Review:

1. The derivative of $y=f(x)$: $f'(x) \overset{\text{defined by}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (if exists).

Alternative notations: $f'(x) = y' = \frac{d}{dx}y = \frac{dy}{dx} = \frac{d}{dx}f = \frac{df}{dx} = Dy = Df$.

$f(x)$ is differentiable at $x=a$ $\Leftrightarrow$ $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

2. Calculation of derivatives:

(1). calculate $\frac{dy}{dx}$ by definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(2). calculate $\frac{dy}{dx}$ by rules of differentiation.

① $(f \pm g)' = f' \pm g'$. ② $(c \cdot f)' = c \cdot f'$. ($c$ is a constant).

③ product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

④ quotient rule: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$.

(3). Calculate $\frac{dy}{dx}$ by the chain rule.

$y=f(u),\ u=g(x) \Rightarrow y=f(g(x))$ and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

Review some notable derivatives: $\frac{d}{dx}\cdot x^n = n\cdot x^{n-1}$ (n is a real number).

$\frac{d}{dx}\sin x = \cos x$. $\frac{d}{dx}\cos x = -\sin x$. $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$.

$\frac{d}{dx}a^x = \ln a \cdot a^x$. $\frac{d}{dx}e^x = e^x$.

We can combine these derivatives with the chain rule.

Take $\frac{d}{dx}e^x = e^x$ as an example.

Suppose $y=f(x)$. Then $\frac{d}{dx}e^{f(x)} = \frac{d}{dx}e^y = \frac{de^y}{dy}\cdot\frac{dy}{dx} = e^y\cdot\frac{dy}{dx}$.

1). Example 1: If $y=c\cdot x$ (c is a constant), then $\frac{d}{dx}e^{cx} = e^{cx}\cdot c = c\cdot e^{cx}$

2). Example 2: If $y=e^x$, then $\frac{d}{dx}e^{e^x} = e^{e^x}\cdot\frac{de^x}{dx} = e^{e^x}\cdot e^x$.

(4). Implicit Differentiation.

Sometimes, y is expressed in an implicit form $F(x,y)=0$:

$F(x,y) = x^3 - y^3 - \sin(x+y) = 0$. $\quad F(x,y) = x^3 + y^3 - xy = 0$.

To calculate $\boxed{\frac{dy}{dx}}$ from $F(x,y)=0$, we take the following 3 steps:

Step 1: Differentiate both sides with respect to x : $\frac{d}{dx}F(x,y) = \frac{d}{dx}(0) = 0$.

Step 2: Calculate $\frac{d}{dx}F(x,y)$ in terms of x, y and $\frac{dy}{dx}$. (use chain / product / quotient rule).

Step 3: solve $\frac{d}{dx}F(x,y)=0$ for $\frac{dy}{dx}$ $\Rightarrow$ express $\boxed{\frac{dy}{dx}}$ in terms of x and y.

Example: $F(x,y) = x^3 + y^3 - xy = 0$.

Step 1: $\frac{d}{dx}F(x,y) = 0$.

Step 2: $\frac{d}{dx}F(x,y) = \frac{d}{dx}x^3 + \frac{d}{dx}y^3 - \frac{d}{dx}xy$.

$$= 3x^2 + 3y^2 \cdot \frac{dy}{dx} - \left(\frac{dx}{dx}\cdot y + x \cdot \frac{dy}{dx}\right).$$

$$= \underbrace{3x^2 + 3y^2 \frac{dy}{dx} - y - x \cdot \frac{dy}{dx}}_{\text{solve for } \frac{dy}{dx}} \overset{\text{(by step 1)}}{=\!=} 0$$

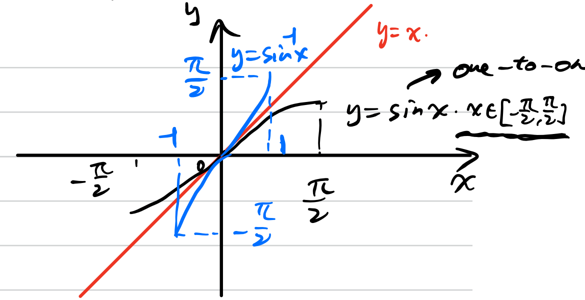
Step 3: $\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$.

defined only for one-to-one functions.

3. Derivatives of the inverse functions

(1). $y = \arc\sin x = \sin^{-1} x$.

$$\boxed{(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}}$$



Recall: ① domain: $[-1, 1]$. range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

② $y = \sin^{-1} x \overset{\text{means}}{\Longrightarrow} \sin y = x$.

To calculate $\boxed{\frac{dy}{dx}}$ we use the implicit differentiation for $F(x,y) = x - \sin y = 0$.

Step 1 $\frac{d}{dx}(x - \sin y) = \frac{d}{dx}(0) = 0$.

