Dynamics of Rigid Bodies II

PHYS1112

Lecture 9

Intended Learning Outcomes

- After this lecture you will learn:
    1) how to calculate the moment of inertia of simple symmetric rigid bodies
    2) the parallel axis theorem to find the moment of inertia about different rotation axis
    3) Vector product
    4) torque, and the Newton's second law in rotational dynamics

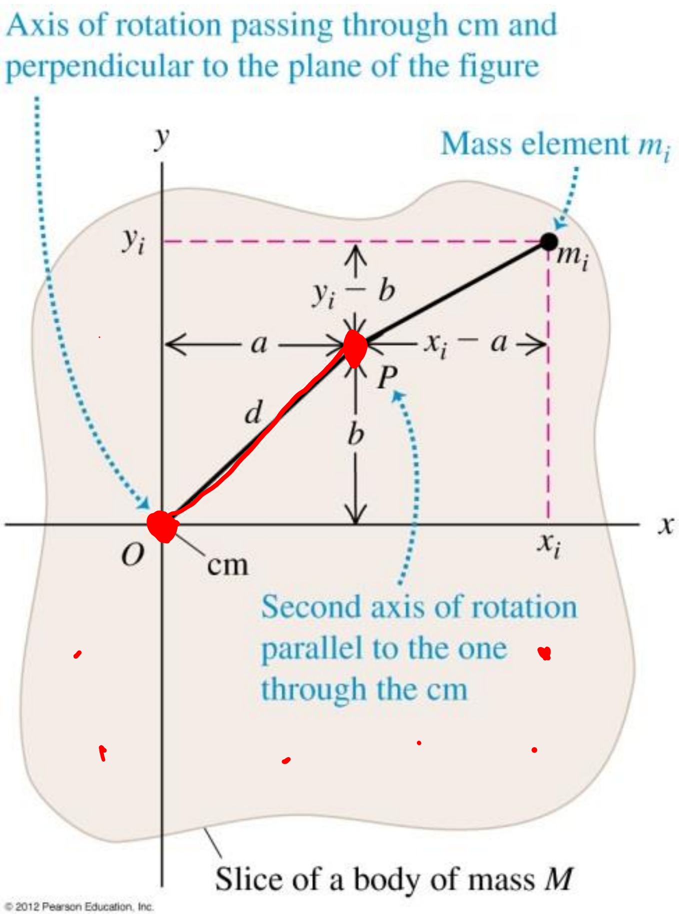
Parallel axis theorem

I_{cm}: moment of inertia about an axis through its CM

I_p: moment of inertia about another axis ∥ to the original one and at ⊥ distance d

$$\boxed{I_p = I_{\text{cm}} + Md^2}$$

Proof: take CM as the origin, rotation axis as the z axis. A point mass m_i in the solid has coordinates (x_i, y_i, z_i)



$$I_{\text{cm}} = \sum m_i \overbrace{\left(x_i^2 + y_i^2\right)}^{\text{square of } \perp \text{ distance of } m_i \text{ to rotation axis}}$$

not $\sum m_i\left(x_i^2 + y_i^2 + z_i^2\right)$

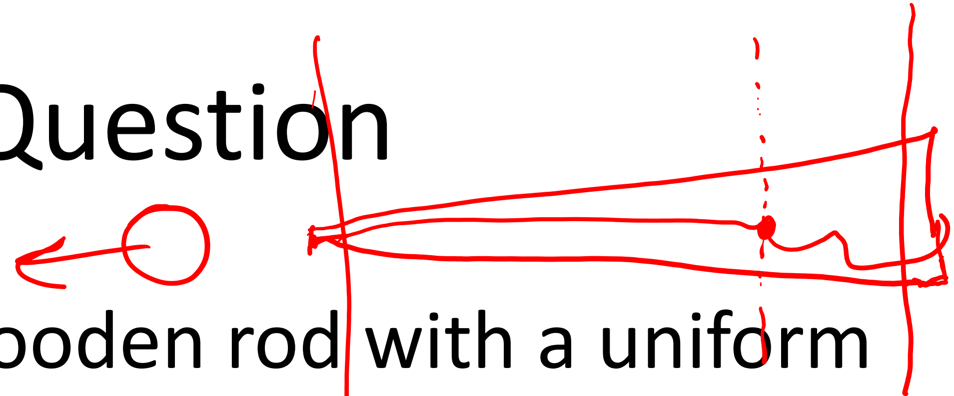
$$I_p = \sum m_i[(x_i - a)^2 + (y_i - b)^2]$$

$$= \underbrace{\sum m_i\left(x_i^2 + y_i^2\right)}_{I_{\text{cm}}} - 2a \overbrace{\underbrace{\sum m_i x_i}_{Mx_{\text{cm}} = 0}}^{=0} - 2b \overbrace{\underbrace{\sum m_i y_i}_{My_{\text{cm}} = 0}}^{=0}$$

$$+ \underbrace{\left(a^2 + b^2\right)}_{d^2} \underbrace{\sum m_i}_{M}$$

$$\frac{\Sigma m_i x_i}{M} = x_{cm} = 0$$

Question



- A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Does it have a larger moment of inertia

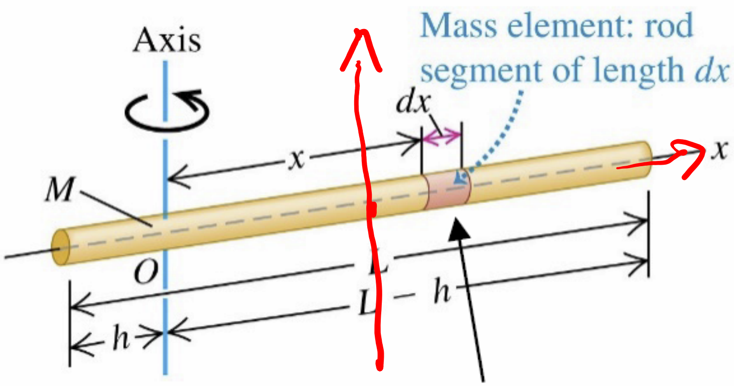
$$I = \sum_i m_i x_i^2 + Md^2$$

① for an axis through the thicker end of the rod and perpendicular to the length of the rod, or

② for an axis through the thinner end of the rod and perpendicular to the length of the rod?

