# Derivatives

## 3.1 Derivatives

**Definition 3.1** *Let $f$ be a function and $a < b$ be numbers. The average rate of change of $f$ from $a$ to $b$ is*

$$\frac{f(b) - f(a)}{b - a}.$$

**Remark 3.2** *The average rate of change of a function $f$ from $a$ to $b$ is the slope of the chord joining the points $(a, f(a))$ and $(b, f(b))$. We are interested in studying the limit as $a$ and $b$ are close together so that the previous chord becomes a tangent.*

**Remark 3.3** *Consider a car moving in a lane. Let $f(x)$ be the distance of the car from the initial position after $x$ seconds. Then, the average rate of change of $f$ from $a$ to $b$ is the average speed of the car within the duration starting from the $a^{th}$ second and ending at the $b^{th}$ second.*

**Definition 3.4** *Let $f$ be a function and $x$ be a number. The derivative of the function $f$ is a function, denoted by $f'$, so that*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

*if such a limit exists. In this case when such a limit exists, we say that the function $f$ is differentiable.*

**Remark 3.5** *Another common notations for the derivative of $f$ at $x$ is*

$$f'(x) = \frac{d}{dx} f(x).$$

*Sometimes, people present a function as $y = f(x)$ and they write its derivative as*

$$\frac{dy}{dx}.$$

*These alternative notations are proved to be convenient when we encounter "chain rule" later.*

**Definition 3.6** *Let $f$ be a function and $a$ be a number. The tangent line (or just tangent) to the graph of $f$ at $a$ is the straight line with slope $f'(a)$ which passes through the point $(a, f(a))$.*

**Example 3.7** *Evaluate the derivative of the function*

$$f(x) = \frac{1}{x} \text{ for all numbers } x \neq 0$$

solution:
For $x \neq 0$ and $0 < |h| < |x|$,

$$\begin{aligned} &\tfrac{f(x+h)-f(x)}{h} \\ &= \tfrac{1}{h}[\tfrac{1}{x+h} - \tfrac{1}{x}] \\ &= \tfrac{x-(x+h)}{hx(x+h)} \\ &= -\tfrac{1}{x(x+h)}. \end{aligned}$$

So,

$$f'(x) = \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$$

for all $x \neq 0$.

**Example 3.8** *Evaluate the derivative of the function*

$$f(x) = \sqrt{x} \text{ for all numbers } x > 0.$$

solution:
For $x > 0$ and $0 < |h| < x$,

$$\begin{aligned} &\tfrac{f(x+h)-f(x)}{h} \\ &= \tfrac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \tfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \tfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \tfrac{1}{(\sqrt{x+h}+\sqrt{x})}. \end{aligned}$$

So,

$$f'(x) = \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

for all $x > 0$.

**Example 3.9** *Evaluate the derivative of $f$ at $0$ if*

$$f(x) = |x| \text{ for all numbers } x$$

solution:
For $h \neq 0$,

$$\begin{aligned} &\frac{f(0+h)-f(0)}{h} \\ &= \frac{|h|}{h}. \end{aligned}$$

But $\lim_{h\to 0} \dfrac{|h|}{h}$ does not exist (why?). So, the function $f$ does not have a derivative at 0. That is, $f$ is not "differentiable" at 0.

**Theorem 3.10** *A differentiable function is continuous.*

proof: difficult

## 3.2 Standard Formula for Derivatives

**Theorem 3.11** *If $a$ is a number and $f(x) = a$ for all $x$. Then, $f'(x) = 0$.*

proof:
For $h \neq 0$ and all $x$,

$$\frac{f(x+h) - f(x)}{h}$$
$$= \frac{a - a}{h}$$
$$= 0$$

So,

$$f'(x) = \lim_{h \to 0} 0 = 0.$$

**Theorem 3.12** *If $n$ is a positive integer and $f(x) = x^n$ for all $x$. Then, $f'(x) = nx^{n-1}$.*

proof:
For $h \neq 0$ and all $x$,

$$\frac{f(x+h) - f(x)}{h}$$
$$= \frac{(x+h)^n - x^n}{h}$$
$$= (x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + ... + (x+h)x^{n-2} + x^{n-1}.$$

