Math2001 Answer to Homework 2

Exercise 1.29 (3)

For $n=1$, we have

$$1^2 = 1 = \frac{1}{6}1\cdot 2\cdot 3.$$

Suppose the equality holds for all positive integers $n-1$. Then

$$\begin{aligned}1^2+2^2+3^2+\cdots+n^2 &= (1^2+2^2+3^2+\cdots+(n-1)^2)+n^2\\ &= \frac{1}{6}(n-1)n(2(n-1)+1)+n^2\\ &= \frac{1}{6}((n-1)n(2n-1)+6n^2)\\ &= \frac{1}{6}n(n+1)(2n+1).\end{aligned}$$

Exercise 1.29 (5)

For $n=1$, we have

$$1^3 = 1 = \frac{1}{4}1^2\cdot 2^2.$$

Suppose the equality holds for all positive integers $n-1$. Then

$$\begin{aligned}1^3+2^3+3^3+\cdots+n^3 &= (1^3+2^3+3^3+\cdots+(n-1)^3)+n^3\\ &= \frac{1}{4}(n-1)^2n^2+n^3\\ &= \frac{1}{4}n^2((n-1)^2+4n)\\ &= \frac{1}{4}n^2(n+1)^2.\end{aligned}$$

Exercise 1.30 (2)

For $n=10$, we have $2^{10}=1024>1000=10^3$.

Suppose $2^n>n^3$ for $n\geq 10$. Then $2^{n+1}=2\cdot 2^n>2n^3$. Since $n\geq 10$ implies

$$\frac{(n+1)^3}{n^3}=\frac{n^3+3n^2+3n+1}{n^3}=1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}<1+\frac{3+3+1}{n}<2,$$

we get $2^{n+1}>2n^3>(n+1)^3$.

Exercise 1.30 (3)

For $n=7$, we have

$$2^7=128>2\cdot 7^2=98.$$

Suppose $2^n>n^3$ for some $n\geq 7$. Then $2^{n+1}=2\cdot 2^n>4n^2$. By $2n^2-(n+1)^2=n^2-2n-1=(n-1)^2-2\geq 6^2-2>0$ for $n\geq 7$, we get $2^{n+1}>4n^2>2(n+1)^2$.

Exercise 1.33 (1)

We have

$$\begin{aligned}a_2&=3\cdot 2-2\cdot 1=4, & a_3&=3\cdot 4-2\cdot 2=8,\\ a_4&=3\cdot 8-2\cdot 4=16, & a_5&=3\cdot 16-2\cdot 8=32.\end{aligned}$$

We conjecture $a_n=2^n$.

Exercise 1.33 (2)

We have $a_0 = 1 = 2^0$ and $a_1 = 2 = 2^1$.

Suppose $a_n = 2^n$ is true for all non-negative integers $< n$ (in fact, for $n-1$ and $n-2$ is enough). Then

$$a_n = 3a_{n-1} - a_{n-2} = 3 \cdot 2^{n-1} - 2 \cdot 2^{n-2} = (3 \cdot 2 - 2)2^{n-2} = 2^n.$$

This proves $a_n = 2^n$ by induction.

Exercise 1.34

We have

$$a_0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0,$$
$$a_1 = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2} - \frac{1}{\sqrt{5}}\frac{1-\sqrt{5}}{2} = 1.$$

Suppose the formula is true for all non-negative integers $< n$. Then

$$\begin{aligned}
&a_n = a_{n-1} + a_{n-2}\\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}+1\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}+1\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+2\sqrt{5}+(\sqrt{5})^2}{2^2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-2\sqrt{5}+(\sqrt{5})^2}{2^2}\right)\\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}.
\end{aligned}$$

This completes the inductive proof.

Exercise 1.36

For $n = 1$, we have $1 = 10^0 1$, for $k = 0$ and $m = 1$.

Suppose the statement holds for all natural numbers $< n$. If n is not divisible by 10, then $n = 10^0 n$ for $k = 0$ and $m = n$. If n is divisible by 10, then $n = 10n'$ for some integer $n' = \frac{n}{10} < n$. By the inductive assumption, we know $n' = 10^{k'}m$ for some integer $k \geq 0$ and m not divisible by 10. Then $n = 10^{k'+1}m$. This completes the inductive proof.

Exercise 1.37

A rational number is $r = \frac{m}{n}$, for integers m, n satisfying $n > 0$. If $n = 1$, then $r = \frac{m}{1}$, with 1 (among $m, 1$) not divisible by 3.

Suppose the statement is true for all denominators $< n$. We try to prove the statement for $r = \frac{m}{n}$. If one of m, n is not divisible by 3, then we are done. If both m, n are divisible by 3, then $m = 3m'$ and $n = 3n'$ for integers m', n' satisfying $n' > 0$. Then $r = \frac{m'}{n'}$, with $n' = \frac{n}{3} < n$. By the inductive assumption, we know $r = \frac{m''}{n''}$ for integers m'', n'', such that one of m', n' is not divisible by 3. This completes the inductive proof.

Exercise 1.39

If $m + n = 2$, then $m = n = 1$, and $N(1, 1) = 1$ is divisible by 1.

Suppose $N(m,n)$ is divisible by $n!$ for $m+n=N-1\geq 2$. Then for $m+n=N$, we have

$$N(m,n)=N(m,n-1)n+N(m-1,n).$$

By $m+(n-1)=(m-1)+n=N-1$, we know $N(m,n-1)$ is divisible by $(n-1)!$ and $N(m-1,n)$ is divisible by $n!$. Then $N(m,n-1)n$ is divisible by $(n-1)!n=n!$, and $N(m,n)$ is also divisible by $n!$. This completes the inductive proof.

