Lecture 25

§7.2 Quadratic Forms

Def: A *quadratic form* on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector $x \in \mathbb{R}^n$ can be computed by an expression of the form $Q(x) = x^T A x$, where A is an $n \times n$ symmetric matrix. The matrix A is called the *matrix of the quadratic form.*

Example: Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Compute $x^T A x$ for the following matrices:

a) $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$

Solution: a) $x^T A x = (x_1 \quad x_2) \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= (x_1 \quad x_2) \begin{pmatrix} 4x_1 \\ 3x_2 \end{pmatrix}$$

$$= 4x_1^2 + 3x_2^2$$

b) $x^T A x = (x_1 \quad x_2) \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= (x_1 \quad x_2) \begin{pmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{pmatrix}$$

$$= x_1 (3x_1 - 2x_2) + x_2 (-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_1x_2 + 7x_2^2$$
$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

Remark: The presence of $-4x_1x_2$ in the quadratic form is due to the -2 entries off the diagonal in the matrix A.

Example: For $\vec{x} \in \mathbb{R}^3$, let $Q(\vec{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$. Write this quadratic form as $\vec{x}^T A \vec{x}$.

Solution: The coefficients of x_1^2, x_2^2, x_3^2 go on the diagonal of A. To make A symmetric, the coefficient of x_ix_j $(i \neq j)$ must split evenly between the (i,j) and (j,i)-entries in A.

It is readily checked that

$$Q(x) = \vec{x}^T A \vec{x} = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Example: Let $Q(\vec{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$. Compute the value of $Q(\vec{x})$ for $\vec{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Solution:
$$Q(-3,1) = (-3)^2 - 8\cdot(-3)\cdot 1 - 5\cdot 1^2 = 28$$
$$Q(2,-2) = 2^2 - 8\cdot 2\cdot(-2) - 5\cdot(-2)^2 = 16$$
$$Q(1,-3) = 1^2 - 8\cdot 1\cdot(-3) - 5\cdot(-3)^2 = -20$$

Remark: 1) Quadratic forms are easier to use when they have no cross-product terms

2) The cross-product term can be eliminated by making a suitable change of variable.

Change of Variable in a Quadratic Form

If $\vec{x}$ represents a variable vector in $\mathbb{R}^n$, then a change of variable is an equation of the form

$$\vec{x} = P\vec{y} \iff \vec{y} = P^{-1}\vec{x}$$

where P is an invertible matrix and $\vec{y}$ is a new variable vector in $\mathbb{R}^n$.

$$\vec{x}^T A \vec{x} = (P\vec{y})^T A P\vec{y} = \vec{y}^T P^T A P \vec{y} = \vec{y}^T (P^T A P) \vec{y}$$

P^TAP — the new matrix of the quadratic form

Remark: Since A is symmetric, there is an orthogonal matrix P such that P^TAP is a diagonal matrix D and the quadratic form $\vec{x}^T A \vec{x} = \vec{y}^T P^T A P \vec{y} = \vec{y}^T D \vec{y}$.

Example: Make a change of variable that transforms the quadratic form $Q(\vec{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form with no cross-product term.

Solution: The matrix of the quadratic form is

$$A = \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix}$$

First, we need to orthogonally diagonalize A.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-5-\lambda) - 16$$

$$= -5+5\lambda-\lambda+\lambda^2-16$$

$$= -21+4\lambda+\lambda^2$$

$$= (\lambda+7)(\lambda-3)$$

Hence the eigenvalues are $\lambda_1=-7$ and $\lambda_2=3$.

For $\lambda_1=-7$, $\begin{pmatrix} 1-(-7) & -4 \\ -4 & -5-(-7) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2x_1 - x_2 = 0$ $\qquad x_1 = \frac{x_2}{2}$ $\qquad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

For $\lambda_2=3$, $\begin{pmatrix} 1-3 & -4 \\ -4 & -5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -4 & 0 \\ -4 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + 2x_2 = 0$ $\qquad x_1 = -2x_2$ $\qquad \vec{v}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

Let $P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$

Then $A = PDP^{-1}$ and $D = P^{-1}AP = P^TAP$

A suitable change of variable is

$\vec{x} = P\vec{y}$, where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Then
$$\begin{aligned} x_1^2 - 8x_1x_2 - 5x_2^2 = x^TAx &= (Py)^TA(Py) \\ &= y^TP^TAPy \\ &= y^TDy \\ &= 3y_1^2 - 7y_2^2 \end{aligned}$$

Theorem: Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variables, $\vec{x} = P\vec{y}$, that transforms the quadratic form $\vec{x}^TA\vec{x}$ into a quadratic form $\vec{y}^TD\vec{y}$ with no cross-product term.

Example: The graph of the equation $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$.
Find a change of variable that removes the cross-product term from the equation.

Solution: The matrix of the quadratic form is

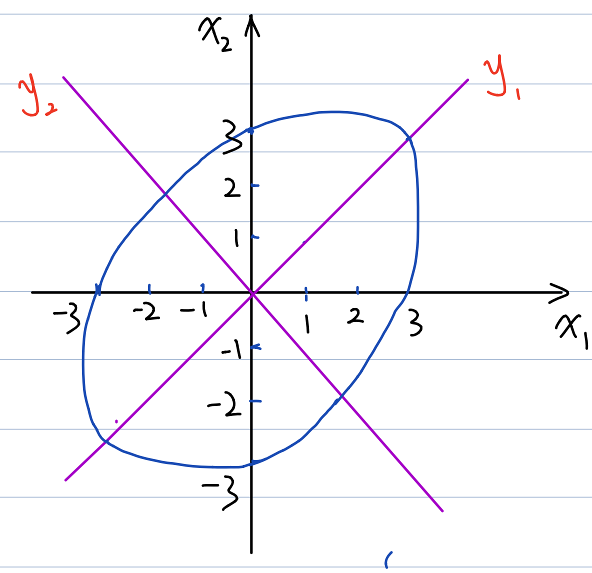
$A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$. The eigenvalues of A turn out to be 3 and 7, with corresponding unit vector

$$\vec{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Let $P = [u_1, u_2] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then

$$P^TAP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}.$$

So the change of variable $\vec{x} = P\vec{y}$ produces the quadratic form $\vec{y}^T D \vec{y} = 3y_1^2 + 7y_2^2$.



Def: A quadratic form Q is

a) positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$

b) negative definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$

c) indefinite if $Q(\vec{x})$ assumes both positive and negative values.

Also, Q is said to be positive semidefinite if $Q(\vec{x}) \geq 0$

for all $\vec{x}$, and to be negative semidefinite if $Q(\vec{x}) \leq 0$ for all $\vec{x}$.

Thm: Quadratic Forms and Eigenvalues

Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\vec{x}^T A \vec{x}$ is:

a) positive definite if and only if the eigenvalues of A are all positive,

b) negative definite if and only if the eigenvalues of A are all negative, or

c) indefinite if and only if A has both positive and negative eigenvalues.

Example: Is $Q(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite?

Solution: The matrix of the form is

$$\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

It is easy to find the eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = 2$ and $\lambda_3 = -1$.

So Q is an indefinite quadratic form, not positive definite.