Math1014 Calculus II
Integration Techniques

1. Practise integration by parts by evaluating some of the following integrals.

(i) $\int t \sin 2t dt$ (ii) $\int p^5 \ln p dp$ (iii) $\int e^{-\theta} \cos 2\theta d\theta$,

(iv) $\int (x^2+1)e^{-x} dx$ (v) $\int_1^{\sqrt{3}} \tan^{-1} \frac{1}{x} dx$ (vi) $\int_1^2 \frac{(\ln x)^2}{x^3} dx$

(vii) $\int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr$ (viii) $\int x^2 \sin 2x dx$

Solution

(i) Let $u = t$ and $v' = \sin 2t$. Then $u' = 1$ and $v = -\frac{1}{2}\cos 2t$. Using integration by parts, we have

$$\int \underbrace{t}_{u} \underbrace{\sin 2t}_{v'} dt = \underbrace{-\frac{1}{2} t \cos 2t}_{uv} - \int \underbrace{-\frac{1}{2}\cos 2t}_{v} \cdot \underbrace{1}_{u'} dt = -\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t + C$$

Or, using the version $\int u dv = uv - \int v du$, we have

$$\int t \sin 2t dt = \int \underbrace{-\frac{1}{2} t}_{u} \, d \underbrace{(\cos 2t)}_{v} = -\frac{1}{2} t \cos 2t - \int \cos 2t \, d(-\frac{1}{2}t)$$

$$= -\frac{1}{2} t \cos 2t + \int \frac{1}{2} \cos 2t \, dt = -\frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t + C$$

(ii)

$$\int p^5 \ln p dp = \int \underbrace{\frac{1}{6} \ln p}_{u} \, \underbrace{dp^6}_{dv} = \frac{1}{6} p^6 \ln p - \int \frac{1}{6} p^6 d \ln p = \frac{1}{6} p^6 \ln p - \int \frac{1}{6} p^6 \cdot \frac{1}{p} dp$$

$$= \frac{1}{6} p^6 \ln p - \int \frac{1}{6} p^5 dp = \frac{1}{6} p^6 \ln p - \frac{1}{36} p^6 + C$$

(iii)

$$\underline{\int e^{-\theta} \cos 2\theta d\theta} = \int \frac{1}{2} e^{-\theta} d \sin 2\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \int \frac{1}{2} \sin 2\theta d e^{-\theta}$$

$$= \frac{1}{2} e^{-\theta} \sin 2\theta - \int \frac{1}{4} e^{-\theta} d \cos 2\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta - \underline{\frac{1}{4} \int e^{-\theta} \cos 2\theta d\theta}$$

i.e.,

$$\frac{5}{4} \int e^{-\theta} \cos 2\theta d\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta + C$$

$$\int e^{-\theta} \cos 2\theta d\theta = \frac{2}{5} e^{-\theta} \sin 2\theta - \frac{1}{5} e^{-\theta} \cos 2\theta + C$$

(iv)

$$\int (x^2+1)e^{-x} dx = \int -(x^2+1) \, de^{-x} = -(x^2+1)e^{-x} + \int e^{-x} \cdot 2x dx = -(x^2+1)e^{-x} - \int 2x de^{-x}$$

$$= -(x^2+1)e^{-x} - 2xe^{-x} + \int 2e^{-x} dx = -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

(v)

$$\int_1^{\sqrt{3}} \tan^{-1} \frac{1}{x} dx = x \tan^{-1} \frac{1}{x} \Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} x \, d \tan^{-1} \frac{1}{x} = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} - \int_1^{\sqrt{3}} x \frac{1}{1+(\frac{1}{x})^2} \cdot \frac{-1}{x^2} dx$$

$$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \int_1^{\sqrt{3}} \frac{x}{x^2+1} dx = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt{3}} = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

(vi)

$$\int_1^2 \frac{(\ln x)^2}{x^3} dx = \int_1^2 -\frac{1}{2}(\ln x)^2\, d(x^{-2}) = -\frac{1}{2}x^{-2}(\ln x)^2\Big|_1^2 + \frac{1}{2}\int_1^2 x^{-2} d(\ln x)^2$$

$$= -\frac{1}{8}(\ln 2)^2 + \int_1^2 x^{-2}(\ln x)\frac{1}{x}dx = -\frac{1}{8}(\ln 2)^2 + \int_1^2 x^{-3}\ln x\, dx$$

$$= -\frac{1}{8}(\ln 2)^2 - \int_1^2 \frac{1}{2}\ln x\, dx^{-2} = -\frac{1}{8}(\ln 2)^2 - \frac{1}{2}x^{-2}\ln x\Big|_1^2 + \int_1^2 \frac{1}{2}x^{-2} d\ln x$$

$$= -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \int_1^2 \frac{1}{2}x^{-3}dx = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 - \frac{1}{4}x^{-2}\Big|_1^2 = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \frac{3}{16}$$

