HKUST – Department of Computer Science and Engineering
COMP 2711: Discrete Math Tools for Computer Science

Fall 2021 Midterm Examination

Date: 29 October 2022 Time: 14:00–16:00

Name: ____________	Student ID: ____________
Email: ____________	Lecture: ____________

Instructions

- This is a closed book exam. It consists of **XX** pages and **Y** questions. You are allowed to bring a one-page cheat sheet, handwritten on both sides.
- Please write your name, student ID, email, lecture section in the space provided at the top of this page.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may also use the back of the pages. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full-page answer. Some can be answered using only a few lines.
- Solutions can be written in terms of binomial coefficients, factorials, and the $C(n,k)$, $P(n,k)$ notation. For example, you can write $\binom{5}{3} + \binom{4}{2}$ instead of 16. Avoid use of nonstandard notation such as ${}_nP_k$ or ${}_nC_k$. Calculators may be used for the exam (but are not necessary).

Problems	1	2	3	4	5	6	7	Total
Points	11	16	14	20	14	12	13	100
Score								

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:       ______________________

Student's Signature:  ______________________
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Problem 1: Propositional Logic [11 pts]

Let s be the statement whose truth table is given below.

p	q	r	s
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	F

(a) [5 pts] Express the statement s in terms of p, q, and r in such a way that only negation ($\neg$) and logical connectives $\wedge$ and $\vee$ are used.

(b) [3 pts] Find an equivalent formulation of s that used only $\neg$ and $\wedge$

(c) [3 pts] Find an equivalent formulation of s that used only $\neg$ and $\rightarrow$

Answer: (a) From row 1,3,7, we can write the truth table like following (Simplify not needed, all the following forms are accepted):

$$
\begin{aligned}
& (\neg \mathbf{p} \wedge \neg \mathbf{q} \wedge \neg \mathbf{r}) \vee (\neg \mathbf{p} \wedge \mathbf{q} \wedge \neg \mathbf{r}) \vee (\mathbf{p} \wedge \mathbf{q} \wedge \neg \mathbf{r}) \\
\equiv\ & \neg r \wedge ((\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)) \\
\equiv\ & \neg r \wedge ((\neg p \wedge (\neg q \vee q)) \vee (p \wedge q)) \\
\equiv\ & \neg r \wedge ((\neg p \wedge T) \vee (p \wedge q)) \\
\equiv\ & \neg \mathbf{r} \wedge (\neg \mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})) \\
\equiv\ & \neg r \wedge ((\neg p \vee p) \wedge (\neg p \vee q)) \\
\equiv\ & \neg \mathbf{r} \wedge (\neg \mathbf{p} \vee \mathbf{q})
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \neg r \wedge (\neg p \vee (p \wedge q)) \\
\equiv\ & \neg \mathbf{r} \wedge \neg(\mathbf{p} \wedge \neg(\mathbf{p} \wedge \mathbf{q})) \\
\equiv\ & \neg \mathbf{r} \wedge \neg(\mathbf{p} \wedge \neg \mathbf{q})
\end{aligned}
$$

(c)

$$
\begin{aligned}
& \neg r \wedge (\neg p \vee (p \wedge q)) \\
\equiv\ & \neg(r \vee \neg(\neg p \vee \neg(\neg p \vee \neg q))) \\
\equiv\ & \neg(\neg \mathbf{r} \rightarrow \neg(\mathbf{p} \rightarrow \neg(\mathbf{p} \rightarrow \neg \mathbf{q}))) \\
\equiv\ & \neg(\neg \mathbf{r} \rightarrow \neg(\mathbf{p} \rightarrow \mathbf{q}))
\end{aligned}
$$

Problem 2: Predicate Logic [16 pts]

Determine the truth value of each of these statements and prove your answer.

