§4.5 The Dimension of a vector space

Thm: If a vector space V has a basis $B = \{\vec{b}_1, \cdots, \vec{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: $\{\vec{u}_1, \cdots, \vec{u}_p\}$ in V with $p > n$

$[\vec{u}_1]_B, \cdots, [\vec{u}_p]_B$ form a linearly dependent set in $\mathbb{R}^n$, because one of the columns in $[[\vec{u}_1]_B, \cdots, [\vec{u}_p]_B]$ is not a pivot column.

Thus there exists $c_1, \cdots, c_p$ not all zeros such that $c_1[\vec{u}_1]_B + \cdots + c_p[\vec{u}_p]_B = \vec{0}$

Hence $[c_1\vec{u}_1 + \cdots + c_p\vec{u}_p]_B = \vec{0}$

So $c_1\vec{u}_1 + \cdots + c_p\vec{u}_p = \vec{0}$, i.e. $\vec{u}_1, \cdots, \vec{u}_p$ are linearly dependent.

Thm: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof: B_1 : a basis of n vectors

B_2 : a basis of m vectors

If $n \neq m$, then either $n < m$ or $n > m$.

For simplicity, assume $n < m$. Then by the previous theorem, vectors in B_2 must be linearly dependent, a contradiction.

Def: If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\vec{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Ex: $\dim \mathbb{R}^n = n$

$\dim \mathbb{P}^d = d+1$

Let $\mathbb{P} = \{$ all polynomials in one variable $\}$

$\dim \mathbb{P} = \infty$

Ex: Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{pmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

Solution:
$$\begin{pmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{pmatrix} = \begin{pmatrix} a \\ 5a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3b \\ 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} 6c \\ 0 \\ -2c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4d \\ -d \\ 5d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 6 & 0 \\ 0 & 15 & -30 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 15 & -30 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns

Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is a basis of H.

Ex: The subspaces of $\mathbb{R}^3$ can be classified by dimension.

0-dimensional subspaces Only the zero subspace

1-dimensional subspaces Any subspace spanned by a single nonzero vector

2-dimensional subspaces Any subspace spanned by two linearly independent vectors. Such subspaces are planes trough the origin

3-dimensional subspaces Only $\mathbb{R}^3$ itself. Any three linearly independent vectors in $\mathbb{R}^3$ span all of $\mathbb{R}^3$.

* Subspaces of a Finite-Dimensional Space

Thm: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \le \dim V$$

Proof: If $H = \{\vec{0}\}$. Clearly $\dim H = 0 \le \dim V$.

If $H \ne \{\vec{0}\}$. Let $S = \{\vec{u}_1, \cdots, \vec{u}_k\}$ be any linearly independent set in H. If S spans H, then S is a basis of H.

If S doesn't span H, there exists $u_{k+1} \in H$ such that $\vec{u}_{k+1}$ is not in $\text{Span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Thus $\{\vec{u}_1, \cdots, \vec{u}_k, \vec{u}_{k+1}\}$ is linearly independent by

(Theorem: An indexed set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ of two or more vectors, with $\vec{v}_1 \ne 0$, is linearly dependent iff some $\vec{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\vec{v}_1, \cdots, \vec{v}_{j-1}$)

So we continue this process, we can get a larger linearly independent set in H. But this process will stop in finite many steps since $\#\{\text{linearly independent vectors in } H\} \le \dim V$.

Then $\dim H \le \dim V$.

Thm (The Basis Theorem)

Let V be a p-dimensional vector space, $p \geq 1$, any linearly independent set of exactly p elements in V is automatically a basis for V.

Any set of exactly p elements that spans V is automatically a basis for V.

Proof. Let $S = \{\vec{v}_1, \cdots, \vec{v}_p\}$ be a linearly independent set.

If S is not a basis, $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\} \subsetneqq V$. Then we can extend $\{\vec{v}_1, \cdots, \vec{v}_p\}$ to a basis of V. Thus $p < \dim V$, a contradiction.

If $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\} = V$. If $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is not linearly independent, then by the spanning set Thm, a subset S' of S will span V. Then $\dim V < p$, a contradiction.

Review:

Thm: The Spanning Set Theorem

Let $S = \{\vec{v}_1, \cdots, \vec{v}_p\}$ be a set in V, and let $H = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$.

a) If one of the vectors in S – say, $\vec{v}_k$ – is a linear combination of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

* The dimension of Nul A and Col A.

