Review : Antiderivative and indefinite integral.

If $\underline{\frac{d}{dx}F(x) = f(x)}$, then ① $f(x)$ is the derivative of $F(x)$.

② $F(x)$ is an antiderivative of $f(x)$

③ all antiderivatives of f have the form $F(x)+\underline{C}$ ← an arbitrary constant

④. we write $\int f(x)\,dx = F(x) + C$.

called "indefinite integral"

$\int f(x)\,dx \overset{\text{defined as}}{=}$ all antiderivatives of $f(x)$.

Finding $\int f(x)\,dx$ is equivalent to finding $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$

Example : Finding $\int (x^2 + \sin x)\,dx$. $\Leftrightarrow$ Finding $F(x)$ such that $\frac{d}{dx}F(x) = x^2 + \sin x$.

Recall that $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$, $\frac{d}{dx}(-\cos x) = \sin x$.

$\Rightarrow \frac{d}{dx}\left(\frac{1}{3}x^3 - \cos x\right) = x^2 + \sin x$.

$\Rightarrow \int (x^2 + \sin x)\,dx = \frac{1}{3}x^3 - \cos x + C$.

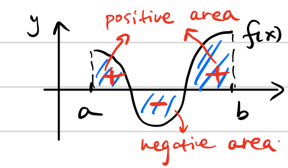
Definite integral.

Suppose that $a < b$.

$\int_a^b f(x)\,dx$ → called "definite integral of f from a to b".

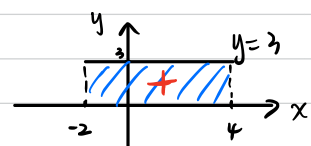
$\int_a^b f(x)\,dx \overset{\text{geometric}}{\underset{\text{meaning}}{=}}$ sum of the signed areas between the graph of f and x-axis over $[a, b]$

signed area: the area above the x-axis is positive.
the area below the x-axis is negative.



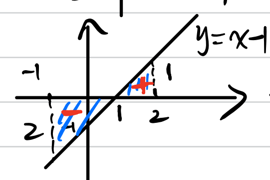
$\int_a^b f(x)\,dx = $ (sum of areas above the x-axis) − (sum of areas below the x-axis)

Example 1: $\int_{-2}^{4} 3\,dx =$



$= 3 \times (4+2) = 3 \times 6 = 18.$

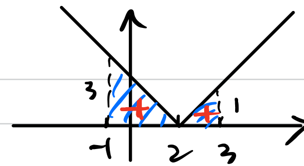
Example 2: $\int_{-1}^{2} (x-1)\,dx =$

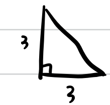


$=$ (area of $\triangle$) − (area of $\triangle$)

$= \left(\frac{1}{2}\cdot 1\cdot 1\right) - \left(\frac{1}{2}\cdot 2\cdot 2\right)$

$= \frac{1}{2} - 2 = -\frac{3}{2}.$

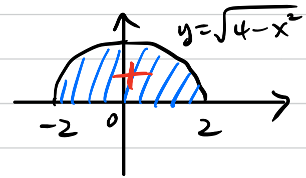
Example 3: $\int_{-1}^{3} |x-2|\, dx =$ 

$=$ area of 

$+$ area of 

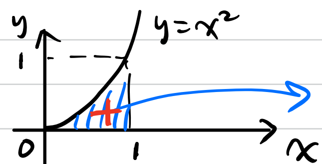
$= \frac{1}{2}\cdot 3\cdot 3 + \frac{1}{2}\cdot 1\cdot 1 = 5$

Recall: $|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2. \end{cases}$

Example 4: $\int_{-2}^{2} \sqrt{4-x^2}\, dx =$ 

$= \frac{1}{2}\cdot\left(\pi\cdot 2^2\right) = 2\pi.$

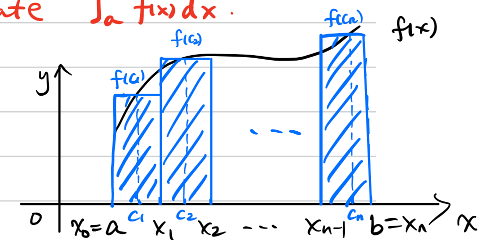
Recall: The graph of $y=\sqrt{4-x^2}$ is an upper semi-circle.
(because $y=\sqrt{4-x^2} \Rightarrow y^2 = 4-x^2 \Rightarrow x^2+y^2=4$)

Example 5: $\int_0^1 x^2\, dx =$ 

How to calculate this area?

Calculation of $\int_a^b f(x)\,dx$ for a general continuous function.

Trick: We use the sum of signed areas of rectangles to approximate $\int_a^b f(x)\,dx$.



