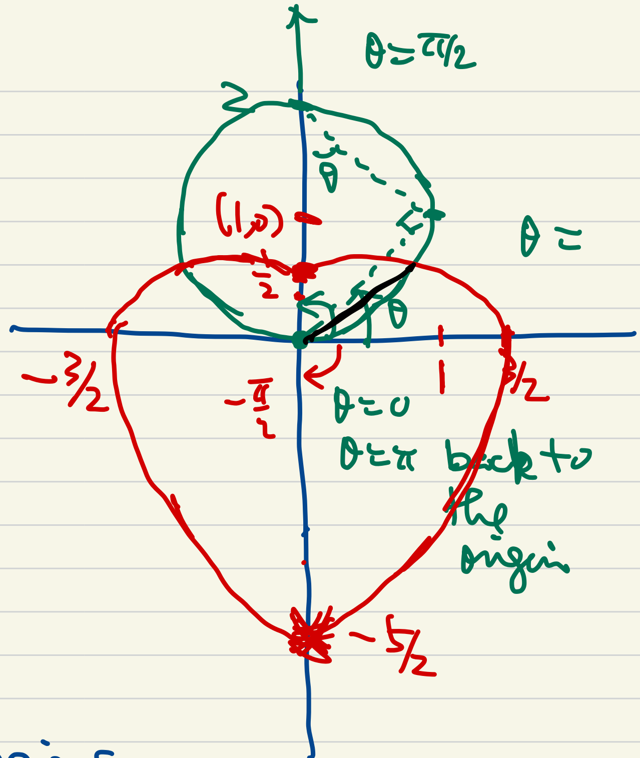
Example

$r = 2\sin\theta$ ←

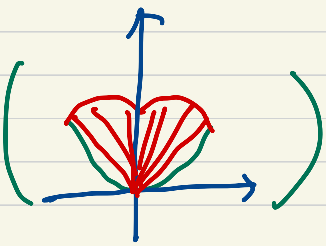
$r = \frac{3}{2} - \sin\theta$ ←

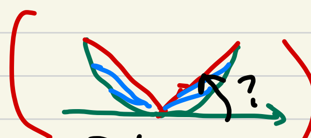
$\theta = \pi/2, \quad r = \frac{3}{2} - 1 = \frac{1}{2}$

$\theta = -\pi/2, \quad r = \frac{3}{2} + 1 = 5/2$



Find the area of the overlapping part of the regions enclosed by these curves.

area  = area 

+ area 

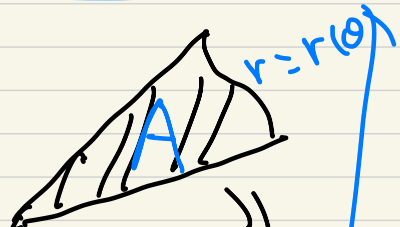
$$= \int_{\pi/6}^{\pi - \pi/6} \frac{1}{2}\left(\frac{3}{2} - \sin\theta\right)^2 d\theta + 2\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2 d\theta$$