Step 2. $\frac{d}{dx}(x - \sin y) = \frac{d}{dx}x - \frac{d}{dx}\sin y \overset{\text{chain rule}}{=} 1 - \cos y \cdot \frac{dy}{dx} = 0$.

Step 3. $1 - \cos y \cdot \boxed{\frac{dy}{dx}} = 0 \Rightarrow \boxed{\frac{dy}{dx}} = \frac{1}{\cos y}$

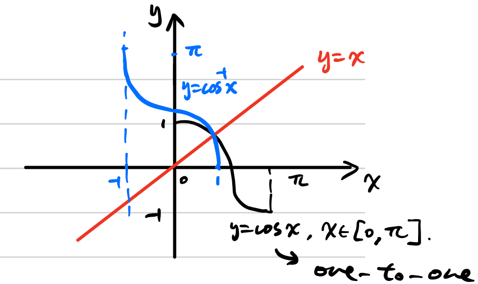
Notice that $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}$. $\Rightarrow \boxed{\frac{dy}{dx}} = \frac{1}{\sqrt{1-x^2}}$.

because $\cos^2 y + \sin^2 y = 1$

and $\cos y \geq 0$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(2). $y = \arccos x = \cos^{-1} x$ $\boxed{(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}}$

Recall: ① domain: $[-1, 1]$. range: $[0, \pi]$.

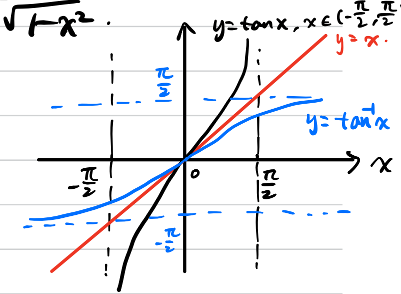


② $y = \cos^{-1} x \xRightarrow{\text{means}} \cos y = x$.

Similarly, $\frac{d}{dx}(x - \cos y) = \frac{d}{dx}(0) \Rightarrow 1 + \sin y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$.

Notice that $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$. $\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

$\sin y \geq 0 \quad y \in [0, \pi]$.



(3). $y = \arctan x = \tan^{-1} x$ $\boxed{(\tan^{-1} x)' = \frac{1}{x^2+1}}$

Recall: ① domain: $(-\infty, +\infty)$. range: $(-\frac{\pi}{2}, \frac{\pi}{2})$.

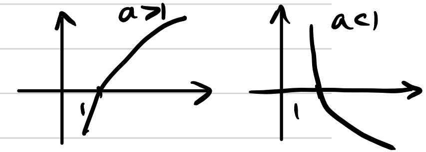
② $y = \tan^{-1} x \xRightarrow{\text{means}} \tan y = x$.

$\frac{d}{dx}(x - \tan y) = \frac{d}{dx}(0) = 0 \Rightarrow 1 - \frac{1}{\cos^2 y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \cos^2 y$.

Notice that $\cos^2 y = \frac{\cos^2 y}{1} = \frac{\cos^2 y}{\sin^2 y + \cos^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$. $\Rightarrow \frac{dy}{dx} = \frac{1}{x^2+1}$.

(4). $y=\log_a x$, $(a>0, a\neq 1)$. $\boxed{(\log_a x)'=\frac{1}{\ln a}\cdot\frac{1}{x}}$ $\boxed{(\ln x)'=\frac{1}{x}}$.

Recall: ① domain $(0,+\infty)$, range $(-\infty,+\infty)$.



② $y=\log_a x \overset{\text{means}}{\Longrightarrow} a^y = x$.

$\frac{d}{dx}(x-a^y)=\frac{d}{dx}(0)=0 \Rightarrow 1-\ln a\cdot a^y\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{1}{\ln a}\cdot\frac{1}{a^y}=\frac{1}{\ln a}\cdot\frac{1}{x}$

Another two definitions for "e": $f(x)=\ln x$. $f'(x)=\frac{1}{x}$.

On the one hand, we have $f'(1)=1$ because $f'(x)=\frac{1}{x}$.

On the other hand, we have

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\ln(1+h)}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\ln(1+h)=\lim_{h\to 0}\ln\left((1+h)^{\frac{1}{h}}\right).$$

Therefore, $\lim_{h\to 0}\ln\left(\underline{(1+h)^{\frac{1}{h}}}\right)=1$

Since $\ln e=1$ and $y=\ln x$ is continuous at $x=e$, we obtain $\boxed{\lim_{h\to 0}(1+h)^{\frac{1}{h}}=e.}$

Let $n=\frac{1}{h}$, then we also have $\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e}$ → "e" can be defined as the value of these two limits.

All these derivatives satisfy the rule (*) in the next page.

To summerize:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}. \quad \frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}.$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\cdot\ln a} \qquad \frac{d}{dx}\ln x = \frac{1}{x}.$$

We can combine these derivatives with the chain rule.