To calculate $\int_a^b f(x)\,dx$, we take the following 3 steps:

$\boxed{\text{Step 1}}$ We divide $[a,b]$ into n small subintervals evenly.

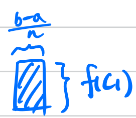
(width of each subinterval is $\frac{b-a}{n}$.)

$x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + 2\cdot\frac{b-a}{n}$, $\cdots$, $x_n = a + n\cdot\frac{b-a}{n} = b$.

$\boxed{\text{Step 2}}$ We choose a representative point in each subinterval:

$c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2]$, $\cdots$, $c_n \in [x_{n-1}, x_n]$.

The choice of each c_i is arbitrary. (Example: c_1 can be any number between x_0 and x_1).

the signed area of 

: an approximation to the signed area between f and x-axis over $[x_0, x_1]$.

In general, the signed area of the i-th rectangle 

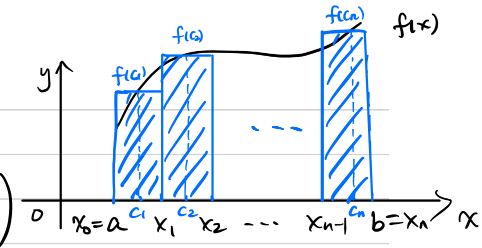
: an approximation to the signed area between f and x-axis over $[x_{i-1}, x_i]$.

$$\Rightarrow \underbrace{\frac{b-a}{n}\cdot f(c_1) + \frac{b-a}{n} f(c_2) + \cdots + \frac{b-a}{n} f(c_n)}_{\text{a sum of signed areas of rectangles.}} \;:\; \text{an approximation to } \int_a^b f(x)\,dx.$$

a sum of signed areas of rectangles. (called a "Riemann sum")

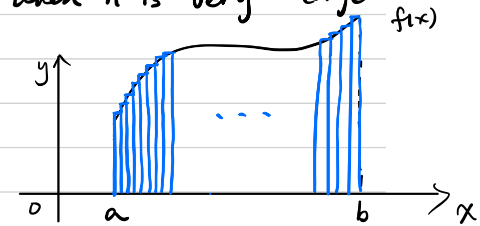
After choosing $c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2]$, $\cdots$, $c_n \in [x_{n-1}, x_n]$, we obtain an approximation of $\int_a^b f(x)\,dx$:

$$\frac{b-a}{n} \cdot f(c_1) + \frac{b-a}{n} f(c_2) + \cdots + \frac{b-a}{n} f(c_n) \quad \left(\text{Riemann sum}\right)$$



If we let $n \to +\infty$, then

when n is very large :



① the width of each subinterval : $\frac{b-a}{n} \to 0$

② the width of each rectangle tends to 0.

③ the number of rectangles tends to $+\infty$.

④ the difference between $\frac{b-a}{n}[f(c_1) + \cdots + f(c_n)]$ and $\int_a^b f(x)\,dx$ becomes smaller and smaller.

$\boxed{\text{Step 3}}$ $$\lim_{n \to +\infty} \frac{b-a}{n}[f(c_1) + f(c_2) + \cdots + f(c_n)] = \int_a^b f(x)\,dx$$

Remark : The limit of a Riemann sum is independent with the choice of each c_i.

(In other words, all Riemann sums have the same value after taking the limit)

Example: Find $\int_0^1 x^2\,dx = \frac{1}{3}$.

Step 1: Divide $[0,1]$ into n subintervals evenly (width of each subinterval $= \frac{1}{n}$)

1st subinterval: $[0, \frac{1}{n}]$, 2-nd subinterval: $[\frac{1}{n}, \frac{2}{n}]$, ..., n-th subinterval: $[\frac{n-1}{n}, 1]$.

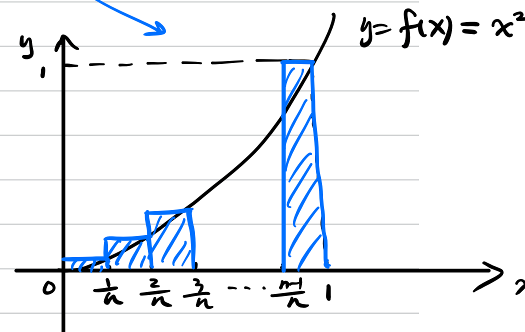
Step 2: We choose each c_i as the right endpoint of the i-th subinterval $[\frac{i-1}{n}, \frac{i}{n}]$:

$c_1 = \frac{1}{n}$, $c_2 = \frac{2}{n}$, ..., $c_n = 1$.

$\Rightarrow$ Riemann sum $\frac{1}{n}[f(c_1) + \cdots + f(c_n)]$: an approximation of $\int_0^1 x^2\,dx$.

sum of signed areas of these rectangles.