At the intersection point,

$$2\sin\theta = 3/2 - \sin\theta$$

$$\sin\theta = \frac{1}{2}$$

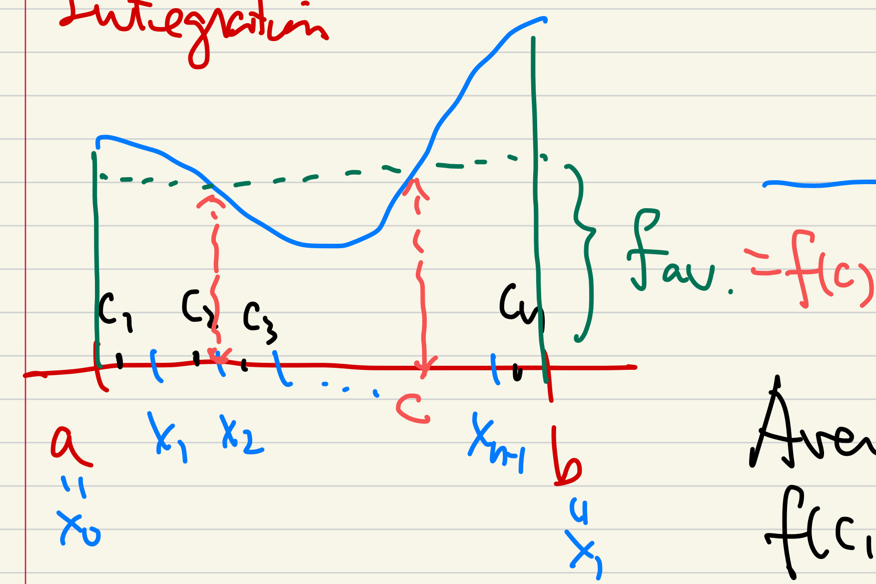
$$\theta = \frac{\pi}{6}$$



$$A = \int_\alpha^\beta \frac{1}{2} r^2(\theta)\, d\theta$$

Average Value of A Continuous Function on [a,b]

"Integration"



$$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}$$

$$\boxed{\int_a^b f(x)\,dx = f_{av} \cdot (b-a)}$$

Average Value of $f(c_1), f(c_2), \dots, f(c_n)$

$$= \frac{f(c_1) + f(c_2) + \dots + f(c_n)}{n}$$

$$= \frac{1}{b-a} \cdot \underbrace{\frac{b-a}{n}\left[f(c_1) + f(c_2) + \dots + f(c_n)\right]}_{\text{a Riemann Sum}}$$

$$\downarrow \; n \to \infty$$

f_{av} = Average function value of f on $[a,b]$ $\overset{\text{Take this as}}{=}$ $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

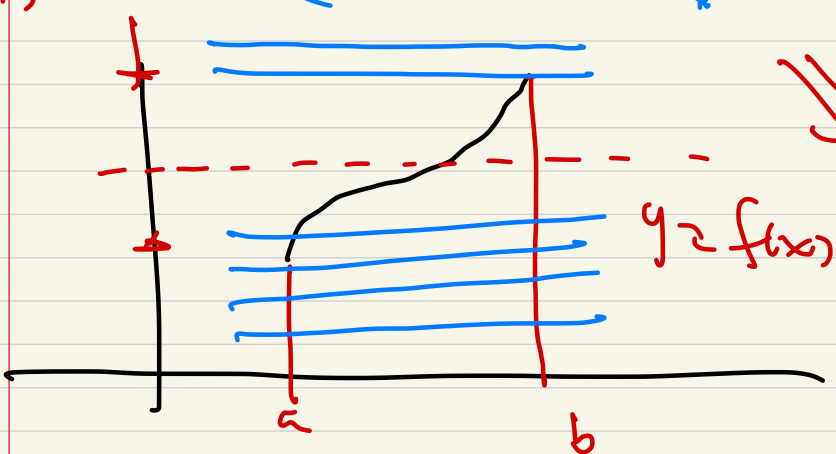
Mean Value Theorem: f is continuous on $[a,b]$

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

for some c in $[a,b]$.

$$\frac{\min\limits_{a\le x\le b} f(x)\,\cancel{(b-a)}}{\cancel{b-a}} \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{a number between}} \le \frac{\max\limits_{a\le x\le b} f(x)\,\cancel{(b-a)}}{\cancel{b-a}}$$

$$\begin{array}{ccccc} \| & & \downarrow & & \| \\ f(c_1) & \le & & \le & f(c_2) \end{array}$$

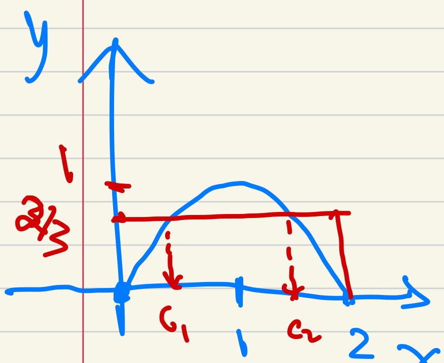


$\Rightarrow$ Intermediate Value Theorem says that this number must also be a function value of f, i.e.