So,

$$f'(x) = \lim_{h \to 0} (x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + ... + (x+h)x^{n-2} + x^{n-1} = nx^{n-1}.$$

**Theorem 3.13** *If $f$ and $g$ are differentiable functions, $(f+g)' = f' + g'$.*

proof:
For $h \neq 0$ and all $x$,

$$\frac{(f+g)(x+h) - (f+g)(x)}{h}$$
$$= \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$
$$= \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

So,

$$f'(x)$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x).$$

**Theorem 3.14** *If $f$ and $g$ are differentiable functions, $(f-g)' = f' - g'$.*

proof:
omitted

**Theorem 3.15** *If $f$ is a differentiable function and $a$ is a number, $(af)' = a(f')$.*

proof:
For $h \neq 0$ and all $x$,

$$\begin{aligned}&\frac{(af)(x+h)-(af)(x)}{h}\\ &= \frac{a(f(x+h))-a(f(x))}{h}\\ &= a\frac{f(x+h)-f(x)}{h}\end{aligned}$$

So,

$$\begin{aligned}&(af)'(x)\\ &= \lim_{h\to 0} a\frac{f(x+h)-f(x)}{h}\\ &= a\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= a(f'(x)).\end{aligned}$$

**Example 3.16** *Let $f(x) = 1+2x+3x^2+4x^3$ for all $x$. Evaluate the derivative of $f$.*

solution:

The derivative of 1 is 0
The derivative of $2x$ is 2 times the derivative of $x$ which is 2
The derivative of $3x^2$ is 3 times the derivative of $x^2$ which is $6x$
The derivative of $4x^3$ is 4 times the derivative of $x^3$ which is $12x^2$

So, the derivative of $f$ is $f'(x) = 2+6x+12x^2$.

**Theorem 3.17 (Product Rule)** *If $f$ and $g$ are differentiable functions, $(fg)' = f'g + fg'$.*

proof:
For $h \neq 0$ and all $x$,

$$\begin{aligned}&\frac{(fg)(x+h)-(fg)(x)}{h}\\ &= \frac{f(x+h)g(x+h)-(f(x)g(x))}{h}\\ &= \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\ &= f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\end{aligned}$$

Now

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ and } \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

exist and are $f'(x)$, $g'(x)$ respectively, and $\lim_{h\to 0} f(x+h) = f(x)$ as $f$ is continuous (follows from the differentiability of $f$),

$$\begin{aligned}
&(fg)'(x)\\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}$$

**Theorem 3.18 (Quotient Rule)** *If $f$ and $g$ are functions, $(\frac{f}{g})' = \frac{gf'-fg'}{g^2}$.*

proof:
Since $f = \frac{f}{g}g$. Taking derivatives on both sides of the equation by product rule yields

$$f' = (\frac{f}{g})'g + \frac{f}{g}g'.$$

Rearrange the equation and we get

$$(\frac{f}{g})' = \frac{gf'-fg'}{g^2}.$$

**Example 3.19** *If $f(x) = \frac{1}{x^7}$ for all $x$. Evaluate the derivative of $f$.*

solution:
The derivative of $f$ is, by the quotient rule,

$$f'(x) = \frac{(0)x^7 - (1)7x^6}{(x^7)^2} = -\frac{7}{x^8}.$$

**Remark 3.20** *In a similar way, we are able to show that if $n$ is a negative integer, the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.*

**Example 3.21** *Find the line tangent to the graph of $f(x) = \frac{2}{x^3}$ at the point $(1,2)$.*

solution:
Since $f'(x) = 2(-3)x^{-4} = -\frac{6}{x^4}$, we have $f'(1) = -6$. The line tangent to the graph of $f$ at $(1,2)$ is the straight line with slope $f'(1) = -6$ which passing through $(1,2)$. That is, the line whose defining equation is $y-2 = -6(x-1)$ or

$$y = -6x + 8.$$

**Example 3.22** *Let $a > 0$, and $C$ is the graph of $f(x) = x^2/4a$. Find the line tangent to $C$ at $(2at, at^2)$. Hence show that this line makes the same angle with the y-axis and the line joining $(0, a)$ and $(2at, at^2)$.*

solution:
Since $f'(x) = x/2a$ for all $x$. The line tangent to $C$ at $(2at, at^2)$ has slope $t$ and so its defining equation is $y - at^2 = t(x - 2at)$, or $y - tx + at^2 = 0$.

