Review of related rates problems:

In a related rates problem, we usually have two functions: $x(t)$ and $y(t)$.

They are closely related to each other.

Question: Given $\frac{d}{dt} y(t)$, how to calculate $\frac{d}{dt} x(t)$?

To calculate $\frac{d}{dt} x(t)$, we take following 2 steps:

Step 1: Find a relation between $x(t)$ and $y(t)$. (e.g. $y(t) = x^2(t)$.)

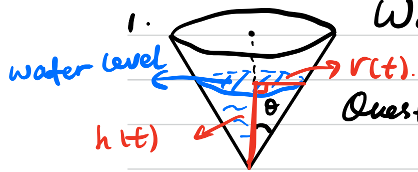
Step 2: Differentiate both sides of this equation with respect to t

and express $\frac{d}{dt} x(t)$ in terms of $\frac{d}{dt} y(t)$.

an inverted conical tank

$V(t)$ increases at a rate of L m^3/s → $\frac{d}{dt}V(t)$.

↑ means

1. 

Water is flowing into this tank at a rate of $\underline{L}$ m^3/s.

Question: How fast is the water level rising when the height of water in this tank is equal to h meters.

$r(t)$: radius of water level at time t. $h(t)$: the height of water in this tank at time t.

$V(t)$: volume of water in this tank at time t. (As t increases, $r(t)$, $h(t)$ and $V(t)$ increase).

Given that $\frac{d}{dt}V(t) = L$, our aim is to find $\frac{d}{dt}h(t)$ when $h(t) = h$.

Step 1: Find a relation between $V(t)$ and $h(t)$.

Notice: $V(t) = \frac{1}{3}h(t)\cdot\pi\cdot r^2(t)$. $\tan\theta = \frac{r(t)}{h(t)}$. ($\theta$ is given)

$$\Rightarrow r(t) = \tan\theta\cdot h(t) \Rightarrow V(t) = \frac{1}{3}\cdot h(t)\cdot\pi\cdot\tan^2\theta\cdot h^2(t) = \frac{\tan^2\theta}{3}\cdot\pi\cdot h^3(t)$$

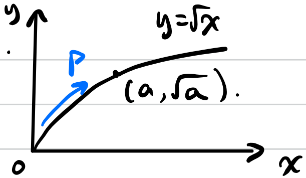
Step 2: Differentiate both sides and express $\frac{d}{dt}h(t)$ in terms of $\frac{d}{dt}V(t)$.

$$\frac{d}{dt}V(t) = \frac{d}{dt}\Big(\underbrace{\frac{\tan^2\theta}{3}\cdot\pi\cdot h^3(t)}\Big) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{\tan^2\theta}{3}\cdot\pi\cdot 3\cdot h^2(t)\cdot\frac{d}{dt}h(t).$$

$$\Rightarrow \frac{d}{dt}h(t) = \frac{1}{\pi\cdot\tan^2\theta}\cdot\frac{1}{h^2(t)}\cdot\frac{d}{dt}V(t).$$

$$\Rightarrow \frac{d}{dt}h(t)\Big|_{h(t)=h} = \frac{1}{\pi\cdot\tan^2\theta}\cdot\frac{1}{h^2}\cdot L = \frac{L}{\pi\cdot\tan^2\theta\cdot h^2}\ m/s.$$

2.



A particle P is moving along the curve $y=\sqrt{x}$. As P passes through the point $(a,\sqrt{a})$, its x-coordinate increases at a rate of L m/s.

$\left.\frac{d}{dt}x(t)\right|_{x(t)=a}$

Question: What is the rate of change of the distance from P to $(0,0)$ when P passes through $(a,\sqrt{a})$?

$x(t)$: the x-coordinate of P at time t. $y(t)$: the y-coordinate of P at time t.
$D(t)$: the distance from P to $(0,0)$ at time t.

Given that $\left.\frac{d}{dt}x(t)\right|_{x(t)=a}=L$, our aim is to find $\left.\frac{d}{dt}D(t)\right|_{x(t)=a}$.

Step 1: Find a relation between $D(t)$ and $x(t)$.

Notice: $y(t)=\sqrt{x(t)}$. $D(t)=\sqrt{x^2(t)+y^2(t)}$. $\Rightarrow D(t)=\sqrt{x^2(t)+x(t)}$.

