MATH2111 Tutorial 7

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1 Applications of Determinant

1. **Theorem (Cramer's Rule)**. Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A} \text{ for } i = 1, 2, \ldots, n$$

   where $A_i(\boldsymbol{b})$ is the matrix obtained from A by replacing column i by the vector $\boldsymbol{b}$.

2. **Theorem (Inverse Formula)**. Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$\text{where } \operatorname{adj} A = (\operatorname{cof} A)^T \text{ and } \operatorname{cof} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

3. **Theorem**. If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

4. **Theorem**. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

   If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

2 Vector Spaces

1. **Definition (Vector Space)**. A vector space is a nonempty set V of objects, called **vectors**, on which are defined two operations, called **addition** and **scalar multiplication** (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

   (a) The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.

   (b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

   (c) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

   (d) There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

   (e) For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

   (f) The scalar multiple of $\mathbf{u}$ by c, denoted by c$\mathbf{u}$, is in V.

   (g) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

   (h) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{v}$

   (i) $c(d\mathbf{u}) = (cd)\mathbf{u}$

   (j) $1\mathbf{u} = \mathbf{u}$

2. **Fact**. For each $\mathbf{u}$ in V and scalar c,

   (a) $0\mathbf{u} = \mathbf{0}$

   (b) $c\mathbf{0} = \mathbf{0}$

   (c) $-\mathbf{u} = (-1)\mathbf{u}$

3. **Definition (Subspace)**. A subspace of a vector space V is a subset H of V that has three properties:

   (a) The zero vector of V is in H.

   (b) H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

   (c) H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

4. **Theorem**. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are in a vector space V, then Span $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a subspace of V.

3 Exercises

1. Use Cramer's rule to solve the following linear system.

$$\begin{cases} x_1 + x_2 = 3 \\ -3x_1 + 2x_3 = 0 \\ x_2 - 2x_3 = 2 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}}$$

Since

$$|A| = \begin{array}{c} \begin{array}{ccc} + & - & \end{array} \\ \begin{vmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} \end{array} = (1) \cdot \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} + (-1) \begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= -2 - 6 = -8$$

$$|A_1(\vec{b})| = \begin{array}{c} \begin{array}{ccc} + & - & \end{array} \\ \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} \end{array} = (3) \cdot \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}$$

$$= -6 + 4 = -2$$

$$|A_2(\vec{b})| = \begin{array}{c} \begin{array}{ccc} + & - & \end{array} \\ \begin{vmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix} \end{array} = (1) \cdot \begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} + (-3) \cdot \begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= -4 - 18 = -22$$

$$|A_3(\vec{b})| = \begin{vmatrix} \overset{+}{1} & 1 & 3 \\ \overset{-}{-3} & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -3$$

By Cramer's Rule,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-2}{-8} \\ \frac{-22}{-8} \\ \frac{-3}{-8} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{3}{8} \end{bmatrix}$$

2. Compute the adjugate of the given matrix, and then use the inverse formula to give A^{-1}.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \det A = \begin{vmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{vmatrix} \end{array} = (1)\cdot\begin{vmatrix} 1 & 4 \\ -3 & 4 \end{vmatrix} + (-2)\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 16 - 14 = 2$$

$$C_{11} = \begin{vmatrix} 1 & 4 \\ -3 & 4 \end{vmatrix} = 16 \qquad C_{12} = -\begin{vmatrix} -3 & 4 \\ 2 & 4 \end{vmatrix} = 20 \qquad C_{13} = \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} = 7$$

$$C_{21} = -\begin{vmatrix} 0 & -2 \\ -3 & 4 \end{vmatrix} = 6 \qquad C_{22} = \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 8 \qquad C_{23} = -\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} = 2 \qquad C_{33} = \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} = 1$$

$$\therefore \operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{2}\begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

3. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Area of image of S

$= |\det A| \cdot$ Area of S

det

$= \left| \begin{vmatrix} 6 & -2 \\ -3 & 2 \end{vmatrix} \right| \cdot \left| \begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix} \right|$ absolute value

$= |6| \cdot |-4| = 24$

4. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{ \text{ area of triangle } \} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Translate triangle to one having the origin as a vertex.

One can subtract the vertex (x_1, y_1) from 3 vertices :

$(0, 0)$, $(x_2 - x_1,\ y_2 - y_1)$, $(x_3 - x_1,\ y_3 - y_1)$.

① LHS :

$$\{\text{Area of triangle}\} = \frac{1}{2}\{\text{area of parallelogram}\}$$

$$= \frac{1}{2} \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

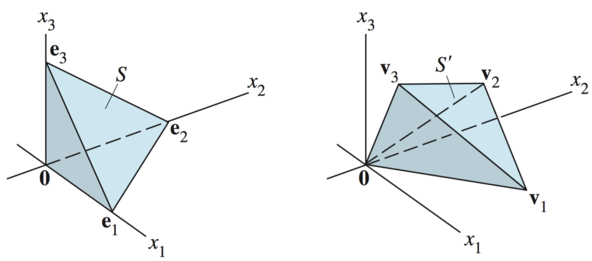
② RHS:

$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \begin{array}{c} \\ R_2 - R_1 \to R_2 \\ \\ R_3 - R_1 \to R_3 \end{array} \frac{1}{2}\det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_3 & 0 \end{bmatrix} \end{array}$$

$$= \frac{1}{2} \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

Therefore, $\{\text{Area of triangle}\} = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

5. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.



a. Describe a linear transformation that maps S onto S'.
b. Find a formula for the volume of the tetrahedron S' using the fact that
$\{$ volume of $S\} = (1/3)\{$ area of base $\} \cdot \{$ height $\}$

a. A linear transformation T which maps S onto S' will map $\vec{e_1}$ to $\vec{v_1}$, $\vec{e_2}$ to $\vec{v_2}$, $\vec{e_3}$ to $\vec{v_3}$,

that is,

$T(\vec{e_1}) = \vec{v_1}$, $T(\vec{e_2}) = \vec{v_2}$, $T(\vec{e_3}) = \vec{v_3}$

The standard matrix A will be:

$$A = [\vec{v_1} \ \vec{v_2} \ \vec{v_3}]$$

b. Area of base of S is $1 \times 1 \times \frac{1}{2} = \frac{1}{2}$.

$$\{\text{volume of } S\} = \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}$$

$$\therefore \{\text{volume of } S'\} = \{\text{volume of } T(S)\}$$

$$= |\det A| \cdot \{\text{volume of } S\}$$

$$= \frac{1}{6} |\det A|$$

6. Let S be a set of 2×2 matrices, whose sum of all diagonal entries is zero. Verify S is a subspace of the vector space of all 2×2 matrices.

let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a + d = 0$.

① S contains 0 ($0 \in M_{2\times2}$) matrix, i.e.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$$

② let $S_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $S_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $S_1, S_2 \in S$

$$S_1 + S_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

Note that $a_1 + a_2 + d_1 + d_2 = (a_1 + d_1) + (a_2 + d_2) = 0 + 0 = 0$

$\therefore$ S is closed under addition.

③ for $t \in \mathbb{R}$, $S_1 \in S$

$$tS_1 = t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ta_1 & tb_1 \\ tc_1 & td_1 \end{bmatrix}$$

note that $ta_1 + td_1 = t(a_1 + d_1) = t \cdot 0 = 0$

$\therefore$ S is closed under multiplication.

$\therefore$ S is a subspace of the vector space of all 2×2 matrices.