Now the line joining $(0, a)$ and $(2at, at^2)$ has slope $(at^2 - a)/2at = t/2 - 1/2t$. Therefore, tangent of the angle subtended by this line the tangent line in the first paragraph is

$$\frac{t - (t/2 - 1/2t)}{1 + t(t/2 - 1/2t)} = \frac{1}{t}$$

which is the same as tangent of the angle subtended by the line tangent to $C$ at $(2at, at^2)$ and the y-axis.

**Theorem 3.23** *If $f(x) = \sin x$ for all $x$. Then, $f'(x) = \cos x$.*

proof:
For $h \neq 0$ and all $x$,

$$\begin{aligned}
&\frac{f(x+h) - f(x)}{h} \\
&= \frac{\sin(x+h) - \sin x}{h} \\
&= \frac{\sin(x + \frac{h}{2} + \frac{h}{2}) - \sin(x + \frac{h}{2} - \frac{h}{2})}{h} \\
&= \frac{\sin(x + \frac{h}{2}) \cos \frac{h}{2} + \cos(x + \frac{h}{2}) \sin \frac{h}{2} - \sin(x + \frac{h}{2}) \cos \frac{h}{2} + \cos(x + \frac{h}{2}) \sin \frac{h}{2}}{h} \\
&= \cos(x + \tfrac{h}{2}) \frac{\sin \frac{h}{2}}{\frac{h}{2}}.
\end{aligned}$$

So,

$$\begin{aligned}
&f'(x) \\
&= \lim_{h \to 0} \cos(x + \frac{h}{2}) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \\
&= \lim_{h \to 0} \cos(x + \frac{h}{2}) \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \\
&= \cos x
\end{aligned}$$

**Theorem 3.24** *If $f(x) = \cos x$ for all $x$. Then, $f'(x) = -\sin x$.*

proof:
similar

**Theorem 3.25** *If $f(x) = \tan x$ for all $x$. Then, $f'(x) = \sec^2 x$.*

proof:
Since $\tan x = \dfrac{\sin x}{\cos x}$. We take derivative by quotient rule and get

$$f'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \sec^2 x.$$

**Theorem 3.26** *If $f(x) = \cot x$ for all $x$. Then, $f'(x) = -\csc^2 x$.*

proof:
Since $\cot x = \dfrac{\cos x}{\sin x}$. We take derivative by quotient rule and get

$$f'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\csc^2 x.$$

**Theorem 3.27** *If $f(x) = \sec x$ for all $x$. Then, $f'(x) = \tan x \sec x$.*

proof:
Since $\sec x = \dfrac{1}{\cos x}$. We take derivative by quotient rule and get

$$f'(x) = \frac{0 \cos x + 1 \sin x}{\cos^2 x} = \tan x \sec x.$$

**Theorem 3.28** *If $f(x) = \csc x$ for all $x$. Then, $f'(x) = -\cot x \csc x$.*

proof:
Since $\csc x = \dfrac{1}{\sin x}$. We take derivative by quotient rule and get

$$f'(x) = \frac{0 \sin x - 1 \cos x}{\sin^2 x} = \cot x \csc x.$$

**Definition 3.29** *The limit*

$$\lim_{h \to 0} \frac{a^h - 1}{h}$$

*is increasing in $a$. It is $0$ when $a = 1$ and the limit is big provided that $a$ is sufficiently large. Define $e$ to be the number such that*

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

**Theorem 3.30** *If $f(x) = e^x$ for all $x$. Then, $f'(x) = e^x$.*

proof:

$$\begin{aligned}
& f'(x) \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^x.
\end{aligned}$$

## 3.3 Chain Rule

**Theorem 3.31 (Chain Rule)** *Let $f$ and $g$ be functions. Then,*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

proof (sloppy):
For all numbers $x$ and $h \neq 0$,

$$\begin{aligned}
&\frac{f \circ g(x+h) - f \circ g(x)}{h} \\
&= \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}
\end{aligned}$$

Thus,

$$\begin{aligned}
&(f \circ g)'(x) \\
&= \lim_{h \to 0} \frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{g(x+h) - g(x)} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{k \to 0} \frac{f(g(x) + k) - f(g(x))}{k} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x))g'(x).
\end{aligned}$$

