Math2001 Answer to Homework 9

Exercise 4.1

(1) $456 \div 123 = 3 \cdots 87$
(2) $123 \div 456 = 0 \cdots 456$
(3) $-456 \div 123 = -4 \cdots 36$
(4) $-456 \div (-123) = 4 \cdots 36$
(5) $-123 \div 456 = -1 \cdots 333$
(6) $1221 \div 33 = 37$

Exercise 4.2

The remainder of the division of $a_1 + a_2$ by b is $r_1 + r_2$.

Exercise 4.3

(1) $x^4 + 2x^3 - 3x + 1 = (x^3 - 3)(x + 2) + 7$.
(2) $3x^5 + 4x^3 - 2x + 5 = (3x^3 - 3x^2 + x + 5)(x^2 + x + 2) - (9x + 5)$.

Exercise 4.4

Suppose $a(x) = (x - x_0)q(x) + r(x)$. Since $x - x_0$ is of degree 1, $r(x)$ is of degree 0, i.e. a constant. $r = r + q(x_0)(x_0 - x_0) = a(x_0)$, thus the remainder of $a(x)$ divided by $x - x_0$ is $a(x_0)$.

Exercise 4.5

$6 = 110_{[2]} = 6_{[16]}$.
$20 = 10100_{[2]} = 14_{[16]}$.
$200 = 11001000_{[2]} = \mathrm{C}8_{[16]}$.
$1024 = 10000000000_{[2]} = 400_{[16]}$.
$12345 = 11000000111001_{[2]} = 3039_{[16]}$.

Exercise 4.6

(1) 8.
(2) 27.
(3) 42.
(4) 273.
(5) 55.
(6) 60.
(7) 4096.
(8) 939.

Exercise 4.7

(1) $10011_{[2]}$.
(2) $111_{[2]}$.
(3) $1001110_{[2]}$.
(4) $10.0\dot{0}\dot{1}_{[2]}$.
(5) $211_{[3]}$.
(6) $22_{[3]}$.
(7) $11110_{[3]}$.

(8) $2.\dot{0}1021\dot{2}_{[3]}$.

Exercise 4.8

(1) $10, 0_{[100]}$.
(2) $1000000_{[10]}$.
(3) $20_{[4]}$.
(4) $617_{[8]}$.
(5) $1110101011011_{[2]}$.
(6) $101011000011000_{[2]}$.
(7) $1210001000_{[3]}$.
(8) $55050_{[6]}$.
(9) $53030_{[10]}$.

Exercise 4.9

$x^4 + 3x^3 - 3x + 1 = y^4 - 6y^3 + 12y^2 - 11y + 7$, where $y = x + 2$.

Exercise 4.10

$3x^5 + 4x^3 - 2x + 5 = (3x - 6)y^2 + (x + 17)y - (9x + 5)$, where $y = x^2 + x + 2$.

Exercise 4.11

Suppose $\deg b(x) = d$. Given two expansions of $a(x)$ in $b(x)$:

$$a(x) = r_k(x)b(x)^k + \ldots + r_1(x)b(x) + r_0(x)$$

and

$$a(x) = t_l(x)b(x)^l + \ldots + t_1(x)b(x) + t_0(x)$$

with $\deg r_0(x), \ldots, \deg r_k(x), \deg t_0(x), \ldots, \deg t_l(x) < d$.

If there is some $r_i(x) \neq t_i(x)$, let n be the largest index such that $r_n(x) - t_n(x) \neq 0$. There is

$$(r_n(x)-t_n(x))b(x)^n = -((r_{n-1}(x)-t_{n-1}(x))b(x)^{n-1}+\ldots+(r_1(x)-t_1(x))b(x)+(r_0(x)-t_0(x))).$$

However, the degree of LHS is not less than nd, while degree of RHS can't be larger than $(d-1)+(n-1)d = nd-1$, thus two sides cannot be the same. Therefore, the two expansions are identical.

Exercise 4.12

(1) $3 \cdot 1053 - 8 \cdot 390 = 39$.
(2) $(-18) \cdot 1053 - 48 \cdot (-390) + 247 = 13$.
(3) $(-1386) \cdot 1053 - 3696 \cdot (-390) + 77 \cdot 247 + 2 \cdot (-500) = 1$.

