Chapter 3 Notes

3.1 Natural number

Definition 3.1.1. The natural number is a set $\mathbb{N}$ satisfying the following properties:

1. There is a special element $1 \in \mathbb{N}$.
2. For any $n \in \mathbb{N}$, there is a unique successor $n' \in \mathbb{N}$.
3. For any $n \in \mathbb{N}$, we have $n' \neq 1$.
4. If $m' = n'$, then $m = n$.
5. If a subset $S \subset \mathbb{N}$ contains 1 and has the property that $n \in S \implies n' \in S$, then $S = \mathbb{N}$.

Proposition 3.1.2. Any natural number other than 1 is a successor.

Definition 3.1.3. The addition $m + n$ of two natural numbers is the operation characterized by

1. $m + 1 = m'$
2. $m + n' = (m + n)'$

Proposition 3.1.4. The addition of natural numbers has the following properties:

1. Cancelation: $m + k = n + k \implies m = n$.
2. Associativity: $m + (n + k) = (m + n) + k$.
3. Commutativity: $m + n = n + m$.

3.2 Integer

$$(m, n) \sim (k, l), \text{ if } m + l = k + n$$

Definition 3.2.1.

$$[m, n] + [k, l] = [m + k, n + l]$$

Proposition 3.2.2. The addition of integers has the following properties:

1. Associativity: $a + (b + c) = (a + b) + c$.
2. Commutativity: $a + b = b + a$.
3. Zero: There is a unique integer 0 satisfying $a + 0 = a = 0 + a$
4. Negative: For any integer a, there is a unique integer $-a$ satisfying $a + (-a) = 0 = (-a) + a$.

We define subtraction of integers by using the negative

$$a - b = a + (-b)$$

3.4 Multiplication

Definition 3.4.1 The multiplication mn of two natural numbers is the operation characterized by

1. $m1 = m$.
2. $mn' = mn + m$.

Proposition 3.4.2 The multiplication of natural numbers has the following properties:

1. Distributivity: $(m+n)k = mk + nk$
2. Associativity: $m(nk) = (mn)k$
3. Commutativity: $mn = nm$

If we expect $(m-n)(k-l) = (mk+nl) - (mk+nk)$, we may define

$$[m,n][k,l] = [mk+nl, mk+nk].$$

Proposition 3.4.3. The multiplication of integers has the following numbers.

1. The multiplication is consistent with the multiplication of natural numbers.
2. Distributivity: $(a+b)c = ac + bc$ and $a(b+c) = ab + ac$.
3. Associativity: $a(bc) = (ab)c$.
4. Commutativity: $ab = ba$.
5. One: $a1 = 1 = 1a$.
6. Zero: $ab = 0 \iff a = 0$ or $b = 0$.
7. Negative: $(-a)b = -ab = a(-b)$.
8. Order: If $a > 0$, then $b > c \iff ab > ac$.

3.5 Rational Number

Definition 3.5.1. The rational numbers is the set $\mathbb{Q}$ of the equivalence classes of pairs (a, b) of integers $a, b \in \mathbb{Z}, b \neq 0$, under the equivalence relation

$$(a,b) \sim (c,d) \iff ad = cd$$

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Proposition 3.5.2. The addition and multiplication of rational numbers have the following properties:

1. The operations are consistent with the operations of integers.
2. Associativity: $r+(s+t)=(r+s)+t, r(st)=(rs)t$.
3. Commutativity: $r+s=s+r, rs=sr$.
4. Distributivity: $(r+s)t=rt+st, r(s+t)=rs+rt$.
5. Zero: The integer 0 is the unique rational number satisfying $r+0=r=0+r$.
6. Negative: For any rational number r, there is a unique rational number $-r$ satisfying $r+(-r)=0=(-r)+r$.
7. One: The integer 1 is the unique rational number satisfying $r1=r=1r$.
8. Reciprocal: For any rational number $r\neq 0$, there is a unique rational number r^{-1} satisfying $rr^{-1}=1=r^{-1}r$.

Proposition 3.5.3. The order of rational numbers has the following properties:

1. For any rational numbers r and s, one of the following mutually exclusive cases happens:
$$r=s, r>s, r<s.$$
2. $r>s$ and $s>t \Longrightarrow r>t$.
3. $r>s \Longrightarrow r+t>s+t$.
4. $r>s \Longrightarrow -r<-s$.
5. If $r>0$, then $s>t \Longleftrightarrow rs>rt$.
6. If $r,s>0$, then $r>s \Longleftrightarrow r^{-1}<s^{-1}$.
7. For any $r>s$, there is t satisfying $r>t>s$.
8. For any $r>0$, there is a natural number n satisfying $n>r>\frac{1}{n}$.

We also define the absolute value of a rational number
$$|r|=\begin{cases} r, & \text{if } r\geq 0\\ -r, & \text{if } r<0\end{cases}.$$
The Absolute value has the following properties.
$$|r+s|\leq|r|+|s|, |rs|=|r||s|, |r|<s \Longleftrightarrow -s<r<s.$$

3.6 Real Number

Definition 3.6.1. A real number is a nonempty subset $X \subset \mathbb{Q}$ of rational numbers satisfying the following:

1. There is $l \in \mathbb{Q}$, such all $r \in X$ satisfy $r > l$.
2. If $s > r \in X$, then $s \in X$.
3. If $r \in X$, then there is $s \in X$ such that $r > s$.