**Remark 3.32** *The chain rule can be presented alternatively as follows: Let $u = g(x)$ and $y = f(u)$ (a way to present the composite function $f \circ g$.). The chain rule becomes*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

**Example 3.33** *Let $y = (2x+1)^4$. Evaluate $\frac{dy}{dx}$.*

solution:
Let $u = 2x + 1$, then $y = u^4$ so that

$$\frac{dy}{du} = 4u^3.$$

On the other hand, $u = 2x + 1$ so that

$$\frac{du}{dx} = 2.$$

Combining these results by chain rule, we have

$$\frac{dy}{dx} = (4u^3)(2) = 8(2x+1)^3.$$

Alternatively, we may let $g(x) = 2x + 1$, $f(x) = x^4$. Then, $f \circ g(x) = (2x+1)^4$ and

$$(f \circ g)'(x) = g'(x)f'(g(x)) = (2)(4(g(x)^3)) = 8(2x+1)^3.$$

**Example 3.34** *Let* $y = \sin(x^2 + 1)$*. Evaluate* $\frac{dy}{dx}$*.*

solution:
Let $u = x^2 + 1$, then $y = \sin u$ so that

$$\frac{dy}{du} = \cos u.$$

On the other hand, $u = x^2 + 1$ so that

$$\frac{du}{dx} = 2x.$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2x \cos u = 2x \cos(x^2 + 1).$$

**Example 3.35** *Let* $y = xe^{2x}$*. Evaluate* $\dfrac{dy}{dx}$*.*

solution:
$y$ is naturally the product of $x$ and $e^{2x}$. By product rule,

$$\frac{d}{dx}xe^{2x} = x\frac{d}{dx}e^{2x} + e^{2x}\frac{d}{dx}x.$$

We know that $\dfrac{d}{dx}x = 1$. So, the issue is to compute $\dfrac{d}{dx}e^{2x}$. Let $u = 2x$ and $v = e^u$. By chain rule,

$$\frac{d}{dx}e^{2x} = \frac{dv}{du}\frac{du}{dx} = 2e^u = 2e^{2x}.$$

Therefore,

$$\frac{d}{dx}xe^{2x} = 2xe^{2x} + e^{2x} = e^{2x}(2x + 1).$$

**Example 3.36** *Let* $p$ *and* $q$ *be integers,* $f(x) = x^{p/q}$*. Evaluate* $f'(x)$*.*

solution:
Let $y = x^{p/q}$. Then, $y^q = x^p$. We have

$$\frac{d}{dx}y^q = \frac{d}{dx}x^p.$$

The right hand side is known to be $px^{p-1}$. Moreover,

$$\frac{d}{dx}y^q = (\frac{dy}{dx})(\frac{d}{dy}y^q) = qy^{q-1}\frac{dy}{dx}.$$

Therefore,

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

or

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

**Remark 3.37** *From the previous example, we see that the derivative of $f(x) = x^k$ is $f'(x) = kx^{k-1}$ for all rational numbers $k$. In fact, such formula is true for all real numbers $k$.*

**Example 3.38** *Find the derivative of $y = \cos e^{e^x}$.*

solution:
Let $v = e^x$, $u = e^v$ so that $y = \cos u$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} \\ &= \frac{d\cos u}{du}\frac{de^v}{dv}\frac{de^x}{dx} \\ &= (-\sin u)(e^v)(e^x) \\ &= -e^{x+e^x}\sin e^{e^x}.\end{aligned}$$

## 3.4 Implicit Differentiation

**Definition 3.39** *An implicitly defined function is a function defined by specifying a relation between $x$ and $f(x)$ for every number $x$.*

**Example 3.40** *A function $y$ is implicitly defined by*

$$x^3 + xy(x) + (y(x))^3 = 1.$$

*Evaluate the derivative of $y$.*

solution:
Take derivatives on both sides of the given equation, we get

$$3x^2 + y + x\frac{dy}{dx} + \frac{d}{dx}y^3 = 0.$$

We may modify the last term by chain rule,

$$3x^2 + y + x\frac{dy}{dx} + \frac{dy^3}{dy}\frac{dy}{dx} = 0.$$

That is

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Rearranging yields

$$\frac{dy}{dx} = -\frac{3x^2 + y}{x + 3y^2}.$$

**Example 3.41** *Let $a > b > 0$ and $C$ be the curve collecting all points $(x, y)$ such that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (*). Find the line tangent to $C$ at $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$.*

solution:
Verify that

$$\frac{1}{a^2}(\frac{a}{\sqrt{a}})^2 + \frac{1}{b^2}(\frac{b}{\sqrt{a}})^2 = 1.$$

