§2.1 Matrix operation

* Sums and scalar Multiples

Ex: Let $A = \begin{pmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$

Then $A+B = \begin{pmatrix} 4+1 & 0+1 & 5+1 \\ -1+3 & 3+5 & 2+7 \end{pmatrix}$

$= \begin{pmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{pmatrix}$

$2A = \begin{pmatrix} 2\times 4 & 2\times 0 & 2\times 5 \\ 2\times(-1) & 2\times 3 & 2\times 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 10 \\ -2 & 6 & 4 \end{pmatrix}$

$A+C$ is undefined.

* Given two matrices A and B of the same size $m\times n$, then the sum $A+B$ is a matrix of size $m\times n$ whose columns are the sums of the corresponding columns of A and B.

Let $A = (\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n) = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ & \cdots & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$,

$B = (\vec{b}_1, \vec{b}_2, \cdots \vec{b}_n) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$

Then $A+B = (\vec{a}_1+\vec{b}_1, \vec{a}_2+\vec{b}_2, \cdots, \vec{a}_n+\vec{b}_n) = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$

* scalar multiple rA is the matrix whose columns are r times the corresponding columns in A

$$rA = r\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22} & \cdots & ra_{2n} \\ \cdots \\ ra_{m1} & ra_{m2} & \cdots & ra_{mn} \end{pmatrix}$$

Thm: Let A, B and C be matrices of the same size, and let r and s be scalars.

(1) $A+B = B+A$

(2) $(A+B)+C = A+(B+C)$

(3) $A+0 = A$ (0 is the matrix of the same size as A whose entries are all zero)

(4) $r(A+B) = rA + rB$

(5) $(r+s)A = rA + sA$

(6) $r(sA) = (rs)A$

Proof: (1) Let $A = (\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n)$, $B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n)$

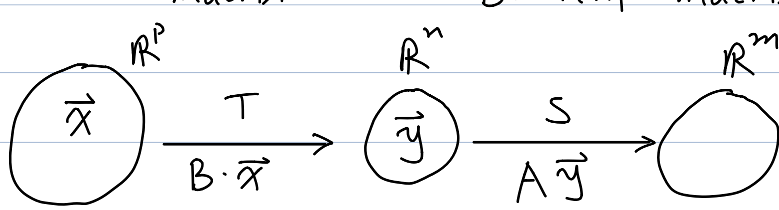
Then $A+B = (\vec{a}_1+\vec{b}_1, \vec{a}_2+\vec{b}_2, \cdots \vec{a}_n+\vec{b}_n)$

$= (\vec{b}_1+\vec{a}_1, \vec{b}_2+\vec{a}_2, \cdots, \vec{b}_n+\vec{a}_n)$

$= B+A$

* Matrix multiplication

A: $m\times n$ matrix $\qquad$ B: $n\times p$ matrix



$(S\circ T)(\vec{x}) = S(T\vec{x}) = A(B\vec{x})$

$A\cdot(B\vec{x})$ is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$

since $A(B(a\vec{u}+b\vec{v})) = A(aB\vec{u} + bB\vec{v})$

$= aA(B\vec{u}) + bA(B\vec{v})$.

So $A(B\vec{x}) = C\cdot\vec{x}$

where C is standard matrix of the linear transformation $S\circ T$.

$C = \left((S\circ T)(\vec{e}_1), \cdots, (S\circ T)(\vec{e}_p)\right)$

Let $B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p)$. Then

$$Be_1 = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{b}_1$$

$\vdots$

$$Be_p = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \vec{b}_p$$

$S\circ T(\vec{e}_1) = A(B\vec{e}_1) = A\vec{b}_1$,

$S\circ T(\vec{e}_2) = A(B\vec{e}_2) = A\vec{b}_2$,

$\cdots$

$S\circ T(\vec{e}_p) = A(B\vec{e}_p) = A\vec{b}_p$.

Def: Define $A\cdot B = C = (A\vec{b}_1, \cdots, A\vec{b}_p)$

Ex: Compute $A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$

Solution: $A \cdot B = (A\vec{b}_1, A\vec{b}_2, A\vec{b}_3)$

$$A\vec{b}_1 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times 4 + 3\times 1 \\ 1\times 4 + (-5)\times 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$$

$$A\vec{b}_2 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2\times 3 + 3\times(-2) \\ 1\times 3 + (-5)\times(-2) \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$$

$$A\vec{b}_3 = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\times 6 + 3\times 3 \\ 1\times 6 - 5\times 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -9 \end{pmatrix}$$

Thus $AB = \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}$

Remark: 1) AB has the same number of rows as A and the same number of columns as B.

2) The number of columns of A must match the number of rows in B in order for a linear combination such as Ab_1 to be defined.

