Review: Antiderivative and indefinite integral.

If $\frac{d}{dx}F(x) = f(x)$, then ① f is the derivative of F

② F is an antiderivative of f.

an arbitrary constant.

③. all antiderivatives of f have the form $F(x)+C$.

④ we write $\int f(x)\,dx = F(x) + C$.

called "indefinite integral" ↙ ↘ denote all antiderivatives of f.

$$\int f(x)\,dx = F(x) + C \iff \frac{d}{dx}F(x) = f(x).$$

Example 1. We have $\int x^2\,dx = \frac{1}{3}x^3 + C$ because $\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2$.

Example 2. We have $\int e^x\,dx = e^x + C$ because $\frac{d}{dx}\left(e^x + c\right) = e^x$.

Summary:

given $f(x)$ $\xrightarrow{\text{differentiation}}$ find $f'(x) = \frac{d}{dx}f(x)$.

derivative of f

Find $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$ $\xleftarrow{\text{anti-differentiation}}$ given $f(x)$

antiderivative of f.

Calculation of indefinite integral : How to find $\int f(x)\,dx$?

$\int f(x)\,dx = F(x) + C \iff \frac{d}{dx}F(x) = f(x)$

find $\int f(x)\,dx \iff$ find $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$.

1. Review some notable indefinite integral.

$\int x^p\,dx = \frac{1}{p+1}\cdot x^{p+1} + C \quad (p \neq -1)$. because $\frac{d}{dx}\left(\frac{1}{p+1}\cdot x^{p+1}\right) = x^p$.

$\int \frac{1}{x}\,dx = \ln|x| + C$ because $\frac{d}{dx}\ln x = \frac{1}{x}$ if $x > 0$.
and $\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$ if $x < 0$.

$\int \cos x\,dx = \sin x + C$ because $\frac{d}{dx}(\sin x) = \cos x$.

$\int \sin x\,dx = -\cos x + C$ because $\frac{d}{dx}(-\cos x) = \sin x$.

$\int a^x\,dx = \frac{a^x}{\ln a} + C$ because $\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = \frac{\ln a \cdot a^x}{\ln a} = a^x$.

$\int e^x\,dx = e^x + C$ because $\frac{d}{dx}(e^x) = e^x$.

2. A useful rule: $\int a f(x) + b\, g(x)\, dx = a \cdot \int f(x)\, dx + b \int g(x)\, dx.$

In other words, if $\int f(x)\, dx = F(x) + C$. and $\int g(x)\, dx = G(x) + C$

then $\int a f(x) + b \cdot g(x)\, dx = a \cdot F(x) + b \cdot G(x) + C.$

Example 1. Find $\int 3x^2 + 4e^x\, dx$.

Recall $\int x^2\, dx = \frac{1}{3} \cdot x^3 + C$. $\int e^x\, dx = e^x + C$.

$$\int 3x^2 + 4e^x\, dx = 3 \cdot \int x^2\, dx + 4 \cdot \int e^x\, dx = x^3 + 4e^x + C$$

Example 2. Find $\int (\sin x - 2\cos x)\, dx$.

Recall $\int \sin x\, dx = -\cos x + C$ $\int \cos x\, dx = \sin x + C$

$$\int (\sin x - 2\cos x)\, dx = \int \sin x\, dx - 2 \cdot \int \cos x\, dx = -\cos x - 2\sin x + C.$$

Example 3. Find $\displaystyle\int \frac{x^3+4\sqrt{x}+3}{x^2}\,dx$.

Notice that $\displaystyle\frac{x^3+4\sqrt{x}+3}{x^2} = x + 4\cdot x^{-\frac{3}{2}} + 3\cdot x^{-2}$,

so
$$\int \frac{x^3+4\sqrt{x}+3}{x^2}\,dx = \int \left(x+4\cdot x^{-\frac{3}{2}}+3\cdot x^{-2}\right)dx = \int x\,dx + 4\cdot\int x^{-\frac{3}{2}}dx + 3\int x^{-2}dx.$$

Recall that $\displaystyle\int x^p\,dx = \frac{1}{p+1}\cdot x^{p+1} + C \quad (p \neq -1).$

$$\int x\,dx = \frac{1}{2}\cdot x^2 + C, \qquad \int x^{-\frac{3}{2}}dx = \frac{1}{-\frac{3}{2}+1}\,x^{-\frac{3}{2}+1} + C = -2\cdot x^{-\frac{1}{2}} + C.$$

$$\int x^{-2}dx = \frac{1}{-2+1}\,x^{-2+1} + C = -x^{-1} + C.$$

$$\Rightarrow \int \frac{x^3+4\sqrt{x}+3}{x^2}\,dx = \frac{1}{2}x^2 + 4\cdot\left(-2\cdot x^{-\frac{1}{2}}\right) + 3\cdot\left(-x^{-1}\right) + C$$

$$= \frac{1}{2}x^2 - 8\cdot x^{-\frac{1}{2}} - 3\cdot x^{-1} + C.$$

3. Initial Value Problem: A problem with a differential equation and an initial condition.

Example 1. Find $F(x)$ if $F'(x) = f(x) = 3\cdot e^{x} - \cos x$ and $F(0) = 5$.

