MATH 2111: Tutorial 10

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Review

- Eigenvectors and eigenvalues
- The characteristic equation
- Similarity

Example 1

Let λ be an eigenvalue of A. Find an eigenvalue of the following matrices.
(1) A^2
(2) $A^3 + A^2$
(3) $A^3 + 2I$
(4) If A is invertible, A^{-1}
(5) If $p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$, define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A (with $A^0 = I$). That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

let $\vec{v}$ be the corresponding eigenvector of A.

(1) Since $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v})$

$= \lambda\cdot(\lambda\vec{v}) = \lambda^2\vec{v}$

$\therefore \lambda^2$ is an eigenvalue of A^2.

(2) Since $(A^3+A^2)\vec{v} = A^3\vec{v} + A^2\vec{v}$

$= A\cdot A^2\vec{v} + A^2\vec{v}$

$= A(\lambda^2\vec{v}) + \lambda^2\vec{v}$

$= \lambda^2(A\vec{v}) + \lambda^2\vec{v}$

$= \lambda^3\vec{v} + \lambda^2\vec{v}$

$= (\lambda^3+\lambda^2)\vec{v}$

$\therefore \lambda^3+\lambda^2$ is an eigenvalue of A^3+A^2.

Actually λ^m is an eigenvalue of A^m for positive integer m.
(can be proved by mathematical induction.)

(3) Since $(A^3+2I)\vec{v} = A^3\vec{v} + 2\vec{v}$

$= \lambda^3\vec{v} + 2\vec{v}$

$= (\lambda^3+2)\vec{v}$

$\therefore \lambda^3+2$ is an eigenvalue of A^3+2I.

(4) Since A is invertible, so $\lambda \neq 0$, otherwise for eigenvector $\vec{v}$,

$\vec{v} = A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v}) \underset{\substack{\uparrow \\ \text{assume} \\ \lambda=0}}{=} A^{-1}\vec{0} = \vec{0}$, however, this is impossible, because eigenvector is nonzero.

We know that $A\vec{v} = \lambda\vec{v}$ with $\lambda \neq 0$, left multiply A^{-1},

$$A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v})$$

$$\Rightarrow \vec{v} = \lambda(A^{-1}\vec{v})$$

$$\Rightarrow A^{-1}\vec{v} = \lambda^{-1}\vec{v}.$$

$\therefore \lambda^{-1}$ is an eigenvalue of A^{-1}.

(5) We know that $A\vec{v} = \lambda\vec{v}$,

$$\begin{aligned} p(A)\vec{v} &= (c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n)\vec{v} \\ &= c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} + \cdots + c_n A^n\vec{v} \\ &= c_0\vec{v} + c_1\lambda\vec{v} + c_2\lambda^2\vec{v} + \cdots + c_n\lambda^n\vec{v} \\ &= (c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n)\vec{v} \\ &= p(\lambda)\vec{v} \end{aligned}$$

$\therefore p(\lambda)$ is an eigenvalue of $p(A)$.

Example 2

Let

$$A = \begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix}$$

Determine whether the following vectors are eigenvectors of A.

(1) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (2) $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(1)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 8 \\ 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of A.

(corresponds to $\lambda = 1$)

(2)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 8 \\ 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is an eigenvector of A.

(corresponds to $\lambda = 2$)

(3)

$$\begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 8 \\ 1 & -2 & -2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\neq c\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is NOT an eigenvector of A.

Example 3

For the given matrix A and the given eigenvalue λ, find the corresponding collection of eigenvectors.

$$A = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix}, \ \lambda = 1$$

$$A - I = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 7 \\ 4 & 9 & 7 \\ -8 & -18 & -14 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 9 & 7 & 0 \\ 4 & 9 & 7 & 0 \\ -8 & -18 & -14 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 9/4 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad x_1 = -\tfrac{9}{4}x_2 - \tfrac{7}{4}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Collection of eigenvectors is :

$$\left\{ s\begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R}, \text{ NOT both zero} \right\}$$

Remark:

- if question asks the eigenspace:

$$\left\{ s\begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

- if asks a basis for eigenspace:

$$\left\{ \begin{bmatrix} -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example 4

Suppose that λ and ρ are two different eigenvalues of the square matrix A. Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{\mathbf{0}\}$

① To show $\{\vec{0}\} \subseteq \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$

Choose $\vec{x} \in \{\vec{0}\}$, then $\vec{x} = \vec{0}$.

