HKUST – Department of Computer Science and Engineering

COMP 2711: Discrete Math Tools for CS – Fall 2017

Serial #

Midterm Examination

Date: October 30, 2017 Time: 19:30 - 21:30

Name: ______________	Student ID: ______________
Email: ______________	Lecture Section: ______________

Instructions

- This is a closed book exam. It consists of 13 pages and 11 questions.
- Please write your name, student ID, email, lecture section and tutorial in the space provided at the top of this page.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Solutions can be written in terms of binomial coefficients, factorials, the $C(n,k)$, $P(n,k)$, and $\binom{n}{k}$ notations. For example, you can write $\binom{5}{3}+\binom{4}{2}$ instead of 16. Avoid using nonstandard notation such as ${}_nP_k$ and ${}_nC_k$. Calculators may be used for the exam (but are not necessary).

Questions	1	2	3	4	5	6	7	8	9	10	11	Total
Score												

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ______________________________

Student's Signature:      ________________________
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Problem 1: [11 pts] Some of the following propositions and predicates are **tautologies**. Give a proof for each of the statements that are tautologies. You do not need to disprove the ones that are not tautologies.

(a) $(p \wedge q) \vee (q \wedge \neg p) \vee \neg q \vee (p \wedge T)$

(b) $(p \rightarrow q) \vee r \vee q$

(c) $\big(\forall x P(x)\big) \rightarrow \big(\exists x (P(x) \rightarrow Q(x))\big)$

(d) For the domain of real numbers,

$$\exists y (\exists x (x^2 = 2) \vee \exists x (x > 2 \wedge x^2 + y^2 = 2))$$

Solution: (a) Tautology

$$\begin{aligned} & (p \wedge q) \vee (q \wedge \neg p) \vee \neg q \vee (p \wedge T) \\ \equiv\ & (q \wedge (p \vee \neg p)) \vee \neg q \vee p \\ \equiv\ & (q \wedge T) \vee \neg q \vee p \\ \equiv\ & (q \vee \neg q) \vee p \\ \equiv\ & T \vee p \\ \equiv\ & T \end{aligned}$$

(b) Not a tautology. It is false when p is true, q is false and r is false. (No counter example required)

(c) Not a tautology. It is false if $Q(x)$ is always false. (No counter example required)

$$\begin{aligned} & (\forall x P(x)) \rightarrow \exists x (P(x) \rightarrow Q(x)) \\ \equiv\ & \exists x \neg P(x) \vee \exists x (\neg P(x) \vee Q(x)) \\ \equiv\ & \exists x \exists y (\neg P(x) \vee \neg P(y) \vee Q(y)) \end{aligned}$$

(d) Tautology.

$$\begin{aligned} & \exists y (\exists x (x^2 = 2) \vee \exists x (x > 2 \wedge x^2 + y^2 = 2)) \\ \equiv\ & \exists y (T \vee \exists x (x > 2 \wedge x^2 + y^2 = 2)) \\ \equiv\ & \exists y (T \vee F) \\ \equiv\ & T \end{aligned}$$

Problem 2: [8 pts] Consider the following predicates:

$C(x)$: x is a child.

$K(x)$: x is kind.

$N(x, y)$: x and y are neighbours.

$L(x)$: x learned.

Write the following sentences using the above predicates, adding quantifiers when necessary:

1. All children are kind.
2. People who are unkind can only have unkind neighbours.
3. People who are not learned are unkind.
4. Mencius, a child, has unkind neighbours.

Solution:

1. $\forall x \ (C(x) \rightarrow K(x))$
2. $\forall x \Big(\neg K(x) \rightarrow \forall y \ (N(x, y) \rightarrow \neg K(y))\Big)$, or equivalently,
   $\forall x \forall y \ \Big((\neg K(x) \land N(x, y)) \rightarrow \neg K(y)\Big)$
3. $\forall x \ (\neg L(x) \rightarrow \neg K(x))$
4. $C(\text{Mencius}) \land \exists x \ (N(x, \text{Mencius}) \land \neg K(x))$

Problem 3: [10 pts] Assuming that any number that is not rational is an irrational number, prove the following:

(a) The sum of any irrational number and any rational number must be an irrational number.

(b) If x^2 is irrational, then x is irrational.

Solution:

(a) We use a proof by contradiction: assume that the sum of some rational number x and some irrational number y is a rational number z. $x = \frac{a}{b}$, and $z = \frac{c}{d}$ for some integers c and d. We have $y = z - x = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{db}$. Since $cb - ad$ and db are integers, this contradicts with the assumption that y is an irrational number.

(b) We use a proof by contraposition to show that if x is rational, then x^2 is rational. Since x is rational, let $x = \frac{a}{b}$ for some integers a and b. $x = (\sqrt{x})^2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$; both a^2 and b^2 are integers. Thus x^2 is rational.

Problem 4: [6 pts]

Prove or disprove the following statements. Assume that a, b, c, d, and m are integers and $m > 1$.

(a) If $a \equiv b \pmod{m}$ and $a \equiv c \pmod{m}$, then $a \equiv b + c \pmod{m}$.

(b) If $a \equiv b \pmod{m}$, then $2a \equiv 2b \pmod{2m}$.