$= f(c)$

for some $c_1 \le c \le c_2$

Example. $f(x) = 2x - x^2$ on $[0, 2]$



$$f_{av} = \frac{1}{2-0}\int_0^2 (2x - x^2)\,dx$$

$$= \frac{1}{2}\left[x^2 - \frac{x^3}{3}\right]_0^2$$

area( ) = area( ) $= \frac{1}{2}\left[4 - \frac{8}{3}\right] = \frac{2}{3}$

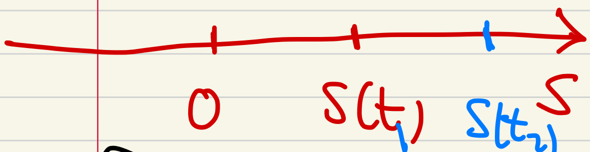
$$\frac{2}{3} = 2c - c^2$$

$$c^2 - 2c + \frac{2}{3} = 0$$

$$c = \frac{2 \pm \sqrt{4 - \frac{8}{3}}}{2} = 1 \pm \sqrt{1 - \frac{2}{3}}$$

$$c = 1 \pm \sqrt{\frac{1}{3}}$$

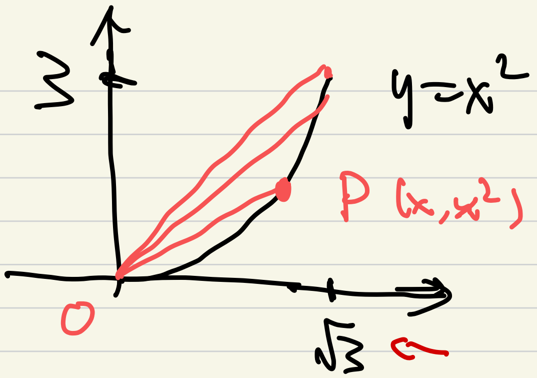
Example $s(t)$ = position function of a particle moving along a line



$$\boxed{\frac{ds}{dt} = v(t)}$$

Then Average value of velocity for $t_1 \le t \le t_2$ $= \frac{1}{t_2 - t_2}\int_{t_1}^{t_2} v(t)\,dt$ $= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$

Example.



Find the average distance from the points on the graph to the origin.

$$f(x) = \sqrt{x^2+(x^2)^2} = \sqrt{x^2+x^4}$$

distance function

$$0 \le x \le \sqrt{3}$$

$$f_{av} = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \sqrt{x^2(1+x^2)}\, dx$$

$$= \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} x\sqrt{1+x^2}\, dx$$

let $u = 1+x^2$

$$= \frac{1}{\sqrt{3}} \left[\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} (1+x^2)^{3/2} \right]_0^{\sqrt{3}}$$

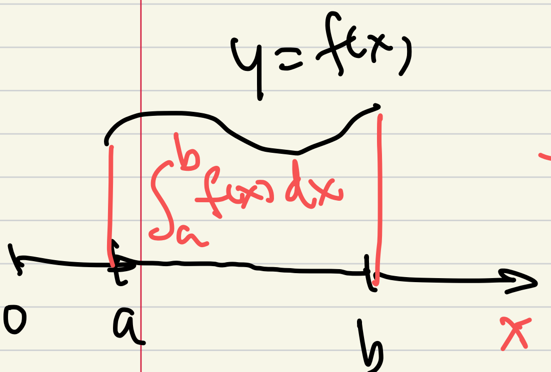
$$= \frac{1}{3\sqrt{3}} [8-1] = \frac{7}{3\sqrt{3}}$$