Since eigenspaces are subspaces,

$\mathcal{E}_A(\lambda)$ and $\mathcal{E}_A(\rho)$ contain the zero vector.

So, $\vec{0} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$.

② To show $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) \subseteq \{\vec{0}\}$

Suppose $\vec{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$,

then $\vec{x}$ is the eigenvector for A corresponds to both λ and ρ.

i.e. $A\vec{x} = \lambda\vec{x}$, $A\vec{x} = \rho\vec{x}$.

$$\therefore \vec{x} = 1 \cdot \vec{x} = \frac{1}{\lambda-\rho}(\lambda-\rho)\vec{x}$$

since λ and ρ are different. $\lambda - \rho \neq 0$

$$= \frac{1}{\lambda-\rho}(\lambda\vec{x} - \rho\vec{x})$$

$$= \frac{1}{\lambda-\rho}(A\vec{x} - A\vec{x})$$

$$= \frac{1}{\lambda-\rho}\vec{0}$$

$$= \vec{0}$$

$\therefore \vec{x} = \vec{0}$. $\Rightarrow \vec{x} \in \{\vec{0}\}$.

Example 5

Find the eigenvalues eigenvalues, eigenspaces, algebraic and geometric multiplicities of the following matrices.

(1) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(1) Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \xlongequal{R_2+R_1 \to R_2} \begin{vmatrix} 1-\lambda & -1 & 1 \\ -\lambda & -\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix}$$

$$= -(-\lambda) \begin{vmatrix} -1 & 1 \\ -1 & 1-\lambda \end{vmatrix} + (-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= \lambda((\lambda-1)+1) - \lambda[(1-\lambda)^2-1]$$

$$= \lambda^2 - \lambda(\lambda^2-2\lambda)$$

$$= \lambda^2 - \lambda^3 + 2\lambda^2$$

$$= 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda)$$

let $\det(A-\lambda I)=0$, we have

$$-\lambda^2(\lambda-3)=0$$

$\therefore$ eigenvalues $\lambda_1=\lambda_2=0$, algebraic multiplicity 2.

$\lambda_3=3$, algebraic multiplicity 1.

① For $\lambda=0$:

$$A-\lambda I = A-0\cdot I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solve $(A-\lambda I)\vec{x}=\vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 = x_2 - x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the eigenspace is:

$$\left\{ s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 2.

② For $\lambda = 3$,

$$A - 3I = \begin{bmatrix}-2 & -1 & 1\\-1 & -2 & -1\\1 & -1 & -2\end{bmatrix}$$

Solve $(A-\lambda I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c}-2 & -1 & 1 & 0\\-1 & -2 & -1 & 0\\1 & -1 & -2 & 0\end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c}-1 & -2 & -1 & 0\\-2 & -1 & 1 & 0\\1 & -1 & -2 & 0\end{array}\right]$$

$$\xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c}-1 & -2 & -1 & 0\\0 & 3 & 3 & 0\\0 & -3 & -3 & 0\end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c}-1 & -2 & -1 & 0\\0 & 1 & 1 & 0\\0 & -1 & -1 & 0\end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \to R_3]{} \left[\begin{array}{ccc|c}-1 & -2 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c}-1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$$\therefore \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

(2) Characteristic polynomial :

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)^2 .$$

let $\det(A-\lambda I) = 0$, we have

$$(3-\lambda)(1-\lambda)^2 = 0$$

$\therefore$ eigenvalues $\lambda_1 = \lambda_2 = 1$, algebraic multiplicity 2.

$\lambda_3 = 3$, algebraic multiplicity 1.

① For $\lambda = 1$,

$$A - I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solve $(A-\lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

② For $\lambda = 3$:

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_2 \to R_2]{-\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + \frac{1}{2}R_2 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the eigenspace is :

$$\left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$\therefore$ the geometric multiplicity is 1.

Remark:

Geometric multiplicity can NEVER EXCEED the algebraic multiplicity.