The given point $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ is on the curve $C$. Consider $C$ (or part of it) as the graph of a function $y$. This function is defined implicitly by (*). We differentiate both sides of (*) and get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

Rearranging yields

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

Thus, the slope of tangent of $C$ at $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ is

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}|_{(x,y)=(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})} = -\frac{b}{a}.$$

Consequently, the line tangent to $C$ at $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ has defining equation

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}(x - \frac{a}{\sqrt{2}})$$

or

$$bx + ay = \sqrt{2}ab.$$

**Example 3.42** *Let $f$ be a function such that*

$$f(x)e^{f(x)-1} = x \quad \textit{for all } x.$$

*Evaluate $f'(1)$.*

solution:
Differentiate both sides of the given relation, we obtain

$$f'(x)e^{f(x)-1} + f(x)f'(x)e^{f(x)-1} = 1 \quad \text{for all } x.$$

Thus,

$$f'(x) = \frac{e^{1-f(x)}}{1+f(x)} \quad \text{for all } x.$$

Observe that $1 \cdot e^{1-1} = 1$. Therefore $f(1) = 1$ and hence

$$f'(1) = \frac{e^{1-f(1)}}{1+f(1)} = \frac{1}{2}.$$

**Theorem 3.43** *Let $f$ be a one-to-one function with inverse $g$. Then,*

$$g'(x) = 1/f'(g(x)) \textit{ for all } x \textit{ unless } f'(g(x)) = 0.$$

proof:
Since $g$ is the inverse of $f$, $f(g(x)) = x$ for all $x$. Differentiate both sides of the equation and apply chain rule, we get

$$f'(g(x))g'(x) = 1.$$

That is,

$$g'(x) = \frac{1}{f'(g(x))}$$

whenever $f'(g(x)) \neq 0$.

**Remark 3.44** *If the function is presented as $y = f(x)$, the inverse function is presented as $x = g(y)$. The previous theorem is saying that the derivative of $g$, that is, $\frac{dx}{dy}$ is related to the derivative of $f$ by*

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

*Where the functions are to be evaluated at the appropriate points.*

**Theorem 3.45** *If $f(x) = \ln x$ for all $x > 0$. Then, $f'(x) = \frac{1}{x}$ for all $x > 0$.*

proof:
Let $g(x) = e^x$ and $f(x) = \ln x$, then

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{e^{f(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

**Example 3.46** *Let $y = \sqrt{e^x x^x \sin x}$. Evaluate $\frac{dy}{dx}$.*

solution:
Since

$$\begin{aligned}
&y = \sqrt{e^x x^x \sin x} \\
&y^2 = e^x x^x \sin x \\
&\ln y^2 = \ln(e^x x^x \sin x) \\
&2\ln y = \ln e^x + \ln x^x + \ln \sin x \\
&2\ln y = x + x\ln x + \ln \sin x,
\end{aligned}$$

we take derivative on both sides of the equation and get

$$\begin{aligned}
&2\frac{d}{dx}\ln y = 1 + \ln x + x\frac{1}{x} + \frac{\cos x}{\sin x} \\
&2\frac{dy}{dx}\frac{d\ln y}{dy} = 2 + \ln x + \cot x \\
&\frac{2}{y}\frac{dy}{dx} = 2 + \ln x + \cot x \\
&\frac{dy}{dx} = \frac{1}{2}(2 + \ln x + \cot x)y \\
&\frac{dy}{dx} = \frac{1}{2}(2 + \ln x + \cot x)\sqrt{e^x x^x \sin x}.
\end{aligned}$$

