MATH 1014 Final Exam
Part II: Long Question Marking Scheme

17. *[12 pts]*

(a) [7 pts]

Green-Yellow Version

$$\int 2x^3 \tan^{-1} x\, dx = \int 2\tan^{-1} x d\left(\frac{x^4}{4}\right)$$

3pts → $$= \frac{1}{2}x^4 \tan^{-1} x - \int \frac{x^4}{2}\frac{1}{1+x^2}\, dx$$

2pts → $$= \frac{1}{2}x^4 \tan^{-1} x - \frac{1}{2}\int (x^2 - 1 + \frac{1}{1+x^2})\, dx$$

$$= \frac{1}{2}(x^4 \tan^{-1} x - \frac{x^3}{3} + x) - \frac{1}{2}\int \frac{1}{1+x^2}\, dx$$

2pts → i.e., $k = \frac{1}{2}$.

White-Orange Version

$$\int x^3 \tan^{-1} x\, dx = \int \tan^{-1} x d\left(\frac{x^4}{4}\right)$$

$$= \frac{1}{4}x^4 \tan^{-1} x - \int \frac{x^4}{4}\frac{1}{1+x^2}\, dx$$

$$= \frac{1}{4}x^4 \tan^{-1} x - \frac{1}{4}\int (x^2 - 1 + \frac{1}{1+x^2})\, dx$$

$$= \frac{1}{4}(x^4 \tan^{-1} x - \frac{x^3}{3} + x) - \frac{1}{4}\int \frac{1}{1+x^2}\, dx$$

i.e., $k = \frac{1}{4}$.

(b) [5 pts]

Green-Yellow Version

$$I = \int_0^1 2x^3 \tan^{-1} x\, dx$$

3 pts → $$= \frac{1}{2}\left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x\right]_0^1$$

2pts → $$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{3} + 1 - \frac{\pi}{4}\right] = \frac{1}{3}$$

White-Orange Version

$$I = \int_0^1 x^3 \tan^{-1} x\, dx$$

$$= \frac{1}{4}\left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x\right]_0^1$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - \frac{1}{3} + 1 - \frac{\pi}{4}\right] = \frac{1}{6}$$

18. [*12 pts*]

(a) [4 pts]

Green-Yellow Version

The series is convergent.

Since $\dfrac{1}{\sqrt{n+2}}$ is a decreasing sequence with $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n+2}} = 0$, the alternating series is convergent by the alternating series test.

White-Orange Version

The series is convergent.

Since $\dfrac{1}{\sqrt{n+1}}$ is a decreasing sequence with $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$, the alternating series is convergent by the alternating series test.

(b) [4 pts]

Green-Yellow Version

The series is convergent.

Since $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{3^n5^n}{n^n}} = \lim_{n\to\infty}\frac{3\cdot 5}{n} = 0 < 1$, the series is convergent by the root test.

White-Orange Version

The series is convergent.

Since $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{2^n5^n}{n^n}} = \lim_{n\to\infty}\frac{2\cdot 5}{n} = 0 < 1$, the series is convergent by the root test.

(c) [4 pts]

Green-Yellow Version

The series is divergent.

Since $\displaystyle\lim_{n\to\infty}\ln\frac{n}{3n+2} = \ln\frac{1}{3} \neq 0$, the series is divergent by the divergence test.

White-Orange Version

The series is divergent.

Since $\displaystyle\lim_{n\to\infty}\ln\frac{n}{2n+3} = \ln\frac{1}{2} \neq 0$, the series is divergent by the divergence test.

1 pt for correct answer

1 pt for correct test

2 pts for correct reasoning with the test chosen.

19. [*14 pts*]

(a) [7 pts]

Green-Yellow Version

By applying the Ratio Test:

3 pts →

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}}{(n+3)6^{n+1}}(x-3)^{n+1}\right|}{\left|\frac{(-1)^{n+1}}{(n+2)6^{n}}(x-3)^{n}\right|} < 1$$

1 pts →

$$\lim_{n\to\infty} \frac{n+2}{6(n+3)}|x-3| < 1$$

2 pts →

$$\frac{1}{6}|x-3| < 1$$

1 pts → The interval is $-3 < x < 9$

White-Orange Version

By applying the Ratio Test:

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}}{(n+3)4^{n+1}}(x-2)^{n+1}\right|}{\left|\frac{(-1)^{n+1}}{(n+2)4^{n}}(x-2)^{n}\right|} < 1$$

$$\lim_{n\to\infty} \frac{n+2}{4(n+3)}|x-2| < 1$$

$$\frac{1}{4}|x-2|^2 < 1$$

The interval is $-2 < x < 6$

(b) [2 pts]

Green-Yellow Version

At the endpoint $x = -3$, the series is

1 pts

$$\sum_{n=0}^{\infty} \frac{1}{n+2}$$

which is divergent.

At the endpoint $x = 9$, the series is

1 pt

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$$

which is convergent.

White-Orange Version

At the endpoint $x = -2$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{n+2}$$

which is divergent.

At the endpoint $x = 6$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$$

which is convergent.

(c) [5 pts]

Green-Yellow Version

1 pt →

$$H'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)6^n}(x-3)^{n+2}$$

1 pt →

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{6^n}(x-3)^{n+1}$$

1 pt →

$$H'(5) = \sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}}{6^n}$$

2 pts →

$$= 2\cdot\frac{1}{1 + \frac{2}{6}} = 3/2$$

White-Orange Version

$$H'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)4^n}(x-2)^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(x-2)^{n+1}$$

$$H'(4) = \sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}}{4^n}$$

$$= 2\cdot\frac{1}{1 + \frac{2}{4}} = 4/3.$$

$$2\cdot\frac{1}{1+\frac{1}{3}} = \cancel{2\cdot\frac{3}{4}} = \cancel{\frac{3}{2}}$$

$$\cancel{\frac{1}{1+\frac{1}{2}} = \frac{2}{3}}$$