MATH 2111: Tutorial 4 Linear Independence and Introduction to Linear Transformations

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Review

- Linear independence/dependence
- Linear independence of columns of matrix
- Characterization of linearly dependent sets
- Conditions for linear dependence
- Transformation

Example 1(a)

Express the general solutions of the following non-homogeneous systems in terms of the given particular solutions.

$$\begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 5 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 4 \\ -x_1 - x_2 + x_3 - x_5 = -1 \end{cases}$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the above linear system.

1(a). For $A\vec{x}=\vec{b}$, let $\vec{p}$ be a particular solution,

$\vec{x}_h$ is any solution of $A\vec{x}=\vec{0}$

Then solution set of $Ax=b$ is of form: $\vec{w}=\vec{p}+\vec{x}_h$

① Solve $A\vec{x}=\vec{0}$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 2 & 2 & 0 & 2 & 4 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[\substack{R_2-2R_1\to R_2 \\ R_3+R_1\to R_3}]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & -6 & -6 & -2 & 0 \\ 0 & 0 & 4 & 4 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-\frac{1}{6}R_2\to R_2 \\ \frac{1}{4}R_3\to R_3}]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow[R_3-R_2\to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \end{array}\right]$$

$$\xrightarrow[6R_3\to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 3 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1-3R_3\to R_1]{R_2-\frac{1}{3}R_3\to R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-3R_2\to R_1} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \leftarrow \text{RREF}$$

x_2, x_4 are free.

Thus $\begin{cases} x_1+x_2+x_4=0 \\ x_3+x_4=0 \\ x_5=0 \end{cases}$

$\therefore$

$$\vec{x}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 & -x_4 \\ x_2 & \\ & -x_4 \\ & x_4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_4 \qquad x_2, x_4 \in \mathbb{R}$$

② Check $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$

Thus, solution set for $A\vec{x} = \vec{b}$ is

$$\vec{w} = \vec{p} + \vec{x}_h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_4 \qquad x_4, x_5 \in \mathbb{R}$$

Example 1(b)

Denote the coefficient matrix as $\boldsymbol{A}$. Use as many columns of $\boldsymbol{A}$ as possible to construct a matrix $\boldsymbol{B}$ with the property that the equation $\boldsymbol{Bx} = 0$ has only the trivial solution. (Solve $\boldsymbol{Bx} = 0$ to verify your work.)

1(b). According to the RREF of A,

the first three columns are pivot columns

If use the corresponding columns of A to form B,

we have:

$$B = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 4 \\ -1 & 1 & -1 \end{bmatrix},$$

If do row reduction on $[B|\vec{0}]$, we get $\left[\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{array}\right]$.

Since $B\vec{x}=\vec{0}$ has no free variable, it has only trivial solution.

If use any additional columns of A to form B, then $B\vec{x}=\vec{0}$

will have free variable, so $B\vec{x}=\vec{0}$ has nontrivial solution.

Example 2

Find conditions on p and q such that the set of vectors

$$\left\{ \underset{\vec{a}_1}{\underset{\parallel}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}}}, \underset{\vec{a}_2}{\underset{\parallel}{\begin{bmatrix} 2 \\ 5 \\ 5 \\ 1 \end{bmatrix}}}, \underset{\vec{a}_3}{\underset{\parallel}{\begin{bmatrix} 1 \\ 2 \\ 4 \\ p \end{bmatrix}}}, \underset{\vec{a}_4}{\underset{\parallel}{\begin{bmatrix} 3 \\ 8 \\ 9 \\ q \end{bmatrix}}} \right\}$$

is linearly independent.

