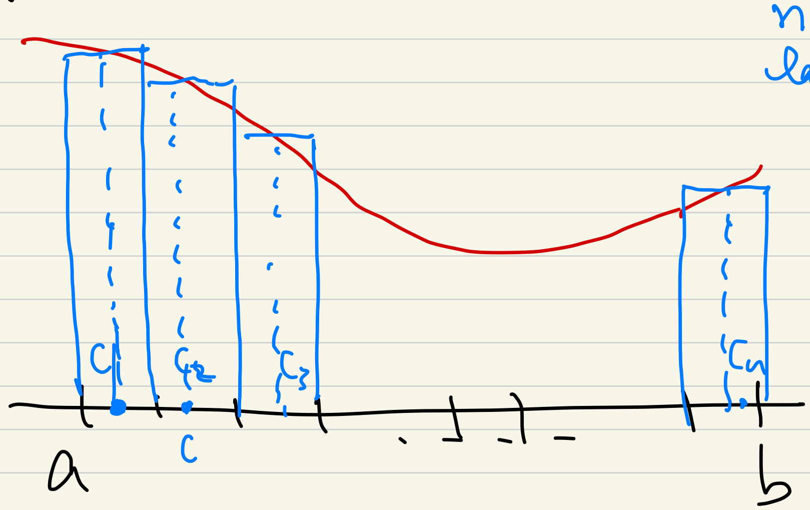
Integration Techniques $\begin{cases} \text{Substitution rule : trigonometric identities.} \\ \text{integration by parts:} \\ \text{Numerical method !} \end{cases}$

$$\int_a^b f(x)\,dx \approx \text{approximate value ?}$$

$$\shortparallel$$

$$\lim_{n\to\infty} \underbrace{\frac{b-a}{n}\left[f(c_1)+f(c_2)\cdots+f(c_n)\right]}_{\text{Riemann Sum}} \approx \frac{b-a}{n}\left[f(c_1)+\cdots \cdots+f(c_n)\right]$$

n is large !!

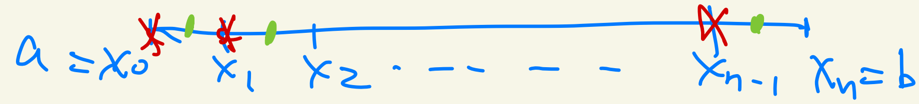


$$\Delta x = \frac{b-a}{n} \quad \frac{b-a}{n} \quad \cdots$$

Some Basic Ways to Choose those $c_1, \dots, c_n$ are:

1. Left-endpoint of the subintervals $\longrightarrow L_n$
2. Right-endpoint of the subintervals $\longrightarrow R_n$
3. Mid-Point of the subintervals $\longrightarrow M_n$

$\int_a^b f(x)\,dx$



$$\approx L_n = \frac{b-a}{n}\left[f(x_0)+f(x_1)+\cdots+f(x_{n-1})\right]$$

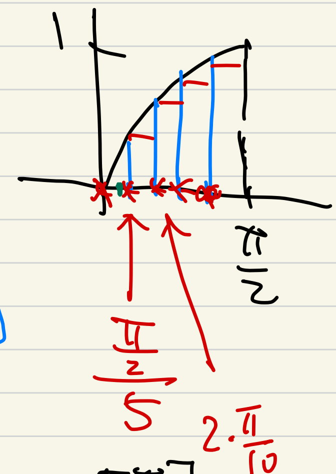
Left-endpoint Rule

$$R_n = \frac{b-a}{n}\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]$$ Right-endpoint Rule

$$M_n = \frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

Midpoint Rule!

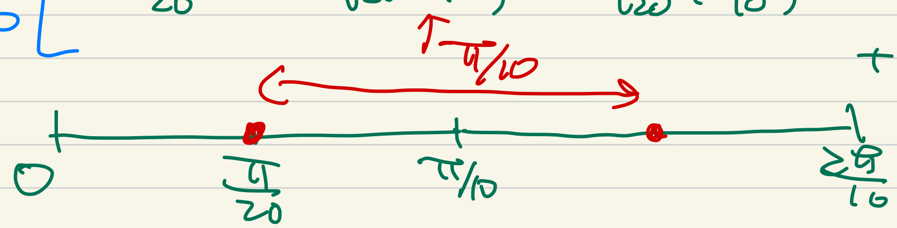
Example. $\int_0^{\pi/2}\sin x\,dx = 1.$



$$L_5 = \frac{\frac{\pi}{2}}{5}\left[\sin 0+\sin\frac{\pi}{10}+\sin\frac{2\pi}{10}+\sin\frac{3\pi}{4}+\sin\frac{4\pi}{10}\right]$$

