§3.3 Cramer's Rule, Volume, and Linear Transformation

Thm: Cramer's Rule

Let A be an invertible $n\times n$ matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x}=\vec{b}$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i=1,2,\cdots,n.$$

A Formula for A^{-1}

Thm: Let A be an invertible $n\times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\boxed{\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}}$$

adj A (adjugate of A)
classical adjoint of A

Remark: This Theorem is useful mainly for theoretical calculation. The formula for A^{-1} permits one to deduce properties of the inverse without actually calculating it.

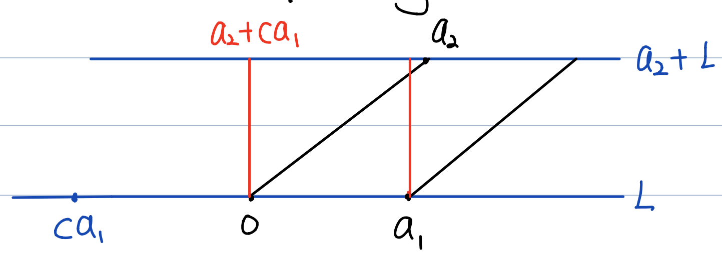
*** Determinants as Area or Volume**

Theorem: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Proof: The theorem is obviously true for any 2×2 diagonal matrix:

$$\left|\det\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right| = |ad| = \left\{\begin{matrix}\text{area of} \\ \text{rectangle}\end{matrix}\right\}$$

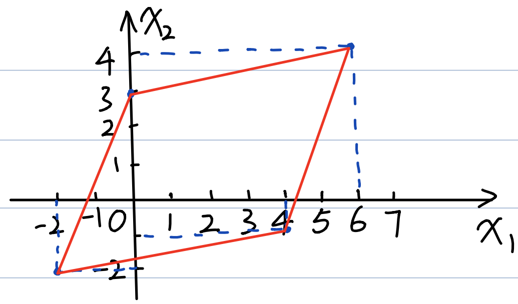
So it is suffice to show that any 2x2 matrix $A = \{\vec{a}_1, \vec{a}_2\}$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

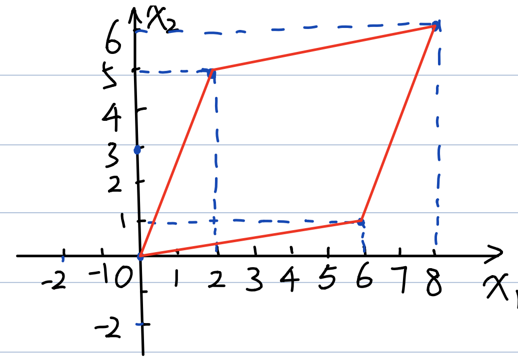


1) Let $\vec{a}_1$ and $\vec{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2$ equals the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2 + ca_1$. Column interchanges do not change the parallelogram at all.

2) The absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another. Such operation suffice to transform A into a diagonal matrix.

Example: Calculate the area of the parallelogram determined by the points $(-2,-2)$, $(0,3)$, $(4,-1)$ and $(6,4)$





Solution: First translate the parallelogram to one having the origin as a vertex.

We subtract the vertex $(-2,-2)$ from each of the four vertices. The new parallelogram has the same area, and its vertices are $(0,0)$, $(2,5)$, $(6,1)$ and $(8,6)$.

This parallelogram is determined by the columns of

$$A = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}$$

Since $|\det A| = |-28|$, the area of the parallelogram is 28.

* Linear Transformations

Theorem: Let $T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det(A)| \cdot \{\text{volume of } S\}$$

Proof: Consider the 2x2 case, with $B = [\vec{b}_1, \vec{b}_2]$. A parallelogram at the origin in $\mathbb{R}^2$ determined by vectors $\vec{b}_1$ and $\vec{b}_2$ has the form

$$S = \{s_1\vec{b}_1 + s_2\vec{b}_2 : 0 \le s_1 \le 1, 0 \le s_2 \le 1\}$$

Then the image of S under T consists of the form

$$T(s_1\vec{b}_1 + s_2\vec{b}_2) = s_1 T(\vec{b}_1) + s_2 T(\vec{b}_2)$$
$$= s_1 A b_1 + s_2 A b_2$$

where $0 \le s_1 \le 1$, $0 \le s_2 \le 1$.

Hence $T(S)$ is the parallelogram determined by the columns of the matrix $[Ab_1 \ \ Ab_2] = AB$, where $B = [b_1 \ \ b_2]$.

$$\{\text{area of } T(S)\} = |\det AB| = |\det A|\,|\det B|$$
$$= |\det A| \cdot \{\text{area of } S\} \qquad \cdots \quad (*)$$

An arbitrary parallelogram has the form $\vec{p} + S$, where $\vec{p}$ is a vector and S is a parallelogram at the origin.

$$\{\text{area of } T(\vec{p}+S)\} = \{\text{area of } T(\vec{p}) + T(S)\}$$
$$= \{\text{area of } T(S)\} \qquad \text{Translation}$$
$$= |\det A| \cdot \{\text{area of } S\} \qquad \text{By } (*)$$
$$= |\det A| \cdot \{\text{area of } \vec{p} + S\} \qquad \text{Translation}$$

Example: Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1$.

