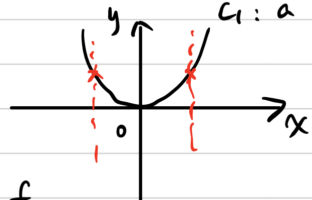
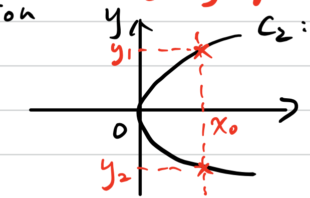
Review

function f: $\quad x \xrightarrow{f} y = f(x)$ $\qquad$ y is uniquely given by x.

vertical line test:



C_1: a function

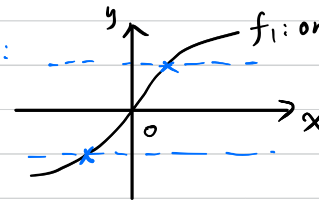


C_2: not a function

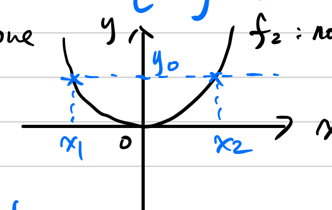
One-to-one function f: $\quad x \xrightarrow{f} y = f(x)$ $\qquad$ y is uniquely given by x.

$\qquad x \xleftarrow{g} y = f(x)$ $\qquad$ x is uniquely given by y

horizontal line test:



f_1: one-to-one



f_2: not one-to-one

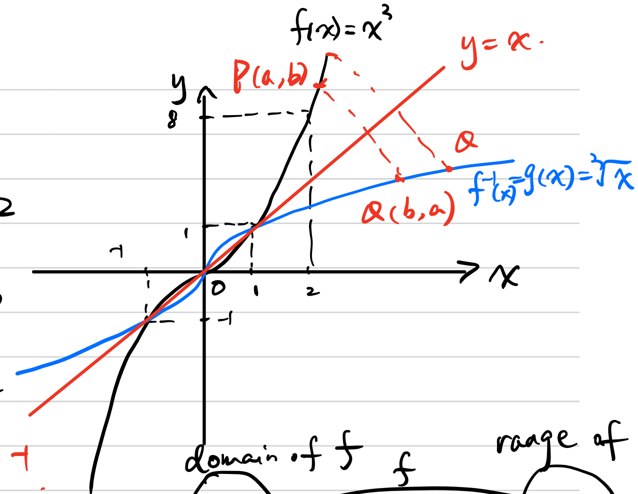
g: the inverse function of f, denoted by f^{-1}.

Notice: 1. Only one-to-one functions have the inverse functions.

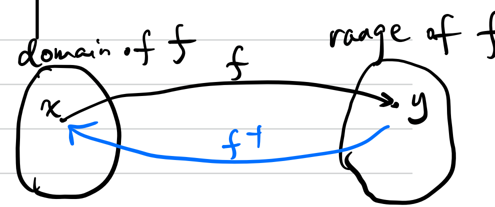
2. f and g are inverse functions to each other.

Example: $y=f(x)=x^3$. one-to-one $f^{-1}(x)=g(x)=\sqrt[3]{x}$.

$x \xrightarrow{f} y=f(x)=x^3$.		$y=f(x)=x^3 \xrightarrow{g} x$	
$-2 \to$	$f(-2)=-8$	$-8 \to$	$g(-8)=-2$
$-1 \to$	$f(-1)=-1$	$-1 \to$	$g(-1)=-1$
$0 \to$	$f(0)=0$	$0 \to$	$g(0)=0$
$1 \to$	$f(1)=1$	$1 \to$	$g(1)=1$
$2 \to$	$f(2)=8$	$8 \to$	$g(8)=2$



Notice: ① domain of f = range of f^{-1}.
range of f = domain of f^{-1}.



② The graph of f and the graph of f^{-1} are symmetric about the line $y=x$.

$y=f(x)$. $x=f^{-1}(y)$.

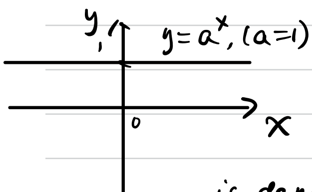
proof of ②: Choose any point $P(a,b)$ on the graph of f.
Then we have $f(a)=b$. $g(b)=a$.
This means the point $Q(b,a)$ is on the graph of g.
Since P and Q are symmetric about $y=x$,
and P can be any point on the graph of f, the graphs of f and g must be symmetric about $y=x$.

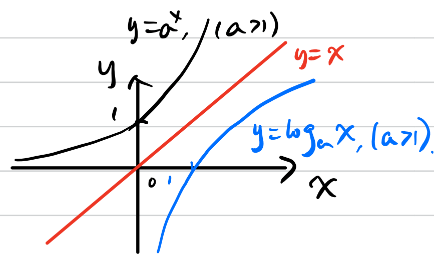
domain: $(-\infty,+\infty)$.
range: $\{1\}$ if $a=1$
$(0,+\infty)$ if $a\neq 1$.