Exercise 4.13

Suppose $d = \gcd(a, b)$. There are $u, v \in \mathbb{Z}$ such that $ua - vb = d$, thus $u(ac) - v(bc) = dc$. Since dc is common divisor of ac and bc, $dc = \gcd(ac, bc)$.

Suppose $e = \gcd(b, c)$ and $f = \gcd(a, e)$. There are $m, n, p, q \in \mathbb{Z}$ such that $mb - nc = e$, $pa - qe = f$. Thus there is $pa - q(mb - nc) = pa - (mq)b - (nq)c = f$. Since f is common divisor of a, b and c, $\gcd(a, b, c) = \gcd(a, e) = \gcd(a, \gcd(b, c))$.

EXERCISE 4.14

(1) $\frac{1}{2}(x^2+x-1)(x^7-x^3)-\frac{1}{2}(x^4+x^3-2)(x^5-x^3+x^2-1)=x^2-1$.

(2) $\frac{1}{4}(x^4+x^3+x-1)(x^7-x^3)-\frac{1}{4}(x^6+x^5+x^4+x^3-2x^2-2)(x^5-x^3+x^2-1)-\frac{1}{2}(x^4-2x+1)=x-1$.

(3) $\frac{1}{2}(x^4+1)-\frac{1}{8}(x^7+2x^6+2x^5+3x^4+x^3-1)(x^7-x^3)+\frac{1}{8}(x^9+2x^8+3x^7+4x^6+x^5-3x^3-4x^2-2x-2)(x^5-x^3+x^2-1)+\frac{1}{4}(x^3+x^2+x+1)(x^4-2x+1)=1$.

EXERCISE 4.16

If $q|n$ for some $q \geqslant \sqrt{n}$, then $\frac{n}{q}|n$ and $\frac{n}{q} \leqslant \sqrt{n}$. Whether $\frac{n}{q}$ is a prime or not, n will be divisible by a prime p not larger than $\sqrt{n}$.

EXERCISE 4.17

$1053 = 3^4 \cdot 13$, $-390 = (-1)\cdot 2\cdot 3\cdot 5\cdot 13$, $247 = 13\cdot 19$,$-500 = (-1)\cdot 2^2\cdot 5^4$. The common divisor is thus 1.

EXERCISE 4.19

(1) Suppose $n = 3p = 5q$, then $n = 6n - 5n = 15(2q-p)$. Thus $15|n$.

(2) True. Suppose $n = 3p = 20q$, then $n = 21n - 20n = 60(7q-p)$. Thus $60|n$.

(3) False. We only have $30|n$.

EXERCISE 4.22

(1) One of three consecutive natural numbers must be divisible by 2, and one of three consecutive natural numbers must be divisible by 3. The product is then divisible by both 2 and 3, hence divisible by 6.

(2) Within four consecutive natural numbers, one must be odd multiples of 2 and another one is even multiples of 2, hence the product is divisible by 8. One of the four numbers must be divisible by 3, so the product is also divisible by 3. Thus the product of four consecutive natural numbers is a multiple of 24.

EXERCISE 4.28

1,3,5,7 are invertible in $\mathbb{Z}/8\mathbb{Z}$. Their product is one.

EXERCISE 4.29

1,7,11,13,17,19,23,29.

EXERCISE 4.30

Such a is coprime with p, thus invertible in $\mathbb{Z}/p\mathbb{Z}$.

EXERCISE 4.31

$p^{e-1}q^{f-1}(p-1)(q-1)$.

EXERCISE 4.37

We have $10^k - 1 = (10-1)(10^{k-1}+\ldots+1) = 9(10^{k-1}+\ldots+1)$, hence 10^k-1 is divisible by 9.

