HKUST

MATH 2111 Matrix Algebra and Applications

2022-23 Sample Final Examination

Name:________________________

Student ID:________________________

Lecture Section:________________________

Directions:

- This is a closed book examination. No Calculator is allowed in this examination.
- DO NOT open the exam booklet until you are instructed to do so.
- DO NOT detach any pages from this exam booklet, except the last two blank sheets.
- Please switch all mobile phones to silent mode. And all electronic communication devices (e.g. laptops, tablets, smart watches, etc.) must be kept away from your body.
- Please write your name, ID number, and lecture section in the space provided above.
- DO NOT use any of your own scratch paper. Write your name on every scratch paper supplied by the examination, and do not take any scratch paper away after the examination.
- When instructed to open the exam booklet, please check that you have **14** pages of **7** questions.
- **Answer all questions**. Show an appropriate amount of work for each problem. If you do not show enough work, you will get only partial credit.
- You may write on the backside of the pages, but if you use the backside, clearly indicate that you have done so.
- Please have pens, papers and, in particular, your student ID ready. We will check your ID during the exam.

Please read the following statement and sign your signature.

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature :

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Question No.	Marks	Out of
Qn. 1		20
Qn. 2		10
Qn. 3		10
Qn. 4		15
Qn. 5		15
Qn. 6		15
Qn. 7		15
Total Marks		100

Qn. 1 (20 marks) Choose a correct option for each question. No justification is required. Each correct answer is worth 2 marks.

(1) Let $A\mathbf{x} = \mathbf{b}$ be a linear system with 20 equations, 15 variables, and 10 basic variables. Then dimRow$A$ is:

(A) 0 (B) 5 (C) 10 (D) 15 (E) 20

(2) Let A be a $p \times q$ matrix with rank $A = q$. Consider the statements:

(I) $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one transformation.
(II) $\mathbf{x} \mapsto A\mathbf{x}$ is an onto transformation.
(III) $\mathbf{x} \mapsto A^T\mathbf{x}$ is a one-to-one transformation.
(IV) $\mathbf{x} \mapsto A^T\mathbf{x}$ is an onto transformation.

The correct statements are:

(A) I, III only (B) I, IV only (C) II, III only (D) II, IV only (E) I, II, III, IV

3) Let A, B, E be $n \times n$ matrices and let E be invertible. Consider the relations:

(I) $A = EB$ (II) $A = BE$ (III) $EA = B$ (IV) $AE = B$

The relation(s) that guarantee(s) A being row-equivalent to B is/are:

(A) I only (B) II only (C) I, III only (D) II, IV only (E) I, II, III, IV

(4) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent set in $\mathbb{R}^3$. Which of the following 3×3 matrices have zero determinant?

(A) $[2\mathbf{v}_1 \quad 3\mathbf{v}_2 \quad 4\mathbf{v}_3]$
(B) $[\mathbf{v}_1 \quad \mathbf{v}_1 + \mathbf{v}_2 \quad \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3]$
(C) $[\mathbf{v}_1 + \mathbf{v}_2 \quad \mathbf{v}_2 + \mathbf{v}_3 \quad \mathbf{v}_3 + \mathbf{v}_1]$
(D) $[\mathbf{v}_1 - \mathbf{v}_2 \quad \mathbf{v}_2 - \mathbf{v}_3 \quad \mathbf{v}_3 - \mathbf{v}_1]$
(E) None of the above.

(5) Which of the following sets is a subspace of $M_{3\times3}$ (the vector space of 3×3 matrices)?

(A) $S = \{A \in M_{3\times3} :$ Nul$A$ contains $\mathbf{e}_1\}$.
(B) $S = \{A \in M_{3\times3} : \det A = 0\}$.
(C) $S = \{A \in M_{3\times3} : A^TA = I_3\}$.
(D) $S = \{A \in M_{3\times3} : A$ is diagonalizable$\}$.
(E) None of the above.

(6) Let A be an $m \times n$ matrix with rank$A = n < m$. Which of the following statements is correct?

(A) dim Row $A >$ dim Col A.

(B) dim Row $A >$ dim Nul A.

(C) dim Nul $A >$ dim Col A.

(D) dim Nul $A =$ dim Col A.

(E) None of the above.

