which iceboat crosses the finish line with more kinetic energy (KE)?

A. The iceboat of mass *m*: it has twice as much KE as the other.

B. The iceboat of mass *m*: it has 4 times as much KE as the other.

C. The iceboat of mass 2*m*: it has twice as much KE as the other.

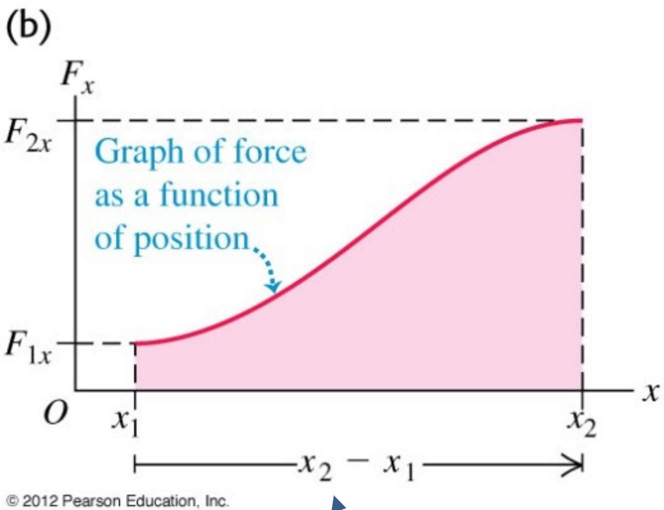
D. The iceboat of mass 2*m*: it has 4 times as much KE as the other.

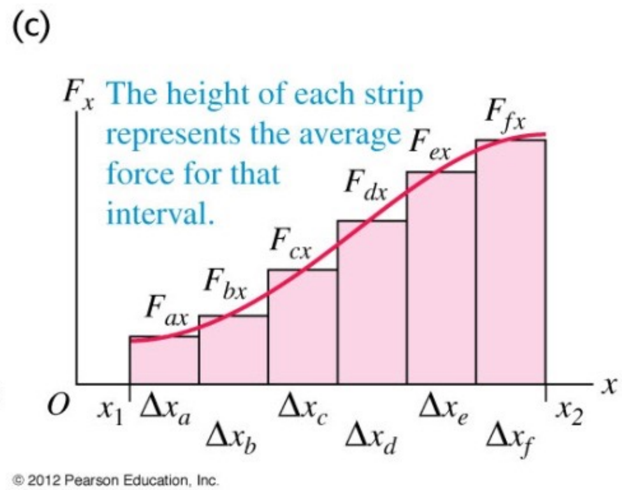
E. They both cross the finish line with the same kinetic energy.

Question:

What if the force is not constant (but still in 1D)?

Need a magical mathematical tool called calculus.





approximate each sub-interval by a constant force

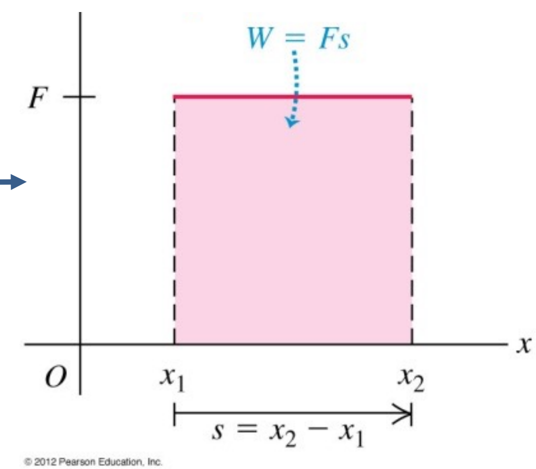
$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$

as $\Delta x \to 0$

W = area under curve

$= \int_{x_1}^{x_2} F_x \, dx$

c.f. constant force



Example: An ideal spring

displacement from natural length (unstretched position)

Hooke's law (Robert Hooke, 1678)

