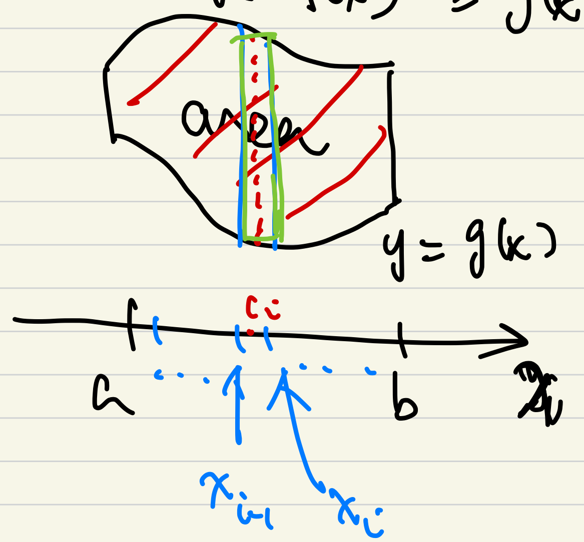
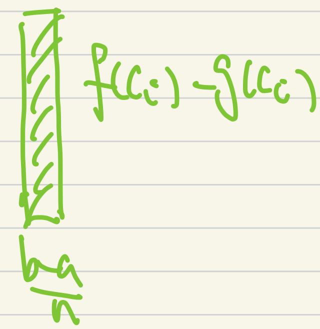
Area Between Curves

①

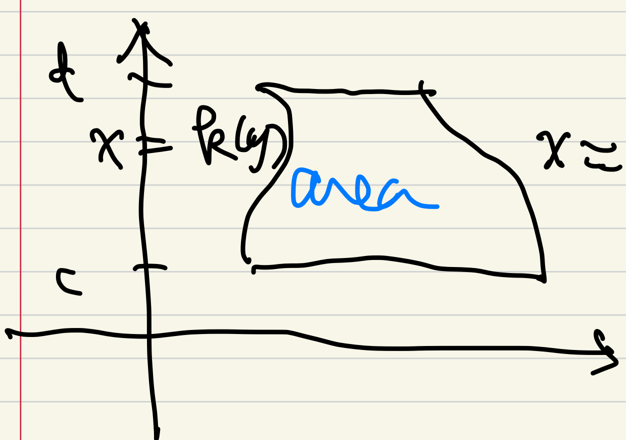


$$\text{area} = \int_a^b [f(x) - g(x)]\,dx$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n}[f(c_i) - g(c_i)]\right]$$

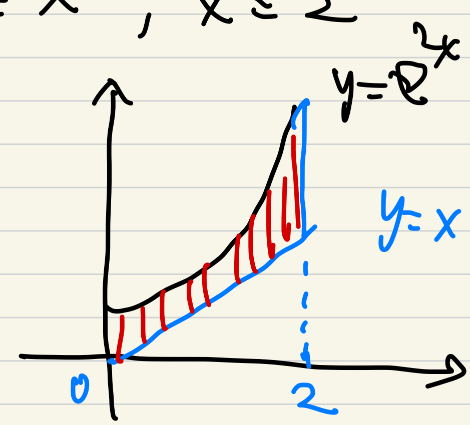


②



$$\text{area} = \int_c^d [h(y) - k(y)]\,dy$$

Example Find the area of the region bounded by $y = e^{2x}$, $y = x$, $x = 2$ and the y-axis.



