Math1014 Calculus II

Week 7-8: Brief Review and Some Practice Problems

Polar Coordinates, Partial Fractions, Numerical Integration

- Get use to using polar coordinates (r, θ) to describe points in the plane, which are related to the rectangular coordinates by
$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$
- When dealing with derivative problems of polar curves $r = r(\theta)$, Chain Rule is useful: $\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta}$.
- When dealing with area or arc length problems in polar coordinates, i.e., when using
$$\text{area} \ = \frac{1}{2}\int_a^b r^2 d\theta, \quad \text{arc length} \ = \int_a^b \sqrt{r^2 + [r'(\theta)]^2}d\theta$$
be careful with the appropriate choice $[a, b]$ of the range of the "polar angle".
- The method of partial fractions is just about breaking up a rational function $f(x)$ into sum of terms like $\frac{A}{(ax+b)^2}$, or $\frac{Ax+B}{a^2(x+b)^2+c^2}$, whose indefinite integrals could be found by standard integration techniques, say by substitution $u = ax + b$, or $x + b = \frac{c}{a}\tan\theta$.
- Numerical integration:
  - how to use rectangles, trapeziums, or quadratic polynomials to approximate integrals;
  - how to use the *error bounds* of the numerical integration methods;

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1. Find the area of the region that lies inside the first curve and outside the second curve given by the following polar equations. (Try sketching the curves first.)

   (i) $r = 3\cos\theta, \ r = 2 - \cos\theta$ (ii) $r = 3\sin\theta, r = 2 - \sin\theta$.

2. Find the slope of the tangent line to the polar curve at the point with angular coordinate $\theta = \frac{\pi}{3}$, and also the length of the polar curve.

   (i) $r = e^{2\theta}, \ 0 \le \theta \le \pi$ (ii) $r = \cos^2\frac{\theta}{2}$.

3. Evaluate the following integrals.

   (i) $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}dx$ (ii) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}dx$ (iii) $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx$,

   (iv) $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$ (v) $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}dx$ (vi) $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}dx$

   (vii) $\displaystyle\int \frac{\cos x}{\sin^2 x + \sin x}dx$ (viii) $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx$ (vi) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x - \cos x}dx$

4. Use (a) the Trapezoidal rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified valued of n. (Round your answers to six decimal places.)

   (i) $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}dx, \quad n = 4$ (ii) $\displaystyle\int_0^1 \sqrt{z}e^{-z}dz, \quad n = 4$

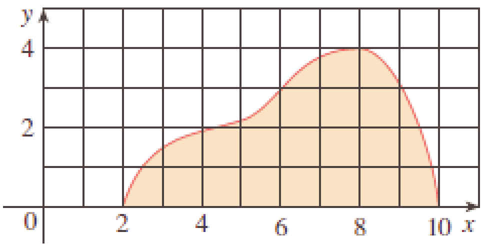
5. Find the approximation T_{10} and M_{10} for $\displaystyle\int_1^2 e^{1/x}dx$, and then estimate the errors in the approximations. How large do we have to choose n so that the approximation T_n and M_n to the integral are accurate to within 0.0001?

6. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2} dx$ is accurate to within 0.00001?

7. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's Rule to estimate the distance the runner covered during those 5 seconds.

$t\,(s)$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$v\,(m/s)$	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.76	10.81	10.81

8. If the region shown in the figure is rotated about the y-axis to form a solid, use Simpson's Rule with $n = 8$ to estimate the volume of the solid.



9. If f is a positive function and $f''(x) < 0$ for $a \leq x \leq b$, show that

$$T_n < \int_a^b f(x)dx < M_n$$

10. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)dx$.