Review:

read as "f prime".

1. The derivative of $y=f(x)$: $f'(x) \overset{\text{defined by}}{=\!=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ (if exists).

$f'(x)$ is also a function of x, and its domain is the set $\{x \mid \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists}\}$

$f'(x)$: y', $\frac{d}{dx}y$, $\frac{dy}{dx}$, $\frac{d}{dx}f$, $\frac{df}{dx}$, Dy, Df.

$f'(a)$: $f'(x)\Big|_{x=a}$, $y'\Big|_{x=a}$, $\frac{dy}{dx}\Big|_{x=a}$, $\frac{df}{dx}\Big|_{x=a}$, $Df\Big|_{x=a}$.

Notice: (1) $f'(a)$ exists $\Longleftrightarrow$ f is differentiable at $x=a$

(2) $f'(a)$ (if exists) $=$ the slope of the tangent line at $x=a$.

$=$ the rate of change of f at $x=a$.

3. Computation of derivatives

(1) Compute $f'(x)$ by definition : $f'(x) \overset{\text{defined by}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Example: $f(x) = x^2$ $\quad f'(x) = 2x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

(2). Compute $f'(x)$ by rules of differentiation. $f', g' \Rightarrow (f\pm g)', c\cdot f', (f\cdot g)', (\frac{f}{g})'$

① $(f\pm g)' = f' \pm g'$. ② $(c\cdot f)' = c\cdot f'$ (c is a constant)

③ $(f\cdot g)' = f'\cdot g + f\cdot g'$. ④ $(\frac{f}{g})' = \frac{f'\cdot g - f\cdot g'}{g^2}$.

Example: $h(x) = \frac{1}{x^2}$. $\Rightarrow h(x) = \frac{f(x)}{g(x)}$. where $f(x) = 1$. $g(x) = x^2$

$$h'(x) = \frac{f'(x)g(x) - f(x)\cdot g'(x)}{g^2(x)} = \frac{0\cdot g(x) - f(x)\cdot 2x}{g^2(x)} = \frac{-2x}{(x^2)^2} = \frac{-2}{x^3}$$

Quotient Rule ④

Review some notable derivatives :

a constant.

$\frac{d}{dx} C = 0$ $\qquad$ $\frac{d}{dx} \cdot x = 1$, $\qquad$ $\frac{d}{dx} x^2 = 2x$.

$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$. $\qquad$ $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$ $\qquad$ $\frac{d}{dx} \frac{1}{x^2} = \frac{-2}{x^3}$.

In general, we have $\frac{d}{dx} \cdot x^n = n \cdot x^{n-1}$. (n can be any real number).

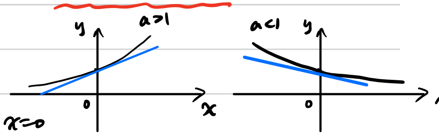
$\frac{d}{dx} \sin x = \cos x$. $\qquad$ $\frac{d}{dx} \cos x = -\sin x$. $\qquad$ $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$

by definition of derivatives $\qquad$ by the Quotient rule ④

Example : $y=f(x)=a^x$ How to calculate $f'(x)$? $\boxed{(a^x)'=\ln a \cdot a^x. \quad (e^x)'=e^x.}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = a^x\cdot \underline{\lim_{h\to 0}\frac{a^h-1}{h}} = \ln a \cdot a^x.$$

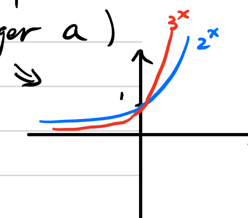
Aim to calculate $\lim_{h\to 0}\frac{a^h-1}{h}$. $\quad \lim_{h\to 0}\frac{a^h-1}{h} = \ln a.$



We can observe that : $\rightarrow$ rate of change of a^x at $x=0$

① $\lim_{h\to 0}\frac{a^h-1}{h} = f'(0)$ exists. ② $f'(0)$ is increasing in a when $a>1$. (a^x grows faster at $x=0$ for larger a)

③ $\lim_{h\to 0}\frac{2^h-1}{h} \approx 0.69$, $\lim_{h\to 0}\frac{3^h-1}{h} \approx 1.10$.



$\Rightarrow$ There exists a unique number $a\in(2,3)$ such that $\lim_{h\to 0}\frac{a^h-1}{h} = 1$.

$\Rightarrow$ We use "e" to denote this number: $\boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1.}$ $\rightarrow$ one of the definitions of e.

$e = 2.71828\cdots$ (irrational number)

Now we can calculate $\lim_{h\to 0}\frac{a^h-1}{h}$ in terms of e and a.

$\rightarrow$ proof : Set $P=\log_b a$. Then $a=b^P=b^{\log_b a}$

Recall that $a=b^{\log_b a}$ for any $a>0$, $b>0$, $b\neq 1$. $\Rightarrow$ $a=e^{\ln a}$. ($\ln a=\log_e a$)

$$\lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{(e^{\ln a})^h-1}{h} = \lim_{h\to 0}\frac{e^{h\cdot\ln a}-1}{h} = \lim_{h\to 0}\left(\frac{e^{h\cdot \ln a}-1}{h\cdot\ln a}\cdot\ln a\right) = \ln a\cdot\boxed{\lim_{h\to 0}\frac{e^{h\cdot\ln a}-1}{h\cdot \ln a}} = \ln a\cdot 1 = \ln a.$$

$$\parallel$$

$\lim_{u\to 0}\frac{e^u-1}{u}$, where $u=h\cdot\ln a$

(3). Compute the derivative by the chain rule.

If $y = f(u)$, $u = g(x)$, then $y = f(u) = f(g(x))$.