Significance of the parallel axis theorem: need formula for I_{cm} only

Example A thin rod with uniform linear density $\rho = M/L$



⚠ Before calculating moment of inertia, must specify rotation axis

$$I = \sum m_i r_i^2 \longrightarrow \int r^2\, dm$$

⊥ distance of m_i to rotation axis

Choose a convenient mass element dm – a segment of length dx at a ⊥ distance x from the axis, and mass $dm = \rho dx$

$(L-h)(L-h)^2$

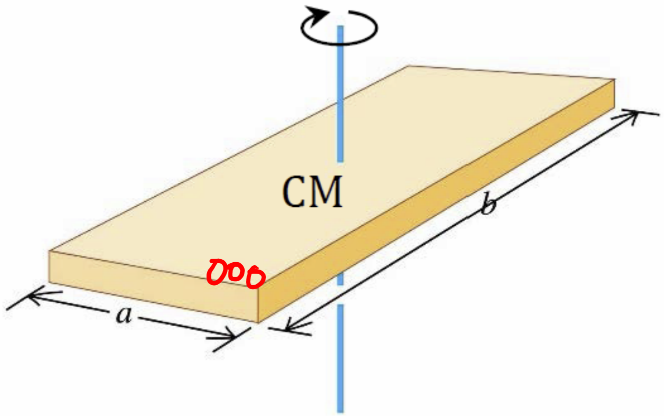
$\int_{-h}^{L-h} x^2 dx = \frac{x^3}{3}\Big|_{-h}^{L-h}$ $\quad (L-h)^3 = L^3 - 3Lh(L-h) - h^3$

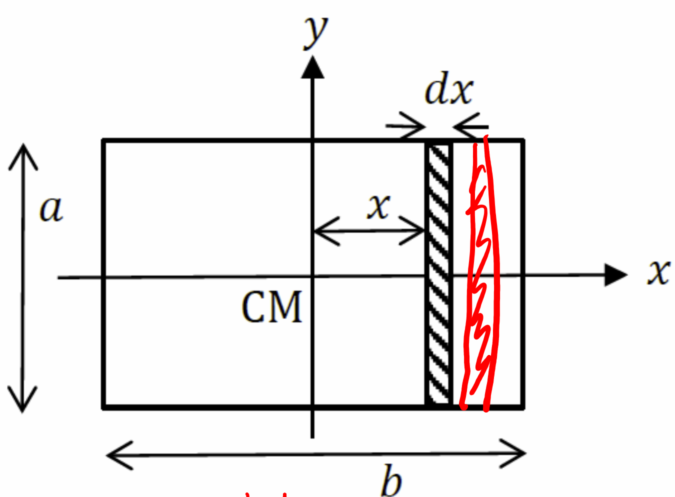
$$I_O = \int_{-h}^{L-h} x^2 \,(\rho dx) = \frac{\rho}{3}\left[(L-h)^3 + h^3\right] = \frac{M}{3}\left(L^2 - 3Lh + 3h^2\right)$$

⚠ Put $h = L/2$, we get $I_{\mathrm{cm}} = ML^2/12$.

⚠ Check the parallel axis theorem $I_O = I_{\mathrm{cm}} + M(\qquad\qquad)^2$

Example A rectangular plate





Choose the mass element dm to be a rod at $\perp$ distance x from the axis. *Why? Because you know its moment of inertia!*

$$dI = \underbrace{\frac{(dm)a^2}{12}}_{} + \underbrace{(dm)x^2}_{}$$

about CM *of the rod*, not of the plate

parallel axis theorem

$\frac{M}{\cancel{b}}\frac{a^2}{12}\cdot \cancel{b}$

Since $dm = \left(\frac{M}{b}\right) dx$

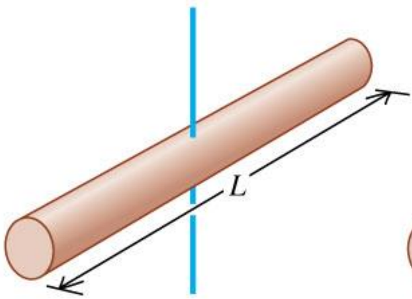
$$I = \int dI = \frac{M}{b}\int_{-b/2}^{b/2}\left[\frac{a^2}{12} + x^2\right] dx = \frac{1}{12}M(a^2 + b^2)$$

$$\frac{M}{b}\int_{-b/2}^{b/2} x^2 dx = \frac{M}{b}\frac{x^3}{3}\Big|_{-b/2}^{b/2} = \frac{M}{b}\cdot 2\frac{b^3}{3\cdot 8}$$

Table 9.2 Moments of Inertia of Various Bodies

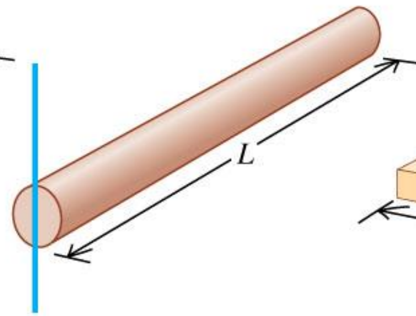
(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$



