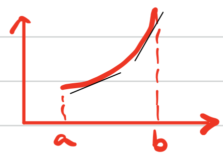
derivative of $f(x)$ derivative of $f'(x)$

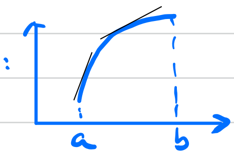
Review: How $\boxed{f'(x)}$ and $\boxed{f''(x)}$ affects the shape of the graph of $y=f(x)$.

(1). $f'(x)>0$ on (a,b) $\Rightarrow$ f is increasing on (a,b)
and the slope of tangent lines is positive.

$f''>0$ on (a,b):



$f''<0$ on (a,b):

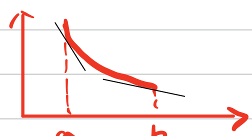


$f'' > 0 \overset{\text{means}}{\Longrightarrow} f'(x)$ is increasing on (a,b)
$\Rightarrow$ the slope of tangent line is increasing.

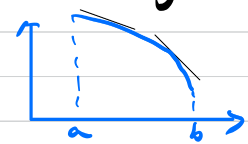
$f''<0 \Rightarrow f'(x)$ is decreasing on (a,b)
$\Rightarrow$ the slope of tangent line is decreasing

(2). $f'(x)<0$ on (a,b) $\Rightarrow$ f is decreasing on (a,b).
and the slope of tangent lines is negative.

$f''>0$ on (a,b):



$f''<0$ on (a,b):



$f'' > 0 \overset{\text{means}}{\Longrightarrow} f'(x)$ is increasing on (a,b)
$\Rightarrow$ the slope of tangent line is increasing.

$f''<0 \Rightarrow f'(x)$ is decreasing on (a,b)
$\Rightarrow$ the slope of tangent line is decreasing

L'Hopital's Rule (A very important trick to calculate limits of $\frac{0}{0}$ type and $\frac{\infty}{\infty}$ type).

We have $\boxed{\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}}$

if ① $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$. $\overset{\text{means}}{\Rightarrow} \lim_{x\to c}\frac{f}{g} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

② $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists or $\lim_{x\to c}\frac{f'(x)}{g'(x)} = \infty$.

($x\to c$ can be replaced by $x\to c^-$ or $x\to c^+$ or $x\to+\infty$ or $x\to-\infty$).

1. Use L'Hopital's Rule to calculate limits of $\frac{0}{0}$ type.

Example 1: $\lim_{x\to 2}\frac{x^3-8}{x^2-4} = 3$.

Method 1: Algebraic trick: cancell the common factor in the numerator and denominator.

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{4+4+4}{4} = 3$$

Method 2: use L'Hopital's Rule.

Define $f(x) = x^3-8$. $g(x) = x^2-4$. $\lim_{x\to 2} f(x) = f(2) = 0$. $\lim_{x\to 2} g(x) = g(2) = 0$. → condition ① is satisfied.

$$\lim_{x\to 2}\frac{f'(x)}{g'(x)} = \lim_{x\to 2}\frac{3x^2}{2x} = \frac{3\times 4}{4} = 3 \xRightarrow[\text{Rule}]{\text{L'Hopital's}} \lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{f'(x)}{g'(x)} = 3$$

↳ condition ② is also satisfied.

Example 2. $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1$

$f(x) = e^x - 1.$ $g(x) = x.$ $f(0) = g(0) = 0.$

$\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ (if exists or equals ∞) $= \lim\limits_{x\to 0} \dfrac{e^x}{1} = 1$

Example 3: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$f(x) = \sin x.$ $g(x) = x.$ $f(0) = g(0) = 0.$

$\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = \lim\limits_{x\to 0} \cos x = 1.$

Remark: L'Hoptial's Rule can be used repetitively.

Example 4: $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3} = -\dfrac{1}{6}$

$f(x) = \sin x - x.$ $g(x) = x^3.$ $f(0) = g(0) = 0.$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \underbrace{\lim_{x\to 0} \frac{\cos x - 1}{3x^2}} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to 0} \frac{(\cos x - 1)'}{(3x^2)'} = \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6} \cdot \lim_{x\to 0} \frac{\sin x}{x} = -\frac{1}{6}$$

$\frac{0}{0}$ type: $\cos x - 1 \to 0$ as $x \to 0$
$3x^2 \to 0$ as $x \to 0$.

2. use L'Hopital's Rule to calculate limits of $\frac{\infty}{\infty}$ type.

Example 1. $\lim\limits_{x\to+\infty} \frac{x}{e^x} = 0.$

$f(x) = x.\quad g(x) = e^x.\qquad \lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = +\infty.$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

In general we have $\lim\limits_{x\to+\infty} \frac{f(x)}{e^x} = 0$ for any polynomial $f(x)$.

$$\text{Example 2: } \lim_{x\to+\infty} \frac{x^3+2x^2+x}{e^x} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to+\infty} \frac{(x^3+2x^2+x)'}{(e^x)'} = \lim_{x\to+\infty} \frac{3x^2+4x+1}{e^x} \rightarrow \frac{\infty}{\infty} \text{ type.}$$

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to+\infty} \frac{(3x^2+4x+1)'}{(e^x)'} = \lim_{x\to+\infty} \frac{6x+4}{e^x}. \rightarrow \frac{\infty}{\infty} \text{ type.}$$

$$\overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to+\infty} \frac{(6x+4)'}{(e^x)'} = \lim_{x\to+\infty} \frac{6}{e^x} = 0.$$

Remark 1. L'Hopital's Rule is valid only if $\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$ exists or equals ∞.