(a) [4 pts] $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R}(x^2 + y^2 \geq 2xy)$

(b) [4 pts] $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R}(x \geq y^2)$

for every propositions $p(x)$ and $q(x)$ on $\mathbb{R}$, are the statements equivalent? If not, give a counter example consisting two propositions $p(x)$ and $q(x)$ which shows the statements are not logically equivalent

(c) [4 pts] $\neg(\exists x \in \mathbb{R}^+(p(x))) \equiv \forall x \in \mathbb{R}(x \leq 0 \lor \neg p(x))$

(d) [4 pts] $(\exists x \in \mathbb{R}(p(x))) \land (\exists y \in \mathbb{R}(q(y))) \equiv \exists z \in \mathbb{R}(p(z) \land q(z))$

Answer: (a) True. $x^2 + y^2 \geq 2xy \leftrightarrow x^2 - 2xy + y^2 \geq 0 \leftrightarrow (x - y)^2 \geq 0$. Since $(x - y)^2 \geq 0$, the statement is true for all x and y.

(b) False. For every x, there exists a y, such as $y = \sqrt{|x| + 1}$, that $x < y^2$.

(c) True. $\neg(\exists x \in \mathbb{R}^+(p(x))) \equiv \neg(\exists x \in \mathbb{R}((x > 0) \land (p(x))) \equiv \forall x \in \mathbb{R}(\neg(x > 0) \lor \neg p(x)) \equiv \forall x \in \mathbb{R}(x \leq 0 \lor \neg p(x))$. (By De Morgan's Law)

(d) False. Take $p(x)$ to be $x < 0$ and $q(x)$ be $x > 0$. There exist in $x \in \mathbb{R}$ that $x < 0$ and exist in $y \in \mathbb{R}$ that $y > 0$, but none of them are both less than 0 and larger than 0

Problem 3: Proof by Contradiction [14 pts]

Given the two premises below:

(1) Every tree in a forest has at least one leaf.

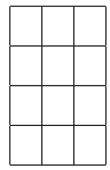
(2) The number of trees in the forest is greater than the number of leaves in any tree in the forest.

Prove that there exist at least two trees in the forest with the same number of leaves.

Answer: Proof by contradiction: Assume that every tree has a different number of leaves.
Order the trees by the number of leaves each has, smallest to largest.
Let $t_1, t_2, \ldots, t_n$ be the number of leaves that tree i has, where n is the number of trees.
Thus, by premise (1), $1 \leq t_1 < t_2 < \cdots < t_n \implies n \leq t_n$
By premise (2), $t_n < n$, but this contradicts $n \leq t_n$.

Problem 4: Counting [20 pts]

A $n \times k$ grid consists of n non-overlapping vertical lines and k non-overlapping horizontal lines. Below is a 4×5 grid:



(a) [6 pts] How many rectangles are there in a $n \times k$ grid? Note that square is a special rectangle, and a rectangle must have non-zero area.

We color every point in the grid blue, red or green. A *monochrome pair* is two vertices on the same horizontal or vertical line with the same color, and a *monochrome rectangle* is a rectangle formed by four vertices with the same color.

(b) [6 pts] How many vertical lines must a grid have to guarantee that each horizontal line have at least one monochrome pair?

(c) [8 pts] Take your answer of (b) to be the number of vertical lines. How many horizontal line must this grid have to guarantee that there is at least one monochrome rectangle in the grid?

Answer: (a) A rectangle can be formed by choosing two horizontal lines and two vertical lines.
There are $\binom{k}{2}$ ways to choose two horizontal lines and $\binom{n}{2}$ ways to choose two vertical lines. Thus, by product principle, there should be $\binom{k}{2} \cdot \binom{n}{2}$ rectangles.

(b) Since there are 3 colors, by the pigeonhole principle, it must have at least $3 + 1 = 4$ vertical line to have at least one monochrome pair in each horizontal line.

(c) For each horizontal line, each color have $\binom{4}{2} = 6$ ways to place the monochrome pair. We have 3 colors, thus, we have $6 \times 3 = 18$ options to form a monochrome pair. By the pigeonhole principle, we need at least $18 + 1 = 19$ horizontal line.

Problem 5: Inclusion and Exclusion [14 pts]

How many solutions are there in $a+b+c=13$, where a, b, c are nonnegative integers and $a \leq 4, b \leq 5, c \leq 6$. Use inclusion-exclusion to calculate the result and don't enumerate the values of possible solutions.

