Qn. 1 (20 marks) Choose a correct option for each question. No justification is required. Each correct answer is worth 2 marks (no deduction for wrong answers).

(1) Let $A\mathbf{x} = \mathbf{b}$ be a linear system with 20 equations, 15 variables, and 10 basic variables. Then $\dim \operatorname{Row} A$ is:

(A) 0 (B) 5 (C)* 10 (D) 15 (E) 20

(2) Let A be a $p \times q$ matrix with $\operatorname{rank} A = q$. Consider the statements:

(I) $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one transformation.
(II) $\mathbf{x} \mapsto A\mathbf{x}$ is an onto transformation.
(III) $\mathbf{x} \mapsto A^T\mathbf{x}$ is a one-to-one transformation.
(IV) $\mathbf{x} \mapsto A^T\mathbf{x}$ is an onto transformation.

The correct statements are:

(A) I, III only (B)* I, IV only (C) II, III only (D) II, IV only (E) I, II, III, IV

(3) Let A, B, E be $n \times n$ matrices and let E be invertible. Consider the relations:

(I) $A = EB$ (II) $A = BE$ (III) $EA = B$ (IV) $AE = B$

The relation(s) that guarantee(s) A being row-equivalent to B is/are:

(A) I only (B) II only (C)* I, III only (D) II, IV only (E) I, II, III, IV

(4) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent set in $\mathbb{R}^3$. Which of the following 3×3 matrices is having zero determinant?

(A) $\begin{bmatrix} 2\mathbf{v}_1 & 3\mathbf{v}_2 & 4\mathbf{v}_3 \end{bmatrix}$.
(B) $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_1 + \mathbf{v}_2 & \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \end{bmatrix}$.
(C) $\begin{bmatrix} \mathbf{v}_1 + \mathbf{v}_2 & \mathbf{v}_2 + \mathbf{v}_3 & \mathbf{v}_3 + \mathbf{v}_1 \end{bmatrix}$.
(D)* $\begin{bmatrix} \mathbf{v}_1 - \mathbf{v}_2 & \mathbf{v}_2 - \mathbf{v}_3 & \mathbf{v}_3 - \mathbf{v}_1 \end{bmatrix}$.
(E) None of the above.

(5) Which of the following sets is a subspace of $M_{3\times 3}$ (the vector space of 3×3 matrices)?

(A)* $S = \{A \in M_{3\times 3} : \operatorname{Nul} A \text{ contains } \mathbf{e}_1\}$.
(B) $S = \{A \in M_{3\times 3} : \det A = 0\}$.
(C) $S = \{A \in M_{3\times 3} : A^TA = I_3\}$.
(D) $S = \{A \in M_{3\times 3} : A \text{ is diagonalizable}\}$.
(E) None of the above.

... to be continued

(6) Let A be an $m \times n$ matrix with $\operatorname{rank} A = n < m$. Which of the following statements is correct?

(A) $\dim \operatorname{Row} A > \dim \operatorname{Col} A$.

(B)* $\dim \operatorname{Row} A > \dim \operatorname{Nul} A$.

(C) $\dim \operatorname{Nul} A > \dim \operatorname{Col} A$.

(D) $\dim \operatorname{Nul} A = \dim \operatorname{Col} A$.

(E) None of the above.

(7) Let A be an $n \times n$ matrix. Which of the following subspaces, if non-zero, must be an eigenspace of A?

(A)* $(\operatorname{Row} A)^{\perp}$ (B) $\operatorname{Row} A$ (C) $(\operatorname{Col} A)^{\perp}$ (D) $\operatorname{Col} A$ (E) none of the previous.

(8) Let A, B be $n \times n$ matrices similar to each other. Which of the following statements is INCORRECT?

(A) A, B have the same determinant.

(B) A, B have the same rank.

(C) A, B have the same nullity.

(D) A, B have the same collection of eigenvalues.

(E)* A, B have the same collection of eigenvectors.

(9) Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{u} \in \mathbb{R}^n$. Consider the statements:

(I) $\operatorname{proj}_W \mathbf{u} \perp (\mathbf{u} - \operatorname{proj}_W \mathbf{u})$.