Solution: E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, beause if $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

and $\vec{x} = A\vec{u}$, then

$$u_1 = \frac{x_1}{a}, \quad u_2 = \frac{x_2}{b}.$$

$\vec{u}$ is in the unit disk if and only if $\vec{x}$ is in E.

$$\begin{aligned} \{\text{area of } E\} &= \{\text{area of } T(D)\} \\ &= |\det(A)| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi \cdot 1^2 = \pi ab \end{aligned}$$

Exercise: Let S be the parallelogram determined by the vectors $\vec{b}_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $A = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$.

Compute the area of the image of S under the mapping $\vec{x} \mapsto A\vec{x}$.

Solution:

$$\begin{aligned} \{\text{area of } S\} &= |\det A| \cdot \left| \det \begin{pmatrix} -2 & -2 \\ 3 & 5 \end{pmatrix} \right| \\ &= \left| [6 \cdot 2 - (-3) \cdot (-3)] \right| \cdot \left| [-2 \cdot 5 - 3 \cdot (-2)] \right| \\ &= |3 \cdot (-4)| \\ &= 12. \end{aligned}$$

§4.1 Vector Spaces and Subspaces

$\mathbb{R}^n$: vectors in $\mathbb{R}^n$ satisfy some properties

(1) we can add two vectors: $\vec{v}+\vec{u}$ for $\vec{v}, \vec{u}$

(2) we can multiply a vector $\vec{v}$ by a number to get another vector: $a\cdot\vec{v}$.

These two operations "+", "$\cdot$" satisfy some rules.

Ex: $P_n = \{$polynomials of one variable x of degree at most $n\}$.

Any element $f(x) \in P_n$ is of the form:

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

For any two elements $f(x), g(x)$ in P_n, we can add them to get a new element $f+g$ in P_n:

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

$$(f+g)(x) = f(x) + g(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n.$$

For any element $f(x)$ in P_n, we can multiply $f(x)$ by a scalar k to get a new element kf in P_n

$$(kf)(x) = k\cdot f(x) = ka_0 + ka_1 x + \cdots + ka_n x^n.$$

Therefore, P_n has two operations "+", "$\cdot$" addition and scalar multiplication. These operations satisfy the same rules as those on $\mathbb{R}^n$.

Ex: $M(m\times n) = \{m\times n$ matrices$\}$

For two elements $M, N \in M(m,n)$, we can add

them to get a new $m \times n$ matrix $M+N$.
We can also multiply M by a scalar k to get a new $m \times n$ matrix kM.
These operations: addition "$+$" and scalar multiplication "$\cdot$" satisfy the same rules as those on $\mathbb{R}^n$.

Now we see $\mathbb{R}^n$, P_n and $M(m \times n)$, even though they look totally different, but they share some properties. We call such spaces with these properties "vector space".

Def: A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in V and for all scalars c and d.

1. The sum of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}+\vec{v}$, is in V.
2. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
3. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
4. There is a zero vector $\vec{0}$ in V such that $\vec{u}+\vec{0} = \vec{u}$.

5. For each $\vec{u}$ in V, there is a vector $-\vec{u}$ in V such that $\vec{u} + (-\vec{u}) = \vec{0}$.

6. The scalar multiple of $\vec{u}$ by c, denoted by $c\vec{u}$, is in V.

7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

9. $c(d\vec{u}) = (cd)\vec{u}$

10. $1 \cdot \vec{u} = \vec{u}$

Ex: Let V be the set of all real-valued functions defined on a set D.
For instance, if $D = \mathbb{R}$, $f(t) = 1+\sin 2t$, $g(t) = 2 + \frac{t}{2}$, then
$(f+g)(t) = 3 + \sin 2t + \frac{t}{2}$ and $(2g)(t) = 4 + t$.

Facts: For each $\vec{u}$ in V and scalar c,

1) $0\vec{u} = \vec{0}$
2) $c \cdot \vec{0} = \vec{0}$
3) $-\vec{u} = (-1)\vec{u}$

Proof: 1) $0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$ By axiom 8

$0\vec{u} + (-0\vec{u}) = [0\vec{u} + 0\vec{u}] + (-0\vec{u})$

$0\vec{u} + (-0\vec{u}) = 0\vec{u} + (0\vec{u} + (-0\vec{u}))$ By axiom 3

$\vec{0} = 0\vec{u} + \vec{0}$ By axiom 5

$0 = 0\vec{u}$ By axiom 4

2) $c\vec{0} = c(\vec{0} + \vec{0})$ By axiom 4

$= c\vec{0} + c\vec{0}$ By axiom 7

$c\vec{0} + (-c\vec{0}) = [c\vec{0} + c\vec{0}] + (-c\vec{0})$

$c\vec{0} + (-c\vec{0}) = c\vec{0} + (c\vec{0} + (-c\vec{0}))$ By axiom 3

$\vec{0} = c\vec{0} + \vec{0}$ By axiom 5

$\vec{0} = c\vec{0}$ By axiom 4

3) $(-1)\vec{u} + \vec{u} = (1 + (-1))\vec{u} = 0\vec{u} = \vec{0}$

$(-1)\vec{u} + \vec{u} + (-\vec{u}) = \vec{0} + (-\vec{u})$

$(-1)\vec{u} + [\vec{u} + (-\vec{u})] = \vec{0} + (-\vec{u})$

$(-1)\vec{u} + \vec{0} = -\vec{u}$

$(-1)\vec{u} = -\vec{u}$