COMP 2711 Discrete Math Tools for Computer Science

2022 Fall Semester - Homework 6

Review: We learned about several special types of graphs: complete graphs K_n, cycles C_n, bipartite graphs (denoted as $G_{(b)}$ here), and complete bipartite graphs $K_{m,n}$. Recall the definitions:

K_n For $V = \{v_1, v_2, \cdots, v_n\}$ $(n \geq 1)$, there is exactly one edge between every pair of vertices in V. K_1 is a single vertex and K_2 is two vertices connected by an edge.

C_n For $V = \{v_1, v_2, \cdots, v_n\}$ $(n \geq 3)$, there is exactly one edge between v_i and v_{i+1} for all $1 \leq i \leq n$, plus exactly one edge from v_n to v_1.

$G_{(b)}$ For $V = \{v_1, v_2, \cdots, v_n\}$ $(n \geq 2)$, it can be partitioned into two disjoint subsets V_1 and V_2 such that $(V_1 \cap V_2 = \varnothing) \wedge (V_1 \cup V_2 = V)$. Every edge connects u_i in V_1 and v_j in V_2.

$K_{m,n}$ For every vertex u_i in $U = \{u_1, u_2, \cdots, u_m\}$, and v_j in $V = \{v_1, v_2, \cdots, v_n\}$ $(m \geq 1, n \geq 1)$, there is exactly one edge connecting u_i and v_j. There are no edges between two vertices in U, and no edges between two vertices in V.

Question 1:
(a) Draw a complete graph K_7.

(b) Draw a bipartite graph $G_{(b)}$ which is not a complete bipartite graph.

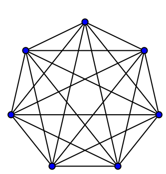
(c) Represent your bipartite graph in (b) by adjacency matrix. Please label the vertices in (b) and declare the order in (c).

(d) Can a complete graph K_n be bipartite? Explain what conditions n must satisfy if it is possible.

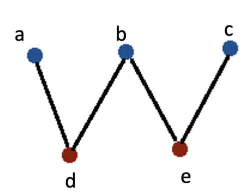
(e) Can a cycle C_n be bipartite? Explain what conditions n must satisfy if it is possible.

(f) Can a complete bipartite graph $K_{m,n}$ have an Euler path but not an Euler circuit? Explain what conditions m and n must satisfy if it is possible.

Answer: (a)



(b)



(c) $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

(d) Yes, only if $n = 2$: the graph with two vertices and one edge. Consider the colouring method to determine if a graph is bipartite. If $n > 2$, then if we mark a vertex as red, all other vertices will be marked as blue, and they will be neighbours of each other, so the colouring contradicts the requirement for bipartite graphs.

(e) Yes, when n is even. Using the colouring method, if vertex 1 is red, then 2 is blue, 3 is red, and so on. Vertex n is blue if n is even, in which case there is no contradiction (all red vertices have blue neighbours, all blue vertices have red neighbours). Vertex n is red if n is odd, but it is a neighbour of vertex 1, so there is a contradiction; no colouring is possible.

(f) For a bipartite graph, the vertices in U all have degree n and the vertices in V all have degree m. We know that there exists an Euler path but not an Euler circuit if there are exactly two vertices of odd degree and all other vertices have even degree. For such a scenario to occur, this means that either $m = 2$ and n is odd, or $n = 2$ and m is odd. So it is possible.

Question 2: Show that a directed multigraph (graphs that may have multiple edges connecting the same vertices) having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

Answer: First suppose that the directed multigraph has an Euler circuit. Since this circuit provides a path from every vertex to every other vertex, the graph must be strongly connected (and hence also weakly connected). Also, we can count the in-degrees and out-degrees of the vertices by following this circuit; as the circuit passes through a vertex, it adds one to the count of both the in-degree (as it comes in) and the out-degree (as it leaves). Therefore the two degrees are equal for each vertex.

Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of the Euler Circuit Theorem and construct an Euler circuit.