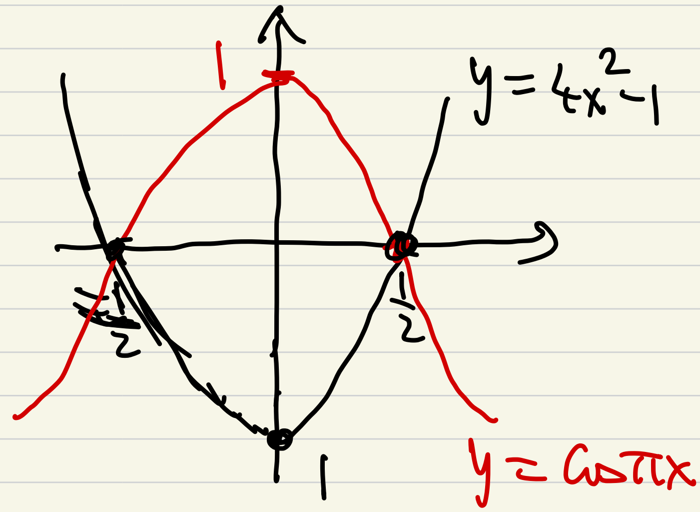
$$\text{area} = \int_0^2 (e^{2x} - x)\,dx$$

$$= \left[\frac{e^{2x}}{2} - \frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{e^4}{2} - 2\right) - \left(\frac{1}{2}\right) = \frac{e^4}{2} - \frac{5}{2} \quad \text{(sq. units)}$$

Example. Find the area enclosed by the curves $y = 4x^2 - 1$ and $y = \cos\pi x$

$$\text{area} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\cos\pi x - (4x^2 - 1)\right] dx$$



$$= \left[\frac{\sin\pi x}{\pi} - \frac{4}{3}x^3 + x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

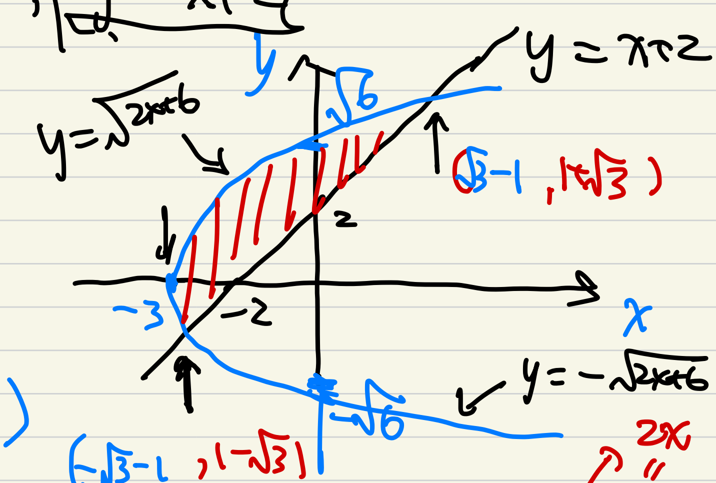
$$= \left[\frac{1}{\pi} - \frac{1}{6} + \frac{1}{2}\right] - \left[-\frac{1}{\pi} + \frac{1}{6} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi} + \frac{2}{3} \quad \text{(sq units)}$$

Example, $y^2 = 2x + 6$, $\boxed{y = x + 2}$ $\leftrightarrow$ $2y = 2x + 4$

$x = \frac{1}{2}y^2 - 3$

$x = 0, \quad y^2 = 6$



area( ) = area( ) + area( )

$$= \int_{-3}^{-\sqrt{3}-1} 2\cdot\sqrt{2x+6}\, dx + \int_{-\sqrt{3}-1}^{\sqrt{3}-1} \left[\sqrt{2x+6} - (x+2)\right] dx$$

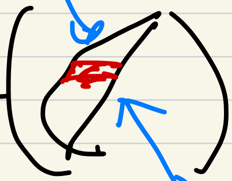
$y^2 = \boxed{2y - 4} + 6$, $2x$

$y^2 - 2y - 2 = 0$

$y = \frac{2 \pm \sqrt{4+8}}{2}$

$y = 1 \pm \sqrt{3}$

$x=\frac{1}{2}y^2-3$

area() $= \displaystyle\int_{1-\sqrt{3}}^{1+\sqrt{3}} \left[y-2-\left(\tfrac{1}{2}y^2-3\right) \right] dy$

