MATH 2111: Tutorial 12
Inner Product and Orthogonality

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- Inner product, length, and orthogonality
- Orthogonal sets
- Orthogonal projections

Determine which pairs of vectors are orthogonal.

$$\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

1. $\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = \begin{bmatrix} 8 & -5 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = 8(-2)+(-5)(-3) = -1 \neq 0$

$$\text{ortho-gonal}\left\{\begin{aligned}
&2.\ \vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = \begin{bmatrix} 12 & 3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12(2)+3(-3)+(-5)3 = 0 \\
&3.\ \vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = \begin{bmatrix} 3 & 2 & -5 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix} = 3(-4)+2(1)+(-5)(-2)+0(6) = 0
\end{aligned}\right.$$

4. $\vec{y}\cdot\vec{z} = \vec{y}^T\vec{z} = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = (-3)1+7(-8)+4(15)+0(-7) = 1 \neq 0$

(1) Verify the parallelogram law for vectors $\mathbf{u}$ and $\mathbf{v}$ $\mathbb{R}^{\mathrm{n}}$:

$$\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2 \| \mathbf{u} \|^2 + 2 \| \mathbf{v} \|^2$$

(2) Let $\mathrm{W} = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{\mathrm{p}}\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_{\mathrm{j}}$,

then for $1 \leq \mathrm{j} \leq \mathrm{p}$, then $\mathbf{x}$ is orthogonal to every vector in W.

(1) $\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$= (\vec{u}\cdot\vec{u} + \cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v}} - \cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v})$

$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

(2) $\forall\ \vec{v}\in W$, we have $\vec{v} = \sum_{i=1}^{p} a_i \vec{v}_i$, where $\vec{a_i}\in\mathbb{R}$

$\vec{x}\cdot\vec{v} = \vec{x}^T\vec{v} = \vec{x}^T\sum_{i=1}^{p} a_i\vec{v}_i = \sum_{i=1}^{p} a_i\,\vec{x}^T\vec{v}_i = \sum_{i=1}^{p} a_i\, 0 = 0.$

$\therefore$ $\vec{x}$ is orthogonal to every vector in W.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

① $\vec{u}_1 \cdot \vec{u}_2 = [3 \ -3 \ 0]\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 3(2) + (-3)2 + 0(-1) = 0$

$\vec{u}_1 \cdot \vec{u}_3 = [3 \ -3 \ 0]\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3(1) + (-3)1 + 0(4) = 0$

$\vec{u}_2 \cdot \vec{u}_3 = [2 \ 2 \ -1]\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2(1) + 2(1) + (-1)4 = 0$

Then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set, then is linearly independent.

And $\dim \mathbb{R}^3 = 3$, thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis.

② There are several ways of doing this part.

Method 1:

$$\vec{x} = \frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|}\vec{u}_1 + \frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|}\vec{u}_2 + \frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|}\vec{u}_3$$

$$= \frac{4}{3}\vec{u}_1 + \frac{1}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3$$

Method 2:

Solve $[\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{x}$.

Note: One can always use this method to find weights when the basis are NOT orthogonal.

i.e. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$

Method 3: Cramer's Rule. . . .

Find the closest point to y in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

From best approximation thm, calculate the $\text{proj}_W(\vec{y})$

(1) $$\hat{\vec{y}} = \text{proj}_W(\vec{y}) = \frac{\vec{y}\cdot\vec{v_1}}{\|\vec{v_1}\|}\vec{v_1} + \frac{\vec{y}\cdot\vec{v_2}}{\|\vec{v_2}\|}\vec{v_2}$$

$$= \frac{6}{12}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

(2) $$\hat{\vec{y}} = \text{proj}_W(\vec{y}) = \frac{\vec{y}\cdot\vec{v_1}}{\|\vec{v}\|}\vec{v_1} + \frac{\vec{y}\cdot\vec{v_2}}{\|\vec{v_2}\|}\vec{v_2}$$

$$= \frac{30}{10}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26}\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that dim $\dim W + \dim W^\perp = n$.

(1) Consider the dot product of any two vectors in this set, there are only 3 possibilities:

- $\vec{w_i} \cdot \vec{w_j} = 0$ because $\{\vec{w_1}, \dots, \vec{w_p}\}$ is an orthogonal set.
- $\vec{v_i} \cdot \vec{v_j} = 0$ because $\{\vec{v_1}, \dots, \vec{v_q}\}$ is an orthogonal set.
- $\vec{w_i} \cdot \vec{v_j} = 0$ because $\vec{w_i} \in W$, $\vec{v_j} \in W^{\perp}$.

(2) By Orthogonal Decomposition thm,

$$\forall \vec{y} \in \mathbb{R}^n, \quad \vec{y} = \underset{\text{in } W}{\underset{\uparrow}{\hat{\vec{y}}}} + \underset{\text{in } W^{\perp}}{\underset{\uparrow}{\vec{z}}}$$

Assume, $\hat{\vec{y}} = \sum_{i=1}^{p} a_i \vec{w_i}$, $\vec{z} = \sum_{i=1}^{q} b_i \vec{v_i}$ for $a_i, b_i \in \mathbb{R}$.

Then, $\vec{y} = \sum_{i=1}^{p} a_i \vec{w_i} + \sum_{i=1}^{q} b_i \vec{v_i}$ is a linear combination of $\{\vec{u_1}, \dots, \vec{u_p}, \vec{v_1}, \dots, \vec{v_q}\}$.

So, this set spans $\mathbb{R}^n$.

(3) According to (2), $p + q = n$.

Also, $\dim W = p$, $\dim W^{\perp} = q$

$\therefore \dim W + \dim W^{\perp} = n$.