– restoring force (i.e., tension in the spring) = $-\,kx$

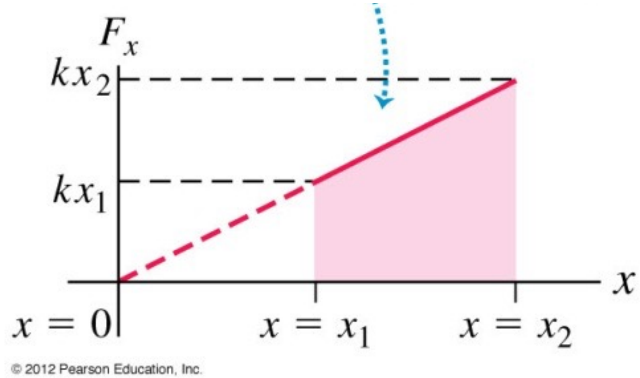
direction opposite to displacement

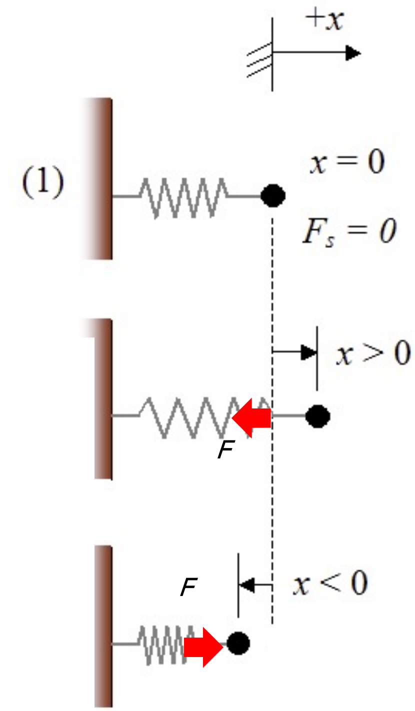
force constant

unit: N/m

Work done by an external force (⚠ not tension in the spring) in stretching a spring from x_1 to x_2, $x_1 < x_2$

$$W = \int_{x_1}^{x_2} F\,dx = k\int_{x_1}^{x_2} x\,dx = \underbrace{\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2}_{\text{area under the line}} > 0$$





In *compressing* from $-x_1$ to $-x_2$, same formula holds, $W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$

1D motion with variable force,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

i.e., $$F = ma = mv\frac{dv}{dx}$$

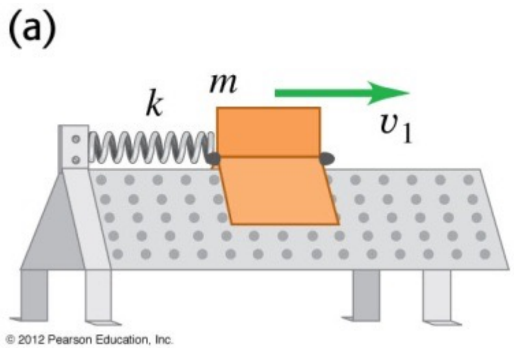
∴ work done by the net external force

$$\int_{x_1}^{x_2} F\,dx = m\int_{x_1}^{x_2} v\frac{dv}{dx}dx = m\int_{v_1}^{v_2} v\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Work-energy theorem works for variable force!

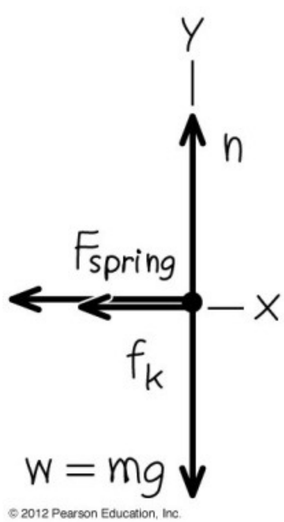
Example 6.7

A glider of mass *m*, and a spring with force constant *k*. Initially the spring is unstretched and the glider is moving with speed v_1. What is the maximum displacement *d* to the right if the frictional coefficient is μ_k?



By the work-energy theorem

$$\underbrace{(-\mu_k mgd)}_{\text{work done by } f_k} + \underbrace{\left(-\int_0^d kx\,dx\right)}_{\text{work done by } F_{\text{spring}}} = \underbrace{0 - \frac{1}{2}mv_1^2}_{\text{change in KE}}$$

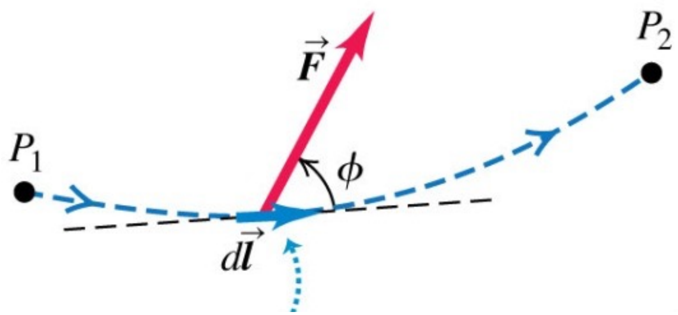


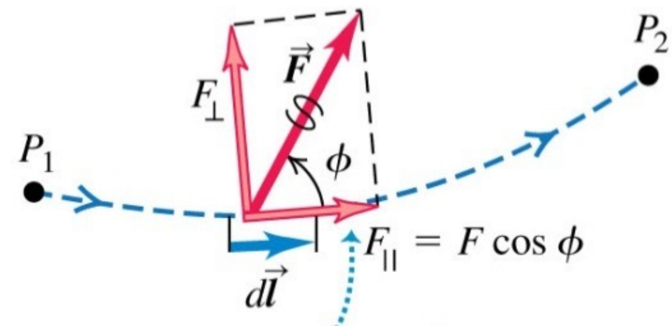
$$\frac{1}{2}kd^2 + \mu_k mgd - \frac{1}{2}mv_1^2 = 0$$

