MATH2111 Tutorial 3

QUAN Xueyang (xquan@connect.ust.hk)
T1A: Tue (18:00-18:50) 2465
T1B: Thu (18:00-18:50) 6591

1 The Matrix Equation

1. **Matrix-Vector Products**. We can multiply an $m \times n$ matrix $\boldsymbol{A}$ by a vector $\boldsymbol{v} \in \mathbb{R}^n$. The result, written $\boldsymbol{A}\boldsymbol{v}$, belongs to $\mathbb{R}^m$. If $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n \in \mathbb{R}^m$ are the columns of $\boldsymbol{A}$ and $v_1, v_2, \cdots, v_n \in \mathbb{R}$ are the entries of $\boldsymbol{v}$, then

$$\boldsymbol{A}\boldsymbol{v} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \cdots + v_n\boldsymbol{a}_n$$

2. **Property of Matrix-Vector Products**. If $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then:

   (a) $\boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{v} \in \mathbb{R}^m$

   (b) $\boldsymbol{A}(c\boldsymbol{v}) = c(\boldsymbol{A}\boldsymbol{v}) \in \mathbb{R}^m$

3. **Theorem**. If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and if $\boldsymbol{b}$ is in $\boldsymbol{R}^m$, the matrix equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

   has the same solution set as the vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

   which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \mid \boldsymbol{b} \end{bmatrix}$$

Existence of solutions to linear systems:

4. **Theorem**. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then the following statements are logically equivalent:

   (☆)

   (a) For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

   (b) Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $\boldsymbol{A}$.

   (c) The columns of $\boldsymbol{A}$ span $\mathbb{R}^m$. i.e. Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = \mathbb{R}^m$

   (d) A has a pivot position in every row.

   Warning: The above theorem is about a coefficient matrix, not an augmented matrix. If an augmented matrix $[\boldsymbol{A} \mid \boldsymbol{b}]$ has a pivot position in every row, then the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ may or may not be consistent.

Example 1.1. Could a set of n vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$ if $n < m$? Explain. Can't

at least $m \le n$

$\{v_1, v_2, \ldots, v_n\}$, $v_i \in \mathbb{R}^m$

$V = [v_1 \ v_2 \ \ldots \ v_n] \in \mathbb{R}^{m\times n}$, $m > n$, m n, V can't have a pivot position in every row.
By thm, $\{v_1, \ldots, v_n\}$ can't span $\mathbb{R}^m$

2 Solution Sets of Linear Systems

1. **Homogeneous Linear Systems**. A system of linear equations is said to be **homogeneous** if it can be written in the form
$$\boldsymbol{Ax} = \boldsymbol{0}$$
where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{0}$ is the zero vector in $\mathbb{R}^m$.

   Note:
   The system $\boldsymbol{Ax} = \boldsymbol{0}$ always has at least one solution, namely, $\boldsymbol{x} = \boldsymbol{0}$ (the zero vector in $\mathbb{R}^n$), and
   (a) this zero solution is called the **trivial solution**.
   (b) the other non-zero solution are called the **nontrivial solution**.

2. **Theorem**. The homogeneous equation $\boldsymbol{Ax} = \boldsymbol{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

3. **Theorem**. Suppose $\boldsymbol{A}$ has k free columns, then the homogeneous equation $\boldsymbol{Ax} = \boldsymbol{0}$ has k free variables, and the general solution can be written as **parametric vector form**
$$\boldsymbol{x} = s_1\boldsymbol{x}_1 + s_2\boldsymbol{x}_2 + \ldots + s_k\boldsymbol{x}_k$$
In other words, the solution set of the homogeneous system is
$$\text{Span}\,\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$$
Since Span$\{\vec{0}\} = \{\vec{0}\}$

   Note: If there are no non-pivot columns (i.e. no free variables), the solution set is just $\{\boldsymbol{0}\}$. (because $x_1\vec{0} = \vec{0}\ \forall x$)

4. **Non-Homogeneous Linear Systems**. Suppose the equation $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent for some given $\boldsymbol{b}$ ($\boldsymbol{b} \neq \boldsymbol{0}$), and let $\boldsymbol{p}$ be a particular solution. Then the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is the set of all vectors of the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is any solution of the homogeneous equation $\boldsymbol{Ax} = \boldsymbol{0}$.

5. **Procedures of Writing a Solution Set (of a consistent system) in Parametric Vector Form**

   (a) Row reduce the augmented matrix to reduced row echelon form (RREF).
   (b) Express each basic variable in terms of any free variables appearing in an equation.
   (c) Write a typical solution $\boldsymbol{x}$ as a vector whose entries depend on the free variables, if any.
   (d) Decompose $\boldsymbol{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

3 Exercises

1. Write the matrix equation as a vector equation, or vice versa.

(a) $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

2. Suppose $\boldsymbol{A}$ is a 3×3 matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$ with the property that $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution. Explain why the columns of $\boldsymbol{A}$ must span $\mathbb{R}^3$.

