Math1014 Final Exam, Spring 2013
MC Answers
`White Version`

Question	1	2	3	4	5	6	7	8	9	10
Answer	c	c	a	c	c	d	a	b	b	e
Question	11	12	13	14	15	16	17	18	19	20
Answer	c	e	c	c	b	d	a	b	c	a

Part II: Long Questions

21. [*10 pts*] Evaluate the following improper integrals.

(a) For any positive integer n, use integration by parts to find a number c_n such that $\int_{-\infty}^{\infty} x^n e^{-x^2}\,dx = c_n \int_{-\infty}^{\infty} x^{n-2}e^{-x^2}\,dx$. Hence or otherwise find the ratio $\dfrac{\int_{-\infty}^{\infty} x^{2n}e^{-x^2}dx}{\int_{-\infty}^{\infty} e^{-x^2}dx}$. [6 pts]

Solution

$$\int_{-\infty}^{\infty} x^n e^{-x^2}dx = \int_{-\infty}^{\infty} -\frac{1}{2}x^{n-1}de^{-x^2}$$

$$= -\frac{1}{2}x^{n-1}e^{-x^2}\Big|_{-\infty}^{\infty} + \frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2}e^{-x^2}dx$$

$$= \frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2}e^{-x^2}dx$$

So $c_n = \frac{n-1}{2}$, and

$$\frac{\int_{-\infty}^{\infty} x^{2n}e^{-x^2}dx}{\int_{-\infty}^{\infty} e^{-x^2}dx} = \frac{\frac{2n-1}{2}\int_{-\infty}^{\infty} x^{2n-2}e^{-x^2}dx}{\int_{-\infty}^{\infty} e^{-x^2}dx}$$

$$= \frac{\frac{2n-1}{2}\frac{2n-3}{2}\int_{-\infty}^{\infty} x^{2n-4}e^{-x^2}dx}{\int_{-\infty}^{\infty} e^{-x^2}dx}$$

$$= \frac{(2n-1)(2n-3)\cdot\dots\cdot 2\cdot 1}{2^n}\cdot\frac{\int_{-\infty}^{\infty} e^{-x^2}dx}{\int_{-\infty}^{\infty} e^{-x^2}dx}$$

$$= \frac{(2n-1)(2n-3)\cdot\dots\cdot 3\cdot 1}{2^n}$$

(b) $\displaystyle\int_2^{\infty} \frac{1}{(x+7)\sqrt{x-2}}\,dx$ [4 pts]

Solution

Make a substitution $t = \sqrt{x-2}$ such that $t^2 = x-2$, $2tdt = dx$. Then

$$\int_2^{\infty} \frac{1}{(x+7)\sqrt{x-2}}\,dx = \int_0^{\infty} \frac{2t}{(t^2+9)t}\,dt$$

$$= \int_0^{\infty} \frac{2}{3}\frac{1}{1+(t/3)^2}\,d(t/3) \qquad \text{(or let } u = 3\tan\theta)$$

$$= \frac{2}{3}\tan^{-1}\frac{t}{3}\Big|_0^{\infty}$$

$$= \frac{2}{3}\cdot\left(\frac{\pi}{2}-0\right) = \frac{\pi}{3}$$

22. [*10 pts*] Determine whether the given series is convergent or divergent. Given brief reason to justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^{\frac{3}{2}}+2}{2n^{\frac{3}{2}}}\right)^n$ [2 pts]

Solution

Convergent by the root test, since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n^{\frac{3}{2}}+2}{2n^{\frac{3}{2}}}\right)^n}=\lim_{n\to\infty}\frac{n^{\frac{3}{2}}+2}{2n^{\frac{3}{2}}}=\frac{1}{2}<1$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ [2 pts]

Solution

Convergent by the alternating series test, since $\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ is a decreasing sequence with 0 as its limit as $n\to\infty$, because $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{-n}}{\sqrt{n}}$ [3 pts]

Solution

Convergent by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty}e^{-n}$, where $e^{-1}<1$, since $\frac{e^{-n}}{\sqrt{n}}\le e^{-n}$ for all $n\ge 1$. Or, use ratio test.

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{\tan^{-1}n}{n(\ln n)^2}$ [3 pts]

Solution

Convergent. By $\tan^{-1}n<\pi$ for all $n\ge 2$, we have $\dfrac{\tan^{-1}n}{n(\ln n)^2}\le\dfrac{\pi}{n(\ln n)^2}$.