(II) $\operatorname{proj}_{(W^{\perp})} \mathbf{u} \perp (\mathbf{u} - \operatorname{proj}_{(W^{\perp})} \mathbf{u})$.

(III) $\operatorname{proj}_W \mathbf{u} \perp \operatorname{proj}_{(W^{\perp})} \mathbf{u}$.

(IV) $(\mathbf{u} - \operatorname{proj}_W \mathbf{u}) \perp (\mathbf{u} - \operatorname{proj}_{(W^{\perp})} \mathbf{u})$.

The correct statements are:

(A) I, II, III only (B) I, II, IV only (C) I, III, IV only (D) II, III, IV only (E)* I, II, III, IV

(10) Let A be an $m \times n$ matrix and let $\mathbf{v}$ be the orthogonal projection of a vector $\mathbf{u} \in \mathbb{R}^m$ onto $\operatorname{Col} A$. Which of the followings is correct?

(A) $A^T \mathbf{u} = \mathbf{0}$.

(B) $A^T \mathbf{v} = \mathbf{0}$.

(C) $A^T(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

(D)* $A^T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$.

(E) None of the above.

... to be continued by Qn. 2

Qn. 2 (10 marks) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find A^{-1}. *[4 marks]*

(b) Find the matrix X such that $AXA^{-1} = B$. *[6 marks]*

Solution:

(a) We perform EROs on the combined matrix $[\ A \mid I_4\ ]$:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-r_3+r_2 \\ -r_4+r_3}]{-r_2+r_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\frac{1}{3}r_3 \\ \frac{1}{4}r_4}]{\frac{1}{2}r_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right]$$

Hence:

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

(b) Since $AXA^{-1} = B \Rightarrow X = A^{-1}BA$, so we compute:

$$\begin{aligned} X = A^{-1}BA &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} A \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

OR:

$$\begin{aligned} X = A^{-1}BA = A^{-1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} &= A^{-1} \begin{bmatrix} 1 & 6 & 18 & 40 \\ 0 & 2 & 9 & 24 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix} \\ = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 6 & 18 & 40 \\ 0 & 2 & 9 & 24 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

... to be continued by Qn. 3

Qn. 3 (10 marks) Consider a linear system $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Check that $A\mathbf{x} = \mathbf{b}$ is inconsistent. *[2 marks]*

(b) Find a least squares solution $\mathbf{x}_0$ to the system $A\mathbf{x} = \mathbf{b}$. *[4 marks]*

(c) Find the distance of $\mathbf{b}$ to $\operatorname{Col} A$. *[4 marks]*

Solution:

(a) Perform EROs on the augmented matrix:

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow[r_1+r_3]{-2r_1+r_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & 2 \end{array}\right] \xrightarrow{3r_2+r_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{array}\right]$$

Since the last column of the augmented matrix contains a pivot position, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(b) The least squares solution $\mathbf{x}_0$ is determined by the normal equation $A^T A\mathbf{x}_0 = A^T\mathbf{b}$. So we compute:

$$A^T A = \begin{bmatrix} 6 & 7 \\ 7 & 14 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

and $$[\, A^T A \mid A^T\mathbf{b} \,] = \left[\begin{array}{cc|c} 6 & 7 & 2 \\ 7 & 14 & 6 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{\frac{1}{7}r_2} \left[\begin{array}{cc|c} 1 & 2 & \frac{6}{7} \\ 6 & 7 & 2 \end{array}\right] \xrightarrow{-6r_1+r_2} \left[\begin{array}{cc|c} 1 & 2 & \frac{6}{7} \\ 0 & -5 & -\frac{22}{7} \end{array}\right] \xrightarrow[-2r_2+r_1]{-\frac{1}{5}r_2} \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{22}{35} \end{array}\right].$$

Hence $\mathbf{x}_0 = \begin{bmatrix} -\frac{2}{5} \\ \frac{22}{35} \end{bmatrix}$.