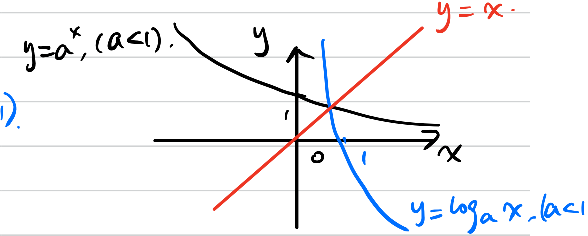
Exponential function: $y=f(x)=a^x$, $a>0$ ("base"). $y=a^x$ is one-to-one when $a\neq 1$!

$$x \xrightarrow{f} y=a^x.$$
$$-2 \to f(-2)=a^{-2}=\frac{1}{a^2}.$$
$$-1 \to f(-1)=a^{-1}=\frac{1}{a}.$$
$$0 \to f(0)=\boxed{a^0=1}.$$
$$1 \to f(1)=a$$
$$2 \to f(2)=a^2$$

$$a^x \xrightarrow{g} x$$
$$a^{-2} \to g(a^{-2})=-2$$
$$a^{-1} \to g(a^{-1})=-1$$
$$1 \to g(1)=0$$
$$a \to g(a)=1$$
$$a^2 \to g(a^2)=2.$$







$g(x) = \log_a x$ (is denoted by). $(a>0, a\neq 1) \to$ logarithmic function: the inverse function of $y=a^x$

$\boxed{y}=g(x)=\log_a x \iff a^{\boxed{y}}=x$. (*).

domain of g: $(0,+\infty)$. range of g: $(-\infty,+\infty)$.

Laws of exponents: $a>0$, $b>0$, $x, y \in (-\infty, +\infty)$.

① $a^x \cdot a^y = a^{x+y}$. ② $\frac{a^x}{a^y} = a^{x-y}$. ③ $(a^x)^y = a^{x \cdot y}$ ④ $a^x \cdot b^x = (a \cdot b)^x$.

⑤. $a^{-x} = \frac{1}{a^x}$ ⑥ $a^{\frac{x}{y}} = \sqrt[y]{a^x}$. ⑦ $a^{-\frac{x}{y}} = \frac{1}{\sqrt[y]{a^x}}$. ⑧ $a^0 = 1$.

Laws of logarithms: $a>0$, $a \neq 1$. $x>0$, $y>0$.

① $\log_a(xy) = \log_a x + \log_a y$. ②. $\log_a \frac{x}{y} = \log_a x - \log_a y$.

③ $\log_a x^r = r \cdot \log_a x$ for any $r \in (-\infty, +\infty)$. ④ $\log_a a = 1$. ⑤. $\log_a 1 = 0$.

Two important formulas: ① $a^{\log_a x} = x$. (proof: Let $p = \log_a x$. Then $a^p \overset{(*)}{=} x$.).

"change of base formula" ② $\log_a b = \frac{\log_c b}{\log_c a}$ for any positive number $c \neq 1$. Example: $\log_2 14 = \frac{\log_3 14}{\log_3 2} = \frac{\log_5 14}{\log_5 2}$.

(proof of ②: Let $m = \log_c a$ and $n = \log_c b$. Then $a \overset{(*)}{=} c^m$. $b \overset{(*)}{=} c^n$.

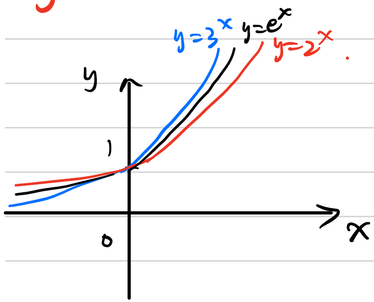
Now we can write $b = c^n \underset{\text{exp law ③}}{=} (c^m)^{\frac{n}{m}} = a^{\frac{n}{m}}$. Then $\log_a b = \log_a a^{\frac{n}{m}} \underset{\text{log law ③}}{=} \frac{n}{m} \log_a a \underset{\text{log law ④}}{=} \frac{n}{m} = \frac{\log_c b}{\log_c a}$.)

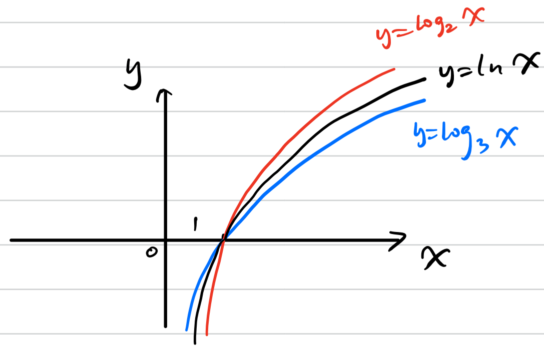
Natural logarithmic function:

When $a = e = 2.71828\cdots$, we call $f(x) = \log_e x = \ln x$

the natural logarithmic function.

$y = \ln x \iff e^y = x$.



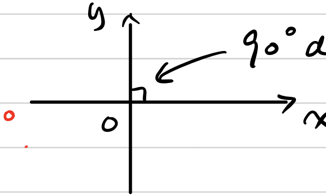


Why we are interested in $y=e^x$ and $y=\ln x$?

Their "derivatives" have a very simple formula.

Trigonometric functions.

1. The radian of an angle:

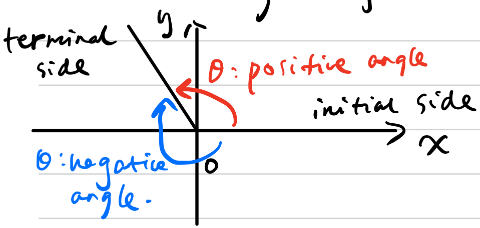


$90°$ degrees $= \frac{\pi}{2}$ rad

In general: $1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3°$.

Example: $2 \text{ rad} = \left(\frac{360}{\pi}\right)^{\circ}$. $\pi \text{ rad} = 180°$

2. The sign of an angle:



A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side.

A negative angle is obtained by rotating the initial side clockwise until it coincides with the terminal side.