Math2001 Answer to Homework 5

EXERCISE 2.52

1. one-to-one.
2. neither.
3. onto and one-to-one.
4. onto and one-to-one.
5. one-to-one.
6. one-to-one.
7. onto and one-to-one.
8. onto.
9. one-to-one.
10. neither.
11. neither.

EXERCISE 2.56

Suppose $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$.

For any element $z \in Z$, there exists $x \in X$ such that $g(f(x)) = (g \circ f)(x) = z$ since $g \circ f$ is onto. Hence we have $f(x) \in Y$ such that $g(f(x)) = z$. Thus $g : Y \to Z$ is onto.

For any x_1, $x_2 \in X$ such that $f(x_1) = f(x_2)$ there is $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, by definition there is $x_1 = x_2$ in X. Thus f is one-to-one.

EXERCISE 2.57

Suppose $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$.

For any $z \in Z$, surjectivity of g implies that there exists $y \in Y$ such that $g(y) = z$. Furtherly, surjectivity of f implies that there exists $x \in X$ such that $f(x) = y$. Thus $g(f(x)) = z$, $g \circ f$ is onto.

For any x_1, $x_2 \in X$ such that $g(f(x_1)) = g(f(x_2))$, injectivity of g implies that $f(x_1) = f(x_2)$. Furtherly, injectivity of f implies that $x_1 = x_2$. Thus $g \circ f$ is one-to-one.

EXERCISE 2.60

1. $f(u, v) = (\frac{2}{7}u + \frac{1}{7}v + \frac{1}{7}, -\frac{3}{7}u + \frac{2}{7}v - \frac{5}{7}) : \mathbb{R}^2 \to \mathbb{R}^2$.
2. $f(u, v, w) = (u, v - u, w - v) : \mathbb{R}^3 \to \mathbb{R}^3$.
3. $f(u, v) = (\frac{u-1}{2}, \frac{1}{4}u^2 + u - \frac{5}{4} - v) : \mathbb{R}^2 \to \mathbb{R}^2$.
4. $f(u) = -\sqrt{\sqrt{u} - 2} : [4, \infty) \to (-\infty, 0]$.

EXERCISE 2.62

$$(x, y) \mapsto \begin{cases} (\sqrt{x^2 + y^2}, \arccos(\frac{x}{\sqrt{x^2+y^2}})) & \text{if } y \geqslant 0 \text{ and } x^2 + y^2 \neq 0 \\ (\sqrt{x^2 + y^2}, 2\pi - \arccos(\frac{x}{\sqrt{x^2+y^2}})) & \text{if } y < 0 \text{ and } x^2 + y^2 \neq 0 \\ (0, 0) & \text{if } x^2 + y^2 = 0 \end{cases}$$

EXERCISE 2.63

Suppose $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$, and their inverses are $\xi : Y \to X, \zeta : Z \to Y, \eta : Z \to X$ resp. if exists.

If f, g are invertible, then $(\xi \circ \zeta) \circ (g \circ f) = \xi \circ (\zeta \circ g) \circ f = \xi \circ f = \mathrm{Id}_X$. Besides, $(g \circ f) \circ (\xi \circ \zeta) = g \circ (f \circ \xi) \circ \zeta = g \circ \zeta = \mathrm{Id}_Z$. Thus $g \circ f$ is invertible with $(g \circ f)^{-1} = \xi \circ \zeta = f^{-1} \circ g^{-1}$.

If $f, g \circ f$ are invertible, then let $\zeta = f \circ \eta$. $\zeta \circ g = f \circ \eta \circ g = f \circ \eta \circ g \circ (f \circ \xi) = f \circ (\eta \circ (g \circ f)) \circ \xi = f \circ \xi = \mathrm{Id}_Y$. Besides, $g \circ \zeta = g \circ f \circ \eta = \mathrm{Id}_Z$. Thus g is invertible with its inverse being $g^{-1} = f \circ \eta$.

If $g, g \circ f$ are invertible, then let $\xi = \eta \circ g$. $\xi \circ f = \eta \circ g \circ f = \mathrm{Id}_X$. Besides, $f \circ \xi = f \circ \eta \circ g = (\zeta \circ g) \circ f \circ \eta \circ g = \zeta \circ (g \circ f \circ \eta) \circ g = \zeta \circ g = \mathrm{Id}_Y$. Thus f is invertible with its inverse being $f^{-1} = \eta \circ g$.

EXERCISE 2.66

1. NO.
2. NO.
3. NO.
4. YES.
5. NO.
6. NO.
7. YES.
8. NO.
9. NO.
10. NO.
11. YES.
12. NO.
13. NO.

EXERCISE 2.70

4. The equivalence classes are parametrized by $\lambda \in \mathbb{R}$ as $C_\lambda = \{(x, y) | x^2 + y = \lambda\}$.
7. The equivalence classes are parametrized by $\alpha \geqslant 0$ as $C_\alpha = \{(x, y) | (x-1)^2 + (y-1)^2 = \alpha\}$.
11. There is only one equivalence class which is $X = \mathbb{Q} - \{0\}$.

EXERCISE 2.69

Reflexivity: Since $x \sim_X x$ and $y \sim_Y y$, there is $(x, y) \sim (x, y)$.

Symmetry: Given $(x_1, y_1) \sim (x_2, y_2)$, there are $x_1 \sim_X x_2$ and $y_1 \sim_Y y_2$. Since $\sim_X$ and $\sim_Y$ are equivalence relations, we have $x_2 \sim_X x_1$ and $y_2 \sim_Y y_1$. By definition, $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity: Given $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, there are $x_1 \sim_X x_2$ and $y_1 \sim_Y y_2$, $x_2 \sim_X x_3$ and $y_2 \sim_Y y_3$, thus $x_1 \sim_X x_3$ and $y_1 \sim_Y y_3$. By definition, $(x_1, y_1) \sim (x_3, y_3)$.

Therefore, $\sim$ is an equivalence relation on $X \times Y$.

If the relation is defined by $x_1 \sim_X x_2$ or $y_1 \sim_Y y_2$, it would fail to have transitivity thus not an equivalence relation.

COUNTEREXAMPLE: Take $X = Y = \mathbb{R}$, $x \sim_X x'$ be $x = x'$ and $y \sim_Y y'$ be $y = y'$. Define $(x_1, y_1) \asymp (x_2, y_2)$ to be $x_1 \sim_X x_2$ or $y_1 \sim_Y y_2$. Then $(0, 0) \asymp (0, 1)$, $(0, 1) \asymp (1, 1)$, but $(0, 0) \not\asymp (1, 1)$.

EXERCISE 2.71

Suppose equivalence classes in X and Y are $\{C_\lambda\}_\lambda$ and $\{C'_\mu\}_\mu$ resp. Then the equivalence clasess under $\sim$ on $X \times Y$ are $\{C_\lambda \times C'_\mu\}_{\lambda,\mu}$.

EXERCISE 2.73

The partition is

$$\mathbb{R}^2 = \{0\} \sqcup \bigsqcup_{\theta \in [0, 2\pi)} L_\theta,$$

where $L_\theta = \{(r\cos\theta, r\sin\theta) | r > 0\}$.

The quotient is $S^1 \sqcup \{0\}$.

Exercise 2.78

$$X/\sim_1 \twoheadrightarrow X/\sim_2 = (X/\sim_1)/\sim_2.$$