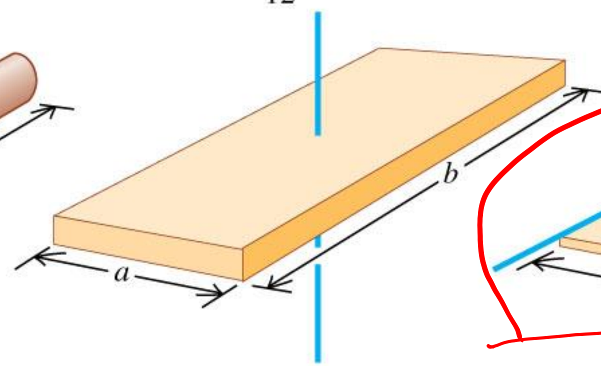
(b) Slender rod, axis through one end

$$I = \frac{1}{3}ML^2$$



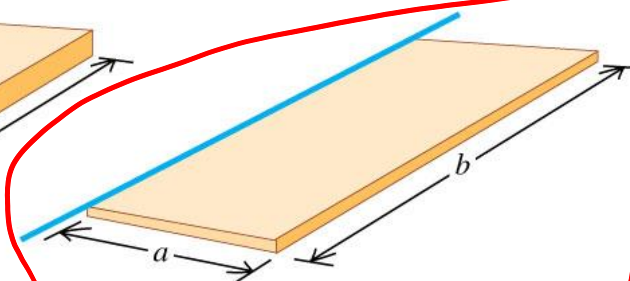
(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$



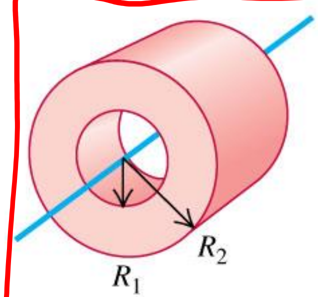
(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



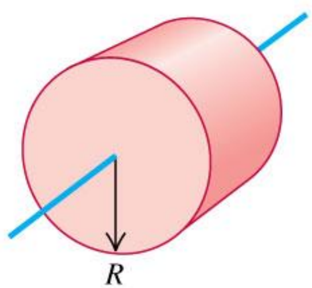
(e) Hollow cylinder

$$I = \frac{1}{2}M({R_1}^2 + {R_2}^2)$$



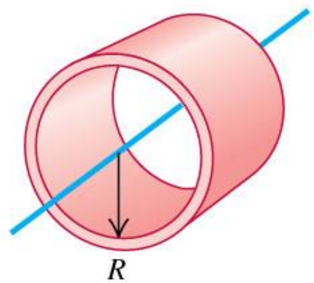
(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$



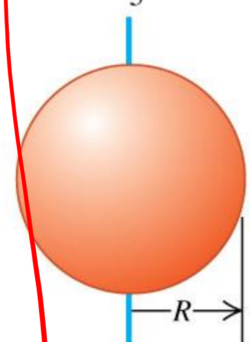
(g) Thin-walled hollow cylinder

$$I = MR^2$$



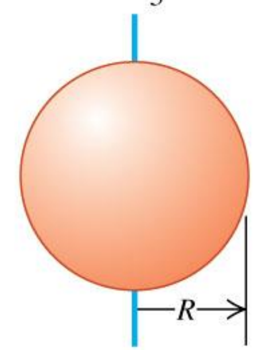
(h) Solid sphere

$$I = \frac{2}{5}MR^2$$



(i) Thin-walled hollow sphere

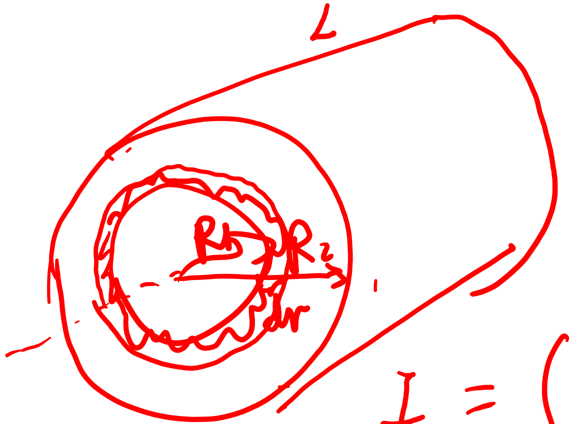
$$I = \frac{2}{3}MR^2$$



✓

$R_1 = 0$

$R_1 = R_2 = R$



$$I = \int r^2\, dm$$

$$dm = \rho\, dV = \rho\, 2\pi r\, dr \cdot L$$

$$I = \int_{R_1}^{R_2} 2\pi\rho L\, r^3\, dr = \frac{2\pi\rho L}{4}\, r^4 \Big|_{R_1}^{R_2}$$

$$= \frac{\pi\rho L}{2}\left(R_2^4 - R_1^4\right)$$

$$\rho = \frac{M}{V} = \frac{M}{\pi\left(R_2^2 - R_1^2\right)\cdot L}$$

$$= \frac{\cancel{\pi}\cancel{L}}{2}\,\frac{M}{\cancel{\pi}\left(R_2^2 - R_1^2\right)\cdot\cancel{L}}\left(R_2^4 - R_1^4\right)$$

$$\boxed{I = \frac{1}{2} M\left(R_2^2 + R_1^2\right)}$$

torque

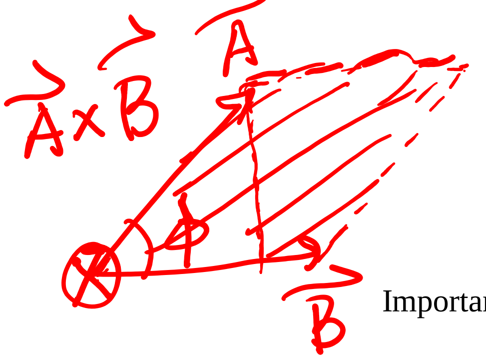
Vector (Cross) Product

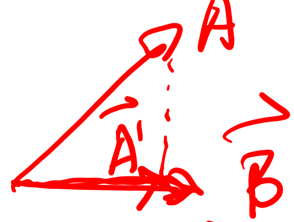
$\vec{A}\cdot\vec{B} = AB\cos\phi$
$= \vec{B}\cdot\vec{A}$