(1) Given $n = a_k 10^k + \ldots + a_1 10 + a_0$ is divisible by 3. Since $10 \equiv 1 \pmod 3$,

$$n \equiv a_k + \ldots + a_1 + a_0 \pmod 3.$$

Hence

$$n \equiv 0 \pmod 3 \Longleftrightarrow a_k + \ldots + a_1 + a_0 \equiv 0 \pmod 3.$$

(2) Likewise, since $10 \equiv 1 \pmod 9$,

$$n \equiv a_k + \ldots + a_1 + a_0 \pmod 9.$$

Hence

$$n \equiv 0 \pmod 9 \Longleftrightarrow a_k + \ldots + a_1 + a_0 \equiv 0 \pmod 9.$$

Exercise 4.38

Given a natural number n and its decimal expansion:

$$n = \sum_{i=0}^{k} a_i 10^i,$$

since $10 \equiv -1 \pmod{11}$, there is

$$n \equiv \sum_{i=0}^{k} a_i 10^i \equiv \sum_{i=0}^{k} (-1)^i a_i \pmod{11},$$

thus

$$n \equiv 0 \pmod{11} \Longleftrightarrow \sum_{i=0}^{k} (-1)^i a_i \equiv 0 \pmod{11}.$$

Exercise 5.1

(1) 50; 33; 20.
(2) 50; 67; 80.
(3) 16; 10; 6.
(4) 26.
(5) 74.
(6) 54.

Exercise 5.2

(1) 23.
(2) 10.
(3) 13.

Exercise 5.3

We prove the Inclusion-Exclusion principle by using induction. Firstly, the principle holds for $n = 2$, i.e. $|X_1 \cup X_2| = |X_1| + |X_2| - |X_1 \cap X_2|$.

Suppose it holds for $X_1, \ldots, X_n$, i.e.

$$|X_1 \cup \cdots \cup X_n| = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^{k-1} |X_{i_1} \cap \cdots \cap X_{i_k}|,$$

then consider the case of $X_1, \dots, X_n, X_{n+1}$, apply the principle in $n = 2$, there is $|X_1 \cup \dots \cup X_n \cup X_{n+1}| = |X_1 \cup \dots \cup X_n| + |X_{n+1}| - |(X_1 \cup \dots \cup X_n) \cap X_{n+1}|$. Furthermore, there is

$$
\begin{aligned}
|X_1 \cup \dots \cup X_n \cup X_{n+1}| =& |X_1 \cup \dots \cup X_n| + |X_{n+1}| - |(X_1 \cup \dots \cup X_n) \cap X_{n+1}| \\
=& |X_1 \cup \dots \cup X_n| + |X_{n+1}| - |(X_1 \cap X_{n+1}) \cup \dots \cup (X_n \cap X_{n+1})| \\
=& \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} |X_{i_1} \cap \dots \cap X_{i_k}| + |X_{n+1}| \\
& - \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} |(X_{i_1} \cap X_{n+1}) \cap \dots \cap (X_{i_k} \cap X_{n+1})| \\
=& \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} |X_{i_1} \cap \dots \cap X_{i_k}| + |X_{n+1}| \\
& - \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} |(X_{i_1} \cap \dots \cap X_{i_k} \cap X_{n+1})| \\
=& \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} |X_{i_1} \cap \dots \cap X_{i_k}| + |X_{n+1}| \\
& + \sum_{k=1}^{n} \sum_{\substack{1 \leqslant i_1 < \dots < i_k \leqslant n, \\ i_{k+1} = n+1}} (-1)^{k} |X_{i_1} \cap \dots \cap X_{i_k} \cap X_{i_{k+1}}| \\
=& (-1)^n |X_1 \cap \dots \cap X_{n+1}| + \\
& \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (-1)^{k-1} (|X_{i_1} \cap \dots \cap X_{i_k}| + |X_{i_1} \cap \dots \cap X_{i_{k-1}} \cap X_{n+1}|) \\
=& \sum_{k=1}^{n+1} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n+1} (-1)^{k-1} |X_{i_1} \cap \dots \cap X_{i_k}|.
\end{aligned}
$$

So far we have proven the Inclusion-Exclusion principle.

EXERCISE 5.6

The proposition is true for $|Y| = 1$.

Suppose it is true for $|Y| = n$, then in the case $|Y| = n + 1$: For an element $y \in Y$, $|X \times Y| = |X \times (Y - \{y\}) \cup X \times \{y\}| = |X \times (Y - \{y\})| + |X \times \{y\}| = |X|(|Y| - 1) + |X| = |X||Y|$.

