MATH2111 Tutorial 1

QUAN Xueyang (xquan@connect.ust.hk)
T1A: Tue (18:00-18:50) 2465
T1B: Thu (18:00-18:50) 6591

1 Linear Systems

1. A **linear equation** with variables $x_1, x_2, \cdots, x_n$ is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_i's$ are coefficients, and b is a constant.

Example 1.1. Determine if the following equations are linear equations.

(a) $x_1 = x_2$
(b) $x_1^2 = x_2^2$
(c) $\sqrt{3}x_1 - x_2 = 1$
(d) $3\sqrt{x_1} - x_2 = 1$
(e) $x_1x_2 + x_1 = e^5$

2. A **linear system** is a collection of linear equations.

3. A **solution** to a linear system is an assignment of values to variables that make all equations in the system simultaneously true.

4. **Consistent / Inconsistent**
Any linear system has

(a) A unique solution, or
(b) Infinitely many solutions, or
(c) No solutions

5. A **matrix** is a rectangular array of numbers. A matrix with m rows and n columns is said to be "$m \times n$" or "m by n".

6. **Coefficient Matrix and Augmented Matrix**
For a system of linear equations,

(a) the **coefficient matrix** is the matrix consist of all the coefficients of the linear equations;

(b) the **augmented matrix** is the matrix consist of all the coefficients and the constants of the linear equations.

Example 1.2. Write down the coefficient matrix and the augmented matrix of the following linear system.

$$\begin{cases} x_1 + x_5 = 1 \\ 3x_2 - x_4 = 4 \\ x_1 + 2x_3 = -3 \end{cases}$$

Example 1.3. Write down the linear system with the following augmented matrix.

$$\left[\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 5 \\ 0 & -2 & 0 & 4 & 1 \end{array}\right]$$

2 Row Reduction and Echelon Forms

1. Here are the 3 **elementary row operations (EROs)** on any matrices which are helpful in solving linear equations:

   - Row Replacement: $cr_j + r_i$
   - Row Interchange: $r_i \leftrightarrow r_j$
   - Row Scaling: $cr_i, c \neq 0$

   Two matrices are **row equivalent** (denoted by $\sim$) if one can be transformed to the other by applying a finite sequence of these row operations.

2. Linear systems with row equivalent augmented matrices have the same solutions (are **equivalent**).

3. Any matrix is in **row echelon form (REF)** if

   (a) The rows with all zero entries must be at the bottom.

   (b) The leading entry (the first non-zero entry on each row) must move to the right by at least one column when going down a row.

(c) All entries in a column below a leading entry must be zeros.

4. A **reduced row echelon form (RREF)** of any matrix is a REF with the extra properties:

   (d) On a non-zero row, the leading entry must be 1.

   (e) On the columns containing the leading entry 1 , the 1 is the only non-zero entry.

5. **Algorithm to get REF from a random matrix**:

   (1) Stop when all entries are zeros, or no visible entry.

   (2) Locate the left-most non-zero column, select a non-zero entry, use row interchange to move it to the top row.

   (3) Use row replacement to make all entries below 0.

   (4) Neglect the top row, and repeat Step 1-3 for the submatrix below that row.

6. **Algorithm from a REF to a RREF**:

   (5) Use row scaling to scale all the leading entries to 1.

   (6) Working from the rightmost leading entries to left, use row replacement to make all entries above each of them 0.

7. Each matrix A is row equivalent to exactly one matrix in reduced echelon form.

8. A pivot position in a matrix **M** is a location in **M** that corresponds to a leading 1 in the reduced echelon form of **M**. A pivot column is a column of **M** that contains a pivot position.

9. **Existence and Uniqueness Theorem**:

   (1) A linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form $[0 \cdots 0|\blacksquare]$ with $\blacksquare \neq 0$.

   (2) If a linear system is consistent, then:

      - it has a unique solution if there are no free variables;
      - it has infinitely many solutions if there are free variables

3 Exercises

1. Write down the linear system with this augmented matrix: $\left[\begin{array}{cc|c} 1 & 2 & 3 \end{array}\right]$

2. Determine the value(s) of h such that the matrix

$$\left[\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right]$$

is the augmented matrix of a consistent linear system.