$\vec{C} = \vec{A}\times\vec{B}$

Magnitude: $C = AB\sin\phi$

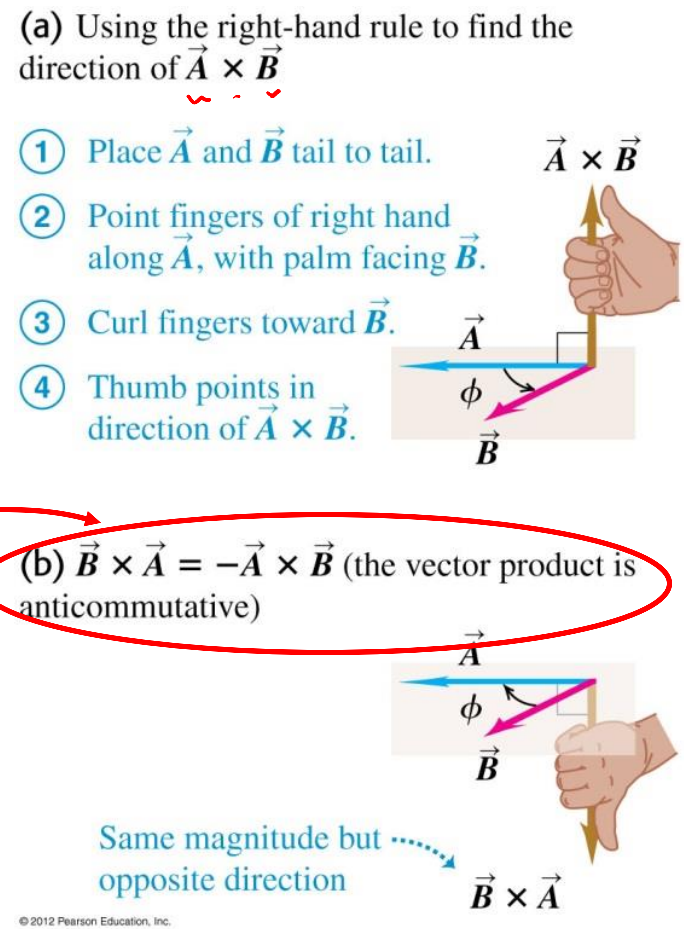
direction determined by *Right Hand Rule*



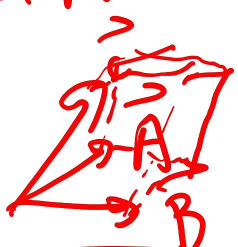


Important!

$\vec{A}\cdot\vec{B}$



$\cancel{\vec{C}\cdot(\vec{A}\times\vec{B})}$?



Special cases:

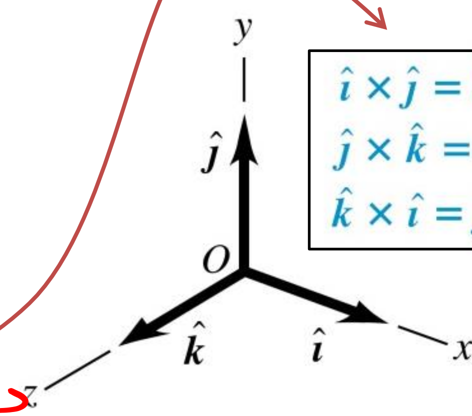
(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $\left|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right| = 0$,

$\vec{A}$ $\vec{B}$

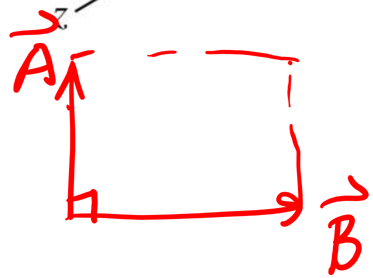
in particular, $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $\left|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right| = AB$

in particular,



$\hat{\imath} \times \hat{\jmath} = \hat{k}$
$\hat{\jmath} \times \hat{k} = \hat{\imath}$
$\hat{k} \times \hat{\imath} = \hat{\jmath}$



In analytical form (no need to memorize)

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$$
$$= \left(A_y B_z - A_z B_y\right)\hat{\imath} + \left(A_z B_x - A_x B_z\right)\hat{\jmath} + \left(A_x B_y - A_y B_x\right)\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

