Math1013 Calculus I **Name:** Kin Nam KOO

Homework-5 : Due 11/07/2021 at 11:59pm HKT

The problems in this homework set cover further techniques in differentiation and the application of derivatives. You need to work on:

(1) implicit differentiation and logarithmic differentiation;
(2) application of derivatives in rate of change and related rate problems.

Give 4 or 5 significant digits for numerical answers. For most problems when entering numerical answers, you can if you wish enter elementary expressions such as 3^2 or 3**2 instead of 9, $\sin(3*pi/2)$ instead of -1, e^(ln(3)) instead of 3, $(1+\tan(3))*(4-\sin(5))$^$6-15/8$ instead of 12748.8657, etc.

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1. (4 points) Find an equation of the line tangent to the curve defined by $x^6+6xy+y^4=8$ at the point $(1,1)$.

$y =$ ______

Solution: Differentiating implicitly with respect to x gives

$$6x^5+6y+6x\frac{dy}{dx}+4y^3\frac{dy}{dx}=0.$$

If $x=1$ and $y=1$, this equation becomes

$$12+10\frac{dy}{dx}=0,$$

and so

$$\frac{dy}{dx}=-\tfrac{12}{10}=-\tfrac{6}{5}.$$

Therefore, the line tangent to the given curve at the point $(1,1)$ is defined by $y=1-\frac{6}{5}(x-1)$, or $y=\frac{11}{5}-\frac{6}{5}x$.

Correct Answers:

- `11/5-6/5*x`

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2. (3 points) Let

$$f(x)=\frac{x^6(x-7)^4}{(x^2+7)^4}$$

Use logarithmic differentiation to determine the derivative.

$f'(x) =$ ______

$f'(7) =$ ______

Correct Answers:

- `x**6*(x-7)**4/(x**2+7)**4*(6/x+4/(x-7)-2*4*x/(x**2+7))`
- `0`

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3. (4 points) If a ball is thrown vertically upward from the roof of 48 foot building with a velocity of 48 ft/sec, its height after t seconds is $s(t)=48+48t-16t^2$.

a.) What is the maximum height the ball reaches?
Answer: ______

b.) What is the velocity of the ball when it hits the ground (height 0)?
Answer: ______

Solution:
SOLUTION

(a) We first compute the veloity function:

$$v(t)=s'(t)=48-32t$$

The maximum height is attained when the velocity is zero, that is, at $t=1.5$.
Substituting this value of t in the position function gives the maximum height:

$$s(1.5)=84$$

(b) The ball hits the ground when $s(t)=0$ and $t>0$ Solving the equation

$$48+48t-16t^2=-16(t^2-3t-3)=0$$

gives $t=\dfrac{3\pm\sqrt{21}}{2}$. Thus the ball hits the ground when $t=\dfrac{3+\sqrt{21}}{2}$.
The velocity of the ball when it hits the ground is

$$v\left(\frac{3+\sqrt{21}}{2}\right)=-16\sqrt{21}$$

Correct Answers:

- `48 + 48*3/2 -4*3^2`
- `-16*(3*3+4*3)^.5`

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4. (4 points) A particle moves according to the law of motion

$$s=t^3-8t^2+5t,\quad t\geq 0,$$

where t is measured in seconds and s in feet.

a.) Find the velocity at time t.
Answer: ______

b.) What is the velocity after 3 seconds?
Answer: ______

c.) When is the particle at rest? Enter your answer as a comma separated list. Enter *None* if the particle is never at rest.
Answer: ______

d.) When is the particle moving in the positive direction?
When $0\leq t<$ ______ and $t>$ ______

Correct Answers:

- `3*t^2-2*8*t+5`
- `3*3^2-2*8*3+5`
- `[8-sqrt(8^2-3*5)]/3, [8+sqrt(8^2-3*5)]/3`
- `(8-sqrt(8^2-3*5))/3`
- `(8+sqrt(8^2-3*5))/3`

5. (5 points) A mass attached to a vertical spring has position function given by $s(t) = 2\sin(3t)$ where t is measured in seconds and s in inches.
Find the velocity at time $t = 4$. ________
Find the acceleration at time $t = 4$. ________

Correct Answers:

- `5.06312375239495`
- `9.65831252400783`

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6. (5 points)

A price p (in dollars) and demand x for a product are related by

$$2x^2 + 6xp + 50p^2 = 23200.$$

If the price is increasing at a rate of 2 dollars per month when the price is 20 dollars, find the rate of change of the demand.

Rate of change of demand = ________

Correct Answers:

- `-21.2`

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7. (5 points) A particle is moving along the curve $y = 2\sqrt{5x+5}$. As the particle passes through the point $(4,10)$, its x-coordinate increases at a rate of 5 units per second. Find the rate of change of the distance from the particle to the origin at this instant.