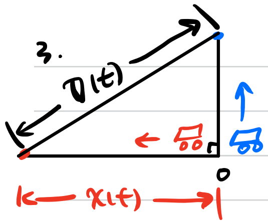
Step 2: Differentiate both sides and express $\frac{d}{dt}D(t)$ in terms of $\frac{d}{dt}x(t)$.

$$\frac{d}{dt}D(t)=\frac{d}{dt}\Big(\sqrt{\underbrace{x^2(t)+x(t)}_{=u}}\Big)\overset{\text{chain}}{\underset{\text{rule}}{=\!=}}\frac{d\sqrt{u}}{du}\cdot\frac{du}{dt}=\frac{1}{2\sqrt{u}}\cdot\frac{d\left(x^2(t)+x(t)\right)}{dt}$$

$$=\frac{1}{2\sqrt{u}}\cdot\left(2\cdot x(t)\cdot\frac{d}{dt}x(t)+\frac{d}{dt}x(t)\right)=\frac{2x(t)+1}{2\sqrt{x^2(t)+x(t)}}\cdot\frac{d}{dt}x(t)$$

$$\Rightarrow \left.\frac{d}{dt}D(t)\right|_{x(t)=a}=\left.\left(\frac{2x(t)+1}{2\sqrt{x^2(t)+x(t)}}\cdot\frac{d}{dt}x(t)\right)\right|_{x(t)=a}=\frac{2a+1}{2\sqrt{a^2+a}}\cdot L \text{ m/s.}$$

3.



$\frac{d}{dt}x(t)$

Red car is moving west from O at a rate of L_1 m/s.
Blue car is moving north from O at a rate of L_2 m/s.

$\frac{d}{dt}y(t)$.

Question: How fast is their distance changing after 5 seconds?

$x(t)$: distance from red car to O at time t. $y(t)$: distance from blue car to O at time t.
$D(t)$: distance from red car to blue car at time t.

Given that $\frac{d}{dt}x(t) = L_1$ and $\frac{d}{dt}y(t) = L_2$, our aim is to find $\frac{d}{dt}D(t)\Big|_{t=5}$.

Step 1: Find a relation between $x(t)$, $y(t)$ and $D(t)$.

Notice: $D^2(t) = x^2(t) + y^2(t)$.

Step 2: Differentiate both sides and express $\frac{d}{dt}D(t)$ in terms of $\frac{d}{dt}x(t)$ and $\frac{d}{dt}y(t)$.

$$\frac{d}{dt}\left(D^2(t)\right) = \frac{d}{dt}\left(x^2(t) + y^2(t)\right).$$

$$\Rightarrow 2\cdot D(t)\cdot\frac{d}{dt}D(t) = 2\cdot x(t)\cdot\frac{d}{dt}x(t) + 2\cdot y(t)\cdot\frac{d}{dt}y(t).$$

$$\Rightarrow \frac{d}{dt}D(t) = \frac{x(t)\cdot\frac{d}{dt}x(t) + y(t)\cdot\frac{d}{dt}y(t)}{D(t)}$$

$$\Rightarrow \frac{d}{dt}D(t)\Big|_{t=5} = \frac{5L_1\cdot L_1 + 5L_2\cdot L_2}{\sqrt{(5L_1)^2+(5L_2)^2}} = \sqrt{L_1^2+L_2^2}\ \text{m/s}.$$

$x(5) = 5L_1$. $y(5) = 5L_2$.

To summarize:

In a <u>related rates problem</u>, we have at least two different rates of change: $\frac{d}{dt}x(t)$ and $\frac{d}{dt}y(t)$, and they are related by an equation.

To calculate $\frac{d}{dt}x(t)$ from $\frac{d}{dt}y(t)$, we take 2 steps:

Step 1: Find a relation between $x(t)$ and $y(t)$.

Step 2: Differentiate both sides with respect to t and express $\frac{d}{dt}x(t)$ in terms of $\frac{d}{dt}y(t)$.

Applications of differentiation.

1. use the differentiation to find maximum and minimum values of $y=f(x)$.

(also called global maximum)

$f(c)$ is the absolute maximum of $f \iff f(c) \geq f(x)$ for all $x \in D$. → domain of f.

(also called global minimum)

$f(c)$ is the absolute minimum of $f \iff f(c) \leq f(x)$ for all $x \in D$.

(also called relative maximum)

$f(c)$ is the local maximum of $f \iff f(c) \geq f(x)$ for all x close to c.

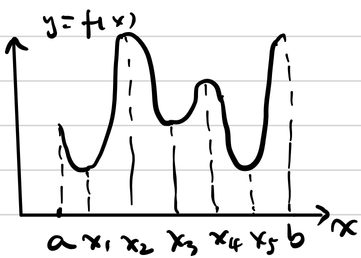
(also called relative minimum)

$f(c)$ is the local minimum of $f \iff f(c) \leq f(x)$ for all x close to c.

Notice: "for all x close to c" means "for all x in some $\boxed{\text{open interval}}$ containing c".

