COMP 2711 Discrete Math Tools for Computer Science

2022 Fall Semester - Homework 3

Question 1: Variant Monty Hall game:

Recall that we gave two explanations for the Monty Hall game. The first is an intuitive one: Our initial guess is correct with probability $\frac{1}{3}$, which is not affected by Monty opening another door, so switching will give us a winning probability of $\frac{2}{3}$. Then we gave a formal proof that $p(W = 1 \mid M = 2) = p(W = 1) = \frac{1}{3}$, where W denotes the winning door and M the door Monty opens.

Now, we organized a Variant Monty Hall Game that have 4 doors. Behind one door is a prize; behind the others, goat. In this game, $p(W = 1) = p(W = 2) = p(W = 3) = p(W = 4) = \frac{1}{4}$. You pick door 1 first. The host will look behind the 4 doors and open one of other doors with a goat randomly with equal probability. He will then give you a chance to change your selection to one of the two remaining doors.

(a) Suppose $M = 2$, i.e., Monty opens door 2. What is $p(W = 1 \mid M = 2)$, $p(W = 3 \mid M = 2)$ and $p(W = 4 \mid M = 2)$? Based on your calculation, should you switch to door 3 or door 4?

Then, suppose that the probabilities of the 4 doors having the prize are $0.4, 0.2, 0.2$, and 0.2, respectively. We pick door 1 as our initial guess. Then Monty will open one of door 2, 3, or 4 without the prize with equal probability.

(b) Suppose $M = 2$, i.e., Monty opens door 2. What is $p(W = 1 \mid M = 2)$, $p(W = 3 \mid M = 2)$ and $p(W = 4 \mid M = 2)$? Based on your calculation, should you switch to door 3 or door 4?

(c) What's the winning probability of your strategy above?

If we pick door 2 as our initial guess,

(d) suppose $M = 1$, i.e., Monty opens door 1. What is $p(W = 2 \mid M = 1)$, $p(W = 3 \mid M = 1)$ and $p(W = 4 \mid M = 1)$? In this case, should you switch? Which door should you switch?

(e) suppose $M = 3$, i.e., Monty opens door 3. What is $p(W = 1 \mid M = 3)$, $p(W = 2 \mid M = 3)$ and $p(W = 4 \mid M = 3)$? In this case, should you switch? Which door should you switch?

(f) What's the winning probability of your strategy above?

Answer: (a) By the Bayes' theorem,

$$p(W = w \mid M = 2) = \frac{p(M = 2 \mid W = w)p(W = w)}{p(M = 2)},$$

where $p(M=2)=\sum_{w=1}^{4} p(M=2 \mid W=w)p(W=w)$. We have $p(W=1)=p(W=2)=p(W=3)=p(W=4)=0.25$, $p(M=2 \mid W=1)=\frac{1}{3}, p(M=2 \mid W=2)=0, p(M=2 \mid W=3)=\frac{1}{2}, p(M=2 \mid W=4)=\frac{1}{2}$. Plugging in these numbers, we obtain $p(M=2)=\frac{1}{3}$ and $p(W=1 \mid M=2)=\frac{1}{4}$, $p(W=3 \mid M=2)=p(W=4 \mid M=2)=\frac{3}{8}$. So, we should switch.

(b) By the Bayes' theorem,

$$p(W=w \mid M=2)=\frac{p(M=2 \mid W=w)p(W=w)}{p(M=2)},$$

where $p(M=2)=\sum_{w=1}^{4} p(M=2 \mid W=w)p(W=w)$. We have $p(W=1)=0.4, p(W=2)=p(W=3)=p(W=4)=0.2$, $p(M=2 \mid W=1)=\frac{1}{3}, p(M=2 \mid W=2)=0, p(M=2 \mid W=3)=\frac{1}{2}, p(M=2 \mid W=4)=\frac{1}{2}$. Plugging in these numbers, we obtain $p(M=2)=\frac{1}{3}$ and $p(W=1 \mid M=2)=0.4$, $p(W=3 \mid M=2)=p(W=4 \mid M=2)=0.3$. So, we should not switch.