Step 3:
$$\int_0^1 x^2\,dx = \lim_{n\to+\infty} \frac{1}{n}[f(c_1) + f(c_2) + \cdots + f(c_n)]$$
$$= \lim_{n\to+\infty} \frac{1}{n}\left[f(\tfrac{1}{n}) + f(\tfrac{2}{n}) + \cdots + f(1)\right]$$
$$= \lim_{n\to+\infty} \frac{1}{n}\left[(\tfrac{1}{n})^2 + (\tfrac{2}{n})^2 + \cdots + (\tfrac{n}{n})^2\right].$$
$$= \lim_{n\to+\infty} \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2).$$
$$= \lim_{n\to+\infty} \frac{1}{n^3} \frac{1}{6}\cdot n\cdot(n+1)\cdot(2n+1).$$
$$= \lim_{n\to+\infty} \frac{1}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right) = \frac{1}{6}\cdot 1\cdot 1\cdot 2 = \frac{1}{3}.$$



Calculation of $1^2+2^2+\cdots+n^2$:

Notice : $n^3-(n-1)^3 = n^3-(n^3-3n^2+3n-1) = 3n^2-3n+1.$

$$\left.\begin{aligned}
n^3-(n-1)^3 &= n^3-(n^3-3n^2+3n-1) = 3n^2-3n+1. \\
(n-1)^3-(n-2)^3 &= 3(n-1)^2-3(n-1)+1. \\
(n-2)^3-(n-3)^3 &= 3\cdot(n-2)^2-3(n-2)+1 \\
&\ \vdots \\
(n=2)\quad 2^3-1^3 &= 3\cdot 2^2-3\cdot 2+1. \\
(n=1)\quad 1^3-0^3 &= 3\cdot 1^2-3\cdot 1+1.
\end{aligned}\right\} n \text{ equations}$$

sum of the left hand side of these equations = sum of the right hand side of these equation

$$\Rightarrow n^3 \cancel{-(n-1)^3} + \cancel{(n-1)^3} - \cancel{(n-2)^3} + \cdots + \cancel{2^3} \cancel{-1} + \cancel{1} - 0 = 3(1^2+2^2+\cdots+n^2) - 3(\underbrace{1+2+\cdots+n}_{\frac{1}{2}\cdot(n+1)\cdot n}) + n.$$

$$\Rightarrow \quad n^3 = 3(1^2+2^2+\cdots+n^2) - 3\cdot\tfrac{1}{2}\cdot(n+1)\cdot n + n$$

$$\Rightarrow \quad 1^2+2^2+\cdots+n^2 = \tfrac{1}{3}n^3 + 3\cdot\tfrac{1}{2}\cdot n\cdot(n+1) - \tfrac{1}{3}n = \tfrac{1}{6}n\cdot(n+1)\cdot(2n+1)$$

Let's consider a different choice of each C_i :

step 1. Recall : $[0,1] \xrightarrow{\text{divided into}} [0,\frac{1}{n}], [\frac{1}{n},\frac{2}{n}], \cdots, [\frac{n-1}{n},1]$.

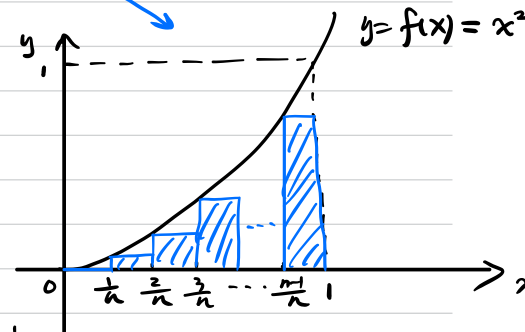
step 2. This time we choose C_i as the left endpoint of the i-th subinterval $[\frac{i-1}{n},\frac{i}{n}]$:

$C_1 = 0, \quad C_2 = \frac{1}{n}, \quad C_3 = \frac{2}{n}, \quad \cdots, \quad C_n = \frac{n-1}{n}$.

step 3: $\int_0^1 x^2 dx = \lim\limits_{n\to+\infty} \underbrace{\frac{1}{n}[f(C_1)+f(C_2)+\cdots+f(C_n)]}$ sum of signed areas of these rectangles.

$$= \lim_{n\to+\infty} \frac{1}{n}\left[0^2+\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]$$

$$= \lim_{n\to+\infty} \frac{1}{n^3}\left[1^2+2^2+\cdots+(n-1)^2\right]$$



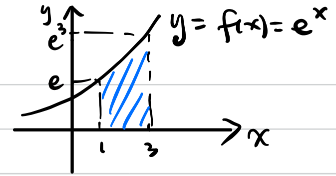
Notice : $1^2+2^2+\cdots+(n-1)^2 = \frac{1}{6}\cdot(n-1)\cdot n\cdot(2n-1)$.

