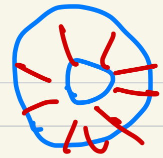
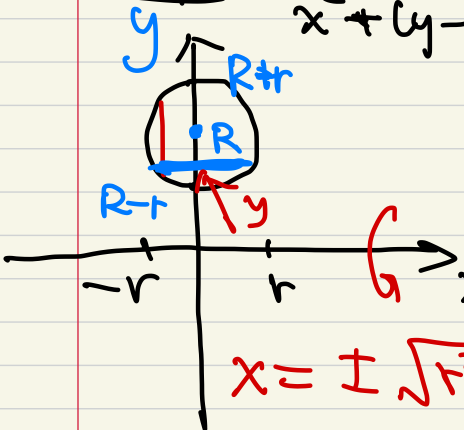
Method of Cylindrical Shells



Example

$x^2+(y-R)^2=r^2$



$x=\pm\sqrt{r^2-(y-R)^2}$

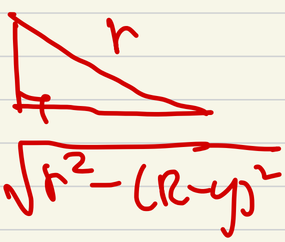
by slicing

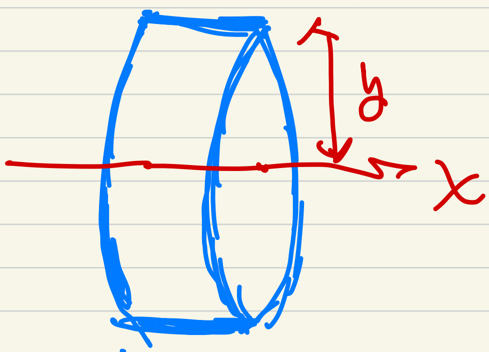
$$\text{volume}=\int_{-r}^{r}\pi\left(R+\sqrt{r^2-x^2}\right)^2-\pi\left(R-\sqrt{r^2-x^2}\right)^2\,dx$$

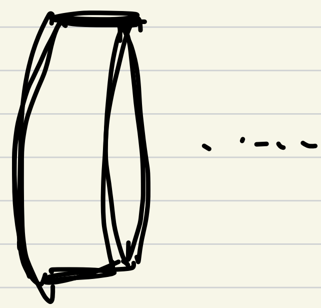
$$=\boxed{2\pi R\cdot\pi r^2}$$

Cylindrical Shell Method.

solid of revolution about the x-axis is a torus.



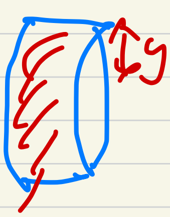




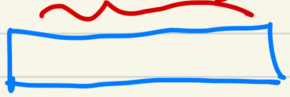
$\underbrace{\qquad}_{2\sqrt{r^2-(y-R)^2}}$

"Sum of thin cylindrical shell volumes as an integral"

$$\text{volume}=\int_{R-r}^{R+r}$$



dy



$$=\int_{R-r}^{R+r}2\pi y\cdot2\sqrt{r^2-(y-R)^2}\,dy\,,$$ Let $u=y-R$, $du=dy$

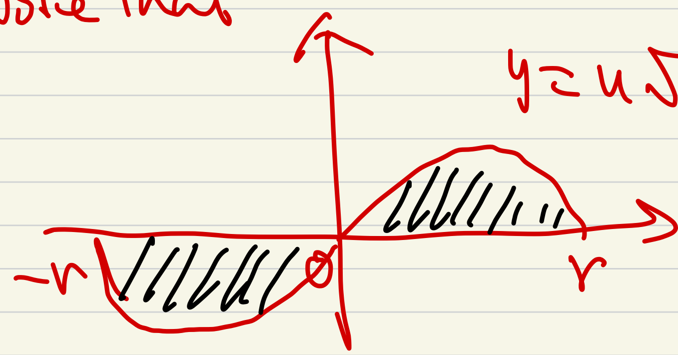
$$=4\pi\int_{-r}^{r}(u+R)\sqrt{r^2-u^2}\,du$$