$x = y-2$

$$\begin{cases} x = y-2 \\ x = \frac{1}{2}y^2-3 \end{cases}$$

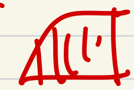
$$= \left[\frac{y^2}{2} - 2y - \frac{1}{6}y^3 + 3y \right]_{1-\sqrt{3}}^{1+\sqrt{3}}$$

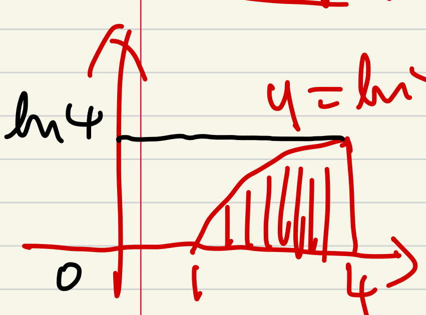
$$= \left[\frac{(1+\sqrt{3})^2}{2} - 2(\cancel{1}+\sqrt{3}) - \frac{1}{6}(1+\sqrt{3})^3 + 3(\cancel{1}+\sqrt{3}) \right]$$

$$- \left[\frac{(1-\sqrt{3})^2}{2} - 2(\cancel{1}-\sqrt{3}) - \frac{1}{6}(1-\sqrt{3})^3 + 3(\cancel{1}-\sqrt{3}) \right]$$

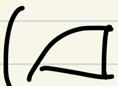
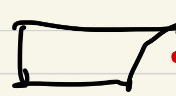
$$= 2\sqrt{3} + \frac{(1+\sqrt{3})^2 - (1-\sqrt{3})^2}{2}$$

$$- \frac{1}{6}\left[(1+\sqrt{3})^3 - (1-\sqrt{3})^3 \right].$$

Example. $\displaystyle\int_1^4 \ln x \, dx$ = area()



(integration by parts) → 1^{st} way!

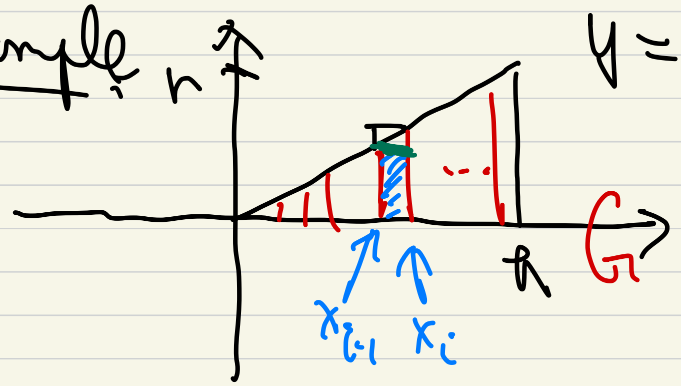
area() =  − 

$$= 4\ln 4 - \int_0^{\ln 4} e^y \, dy = 4\ln 4 - \left[e^y \right]_0^{\ln 4}$$

$$= 4\ln 4 - 4 + 1$$

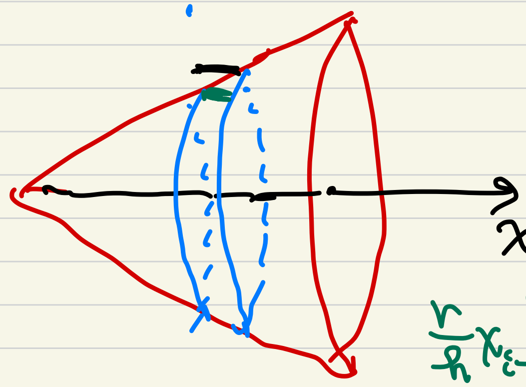
Volume by Slicing :

