Math 2001 Final Exam

December 8, 2022

Your Name ________________________
Student Number ____________________

1. You can use and quote anything from my lecture note.
2. Show all your work. Cross off (instead of erase) the undesired part.
3. Provide all the details, especially proofs. Your reason counts most of the points.
4. Please feel free to raise your questions.

Number	Score
1	
2	
3	
4	
5	
Total	

(1) (20 points) Solve the system of linear equations

$$
\begin{aligned}
(2+i)z_1 + (1-3i)z_2 &= -1-i, \\
(2-3i)z_1 + (1+i)z_2 &= 7-i,
\end{aligned}
$$

(2) (20 points) Show that 34 and 19 are coprime (i.e., the only common divisor is 1). Then find integers u and v satisfying $34u + 19v = 1$. Then calculate $\overline{3} \div \overline{19}$ in $\mathbb{Z}_{34}$.

(3) (20 points) People have preference for colors. We know the following

1. 16 people like red.
2. 28 people like red or blue.
3. 8 people like red, but hate blue.
4. 3 people like red and blue and green.
5. 6 people like red and green.
6. 7 people like green, but hate red and blue.

Find the following numbers

1. How many people are there?
2. How many people like blue?
3. how many people like red and blue?
4. how many people like red or green, but hate blue?
5. how many people like red and green, but hate blue?

(4) (20 points) Identify the size of the sets as finite, or $|\mathbb{N}|$, or $|\mathbb{R}|$, or $|\mathcal{P}(\mathbb{R})|$. Just present your answer at the end of sets. No reason needed.

1. natural numbers divisible by 300 and 750, but not divisible by 210.
2. natural numbers dividing 300 and 750, but not dividing 210.
3. natural numbers dividing 300 and 750, but not divisible by 210.
4. $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$.
5. $\{(r, s) \in \mathbb{Q} \times \mathbb{Q} \colon r < s\}$.
6. all subsets of $\mathbb{Z}$, consisting of only even numbers.
7. all strictly increasing sequences of integers.
8. all strictly increasing sequences of real numbers.
9. all finite subsets of $\mathbb{Q}$.
10. all infinite subsets of $\mathbb{N}$ not containing prime numbers.

(5) (20 points) Suppose X is countable, and Y is not countable. Prove that $Y - X$ is not countable. How about $X - Y$?

Answer to Math 2001 Final, Autumn 2022
not absolutely guaranteed to be correct

(1) Multiplying $2-3i$ to the first equation, multiplying $2+i$ to the second equation, and subtracting the two, we get

$$[(2-3i)(1-3i)-(2+i)(1+i)]z_2 = (2-3i)(-1-i)-(2+i)(7-i).$$

By

$$\begin{aligned}(2-3i)(1-3i)-(2+i)(1+i) &= (-7-9i)-(1+3i) = -8-12i,\\ (2-3i)(-1-i)-(2+i)(7-i) &= (-5+i)-(15+5i) = -20-4i,\end{aligned}$$

we get

$$z_2 = \frac{-20-4i}{-8-12i} = \frac{5+i}{2+3i} = \frac{(5+i)(2-3i)}{(2+3i)(2-3i)} = \frac{13-13i}{2^2+3^2} = 1-i.$$

Substituting into the first equation, we get

$$(2+i)z_1 = (-1-i)-(1-3i)(1-i) = (-1-i)-(-2-4i) = 1+3i.$$

Then

$$z_1 = \frac{1+3i}{2+i} = \frac{(1+3i)(2-i)}{(2+i)(2-i)} = \frac{5+5i}{2^2+1^2} = 1+i.$$

(2) We carry out the Euclidean algorithm

$$\begin{aligned}34 &= 1\times 19+15,\\ 19 &= 1\times 15+4,\\ 15 &= 3\times 4+3,\\ 4 &= 1\times 3+1.\end{aligned}$$

This shows $\gcd(34,19)=1$. In other words, the two numbers are coprime.

The calculation also gives

$$\begin{aligned}1 &= 4-1\times 3 = 4-1\times(15-3\times 4)\\ &= 4\times 4-1\times 15 = 4\times(19-1\times 15)-1\times 15\\ &= 4\times 19-5\times 15 = 4\times 19-5\times(34-1\times 19)\\ &= -5\times 34+9\times 19.\end{aligned}$$

This implies

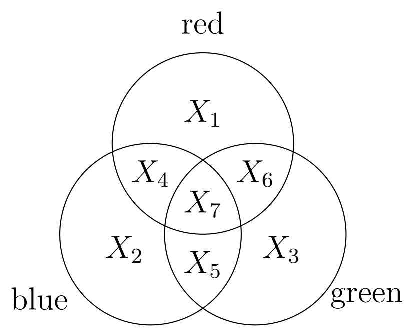
$$\bar{1} = \bar{9}\times\overline{19} \text{ in } \mathbb{Z}_{34}.$$

Then we have

$$\bar{3}\div\overline{19} = \bar{3}\times\bar{9} = \overline{27} \text{ in } \mathbb{Z}_{34}.$$

(3) The whole picture is seven disjoint pieces X_i. Let x_i be the number of people in X_i.

1. $y_1 = x_1 + x_4 + x_6 + x_7 = 16$ people like red.
2. $y_2 = x_1 + x_2 + x_4 + x_5 + x_6 + x_7 = 28$ people like red or blue.
3. $y_3 = x_1 + x_6 = 8$ people like red, but hate blue.
4. $y_4 = x_7 = 3$ people like red and blue and green.
5. $y_5 = x_6 + x_7 = 6$ people like red and green.
6. $y_6 = x_3 = 7$ people like green, but hate red and blue.



1. Total number of people is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = y_2 + y_6 = 35$.
2. Number of people like blue is $x_2 + x_4 + x_5 + x_7 = y_2 - y_3 = 20$.
3. Number of people like red and blue is $x_4 + x_7 = y_1 - y_3 = 8$.
4. Number of people like red or green, but hate blue, is $x_1 + x_3 + x_6 = y_3 + y_6 = 15$.
5. Number of people like red and green, but hate blue, is $x_6 = y_5 - y_4 = 3$.

(4)

1. natural numbers divisible by 300 and 750, but not divisible by 210. $|\mathbb{N}|$
2. natural numbers dividing 300 and 750, but not dividing 210. Finite
3. natural numbers dividing 300 and 750, but not divisible by 210. Finite
4. $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$. $|\mathbb{R}|$
5. $\{(r, s) \in \mathbb{Q} \times \mathbb{Q} \colon r < s\}$. $|\mathbb{N}|$
6. all subsets of $\mathbb{Z}$, consisting of only even numbers. $|\mathbb{R}|$
7. all strictly increasing sequences of integers. $|\mathbb{R}|$
8. all strictly increasing sequences of real numbers. $|\mathbb{R}|$

9. all finite subsets of $\mathbb{Q}$. $|\mathbb{N}|$

10. all infinite subsets of $\mathbb{N}$ not containing prime numbers. $|\mathbb{R}|$

(5) We have $Y = (Y \cap X) \cup (Y - X)$. Since X is countable, by Proposition 5.3.3, the subset $Y \cap X \subset X$ is still countable.

If we also know $Y - X$ is countable, then Y is a union of two countable sets. By Proposition 5.3.3, we know Y is countable. Since Y is assumed to be uncountable, this proves that $Y - X$ is uncountable.

Since X is countable, by Proposition 5.3.3, we know the subset $X - Y \subset X$ is still countable.