To calculate $\frac{dy}{dx}$, we use the chain rule:

$$\underbrace{\frac{dy}{dx}}_{\text{derivative of } y \text{ with respect to } x} = \underbrace{\frac{dy}{du}}_{\text{derivative of } y \text{ with respect to } u} \cdot \underbrace{\frac{du}{dx}}_{\text{derivative of } u \text{ with respect to } x}.$$

Example 1. Find $\frac{dy}{dx}$ if $y = (3x+1)^5$.

$y = f(u) = u^5$, $u = g(x) = 3x+1$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{du^5}{du} \cdot \frac{d(3x+1)}{dx} = 5u^4 \cdot 3 = 15u^4 = 15(3x+1)^4.$$

Example 2. Find $\frac{dy}{dx}$ if $y = \sin(x^2+e^x)$ → a special case of Example 3.

$y = f(u) = \sin(u)$, $u = g(x) = x^2 + e^x$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d\sin u}{du} \cdot \frac{d(x^2+e^x)}{dx} = \cos u \cdot (2x+e^x) = \cos(x^2+e^x) \cdot (2x+e^x)$$

Example 3. Suppose $y = g(x)$. Calculate $\frac{d}{dx}\sin y$ (→ a function of x.) in terms of x, y and $\frac{dy}{dx}$.

$z = \sin y$ $\quad y = f(x)$.

$$\frac{d}{dx}\sin y = \frac{d}{dx} z = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{d\sin y}{dy} \cdot \frac{dy}{dx} = \cos y \cdot \frac{dy}{dx}.$$

Similarly, by the chain rule we obtain:

$$\frac{d}{dx} \cdot \cos y = \frac{d\cos y}{dy} \cdot \frac{dy}{dx} = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx} \cdot y^n = \frac{dy^n}{dy} \cdot \frac{dy}{dx} = n \cdot y^{n-1} \cdot \frac{dy}{dx}. \qquad (n \text{ is a real number})$$

$$\frac{d}{dx} a^y = \frac{da^y}{dy} \cdot \frac{dy}{dx} = \ln a \cdot a^y \cdot \frac{dy}{dx},$$

$$\frac{d}{dx} e^y = \frac{de^y}{dy} \cdot \frac{dy}{dx} = e^y \cdot \frac{dy}{dx}$$

We can make the "chain" longer:

$y=f(u),\quad u=g(x),\quad x=h(t). \Rightarrow \frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}$

proof: We use the chain rule twice:

$y=f(u),\quad u=g(h(t)). \Rightarrow \frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dt}.$

$u=g(x).\quad x=h(t) \Rightarrow \frac{du}{dt}=\frac{du}{dx}\cdot\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}.$

Example: Find $\frac{dy}{dx}$ if $y=\cos(e^{\sin t})$.

$y=f(u)=\cos u.\quad u=g(x)=e^x,\quad x=h(t)=\sin t$

$$\frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}=\frac{d\cos u}{du}\cdot\frac{de^x}{dx}\cdot\frac{d\sin t}{dt}=-\sin u\cdot e^x\cdot\cos t=-\sin(e^{\sin t})\cdot e^{\sin t}\cdot\cos t.$$

(4). Implicit Differentiation.

Two different ways to express a function $y = f(x)$:

① express y explicitly in terms of x :

$y = x^2$, $y = \sin x$, $y = x^2 \sin x$. $\Rightarrow$ How to find $\frac{dy}{dx}$: definition. rules $(+, -, \times, \div)$.
the chain rule

②. express y implicitly by a relation between x and y :

$x^3 - y^3 - \sin(x+y) = 0$, $\Rightarrow$ How to find $\frac{dy}{dx}$?

(Given an input x, we solve the equation for y.
If y is uniquely determined by x, then y is a function of x.)

Example 1. $\boxed{\text{Compute } \frac{dy}{dx} \text{ in terms of } x \text{ and } y}$ if $x^3 - y^3 - \sin(x+y) = 0$.

Step 1: Differentiate both sides with respect to x.

$$\frac{d}{dx}\left(x^3 - y^3 - \sin(x+y) \right) = \frac{d}{dx}(0) = 0 \qquad (*)$$

If $z = h(u)$, $u = g(x)$.
$\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx}$.

Step 2: Regard y as $y = f(x)$ and use the chain rule to compute the derivative.

① $\frac{d}{dx} x^3 = 3x^2$

② By the chain rule: $\frac{d}{dx} y^3 = \frac{dy^3}{dy} \cdot \frac{dy}{dx} = 3y^2 \cdot \frac{dy}{dx}$.

③ $\sin(x+y) = \sin(x + f(x))$: a function of x.

Define $z = \sin(u)$. $u = x + f(x) = x + y$ $\Rightarrow$ $z = \sin(x + f(x)) = \sin(x+y)$

$$\frac{d}{dx}\left(\sin(x+y)\right) = \underset{\text{chain rule}}{\frac{dz}{dx}} = \frac{dz}{du} \cdot \frac{du}{dx} = \frac{d \sin u}{du} \cdot \frac{d(x+f(x))}{dx} = \cos u \cdot \left(1 + \frac{dy}{dx}\right) = \cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right).$$

$\Rightarrow$ $(*)$ becomes $3x^2 - 3y^2 \cdot \boxed{\frac{dy}{dx}} - \cos(x+y)\left(1 + \boxed{\frac{dy}{dx}}\right) = 0$

Step 3: solve for $\frac{dy}{dx}$.

$$\left(-3y^2 - \cos(x+y)\right) \boxed{\frac{dy}{dx}} = \cos(x+y) - 3x^2. \Rightarrow \boxed{\frac{dy}{dx}} = \frac{\cos(x+y) - 3x^2}{-3y^2 - \cos(x+y)}.$$