Review: L'Hopital's Rule

We have $\boxed{\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}}$ ($x\to c$ can be replaced by $x\to c^-$ or $x\to c^+$ or $x\to+\infty$ or $x\to-\infty$).

if ① $\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}g(x)=0$ or $\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}g(x)=\infty$.

② $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$ exists or $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}=\infty$.

We can use L'Hopital's Rule to calculate limits.

① $\frac{0}{0}$ type: $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{x}$ ② $\frac{\infty}{\infty}$ type: $\lim\limits_{x\to+\infty}\frac{3x^2+4x}{e^x}$.

Example: $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{x}\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}}\lim\limits_{x\to 0}\frac{(e^x-e^{-x})'}{(x)'}=\lim\limits_{x\to 0}\frac{e^x-(-e^{-x})}{1}=\lim\limits_{x\to 0}\frac{e^x+e^{-x}}{1}=2.$

③ $0\cdot\infty$ type: $\lim\limits_{x\to 0^+}x^2\ln x=\underbrace{\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}}}_{\to\frac{\infty}{\infty}\text{ type}}\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}}\lim\limits_{x\to 0^+}\frac{(\ln x)'}{(\frac{1}{x^2})'}=\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-2}{x^3}}=\lim\limits_{x\to 0^+}\frac{x^2}{-2}=0.$

Take logarithm ← ④ 1^∞ type $\lim\limits_{x\to+\infty}(1+\frac{2}{x})^x$ ⑤ 0^0 type $\lim\limits_{x\to 0^+}x^{\tan x}$. ⑥ ∞^0 type $\lim\limits_{x\to+\infty}x^{\frac{1}{x}}$.

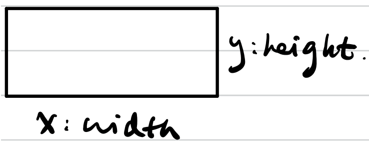
Example: $\lim\limits_{x\to+\infty}x^{\frac{1}{x}}$. We first calculate $\lim\limits_{x\to+\infty}\ln(x^{\frac{1}{x}})=0\Rightarrow\lim\limits_{x\to+\infty}x^{\frac{1}{x}}=1.$

$\lim\limits_{x\to+\infty}\ln(x^{\frac{1}{x}})=\underbrace{\lim\limits_{x\to+\infty}\frac{1}{x}\ln x}_{0\cdot\infty\text{ type}}=\underbrace{\lim\limits_{x\to+\infty}\frac{\ln x}{x}}_{\frac{\infty}{\infty}\text{ type}}\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=}}\lim\limits_{x\to+\infty}\frac{(\ln x)'}{(x)'}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{1}=\lim\limits_{x\to+\infty}\frac{1}{x}=0$

Optimization Problems: The problems of find the absolute maximum or absolute minimum $\rightarrow$ use derivatives.

Example 1. Suppose that the perimeter of a rectangle is L meters.

Question: What is the largest possible area of this rectangle?



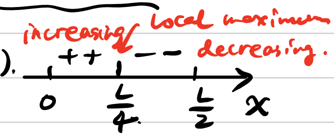
Perimeter: $L = 2(x+y)$. $\Rightarrow y = \frac{L}{2} - x$.

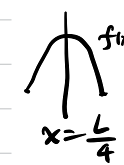
Area: $A = x \cdot y = x\left(\frac{L}{2} - x\right)$. $x \in \left(0, \frac{L}{2}\right)$.

Our aim is to maximize the area. $\Rightarrow$ maximize $f(x) = x\left(\frac{L}{2} - x\right)$ for $x \in \left(0, \frac{L}{2}\right)$.

Step 1: Calculate $f'(x)$ and find all critical numbers. $\underbrace{f'(x) = \frac{L}{2} - 2x}$. critical number: $\frac{L}{4}$.

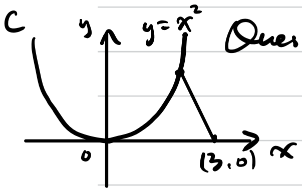
Step 2: Find the absolute maximum or minimum. sign of $f'(x)$.





$\Rightarrow f\left(\frac{L}{4}\right) = \frac{L^2}{16}$ is the largest area.

Example 2. Suppose that C is the curve of $y=x^2$.



Question: Which point on C is closest to (3, 0). ?

point on C : (x, x^2). $x \in \mathbb{R}$.

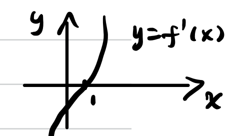
its distance from (3, 0) is $d=\sqrt{(x-3)^2+(x^2-0)^2}$.

It is more convenient to minimize d^2 because the calculation is much simpler.

Our aim is to minimize d. $\overset{\text{equivalent}}{\Longrightarrow}$ minimize $d^2 = f(x) = (x-3)^2+(x^2-0)^2 = (x-3)^2+x^4$.

Step 1: Calculate $f'(x)$ and find critical numbers. $f'(x)=2(x-3)+4\cdot x^3 = 4x^3+2x-6$.

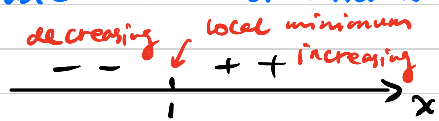
Notice: ① $f'(1)=4+2-6=0$. ② $f'(x)$ is increasing on $(-\infty,+\infty)$. (because $f''(x)=12\cdot x^2+2>0$).

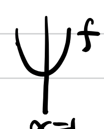


$\Rightarrow f'(x)$ has only one root: $x=1$. $\Rightarrow$ critical number: $x=1$.