(vii)

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr = \int_0^1 r^2 d(4+r^2)^{1/2} = r^2(4+r^2)^{1/2}\Big|_0^1 - \int_0^1 (4+r^2)^{1/2} dr^2$$

$$= \sqrt{5} - \frac{2}{3}(4+r^2)^{3/2}\Big|_0^1 = \frac{16-7\sqrt{5}}{3}$$

(viii)

$$\int x^2 \sin 2x dx = \int -\frac{1}{2}x^2 d\cos 2x = -\frac{1}{2}x^2\cos 2x - \int \cos 2x d(-\frac{1}{2}x^2) = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x dx$$

$$= -\frac{1}{2}x^2\cos 2x + \int \frac{1}{2}x d\sin 2x = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x dx$$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

2. Use integration by parts to prove the reduction formula:

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx\ .$$

Solution For any integer $n \geq 2$,

$$\underline{\int \sec^n x dx} = \int \sec^{n-2} x \sec^2 x dx = \int \sec^{n-2} x d\tan x = \sec^{n-2} x \tan x - \int \tan x\, d\sec^{n-2} x$$

$$= \sec^{n-2} x \tan x - \int (n-2)\tan^2 x \sec^{n-2} x dx = \sec^{n-2} x\tan x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\underline{\int \sec^n x dx} + (n-2)\int \sec^{n-2} x dx$$

i.e.,

$$(n-1)\int \sec^n x dx = \sec^{n-2} x\tan x + (n-2)\int \sec^{n-2} x dx$$

$$\int \sec^n x dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x dx$$

3. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

Solution

$$\int_0^a f(x)g''(x)dx = \int_0^a f(x)dg'(x) = f(x)g'(x)\Big|_0^a - \int_0^a g'(x)df(x)$$

$$= f(a)g'(a) - f(0)g'(0) - \int_0^a g'(x)f'(x)dx = f(a)g'(a) - \int_0^a f'(x)dg(x)$$

$$= f(a)g'(a) - f'(x)g(x)\Big|_0^a + \int_0^a g(x)df'(x) = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

4. Use suitable trigonometric identities and substitutions to evaluate the following integrals.

(i) $\displaystyle\int_0^{\pi/2} \sin^2(2\theta)d\theta$ (ii) $\displaystyle\int \frac{\sin^2(\sqrt{x})}{\sqrt{x}}dx$ (iii) $\displaystyle\int_0^{\pi} \sin^2 t\cos^4 t dt$

(iv) $\displaystyle\int \cos^2 x\sin 2x dx$ (v) $\displaystyle\int \tan^2(2x)\sec^5(2x)dx$ (vi) $\displaystyle\int_0^{\pi/3} \tan^5 x\sec^6 x dx$

Solution

(i) $\displaystyle\int_0^{\pi/2} \sin^2(2\theta)d\theta = \int_0^{\pi/2} \frac{1-\cos 4\theta}{2}d\theta = \Big[\frac{\theta}{2} - \frac{1}{8}\sin 4x\Big]_0^{\pi/2} = \frac{\pi}{4}$

(Or, apply integration by parts to $\displaystyle\int_0^{\pi/2} \sin^2 2\theta d\theta = \int_0^{\pi/2} -\frac{1}{2}\sin 2\theta d\cos 2\theta$.)

(ii) $\displaystyle\int \frac{\sin^2(\sqrt{x})}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} \int 2\sin^2 u du = \int (1-\cos 2u)du = u - \frac{1}{2}\sin 2u + C = \sqrt{x} - \frac{1}{2}\sin 2\sqrt{x} + C$

(iii) $\displaystyle\int_0^{\pi} \sin^2 t\cos^4 t dt = \int_0^{\pi} (\sin t\cos t)^2\cos^2 t dt = \int_0^{\pi} \frac{1}{4}\sin^2 2t\cos^2 t dt = \int_0^{\pi} \frac{1}{16}[\sin 3t + \sin t]^2 dt$