(7) Let A be an $n \times n$ matrix. If 0 is an eigenvalue of A, then which of the following subspaces, if non-zero, must be an eigenspace of A?

(A) $(\text{Row } A)^{\perp}$ (B) Row A (C) $(\text{Col } A)^{\perp}$ (D) Col A (E) None of the pervious.

(8) Let A, B be an $n \times n$ matrix similar to each other. Which of the following statements is INCORRECT?

(A) A, B have the same determinant.

(B) A, B have the same rank.

(C) A, B have the same nullity.

(D) A, B have the same collection of eigenvalues.

(E) A, B have the same collection of eigenvectors.

(9) Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{u} \in \mathbb{R}^n$. Consider the statements:

(I) $\text{proj}_W \mathbf{u} \perp (\mathbf{u} - \text{proj}_W \mathbf{u})$.

(II) $\text{proj}_{W^\perp} \mathbf{u} \perp (\mathbf{u} - \text{proj}_{(W^\perp)} \mathbf{u})$

(III) $\text{proj}_W \mathbf{u} \perp \text{proj}_{W^\perp} \mathbf{u}$

(IV) $(\mathbf{u} - \text{proj}_W \mathbf{u}) \perp (\mathbf{u} - \text{proj}_{(W^\perp)} \mathbf{u})$

The correct statements are:

(A) I, II, III only (B) I, II, IV only (C) I, III, IV only (D) II, III, IV only (E) I, II, III, IV

(10) Let A be an $m \times n$ matrix and let $\mathbf{v}$ be the orthogonal projection of a vector $\mathbf{u} \in \mathbb{R}^n$ onto Col$A$. Which of the followings is correct?

(A) $A^T\mathbf{u} = \mathbf{0}$.

(B) $A^T\mathbf{v} = \mathbf{0}$.

(C) $A^T(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

(D) $A^T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$.

(E) None of the above.

Qn. 2 (10 marks) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Find the matrix X such that $AXA^{-1} = B$.

Qn. 3 (10 marks) Consider a linear system $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Check that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(b) Find a least squares solution $\mathbf{x}_0$ to the system $A\mathbf{x} = \mathbf{b}$.

(c) Find the distance of $\mathbf{b}$ to Col$A$.

Qn. 4(15 marks) Consider $\mathbb{P}_2$, the vector space of polynomials with degree at most 2. Let:

$$\mathcal{B} = \{1+t, t+t^2, t^2+1\}, \qquad p(t) = 1+t+t^2, \qquad q(t) = 2+t-t^2.$$

(a) (4 marks) Verify that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) (6 markds) Find the coordinate vectors of $p(t), q(t)$ relative to basis $\mathcal{B}$.

(c) (2 marks) Let $[r(t)]_{\mathcal{B}} = [1\ 2\ 1]^T$. Find the polynomial $r(t)$.

(d) (3 marks) Does the $r(t)$ in (c) belong to $\text{Span}\{p(t), q(t)\}$? Why or Why not?

Qn. 5 (15 marks) Let:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

(a) (10 marks) Diagonalize A, namely, find invertible matrices P, P^{-1} and a diagonal matrix D such that $A = PDP^{-1}$.

(b) (5 marks) Find a general formula of A^n.

Qn. 6 (15 marks) Let:

$$A = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

(a) (4 marks) Show that $\mathbf{v}, \mathbf{w}$ are both eigenvectors of A. Write down also their eigenvalues.

(b) (8 marks) Find an orthonormal basis for the eigenspace of A containing (i) $\mathbf{v}$ (ii) $\mathbf{w}$ respectively. [Note: your orthonormal basis should start with the vector $\mathbf{v}$ (or $\mathbf{w}$).]

(c) (3 marks) Find an orthogonal matrix P such that $D = P^T AP$ is a diagonal matrix. Write down also the diagonal matrix D.

Qn. 7 (15 marks) Let:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}, \quad U = \mathrm{Row} A, \quad W = \mathrm{Nul} A, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(a) (6 marks) Find $\mathrm{proj}_U \mathbf{v}$.

(b) (4 marks) Find $\mathrm{proj}_W \mathbf{v}$.

(c) (5 marks) Let B denote the standard matrix of the orthogonal projection transformation proj_U, and let C denote the standard matrix of the orthogonal projection transformation $proj_W$. Find $B + C$.