3x3

linear algebra

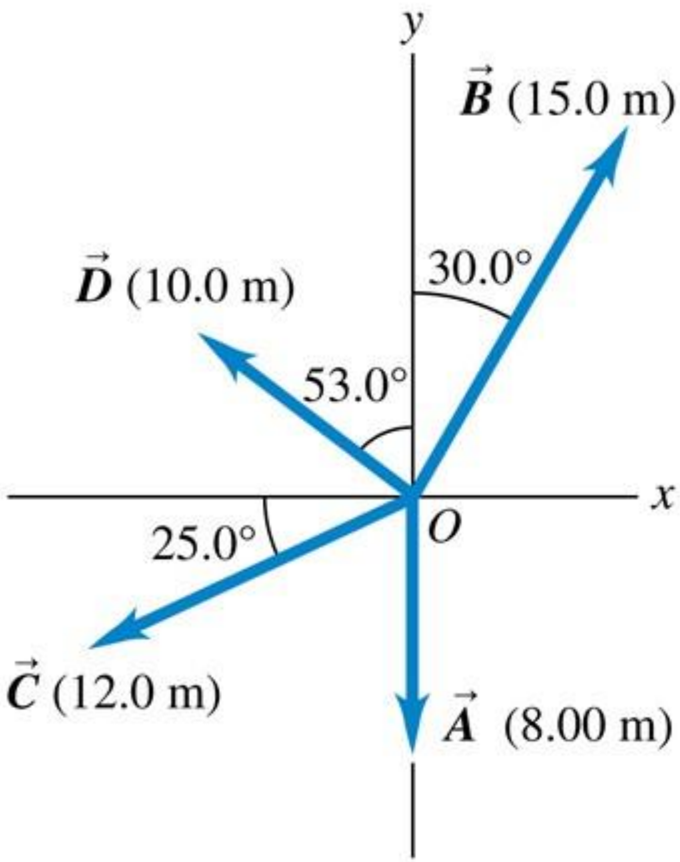
don't worry if you have not learnt determinants in high school

matrix?

big data
nxn

Q1.14

Consider the vectors shown. What is the cross product $\vec{A} \times \vec{C}$?



A. $(96.0\ \text{m}^2)\sin 25.0°\hat{\boldsymbol{k}}$

B. $(96.0\ \text{m}^2)\cos 25.0°\hat{\boldsymbol{k}}$

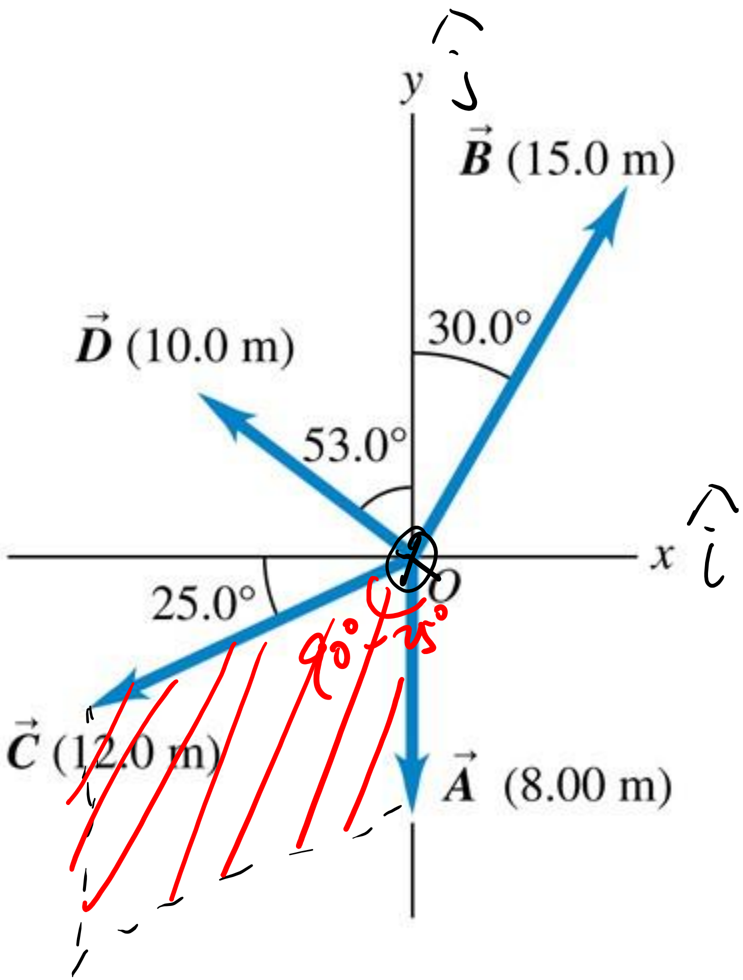
C. $-(96.0\ \text{m}^2)\sin 25.0°\hat{\boldsymbol{k}}$

D. $-(96.0\ \text{m}^2)\cos 25.0°\hat{\boldsymbol{k}}$

E. none of these

A1.14

Consider the vectors shown. What is the cross product $\vec{A} \times \vec{C}$?