**Example 3.47** *Evaluate the derivative of* $y = \tan^{-1} x$.

solution:
Since $y = \tan^{-1} x$, $\tan y = x$ and $-\pi/2 \leq y \leq \pi/2$. Hence,

$$\begin{aligned}
\frac{d}{dx}\tan y &= \frac{d}{dx}x \\
\frac{dy}{dx}\frac{d\tan y}{dy} &= 1 \\
\sec^2 y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
\frac{dy}{dx} &= \frac{1}{1+\tan^2 y} \\
\frac{dy}{dx} &= \frac{1}{1+x^2}.
\end{aligned}$$

**Example 3.48** *Let* $y = \sin^{-1} x$. *Evaluate* $\frac{dy}{dx}$.

solution:
Since $y = \sin^{-1} x$, $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$. Hence,

$$\begin{aligned}
\frac{d}{dx}\sin y &= \frac{d}{dx}x \\
\frac{dy}{dx}\frac{d\sin y}{dy} &= 1 \\
\cos y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1-\sin^2 y}} \text{ since } -\pi/2 \leq y \leq \pi/2 \text{ and thus } \cos y \geq 0 \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

## 3.5 Higher Derivatives

**Definition 3.49** *Let $f$ be a function, the derivative of $f'$ is called the second derivative of $f$. Its symbol is $f''$. The derivative of $f''$ is called the third derivative of $f$. Its symbol is $f'''$. Similarly, the $n^{th}$ derivative of $f$ is the derivative of the $(n-1)^{th}$ derivative of $f$ and the symbol for the $n^{th}$ derivative of $f$ is $f^{(n)}$.*

**Remark 3.50** *If a function is presented as $y = f(x)$, its $n^{th}$ derivative is also written as* $\dfrac{d^n y}{dx^n}$.

**Example 3.51** *Evaluate the second derivative of* $f(x) = e^{x^2}$.

solution:
By chain rule,

$$f'(x) = 2xe^{x^2}.$$

Then, by product rule and chain rule,

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = e^{x^2}(2 + 4x^2).$$

**Example 3.52** *Evaluate the first seven derivatives of* $f(x) = 4x^4 + 3x^3 + 2x^2 + x$.

solution:

$$\begin{aligned} f(x) &= 4x^4 + 3x^3 + 2x^2 + x \\ f'(x) &= 12x^3 + 9x^2 + 4x + 1 \\ f''(x) &= 36x^2 + 18x + 4 \\ f'''(x) &= 72x + 18 \\ f^{(4)}(x) &= 72 \\ f^{(5)}(x) &= 0 \\ f^{(6)}(x) &= 0 \\ f^{(7)}(x) &= 0. \end{aligned}$$

**Remark 3.53** *If $f$ is a polynomial of degree $d$, the $n^{th}$ derivative of $f$ is* 0 *whenever $n > d$.*

## 3.6 Rates of Changes

**Definition 3.54** *Suppose that there is a particle moving in a line. Define $f(t)$ =distance of the particle from a certain fixed point on the line at time $t$. Then, $f'(t)$ is called the velocity of the particle at time $t$ and $f''(t)$ is called the acceleration of the particle at time $t$.*

**Example 3.55** *A particle is moving in a line such that it is located $x(t) = 2e^{-3t}\sin 4t$ meters away from its initial position after $t$ seconds. Evaluate its initial velocity and initial acceleration.*

solution:
Since

$$\begin{aligned} x'(t) &= -6e^{-3t}\sin 4t + 8e^{-3t}\cos 4t \\ x''(t) &= -14e^{-3t}\sin 4t - 48e^{-3t}\cos 4t \end{aligned}$$

we have

$$\begin{aligned} x'(0) &= 8 \\ x''(0) &= -48. \end{aligned}$$

Thus, the initial velocity of the particle is $8ms^{-1}$, its initial acceleration is $-48ms^{-2}$.

**Definition 3.56** *When there is is a quantity changing over time and $f(t)$ is that quantity at time $t$. $f'(t)$ is called the rate of change of the given quantity at time $t$.*

**Example 3.57** *In a chemical reaction chamber, the concentration of a certain reactant $A$ after $t$ seconds is $[A](t) = \dfrac{1}{k(t+c)}$ $M$. Here $k$ and $c$ are two fixed positive numbers. Show that the rate of change of the concentration of $A$ is proportional to the square of the concentration of $A$.*

solution:
The rate of change of the concentration of $A$ (rate of reaction) is

$$\begin{aligned}&\frac{d}{dt}[A]\\&= \frac{d}{dt}\frac{1}{k(t+c)}\\&= -\frac{1}{k(t+c)^2}\\&= -k[A]^2.\end{aligned}$$

Thus, the assertion is proved.

