§3.2 Properties of Determinants

Theorem: Row Operations

Let A be a square matrix.

a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \cdots & & \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

b) If two rows of A are interchanged to produce B, then $\det B = -\det A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Example: Compute $\det A$, where $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$

Solution: $\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & -4 & -2 \\ -12 & 10 & 10 \\ 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 3 & -4 & -2 \\ 0 & -6 & 2 \\ 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot 3 \begin{vmatrix} -6 & 2 \\ -3 & 2 \end{vmatrix}$$

$$= 6 \cdot (-6 \times 2 - 2 \times (-3))$$

$$= 6 \cdot (-12 + 6)$$

$$= -36$$

Example: Compute $\det A$, where $A = \begin{pmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{pmatrix}$

Solution: $\det A = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0$

Thm: A square matrix A is invertible if and only if $\det(A) \neq 0$

Proof: Suppose A has been reduced to an echelon form U by row replacement and row interchanges. If there are r interchanges, then $\det A = (-1)^r \det U = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U), & A \text{ is invertible} \\ 0, & A \text{ is not invertible.} \end{cases}$

$\boxed{\text{Corollary: } \det A = 0 \text{ if the rows of } A \text{ are linearly dependent.}}$

Remark: linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

* Column Operations

The following Theorem shows that column operations have the same effects on determinants as row operations.

Thm: If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Example: Compute $\det A$ where $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 5 & 10 & 6 \end{pmatrix}$

Solution: Since $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$,

$\det A = 0$

* Determinants and Matrix Products

Thm: Multiplicative Property

If A and B are $n\times n$ matrices, then

$\det AB = (\det A)\cdot(\det B)$.

Ex: Verify the theorem for $A = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

Solution: $AB = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}\cdot\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 25 & 20 \\ 14 & 13 \end{pmatrix}$

$\det AB = 25\cdot 13 - 14\cdot 20 = 325 - 280 = 45$

$\det A = 6\cdot 2 - 3 = 9$

$\det B = 4\cdot 2 - 3\cdot 1 = 5$

So $\det AB = \det A \cdot \det B$

Remark: $\det(A+B) \neq \det A + \det B$

Exercise: Use determinants to decide if the set of vectors $\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix}$

is linearly independent.

Solution:
$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{vmatrix} = - \begin{vmatrix} 2 & 7 & -2 \\ 6 & 0 & -5 \\ 4 & -7 & -3 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 7 & -2 \\ 0 & -21 & 1 \\ 0 & -21 & 1 \end{vmatrix} = 0$$

So the set of vectors is linearly dependent.

Ex: Show that if A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

Proof: If A is invertible, then $AA^{-1} = I$.

Hence $\det(AA^{-1}) = \det I$

Since $\det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$,

$\det I = 1$

$\det(A)\det(A^{-1}) = 1$.

Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$

* A Linearity Property of the Determinant Function

1) $\det(\vec{a}_1, \vec{a}_2, \cdots \vec{a}_{j-1}, c\vec{a}_j, \vec{a}_{j+1}, \cdots \vec{a}_n) = c\det(\vec{a}_1, \cdots \vec{a}_{j-1}, \vec{a}_j, \vec{a}_{j+1}, \cdots \vec{a}_n)$

That is,
$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & ca_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & ca_{2j} & \cdots & a_{2n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & ca_{ij} & \cdots & a_{in} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & ca_{nj} & \cdots & a_{nn} \end{pmatrix} = c\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

for all scalars c.