$\displaystyle= \int_0^{\pi} \frac{1}{16}(\sin^2 3t + 2\sin 3t\sin t + \sin^2 t)dt = \int_0^{\pi} \frac{1}{16}(\frac{1}{2} - \frac{1}{2}\cos 6t + \cos 2t - \cos 4t + \frac{1}{2} - \frac{1}{2}\cos 4t)dt = \frac{\pi}{16}$

(iv) $\displaystyle\int \cos^2 x\sin 2x dx = \int \frac{1}{2}(1+\cos 2x)\sin 2x dx = \int (\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x dx = -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 4x + C$

(v)

$$\int \tan^2(2x)\sec^5(2x)dx \overset{u=2x}{=} \int \frac{1}{2}\tan^2 u\sec^5 u du = \frac{1}{2}\int (\sec^2 u - 1)\sec^5 u du = \frac{1}{2}\int \sec^7 u du - \frac{1}{2}\int \sec^5 u du$$

Using the reduction formula,

$$\int \sec^n u du = \frac{1}{n-1}\sec^{n-2} u\tan u + \frac{n-2}{n-1}\int \sec^{n-2}\sec u du$$

we have

$$\int \sec^7 u du - \int \sec^5 u du = \frac{1}{6}\sec^5 u\tan u - \frac{1}{6}\int \sec^5 u = \frac{1}{6}\sec^5 u\tan u - \frac{1}{24}\sec^3 u\tan u - \frac{3}{24}\int \sec^3 u du$$

$$= \frac{1}{6}\sec^5 u\tan u - \frac{1}{24}\sec^3 u\tan u - \frac{3}{48}\sec u\tan u - \frac{3}{48}\int \sec u du$$

$$= \frac{1}{6}\sec^5 u\tan u - \frac{1}{24}\sec^3 u\tan u - \frac{3}{48}\sec u\tan u - \frac{3}{48}\ln|\sec u + \tan u| + C$$

i.e.,

$$\int \tan^2(2x)\sec^5(2x)dx = \frac{1}{12}\sec^5 2x\tan 2x - \frac{1}{48}\sec^3 2x\tan 2x - \frac{1}{32}\sec 2x\tan u - \frac{1}{32}\ln|\sec 2x + \tan 2x| + C$$

(vi) Let $u = \sec x$, such that $du = \sec x\tan x dx$. Then

$$\int_0^{\pi/3} \tan^5 x\sec^6 x dx = \int_0^{\pi/3} \tan^4 x\sec^5 x\sec x\tan x dx = \int_0^{\pi/3} (\sec^2 x - 1)^2\sec^5 x\sec x\tan x dx$$

$$= \int_1^2 (u^2-1)^2 u^5 du = \int_1^2 (u^9 - 2u^7 + u^5)du = \Big[\frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6\Big]_1^2 = \frac{981}{20}$$

5. Find the volume obtained by rotating the region bounded by the curves $y = \sec x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{3}$ about the line $y = -1$.

Solution

$$\text{volume } = \int_0^{\pi/3} \pi[(1+\sec x)^2 - (1+\cos x)^2]dx = \pi\int_0^{\pi/3} [2\sec x + \sec^2 x - 2\cos x - \cos^2 x]dx$$

$$= \pi\int_0^{\pi/3} [2\sec x + \sec^2 x - 2\cos x - \frac{1}{2} - \frac{1}{2}\cos 2x]dx$$

$$= \pi\Big[2\ln|\sec x + \tan x| + \tan x - 2\sin x - \frac{x}{2} - \frac{1}{4}\sin 2x\Big]_0^{\pi/3} = \pi\Big[2\ln|2+\sqrt{3}| - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\Big]$$

6. Use suitable trigonometric identities to help show that:

(i) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ for any integers m, n.

(ii) A *finite Fourier series* is given by the sum

$$f(x) = \sum_{i=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin(2x) + \cdots + a_N \sin(Nx) \ .$$

Show that the m-th coefficient a_m is given by the formula $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x) \sin mx dx$.

Solution

(i) For any positive integers $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]dx$$

$$= \frac{1}{2}\Big[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\Big]_{-\pi}^{\pi} = 0$$

and

$$\int_{-\pi}^{\pi} \sin nx \sin nx dx = \int_{-\pi}^{\pi} [\frac{1}{2} - \frac{1}{2}\cos 2nx]dx = \Big[\frac{x}{2} - \frac{1}{4n}\sin 2nx\Big]_{-\pi}^{\pi} = \pi$$

(ii) Multiply both sides by $\sin mx$:

$$f(x)\sin mx = a_1 \sin x \sin mx + a_2 \sin 2x \sin mx + \cdots + a_m \sin mx \sin mx + \cdots + a_N \sin Nx \sin mx$$

Using part (i), integrate both sides to get

$$\int_{-\pi}^{\pi} f(x)\sin mx dx \int_{-\pi}^{\pi} a_1 \sin x \sin mx dx + \cdots + \int_{-\pi}^{\pi} a_m \sin mx \sin mx dx + \cdots + \int_{-\pi}^{\pi} + a_N \sin Nx \sin mx dx$$

$$= a_m \pi$$

i.e., $a_m = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin mx dx$.