Step 2. Find absolute maximum or minimum.

sign of $f'(x)$:



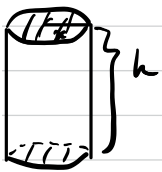


$x=1$ minimizes $f(x)$

$\Rightarrow$ The point (1, 1) is closest to the point (3. 0).

Example 3. Suppose that the volume of a cylindrical can is L m^3.

Question: What is the smallest possible area of this can?



Volume : $L = \pi \cdot x^2 \cdot h.$ $\Rightarrow h = \dfrac{L}{\pi \cdot x^2}.$

Area : $A = \pi \cdot x^2 + \pi \cdot x^2 + 2\pi x \cdot h = \pi \cdot x^2 + \pi \cdot x^2 + 2\pi x \cdot \dfrac{L}{\pi \cdot x^2}.$

$= 2\pi x^2 + \dfrac{2L}{x}.$

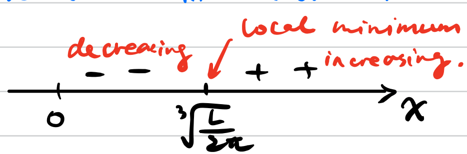
Our aim is to minimize the area $\Rightarrow$ minimize $f(x) = 2\pi x^2 + \dfrac{2L}{x}$ for $x > 0$.

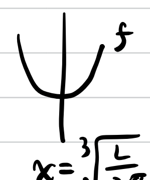
Step 1: Calculate $f'(x)$ and find critical numbers.

$f'(x) = 4\pi \cdot x - \dfrac{2L}{x^2} = \dfrac{4\pi \cdot x^3 - 2L}{x^2}$ critical number $x = \sqrt[3]{\dfrac{L}{2\pi}}.$

Step 2. Find absolute maximum or minimum.

sign of $f'(x)$:





$\Rightarrow f\left(\sqrt[3]{\dfrac{L}{2\pi}}\right)$ is the smallest area.

Antiderivatives

Definition: If $F'(x) = f(x)$, then $f(x)$ is the derivative of $F(x)$,

and $F(x)$ is called an antiderivative of $f(x)$.

Example: $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because $\frac{d}{dx}(x^2) = 2x$.

$G(x) = x^2 + 1$ is also an antiderivative of $f(x) = 2x$ because $\frac{d}{dx}(x^2+1) = 2x$.

In general, if $F(x)$ is an antiderivative of $f(x)$,

then all antiderivatives of $f(x)$ have the form $F(x) + C$. → an arbitrary constant

Example: $x^2 + C$ is an antiderivative of $2x$ for any constant C.

All antiderivatives of $2x$ have the form $x^2 + C$, where C is a constant.

Example: $x^3 + C$ is an antiderivative of $3x^2$ for any constant C

All antiderivatives of $3x^2$ have the form $x^3 + C$, where C is a constant.

The antiderivative of $f(x)$ is determined by $f(x)$ up to a constant.

called "indefinite integral".

$$\underbrace{\int f(x)\,dx}_{} \overset{\text{denote}}{=} \text{all antiderivatives of } f(x).$$

$$= \underbrace{F(x)}_{\text{one antiderivative of } f(x)} + \underbrace{C}_{\text{an arbitrary constant}}.$$

Example: $\int x\,dx = \frac{1}{2}\cdot x^2 + C$ because $\frac{d}{dx}\left(\frac{1}{2}\cdot x^2 + C\right) = x$.

Example: $\int x^2\,dx = \frac{1}{3}\cdot x^3 + C$ because $\frac{d}{dx}\left(\frac{1}{3}\cdot x^3 + C\right) = x^2$.

In general, $\int x^p\,dx = \frac{1}{p+1}\cdot x^{p+1} + C$ for $p \neq -1$. because $\frac{d}{dx}\left(\frac{1}{p+1}\cdot x^{p+1} + C\right) = x^p$.

$\int \frac{1}{x}\,dx = \ln x + C$ if $x > 0$ because $\frac{d}{dx}(\ln x + C) = \frac{1}{x}$.

Example: $\int \cos x\,dx = \sin x + C$ because $\frac{d}{dx}(\sin x + C) = \cos x$.

$\int \sin x\,dx = -\cos x + C$ because $\frac{d}{dx}(-\cos x + C) = \sin x$.

Rule: If $\int f(x)\,dx = F(x) + C$ and $\int g(x)\,dx = G(x) + C$.

then $\int [a f(x) + b \cdot g(x)]\,dx = a \cdot F(x) + b \cdot G(x) + C$.

(proof: $\frac{d}{dx}[a F(x) + b \cdot G(x) + C] = a \frac{d}{dx} F(x) + b \cdot \frac{d}{dx} G(x) = a f(x) + b \cdot g(x)$).

Example: Find $\int (3x^4 + x^2 + 2 + 4\sin x)\,dx$.

We need to find an antiderivative of x^4, x^2, 1 and $\sin x$.

Notice $\frac{d}{dx}\left(\frac{1}{5}x^5\right) = x^4$. $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$.

$\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(-\cos x) = \sin x$.

Therefore, $\int (3x^4 + x^2 + 2 + 4\sin x)\,dx$

$$= 3 \cdot \frac{1}{5} \cdot x^5 + \frac{1}{3} \cdot x^3 + 2 \cdot x + 4 \cdot (-\cos x) + C.$$

$$= \frac{3}{5}x^5 + \frac{1}{3}x^3 + 2x - 4\cos x + C.$$