MATH 2111: Tutorial 3
Matrix Equation and Solution Sets of Linear Systems

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- Relationship between matrix equation and vector equation
- Computation of matrix equation
- Homogeneous Linear Systems
- Parametric Vector Form

Write the matrix equation as a vector equation, or vice versa

$$\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

1. $5 \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3 \cdot \begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

(1) Could a set of n vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$ if $n < m$? Explain.

(2) Suppose A is a 3×3 matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution. Explain why the columns of Amust span $\mathbb{R}^3$.

1. Can't n vectors form $A \in R^{m\times n}$.
By Theorem 4, A should has a pivot position in every row.
Then A should have at least $m>n$ columns.

2. The reduced echelon form of A should be $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$. And for each row, A has a pivot position, due to Theorem 4, the columns of A span R^3.

Determine if the columns of the matrix span $\mathbb{R}^4$

$$\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & \frac{2}{5} & -\frac{11}{5} & -\frac{29}{5} \\ 0 & -\frac{8}{5} & -\frac{29}{5} & -\frac{81}{5} \\ 0 & -\frac{8}{5} & \frac{44}{5} & \frac{116}{5} \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & \frac{2}{5} & -\frac{11}{5} & -\frac{29}{5} \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only three columns of A have pivots, due to Theorem 4, the columns of matrix don't span R^4

Determine if the system has a nontrivial solution

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

1. $\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

x_3 is free, the system has a nontrivial solution.

2. $\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$

no free variable, the system has no nontrivial solution

Describe all solutions of $\boldsymbol{A x = 0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(1) Suppose $\mathbf{w}, \boldsymbol{p}$ are two solutions of the equation $A\boldsymbol{x} = \boldsymbol{b}$ and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\boldsymbol{x} = \mathbf{0}$.

(2) Suppose $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. Explain why the solution is unique precisely when $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

1. $Ap=b$, $Aw=b$,

$Av_h = A(w-p) = Aw - Ap = b - b = 0$

Then v_h is a solution of $Ax=0$

2. Let the solution be p, by Theorem 6, the solution set is $w = p + v_h$, where v_h is one solution of $Ax=0$

Then w is unique $\Leftrightarrow$ v_h is unique.

$\Leftrightarrow$ $Ax=0$ has only the trivial solution.