7. Evaluate the following integrals by suitable trigonometric substitutions.

(i) $\displaystyle\int_0^2 x^2\sqrt{x^2+4}dx$ (ii) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}dx$, where $a > 0$ is a constant. (iii) $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$,

(iv) $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx$ (v) $\displaystyle\int \frac{x^2}{(x^2+a^2)^{3/2}}dx$ by the substitution $x = a\sinh t$.

Solution

(i) Let $x = 2\tan u$, such that $dx = 2\sec^2 u du du$. Then $u = 0$ when $x = 0$, and $u = \frac{\pi}{4}$ when $x = 2$. So,

$$\int_0^2 x^2\sqrt{x^2+4}dx = \int_0^{\frac{\pi}{4}} 4\tan^2 u\sqrt{4\tan^2 u + 4}\ 2\sec^2 u du = \int_0^{\frac{\pi}{4}} 16\tan^2 u \sec^3 u du$$

$$= \int_0^{\frac{\pi}{4}} \frac{16}{3}\tan u \ d\sec^3 u = \frac{16}{3}\tan u \sec^3 u\Big|_0^{\frac{\pi}{4}} - \frac{16}{3}\int_0^{\frac{\pi}{4}} \sec^5 u du$$

$$= \frac{32\sqrt{2}}{3} - \frac{16}{3}\Big[\frac{1}{4}\sec^3 u \tan u\Big|_0^{\pi/4} + \frac{3}{4}\int_0^{\frac{\pi}{4}} \sec^3 u du\Big] = 8\sqrt{2} - 4\Big[\frac{1}{2}\sec u \tan u\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec u du\Big]$$

$$= 6\sqrt{2} - 2\ln|\sec u + \tan u|\Big|_0^{\frac{\pi}{4}} = 6\sqrt{2} - 2\ln(\sqrt{2}+1)$$

(ii) Let $x = a\sec u$, $dx = a\sec u\tan u du$. Then

$$\int \frac{\sqrt{x^2-a^2}}{x^4}dx = \int \frac{\sqrt{a^2\sec^2 u - a^2}}{a^4\sec^4 u} a\sec u\tan u du = \frac{1}{a^2}\int \sin^3 u\cos u du$$

$$= \frac{1}{3a^2}\sin^3 u + C = \frac{(x^2-a^2)^{3/2}}{3a^2x^3} + C$$

(iii) Let $x = \frac{1}{3}\sec u$, $dx = \frac{1}{3}\sec u\tan u du$, and hence

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3^4\sec u\tan u du}{\sec^5 u\tan u} = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4 u du = \frac{567}{64}\sqrt{3} + \frac{81}{32}\pi - \frac{81}{4}$$

(Try the induction formula for $\cos^n x$!)

(iv) Let $2x-1 = 2\sin u$, $dx = \cos u du$.

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx = \int \frac{x^2}{(4-(2x-1)^2)^{3/2}}dx = \int \frac{(\sin u + \frac{1}{2})^2\cos u}{8\cos^3 u}du$$

$$= \frac{1}{8}\int (\tan^2 u + \tan u\sec u + \frac{1}{4}\sec^2 u)du = \frac{1}{8}\int (\sec^2 u - 1 + \tan u\sec u + \frac{1}{4}\sec^2 u)du$$

$$= \frac{1}{8}\left[\frac{5}{4}\tan u + \sec u - u\right] + C$$

$$= \frac{5}{32}\frac{2x-1}{\sqrt{3+4x-4x^2}} + \frac{2}{8\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\frac{2x-1}{2} + C$$

$$= \frac{1}{32}\frac{10x+3}{\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\frac{2x-1}{2} + C$$

(v) Let $x = a\sinh t$, $dx = a\cosh t dt$.

