Math201 Answer to Homework 1

EXERCISE 1.1
1. T,
2. F,
3. T,
4. F,
5. T,
6. T, when $n \leqslant 2$; F, when $n > 3$,
7. T, when $a > 0$; F, when $a \leqslant 0$.

EXERCISE 1.2
1, 3 and 5 are equivalent; 2 and 4 are equivalent.
1, 3, 5 are opposite to 2, 4.

EXERCISE 1.4
1. T. 2. T. 3. F. 4. T. 5. T. 6. F. 7. T.

EXERCISE 1.5
3, 4 and 6 are equivalent to "n is divisible by 6".
9 is opposite.

EXERCISE 1.6
1. 10 is not divisible by 2 or 5.
2. 10 is divisible by 3 or 7.
3. There are integers m and n, such that $10 = m^2 + n^2$.
4. x=y.
5. one of l, m, n is odd and one of l, m, n is even.
6. l, m, n are odd or l, m, n are even.
7. A number is nonnegative and doesn't have square roots.
8. ($y \geqslant x$ and $y \geqslant z$) or ($y \leqslant x$ and $y \leqslant z$).
9. $x \leqslant y$ or $y \leqslant z$.
10. $x \leqslant y \leqslant z$.
11. $x \leqslant y$ or ($x \leqslant z$ and $y \leqslant z$).

EXERCISE 1.7
1. T. 2. F. 3. T. 4. F.

EXERCISE 1.9
1 and 4, 2 and 3, 5 and 8, 6 and 7 are opposite.

EXERCISE 1.11
1. $m + n$ is even, if m, n are even; m, n are even only if $m + n$ is even.
That m, n are even is sufficient condition for $m + n$ to be even.
That $m + n$ is even is necessary condition for m, n to be even.

2. xy is positive, if x, y are positive; x, y are positive only if xy is positive.

That x, y are positive is sufficient condition for xy to be positive.
That xy is positive is necessary condition for x, y to be positive.

3. $x = 1$ or 2 if $x^2 - 3x + 2 = 0$; $x^2 - 3x + 2 = 0$ only if $x = 1$ or 2.
That $x^2 - 3x + 2 = 0$ is sufficient condition for $x = 1$ or 2.
That $x =$1 or 2 is necessary condition for $x^2 - 3x + 2 = 0$.

6. We exercise if we become healthy; we will become healthy only if we exercise.
Becoming healthy is sufficient condition for us to exercise.
Exercising is necessary condition for us to become healthy.

9. We are ready if we go. We will go only if we are ready.
That we go is the sufficient condition for us to be ready.
That we are ready is necessary condition for us to go.

10. No fish if no water; no water only if no fish.
No water is the sufficient condition of no fish.
No fish is the necessary condition of no water.

Exercise 1.12
1. If $m + n$ is odd, then one of m, n is odd.
2. If xy is non-positive, then one of x, y is non-positive.
3. If $x^2 - 3x + 2 \neq 0$, then $x \neq 1$ and 2.
6. If we don't exercise then we will become unhealthy.
9. If we are not ready then we don't go.
10. If there is fish then there is water.

Exercise 1.13
1. If $xy < 0$ and one of x, y is negative, then $x < 0$ and $y < 0$.
2. If $x^2 - 3x + 2 = 0$, then $x = 1$ or 2.
3. If a real number is not positive then it is zero or negative.
4. If a person is not male then she is female.
5. If it is too expansive and we buy it, then we get more money.
6. If we get ill then we can find a doctor or take medicine by ourselves.
7. If we want to travel then we take bus or train or plane.

Exercise 1.15
1. n is a multiple of 2 and 3.
2. n is not a multiple of 2 and 3.
3. $n < 10$ and x_n is odd.
4. $m > n$ and $x_m > x_n$.
5. $m < n$ and $x_m \leqslant n$.

Exercise 1.16
1. T. 2. T. 3. F. 4. F. 5. T. 6. T. 7. F. 8. T. 9. T. 10. T.

Exercise 1.17

1. T. 2. F. 3. F. 4. T. 5. T. 6. F. 7. T.

Exercise 1.18
1. F. 2. T. 3. T. 4. F. 5. F. 6. T. 7. T.