$$= \frac{7\sqrt{3}}{9}$$

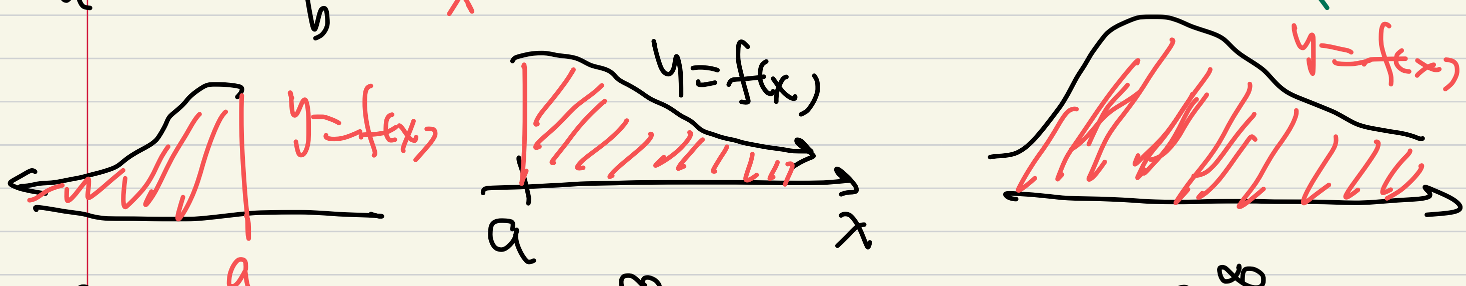
Recall

Improper Integrals: $\Leftrightarrow$ Deal with infinite intervals or unbounded functions.

(1)

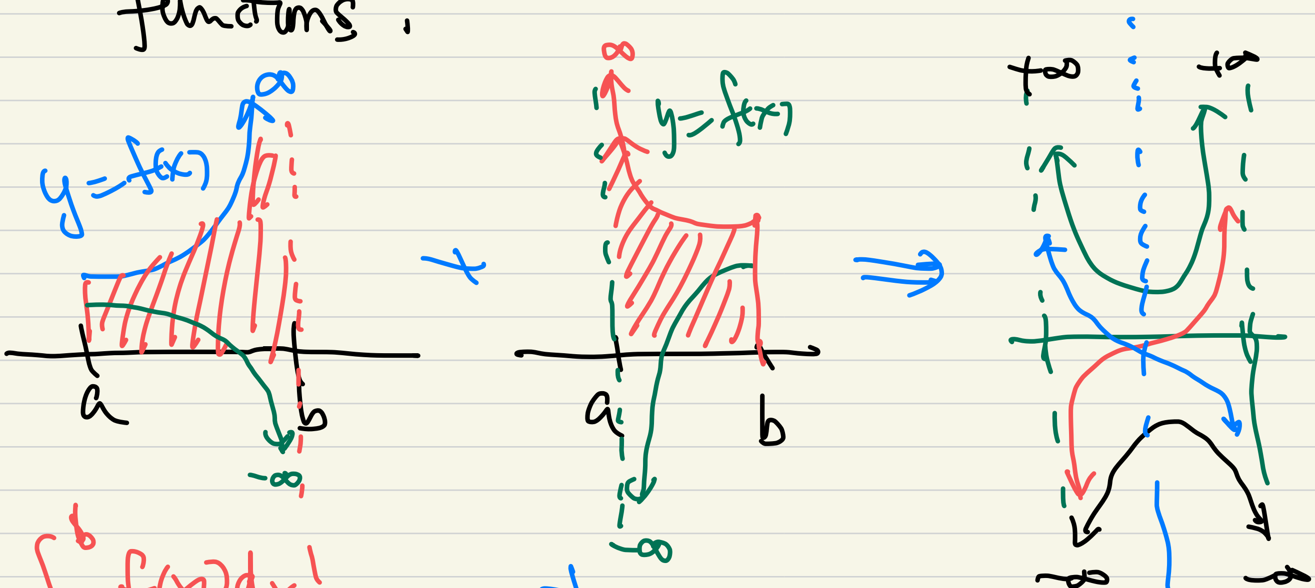


$\longrightarrow$ How about integrating continuous function on infinite intervals?



$$\int_{-\infty}^{a} f(x)\,dx \;+\; \int_a^{\infty} f(x)\,dx \;\Longrightarrow\; \int_{-\infty}^{\infty} f(x)\,dx$$

(2) How about integrating unbounded functions?



$$\int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx$$