EXERCISE 5.7

Let $\mathcal{A}(n)$ be the subset of $\mathcal{P}(\{1, 2, \dots, n\})$ consists of subsets with even number of elements, and $\mathcal{B}(n)$ be the subset of $\mathcal{P}(\{1, 2, \dots, n\})$ consists of subsets with even number of elements. So there is $E(n) = |\mathcal{A}(n)|$ and $O(n) = |\mathcal{B}(n)|$.

$\mathcal{A}(n)$ has a decomposition into two parts, the first part $\mathcal{A}_1(n)$ consists of those subsets of $\{1, 2, \dots, n\}$ not containing n and the second part $\mathcal{A}_2(n)$ consists of those subsets of $\{1, 2, \dots, n\}$ containing n. Then it can be noticed that $\mathcal{A}_1(n) = \mathcal{A}(n - 1)$ and $\mathcal{A}_2(n) = \mathcal{B}(n - 1)$. Since $\mathcal{A}_1(n) \cap \mathcal{A}_2(n) = \emptyset$, $E(n) = |\mathcal{A}_1(n)| + |\mathcal{A}_2(n)| = E(n - 1) + O(n - 1)$. Likewise, we can show $O(n) = O(n - 1) + E(n - 1)$ in the same way.

Then we have $E(n) = O(n-1) + E(n-1) = O(n)$. Together with the fact that $E(n) + O(n) =$

2^n, we have $E(n) = O(n) = 2^{n-1}$.

Exercise 5.9

Assume that there is an onto map f from $\{1, 2, \ldots, n\}$ to $\mathbb{N}$, then the image $\mathbb{N}$ has at most n elements. Since $\mathbb{N}$ is a total order set, there is a maximal element in $\mathbb{N}$, which is a contradiction to Peano axioms. Therefore, such f doesn't exist.

Exercise 5.10

There is a bijection f between X and Y, and another bijection g between Y and Z. Then $g \circ f : X \to Z$ is also a bijection. By definition we have $|X| = |Z|$.

Exercise 5.11

There is a bijection f between X_1 and X_2, and another bijection g between Y_1 and Y_2. Then $(f, g) : X_1 \times X_2 \to Y_1 \times Y_2$ is a bijection. By definition we have $|X_1 \times X_2| = |Y_1 \times Y_2|$.

Exercise 5.12

Given a bijection $f : X \to Y$, there is an induced bijection $\mathcal{P}(f)$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined as

$$\mathcal{P}(f)(U) = \{f(u); u \in U \subset X\}.$$

Hence $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$.

Exercise 5.13

(1) By the maps $f_1 : \mathbb{R} \to \text{Unit Circle}$, $t \mapsto \exp(\frac{it}{1+|t|})$ and $g_1 : \text{Unit Circle} \to \mathbb{R}$, $e^{i\theta} \mapsto \theta$ where $\theta \in [-\pi, \pi)$, it can be concluded that $|\mathbb{R}| = |\text{Unit CIrcle}|$.

(2) By the maps $f_2 : \mathbb{R}^2 \to \text{Open Unit Disk}$, $v \mapsto \frac{v}{1+|v|}$ and $g_2 : \text{Open Unit Disk} \to \mathbb{R}^2$, $w \mapsto w$, it can be concluded that $|\mathbb{R}^2| = |\text{Open Unit Disk}|$. Since $|\mathbb{R}^2| = |\mathbb{R}|$, $|\mathbb{R}| = |\text{Open Unit Disk}|$.

(3) By the maps $f_3 : \text{Open Unit Disk} \to \text{Closed Unit Square}$, $v \mapsto v$ and $g_3 : \text{Closed Unit Square} \to \text{Open Unit Disk}$, $w \mapsto \frac{1}{2}w$, it can be concluded that $|\text{Open Unit Disk}| = |\text{Closed Unit Square}|$.