A. $(96.0\ \mathrm{m}^2)\sin 25.0°\,\hat{\boldsymbol{k}}$

B. $(96.0\ \mathrm{m}^2)\cos 25.0°\,\hat{\boldsymbol{k}}$

C. $-(96.0\ \mathrm{m}^2)\sin 25.0°\,\hat{\boldsymbol{k}}$

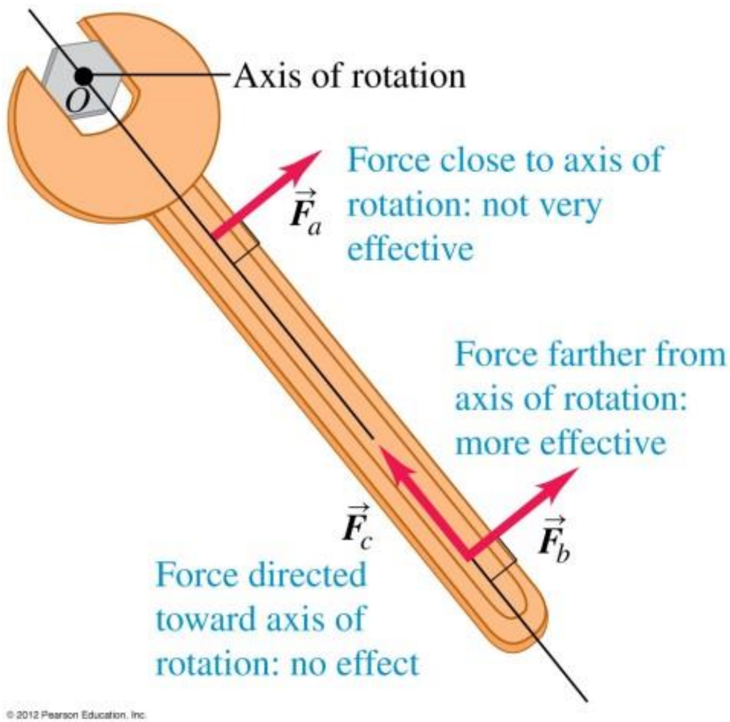
D. $-(96.0\ \mathrm{m}^2)\cos 25.0°\,\hat{\boldsymbol{k}}$

E. none of these

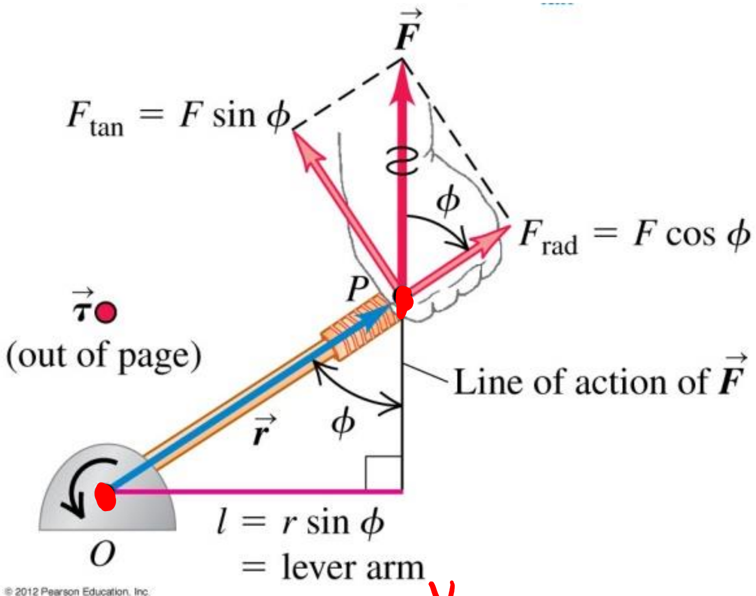
$|A|\cdot|C|\cdot\sin(90°-25°) = \cos 25°$

Torque

Besides magnitude and direction, the **line of action** of a force is important because it produces rotation effect.



$\vec{\boldsymbol{F}}_a$ and $\vec{\boldsymbol{F}}_b$ have the same magnitudes and directions, but different line of action: they produce different physical effects – which force would you apply if you were to tighten/loosen the screw?



"$\vec{r} \perp$ axis"

Define **torque** about a point O as a vector $\vec{r}$: "leverarm vector"

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} = -\vec{F} \times \vec{r}$$

$\vec{\tau}$ is $\perp$ to both $\vec{r}$ and $\vec{F}$

Magnitude:

$$\tau = r\underbrace{(F\sin\phi)}_{\text{component of } \vec{F} \perp \text{ to } \vec{r}} = \underbrace{(r\sin\phi)}_{\perp \text{ distance from } O \text{ to line of actions of } \vec{F}}F$$

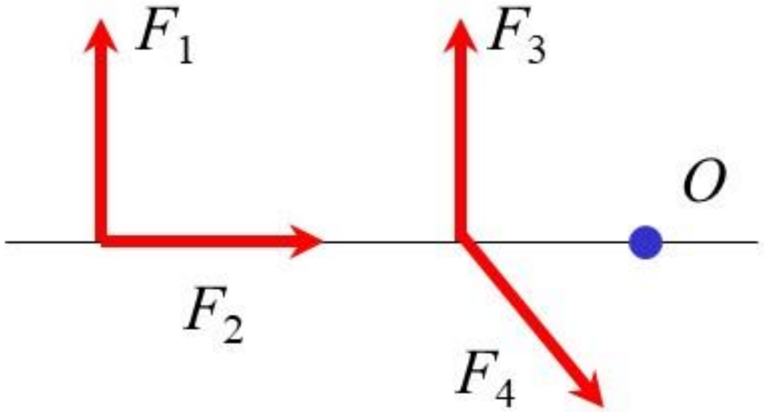
Direction gives the sense of rotation about O through the right-hand-rule.

Notation: $\odot$ out of the plane
$\otimes$ into the plane

SI unit for torque: Nm (just like work done)

Q10.2

Which of the four forces shown here produces a torque about O that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

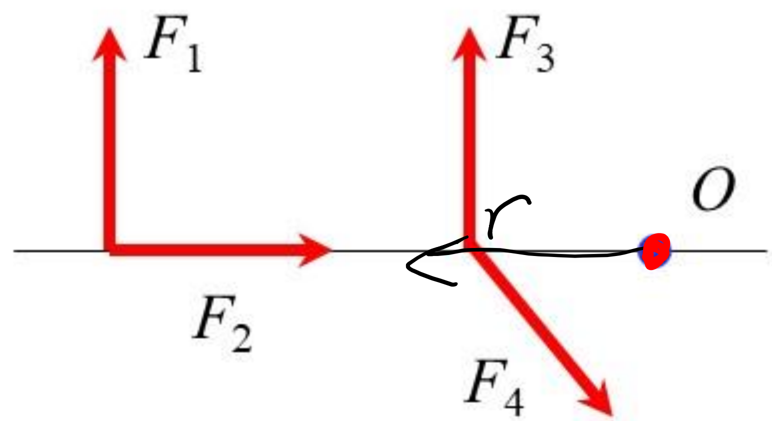
C. F_3

D. F_4

E. more than one of these

A10.2

Which of the four forces shown here produces a torque about O that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

