MATH2111 Tutorial 9

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1 Coordinate System

1. **Theorem (The Unique Representation Theorem)**. Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\boldsymbol{x} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \cdots + c_n\boldsymbol{b}_n$$

2. **Definition**. Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis B (or the B-coordinates of $\mathbf{x}$) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$. And the coordinate vector of $\mathbf{x}$ relative to B, or the B-coordinate vector of $\mathbf{x}$ is written as

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

3. **Change of coordinates matrix**.
Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis of $\mathbb{R}^n$. Let $P_B = [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n]$.
$\boldsymbol{v} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + \ldots + c_n\boldsymbol{b}_n$ if and only if

$$\boldsymbol{v} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = P_B[\boldsymbol{v}]_B.$$

P_B is called the **change-of-coordinates matrix** from B to the standard basis in $\mathbb{R}^n$.
Since $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is linearly independent, P_B is invertible. Thus

$$[\boldsymbol{v}]_B = P_B^{-1}\boldsymbol{v}.$$

4. **Theorem (the coordinate mapping)**. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

2 Dimension of a Vector Space

1. **Theorem**. If a vector space V has a basis $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

2. **Theorem**. If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

3. **Definition**. If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is **defined to be zero**. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

4. **Theorem (Basis Extension Theorem).**. Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finitedimensional and
$$\dim H \leq \dim V$$

5. **Theorem (The Basis Theorem)**. Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

6. **Fact**.
   (a) dim Nul A = number of free variables in $A\mathbf{x} = \mathbf{0}$.
   (b) dim Col A = number of pivot colums of A.

7. **Definition**. Let A be an $m \times n$ matrix.
   (a) The dimension of Nul A is called the nullity of A.
   (b) The dimension of Col A is called the column rank of A.
   (c) The dimension of Row A is called the row rank of A.

3 Rank of a Matrix

3.1 Row Space

1. **Definition (Row Space)**. The row space of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A.
$$\text{Row } A = \text{Span}\,\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$$
where $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the row vectors of the matrix A.

2. **Theorem**. The row space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

3. **Theorem**.
   (a) Row A = Col $A^\top$
   (b) Suppose A is row equivalent to B, then Row A = Row B.

3.2 Rank

1. **Definition**. The **rank** of A is the dimension of the column space of A.

2. **Theorem (Rank Theorem)**. Let A be an $m \times n$ matrix. Suppose A has p pivot positions. Then

   (a) nullity of $A = n - p$

   (b) column rank of $A = p$

   (c) row rank of $A = p$

   Therefore, by defining Rank A = column rank of A = row rank of A, we have

   $$\text{nullity } A + \text{rank } A = n, \text{ the number of columns.}$$

3. **Theorem (Dimension Theorem)**. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then

   $$\dim \ker T + \dim \operatorname{range} T = n$$

The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent.
(a) A is an invertible matrix.
(b) A is row equivalent to the $n \times n$ identity matrix.
(c) A has n pivot positions.
(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A form a linearly independent set.
(f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
(j) There is an $n \times n$ matrix C such that $CA = I$.
(k) There is an $n \times n$ matrix D such that $AD = I$.
(l) A^T is an invertible matrix.
(m) The columns of A form a basis of R^n
(n) Col $A = \mathbb{R}^n$
(o) $\dim \text{Col } A = n$
(p) rank $A = n$
(q) Nul $A = \{\mathbf{0}\}$
(r) $\dim \text{Nul } A = 0$
(s) $\det(A) \neq 0$

4 Exercises

1. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

(1) $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(2) $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

2.

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$$

(1). Show that the set $\mathcal{B}$ is a basis of $\mathbb{R}^3$
(2). Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
(3). Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$

3. For each subspace, find a basis, and state the dimension.

(1) $\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$

(2) $\left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

4. Determine the dimensions of Nul A and Col A for the matrices.

(1)

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Assume that the matrix A is row equivalent to B. Without calculations, list $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$. Then find bases for $\operatorname{Col} A, \operatorname{Row} A$, and $\operatorname{Nul} A$.

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$