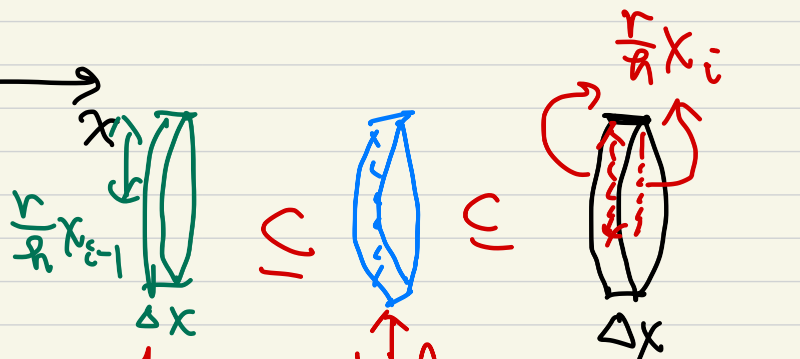
Example :



$$y = \frac{r}{h}x$$

notice the area about x-axis to generate a solid cone





Volume

$$\sum \pi\left(\frac{r}{h}x_{i-1}\right)^2 \Delta x \le \sum V_i \le \sum \pi\left(\frac{r}{h}x_i\right)^2 \Delta x$$

$n \to \infty$ $\qquad$ $n \to \infty$

$$\int_0^h \pi \frac{r^2}{h^2}x^2\,dx \le \sum V_i \le \int_0^h \pi \frac{r^2}{h^2}x^2\,dx$$

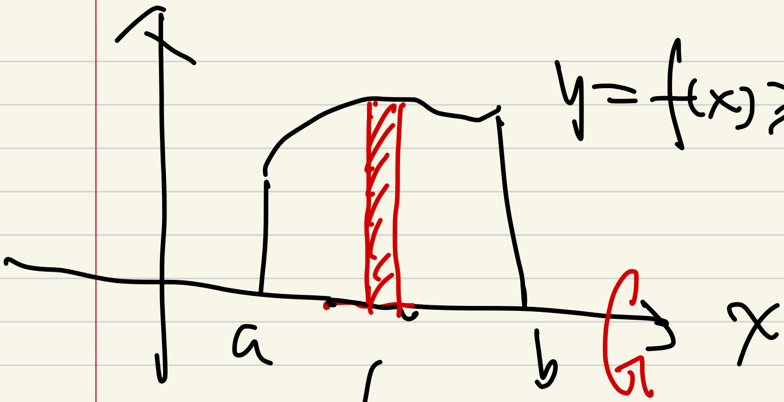
Volume of the cone

$$\parallel$$

$$\int_0^h \pi\left(\frac{r}{h}\right)^2 x^2\,dx$$

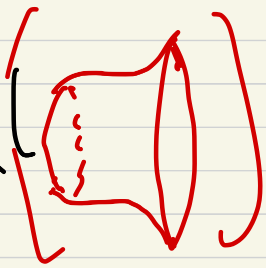
$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{1}{3}\pi r^2 h$$

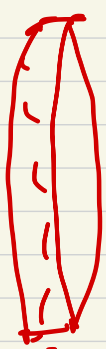
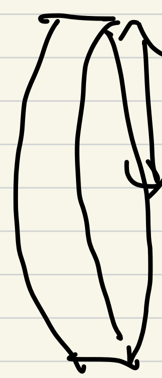


$y = f(x) \geq 0$

$\downarrow$ solid of revolution

volume() $= \int_a^b \pi [f(x)]^2 \, dx$

$\uparrow$ $n \to \infty$

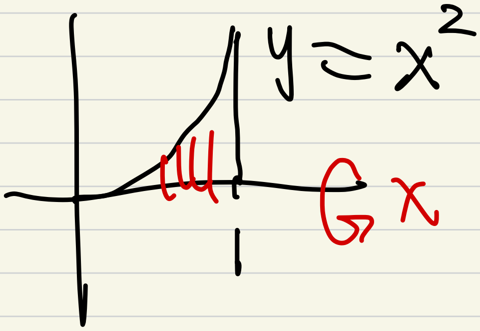
$\sum$ vol  $\min_{x_{i-1} \leq x \leq x_i} f(x)$ $\leq \sum$ vol  $\leq \sum$ vol  $\max_{x_{i-1} \leq x \leq x_i} f$

