# Partial Fractions

$$\int \boxed{\frac{p(x)}{q(x)}}\, dx \quad \longleftrightarrow$$

a proper rational function

It can be turned into integrals of the form

$$\longrightarrow \int \boxed{\frac{1}{(x-a)^k}}\, dx \quad \checkmark \text{ easy}$$

or

$$\longrightarrow \int \boxed{\frac{Ax+B}{[\underbrace{x^2+bx+c}]^k}}\, dx$$

after completing the square!

$$(x-\alpha)^2+\beta^2$$

$$\int \frac{\boxed{A(x-\alpha)} + (B-\alpha A)}{[(x-\alpha)^2+\beta^2]^k}$$

$$\frac{A}{2}\int \frac{\cancel{2}\,2(x-\alpha)}{[(x-\alpha)^2+\beta^2]}\, dx \quad + (B-\alpha A)\int \frac{dx}{[(x-\alpha)^2+\beta^2]^k}$$

$$\Vert \quad u = x-\alpha$$

$$\frac{A}{2}\ln|(x-\alpha)^2+\beta^2|$$

$$(x-\alpha) = \beta\tan\theta$$

$$\frac{1}{\beta^{2k-1}}\int \cos^{2k-2}\theta\, d\theta = \int \frac{\beta\cdot\sec^2\theta}{[\beta^2\sec^2\theta]^k}\, d\theta$$

a standard integral!

Example:

$$\int \frac{\boxed{dx}}{\sqrt{x}-1}$$

$u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2}$

$2\,du = \frac{1}{\sqrt{x}} dx = \frac{dx}{u}$

$dx = 2u\,du$

$$= \int \boxed{\frac{2u}{u-1}}\, du$$

not proper

$$= \int \left[2 + \underbrace{\frac{2}{u-1}}_{\text{proper}}\right] du = 2u + 2\ln|u-1| + C$$

$$= 2\sqrt{x} + 2\ln|\sqrt{x}-1| + C$$

In general, if

$\frac{p(x)}{q(x)}$ is not proper

(degree $p(x) \geq$ degree $q(x)$)

$$\xrightarrow{\text{long division}} g(x)\, q(x) + \underbrace{\frac{r(x)}{q(x)}}_{\text{proper.}}$$

$$\begin{array}{r} g(x) \\ q(x)\overline{)\, p(x)} \\ \underline{q(x)g(x)} \\ r(x) \end{array}$$

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Example $\int \ln(x^2+1)\, dx$ ← integration by parts

$$= x\ln(x^2+1) - \int \boxed{x \cdot \frac{2x}{x^2+1}}\, dx$$

not proper.

$\frac{d}{dx}\ln(x^2+1) = \frac{2x}{x^2+1}$

$\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1}$

$$= x\ln(x^2+1) - \int \left(2 - \frac{2}{x^2+1}\right) dx$$

proper.

$$= x\ln(x^2+1) - 2x + 2\tan^{-1}x + C.$$

Example:

$$\rightarrow \int \frac{1}{(e^x-1)(2e^x+1)} \boxed{dx}$$

Let

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$du = u\,dx$

$\frac{du}{u} = dx$

$$= \int \frac{1}{(u-1)(2u+1)} \frac{du}{u}$$

$$= \int \frac{-1}{u} + \frac{1/3}{u-1} + \frac{4/3}{2u+1} \, du$$

$$= -\ln|u| + \frac{1}{3}\ln|u-1| + \frac{2}{3}\ln|2u+1| + C$$

$$\frac{1}{u(u-1)(2u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{2u+1}$$

$$1 = A(u-1)(2u+1) + Bu(2u+1) + Cu(u-1)$$

$u = 1 \Rightarrow 1 = B \cdot 3, \quad B = 1/3$

$u = 0 \Rightarrow 1 = A(-1)(1), \quad A = -1$

$u = -\frac{1}{2} \Rightarrow 1 = C \cdot \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$

$C = \frac{4}{3}$

Example

(See the last page for a cleaner version)

$$\int \underbrace{R(\overset{\frac{1-u^2}{1+u^2}}{\cos\theta}, \overset{\frac{2u}{1+u^2}}{\sin\theta})}_{\text{a natural expression in } \cos\theta \text{ and } \sin\theta} \overset{\frac{2du}{1+u^2}}{d\theta} = \int \text{natural function } du$$

Let $\boxed{\tan\frac{\theta}{2} = u}$

$R(x,y) = \dfrac{\text{polynomial in } x,y}{\text{polynomial in } x,y}$

$R(x,y) = \dfrac{x}{x+y^2}$

$$\int \boxed{\frac{\cos\theta}{\sin\theta + \cos^2\theta}}\, d\theta$$

$+\sin^2\theta$

$$= \int \frac{\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2} + \frac{4u^2}{(1+u^2)^2} + \frac{(1-u^2)^2}{(1+u^2)^2}} \frac{2du}{1+u^2}$$