____

Solution:

SOLUTION

Let D be the distance from the origin $(0,0)$ to the point on the curve $y = 2\sqrt{5x+5}$.

$$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + (2\sqrt{5x+5})^2} = \sqrt{x^2 + 4(5x+5)} = \sqrt{x^2+20x+20}$$

Differentiating with respect to t, we get:

$$\frac{dD}{dt} = \frac{1}{2}(x^2+20x+20)^{-1/2}(2x+20)\frac{dx}{dt} = \frac{2x+20}{2\sqrt{x^2+20x+20}}\frac{dx}{dt}$$

Substituting $x = 4$ and $\frac{dx}{dt} = 5$ gives

$$\frac{dD}{dt} = \frac{140}{2\sqrt{116}} \text{units per second}$$

Correct Answers:

- `6.49933683619682`

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8. (5 points)

Air is being pumped into a spherical balloon so that its volume increases at a rate of $40\text{cm}^3/\text{s}$. How fast is the surface area of the balloon increasing when its radius is 16cm? Recall that a ball of radius r has volume $V = \dfrac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$.

____

Solution:

Solve $S = 4\pi r^2$ for r:

$$r = \left(\frac{S}{4\pi}\right)^{1/2} = \frac{S^{1/2}}{2\pi^{1/2}}$$

Then $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\dfrac{S^{3/2}}{2^3\pi^{3/2}} = \dfrac{S^{3/2}}{6\pi^{1/2}}$

Now differentiate both sides of $V = \dfrac{S^{3/2}}{6\pi^{1/2}}$ with respect to time (t):

$$V' = \frac{\frac{3}{2}S^{1/2}S'}{6\pi^{1/2}} = \frac{S^{1/2}S'}{4\pi^{1/2}}$$

We are given that $V' = 40$.

Also, if $r = 16$, then $S = 4\pi\cdot 16^2 = 1024\pi$.

Thus we have $40 = \dfrac{32\pi^{1/2}S'}{4\pi^{1/2}} = \dfrac{32S'}{4}$

$$S' = \frac{40\cdot 4}{32} = 5$$

Correct Answers:

- `5`

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9. (5 points) At noon, ship A is 20 nautical miles due west of ship B. Ship A is sailing west at 25 knots and ship B is sailing north at 16 knots. How fast (in knots) is the distance between the ships changing at 7 PM?

The distance is changing at ____________ knots.

(Note: 1 knot is a speed of 1 nautical mile per hour.)

Correct Answers:

- `29.6475`

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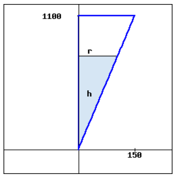
10. (5 points) Water is leaking out of an inverted conical tank at a rate of 10000.0 cubic centimeters per min at the same time that water is being pumped into the tank at a constant rate. The tank has height 11.0 meters and the diameter at the top is 3.0 meters. If the water level is rising at a rate of 15.0 centimeters per minute when the height of the water is 4.5 meters, find the rate at which water is being pumped into the tank in cubic centimeters per minute. ________

Note: Let "R" be the unknown rate at which water is being pumped in. Then you know that if V is volume of water, $\frac{dV}{dt} = R - 10000.0$. Use geometry (similar triangles?) to find the relationship between the height of the water and the volume of the water at any given time. Recall that the volume of a cone with base radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

Solution:

SOLUTION

A cross section of the tank through the vertical center to the side is approximated below (not necessarily to scale). Note that the height and the radius of the tank are measured in cm.



If R is the unknown rate at which water is being pumped in and V is volume of water, then $\frac{dV}{dt} = R - 10000.0$, where $V = \frac{1}{3}\pi r^2 h$ is the volume at time t.
By similar triangles, $\dfrac{r}{150} = \dfrac{h}{1100}$. So $r = \frac{3}{22}h$.
Substituting this expression for r into the formula for the volume, we get

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\tfrac{3}{22}h\right)^2 h = \tfrac{3}{484}\pi h^3$$

Differentiating with respect to t, gives

$$\frac{dV}{dt} = \tfrac{9}{484}\pi h^2 \frac{dh}{dt}$$

When $h = 450$, $\frac{dh}{dt} = 15$, so

$$\frac{dV}{dt} = \tfrac{9}{484}\pi(450)^2 15 = R - 10000.0$$

Solving for R gives:

$$R = 10000.0 + \tfrac{9}{484}\pi(450)^2 15 \approx 187444.8 \text{ cm}^3/\text{min}$$

Correct Answers:

- `187444.809302686`

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11. (5 points) A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the front of the boat, which is 10 feet below the level of the pulley.

If the rope is pulled through the pulley at a rate of 16 ft/min, at what rate will the boat be approaching the dock when 110 ft of rope is out?

The boat will be approaching the dock at ______ ft/min.

Correct Answers:

- `16.0665`