Example:

$D=[a,b]$.



absolute maximum: $f(x_2)$ $f(b)$. (Highest points).

absolute minimum: $f(x_1)$ $f(x_5)$. (Lowest points).

local maximum: $f(x_2)$ $f(x_4)$.

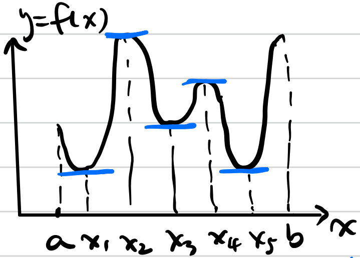
local minimum: $f(x_1)$, $f(x_3)$, $f(x_5)$.

Notice: The endpoints $f(a)$ and $f(b)$ are not local maximum or local minimum.

The tangent line at $x=c$ is horizontal.
$\Uparrow$ means

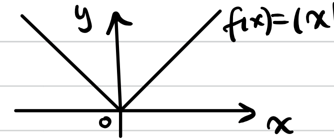
Fermat's Theorem: $\left.\begin{array}{l} f(c) \text{ is a local maximum or minimum} \\ f'(c) \text{ exists} \end{array}\right\} \Rightarrow f'(c)=0$.

Example 1:



$f'(x_1)=f'(x_2)=f'(x_3)=f'(x_4)=f'(x_5)=0$.

Example 2.

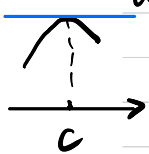


$f(0)$ is the local minimum.

$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{|h|}{h}$ does not exist.

Fermat's Theorem does not apply in this case.

Proof of Fermat's Theorem: Let's consider a local maximum at c



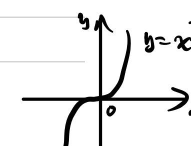
We observe that:

(1) Since $f'(c)$ exists, we have $f'(c)=\lim\limits_{h\to 0^-}\dfrac{f(c+h)-f(c)}{h}=\lim\limits_{h\to 0^+}\dfrac{f(c+h)-f(c)}{h}$.

$\left.\begin{array}{l} \text{(2). If } h\to 0^-, \text{ then } \dfrac{f(c+h)-f(c)}{h}\geq 0. \Rightarrow \lim\limits_{h\to 0^-}\dfrac{f(c+h)-f(c)}{h}\geq \lim\limits_{h\to 0^-} 0=0 \Rightarrow f'(c)\geq 0. \\ \text{(3). If } h\to 0^+, \text{ then } \dfrac{f(c+h)-f(c)}{h}\leq 0. \Rightarrow \lim\limits_{h\to 0^+}\dfrac{f(c+h)-f(c)}{h}\leq \lim\limits_{h\to 0^+} 0=0 \Rightarrow f'(c)\leq 0. \end{array}\right\} \Rightarrow f'(c)=0.$

Notice: $f'(c)=0 \not\Rightarrow c$ is a local maximum or minimum.

Example: $y=f(x)=x^3$. $f'(x)=3x^2$. $f'(0)=0$. However, 0 is not the local maximum or minimum.



Application of Fermat's Theorem: find the absolute maximum/minimum.

Let $f(x)$ be a continuous function on a closed interval $[a,b]$.

Definition: c is a critical number of f if $f'(c)=0$ or $f'(c)$ does not exist.

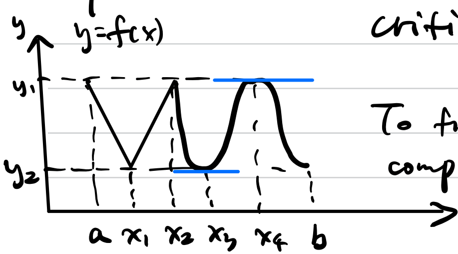
To find absolute maximum/minimum of f on $[a,b]$, we take 2 steps:

Step 1: Find the values of f at all critical numbers in (a,b).

Step 2: Calculate the endpoints $f(a)$ and $f(b)$.

Then, absolute maximum = the largest of the values from step 1 and 2.
absolute minimum = the smallest of the values from step 1 and 2.

Example:



critical numbers $\begin{cases} x_1 \quad x_2 & (f' \text{ does not exist}). \\ x_3 \quad x_4. & (f'=0). \end{cases}$

To find absolute maximum and minimum, we only need to compare the value of $f(x_1)$. $f(x_2)$. $f(x_3)$. $f(x_4)$. $f(a)$. $f(b)$.

Answer: Absolute maximum: $f(a)$. $f(x_2)$. $f(x_4)$.

Absolute minimum: $f(b)$. $f(x_1)$. $f(x_3)$.