Take $\frac{d}{dx}\ln x = \frac{1}{x}$ as an example.

Suppose $y=f(x)$. Then $\frac{d}{dx}\ln f(x) = \frac{d}{dx}\ln y \overset{\text{chain rule}}{=} \frac{d\ln y}{dy}\cdot\frac{dy}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$

1). Example 1: If $y=x^2+1$, then $\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot\frac{d(x^2+1)}{dx} = \frac{2x}{x^2+1}$

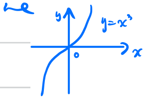
2). Example 2: If $y=\ln x$, then $\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\cdot\frac{d\ln x}{dx} = \frac{1}{\ln x}\cdot\frac{1}{x} = \frac{1}{x\ln x}$

(5) The general formula for derivatives of inverse functions:

$\left.\begin{array}{l} f(x) \text{ is one-to-one} \\ g(x) \text{ is the inverse function of } f(x) \end{array}\right\} \Rightarrow \dfrac{d}{dx} g(x) = \dfrac{1}{\underbrace{f'(g(x))}}$. $(*)$

→ derivative of f at $g(x)$.

Example: $g(x) = x^{\frac{1}{3}}$ is the inverse function of $\underbrace{f(x) = x^3}$. → one-to-one



$$x \underset{g}{\overset{f}{\rightleftarrows}} x^3$$

On the one hand, by the rule $(x^n)' = n \cdot x^{n-1}$, we have $(x^{\frac{1}{3}})' = \underline{\underline{\frac{1}{3} \cdot x^{-\frac{2}{3}}}}$. → same result.

On the other hand, by rule $(*)$, we have $\dfrac{d}{dx} g(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{\underbrace{3 \cdot (g(x))^2}} = \dfrac{1}{3 \cdot x^{\frac{2}{3}}} = \underline{\underline{\frac{1}{3} \cdot x^{-\frac{2}{3}}}}$

→ derivative of $y = x^3$ at $g(x)$.

Proof of $(*)$: Notice that $\underline{f(g(x)) = x}$.

$$x \underset{g}{\overset{f}{\rightleftarrows}} y$$

$f(x) = y$. $g(y) = x$

→ $f(g(y)) = y$.

Differentiate both sides: $\underline{\dfrac{d}{dx} f(g(x)) = \dfrac{d}{dx}(x) = 1}$.

By the chain rule: $\dfrac{d}{dx} f(\underbrace{g(x)}_{=u}) = \dfrac{d f(u)}{du} \cdot \dfrac{du}{dx} = f'(u) \cdot g'(x)$

$\Rightarrow f'(u) \cdot g'(x) = 1 \Rightarrow g'(x) = \dfrac{1}{f'(u)} = \dfrac{1}{f'(g(x))}$.

4. Logarithmic differentiation: A trick to calculate $\frac{dy}{dx}$ if y is in the forms $\frac{f_1(x)\cdots f_n(x)}{g_1(x)\cdots g_m(x)}$ or $f(x)^{g(x)}$.

(e.g. $y=(1+x)(1+x^2)(1+x^4)$. $y=\frac{(x+1)^5\sqrt{x^2+1}}{(3x+4)^3(x^2-1)}$. $y=x^x$).

Example 1: Find $\left.\frac{dy}{dx}\right|_{x=1}$ if $y=(1+x)(1+x^2)(1+x^4)$.

Step 1: Take natural logarithms of both sides of $y=f(x)$

$$\ln y = \ln\left((1+x)(1+x^2)(1+x^4)\right) \overset{\text{laws of logarithms.}}{=} \ln(1+x)+\ln(1+x^2)+\ln(1+x^4).$$

Step 2: Differentiate both sides with respect to x.

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\ln(1+x)+\ln(1+x^2)+\ln(1+x^4)\right)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}\ln(1+x)+\frac{d}{dx}\ln(1+x^2)+\frac{d}{dx}\ln(1+x^4) \overset{\text{chain rule}}{=} \frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}.$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left(\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}\right).$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 8\cdot\left(\frac{1}{2}+\frac{2}{2}+\frac{4}{2}\right) = 8\cdot\frac{7}{2} = 28.$$

Example 2. Find $\frac{dy}{dx}$ if $y = x^x$.

Step 1. Take natural logarithms of both sides of $y = f(x)$.

$$\ln y = \ln(x^x) \overset{\text{Laws of logarithms}}{=} x \cdot \ln x.$$

Step 2. Differentiate both sides with respect to x.

$$\frac{d}{dx}\ln y = \frac{d}{dx}(x \cdot \ln x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}x \cdot \ln x + x \cdot \frac{d}{dx}\ln x = \ln x + 1.$$

$$\Rightarrow \frac{dy}{dx} = y \cdot (\ln x + 1) = x^x \cdot (\ln x + 1).$$