MATH2111 Tutorial 11

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1 Diagonalization

1. **Definition**. A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix. i.e. If $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

2. **Theorem (The Diagonalization Theorem)**.

   (a) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

   (b) $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

3. **Procedures to Diagonalize a Matrix** A.

   (a) Find all the eigenvalues and the corresponding eigenvectors of A.

   (b) Construct D from the eigenvalues in step (a) to fill all the diagonal entries in D.

   (c) Construct P from the corresponding eigenvectors in step (a) to form the columns of P.

4. **Theorem**. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

5. **Theorem**. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

   (a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

   (b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if

      i. the characteristic polynomial factors completely into linear factors and

      ii. the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

   (c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

2 Exercises

1. Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix. Given λ and ρ are two distinct eigenvalues of A. Show that eigenspaces of λ and ρ are orthogonal. Namely, for any vectors $\mathbf{x}_1 \in \mathcal{E}_\rho(A), \mathbf{x}_2 \in \mathcal{E}_\lambda(A)$, it has $\mathbf{x}_1^\top \mathbf{x}_2 = 0$.

1. For any $\vec{x}_1 \in \mathcal{E}_\rho(A)$, $\vec{x}_2 \in \mathcal{E}_\lambda(A)$, we have

$$A\vec{x}_1 = \rho\vec{x}_1 , \quad A\vec{x}_2 = \lambda\vec{x}_2 .$$

$\therefore \lambda \vec{x}_1^T\vec{x}_2 = \vec{x}_1^T(\lambda\vec{x}_2) = \vec{x}_1^T A\vec{x}_2$ is a scalar.

$\rho \vec{x}_2^T\vec{x}_1 = \vec{x}_2^T(\rho\vec{x}_1) = \vec{x}_2^T A\vec{x}_1$ is a scalar.

Also, $\vec{x}_1^T A\vec{x}_2 = (\vec{x}_1^T A\vec{x}_2)^T = \vec{x}_2^T A^T\vec{x}_1 = \vec{x}_2^T A\vec{x}_1$

since A is symmetric.

and $\vec{x}_1^T\vec{x}_2 = (\vec{x}_1^T\vec{x}_2)^T = \vec{x}_2^T\vec{x}_1$

Thus, $\lambda\vec{x}_1^T\vec{x}_2 = \rho\vec{x}_1^T\vec{x}_2$

$\therefore (\lambda-\rho)\vec{x}_1^T\vec{x}_2 = 0$.

Since $\lambda \neq \rho$, $\vec{x}_1^T\vec{x}_2 = 0$.

2. Given $A \in \mathbb{R}^{n\times n}$ and its characteristic function $f(\lambda) = \lambda^2(\lambda+1)(\lambda-1)(3-\lambda)^{n-4}$.
(1) Write down eigenvalues and their multiplicities.
(2) What is characteristics function of matrix $A + 2I$?

2. (1) let $f(\lambda) = 0$, it has eigenvalues:

$\lambda = 0$, multiplicities 2

$\lambda = -1$, multiplicity 1

$\lambda = 1$, multiplicity 1

$\lambda = 3$, multiplicities $n-4$

(2) By definition, $f(\lambda) = \det(A - \lambda I)$

For $\det(A + 2I - \lambda I) = \det(A - (\lambda - 2) I)$

$$= f(\lambda - 2)$$

$$= (\lambda-2)^2 (\lambda-1)(\lambda-3)(5-\lambda)^{n-4}$$

Recall from last tutorial, eigenvalues of $A + 2I$:

$\lambda = 2$, multiplicities 2

$\lambda = 1$, multiplicity 1

$\lambda = 3$, multiplicity 1

$\lambda = 5$, multiplicities $n-4$

3. Suppose $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(1) Find out characteristics function of A.

(2) Determine whether A is diagonalizable.

3\. (1) $\det(A-\lambda I) = (4-\lambda)(1-\lambda)^3$

(2) No.

For $\lambda = 1$, we check the $\dim \mathcal{E}_1(A)$:

$$A - I = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A-I)$$

$\therefore \operatorname{rank}(A-I) = 3$

Then, $\dim \mathcal{E}_1(A) = 4 - 3 = 1$.

However, multiplicity of $\lambda = 1$ is 3 $(\neq 1)$.

Thus, not diagonalizable.