Exercise 1.19
A function $f(x)$ is not uniformly continuous if there exists $\epsilon_0 > 0$, for any $\delta > 0$, there exist x_0 and y_0 such that $|x_0 - y_0| < \delta$ but $|f(x_0) - f(y_0)| \geqslant \epsilon_0$.

Exercise 1.20
A sequence $\{x_n\}$ is not Cauchy if there exists $\epsilon_0 > 0$ such that for any N, there are $m_0, n_0 > N$ satisfly $|x_{m_0} - x_{n_0} \geqslant \epsilon_0|$.

Exercise 1.21
1. There is $n > 0$ such that for any m, m doesn't divide n.
2. For any m and n $m^3 + 2mn + n^2 \neq 0$.
3. There is a moment that you cannot fool all people.
4. Some people don't make mistakes at anytime.
5. There is a price which is so high that no one would pay for it.
6. All students show up in some lectures.
7. Some students in the class don't love math.
8. Some students in the class are not 19 years old or not born in HK.
9. All students are not from Beijing or Shanghai.

Exercise 1.22
1. There is m such that for any n $A(m,n)$ doesn't happen.
2. There is m such that for any $n > m$ $A(m,n)$ happens.
3. For any m there is $n > m$ such that $A(m,n)$ does not happen.
4. There is $\epsilon > 0$ such that for any n, there is m satisfying $2m \geqslant n+1$ that we don't have $A(m,n,\epsilon)$.
5. For any $\epsilon > 0$ and any m, we don't have $A(m,n)$ for some n.
6. For any a, b, n, $A(a,b,n)$ doesn't happen.

Exercise 1.23
1. Suppose n is even. Then $n = 2k$ for some integer k. Then $n + 3 = 2k + 3 = 2(k+1) + 1$ is odd.
Suppose $n+3$ is odd. Then $n+3 = 2k+1$ for some integer k. Then $n = (2k+1) - 3 = 2(k-2)$ is even.

2. Suppose n is even , then $n = 2k$ for some integer k. Then $n(n+1) = 2k(2k+1)$ is even.
Suppose n is odd, then $n = 2k+1$ for some integer k. Then $n(n+1) = (2k+1)(2k+2) = 2(k+1)(2k+1)$ is even.

3. By n odd, we have $n = 2k+1$ for some integer k. Then $n^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. By Exercise 1.23 (2), we have $k(k+1) = 4m$ for some integer m. Therefore $n^2 = 4 \cdot 2m + 1 = 8m + 1$.

Exercise 1.24
For $m \leqslant n$,

$$m + (m+1) + (m+2) + \cdots + (n-1) + n = m(n-m+1) + 1 + 2 + \cdots + (n-m).$$

By Example 1.4.2 we have

$$1 + 2 + \cdots + (n-m) = \frac{(n-m+1)(n-m)}{2}.$$

Then

$$\begin{aligned} m + (m+1) + (m+2) + \cdots + (n-1) + n &= m(n-m+1) + 1 + 2 + \cdots + (n-m) \\ &= m(n-m+1) + \frac{(n-m+1)(n-m)}{2} \\ &= \frac{(n-m+1)(n+m)}{2}. \end{aligned}$$

Exercise 1.25
2.
(if part):
Suppose $n = 3p + r$ where $r = 0, 1$ or 2. Then $n^2 = 9p^2 + 6pr + r^2$.
Since n^2 and $9p^2$ are divisible by 9, $6pr + r^2$ is divisible by 9.
If $r = 1$ or 2, then $6pr + r^2$ is not divisible by 3, thus not divisible by 9.
Thus r can only be 0, which means $n = 3p$ for some integer p. Thus n is divisible by 3.

(only if part):
By n is divisible by 3, $n = 3p$ for some integer p.
Then $n^2 = 9p^2$. Thus n^2 is divisible by 9.

4.
If n^2 is divisible by 6, then n^2 is divisible by 2 and 3.
Then by Exercise 1.25 (1, 3), we have n is divisible by 2 and 3.
Thus n is divisible by 6.