Answer:

$$\begin{aligned} U &= \{(a,b,c)|a+b+c=13, a,b,c \geq 0\} \\ A &= \{(a,b,c)|a+b+c=13, a>4, b,c \geq 0\} \\ B &= \{(a,b,c)|a+b+c=13, b>5, a,c \geq 0\} \\ C &= \{(a,b,c)|a+b+c=13, c>6, a,b \geq 0\} \end{aligned}$$

So the result of our question is $|\overline{A} \cap \overline{B} \cap \overline{C}| = |U| - |A \cup B \cup C|$. And we use inclusion-exclusion to count $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

We know that the number of r-combinations from a set with n elements when repetition of elements is allowed is $\binom{n+r-1}{r}$, which means that the number of solutions of $a+b+c=n$, where $a, b, c \leq 0$, is $\binom{n+2}{2}$

$$|U| = \binom{15}{2} = 105$$

Since we can rewrite the set A to be,

$$A = \{(a,b,c)|a+b+c=13, a>4, b,c \leq 0\} = \{(a'+5,b,c)|a'+b+c=8, a',b,c \leq 0\}$$

$$|A| = \binom{10}{2} = 45$$

In the same way, we can calculate set B, C as well as their intersections.

$$\begin{aligned} |B| &= \binom{9}{2} = 36 \\ |C| &= \binom{8}{2} = 28 \\ |A \cap B| &= \binom{4}{2} = 6 \\ |A \cap C| &= \binom{3}{2} = 3 \\ |B \cap C| &= \binom{2}{2} = 1 \end{aligned}$$

While the set $A \cap B \cap C$ should satisfy $a+b+c = 13$ where $a > 4, b > 5, c > 6$, there is no solution for this equation. So we have,

$$|B| = \binom{9}{2} = 36$$
$$|C| = \binom{8}{2} = 28$$
$$|A \cap B| = \binom{4}{2} = 6$$
$$|A \cap C| = \binom{3}{2} = 3$$
$$|B \cap C| = \binom{2}{2} = 1$$

While the set $A \cap B \cap C$ should satisfy $a+b+c = 13$ where $a > 4, b > 5, c > 6$, there is no solution for this equation. So we have,

$$|A \cap B \cap C| = 0$$

In all, the result is,

$$\begin{aligned} & |\overline{A} \cap \overline{B} \cap \overline{C}| \\ = \; & |U| - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) \\ = \; & 105 - (45 + 36 + 28 - 6 - 3 - 1 + 0) \\ = \; & 6 \end{aligned}$$

Problem 6: Probability and Conditional Probability [12 pts]

Tony lives in Kwun Tong and he can go to North Point in three different ways: by MTR, by ferry, or by bus. There is a 50% chance that he will be late when he takes the ferry. When he takes the bus, there is a 20% chance that he will be late. The probability that he is late when he takes the MTR is only 5%.

(a) [6 pts] Suppose the priori probability for Tony to take each of the three forms of transportation is 1/3. Then given the fact that Tony is late today, what's the probability that he took the ferry today?

(b) [6 pts] Suppose the priori probabilities for Tony to take the ferry is 30%, the bus 10%, and the MTR 60%. Then given the fact that Tony arrived on time today, what is the probability that he took the ferry today?

Answer: Let L be the event that Tony is late, and let M, F, and B be the events that he went by MTR, ferry, and bus, respectively. We are told that $p(L|M) = 0.05, p(L|F) = 0.5, p(L|B) = 0.2$. We are asked to find $p(F|L)$.