$\text{area}(\text{half disk})=\pi r^2/2$

$$=4\pi\boxed{\int_{-r}^{r}u\sqrt{r^2-u^2}\,du}+4\pi\boxed{\int_{-r}^{r}\sqrt{r^2-u^2}\,du}=2\pi R\cdot\pi r^2$$

odd function $\to 0$

Note that



$$\int_{-r}^{r} u\sqrt{r^2-u^2}\,du = 0$$

Example

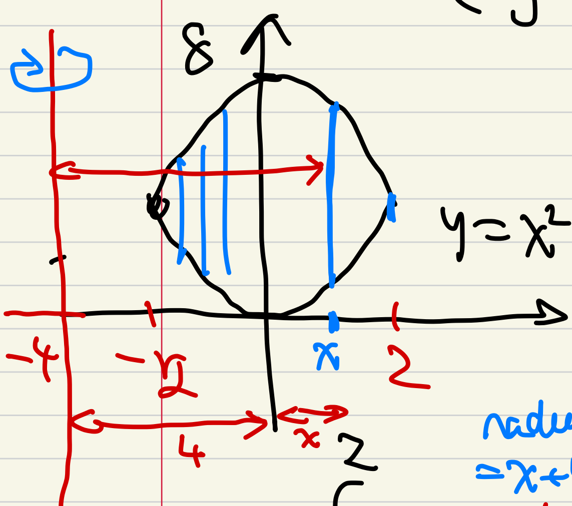
$$\begin{cases} y = 8 - x^2 \\ y = x^2 \end{cases}$$

Intersection Points:

$8 - x^2 = x^2$

$x^2 = 4$

$\begin{cases} x = 2 \\ y = 4 \end{cases}$ $\quad$ $\begin{cases} x = -2 \\ y = 4 \end{cases}$

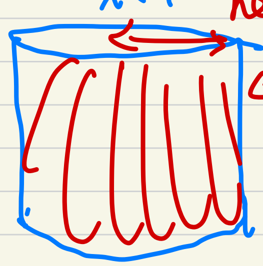


Rotate the enclosed area about the line $x = -4$.

radius $= x+4$

height $= 8 - x^2 - x^2 = 8 - 2x^2$

dx

Volume $= \int_{-2}^{2}$ 

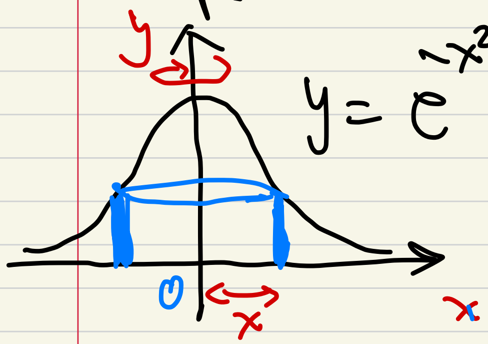
$$= \int_{-2}^{2} 2\pi(x+4)\cdot(8-2x^2)\,dx$$

$$= 4\pi \int_{-2}^{2} (x+4)(4-x^2)\,dx = 4\pi \int_{-2}^{2} (4x - x^3 + 16 - 4x^2)\,dx$$

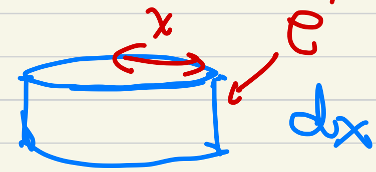
$$= 4\pi\left[\underline{\underline{2x^2 - \frac{x^4}{4}}} + 16x - \frac{4}{3}x^3\right]_{-2}^{2} = 4\pi\cdot\left[32 - \frac{32}{3} + 32 - \frac{32}{3}\right]$$

$$= 4\pi\cdot\frac{128}{3}$$

Example :



$y = e^{-x^2}$

vol() $= \int_0^\infty$ 

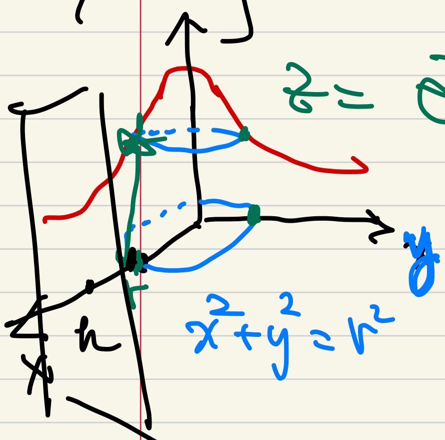
$$= \int_0^\infty 2\pi x\, e^{-x^2}\, dx$$

$$= \left[-\pi e^{-x^2} \right]_0^\infty$$

$$= \underbrace{\lim_{x\to\infty} -\pi e^{-x^2}}_{0} + \pi = 0 + \pi$$

$$\boxed{\begin{aligned} I^2 &= \pi \\ I &= \sqrt{\pi} \end{aligned}}$$

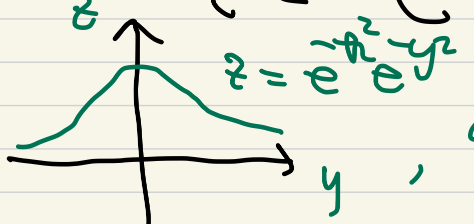
What if we conside the volume by the method of slicing?