Δx $\qquad$ Δx $\qquad$ Δx

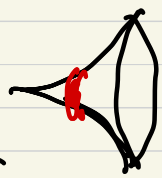
$\downarrow n \to \infty$ $\qquad\qquad$ $\downarrow n \to \infty$

$$\int_a^b \pi [f(x)]^2 \, dx \qquad\qquad \int_a^b \pi [f(x)]^2 \, dx$$

Example

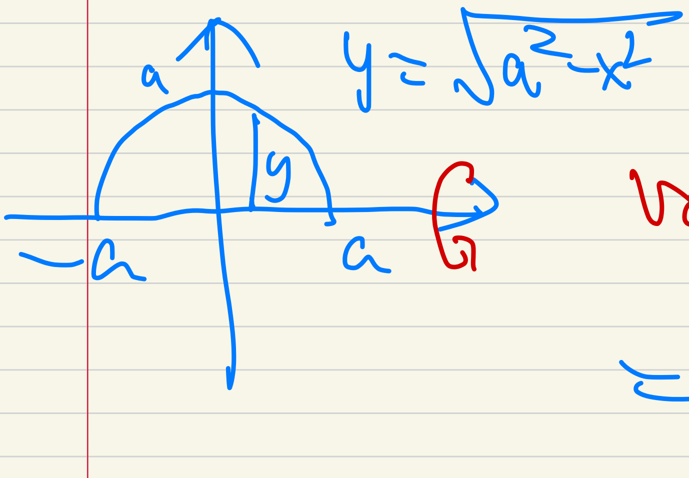


$\longrightarrow$ solid of revolution about the x-axis

vol() $= \displaystyle\int_0^1 \pi \left[x^2\right]^2 dx$

$$= \left[\frac{\pi x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

Example. Volume of a sphere

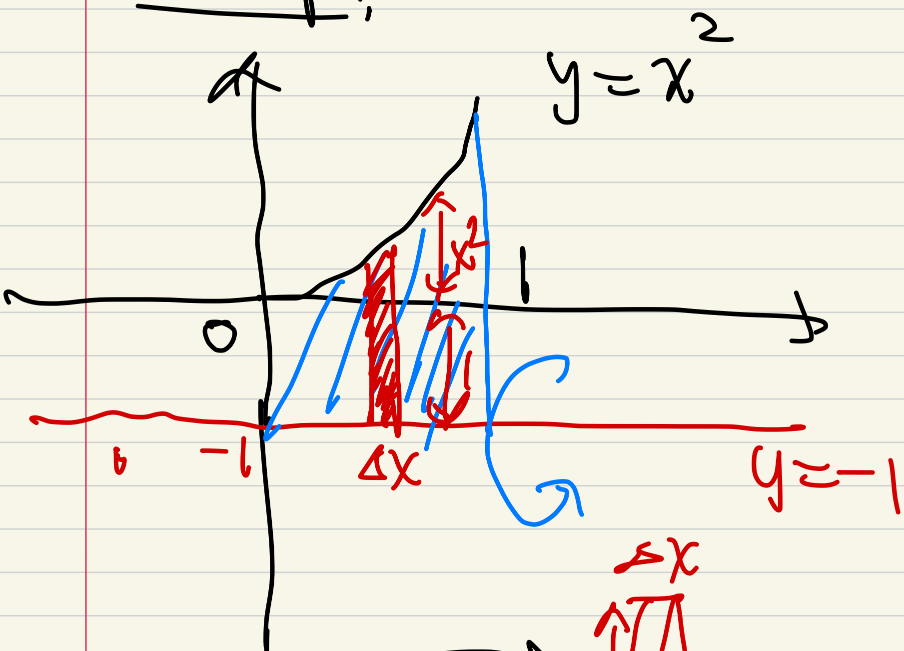


vol() $= \displaystyle\int_{-a}^{a} \pi y^2 \, dx$

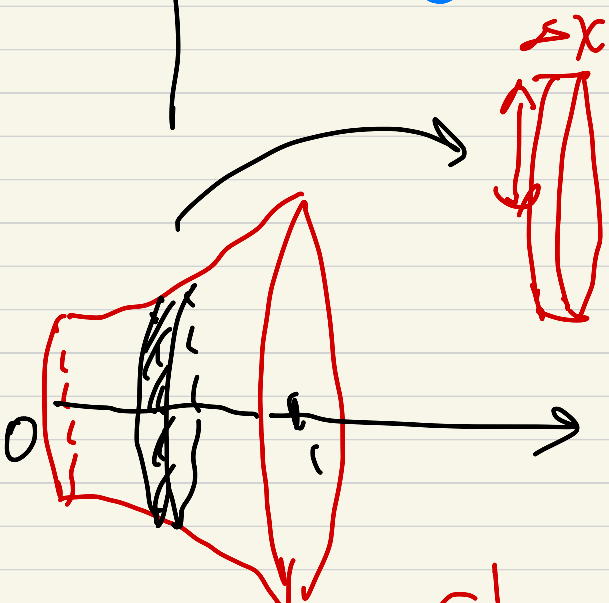
$$= \int_{-a}^{a} \pi (a^2 - x^2)\, dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4\pi a^3}{3}$$

