§1.4 The matrix equation $Ax = b$

Def: A: $m \times n$ matrix with column $\vec{a}_1, \cdots, \vec{a}_n \in \mathbb{R}^m$ i.e. $A = [\vec{a}_1, \cdots, \vec{a}_n]$, $\vec{x}$ is a vector in $\mathbb{R}^n$.

$$A \cdot \vec{x} = [\vec{a}_1, \cdots, \vec{a}_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \triangleq x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

i.e. $A\vec{x}$ is a linear combination of the columns of A with $x_1, x_2, \cdots, x_n$ as weights.

Example: Find a) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

Solution: a) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2 \\ -5 \end{pmatrix} + 7\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -15 \end{pmatrix} + \begin{pmatrix} -7 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

b) $\begin{pmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} + 7\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 8 \\ 32 \\ -20 \end{pmatrix} + \begin{pmatrix} -21 \\ 0 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 8 - 21 \\ 32 + 0 \\ -20 + 14 \end{pmatrix} = \begin{pmatrix} -13 \\ 32 \\ -6 \end{pmatrix}$$

Example: Write the linear combination $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3$ as a matrix times a vector.

Solution: $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3 = [\vec{v}_1, \vec{v}_2, \vec{v}_3] \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$

Thm: $A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$, $\vec{a}_1, \cdots, \vec{a}_n \in \mathbb{R}^m$, $\vec{b} \in \mathbb{R}^m$, $\vec{x} \in \mathbb{R}^n$

The equation $A\vec{x} = \vec{b}$ has the same solution set as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $[\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n, \vec{b}]$.

* Existence of solutions

The equation $A\vec{x} = \vec{b}$ has a solution if and only if b is a linear combination of the columns of A.

Example 3: Let $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Is the equation $A\vec{x} = \vec{b}$ consistent for all possible b_1, b_2, b_3?

Solution: Row reduce the augmented matrix for $Ax = b$:

$$\begin{pmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{pmatrix} \xrightarrow[③+3①]{②+4①} \begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{pmatrix}$$

$$\xrightarrow{③-\frac{1}{2}②}\begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_3+3b_1-\frac{1}{2}(b_2+4b_1) \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_3-\frac{1}{2}b_2+b_1 \end{pmatrix}$$

The equation $A\vec{x}=\vec{b}$ is only consistent for $\vec{b}$ satisfying $b_3-\frac{1}{2}b_2+b_1=0$. Thus the answer is no.

Thm: A: $m\times n$ matrix. The following statements are equivalent.

(a) For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x}=\vec{b}$ has a solution.

(b). Each $\vec{b}\in\mathbb{R}^m$ is a linear combination of the column vectors of A.

(c) The column vectors of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Proof: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) come from the definition.

Now we prove (a) $\Leftrightarrow$ (d).

Assume $[A,\vec{b}]\xrightarrow{\text{row operation}}[U,\vec{d}]$

U: echelon form of A.

If (d) is true, $U=\begin{pmatrix} \square & * & * & \cdots & * & * \\ 0 & \square & * & \cdots & * & * \\ 0 & 0 & \square & \cdots & * & * \\ & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \square & * \end{pmatrix}$

Thus there is no pivot in the last column. Hence $A\vec{x} = \vec{b} \Leftrightarrow U\vec{x} = \vec{d}$ has a solution.

If (d) is false, then the last row of U is zero, thus for $\vec{d}$ whose last entry is not zero, $U\vec{x} = \vec{d}$ has no solution, thus $A\vec{x} = \vec{b}$ has no solution.

* Computation of AX.

Example: Compute AX, where $A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Solution: From the definition

$$\begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + x_3\begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{pmatrix} + \begin{pmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{pmatrix}$$

Remark: $\begin{pmatrix} 2 & 3 & 4 \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \end{pmatrix}$

Row vector

$$\begin{pmatrix} -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 + 5x_2 - 3x_3 \end{pmatrix}$$ Rule for computing $A\vec{x}$

* Properties of the Matrix-Vector Product $A\vec{x}$

If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^n$, and c is a scalar, then

(a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

(b) $A(c\vec{u}) = c(A\vec{u})$

Proof: Let $A = (\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$$\begin{aligned} A(\vec{u}+\vec{v}) &= (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2 + \cdots + (u_n+v_n)\vec{a}_n \\ &= u_1\vec{a}_1 + u_2\vec{a}_2 + \cdots + u_n\vec{a}_n + v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n \\ &= A\vec{u} + A\vec{v} \end{aligned}$$

§1.5 Solution sets of linear systems

- Homogeneous Linear systems

$$A\vec{x} = \vec{0}$$

It always has a (trivial) solution $\vec{x} = \vec{0}$.