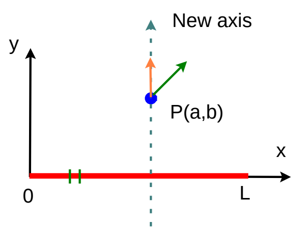
$$\int \frac{x^2}{(x^2+a^2)^{3/2}}dx = \int \frac{a^2\sinh^2 t}{(a^2\sinh^2 t + a^2)^{3/2}} a\cosh t dt$$

$$= \int \frac{\sinh^2 t}{\cosh^2 t}dt = \frac{\cosh^2 t - 1}{\cosh^2 t}dt = \int (1 - \frac{1}{\cosh^2 t})dt = t - \frac{\sinh t}{\cosh t} + C$$

$$= \sinh^{-1}\frac{x}{a} - \frac{\frac{x}{a}}{\sqrt{1+\frac{x^2}{a^2}}} + C = \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$$

8. A charge rod of length L produces an electric field at a point $P(a, b)$ *which has a vertical component* given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0(x^2+b^2)^{3/2}}dx$$



where λ is the charge density per unit length on the rod and ϵ_0 is the free space permittivity. Evaluate the integral. [Recall that by Coulomb's Law, the **magnitude** of the force on a test charge (of 1 coulomb) at a distance of r away from another charge q is given by $\frac{q}{4\pi\epsilon_0 r^2}$. So, consider a tiny piece of the charge rod and the resulting electrostatic force on the test charge at P.]

Solution Just use the substitution $x = b\tan\theta$, such that $dx = b\sec^2\theta d\theta$, and

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0(x^2+b^2)^{3/2}}dx = \int_{-\tan^{-1}\frac{a}{b}}^{\tan^{-1}\frac{L-a}{b}} \frac{\lambda b}{4\pi\epsilon_0 b^3\sec^3\theta} b\sec^2\theta d\theta$$

$$= \int_{-\tan^{-1}\frac{a}{b}}^{\tan^{-1}\frac{L-a}{b}} \frac{\lambda}{4\pi\epsilon_0 b}\cos\theta d\theta = \frac{\lambda}{4\pi\epsilon_0 b}\sin\theta\Big|_{-\tan^{-1}\frac{a}{b}}^{\tan^{-1}\frac{L-a}{b}} = \frac{\lambda}{4\pi\epsilon_0 b}\left[\frac{L-a}{\sqrt{(b^2+(L-a)^2}} + \frac{a}{\sqrt{a^2+b^2}}\right]$$

9. (Integration by parts.) Suppose that f is a positive function such that f' is continuous.

(i) How is the graph of $y = f(x)\sin nx$ related to the graph of $y = f(x)$? What happens as $n \longrightarrow \infty$? (Try $f(x) = x^2$, and $n = 2, 3, 4$ as starting examples.)

(ii) Make a guess as to the value of the limit $\lim\limits_{n\to\infty} \int_0^1 f(x)\sin nx dx$ based on graphs of the integrand.

(iii) Using integration by parts, confirm the guess you made in part (b). [Use the fact that, since f' is continuous, there is a constant M such that $|f'(x)| \leq M$ for $0 \leq x \leq 1$.]

Solution

(i) $y = f(x)\sin nx$ oscillates up and down between $y = f(x)$ and $y = -f(x)$, and oscillates more often as n is getting larger and larger.

(ii) $\lim\limits_{n\to\infty} \int_0^1 f(x)\sin nx dx = 0$, since the +ve and -ve areas seem to cancel each other.

(iii)

$$\lim_{n\to\infty} \int_0^1 f(x)\sin nx dx = \lim_{n\to\infty} \int_0^1 -\frac{1}{n} f(x) d\cos nx = \lim_{n\to\infty} \left[-\frac{1}{n} f(x)\cos nx \Big|_0^1 + \frac{1}{n}\int_0^1 f'(x)\cos nx dx \right]$$

$$= \lim_{n\to\infty} \left[-\frac{1}{n} f(1)\cos n + \frac{1}{n} f(0) + \frac{1}{n}\int_0^1 f'(x)\cos nx dx \right] = \lim_{n\to\infty} \frac{1}{n}\int_0^1 f'(x)\cos nx dx$$

However, since f' is continuous, there is a constant M such that $|f'(x)| \leq M$ for $0 \leq x \leq 1$, and hence

$$\left| \frac{1}{n}\int_0^1 f'(x)\cos nx dx \right| \leq \frac{1}{n}\int_0^1 |f'(x)| dx \leq \frac{1}{n}\int_0^1 M dx = \frac{M}{n} \longrightarrow 0 \text{ as } n \to \infty$$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\int_0^1 f'(x)\cos nx dx = 0$$