3) $$\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2\times 4 + 3\times 1 & 2\times 3 + 3\times(-2) & 2\times 6 + 3\times 3 \\ 1\times 4 + (-5)\times 1 & 1\times 3 + (-5)\times 2 & 6\times 1 + 3\times(-5) \end{pmatrix}$$

Example: If A is a 3X5 matrix and B is a 5X2 matrix, what are the sizes of AB and BA, if they are defined.

Solution: 1) AB is defined and is a 3×2 matrix.

2) BA is not defined

Row - Column Rule for computing AB

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$

Example: Find the entries of AB, where

$$A = \begin{pmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{pmatrix} \begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times4 + (-5)\times7 + 0\times3 & 2\times(-6) + (-5)\times1 + 0\times2 \\ -1\times4 + 3\times7 + (-4)\times3 & -1\times(-6) + 3\times1 + (-4)\times2 \\ 6\times4 + (-8)\times7 + (-7)\times3 & 6\times(-6) + (-8)\times1 + (-7)\times2 \\ -3\times4 + 0\times7 + 9\times3 & -3\times(-6) + 0\times1 + 9\times2 \end{pmatrix}$$

$$= \begin{pmatrix} -27 & -17 \\ 5 & 1 \\ -53 & -58 \\ 15 & 36 \end{pmatrix}$$

* Properties of Matrix Multiplication

Let A be an $m\times n$ matrix, B and C have sizes for which the indicated sums and products are defined

(1) $A(BC) = (AB)C$ associative law for multiplication

(2) $A(B+C) = AB + AC$ left distributive law

(3) $(B+C)A = BA + CA$ right distributive law

(4) $r(AB) = (rA)\cdot B = A(rB)$ for any scalar r

(5) $I_m A = A = A I_n$

I_m : $m \times m$ identity matrix $\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & \cdots & & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$

Write $C = (\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_p)$

$A(BC) = A(B\vec{c}_1, B\vec{c}_2, \cdots, B\vec{c}_p) = [A(B\vec{c}_1), \cdots, AB\vec{c}_p]$

composition of linear transformations (which are functions)

$= ((AB)(\vec{c}_1), \cdots, (AB)\vec{c}_p)$ ← composition of functions is associative

$= (AB)(\vec{c}_1, \cdots, \vec{c}_p)$

$= (AB)C$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $AB \neq BA$

$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Warning: In general, $AB \neq BA$

In general, $AB = AC$ doesn't imply $B = C$

In general, $AB = 0$ doesn't imply $A = 0$ or $B = 0$.

Exercise: Let $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix}$, and

$C = \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix}$. Verify that $AB = AC$ and

yet $B \neq C$.

Solution: $AB = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} 2\times 8 - 3\times 5 & 2\times 4 - 3\times 5 \\ -4\times 8 + 6\times 5 & -4\times 4 + 6\times 5 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$

$AC = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2\times 5 + (-3)\times 3 & 2\times(-2) + (-3)\times 1 \\ -4\times 5 + 6\times 3 & -4\times(-2) + 6\times 1 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$

Hence $AB = AC$. yet $B \neq C$

* Powers of a Matrix

If A is an $n\times n$ matrix, k: a positive integer

$A^k = \underbrace{A \cdots A}_{k \text{ times}}$, $A^0 = I_n$

* Transpose of a matrix

A: an $n\times m$ matrix

A^T: called the transpose of A, is an $m\times n$ matrix whose ith row is the ith column of A.

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$B = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}, \quad B^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$$

Properties of transpose

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) For any scalar, $(rA)^T = rA^T$

(4) $(AB)^T = B^T A^T$

The transpose of a product of matrices equals the product of their transposes in the reverse order.

Exercise: Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors too. Let

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Compute $(A\vec{x})^T$, $\vec{x}^T A^T$, $\vec{x}\vec{x}^T$, and $\vec{x}^T\vec{x}$. Is $A^T\vec{x}^T$ defined?

Solution: $A\vec{x} = \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\times 5 + (-3)\times 3 \\ -2\times 5 + 4\times 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

So $(A\vec{x})^T = (-4, 2)$

$$\vec{x}^T A^T = \begin{pmatrix} 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 5\times1+3\times(-3) & 5\times(-2)+3\times4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \end{pmatrix}$$

The quantities $(A\vec{x})^T$ and $\vec{x}^T A^T$ are equal.

$$\vec{x}\vec{x}^T = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 15 \\ 15 & 9 \end{pmatrix}$$

$$\vec{x}^T\vec{x} = \begin{pmatrix} 5 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 25 + 9 = 34$$

$A^T \vec{x}^T$ is not defined since A^T is 2×2 matrix and $\vec{x}^T$ is 1×2 matrix. $\vec{x}^T$ doesn't have two rows to match the two columns of A^T.