(a) We are to assume here that $p(M) = p(F) = p(B) = 1/3$. Then by the generalized version of Bayes' theorem,

$$\begin{aligned} p(F|L) &= \frac{p(L|F)p(F)}{p(L|M)p(M) + p(L|F)p(F) + p(L|B)p(B)} \\ &= \frac{(0.5)(1/3)}{(0.05)(1/3) + (0.5)(1/3) + (0.2)(1/3)} \\ &= \frac{2}{3} \end{aligned}$$

(b) Now we are to assume here that $p(M) = 0.6, p(F) = 0.3, p(B) = 0.1$. Then by the generalized version of Bayes' theorem,

$$\begin{aligned} p(F|\overline{L}) &= \frac{p(\overline{L}|F)p(F)}{p(\overline{L}|M)p(M) + p(\overline{L}|F)p(F) + p(\overline{L}|B)p(B)} \\ &= \frac{(1 - p(L|F))p(F)}{(1 - p(L|M))p(M) + (1 - p(L|F))p(F) + (1 - p(L|B))p(B)} \\ &= \frac{(1 - 0.5)(0.3)}{(1 - 0.05)(0.6) + (1 - 0.5)(0.3) + (1 - 0.2)(0.1)} \\ &= \frac{3}{16} \end{aligned}$$

Problem 7: Random Variables [13 pts]

Let X be a discrete random variable taking values from $\mathbb{N}$ with the following PMF (probability mass function),

$$p(X = k) = \begin{cases} 0.1 & \text{if k=1} \\ 0.2 & \text{if k=2} \\ 0.3 & \text{if k=3} \\ 0.4 & \text{if k=4} \\ 0 & \text{otherwise} \end{cases}$$

(a) [3 pts] Calculate $E(X)$ and $V(X)$.

(b) [3 pts] If $Y = X - 1$, calculate $E(Y)$ and $V(Y)$.

(c) [3 pts] If $Z = X(X - 1)$, calculate $E(Z)$.

(d) [4 pts] Show that $E(X) = \sum_{k=1}^{\infty} p(X \geq k)$. If X is a discrete random variable taking values from $\mathbb{N}$ with an arbitrary PMF, does this equality still holds? Prove it or give a counter example.

Answer: (a)

$$\begin{aligned} E(X) &= \sum_{k=1}^{4} k \cdot p(X = k) \\ &= 1 \cdot 0.1 + 2 \cdot 0.2 + 3 \cdot 0.3 + 4 \cdot 0.4 \\ &= 3 \\ E(X^2) &= \sum_{k=1}^{4} k^2 \cdot p(X = k) \\ &= 1 \cdot 0.1 + 4 \cdot 0.2 + 9 \cdot 0.3 + 16 \cdot 0.4 \\ &= 10 \\ V(X) &= E(X^2) - E(X)^2 = 1 \end{aligned}$$

(b)

$$\begin{aligned} E(Y) &= E(X - 1) = E(X) - 1 = 2 \\ V(Y) &= V(X - 1) = V(X) = 1 \end{aligned}$$

(c)

$$E(Z) = E(X(X-1)) = E(X^2 - X) = E(X^2) - E(X) = 7$$

(d)

$$p(X \geq k) = \begin{cases} 1 & \text{if k=1} \\ 0.9 & \text{if k=2} \\ 0.7 & \text{if k=3} \\ 0.4 & \text{if k=4} \\ 0 & \text{if k}\geq 5 \end{cases}$$

And we know that,

$$\sum_{k=1}^{\infty} p(X \geq k) = 1 + 0.9 + 0.7 + 0.4 + 0 = 3 = E(X)$$

For any PMF, the equation still exists. Since X is a discrete variable from $\mathbb{N}$,

$$\begin{array}{rcr} p(X \geq 1) & = & p(X = 1) + p(X = 2) + p(X = 3) + \cdots \\ p(X \geq 2) & = & p(X = 2) + p(X = 3) + \cdots \\ p(X \geq 3) & = & p(X = 3) + \cdots \\ \cdots\cdots & & \end{array}$$

And we sum all the equations up. The left side is $\sum_{k=1}^{\infty} p(X \geq k)$. In the right side, each term $p(X = i)$ will occur i times (in the first i equations). Their sum equals to the definition of expectation. Thus we have,

$$\sum_{k=1}^{\infty} p(X \geq k) = \sum_{k=1}^{\infty} k \cdot p(X = k) = E(X)$$