Solution: (a) False. Set $a = 1, b = 3, c = 5, m = 2$ as a counter example. We have $1 \equiv 3 \pmod{2}$, $1 \equiv 5 \pmod{2}$, but $1 \not\equiv 5 \pmod{2}$.

(b) True. $a \equiv b \pmod{m}$ implies $m \mid a - b$, i.e., $a - b = km$ for some integer k. Multiplying both sides by 2 gives $2a - 2b = 2mk$. Therefore $2a \equiv 2b \pmod{2m}$.

Problem 5: [10 pts]

(a) Use the Euclidean algorithm to find gcd(15, 221). Show the steps.

(b) Find $x \in Z_{221}$ such that $15x \equiv 2 \pmod{221}$. Show your workings.

Solution: (a) $\gcd(221, 15)$:

$$\begin{aligned} 221 &= 15 \cdot 14 + 11 \\ 15 &= 11 \cdot 1 + 4 \\ 11 &= 4 \cdot 2 + 3 \\ 4 &= 3 \cdot 1 + 1 \\ 3 &= 1 \cdot 3 + 0 \end{aligned}$$

So, $\gcd(221, 15) = 1$.

(a) Rewriting:

$$\begin{aligned} 11 &= 221 - 15 \cdot 14 \\ 4 &= 15 - 11 \cdot 1 \\ 3 &= 11 - 4 \cdot 2 \\ 1 &= 4 - 3 \cdot 1 \end{aligned}$$

Substituting:

$$\begin{aligned} 1 &= 4 - (11 - 4 \cdot 2) \cdot 1 \\ &= 4 \cdot 3 - 11 \cdot 1 \\ &= (15 - 11 \cdot 1) \cdot 3 - 11 \cdot 1 \\ &= 15 \cdot 3 - 11 \cdot 4 \\ &= 15 \cdot 3 - (221 - 15 \cdot 14) \cdot 4 \\ &= 15 \cdot 59 - 221 \cdot 4 \end{aligned}$$

Therefore, the inverse of 15 modulo 221 is 59.

$15^{-1} \cdot 15x \equiv 15^{-1} \cdot 2 \pmod{221}$.
$x \equiv 59 \cdot 2 \equiv 118 \pmod{221}$.

Problem 6: [12 pts]

Five couples go to the movies together and sit in a row of ten seats. In how many ways can the 10 people be arranged if

(a) They may sit in any order.

(b) All the men sit together and all the women sit together.

(c) Each couple sits together (i.e., for each couple, the two people are in adjacent seats).

(d) One couple is arguing and they refuse to sit together. The other couples can sit in any way (i.e., together or not).

Solution:

(a) $10!$

(b) $2 \cdot 5!^2$

(c) $5! \cdot 2^5$

(d) $10! - 9 \cdot 8! \cdot 2$

Problem 7: [7 pts] A computer randomly prints three-digit codes, with no repeated digits in any code (for example, 387, 072, 760). What is the minimum number of codes that must be printed in order to guarantee that at least six of the codes are identical?

Solution: There are $10 \cdot 9 \cdot 8 = 720$ different codes. By the generalized pigeonhole principle, to guarantee that at least six identical codes will be printed, the minimum number n of printed codes is such that $\lceil n/720 \rceil = 6$, which is $n = 5 \cdot 720 + 1 = 3601$.

Problem 8: [6 pts]

(a) Find the coefficient of x^7y^5 in the expansion of $(3x - y)^{12}$.

(b) Find the number of terms in the expansion of $(5a + 8b)^{15}$.

Solution: (a) $-\binom{12}{7}3^7$.

(b) 16.

Problem 9: [8 pts]

Six women and nine men are on the faculty of a department. The department needs to form a committee of five members.

(a) How many committees are possible if the committee must have two women and three men?

(b) How many committees are possible if the committee must have at least two women?

(c) How many committees are possible if the committee must consist of all women and all men?

Solution: (a) $\binom{6}{2}\binom{9}{3}$

(b) $\binom{6}{2}\binom{9}{3} + \binom{6}{3}\binom{9}{2} + \binom{6}{4}\binom{9}{1} + \binom{6}{5}\binom{9}{0}$

(c) $\binom{6}{5} + \binom{9}{5}$

Problem 10: [12 pts]

Assume you have 50 candies and four boxes, labeled A, B, C, and D.

(a) How many ways can you put the candies in the boxes, assuming that the candies are all different?

(b) How many ways can you put the candies in the boxes, assuming that the candies are identical?

(c) How many ways can you put the candies in the boxes, assuming that the candies are identical and each box must have at least five candies put into it?

Solution: (a) 4^{50}

(b) $\binom{50+4-1}{3}$

(c) $\binom{35+4-1}{3}$

Problem 11: [10 pts]

Give a combinatorial proof of $\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}$.

Solution: $\binom{n}{k}\binom{k}{1}$: $\binom{n}{k}$ is the number of ways to select k people into a club from n people. $\binom{k}{1}$ is the number of ways to select a leader from the k people club. Therefore, this is the number of ways to form a k people club and assign one of them to be club leader.

$\binom{n}{1}\binom{n-1}{k-1}$: $\binom{n}{1}$ is the number of ways to select a person into a club and acts as club leader from n people. $\binom{n-1}{k-1}$ is the number of ways to select $k-1$ people into the club.

Therefore, both expression counts the number of ways to form a k people club and assign one of them to be club leader.