3. Determine which matrices are in reduced echelon form and which others are only in echelon form.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

RREF RREF X not echelon only REF

Q1 System of linear equations:

$$\{ x_1 + 2x_2 = 3$$

Q2. $$\underset{A}{\left[\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right]} \xrightarrow{R_2 - 2R_1 \to R_2} \underset{B}{\left[\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right]}$$

$A \sim B$ row equivalent, they have the same solution set.

if $-8-2h=0$, i.e. $h=-4$, then last row gives $0=16$. inconsistent.

if $-8-2h \neq 0$, i.e. $h \neq -4$, the system has a unique solution, also consistent

Q3. REF: Check 3 things: ① all zero rows are at the bottom.

② leading entry are to the right at least one column when going down a row.

③ All entries in a column below a leading entry is 0.

RREF: two more: ④ leading entry is 1 ⑤ on that column, 1 is the only

4. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols $\blacksquare$, $*$ and 0 , where the leading entries ($\blacksquare$) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

5. Transform the following matrices to Reduced Row Echelon Form.

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 3 & 6 & 2 & 2 & 0 & 4 \\ 4 & 8 & 2 & 6 & -5 & 7 \end{bmatrix}$$

Q4. Suppose: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

① if $a \neq 0$, then $a = \blacksquare$, $\begin{cases} c \neq 0 \;\times \Rightarrow \begin{bmatrix} \blacksquare & * \\ * & * \end{bmatrix} \text{ not an echelon form.} \\ c = 0, \; d \text{ is either } \blacksquare \text{ or } 0 \;\checkmark \end{cases}$

Thus, two echelon forms: $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$, $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$

② if $a = 0$, $\begin{cases} c \neq 0 \Rightarrow \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \times \\ c = 0, \text{ then } \begin{cases} b = \blacksquare, \begin{cases} d = 0, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix} \checkmark \\ d \neq 0, \begin{bmatrix} 0 & \blacksquare \\ 0 & * \end{bmatrix} \times \end{cases} \\ b = 0, \text{ it's a zero matrix } (a=0, b=0) \end{cases} \end{cases}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ meaningless.

one echelon form: $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

R5:

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 3 & 6 & 2 & 2 & 0 & 4 \\ 4 & 8 & 2 & 6 & -5 & 7 \end{bmatrix}$$

$$\xrightarrow{R_2-3R_1\to R_2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 0 & 0 & -1 & 5 & -7.5 & 2.5 \\ 4 & 8 & 2 & 6 & -5 & 7 \end{bmatrix}$$

subtract R_1 multiplied by 3 from R_2

$$\xrightarrow{R_3-4R_1\to R_3} \begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 0 & 0 & -1 & 5 & -7.5 & 2.5 \\ 0 & 0 & -2 & 10 & -15 & 5 \end{bmatrix}$$

subtract R_1 multiplied by 4 from R_3

$$\xrightarrow{R_3-2R_2\to R_3} \begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 0 & 0 & -1 & 5 & -7.5 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

$$\xrightarrow{-R_2\to R_2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2.5 & 0.5 \\ 0 & 0 & 1 & -5 & 7.5 & -2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2\to R_1} \begin{bmatrix} 1 & 2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 1 & -5 & 7.5 & -2.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{RREF.}$$

6. Consider the following linear systems.

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ 2x_1 + 5x_2 + 2x_3 + 8x_4 = 1 \\ 3x_1 + 5x_2 + 4x_3 + 9x_4 = -5 \end{cases}$$

(a) Write down the augmented matrix of the linear system.
(b) Get the reduced echelon form of the augmented matrix using EROs.
(c) Solve the linear system.

Q6 (a).

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 5 & 2 & 8 & 1 \\ 3 & 5 & 4 & 9 & -5 \end{array}\right]$$

(b)

$$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 & -5 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \to R_3]{} \left[\begin{array}{cccc|c} \underline{\underline{1}} & 2 & 1 & 3 & 0 \\ 0 & \underline{\underline{1}} & 0 & 2 & 1 \\ 0 & 0 & \underline{\underline{1}} & 2 & -4 \end{array}\right] \leftarrow \text{REF}$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & -4 \end{array}\right] \leftarrow \text{RREF}$$

(c)

$$\begin{cases} x_1 \quad\quad -3x_4 = 2 \\ \quad x_2 \quad +2x_4 = 1 \\ \quad\quad x_3 + 2x_4 = -4 \end{cases} \Rightarrow \begin{cases} x_1 = 2 + 3x_4 \\ x_2 = 1 - 2x_4 \\ x_3 = -4 - 2x_4 \\ x_4 = x_4 \quad \text{free.} \end{cases}$$