$$z = e^{-r^2} = e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$$

A slice of the solid at $x=k$

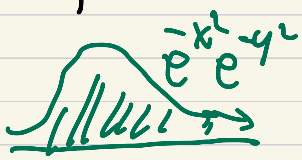
$$z = e^{-k^2} e^{-y^2}$$



, area of the slice $= \int_{-\infty}^{\infty} e^{-k^2} e^{-y^2}\, dy$

$$= e^{-k^2} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\, dy}_{I}$$

$$= e^{-k^2} I$$

Volume of the solid $= \int_{-\infty}^{\infty}$  dx

$$= \int_{-\infty}^{\infty} e^{-x^2} \cdot I\, dx = I \int_{-\infty}^{\infty} e^{-x^2} dx$$

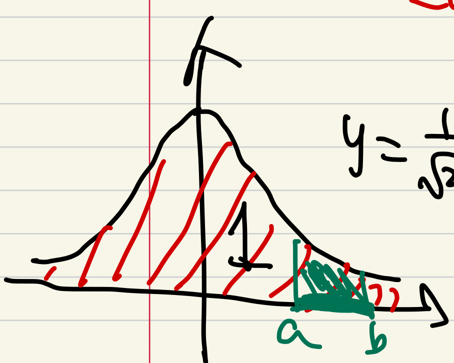
$$= I^2$$

i.e. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Let $x = t/\sqrt{2}$, $dx = \frac{1}{\sqrt{2}}\,dt$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-t^2/2}\,\frac{1}{\sqrt{2}}\cdot dt$$

i.e. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = 1.$



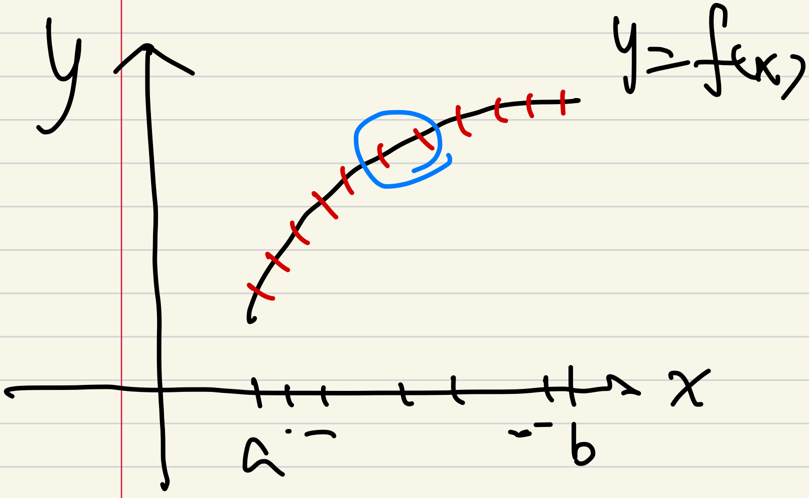
$y = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ = probability density function of the standard normal distribution.

$$P(a < X < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt.$$

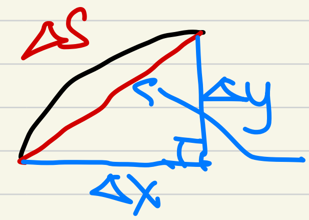
Probability

a standard normal random number

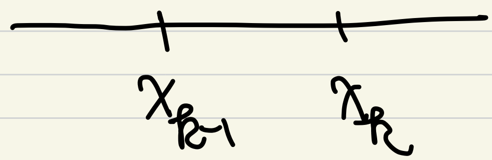
Arc length of the graph of $y = f(x)$



L = length of the graph



$\sqrt{(\Delta x)^2 + (\Delta y)^2}$



$$\Delta x = \frac{b-a}{n}$$

tiny arc length

$$\Delta S \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

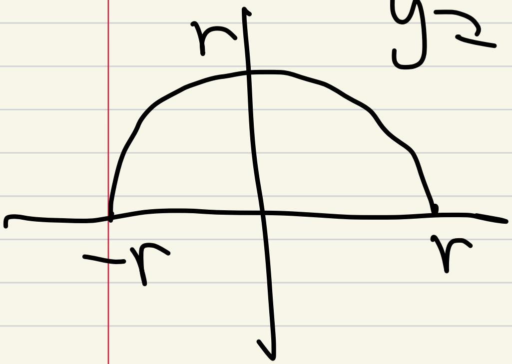
summing by "integration"

$\left(\frac{dy}{dx}\right)^2$ as $\Delta x \to 0$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

"extension of Pythagoras Theorem"

Example



$$y = \sqrt{r^2 - x^2}, \quad \frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \cdot 2x$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\text{arc length} = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= r \int_{-r}^{r} \frac{1}{\sqrt{r^2 - x^2}}\, dx$$