Example



Rotate the area about the line $y = -1$



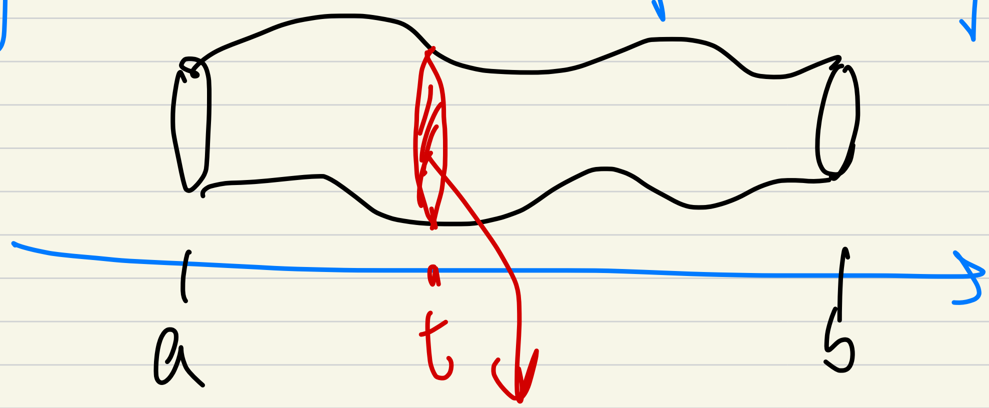
radius at x
$= x^2 + 1$

$$\text{volume} = \int_0^1 \pi \left[x^2+1\right]^2 dx$$

$$= \pi \int_0^1 (x^4 + 2x^2 + 1)\, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{2}{3}x^3 + x\right]_0^1 = \pi\left(\frac{1}{5} + \frac{2}{3} + 1\right)$$

General method of slicing



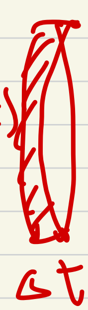
Cross section area at t

t: x-axis, y-axis or other line

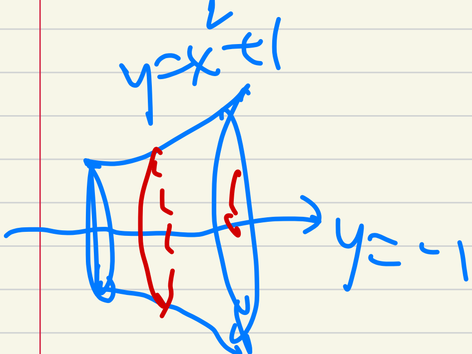
volume of the solid $= \displaystyle\int_a^b \underbrace{A(t)\,dt}$

summing as an integral

thin volume $A(t)$



$\approx A(t)\Delta t$



volume $= \displaystyle\int_0^1 \quad dx$

x^2+1

area

$\pi[x^2+1]^2$