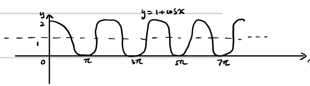
Example: $f(x)=x+\sin x$. $g(x)=x$. $c=+\infty$. $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}g(x)=+\infty$.

$\sin x\in[-1,1]$.

exists $\leftarrow$ $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to+\infty}\dfrac{x+\sin x}{x}=\lim\limits_{x\to+\infty}\left(1+\dfrac{\sin x}{x}\right)=1+\underbrace{\lim\limits_{x\to+\infty}\dfrac{\sin x}{x}}_{=0}=1+0=1.$

x can be arbitrary large

does not exist $\leftarrow$ $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}=\lim\limits_{x\to+\infty}\dfrac{1+\cos x}{1}=\underbrace{\lim\limits_{x\to+\infty}(1+\cos x)}$. does not exist.



In this case L'Hopital's Rule does not apply because $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}$ does not exist and it is not equal to ∞.

Remark 2: Why is L'Hopital's Rule $\left(\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}\right)$ correct?

Let's consider a special case where f', g' are continuous, $\lim\limits_{x\to c}f(x)=f(c)=0$ and $\lim\limits_{x\to c}g(x)=g(c)=0$.

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{h\to0}\frac{f(c+h)}{g(c+h)}=\lim_{h\to0}\frac{f(c+h)-f(c)}{g(c+h)-g(c)}=\lim_{h\to0}\frac{\frac{f(c+h)-f(c)}{h}}{\frac{g(c+h)-g(c)}{h}}=\frac{\lim\limits_{h\to0}\frac{f(c+h)-f(c)}{h}}{\lim\limits_{h\to0}\frac{g(c+h)-g(c)}{h}}=\frac{f'(c)}{g'(c)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Change "x" to "c+h"

because we assume $f(c)=g(c)=0$.

definition of derivative.

because we assume f' and g' are continuous.

trick: take logarithms and convert it to $0\cdot\infty$ type

3. Use L'Hopital's Rule to calculate limits of $\boxed{0\cdot\infty \text{ type}}$, $\boxed{1^{\infty} \text{ type}}$, $\boxed{0^{0} \text{ type}}$, $\boxed{\infty^{0} \text{ type}}$.

① $0\cdot\infty$ type. convert $0\cdot\infty$ type to $\frac{\infty}{\infty}$ or $\frac{0}{0}$ type, then use L'Hopital's Rule.

Example 1. $\lim_{x\to 0^+} x\ln x = 0$

We consider $\lim_{x\to 0^+} x\ln x$ as $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$. $\frac{\infty}{\infty}$ type.

$f(x) = \ln x$. $g(x) = \frac{1}{x}$.

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} \overset{\text{L'Hopital's Rule}}{=\!=} \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$$

In general we have $\lim_{x\to 0^+} x^p \cdot \ln x = 0$ for any constant $p > 0$.

② 1^{∞} type. take logarithm and convert it to $0\cdot\infty$ type.

Example: $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Let $y=(1+x)^{\frac{1}{x}}$. Then $\ln y = \frac{1}{x}\cdot \ln(1+x)$.

Now we calculate $\lim\limits_{x\to 0}\ln y$.

$$\lim_{x\to 0}\ln y = \underbrace{\lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x)}_{\to\, 0\cdot\infty \text{ type}} = \underbrace{\lim_{x\to 0}\frac{\ln(1+x)}{x}}_{\to\, \frac{0}{0} \text{ type.}} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to 0}\frac{(\ln(1+x))'}{(x)'}$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1.$$

$\lim\limits_{x\to 0}\ln y = 1 \overset{\text{means}}{\Longrightarrow} \lim\limits_{x\to 0} y = e$ (because $\ln x$ is continuous at $x=e$ and $\ln e = 1$).

Similarly we can obtain that $\lim\limits_{x\to 0}(1+px)^{\frac{1}{x}} = e^{p}$ for any constant $p\neq 0$.

③ 0^0 type take logarithm and convert it to $0\cdot\infty$ type.

Example: $\lim\limits_{x\to 0^+} x^x = 1$

Let $y = x^x$ Then $\ln y = x\ln x$.

$\to 0\cdot\infty$ type.

$$\lim_{x\to 0^+} \ln y = \underbrace{\lim_{x\to 0^+} x\ln x}\; = 0.$$

$$\lim_{x\to 0^+} \ln y = 0 \overset{\text{means}}{\Longrightarrow} \lim_{x\to 0^+} y = 1.$$

④ ∞^0 type. take logarithm and convert it to $0\cdot\infty$ type.

Example: $\lim\limits_{x\to +\infty} x^{\frac{1}{x}} = 1.$

Let $y = x^{\frac{1}{x}}$. Then $\ln y = \frac{1}{x}\cdot\ln x$.

$\to \frac{\infty}{\infty}$ type

$\to 0\cdot\infty$ type

$$\lim_{x\to+\infty} \ln y = \underbrace{\lim_{x\to+\infty} \frac{1}{x}\cdot\ln x}\; = \underbrace{\lim_{x\to+\infty} \frac{\ln x}{x}} \overset{\text{L'Hopital's}}{\underset{\text{Rule}}{=\!=}} \lim_{x\to+\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to+\infty} \ln y = 0 \overset{\text{means}}{\Longrightarrow} \lim_{x\to+\infty} y = 1.$$