Note that the series $\displaystyle\sum_{n=2}^{\infty}\frac{\pi}{n(\ln n)^2}$ is convergent by the Integral Test:

$$\int_2^{\infty}\frac{\pi}{x(\ln x)^2}dx=\left[-\frac{\pi}{2\ln x}\right]_2^{\infty}=\frac{\pi}{2\ln 2}<\infty$$

Hence by the Comparison Test, the given series converges.

23. [*10 pts*] Consider a power series $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n^{2n+1}}{(2n+1)!}(x-1)^{2n+1}$.

(a) Find the radius of convergence of the given power series. [8 pts]

Solution

By applying the Ratio Test,

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{2(n+1)+1}}{(2(n+1)+1)!}|x-1|^{2(n+1)+1}}{\frac{n^{2n+1}}{(2n+1)!}|x-1|^{2n+1}} = |x-1|^2 \lim_{n\to\infty} \frac{(n+1)^{2n+3}}{(2n+3)(2n+2)n^{2n+1}}$$

$$= |x-1|^2 \lim_{n\to\infty} \frac{(n+1)^2(n+1)^{2n+1}}{(2n+3)(2n+2)n^{2n+1}}$$

$$= |x-1|^2 \lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)(2n+2)} \cdot (1+\frac{1}{n})^{2n}(1+\frac{1}{n})$$

$$= \frac{e^2}{4}|x-1|^2 < 1$$

Hence from $|x-1| < \frac{2}{e}$, the radius of convergence is $\frac{2}{e}$.

(b) Find the Taylor Series for the **derivative function** f' centered at 1. [2 pts]

Solution

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n^{2n+1}}{(2n)!}(x-1)^{2n}$$

24. [*10 pts*] A cone container of top radius 4 m and height 8 m is fully filled with water. A ball of radius 3 m is inserted slowly as far as possible into the container to expel as much water as possible.

(a) Find the amount of water spilled out of the container. [5 pts]

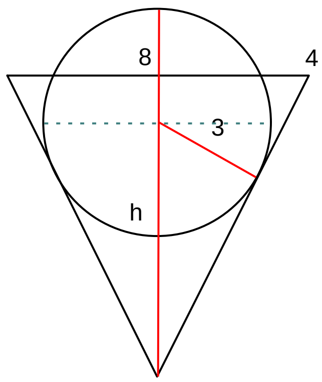
Solution

The amount of water expelled is the volume of the part of the ball inside the container, sitting on the surface of the container. If h is the distance of the center of the ball from the bottom tip of the container,

$$\frac{h}{3} = \frac{\sqrt{8^2+4^2}}{4} \iff h = 3\sqrt{5}$$

The amount of water spilled out is:

$$V = \frac{2}{3}\pi(3)^3 + \int_0^{8-3\sqrt{5}} \pi(9-y^2)dy$$
$$= 18\pi + \pi\left[9y - \frac{1}{3}y^3\right]_0^{8-3\sqrt{5}}$$
$$= 18\pi + \pi(72 - 27\sqrt{5} - \frac{1}{3}(8-3\sqrt{5})^3) \qquad (\mathrm{m}^3)$$



Or

$$V = \frac{4}{3}\pi(3)^3 - \int_{8-3\sqrt{5}}^{3}(9-y^2)dy$$

Or whatever correct way to do it.

(b) Take your answer in part (a), or just denote it by V, and remove the ball from the cone container. Express the work required to pump all the remaining water (water density denoted by ρ, gravity acceleration denoted by g) to the top of the container by an integral. **You do not need to evaluate the integral.** [5 pts]

Solution

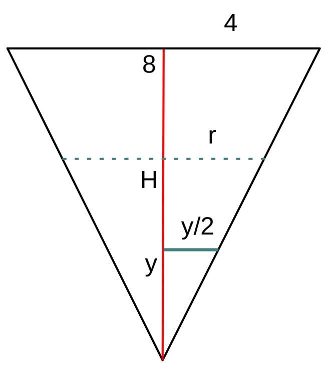
After removing the ball, the depth of water remaining in the container H and the radius of the surface of the water r satisfies

$$\frac{r}{H} = \frac{4}{8} \iff r = \frac{1}{2}H$$

The volume of the water in the cone is

$$\frac{1}{3}\pi(4)^2 \cdot 8 - V = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{\pi}{12}H^3$$

$$H = \sqrt[3]{\frac{4(128\pi - 3V)}{\pi}}$$



The work required to pump the water to the top of the container is

$$W = \int_0^H \rho g(8-y)\pi\left(\frac{y}{2}\right)^2 dy \qquad \text{in joules}$$