$\sin^2\theta$

$\dfrac{(1+u^2)^2 - 4u^2}{(1+u^2)^2} \neq -\sin^2\theta$

$$= \int \frac{2(1-u^2)}{2u(1+u^2) + (1+u^2)^2} du$$

$$= \int \frac{2-2u^2}{(1+u^2)[2u+1+u^2]} du$$

$$= \int \frac{2-2u^2}{(1+u^2)(u+1)^2} du$$

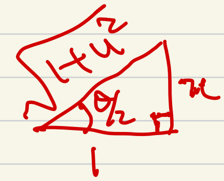
$$= \int \left[\frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{Cu+D}{1+u^2}\right] du$$

Find $A, B, C, D$. (Exercise)

$$= A\ln|u+1| - B(u+1)^{-1} + \frac{C}{2}\ln|1+u^2| + D\tan^{-1}u + C$$

Let

$u = \tan\frac{\theta}{2}$




$\dfrac{du}{d\theta} = \dfrac{1}{2}\sec^2\dfrac{\theta}{2}$

$du = \dfrac{1}{2}(\underbrace{1+u^2}_{1+\tan^2\frac{\theta}{2}})\, d\theta$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$= \dfrac{2}{1+u^2} - 1 = \dfrac{1-u^2}{1+u^2}$

$\boxed{\sin\theta} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2 \cdot \dfrac{u}{\sqrt{1+u^2}} \cdot \dfrac{1}{\sqrt{1+u^2}}$

$$= A\ln\left|\tan\frac{\theta}{2}+1\right| - \frac{B}{\tan\frac{\theta}{2}+1} + \frac{C}{2}\ln\left(1+\tan^2\frac{\theta}{2}\right) + D\frac{\theta}{2} + C_1$$

Remark:

$$\frac{p(x)}{(x-a_1)^{k_1}\cdots(x-a_n)^{k_n}} \quad \deg p < k_1+\cdots+k_n$$

where $a_1, \ldots, a_n$ are distinct numbers.

sum of terms like $\frac{A}{(x-a)^k}$

$$\frac{p(x)}{(x-a_1)^{k_1}\, q(x)} \quad \text{where } q(a_1) \neq 0$$

if we can do this:

$$= \frac{A_1}{(x-a)^{k_1}} + \frac{p_1(x)}{(x-a_1)^{k_1-1}\, q(x)}$$

Yes

continuing with the process

$$\frac{A_2}{(x-a_1)^{k_1-2}} + \frac{p_2(x)}{(x-a_1)^{k_1-2}\, q(x)}$$

$$\vdots$$

$$\frac{A_2}{(x-a_1)^{k_1-2}} + \cdots + \frac{A_{k_1}}{x-a_1} + \frac{p_{k_1}(x)}{q(x)}$$

keep on going

We Need

$$p(x) = A_1 q(x) + p_1(x)(x-a_1)$$

Plug in $x=a_1$

$$p(a_1) = A_1 \boxed{q(a_1)}^{\neq 0} + p_1(a_1)\underbrace{(a_1-a_1)}_{0}$$

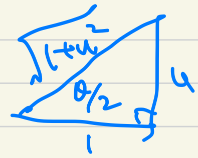
$$A_1 = \frac{p(a_1)}{q(a_1)}$$

Example $\displaystyle\int \frac{\cos\theta}{\sin\theta+\sin^2\theta+\cos^2\theta}\,\boxed{d\theta}$

(which is actually 1, just for correcting an error in the lecture).

$$= \int \frac{\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2}+\frac{4u^2}{(1+u^2)^2}+\frac{(1-u^2)^2}{(1+u^2)^2}}\,\frac{2\,du}{1+u^2}$$

Let $\tan\frac{\theta}{2} = u$




$\dfrac{du}{d\theta} = \dfrac{1}{2}\sec^2\dfrac{\theta}{2} = \dfrac{1+u^2}{2}$

$\dfrac{2\,du}{1+u^2} = d\theta$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+u^2} - 1$

$\cos\theta = \dfrac{1-u^2}{1+u^2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\cdot\dfrac{u}{\sqrt{1+u^2}}\cdot\dfrac{1}{\sqrt{1+u^2}}$

$= \boxed{\dfrac{2u}{1+u^2}}$

$$= \int \frac{2(1-u^2)}{\frac{2u}{1+u^2}+\frac{(1+u^2)^2}{(1+u^2)^2}}\,\frac{du}{(1+u^2)^2}$$

$$= \int \frac{2-2u^2}{2u(1+u^2)+(1+u^2)^2}\,du$$

$$= \int \frac{2-2u^2}{(1+u^2)(1+2u+u^2)}\,du$$

$$= \int \frac{2-2u^2}{(1+u^2)(u+1)^2}\,du$$

$$= \int\left[\frac{A}{u+1}+\frac{B}{(u+1)^2}+\frac{Cu+D}{u^2+1}\right]du$$

Find A, B, C, D!

$$= A\ln|u+1| - B(u+1)^{-1} + \frac{C}{2}\ln(u^2+1) + D\tan^{-1}u + C_1$$

$$= A\ln\left|\tan\frac{\theta}{2}+1\right| - \frac{B}{\tan\frac{\theta}{2}+1} + \frac{C}{2}\ln\left(1+\tan^2\frac{\theta}{2}\right) + \frac{D\theta}{2} + C_1$$