$x = r\sin\theta$

$dx = r\cos\theta\, d\theta$

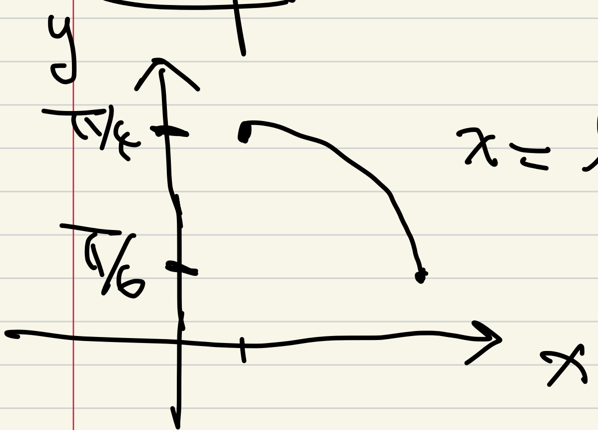
$$= r \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{r\cos\theta}} \cdot \cancel{r\cos\theta}\, d\theta$$

$$= r\,[\theta]_{-\pi/2}^{\pi/2}$$

$$= \pi r$$

$$\boxed{\text{Length of a circle} = 2\pi r}$$

Example: $x = \ln \cos y$, $\frac{\pi}{6} \le y \le \frac{\pi}{4}$.



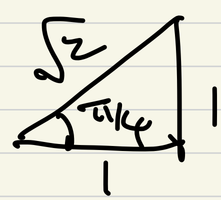
$x = \ln \cos y$, $\frac{dx}{dy} = \frac{-\sin y}{\cos y}$

$= -\tan y$

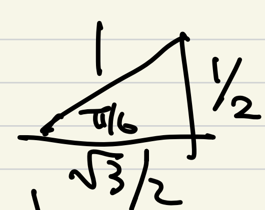
arc length $= \int_{\pi/6}^{\pi/4} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_{\pi/6}^{\pi/4} \sqrt{\underbrace{1 + (-\tan y)^2}_{\sec^2 y}}\, dy$

$= \int_{\pi/6}^{\pi/4} \sec y \, dy$

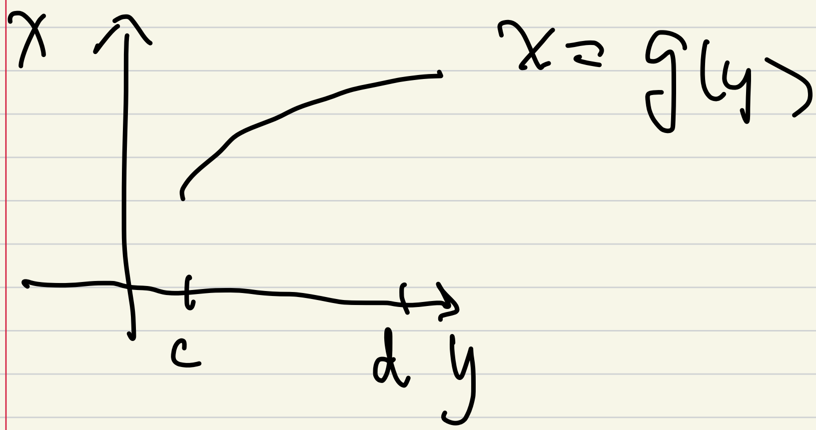


$= \Big[\ln|\sec y + \tan y|\Big]_{\pi/6}^{\pi/4}$



$= \ln(\sqrt{2} + 1) - \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right|$

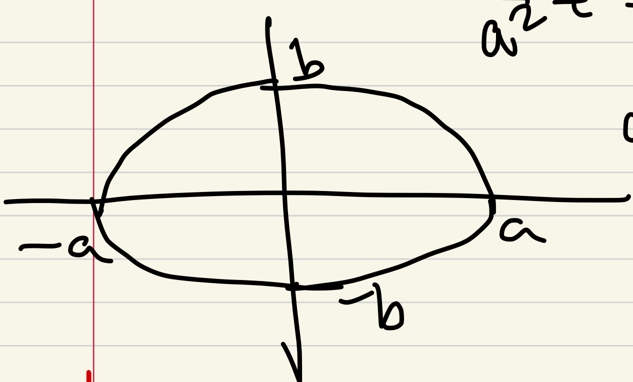
$= \ln(\sqrt{2} + 1) - \ln\sqrt{3}$



arc length

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Ellipse



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a>b>0$.

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{dy}{dx}=b\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\left(-\frac{x}{a^2}\right)$$

$$L=2\int_{-a}^{a}\sqrt{1+\frac{b^2}{a^2}\,\frac{x^2}{1-\frac{x^2}{a^4}}}\,dx$$