4. Diagonalize the following matrix, if possible,

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

4. Step 1: Find eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 4-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(4-\lambda)(2-\lambda)-1\right] + 1\cdot\left[0-(4-\lambda)\right]$$

$$= (4-\lambda)\left[\lambda^2-6\lambda+8-2\right]$$

$$= (4-\lambda)\left[\lambda^2-6\lambda+6\right]$$

(root of $ax^2+bx+c=0$: $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$)

let $\det(A-\lambda I)=0$, we have

$\lambda_1 = 4$, $\lambda_2 = 3+\sqrt{3}$, $\lambda_3 = 3-\sqrt{3}$.

Step 2: Find corresponding eigenvectors.

① for $\lambda_1 = 4$

$$(A-\lambda_1 I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{cases}$$

$\therefore$ one of the eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

② for $\lambda_2 = 3+\sqrt{3}$,

$$A-\lambda_2 I = \begin{bmatrix} 1-\sqrt{3} & 0 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -1-\sqrt{3} \end{bmatrix} \overset{\substack{(1+\sqrt{3})R_1 \\ (1+\sqrt{3})R_2}}{\sim} \begin{bmatrix} -2 & 0 & 1+\sqrt{3} \\ 0 & -2 & 1+\sqrt{3} \\ 1 & 1 & -1-\sqrt{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1+\sqrt{3}}{2} \\ 0 & 1 & -\frac{1+\sqrt{3}}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

∴ one of the eigenvectors :

$$\vec{V_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

③ for $\lambda_3 = 3-\sqrt{3}$,

$$A-\lambda_3 I = \begin{bmatrix} 1+\sqrt{3} & 0 & 1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -1+\sqrt{3} \end{bmatrix} \overset{\substack{(-1+\sqrt{3})R_1 \\ (-1+\sqrt{3})R_2}}{\sim} \begin{bmatrix} 2 & 0 & -1+\sqrt{3} \\ 0 & 2 & -1+\sqrt{3} \\ 1 & 1 & -1+\sqrt{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-1+\sqrt{3}}{2} \\ 0 & 1 & \frac{-1+\sqrt{3}}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

∴ one of the eigenvectors :

$$\vec{V_3} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$ is linearly independent.

Step 3: Construct P.

$$P = [\vec{v_1} \ \vec{v_2} \ \vec{v_3}] = \begin{bmatrix} -1 & \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ 1 & \frac{1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

Step 4: Construct D.

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3+\sqrt{3} & 0 \\ 0 & 0 & 3-\sqrt{3} \end{bmatrix}$$

Thus, $A = PDP^{-1}$. (you can check this by checking $AP = PD$)

5. Determine range of α such that the following matrix is similar to some real diagonal matrix,

$$A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}.$$

5. Compute eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \alpha \\ \alpha & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - \alpha^2 \qquad \alpha = 1-\lambda$$

① when $\alpha \neq 0$, $\det(A - \lambda I) = 0$ $\qquad \lambda = \lambda - 1$

$$\Rightarrow \lambda = 1 \pm \alpha$$

in this case, A has 2 distinct eigenvalues, A is diagonalizable.

② when $\alpha = 0$, A is a diagonal matrix.

Remark :

Given λ and ρ are two distinct eigenvalues of matrix $A \in \mathbb{R}^{n \times n}$. Suppose x_1 is an eigenvector corresponding to λ and x_2 is an eigenvector corresponding to ρ, namely,

$$Ax_1 = \lambda x_1, \quad Ax_2 = \rho x_2.$$

Then $x_1 + x_2$ is not eigenvector of A.

If a matrix A can be diagonalized, that is,

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

then:

$$AP = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

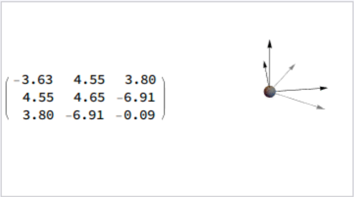
Writing P as a block matrix of its column vectors $\boldsymbol{\alpha}_i$

$$P = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \cdots \quad \boldsymbol{\alpha}_n],$$

the above equation can be rewritten as

$$A\boldsymbol{\alpha}_i = \lambda_i \boldsymbol{\alpha}_i \qquad (i = 1, 2, \dots, n).$$

So the column vectors of P are right eigenvectors of A, and the corresponding diagonal entry is the corresponding eigenvalue. The invertibility of P also suggests that the eigenvectors are linearly independent and form a basis of F^n. This is the necessary and sufficient condition for diagonalizability and the canonical approach of diagonalization. The row vectors of P^{-1} are the left eigenvectors of A.



The diagonalization of a symmetric matrix can be interpreted as a rotation of the axes to align them with the eigenvectors.