2. Denote

$$A=\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \\ 0 & 1 & p & q \end{bmatrix}, \qquad x_1\vec{a_1}+x_2\vec{a_2}+x_3\vec{a_3}+x_4\vec{a_4}=0 \iff A\vec{x}=\vec{0}$$

By thm, we need to gaurantee $A\vec{x}=\vec{0}$ has only trivial solution.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 5 & 2 & 8 & 0 \\ 3 & 5 & 4 & 9 & 0 \\ 0 & 1 & p & q & 0 \end{array}\right]$$

$$\xrightarrow[R_3-3R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & p & q & 0 \end{array}\right]$$

$$\xrightarrow[R_4-R_2\to R_4]{R_3+R_2\to R_3} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & p & q-2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4-pR_3\to R_4]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & q-2-2p & 0 \end{array}\right] \leftarrow \text{REF}$$

The matrix already has pivots in columns 1,2,3, to have a pivot in column 4, we must have $q-2-2p\neq 0$.

Consider matrix $\boldsymbol{A}$,

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 2 & 8 \\ 3 & 5 & 4 & 9 \end{bmatrix},$$

Find a vector which is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and also in Span $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$, or explain why such a vector cannot exist.

(Given $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $\boldsymbol{Ax} = 0$.)

3. $\vec{v} \in \text{Span}\{\vec{a_1}, \vec{a_2}\}$ if $\vec{v} = c_1\vec{a_1} + c_2\vec{a_2}$ for some c_1, c_2.

$\vec{v} \in \text{Span}\{\vec{a_3}, \vec{a_4}\}$ if $\vec{v} = c_3\vec{a} + c_4\vec{a}$ for some c_3, c_4.

$\therefore c_1\vec{a_1} + c_2\vec{a_2} = \vec{v} = c_3\vec{a_3} + c_4\vec{a_4}$

$\therefore c_1\vec{a_1} + c_2\vec{a_2} - c_3\vec{a_3} - c_4\vec{a_4} = 0$

$\therefore A\begin{bmatrix} c_1 \\ c_2 \\ -c_3 \\ -c_4 \end{bmatrix} = \vec{0}$

Since $\begin{bmatrix} 3 \\ -2 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.

$\therefore$ take $c_1 = 3$, $c_2 = -2$, $c_3 = 2$, $c_4 = -1$

$\therefore \vec{v} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -1 \end{bmatrix}$, which is also $2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$

Example 4

State whether each of the following statement is true or false. (If it is true, give a brief justification; if it is false, give a counterexample.)

(a) If $\boldsymbol{A}\begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = 0$, then $\boldsymbol{A}\boldsymbol{e}_4$ is a linear combination of the first three columns of $\boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a 4×3 matrix with columns $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and suppose $\boldsymbol{b}$ is a vector in $\mathbb{R}^4$ such that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{b}\}$ is linearly dependent. Then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

4 (a). True.

Let $\vec{a_i}$ be the i-th column of A.

$A\begin{bmatrix}4\\0\\2\\-3\end{bmatrix} = \vec{0}$ means $4\vec{a_1} + 2\vec{a_3} - 3\vec{a_4} = \vec{0}$,

$\therefore A\vec{e_4} = \vec{a_4} = \frac{4}{3}\vec{a_1} + \frac{2}{3}\vec{a_3}$.

$\vec{e_i} := \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \leftarrow$ ith row.

4(b). False.

Take $\vec{a_1} = \vec{e_1}$, $\vec{a_2} = \vec{e_2}$, $\vec{a_3} = \vec{e_1} + \vec{e_2}$, and $\vec{b} = \vec{e_3}$

Easy to check these 4 vectors are linearly dependent ($\vec{a_3} = \vec{a_1} + \vec{a_2}$).

However, $\vec{b}$ can't be expressed as a linear combination of $\vec{a_1}$, $\vec{a_2}$ and $\vec{a_3}$.

Remark: linear dependence only implies some vector (NOT necessarily all) can be expressed as a linear combination of rest vectors.

Example 5

Consider

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ 0 \\ 3x_1 - x_2 \end{bmatrix}$$

(a) What is the domain of F?

(b) Find the image of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ under F.

5(a). domain: $\mathbb{R}^3$

(b). $F\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1+0\\0\\0\\3\times 1-0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\3\end{bmatrix}$