Step 1: Find $\int f(x)dx$. (If $\int f(x)dx = H(x) + C$, then $F(x) = H(x) + C$ for some constant C).

Step 2: Solve $F(0) = 5$ for $C \Rightarrow F(x) = H(x) + C$.

Step 1: Notice $\int (3e^{x} - \cos x)\,dx = 3\cdot\int e^{x}dx - \int \cos x\,dx$.

Recall $\int e^{x}dx = e^{x} + C$. $\int \cos x\,dx = \sin x + C$.

$\Rightarrow \int (3e^{x} - \cos x)\,dx = 3\cdot e^{x} - \sin x + C$

$\Rightarrow F(x) = 3\cdot e^{x} - \sin x + C$ for some constant C (because $F(x)$ is an antiderivative of f).

Step 2. Solve $F(0) = 5$ for C.

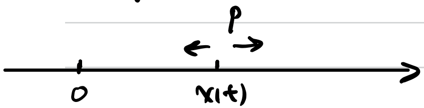
$F(0) = 3\cdot e^{0} - \sin 0 + C = 3 + C = 5 \Rightarrow C = 2. \Rightarrow F(x) = 3\cdot e^{x} - \sin x + 2$.

Summary: If we know $F'(x)$ and $F(a) = b$, then we can find $F(x)$.

the value of F at some point.

Recall $\int x^p\,dx = \frac{1}{p+1}\cdot x^{p+1} + C \ (p \neq -1)$

Example 2. A particle P is moving in a straight line.



$x(t)$: position of P at time t.

$v(t) = \frac{d}{dt}x(t)$: velocity of P at time t.

$a(t) = \frac{d}{dt}v(t)$: acceleration of P at time t.

Question: Find $x(t)$ if $a(t) = 6t+4$, $v(0) = -6$ and $x(0) = 9$.

(1). Notice: $\underline{\frac{d}{dt}v(t) = a(t) = 6t+4}$ and $\underline{v(0) = -6}$ $\Rightarrow$ Find $v(t)$.

Step 1: $\int a(t)\,dt = \int (6t+4)\,dt = 3t^2+4t+C \Rightarrow v(t) = 3t^2+4t+C$ for some constant C.

Step 2: Solve $v(0) = -6$ for C $\Rightarrow v(0) = C = -6 \Rightarrow v(t) = 3t^2+4t-6$.

(2). Notice: $\underline{\frac{d}{dt}x(t) = v(t) = 3t^2+4t-6}$ and $\underline{x(0) = 9}$ $\Rightarrow$ Find $x(t)$.

Step 1: $\int v(t)\,dt = \int (3t^2+4t-6)\,dt = t^3+2t^2-6t+C \Rightarrow x(t) = t^3+2t^2-6t+\underline{C}$ for some constant C.

Step 2: Solve $x(0) = 9$ for C $\Rightarrow x(0) = C = 9. \Rightarrow x(t) = t^3+2\cdot t^2-6t+9$.

Summary: If we know $F'(x)$ and $F(a) = b$, then we can find $F(x)$.

4. Substitution Rule: If $u=g(x)$, then $\underline{\int f(g(x))\cdot g'(x)\,dx} = \underline{\int f(u)\,du}$.

→ difficult to calculate → easy to calculate.

idea: We make a substitution $u=g(x)$ to replace a complicated integral by a simpler integral.

Example 1: Find $\int 2x\cdot\sqrt{x^2+1}\,dx$.

Recall $\int x^p\,dx = \frac{1}{p+1}x^{p+1}+C$ $(p\neq -1)$

Let $u=g(x)=x^2+1$, $f(x)=\sqrt{x}$.

Then $\int 2x\sqrt{x^2+1}\,dx = \int g'(x)\,f(g(x))\,dx \overset{\text{substitution Rule}}{=} \int f(u)\,du = \int \sqrt{u}\,du = \frac{2}{3}\cdot u^{\frac{3}{2}}+C$

return to variable x ← $= \frac{2}{3}(x^2+1)^{\frac{3}{2}}+C$.

Substitution Rule follows from Chain Rule.

Recall Chain Rule: If $y=F(u)$, $u=g(x)$ then $y=F(u)=F(g(x))$ and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Notice: $\frac{dy}{dx}=\frac{d}{dx}F(g(x))$, $\frac{dy}{du}=F'(u)=F'(g(x))$, $\frac{du}{dx}=g'(x)$

Therefore $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \iff \frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x)$

$\overset{\text{means}}{\Longrightarrow} \int F'(g(x))\cdot g'(x)\,dx = F(g(x))+C = F(u)+C$.

