**COMP 2711 Discrete Mathematical Tools for Computer Science**
**2022 Fall Semester – Tutorial 4**

**Question 1:** How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

**Solution :** If we ignore the fact that the table is round and just count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements. However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so this overcounts by a factor of 4. (For example, the sequence Mary-Debra-Cristina-Julie gives the same circular seating as the sequence Julie-Mary-Debra-Cristina.) Therefore the answer is $5040/4 = 1260$.

**Question 2:** Give a combinatorial proof for the following statement for $n \geq 2$,

$$\binom{n}{2}^2 \leq \binom{2n}{4}.$$

Note: An algebraic proof of this statement will not be accepted as a solution.

**Solution :** Consider $n$ men and $n$ women, i.e., a total of $2n$ people. Let $A$ be the set of possible outcomes which is counted by the left expression and set $B$ be the set of possible outcomes which is counted by the right expression.

- Left side: number of ways to select 2 men and 2 women from the $2n$ people.
- Right side: number of ways to select 4 people from the $2n$ people.

Obviously, set $B$ is a superset of $A$ since set $B$ contains all possible outcomes with exactly 2 men and 2 women as well as other possible outcomes such as 3 men and 1 woman, 4 men, etc., when $n > 2$. Thus, $|A| \leq |B| \Rightarrow \binom{n}{2}^2 \leq \binom{2n}{4}$.

**Question 3:**

(a) Let $n$ and $r$ be positive integers. Explain why the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = r$, where $x_i$ is a nonnegative integer for $i = 1, 2, 3, \cdots, n$, equals the number of $r$-combinations with repetition of a set with $n$ elements.

(b) How many solutions in nonnegative integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$?

(c) How many solutions in positive integers are there to the equation in part (b)?

**Solution :** (a) To count the number of solutions, we note that a solution corresponds to a way of selecting $r$ items from a set with $n$ elements so that $x_1$ items of type one, $x_2$ items of type two, $\cdots$, and $x_n$ items of type $n$ are chosen. Hence, the number of solutions is equal to the number of $r$-combinations with repetition allowed from a set with $n$ elements, and there are $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ solutions.

(b) From (a), $\binom{4+17-1}{17}$.

(c) Let $x_i = x_i' + 1$; thus $x_i'$ is the value that $x_i$ has in excess of its required 1. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2' + x_3' + x_4' = 17 - 4 = 13$. Hence, there are $\binom{4+13-1}{13}$ solutions.

**Question 4:** In how many ways can a dozen books be placed on four distinguishable shelves

(a) if the books are indistinguishable copies of the same title?

(b) if no two books are the same, and the positions of the books on the shelves matter? [*Hint:* Break this into 12 tasks, placing each book separately. Start with the sequence $1, 2, 3, 4$ to represent the shelves. Represent the books by $b_i$, $i = 1, 2, \ldots, 12$. Place $b_1$ to the right of one of the terms in $1, 2, 3, 4$. Then successively place $b_2, b_3, \ldots,$ and $b_{12}$.]

**Solution :** (a) All that matters is the number of books on each shelf, so the answer is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$, where $x_i$ is being viewed as the number of books on shelf $i$. The answer is therefore $\binom{4+12-1}{12} = \binom{15}{12} = 455$.

(b) No generality is lost if we number the books $b_1, b_2, \ldots, b_{12}$ and think of placing book $b_1$ , then placing $b_2$ , and so on. There are clearly 4 ways to place $b_1$ , since we can put it as the first book (for now) on any of the shelves. After $b_1$ is placed, there are 5 ways to place $b_2$ , since it can go to the right of $b_1$ or it can be the first book on any of the shelves. We continue in this way: there are 6 ways to place $b_3$ (to the right of $b_1$ , to the right of $b_2$ , or as the first book on any of the shelves), 7 ways to place $b_4, \ldots, 15$ ways to place $b_{12}$. Therefore the answer is the product of these numbers $4 \cdot 5 \cdots 15 = 217,945,728,000$.

**Question 5:** Give a formula for the coefficient of $x^k$ in the expansion of $(x^2 - 1/x)^{100}$, where $k$ is an integer.

**Solution :** Let us apply the binomial theorem to the given binomial:

$$\begin{aligned}(x^2 - x^{-1})^{100} &= \sum_{j=0}^{100} \binom{100}{j} (x^2)^{100-j}(-x^{-1})^j \\ &= \sum_{j=0}^{100} \binom{100}{j} (-1)^j x^{200-2j-j} = \sum_{j=0}^{100} \binom{100}{j} (-1)^j x^{200-3j}\end{aligned}$$

Thus the only nonzero coefficients are those of the form $200-3j$ where $j$ is an integer between 0 and 100, inclusive, namely $200, 197, 194, \ldots, 2, -1, -4, \ldots, -100$. If we denote $200 - 3j$ by $k$, then we have $j = (200 - k)/3$. This gives us our answer. The coefficient of $x^k$ is zero for $k$ not in the list just given (namely those values of $k$ between $-100$ and 200, inclusive, that are congruent to 2 modulo 3), and for those values of $k$ in the list, the coefficient is $(-1)^{(200-k)/3}\binom{100}{(200-k)/3}$.

**Question 6:** Prove that if $n$ and $k$ are integers with $1 \le k \le n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

(a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with $k$ elements from a set with $n$ elements and then an element of this subset.]

(b) using an algebraic proof based on the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

**Solution :** (a) We show that each side counts the number of ways to choose from a set of $n$ elements a subset with $k$ elements and a distinguished element of that set. For the left-hand side, first choose the $k$-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the $k$ elements in this subset to be the distinguished element (this can be done in $k$ ways). For the right-hand side, first choose the distinguished element out of the entire $n$-set (this can be done in $n$ ways), and then choose the remaining $k-1$ elements of the subset from the remaining $n - 1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways).

(b) This is straightforward algebra:

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

**Question 7:** Use combinatorial proof to show that

$$\binom{2n-5}{10} = \sum_{k=0}^{10} \binom{n-3}{k}\binom{n-2}{10-k}$$

**Solution :** Suppose that there are $n$ boys and $n$ girls. The total number of ways to choose 15 people with 3 particular girls and 2 particular boys must be selected is $\binom{2n-5}{10}$, the left-hand side.

Another way to count the same set is as following. The 3 particular girls and the 2 particular boys are selected. We remain to choose 10 people. Such 10 people could be all are girls, 9 girls and 1 boy, 8 girls and 2 boys, and so on. Therefore, the right-hand side, $\sum_{k=0}^{10} \binom{n-3}{k}\binom{n-2}{10-k}$, counts this.

Because both sides of the identity count the same set, they must be equal.