Fact: 1) The dim Nul A = number of free variables in the equation $A\vec{x}=\vec{0}$

2) The dim Col A = # of pivot columns of A.

Ex: Find the dim Nul A and dim Col A

$$A=\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Solution: $[A\ \vec{0}\,] = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix}$

$$\sim \begin{array}{c} \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{free } x_2 & \text{pivot} & \text{free } x_4 & \text{free } x_5 \end{array} \end{array}$$

Thus dim Nul A = 3

dim Col A = 2

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{pmatrix}$$

$$= x_2\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix} + x_5\begin{pmatrix}-3\\0\\2\\0\\1\end{pmatrix}$$

Ex: Decide whether each statement is True or False, and give a reason for each answer.

Here V is a nonzero finite-dimensional vector space.

a) If $\dim V = p$ and if S is a linearly dependent subset of V, then S contains more than p vectors.

b) If S spans V and if T is a subset of V that contains more vectors than S, then T is linearly dependent.

Solution: a) False.

Let $V = \mathbb{R}^3$. Then $p=3$

$$S = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix} \right\}$$

S contains 2 vectors, which is less than 3, but $\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}$, they are linearly dependent.

b) True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. So T is linearly dependent.

Ex: Let H and K be subspaces of a vector space V.
Prove that 1) $H \cap K$ is also a subspace of V.
2) $\dim(H \cap K) \le \dim H$.

Proof: 1) Let $\vec{u} \in H \cap K$, $\vec{v} \in H \cap K$.
Then $\vec{u} \in H$, $\vec{u} \in K$, $\vec{v} \in H$, $\vec{v} \in K$.
Since H and K are subspaces of a vector space V, for any scalars $a, b \in \mathbb{R}$, we have

$$a\vec{u} + b\vec{v} \in H, \quad a\vec{u} + b\vec{v} \in K.$$

Hence $a\vec{u} + b\vec{v} \in H \cap K$.
which implies $H \cap K$ is a subspace of V.

2) Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ be a basis for $H \cap K$.
Notice $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ is a linearly independent subset of H, hence it can be expanded, if necessary, to a basis for H. Since the dimension of a subspace is just the number of vectors in a basis, it follows that

$$\dim(H \cap K) = p \le \dim H.$$

§4.6 Rank

* The Row Space

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \stackrel{\triangle}{=} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{pmatrix}$$

Row Space of A (denoted by Row A) = Span$\{\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m\}$ is a subspace of $\mathbb{R}^n$.

Thm: If two matrices A and B are row equivalent, then their row spaces are the same.
If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Proof: If B is obtained from A by row operations, the rows of B are linear combination of the rows of A.

$\Rightarrow$ Any linear combination of the rows of B is automatically a linear combination of the rows of A.

$\Rightarrow$ Row B $\subseteq$ Row A.

Since row operation is reversible, Row A $\subseteq$ Row B.

Therefore, Row A = Row B.

Ex: Find basis for Row A, Col A, Nul A

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

Solution:

$$A \sim B = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for Row A: $\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$

Basis for Col A $\left\{\begin{pmatrix}-2\\1\\3\\1\end{pmatrix},\begin{pmatrix}-5\\3\\11\\7\end{pmatrix},\begin{pmatrix}0\\1\\7\\5\end{pmatrix}\right\}$

Nul A: $\begin{cases} x_1+x_3+x_5=0\\ x_2-2x_3+3x_5=0\\ x_4-5x_5=0\end{cases}$

$\Rightarrow \begin{cases} x_1=x_3-x_5\\ x_2=2x_3-3x_5\\ x_4=5x_5\end{cases}$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_3-x_5\\2x_3-3x_5\\x_3\\5x_5\\x_5\end{pmatrix}=\begin{pmatrix}x_3\\2x_3\\x_3\\0\\0\end{pmatrix}+\begin{pmatrix}-x_5\\-3x_5\\0\\5x_5\\x_5\end{pmatrix}$$

$$=x_3\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix}$$

Basis for Nul A: $\left\{\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix}\right\}$

Remark: Row operations may change the linear dependence relations among the rows of a matrix.

The first three rows of B are linearly independent.

while the first three row of A are linearly dependent. $\vec{r}_3 = 2\vec{r}_1 + 7\vec{r}_2$.