(c) The distance is given by $||\mathbf{b} - A\mathbf{x}_0||$. So we compute:

$$\mathbf{b} - A\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{5} \\ \frac{22}{35} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{6}{7} \\ \frac{38}{35} \\ \frac{36}{35} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ -\frac{3}{35} \\ -\frac{1}{35} \end{bmatrix}$$

Hence distance $= ||\mathbf{b} - A\mathbf{x}_0|| = \sqrt{\frac{1}{35}}$.

[Note: $A\mathbf{x}_0$ is exactly the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} A$.]

... to be continued by Qn. 4

Qn. 4 (15 marks) Consider $\mathbb{P}_2$, the vector space of polynomials with degree at most 2. Let:

$$\mathcal{B} = \{1+t,\ t+t^2,\ t^2+1\}, \quad p(t) = 1+t+t^2, \quad q(t) = 2+t-t^2.$$

(a) Verify that $\mathcal{B}$ is a basis for $\mathbb{P}_2$. *[4 marks]*

(b) Find the coordinate vectors of $p(t)$, $q(t)$ relative to basis $\mathcal{B}$. *[6 marks]*

(c) Let $[r(t)]_{\mathcal{B}} = [\,1 \;\; 2 \;\; 1\,]^T$. Find the polynomial $r(t)$. *[2 marks]*

(d) Does the $r(t)$ in (c) belong to Span $\{p(t), q(t)\}$? Why or why not? *[3 marks]*

Solution:

(a) (i) Check that $\mathcal{B}$ is linearly independent: Consider the vector equation:

$$c_1(1+t) + c_2(t+t^2) + c_3(t^2+1) = 0(t) \quad \leftrightarrow \quad (c_1+c_3)\cdot 1 + (c_1+c_2)\cdot t + (c_2+c_3)\cdot t^2 = 0(t).$$

By equating coefficients of $1, t, t^2$, we get:

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

So $\mathcal{B}$ is linearly independent by definition.

(ii) Check that $\operatorname{Span}\mathcal{B} = \mathbb{P}_2$: For any polynomial $a + bt + ct^2 \in \mathbb{P}_2$, we try to solve:

$$\begin{aligned} &x_1(1+t) + x_2(t+t^2) + x_3(t^2+1) = a + bt + ct^2 \\ &\leftrightarrow (x_1+x_3)\cdot 1 + (x_1+x_2)\cdot t + (x_2+x_3)\cdot t^2 = a + bt + ct^2 \end{aligned}$$

By equating coefficients of $1, t, t^2$, we get:

$$\begin{cases} x_1 + x_3 = a \\ x_1 + x_2 = b \\ x_2 + x_3 = c \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 = \frac{1}{2}(a+b-c) \\ x_2 = \frac{1}{2}(-a+b+c) \\ x_3 = \frac{1}{2}(a-b+c) \end{cases}$$

which is always consistent. So $\mathcal{B}$ can span $\mathbb{P}_2$.

As $\mathcal{B}$ is both linearly independent, spanning $\mathbb{P}_2$, it will form a basis for $\mathbb{P}_2$.

[Note: (i) or (ii) can be replaced by checking number of vectors in $\mathcal{B} = \dim \mathbb{P}_2 = 3$.]

(b) To find coordinate vectors relative to basis $\mathcal{B}$, we need to determine the coordinates x_1, x_2, x_3 in the linear combination:

$$x_1(1+t) + x_2(t+t^2) + x_3(t^2+1) = p(t) \;\text{ or }\; q(t).$$

By (a)(ii), we have the solutions immediately:

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad [q(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(c) By definition, the vector $r(t)$ should be given by:

$$[r(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \leftrightarrow \quad r(t) = 1\cdot(1+t) + 2\cdot(t+t^2) + 1\cdot(t^2+1) = 2 + 3t + 3t^2.$$

(d) Using the $\mathcal{B}$-coordinate mapping, the problem can be transformed into the same query on the corresponding coordinate vectors:

$$\text{Is } [r(t)]_{\mathcal{B}} \in \text{Span}\,\{[p(t)]_{\mathcal{B}}, [q(t)]_{\mathcal{B}}\}? \quad \text{i.e. Is } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}?$$

We check the consistency of the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} \frac{1}{2} & 2 & 1 \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & 1 \end{array}\right] \xrightarrow[-r_1+r_3]{-r_1+r_2} \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

which is inconsistent. So we can conclude that $r(t) \notin \text{Span}\,\{p(t), q(t)\}$.