$\overset{\text{denote } F' \text{ by } f}{\Longrightarrow} \int f(g(x))\cdot g'(x)\,dx = F(u)+C = \int f(u)\,du$

A easy way to memorize Substitution Rule: We simply conside $g'(x)\,dx$ as du because $g'(x)=\frac{d}{dx}g(x)=\frac{du}{dx}$

Summary: Given $h(x)$, if we can find $u=g(x)$ and $f(x)$ such that $h(x) = g'(x) \cdot f(g(x))$

then Substitution Rule gives $\int h(x)dx = \int g'(x) \cdot f(g(x))\, dx \overset{\text{substitution}}{\underset{\text{Rule}}{=\!=}} \int f(u)\, du$.

Trick to find $u=g(x)$ and $f(x)$:

① We consider $h(x)$ as a product of two functions: $h(x) = h_1(x) \cdot h_2(x)$.

② Then we find $u=g(x)$ such that $h_1(x) = g'(x)$ and $h_2(x) = f(u)$ where f is a simple function.

(The simpler the function $f(x)$ is, the easier it is to calculate $\int f(u)du$.)

Example 2. Find $\int \tan x\, dx$.

Notice: $\int \tan x\, dx = \int \sin x \cdot \frac{1}{\cos x}\, dx$.

Let $u = g(x) = -\cos x$ and $f(x) = \frac{-1}{x}$. (Note: $\int \frac{1}{u} du$ is easy to calculate)

$$\int \tan x\, dx = \int \sin x \frac{1}{\cos x}\, dx = \int g'(x) \cdot f(g(x))\, dx.$$

$$\overset{\text{substitution}}{\underset{\text{Rule}}{=\!=}} \int f(u)\, du = \int \frac{-1}{u} du = -\int \frac{1}{u} du = -\ln|u| + C$$

use $\int \frac{1}{u} du = \ln|u| + C$

$$= -\ln|-\cos x| + C = -\ln|\cos x| + C$$

return to the variable x

More examples for Substitution Rule:

Example 3: Find $\int x^3 \cos(x^4+1)\, dx$.

Notice: $\int x^3 \cdot \cos(x^4+1)\, dx = \frac{1}{4}\int 4x^3 \cdot \cos(x^4+1)\, dx$

Let $u = g(x) = x^4+1$ $\quad f(x) = \cos x$. (Note: $\int \cos u \, du$ is easy to calculate.)

Then $\int x^3 \cdot \cos(x^4+1)\, dx = \frac{1}{4}\int 4x^3 \cdot \underline{\underline{\cos(x^4+1)}}\, dx = \frac{1}{4}\int g'(x) \cdot \underline{\underline{f(g(x))}}\, dx$

$$\overset{\text{substitution}}{\underset{\text{Rule}}{=}} \frac{1}{4}\int \cos u \, du = \frac{1}{4}\sin u + C \underset{\text{return to the variable } x}{=} \frac{1}{4}\sin(x^4+1) + C.$$

Example 4: Find $\int e^{3x+2}\, dx$.

Notice: $\int e^{3x+2}\, dx = \frac{1}{3}\int 3 \cdot e^{3x+2}\, dx$

Let $u = g(x) = 3x+2$ and $f(x) = e^x$ (Note: $\int e^u \, du$ is easy to calculate.)

Then $\int e^{3x+2}\, dx = \frac{1}{3}\int 3\, \underline{\underline{e^{3x+2}}}\, dx = \frac{1}{3}\int g'(x) \cdot \underline{\underline{f(g(x))}}\, dx$

$$\overset{\text{substitution}}{\underset{\text{Rule}}{=}} \frac{1}{3}\int f(u)\, du = \frac{1}{3}\int e^u \, du = \frac{1}{3}e^u + C \underset{\text{return to the variable } x}{=} \frac{1}{3}e^{3x+2} + C.$$

Exaple 5. Find $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

Notice: $\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int x \cdot \frac{x^2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int 2x \cdot \frac{x^2}{\sqrt{x^2+9}}\,dx$

Let $u = g(x) = x^2+9$.

Then $\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int 2x \cdot \frac{x^2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int g'(x) \cdot \frac{g(x)-9}{\sqrt{g(x)}}\,dx$

Let $f(x) = \frac{x-9}{\sqrt{x}} = x^{\frac{1}{2}} - 9 \cdot x^{-\frac{1}{2}}$ (Note: $\int u^p\,du$ is easy to calculate)

Then $\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int g'(x) \cdot f(g(x))\,dx$

$$\overset{\text{substitution}}{\underset{\text{Rule}}{=}} \frac{1}{2}\int f(u)\,du = \frac{1}{2}\int \left(u^{\frac{1}{2}} - 9 \cdot u^{-\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,dx - \frac{9}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} - \frac{9}{2}\cdot 2 \cdot u^{\frac{1}{2}} + C$$

use $\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C \quad (p \neq -1)$

$$= \frac{1}{3}(x^2+9)^{\frac{3}{2}} - 9 \cdot (x^2+9)^{\frac{1}{2}} + C.$$

return to the variable x