1. (a) $A\vec{x} = \vec{b}$, $[\vec{a_1} \ \vec{a_2} \ \vec{a_3} \ \vec{a_4}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{b}$

$\Rightarrow x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3} + x_4\vec{a_4} = \vec{b}$

$\therefore \ 5 \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3 \cdot \begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

2. $A\vec{x} = \vec{b}$ has a unique solution,

RREF of A should be $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$.

Since $A \in \mathbb{R}^{3\times3}$, A has a pivot position for each row,

By theorem (A), columns of A span $\mathbb{R}^3$.

3. Determine if the columns of the matrix span $\mathbb{R}^4$

$$\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$$

3.

$$\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 - \frac{4}{5}R_1 \to R_3 \\ R_4 + \frac{9}{5}R_1 \to R_4}]{R_2 - \frac{6}{5}R_1 \to R_2} \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & \frac{2}{5} & -\frac{11}{5} & -\frac{29}{5} \\ 0 & \frac{8}{5} & -\frac{29}{5} & -\frac{81}{5} \\ 0 & -\frac{8}{5} & \frac{44}{5} & \frac{116}{5} \end{bmatrix}$$

$$\xrightarrow[R_4 + 4R_2 \to R_4]{R_3 - 4R_2 \to R_3} \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & \frac{2}{5} & -\frac{11}{5} & -\frac{29}{5} \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A only has 3 pivot columns, by theorem (4),

colums of A don't span $\mathbb{R}^4$.

4. Determine if the system has a nontrivial solution.

(1) $\begin{cases} 2x_1 - 5x_2 + 8x_3 = 0 \\ -2x_1 - 7x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + 7x_3 = 0 \end{cases}$

(2) $\begin{cases} x_1 - 3x_2 + 7x_3 = 0 \\ -2x_1 + x_2 - 4x_3 = 0 \\ x_1 + 2x_2 + 9x_3 = 0 \end{cases}$

(1) $$\left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right] \xrightarrow[R_3-2R_1 \to R_3]{R_2+R_1 \to R_2} \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[R_3+R_2 \to R_3]{} \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free, (x_1, x_2 are basic variables), the system has a nontrivial solution.

(2) $$\left[\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right] \xrightarrow[R_3-R_1 \to R_3]{R_2+2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{array}\right] \xrightarrow{R_3+R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right]$$

No free variables, the system has no nontrivial solution.

$A\vec{x} = \vec{0}$, only $\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5. Describe all solutions of $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ in parametric vector form.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A \mid \vec{b}] = \left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 + 8R_3 \to R_2} \left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{} \left[\begin{array}{cccccc|c} 1 & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF.}$$

x_2, x_4, x_5 free

x_1, x_3, x_6 basic

Thus, $\begin{cases} x_1 + 5x_2 + 8x_4 + x_5 = 0 \\ x_3 - 7x_4 + 4x_5 = 0 \\ x_6 = 0 \\ 0 = 0 \end{cases}$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

x_2, x_4, x_5 are arbitrary.

6. (1) Suppose $\boldsymbol{w}$, $\boldsymbol{p}$ are two solutions of the equation $\boldsymbol{Ax} = \boldsymbol{b}$ and define $\boldsymbol{v}_h = \boldsymbol{w} - \boldsymbol{p}$.Show that $\boldsymbol{v}_h$ is a solution of $\boldsymbol{Ax} = \boldsymbol{0}$.
(2) Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution. Explain why the solution is unique precisely when $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

(1) $A\vec{w} = \vec{b}$, $A\vec{p} = \vec{b}$, $\vec{v}_h = \vec{w} - \vec{p}$

$\therefore A\vec{v}_h = A(\vec{w} - \vec{p}) = A\vec{w} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$

$\therefore \vec{v}_h$ is a solution of $A\vec{x} = \vec{0}$.

(2) Suppose $A\vec{x} = \vec{b}$ has two solutions $\vec{u}, \vec{v}$,

so, $A\vec{u} = \vec{b}$, $A\vec{v} = \vec{b}$

Then, by linearity, $A(\vec{u} - \vec{v}) = A\vec{u} - A\vec{v} = \vec{b} - \vec{b} = \vec{0}$

Since $A\vec{x} = \vec{0}$ has only trivial solution, which means $\vec{u} - \vec{v} = \vec{0}$, thus, $\vec{u} = \vec{v}$