(c) From (b), symmetrically, if $M=3$ or $M=4$, we still should not change. $p(M=3)=p(M=4)=p(M=2)=\frac{1}{3}$. $p(W=1 \mid M=3)=p(W=1 \mid M=4)=0.4$. Let $\mathcal{E}$ denote the event that you win. We have $p(\mathcal{E})=\sum_{m=2}^{4} p(M=m)p(\mathcal{E} \mid M=m)=0.4 \cdot \frac{1}{3}+0.4 \cdot \frac{1}{3}+0.4 \cdot \frac{1}{3}=0.4$.

(d) By the Bayes' theorem,

$$p(W=w \mid M=1)=\frac{p(M=1 \mid W=w)p(W=w)}{p(M=1)},$$

where $p(M=1)=\sum_{w=1}^{4} p(M=1 \mid W=w)p(W=w)$. We have $p(W=1)=0.4, p(W=2)=p(W=3)=p(W=4)=0.2$, $p(M=1 \mid W=1)=0, p(M=1 \mid W=2)=\frac{1}{3}, p(M=1 \mid W=3)=\frac{1}{2}, p(M=1 \mid W=4)=\frac{1}{2}$. Plugging in these numbers, we obtain $p(M=1)=\frac{4}{15}$ and $p(W=2 \mid M=1)=\frac{1}{4}$, $p(W=3 \mid M=1)=p(W=4 \mid M=1)=\frac{3}{8}$. So, we should switch to door 3 or door 4.

(e) Same as above, but with $p(M=3 \mid W=1)=\frac{1}{2}, p(M=3 \mid W=2)=\frac{1}{3}, p(M=3 \mid W=3)=0, p(M=3 \mid W=4)=\frac{1}{2}$. Plugging in these numbers, we obtain $p(M=3)=\frac{11}{30}$ and $p(W=1 \mid M=3)=\frac{6}{11}$, $p(W=2 \mid M=3)=\frac{2}{11}$, and $p(W=4 \mid M=3)=\frac{3}{11}$. So, we should switch to door 1. This time, the probability that our initial guess is correct has decreased.

(f) From (e), symmetrically, if $M=4$, we should switch to door 1 as well. $p(M=3)=p(M=4)=\frac{11}{30}$. From (d), $p(M=1)=\frac{4}{15}$.
$p(W=3 \mid M=1)=p(W=4 \mid M=1)=\frac{3}{8}$.
$p(W=1 \mid M=3)=p(W=1 \mid M=4)=\frac{6}{11}$.
Let $\mathcal{E}$ denote the event that you win. We have $p(\mathcal{E})=\sum_{m=1,3,4} p(M=m)p(\mathcal{E} \mid M=m)=\frac{4}{15} \cdot \frac{3}{8}+\frac{11}{30} \cdot \frac{6}{11} \times 2=0.5$.

Question 2: A marketing representative from a company randomly approaches people on a random street in Hong Kong until he finds a person who is a customer of his company. Let p, the probability that he succeeds in finding such a person, equal 0.2. And let X denote the number of people (including the customer) he approaches until he finds his first success.

(a) What is the probability that the marketing representative approaches exactly 4 people (including the customer) until he finds one customer of his company?

(b) What is the probability that the marketing representative approaches more than 4 people (including the customer) until he finds one customer of his company?

(c) Calculate $E(X)$

Answer: This problem is modeled using the geometric distribution.

(a) $p(X = 4) = (1 - p)^3 p = 0.8^3 \times 0.2 = 0.1024$.