C. F_3

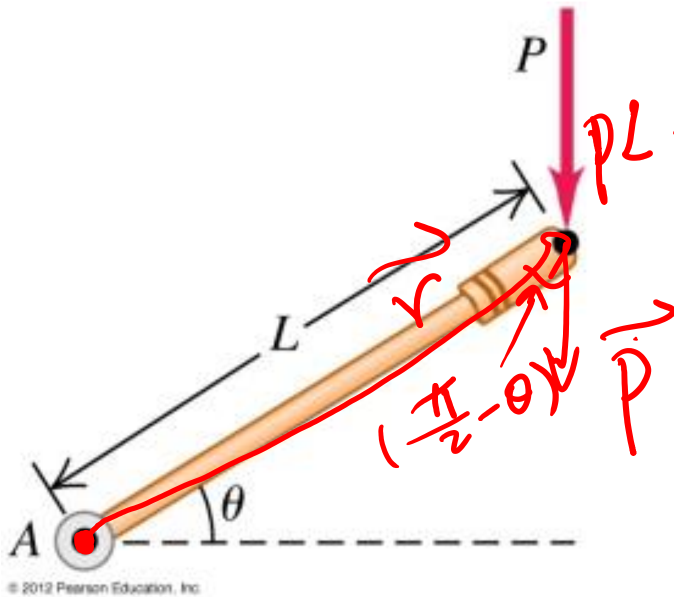
✔ D. F_4

E. more than one of these

$\odot$ $\vec{r} \times \vec{F}$

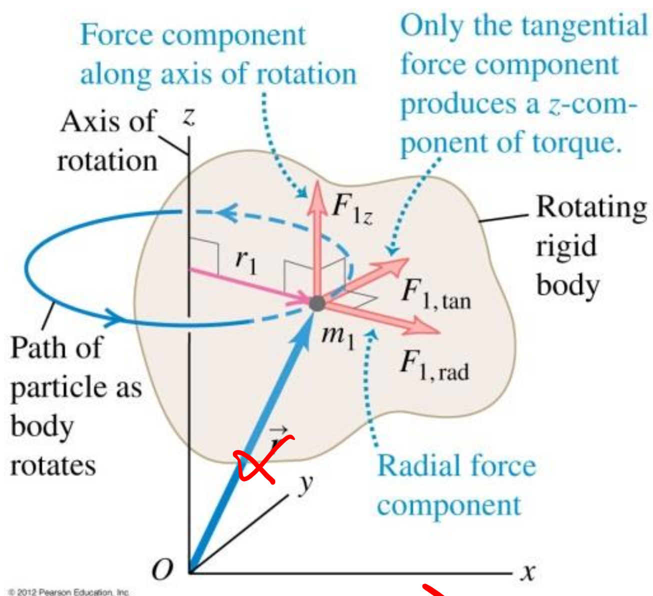
Question

A force P is applied to one end of a lever of length L. The magnitude of the torque of this force about point A is ($PL \sin\theta$ / $PL \cos\theta$ / $PL \tan\theta$)



$PL \sin(\frac{\pi}{2} - \theta)$

Suppose a rigid body is rotating about a fixed axis which we arbitrarily call the *z* axis.
m_1 is a small part of the total mass.



$F_{1,\text{rad}}$, $F_{1,\tan}$, and $F_{1,z}$ are the 3 components of the total force acting on m_1

Only $F_{1,\tan}$ produces the desired rotation, $F_{1,\text{rad}}$ and $F_{1,z}$ produce some other effects which are irrelevant to the rotation about the *z* axis.

$$F_{1,\tan} = m_1 a_{1,\tan} = m_1(r_1\alpha_z)$$

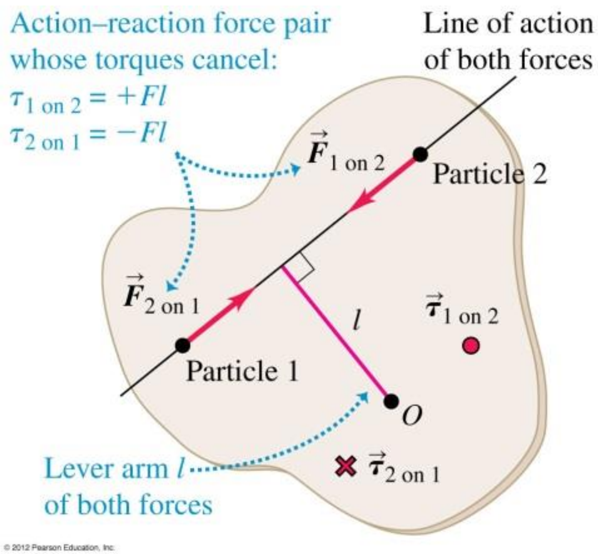
$$\underbrace{F_{1,\tan} r_1}_{\text{torque on } m_1 \text{ about } z,\ \tau_{1z}} = m_1 r_1^2 \alpha_z$$

$$\text{“}\tau = \vec{r}_1 \times \vec{F}\text{”} = \vec{r}_1 \times (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) = \vec{r}_1 \times F_x\hat{i} + \vec{r}_1 \times F_y\hat{j} + \vec{r}_1 \times F_z\hat{k}$$

Sum over all mass in the body, since they all have the same α_z

$$\boxed{\sum \tau_{iz} = \left(\sum m_r r_i^2\right)\alpha_z = I\alpha_z}$$

$$\Sigma F_i = Ma$$



Need to consider torque due to external forces only. Internal forces (action and reaction pairs) produce equal and opposite torques which have no net rotational effect.