$$\int_0^1 x^2 dx = \lim_{n\to+\infty} \frac{1}{n^3}\cdot\frac{1}{6}(n-1)\cdot n(2n-1)$$

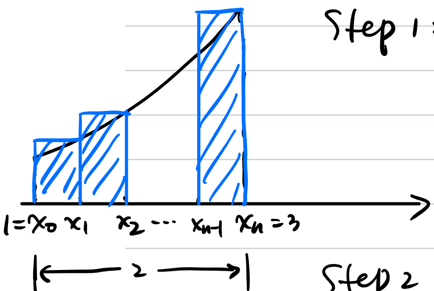
$$= \lim_{n\to+\infty} \frac{1}{6}\cdot\left(1-\frac{1}{n}\right)\cdot 1\cdot\left(2-\frac{1}{n}\right) = \frac{2}{6} = \frac{1}{3}.$$

After taking the limit, all Riemann sums have the same value.

Another example : Find $\int_1^3 e^x \, dx =$



Trick : use sum of signed areas of rectangles (Riemann sum) to approximate $\int_1^3 e^x \, dx$.



Step 1 : Let's divide $[1, 3]$ into n small subintervals evenly:

(width of each subinterval $= \frac{3-1}{n} = \frac{2}{n}$).

$x_0 = 1$, $x_1 = 1 + \frac{2}{n}$, $x_2 = 1 + 2 \cdot \frac{2}{n}$, $x_3 = 1 + 3 \cdot \frac{2}{n}$, ..., $x_n = 1 + n \cdot \frac{2}{n} = 3$.

Step 2 : We choose each c_i as the right endpoint of i-th subinterval $[x_{i-1}, x_i]$.

$c_1 = x_1 = 1 + \frac{2}{n}$, $c_2 = x_2 = 1 + 2 \cdot \frac{2}{n}$, $c_3 = x_3 = 1 + 3 \cdot \frac{2}{n}$, ..., $c_n = x_n = 3$

$\Rightarrow$ Riemann sum $\frac{2}{n}\left[f(c_1) + \cdots + f(c_n) \right]$: an approximation of $\int_1^3 e^x dx$.

Step 3 : $\int_1^3 e^x \, dx = \lim\limits_{n \to +\infty} \frac{2}{n}\left[f(c_1) + \cdots + f(c_n) \right]$

$$= \lim_{n \to +\infty} \frac{2}{n} \left(e^{1+\frac{2}{n}} + e^{1+2\cdot\frac{2}{n}} + e^{1+3\cdot\frac{2}{n}} + \cdots + e^{1+n\cdot\frac{2}{n}} \right).$$

$$\int_1^3 e^x\,dx = \lim_{n\to+\infty} \frac{2}{n}\left(e^{1+\frac{2}{n}} + e^{1+2\cdot\frac{2}{n}} + e^{1+3\cdot\frac{2}{n}} + \cdots + e^{1+n\cdot\frac{2}{n}}\right).$$

$$= \lim_{n\to+\infty} \frac{2}{n}\cdot e\cdot \underbrace{\left(e^{\frac{2}{n}} + e^{2\cdot\frac{2}{n}} + e^{3\cdot\frac{2}{n}} + \cdots + e^{n\cdot\frac{2}{n}}\right)}_{\text{a geometric series.}}.$$

The formuler for the sum of a geometric series. : $r + r^2 + r^3 + \cdots + r^n = \dfrac{r(r^n-1)}{r-1}$ $(r \neq 1)$

$$\int_1^3 e^x\,dx = \lim_{n\to+\infty} \frac{2}{n}\cdot e\cdot \frac{e^{\frac{2}{n}}\left[\left(e^{\frac{2}{n}}\right)^n - 1\right]}{e^{\frac{2}{n}}-1} = \lim_{n\to+\infty} \frac{2}{n}\cdot e\cdot \frac{e^{\frac{2}{n}}(e^2-1)}{e^{\frac{2}{n}}-1}$$

$$= e(e^2-1)\lim_{n\to+\infty} \frac{2}{n}\cdot\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1} = e(e^2-1)\lim_{n\to+\infty}\frac{\boxed{e^{\frac{2}{n}}}\to 1}{\boxed{\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}}\to 1} = e(e^2-1).$$

detail

Notice: $\displaystyle \lim_{n\to+\infty}\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}} \underset{\substack{\text{change of variable}\\ \text{Let } t=\frac{2}{n}}}{=} \lim_{t\to 0}\frac{e^t-1}{t} \underset{\text{L'Hopital's Rule}}{=} \lim_{t\to 0}\frac{(e^t-1)'}{(t)'} = \lim_{t\to 0}\frac{e^t}{1} = 1.$