Exercise 1.40

We first induct on n. For $n=1$, we know $N(m,1)=m$ is divisible by $1!$. Now we assume $N(m,n-1)$ is divisible by $(n-1)!$ for all m.

Next we fix n and further induct on m. For $m=1$, we know $N(1,n)=n!$ is divisible by $n!$. Now we assume $N(m-1,n)$ is divisible by $n!$. This means we know $N(m,n-1)$ is divisible by $(n-1)!$ and $N(m-1,n)$ is divisible by $n!$. Then both $N(m,n-1)n$ $N(m-1,n)$ are divisible by $n!$. This implies

$$N(m,n)=N(m,n-1)n+N(m-1,n)$$

is divisible by $n!$. This completes the double induction.

Exercise 2.4

The only subset of $\emptyset$ is $\emptyset$.

Exercise 2.7

First, $S_{r'}\subset B_r$ if and only if $r'\leq\frac{r}{\sqrt{3}}$. In dimension n, $S_{r'}\subset B_r$ if and only if $r'\leq\frac{r}{\sqrt{n}}$.

Suppose $r'\leq\frac{r}{\sqrt{3}}$. Then $(x,y,z)\in S_{r'}$ satisfies $x^2+y^2+z^2\leq 3r'^2\leq r^2$. This means $(x,y,z)\in B_r$. Therefore $S_{r'}\subset B_r$.

Conversely, suppose $S_{r'}\subset B_r$. Then $(r',r',r')\in S_{r'}$ implies $(r',r',r')\in B_r$. Therefore $r'^2+r'^2+r'^2\leq r^2$. This means $r'\leq\frac{r}{\sqrt{3}}$.

Second, $B_r\subset S_{r'}$ if and only if $r\leq r'$. In dimension n, $B_r\subset S_{r'}$ if and only if $r\leq r'$.

Suppose $r\leq r'$. Then $(x,y,z)\in B_r$ satisfies $x^2+y^2+z^2\leq r^2\leq r'^2$. Therefore $x^2\leq r'^2$, and $y^2\leq r'^2$, and $z^2\leq r'^2$. This implies $|x|<r'$, and $|y|<r'$, and $|z|<r'$. Therefore $(x,y,z)\in S_{r'}$, and proves $B_r\subset S_{r'}$.

Conversely, suppose $B_r\subset S_{r'}$. Then $(r,0,0)\in B_r$ implies $(r,0,0)\in S_{r'}$. This implies $r=|r|\leq r'$.

Exercise 2.8

For any $(x,y)\in G_r$, we have $g(x,y)<r$. By $f(x,y)\leq g(x,y)$, this implies $f(x,y)<r$. Therefore $(x,y)\in F_r$. We conclude $G_r\subset F_r$.

Exercise 2.9

(1) $c\leq a<b\leq d$.
(2) $c\leq a<b\leq d$.
(3) $c<a<b<d$.
(4) $c\leq a<b\leq d$.
(5) $c\leq a<b\leq d$.
(6) $c<a<b\leq d$.

Exercise 2.10

$S_{m,n}\subset S_{m',n'}$ if and only if $m'\leq m\leq n\leq n'$.

Sufficiency: If $m'\leq m\leq n\leq n'$, then any $k\in S_{m,n}$ satisfies $m'\leq m\leq k\leq n\leq n'$, which means $k\in S_{m',n'}$. Therefore $S_{m,n}\subset S_{m',n'}$.

Necessity: If $S_{m,n}\subset S_{m',n'}$, then $m,n\in S_{m,n}$ are in $S_{m',n'}$. Therefore $m'\leq m\leq n\leq n'$.

Exercise 2.11 (3)

We wish $|x+1| < 0.2$ imply $|x^2 - 1| < \epsilon$.

If $|x+1| < 0.2$, then $-1.2 < x < -0.8$. This implies $-2.2 < x - 1 < -1.8$, and $|x-1| < 2.2$. Therefore $|x^2 - 1| = |x+1||x-1| < 0.2 \cdot 2.2 = 0.44$. We may choose any $\epsilon \geq 0.44$. Therefore taking $\epsilon = 0.44$ is enough, and taking $\epsilon = 1$ is also enough.

Exercise 2.12

Take $\delta = \min\{\frac{\epsilon}{3}, 1\} > 0$. Then $|x-1| < \delta$ implies $|x+1| = |(x-1)+2| < |x-1|+2 = \delta+2 \leq 3$. Then $|x-1| < \delta$ further implies

$$|x^2 - 1| = |x+1||x-1| < \delta \cdot 3 \leq \frac{\epsilon}{3} \cdot 3 = \epsilon.$$

Exercise 2.14

Take $n = 10000$. Then $m > n$ implies

$$\frac{m}{m^2+1} < \frac{m}{m^2} = \frac{1}{m} < \frac{1}{10000} = 0.0001.$$

Exercise 2.16 (2)

Taking $a = b = 1$, we get

$$2^n = (1+1)^n = \sum_{k \leq n} \binom{n}{k} = \sum_{\text{odd } k \leq n} \binom{n}{k} + \sum_{\text{even } k \leq n} \binom{n}{k}.$$

Taking $a = 1$ and $b = -1$, we get

$$0 = (1+(-1))^n = \sum_{k \leq n} \binom{n}{k} (-1)^k = \sum_{\text{even } k \leq n} \binom{n}{k} - \sum_{\text{odd } k \leq n} \binom{n}{k}.$$

From the two equalities, we get

$$\sum_{\text{even } k \leq n} \binom{n}{k} = \sum_{\text{odd } k \leq n} \binom{n}{k} = 2^{n-1}.$$