(4) By the maps $f_4 : \text{Unit Sphere} \to \text{Open Unit Disk}$, $(x, y, z) \mapsto (\frac{1}{3}(x + \frac{2}{3}\operatorname{sgn}(z)), \frac{1}{3}y)$, and $g_4 : \text{Open Unit Disk} \to \text{Unit Sphere}$, $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$, it can be concluded that $|\text{Open Unit Disk}| = |\text{Unit Sphere}|$.

Exercise 5.14

Consider $f : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by $f(A, B) = \{2a; a \in A\} \cup \{2b - 1; b \in B\}$. This is a bijection, thus $|\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.

Exercise 5.15

By definition, there are injections $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is also an injection, thus $|X| \leqslant |Z|$ by definition.

Exercise 5.16

WLOG, let $Y = \{0, 1, 2, \ldots, p - 1\}$. Given $\alpha \in \operatorname{Map}(\mathbb{N}, Y)$, define $f : \operatorname{Map}(\mathbb{N}, Y) \to [0, 1]$ as

$$f(\alpha) = \sum_{k \geqslant 1} \alpha(k) p^{-k}.$$

This is a bijection, thus $|\operatorname{Map}(\mathbb{N}, Y)| = |[0,1]| = |\mathbb{R}|$.

Exercise 5.19

To define an injection $f : \mathbb{N} \to X$, first arbitrarily pick an element $a_1 \in X$, and assign it to 1. Suppose we have already define $f(1), \ldots, f(n)$, then define $f(n+1)$ as an arbitrary element in $X - \{f(1), f(2), \ldots, f(n)\}$. In this way we complete the definition of $f : \mathbb{N} \to X$.

Exercise 5.20

Firstly, $X - A$ must be infinite since otherwise $|X| = |A| + |X - A|$ would be finite. Then $X - A$ is at most countable since $X - A$ is a subset of X.

Exercise 5.21

(1) $\mathbb{Q}[x] = \mathbb{Q} \oplus \mathbb{Q}x \oplus \cdots \oplus \mathbb{Q}x^n \oplus \cdots$. Thus $\mathbb{Q}[x]$ is countable.

(2) Consider a map $\varphi : \mathbb{R} \to \mathcal{C}[0,1]$ defined as $\varphi(x) = ([0,1] \to \{x\})$, a real number being sent to a constant function on $[0,1]$. φ is injective thus $\mathcal{C}[0,1]$ is uncountable.

(3) Uncountable by Exercise 5.16.

(4) Countable.

(5) same as $\mathbb{Q}^2$, thus countable.

(6) Uncountable.

(7)Uncountable.

Exercise 5.22

A continuous function is determined by its value on $\mathbb{Q}$. Suppose $\mathbb{Q} = \{n_1, n_2, \ldots, n_k, \ldots\}$, then

$$\mathcal{C}(\mathbb{R}) = \bigcup_{k \geqslant 1} \mathbb{R} \times \{n_1, \ldots, n_k\}.$$

Thus $|\mathcal{C}(\mathbb{R})| = |\mathbb{R}|$.

Exercise 5.23

Denote the set of integrable functions on $[0,1]$ by $\mathcal{R}[0,1]$, and the set of bounded integrable functions on $[0,1]$ by $\mathcal{R}[0,1]_b$.

Define the length of subsets of $[0,1]$ of the form

$$\mathcal{A} = \bigsqcup_{k=1}^{\infty} (a_k, b_k), \text{ where } a_1 < b_1 < \cdots < a_k < b_k < \cdots$$

to be

$$l(\mathcal{A}) := \sum_{k \geqslant 1} (b_k - a_k).$$

For a general subset $S \subset (0,1)$, the length is defined as

$$l(S) := \inf_{S \subset \mathcal{A}} l(\mathcal{A}).$$

First we make the assertion that a bounded function f is (Riemannian) integrable if and only if the set consisting of discontinuity points of f has zero length.

Consider the Cantor set $\mathcal{C}$ and the bijection $\varphi : \mathbb{R} \to \mathcal{C}$. Given $K \in \mathcal{P}(\mathbb{R})$, the characteristic function $\chi_{\varphi(\frac{2}{\pi}\arctan(K))}$ is integrable by our assertion. Thus $|\mathcal{P}(\mathbb{R})| \leqslant |\mathcal{R}[0,1]_b|$.