$$\Rightarrow \quad d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

3D motion with variable force

Idea: break up the path into very short segments so that in each segment, $\vec{\boldsymbol{F}}$ is approximately constant





v along tangential direction (why?)

work done in this small segment $dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = F_\parallel dl = mv\dfrac{dv}{dl}dl = mvdv$

total work done = sum over all segments

$$W_{tot} = \sum \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} \to \int_{P_1}^{P_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} mvdv$$

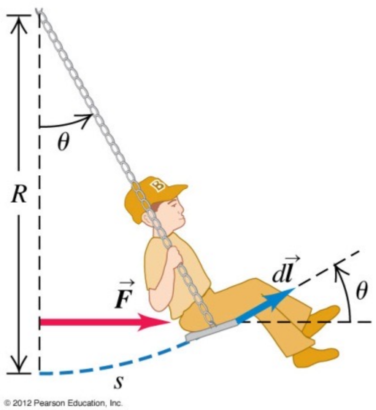
$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

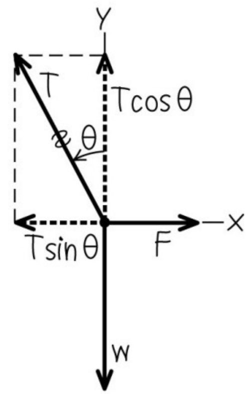
Conclusion: work-energy theorem holds for motion along a curve under variable force.

Example 6.8

Apply a horizontal force $\vec{\boldsymbol{F}}$ to push the swing up from $\theta = 0$ to θ_0

Assumption: $\vec{\boldsymbol{F}}$ is just enough to push it up so that the swing is in equilibrium any t





$$\sum F_x = F - T\sin\theta = 0$$
$$\sum F_y = T\cos\theta - w = 0$$
$$\Rightarrow \; T = w\sec\theta$$
$$F = w\tan\theta$$

Work done by net force, W_{net} = ____

Work done by $\vec{\boldsymbol{T}}$, W_T = ______ $(\because \vec{\boldsymbol{T}} \perp d\vec{\boldsymbol{l}})$

Work done by $\vec{\boldsymbol{F}}$,

$$W_F = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} F\cos\theta\, dl = \int_0^{\theta_0} w\tan\theta\cos\theta\, Rd\theta = wR(1-\cos\theta_0)$$

Work done by $\vec{\boldsymbol{w}}$ $W_w = \int \vec{\boldsymbol{w}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} w\cos\left(\frac{\pi}{2}+\theta\right) dl =$

$$-\int_0^{\theta_0} w\sin\theta\, Rd\theta = -wR(1-\cos\theta_0)$$

Check that $W_{\text{net}} = W_T + W_F + W_w$

Power

Average over a period Δt, $P_{av} = \frac{\Delta W}{\Delta t}$

Instantaneous power ($\Delta t \to 0$), $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$

SI unit: watt 1 W = 1 J/s

Another unit of *energy* besides J – kilowatt hour, common in electric bills

1 KWh = (10^3 J/s)(3600 s) = 3.6×10^6 J

If $\vec{\boldsymbol{F}}$ is the force that do work (can be constant or variable), workdone during Δt is $\Delta W = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{s}}$

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{s}}}{\Delta t} = \vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$$

Conclusion:

Instantaneous power for a force doing work on a particle

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$$

Force that acts on particle

Velocity of particle

Q6.10

An object is initially at rest. A net force (which always points in the same direction) is applied to the object so that the *power* of the net force is constant. As the object gains speed,

A. the magnitude of the net force remains constant.

B. the magnitude of the net force increases.

C. the magnitude of the net force decreases.

D. not enough information given to decide

A6.10

An object is initially at rest. A net force (which always points in the same direction) is applied to the object so that the *power* of the net force is constant. As the object gains speed,

A. the magnitude of the net force remains constant.

B. the magnitude of the net force increases.

C. the magnitude of the net force decreases.

D. not enough information given to decide