2) $\det(\vec{a}_1, \cdots \vec{a}_{j-1}, \vec{a}_j + \vec{b}_j, \vec{a}_{j+1}, \cdots \vec{a}_n) = \det(\vec{a}_1, \cdots, \vec{a}_j, \cdots \vec{a}_n) + \det(\vec{a}_1, \cdots, \vec{b}_j, \cdots \vec{a}_n)$

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j}+b_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j}+b_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj}+b_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & \cdots & b_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_{2j} & \cdots & a_{2n} \\ & & \cdots \\ a_{n1} & a_{n2} & \cdots & b_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Exercise: Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find the determinant

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}.$$

Solution: $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 0$$

$= 2 \times 7$

$= 14$

§3.3 Cramer's rule, Volume, and Linear Transformations

For any $n \times n$ matrix A and any $\vec{b}$ in $\mathbb{R}^n$, Let $A_i(b)$ be the matrix obtained from A by replacing column i by the vector $\vec{b}$.

$$A = (\vec{a}_1, \cdots \vec{a}_{i-1}, \vec{a}_i, \vec{a}_{i+1}, \cdots \vec{a}_n)$$

$$A_i(\vec{b}) = (\vec{a}_1, \cdots \vec{a}_{i-1}, \underset{\text{col } i}{\vec{b}}, \vec{a}_{i+1}, \cdots \vec{a}_n)$$

Thm: Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by

$$\vec{x}_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \cdots, n.$$

Proof: Let $I = (\vec{e}_1, \cdots, \vec{e}_n)$

$$I_i(\vec{x}) = (\vec{e}_1, \cdots, \vec{x}, \cdots, \vec{e}_n).$$

Then
$$\begin{aligned} A \cdot I_i(\vec{x}) &= A(\vec{e}_1, \cdots, \vec{x}, \cdots, \vec{e}_n) \\ &= (A\vec{e}_1, \cdots A\vec{x}, \cdots A\vec{e}_n) \\ &= (\vec{a}_1, \cdots, \vec{b}, \cdots \vec{a}_n) \\ &= A_i(\vec{b}) \end{aligned}$$

$$\det(A \cdot I_i(\vec{x})) = \det A_i(\vec{b})$$

$$\det(A \cdot I_i(\vec{x})) = \det A \cdot \det(I_i(\vec{x}))$$

$$\det(I_i(\vec{x})) = \begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & \cdots & x_i & \cdots & 0 \\ & & & & & \\ 0 & 0 & & x_n & \cdots & 1 \end{vmatrix}$$

$$= (-1)^{i+i} x_i \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{vmatrix} = x_i$$

Hence $x_i \cdot \det A = \det A_i(\vec{b})$

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

Example: Use Cramer's rule to solve the system

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

Solution: $A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$A_1(\vec{b}) = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$, $A_2(\vec{b}) = \begin{pmatrix} 3 & 6 \\ -5 & 8 \end{pmatrix}$

$\det A = 3 \cdot 4 - (-2) \cdot (-5) = 2$

$\det A_1(\vec{b}) = 6 \cdot 4 - (-2) \cdot 8 = 40$

$\det A_2(\vec{b}) = 3 \cdot 8 - 6 \cdot (-5) = 54$

By cramer's rule,

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det A} = \frac{40}{2} = 20$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det A} = \frac{54}{2} = 27$$

Example: Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$\begin{cases} 3s x_1 - 2x_2 = 4 \\ -6x_1 + s x_2 = 1 \end{cases}$$

Solution: $A = \begin{pmatrix} 3s & -2 \\ -6 & s \end{pmatrix}$, $A_1(\vec{b}) = \begin{pmatrix} 4 & -2 \\ 1 & s \end{pmatrix}$, $A_2(\vec{b}) = \begin{pmatrix} 3s & 4 \\ -6 & 1 \end{pmatrix}$

Since $\det(A) = 3s^2 - 12 = 3(s-2)(s+2)$,

the system has a unique solution precisely when $s \neq \pm 2$. For such an s, the solution is

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{4s+2}{3(s-2)(s+2)}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3s+24}{3(s-2)(s+2)} = \frac{s+8}{(s+2)(s-2)}$$

A Formula for A^{-1}

Thm: Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \boxed{\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}}$$

adj A

(adjugate of A)

classical adjoint of A

where $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Ex: Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$$

Solution: $C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$\operatorname{adj} A = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}$$

$$(\operatorname{adj} A) \cdot A = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} = 14I$$

So $\det A = 14$ and

$$A^{-1} = \frac{1}{\det A}(\operatorname{adj} A) = \frac{1}{14} \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{pmatrix}$$