Conversely, given an integrable function f, its graph is a subset of $\mathbb{R}^2$, thus $|\mathcal{R}[0,1]| \leqslant |\mathcal{P}(\mathbb{R}^2)| = |\mathcal{P}(\mathbb{R})|$.

$\mathcal{R}[0,1]_b \subset \mathcal{R}[0,1]$. Therefore, there is $|\mathcal{R}[0,1]_b| = |\mathcal{R}[0,1]| = |\mathcal{P}(\mathbb{R})|$.

Now we prove the assertion: Let $\mathcal{D}(f) = \{x \in (0,1); f \text{ is discontinuous at } x\}$.

- Given a bounded function f with $|f| < M$ and $l(\mathcal{D}(f)) = 0$. For any $\epsilon > 0$, there is $\mathcal{A} \subset [0,1]$ such that $l(\mathcal{A}) < \epsilon$. Since f restricted on $[0,1] - \mathcal{A}$ is continuous, it is integrable, then there is $\delta > 0$ such that for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ with $|P| < \delta$ the difference between upper Darboux sum and lower Darboux sum is smaller than ϵ:

$$\sum_{k=1}^{n}(\sup_{[x_{k-1},x_k]} f|_{[0,1]-\mathcal{A}} - \inf_{[x_{k-1},x_k]} f|_{[0,1]-\mathcal{A}})l([x_{k-1},x_k] \cap ([0,1]-\mathcal{A})) < \epsilon.$$

  Then
$$\begin{aligned}
&\sum_{k=1}^{n}(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f) \cdot |x_{k-1} - x_k| \\
\leqslant&\sum_{k=1}^{n}(\sup_{[x_{k-1},x_k]} f|_{\mathcal{A}} - \inf_{[x_{k-1},x_k]} f|_{\mathcal{A}}) \cdot l([x_{k-1},x_k] \cap \mathcal{A}) + \\
&\sum_{k=1}^{n}(\sup_{[x_{k-1},x_k]} f|_{[0,1]-\mathcal{A}} - \inf_{[x_{k-1},x_k]} f|_{[0,1]-\mathcal{A}})l([x_{k-1},x_k] \cap ([0,1]-\mathcal{A})) \\
\leqslant&2M \cdot l(\mathcal{A}) + \epsilon \\
=&(2M+1)\epsilon.
\end{aligned}$$

  Thus f is integrable.

- Suppose $l(\mathcal{D}(f)) > 0$, then $\mathcal{D}(f)$ has to be uncountable. Being discontinuous at x means there is $\epsilon_x > 0$ such that for any $\delta > 0$, there is x_δ such that $|x - x_\delta| < \delta$ and $|f(x) - f(x_\delta)| > \epsilon_x$.

  Let $\mathcal{D}(f)_n = \{x \in \mathcal{D}(f) | \epsilon_x > \frac{1}{n}\}$, then there must be some n_0 such that $l(\mathcal{D}(f)_{n_0}) > 0$. This means there are infinitely many $x \in \mathcal{D}(f)$ such that $\epsilon_x > \frac{1}{n_0}$, thus this limit is larger than $\frac{1}{n_0}N$ for any positive integer N.

  Now consider $\epsilon_0 = l(\mathcal{D}(f)_{n_0})/n_0$. For any $\delta > 0$, there is a partition $P_\delta = \{t_0, t_1, \ldots, t_m\}$ of $[0,1]$ such that $|P| < \delta$ and P_δ is a refinement of partition at endpoints of $\mathcal{A}$, where $\mathcal{A}$ containing $\mathcal{D}(f)_{n_0}$ with $l(\mathcal{A}) < l(\mathcal{D}(f)_{n_0}) + \frac{1}{n_0}$. Then the difference of upper sum and lower sum satisfies the following:

$$\sum_{k=1}^{n}(\sup_{[t_{k-1},t_k]} f - \inf_{[t_{k-1},t_k]} f) \cdot |t_{k-1} - t_k| > \frac{l(\mathcal{A})}{n_0} > \frac{l(\mathcal{D}(f)_{n_0})}{n_0} = \epsilon_0.$$

  Thus f is not integrable.