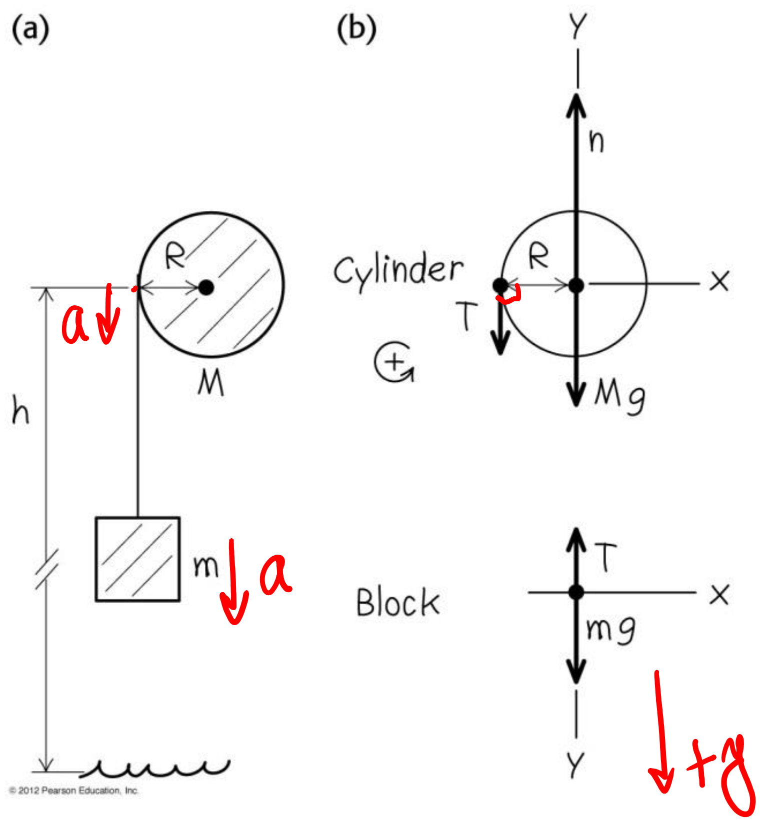
Conclusion: for rigid body rotation about a fixed axis,

$$\boxed{\sum \tau_{\text{ext}} = I\alpha}$$

c.f. Newton's second law $\sum \vec{\boldsymbol{F}}_{\text{ext}} = M\vec{\boldsymbol{a}}$

Example

Pulley rotates about a fixed axis. What is the acceleration *a* of the block?



For the cylinder

$$\underbrace{TR}_{\text{torque due to } T} = \underbrace{\left(\tfrac{1}{2}MR^2\right)}_{\text{moment of inertia of cylinder}} \underbrace{\left(\frac{a}{R}\right)}_{\text{angular acceleration}}$$

i.e. $T = \frac{1}{2}Ma$

For the block

$$mg - T = ma$$

Therefore

$$\boxed{a = \frac{g}{1 + M/2m}}$$

$v_0 = 0$

Suppose the block is initially at rest at height *h*. At the moment it hits the floor:

$$v^2 = \overset{v_0^2}{0} + \overset{2ah}{2\left(\frac{g}{1 + M/2m}\right)h} \implies \boxed{v = \sqrt{\frac{2gh}{1 + M/2m}}}$$

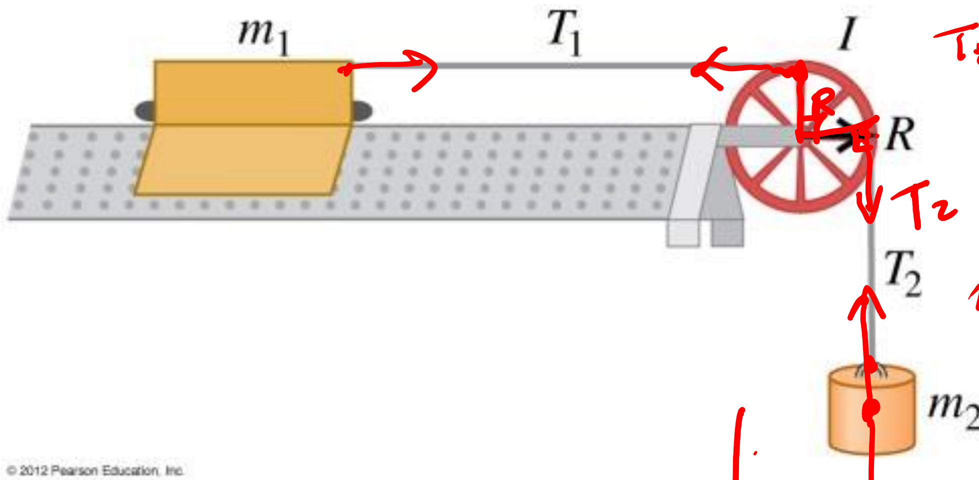
c.f. lecture 8 in which we get the same result using energy conservation.

Question

Mass m_1 slides on a frictionless track. The pulley has moment of inertia *I* about its rotation axis, and the string does not slip nor stretch. When the hanging mass m_2 is released, arrange the forces T_1, T_2, and $m_2 g$ in increasing order of magnitude.

$T_1 = m_1 a$

$T_2 R - T_1 R = I \frac{a}{R} > 0$



$m_2 g - T_2 = m_2 a > 0$

$m_2 g > T_2 > T_1$