Exercise 1.26
1.
(if part):
Since n is divisible by 5 and 7, $n = 5p$ and $n = 7q$ for some integers p, q.
Then there is

$$p = 3 \cdot 5p - 2 \cdot 7p = 3 \cdot 7q - 2 \cdot 7p = 7(3q - 2p).$$

Then $n = 5p = 35(3q - 2p)$. Thus n is divisible by 35.

(only if part):
35 is divisible by 5 and 7. Thus $n = 35k$ for some integer k implies that $n = 5 \cdot 7k$ and $n = 7 \cdot 5k$.

2.
(if part):
The fact n is divisible by 2, 3 and 5 tells us that $n = 2p = 3q = 5r$ for some integers p, q, r.

Then there is $p = 3p - 2p = 3p - 3q = 3(p - q)$ since $2p = 3q = n$.
Then $n = 2p = 6(p - q)$.
There is also $r = 6r - 5r = 6r - 6(p - q) = 6(r - p + q)$ since $5r = 6(p - q) = n$.
Then $n = 5r = 5 \cdot 6(r - p + q) = 30(r - p + q)$. Thus n is divisible by 30.

(only if part):
By n is divisible by 30, $n = 30k$ for some integer k.
Then $n = 2 \cdot 15k = 3 \cdot 10k = 5 \cdot 6k$.
These imply n is divisible by 2, 3 and 5.

3.
(if part):
By n is divisible by 6 and 15, we know that $n = 6p = 15q$ for some integers p, q.
Then there is:

$$\begin{aligned} n &= 2 \cdot 3p \\ &= 2 \cdot (15p - 2 \cdot 6p) \\ &= 2 \cdot (15p - 2 \cdot 15q) \\ &= 30 \cdot (p - 2q). \end{aligned}$$

Thus n is divisible by 30.

(only if part):
By n is divisible by 30, $n = 30k$ for some integer k.
Then $n = 6 \cdot 5k = 15 \cdot 2k$.
These imply n is divisible by 6 and 15.

Exercise 1.27(5)
Assume that $\sqrt{2} + \sqrt{3}$ is rational, then $(\sqrt{2} + \sqrt{3})^2$ is rational.
This implies that $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is rational.
If $5 + 2\sqrt{6}$ is rational then $\sqrt{6}$ is rational.
However, by Exercise 1.27(3) $\sqrt{6}$ is irrational, which is a contradiction. Thus $\sqrt{2} + \sqrt{3}$ is irrational.

Exercise 1.28
1.
(if part):
$m = 2k + 1$ and $n = 2l + 1$ for some integers k, l.
Then $mn = (2k + 1)(2l + 1) = 4kl + 2(k + l) + 1 = 2(2kl + k + l) + 1$.

(only if part):
We prove the contrapostive.
Without loss of generality, we assume that m is even.
Then $m = 2k$ for some integer k.
$mn = 2kn = 2 \cdot kn$ which is even.

2.
If n is even, $n = 6k + 2r$ for some integer k and $r = -1$, 0 or 1.

Then $n^3-n = 8(27k^3+27k^2r+9kr^2+r^3)-6k-2r = 6(36k^3+36k^2r+12kr^2-k)+2(4r^3-r)$. For $r=-1$, 0 or 1, $4r^3-r$ is a multiple of 3. Thus n^3-n is a multiple of 6.

3.
If n is odd, then $n=2m+1$ for some integer m.
Then $n^2=4(m^2+m)+1$. Here $k=m^2+m$
If n is even, then $n=2m$ for some integer m.
Then $n^2=4m^2$. Here $k=m^2$.

(4) Suppose three consecutive integers are $n-1, n$ and $n+1$.

Then their square sum is $(n-1)^2+n^2+(n+1)^2=3n^2+2$.

By Exercise 1.28(3), n^2 is of the form $4k$ or $4k+1$ for some integer k. Thus $3n^2+2$ is either of the form $12k+2$ or $12k+5$ for some integer k. Thus the sum of the square of three consecutive integers cannot be of the form $12k-1$ for some integer k.