**Example 3.58** *Water is pumped into an inverted conical vessel with semi-vertical angle $\pi/6$ at a rate of $R\, m^3 s^{-1}$. Evaluate the rate at which water level in the vessel is raising at the moment when the water level is at $H$ meters.*

solution:
Let $V(t)$ be the volume of water in the vessel after $t$ seconds. $h(t)$ is the water level in the vessel after $t$ seconds. Then,

$$V(t) = \frac{1}{3}h(t)\pi(h(t)\tan\frac{\pi}{6})^2 = \frac{\pi}{9}h(t)^3 \text{ for all } t.$$

Taking derivatives on both sides of the equation yields

$$V'(t) = \frac{\pi}{3}h(t)^2 h'(t).$$

Suppose that after $t_0$ seconds, the water level in the vessel is $H$ meters. That is, $h(t_0) = H$. It is given that $V'(t) = R$ for all $t$. In particular, $V'(t_0) = R$. We evaluate the previous equation at $t_0$ and it yields

$$R = \frac{\pi}{3}H^2 h'(t_0)$$

or

$$h'(t_0) = \frac{3R}{\pi H^2}.$$

The water level is raising at a rate of $\dfrac{3R}{\pi H^2}$ $ms^{-1}$ at the moment when the water level is $H$ meters.

**Example 3.59** *One is keeping an eye on a car which is moving in a straight line. At the moment when the car is closest to the guy, the car is $D$ meters away from the guys and it is moving at a speed of $V$ meters per second. Evaluate the rate at which the guy turns his head at that moment.*

solution:
Let $O$ be the point at which the car is closest to the guy. $x(t)$ is the distance between the car and $O$ after $t$ seconds. $\theta(t)$ is the angle that the line joining the guy and $O$ makes with the line joining the guy and the car. Then,

$$x(t) = D\tan\theta(t) \text{ for all } t.$$

Taking derivatives on both sides of the equation yields

$$x'(t) = D\theta'(t)\sec^2\theta(t).$$

In other words,

$$\theta'(t) = \frac{1}{D}x'(t)\cos^2\theta(t).$$

Suppose that after $t_0$ seconds, the car is located at $O$. That is, $x(t_0) = 0$ and $\theta(0) = 0$. Evaluating the last line at $t_0$ yields

$$\theta'(t_0) = \frac{1}{D}x'(t_0)\cos^2\theta(t_0) = \frac{V}{D}.$$

Consequently, the guy has his head turning at $\frac{V}{D}$ radians per second when the car is closest to him.

**Example 3.60** *Air is pumped into a spherical balloon at a rate of $R$ $m^3s^{-1}$. Evaluate the rate at which the surface area of this balloon is changing when its surface area is $S_0$ $m^2$.*

solution:
Let $V(t)$, $S(t)$, $r(t)$ be the volume, surface area and radius of this balloon respectively after $t$ seconds. Then,

$$\begin{aligned} &V(t) = \tfrac{4\pi}{3}(r(t))^3 \\ &S(t) = 4\pi(r(t))^2 \text{ for all } t. \end{aligned}$$

Eliminating $r(t)$ yields

$$V(t) = \frac{1}{6\sqrt{\pi}}S(t)^{3/2} \text{ for all } t.$$

Differentiate both sides and we obtain

$$V'(t) = \frac{1}{4\sqrt{\pi}}\sqrt{S(t)}S'(t) \text{ for all } t.$$

Now let the surface area of the balloon be $S_0$ after $t_0$ seconds. Then

$$V'(t_0) = \frac{1}{4\sqrt{\pi}}\sqrt{S(t_0)}S'(t_0),$$

or

$$R = \frac{1}{4}\sqrt{\frac{S_0}{\pi}}S'(t_0)$$

so that

$$S'(t_0) = 4R\sqrt{\frac{\pi}{S_0}}.$$

Consequently, the rate at which surface area of the balloon is changing is $4R\sqrt{\frac{\pi}{S_0}}\ m^2s^{-1}$.