The homogeneous linear system $A\vec{x} = \vec{0}$ has nontrivial solution if and only if the system has at least one free variable.

Example: Determine if the following homogeneous system has a nontrivial solution.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

Then describe the solution set.

Solution: $\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \end{cases}$$

x_3 is a free variable

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

* Parametric Vector Form $\vec{x} = s\vec{u} + t\vec{v}$ $(s, t \in \mathbb{R})$

* Solutions of Nonhomogeneous Systems.

Example: Describe all solutions of $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

Solution:
$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

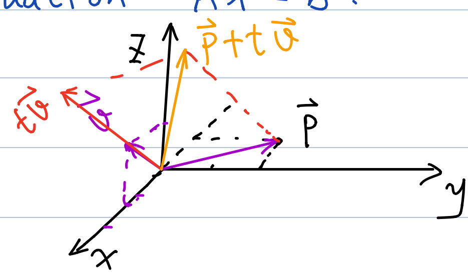
$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ 0 = 0 \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}}$$

$\vec{x} = \vec{p} + t\vec{v}$ where $t = x_3$

$\vec{x} = t\vec{v}$ are solutions of the homogeneous equation $A\vec{x} = \vec{0}$.

$\vec{x} = \vec{p}$ is a solution of the non-homogeneous equation $A\vec{x} = \vec{b}$.



Thm: Suppose $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$ and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + t\vec{v}_n$, where $\vec{v}_n$ is any solution of $A\vec{x} = \vec{0}$.

Exercise: Describe the solutions of the following system in parametric vector form.

$$\begin{cases} x_1 + 3x_2 + x_3 = 1 \\ -4x_1 - 9x_2 + 2x_3 = -1 \\ \quad\ \ -3x_2 - 6x_3 = -3 \end{cases}$$

Solution:

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & 3 & -6 & -3 \end{pmatrix} \xrightarrow{\text{②}+4\text{①}} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{pmatrix}$$

$$\xrightarrow{\text{③}+\text{②}} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{②}/3} \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{①}-3\text{②}} \begin{pmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 \quad - 5x_3 = -2 \\ \quad x_2 + 2x_3 = 1 \\ \qquad\quad 0 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 5x_3 - 2 \\ x_2 = -2x_3 + 1 \\ x_3 = x_3 \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_3 - 2 \\ -2x_3 + 1 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Summary: Writing a solution set (of a consistent system) in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\vec{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose x into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Exercise: Describe all solutions of $A\vec{x} = \vec{0}$ in parametric form, where A is row equivalent to the given matrix

$$\begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: We perform row operations on the coefficient matrix to the reduced echelon form as follows:

$$\begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{①+(-3)③} \begin{pmatrix} 1 & -4 & -2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{①+2②} \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the original system is reduced to

$$\begin{cases} x_1 - 4x_2 + 5x_6 = 0 \\ x_3 - x_6 = 0 \\ x_5 - 4x_6 = 0 \\ 0 = 0 \end{cases}$$

It is clear that x_2, x_4, x_6 are free variables, so that the system has nontrivial solutions.

$$\begin{cases} x_1 = 4x_2 - 5x_6 \\ x_2 = x_2 \\ x_3 = x_6 \\ x_4 = x_4 \\ x_5 = 4x_6 \\ x_6 = x_6 \end{cases}$$

That is $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4x_2 & & -5x_6 \\ x_2 & & \\ & & x_6 \\ & x_4 & \\ & & 4x_6 \\ & & x_6 \end{pmatrix}$

$$= x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

where x_2, x_4 and x_6 are arbitrary.