(b) $p(X > 4) = 1 - p(X \leq 4) = 1 - \sum_{k=1}^{4} (1-p)^{k-1} p = 1 - \sum_{k=1}^{4} 0.8^{k-1} 0.2 = 0.4096$

(c) $E(X) = \frac{1}{p} = \frac{1}{0.2} = 5$

Question 3: Chisato and Takina just become roommates, and they play rock, paper, scissors (R, P, S) to decide who is responsible for the housework duty everyday. Each of them will choose one symbol in R, P, S. R beats S, S beats P, P beats R. If both of them choose the same symbol, it is a tie.

However, Chisato has an ability in dynamic vision. She can predict if Takina is going to choose rock (R) from Takina's subtle movements. Assume that when Takina is going to choose rock (R), Chisato has 60% chance to predict correctly. When Takina is going to choose paper (P) or scissors (S), Chisato has 80% chance to guess that Takina is NOT going to choose rock (R) (Chisato has 20% chance to predict that Takina is going to choose R while Takina is not going to choose R).

Thus, Chisato has a strategy. She will only choose P or S. When she predict that Takina is going to choose R, she will choose P. When she predict that Takina is NOT going to choose R, she will choose S.

Takina doesn't know Chisato's strategy, so she just randomly choose one of three symbols with equal probability.

(a) Given that Takina choose R, what's the probability that Chisato choose P?

(b) What's the probability that Chisato choose S?

(c) What's the probability that Chisato wins in one turn of the game? (When both of them choose the same symbol, this turn will be regarded as a tie)

(d) What's the probability that Takina wins in one turn of the game? (When both of them choose the same symbol, this turn will be regarded as a tie)

To decide who is responsible for the housework duty today, when both of them choose the same symbol, the game will continue until someone wins. The loser needs to do the housework today.

(e) What's the probability that Takina needs to do the housework today (What's the probability that Chisato wins at the end)?

(f) Denote X as the number of dates that Takina needs to do the housework in 365 days. What's $E(X)$?

(g) What's $V(X)$?

Answer: (a) When Takina choose R, Chisato has 60% chance to predict correctly. In this case, she will choose P. When she make a wrong prediction, she will choose S. Denote C as Chisato's choice and T as Takina's choice. $p(C = P \mid T = R) = 0.6$

(b) Same as (a), we have
$p(C = P \mid T = R) = 0.6$, $p(C = S \mid T = R) = 1 - 0.6 = 0.4$,
$p(C = S \mid T = P) = 0.8$, $p(C = P \mid T = P) = 0.2$,
$p(C = S \mid T = S) = 0.8$, $p(C = P \mid T = S) = 0.2$
$p(C = S) = p(C = S \cap T = R) + p(C = S \cap T = P) + p(C = S \cap T = S) = p(C = S \mid T = R)p(T = R) + p(C = S \mid T = P)p(T = P) + p(C = S \mid T = S)p(T = S) = 0.4 \times \frac{1}{3} + 0.8 \times \frac{1}{3} + 0.8 \times \frac{1}{3} = \frac{2}{3}$

(c) Denote the event that Chisato wins as $W = C$, $p(W = C) = p(C = P \cap T = R) + p(C = S \cap T = P) = p(C = P \mid T = R)p(T = R) + p(C = S \mid T = P)p(T = P) = 0.6 \times \frac{1}{3} + 0.8 \times \frac{1}{3} = \frac{7}{15}$

(d) Denote the event that Takina wins as $W = T$, $p(W = T) = p(C = S \cap T = R) + p(C = P \cap T = S) = p(C = S \mid T = R)p(T = R) + p(C = P \mid T = S)p(T = S) = 0.4 \times \frac{1}{3} + 0.2 \times \frac{1}{3} = \frac{1}{5}$

(e) Denote the event that there is a tie in one turn as $W = F$, $p(W = F) = \frac{1}{3}$. Denote the event that Chisato wins at the end as C.