OR, using the standard representations of the three polynomials, we are asking the consistency of:

$$2 + 3t + 3t^2 = a(1 + t + t^2) + b(2 + t - t^2) \quad \leftrightarrow \quad \begin{cases} a + 2b = 2 \\ a + b = 3 \\ a - b = 3 \end{cases}$$

Again, the system is inconsistent.

... to be continued by Qn. 5

Qn. 5 (15 marks) Let:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Diagonalize A, namely, find invertible matrices P, P^{-1} and a diagonal matrix D such that $A = PDP^{-1}$. *[10 marks]*

(b) Find a general formula of A^n. *[5 marks]*

Solution:

(a) First we solve the characteristic equation of A for eigenvalues:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$
$$= (2-\lambda)[(2-\lambda)^2 - 1] = (2-\lambda)(1-\lambda)(3-\lambda).$$

So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Next, for each eigenvalue, we find a basis for the corresponding eigenspace:

(i) $\lambda_1 = 1$: We solve $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the eigenspace of $\lambda_1 = 1$ is:

$$\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}.$$

(ii) $\lambda_2 = 2$: We solve $(A - 2I)\mathbf{x} = \mathbf{0}$.

$$A - 2I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the eigenspace of $\lambda_2 = 2$ is:

$$\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

(iii) $\lambda_3 = 3$: We solve $(A - 3I)\mathbf{x} = \mathbf{0}$.

$$A - 3I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the eigenspace of $\lambda_3 = 3$ is:

$$\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}.$$

Hence we can construct the matrices P and D, and compute the matrix P^{-1} as:

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Then $A = PDP^{-1}$.

(b) Since $A^n = PD^nP^{-1}$, so we compute:

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(3^n+1) & 0 & \frac{1}{2}(3^n-1) \\ 0 & 2^n & 0 \\ \frac{1}{2}(3^n-1) & 0 & \frac{1}{2}(3^n+1) \end{bmatrix}. \end{aligned}$$

... to be continued by Qn. 6

Qn. 6 (15 marks) Let:

$$A = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

(a) Show that $\mathbf{v}$, $\mathbf{w}$ are both eigenvectors of A. Write down also their eigenvalues. [*4 marks*]

(b) Find an orthonormal basis for the eigenspace of A containing (i) $\mathbf{v}$ (ii) $\mathbf{w}$ respectively.
[Note: Your orthonormal basis should start with the vector $\mathbf{v}$ (or $\mathbf{w}$).] [*8 marks*]

(c) Find an orthogonal matrix P such that $D = P^T AP$ is a diagonal matrix. Write down also the diagonal matrix D. [*3 marks*]

Solution:

(a) By direct checking:

$$A\mathbf{v} = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} = 5\mathbf{v}, \quad A\mathbf{w} = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix} = 3\mathbf{w},$$

so $\mathbf{v}$ is an eigenvector of A corresponding to eigenvalue 5,
and $\mathbf{w}$ is an eigenvector of A corresponding to eigenvalue 3.

(b) (i) For eigenspace corresponding to eigenvalue 5, we consider:

$$A - 5I = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\dim \operatorname{Nul}(A-5I) = 2$ and it has a basis $\{(1,0,0,1)^T, (0,1,1,0)^T\}$. To fulfill the requirement of the question, we switch to another basis $\{\mathbf{v}, (1,0,0,1)^T\}$ and apply Gram-Schmidt process:

$$\mathbf{u}_1 = \mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}{|| \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} ||^2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Note that $||\mathbf{u}_1|| = 1 = ||\mathbf{u}_2||$, so an orthonormal basis (containing $\mathbf{v}$) for the eigenspace $\operatorname{Nul}(A - 5I)$ can be chosen as:

$$\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \}.$$

(ii) Similarly, for eigenspace corresponding to eigenvalue 3, we consider:

$$A - 3I = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\dim \mathrm{Nul}\,(A-3I) = 2$ and it has a basis $\{(-1,0,0,1)^T, (0,-1,1,0)^T\}$. To fulfil the requirement of the question, we switch to another basis $\{\mathbf{w}, (-1,0,0,1)^T\}$ and apply Gram-Schmidt process:

$$\mathbf{u}_1' = \mathbf{w} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}}{|| \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} ||^2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Note that $||\mathbf{u}_1'|| = 1 = ||\mathbf{u}_2'||$, so an orthonormal basis (containing $\mathbf{w}$) for the eigenspace $\mathrm{Nul}\,(A-3I)$ can be chosen as:

$$\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \}.$$

(c) Since A is symmetric, the required orthogonal matrix P can be formed by putting the unit vectors in the orthonormal bases in (b) column by column:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad D = P^T A P = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

... to be continued by Qn. 7

Qn. 7 (15 marks) Let:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}, \quad U = \operatorname{Row} A, \quad W = \operatorname{Nul} A, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Find $\operatorname{proj}_U \mathbf{v}$. *[6 marks]*

(b) Find $\operatorname{proj}_W \mathbf{v}$. *[4 marks]*

(c) Let B denote the standard matrix of the orthogonal projection transformation proj_U, and let C denote the standard matrix of the orthogonal projection transformation proj_W.
Find $B + C$. *[5 marks]*

Solution:

(a) To compute $\operatorname{proj}_U \mathbf{v}$, we need an orthogonal basis for $U = \operatorname{Row} A$. Take $\{(1, 2, 2, 1)^T, (2, 1, 1, 2)^T\}$ as a basis for U and apply the Gram-Schmidt process:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} ||} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{8}{10} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{3}{5} \\ -\frac{3}{5} \\ \frac{6}{5} \end{bmatrix}.$$

May take $\{\mathbf{u}_1, \frac{3}{5}\mathbf{u}_2\}$ and an orthogonal basis for U, namely:

$$\{\mathbf{u}_1', \mathbf{u}_2'\} = \{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \}.$$

Using the orthogonal basis for U, we compute:

$$\operatorname{proj}_U \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1'}{||\mathbf{u}_1'||^2} \mathbf{u}_1' + \frac{\mathbf{v} \cdot \mathbf{u}_2'}{||\mathbf{u}_2'||^2} \mathbf{u}_2' = \frac{15}{10} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \frac{5}{10} \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}.$$

(b) Since $W = \operatorname{Nul} A = (\operatorname{Row} A)^\perp = U^\perp$, and the orthogonal decomposition of $\mathbf{v}$ w.r.t. U is actually $\operatorname{proj}_U \mathbf{v} + \operatorname{proj}_{(U^\perp)} \mathbf{v}$, so we get:

$$\operatorname{proj}_W \mathbf{v} = \operatorname{proj}_{(U^\perp)} \mathbf{v} = \mathbf{v} - \operatorname{proj}_U \mathbf{v} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

(*Or by direct computation...*)

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

So an orthogonal basis for $W = \operatorname{Nul} A$ can be chosen as $\{\mathbf{w}_1, \mathbf{w}_2\} = \{(0,-1,1,0)^T, (-1,0,0,1)^T\}$. Hence:

$$\operatorname{proj}_W \mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}_1}{||\mathbf{w}_1||^2}\mathbf{w}_1 + \frac{\mathbf{v}\cdot\mathbf{w}_2}{||\mathbf{w}_2||^2}\mathbf{w}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

(c) Since for every $\mathbf{x} \in \mathbb{R}^4$, we have $\mathbf{x} = \operatorname{proj}_U \mathbf{x} + \operatorname{proj}_W \mathbf{x}$, so the standard matrix of the sum of orthogonal projection transformations $\operatorname{proj}_U + \operatorname{proj}_W$ is just the same as the identity transformation. Hence we must have $B + C = I_4$.

(*Or by direct computation...*)

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Thus $B + C = I_4$.

— END —