COMP 2711 Discrete Mathematical Tools for Computer Science
2022 Fall Semester – Tutorial 10

Question 1: What is the largest n for which one can solve within one second a problem using an algorithm that requires $f(n)$ bit operations, where each bit operation is carried out in 10^{-9} seconds, with these functions $f(n)$?

(a) $\log n$

(b) n

(c) n^2

(d) 2^n

(e) $n!$

Solution : Each bin operation requires 10^{-9} seconds, we want to know for what value of n there will be at most 10^9 bit operations required. Thus we need to set the expression equal to 10^9, solve for n, and round down if necessary.

(a) Solving $\log n = 10^9$, we get $n = 2^{10^9}$ (recalling that "log" means logarithm base 2). By taking $\log_{10}$ of both sides, we find that this number is approximately equal to $10^{300000000}$.

(b) Clearly $n = 10^9$.

(c) Solving $n^2 = 10^9$ gives $n = 10^{4.5}$, which is 31622 when rounded down.

(d) Solving $2^n = 10^9$ gives $n = \log(10^9) \approx 29.9$. Rounding gives the answer, 29.

(e) The quickest way to find the largest value of n such that $n! \leq 10^9$ is simply try a few values of n. We find that $12! \approx 4.8 \cdots 10^8$ while $13! \approx 6.2 \cdot 10^9$, so the answer is 12.

Question 2: Describe the worst-case time complexity, measured in terms of comparisons, of the following algorithm.

procedure *ternary search*(x : integer, $a_1, a_2, \ldots, a_n$: increasing integers)

$i := 1$

$j := n$

while $i < j - 1$

  $l := \lfloor (i + j)/3 \rfloor$

  $u := \lfloor 2(i + j)/3 \rfloor$

  if $x > a_u$ **then** $i := u + 1$

else if $x > a_l$ **then**
$i := l + 1$
$j := u$
else $j := l$
if $x = a_i$ **then** $location := i$
else if $x = a_j$ **then** $location := j$
else $location := 0$
return $location$

Solution : We will count comparisons of elements in the list to x (This ignores comparisons of subscripts, but since we are only interested in a big-O analysis, no harm is done). Furthermore, we will assume that the number of elements in the list is a power of 3, say $n = 3^k$. Just as in the case of binary search, we need to determine the maximum number of times the while loop is iterated. Each pass through the loop cuts the number of elements still being considered (those whose subscripts are from i to j) by a factor of 3. Therefore after k iterations, the active portion of the list will have length 1; that is, we will have $i = j$. The loop terminates at this point. Now each iteration of the loop requires two comparisons in the wort case (one with a_u and the one with a_l). Two more comparisons are needed at the end. Therefore the number of comparisons is $2k + 2$, which is $O(k)$. But $k = \log_3 n$, which is $O(\log n)$ since logarithms to different bases differ only by multiplicative constants, so the time complexity of this algorithm (in all cases, not just the worst case) is $O(\log n)$.

Question 3: Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Solution : Let $P(m)$ be the proposition that $(2m - 1)^2 - 1$ is divisible by 8 ($m \geq 1$).

Basis: $P(1)$ is true because $8 \mid 0$.

Inductive step: Assume that $P(k)$ is true, i.e., $(2k - 1)^2 - 1$ is divisible by 8.

We have

$$\begin{aligned} P(k+1) &= [(2(k+1) - 1]^2 - 1 \\ &= (2k+1)^2 - 1 \\ &= (2k-1)^2 + 8k - 1 \\ &= [(2k-1)^2 - 1] + 8k. \end{aligned}$$

By the induction hypothesis, $P(k + 1)$ is true because both terms on the right-hand side are divisible by 8. This shows that $P(m)$ is true for all positive integers m, so $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Question 4: You are given an infinite supply of 3-cent and 7-cent postage stamps.

Note that some integer postage values can be formed using these stamps and some cannot. For example, it is impossible to form 8 cents of postage value using these stamps but 27 cents can be formed, e.g., by using three 7-cent stamps and two 3-cent stamps.

Which integer postage values can and which cannot be formed using just 3-cent and 7-cent postage stamps? You must prove the correctness of your answer.

Hint: Use induction

Solution : The answer is that all positive integer values except for $1, 2, 4, 5, 8, 11$ are possible.

It is easy to see that $1, 2, 4, 5, 8, 11$ cannot be formed using just 3-cent and 7-cent postage stamps.

Now we want to prove all other values can be formed using just 3-cent and 7-cent postage stamps by induction.
Base case: It is easy to see that $3, 6, 7, 12, 13, 14$ can be formed using just 3-cent and 7-cent postage stamps.

Induction hypothesis: Assume the statement is true for $n-2, n-1$ and n where $n \geq 14$.

Induction step: By the induction hypothesis, we have $n-2 = 3a+7b$ for some integers a, b. Then

$$n+1 = n-2+3 = 3a+7b+3 = 3(a+1)+7b$$

So, the statement is also true for $n+1$.

By the principle of Mathematical Induction, the statement is true for all $n \geq 12$.

Question 5: Consider the following statement.

Given a positive integer n. For any non-negative integer m, there exist integers q and r such that

$$m = qn + r, \quad 0 \leq r < n$$

(a) Prove the statement by weak induction.

(b) Prove the statement by strong induction.

Solution : (a) **Base case:** $m < n$, we set $q = 0$ and $r = m$.

Induction hypothesis: Assume the statement is true for $m-1$.

Induction step: By the induction hypothesis, we have $m-1 = q'n+r'$ for some integers q', r' where $0 \le r' < n$. So, for $m \ge n$, we have

$$m = q'n + r' + 1 \qquad 0 \le r' < n$$

case 1: $r' < n - 1$. In this case, we set $q = q'$ and $r = r' + 1$, then the statement is true for m.

case 2: $r' = n - 1$. In this case, we set $q = q' + 1$ and $r = 0$, then the statement is true for m.

By the principle of Mathematical induction, we conclude that the statement is true for any non-negative integer m.

(b) **Base case:** $m < n$, we set $q = 0$ and $r = m$.

Induction hypothesis: Assume the statement is true for $0 \le w < m$.

Induction step: Consider $m \ge n$.

$$\begin{aligned} m &= m - n + n \\ &= q'n + r' + n, \qquad 0 \le r' < n \text{ (By induction hypothesis)} \\ &= (q' + 1)n + r' \end{aligned}$$

So, the statement is true for m by setting $q = q' + 1$ and $r = r'$.

By the principle of Mathematical Induction, we conclude that the statement is true for all non-negative integer m.