$$\begin{aligned}
p(C) &= p(W = C) + p(W = F)p(W = C) + p(W = F)^2 p(W = C) + \cdots \\
&= p(W = C)(1 + p(W = F) + p(W = F)^2 + \cdots) \\
&= p(W = C) \cdot lim_{n\to\infty} \frac{1 \cdot (1 - p(W = F)^n)}{1 - p(W = F)} \\
&= \frac{7}{15} \cdot lim_{n\to\infty} \frac{1 \cdot (1 - \frac{1}{3}^n)}{1 - \frac{1}{3}} \\
&= \frac{7}{10}
\end{aligned}$$

(f) X follows the binomial distribution where $n = 365$ and $p = 0.7$, thus, $E(X) = np = 255.5$

(g) For binomial distribution, $V(X) = np(1 - p) = 76.65$

Question 4: A bag contains 80 fair coins and 20 magic coins. Flipping a fair coin results in a head with probability 0.5, whereas flipping a magic coin results in a head with probability 1.0.

(a) Consider randomly drawing one coin from the bag and flipping it. What is the probability of seeing a head?

(b) Consider drawing <u>two coins</u> from the bag with replacement and flipping <u>each</u> of them <u>once</u>. Let

$$X_1 = \begin{cases} 1 & \text{if first flip a head} \\ 0 & \text{if first flip a tail} \end{cases}, \quad X_2 = \begin{cases} 1 & \text{if second flip a head} \\ 0 & \text{if second flip a tail} \end{cases}.$$

Are X_1 and X_2 independent?

(c) Let X denote the number of heads you see. Note that X can be 0,1,2. What is $E(X)$? What is $V(X)$?

(d) Now consider drawing <u>one coin</u> from the bag and flipping it <u>twice</u>. Let

$$Y_1 = \begin{cases} 1 & \text{if first flip a head} \\ 0 & \text{if first flip a tail} \end{cases}, \quad Y_2 = \begin{cases} 1 & \text{if second flip a head} \\ 0 & \text{if second flip a tail} \end{cases}.$$

Are Y_1 and Y_2 independent?

Answer: (a) Let H = "Getting a head"; K = "Drawing a fair coin"; $\overline{K}$ = "Drawing a magic coin".

$$\begin{aligned} p(H) &= p(H \cap K) + p(H \cap \overline{K}) \\ &= p(H|K)P(K) + p(H|\overline{K})p(\overline{K}) \\ &= 0.5 \times 0.8 + 1.0 \times 0.2 = 0.6 \end{aligned}$$

(b) Yes. Recall: X, Y are independent if and only if for all values x, y,

$$p\Big((X = x) \wedge (Y = y)\Big) = p(X = x) \cdot p(Y = y)$$

(c) For $i \in \{1, 2\}$,

$$E(X_i) = 1 \cdot (0.6) + 0 \cdot (0.4) = 0.6.$$

Thus,

$$E(X) = E(X_1 + X_2) = E(X_1) + E(X_2) = 1.2.$$

For $i \in \{1, 2\}$,

$$E(X_i^2) = 1^2 \cdot (0.6) + 0^2 \cdot (0.4) = 0.6.$$

Thus,

$$V(X_i) = E(X_i^2) - E(X_i)^2 = 0.6 - 0.6^2 = 0.24.$$

Finally,

$$V(X) = V(X_1 + X_2) = V(X_1) + V(X_2) = 0.48.$$

(d) The two variables are not independent. For example,

$$\begin{aligned} p(H_1 \cap H_2) &= p(H_1 \cap H_2 \cap K) + p(H_1 \cap H_2 \cap \overline{K}) \\ &= p(H_1 \cap H_2 | K) p(K) + p(H_1 \cap H_2 | \overline{K}) p(\overline{K}) \\ &= 0.25 \times 0.8 + 1.0 \times 0.2 \\ &= 0.4 \\ &\neq 0.6 \cdot 0.6 = p(H_1) p(H_2). \end{aligned}$$