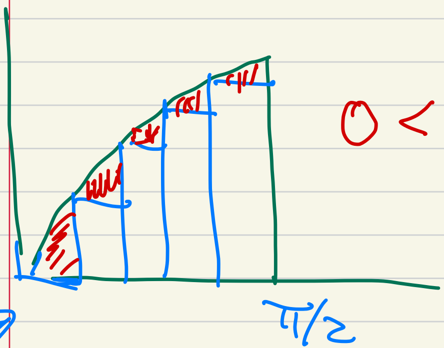
$$R_5 = \frac{\pi}{10}\left[\sin\frac{\pi}{10}+\sin\frac{2\pi}{10}+\sin\frac{3\pi}{10}+\sin\frac{4\pi}{10}+\sin\frac{5\pi}{10}\right]$$

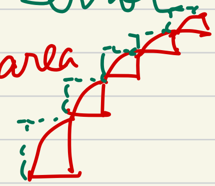
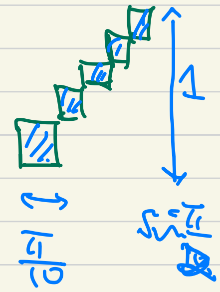
$$M_5 = \frac{\pi}{10}\left[\sin\frac{\pi}{20}+\sin\left(\frac{\pi}{20}+\frac{\pi}{10}\right)+\sin\left(\frac{\pi}{20}+\frac{2\pi}{10}\right)+\sin\left(\frac{\pi}{20}+\frac{3\pi}{10}\right)+\sin\left(\frac{\pi}{20}+\frac{4\pi}{10}\right)\right]$$



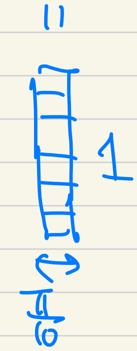
$$\boxed{\int_a^b f(x)\,dx - L_n = E_{L_n}}$$

How large is the approximation error?



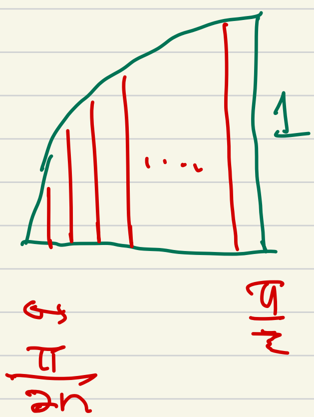
$0 < \int_a^b f(x)\,dx - L_5 =$ area  $<$ 

an underestimate of the integral

$=$ 

$$0 < \int_a^b f(x)\,dx - L_5 < \frac{\pi}{10}$$

$$0 < \overbrace{\int_a^b f(x)\,dx - L_n}^{E_{L_n}} < \frac{\pi}{2n} \to 0 \text{ as } n \to \infty.$$



If we want $E_{L_n} < 10^{-5}$, how large an n we should choose?

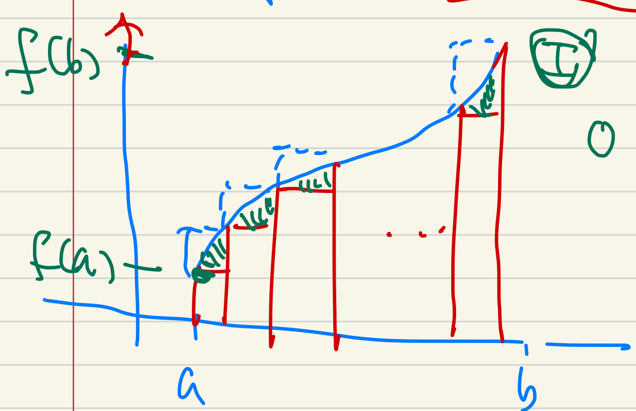
Just take the smallest integer n such that $\frac{\pi}{2n} < 10^{-5}$

$$n > 10^5 \cdot \frac{\pi}{2}$$

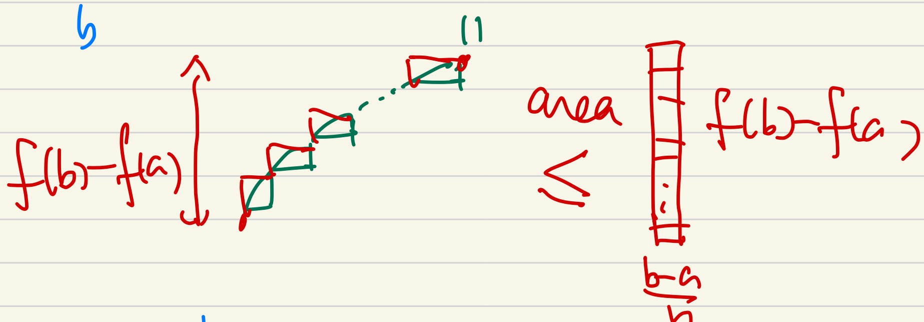
$$n = \left[10^5 \cdot \frac{\pi}{2}\right] + 1,$$

where $[x]$ = smallest integer $\leq x$.

① If f is increasing (decreasing) on $[a, b]$,



(I) $$0 < \underbrace{\int_a^b f(x)\,dx - L_n}_{E_{L_n}} \le \frac{(b-a)}{n}[f(b)-f(a)]$$



(II) $$0 < R_n - \int_a^b f(x)\,dx < \frac{b-a}{n}[f(b)-f(a)]$$

overestimates the integral

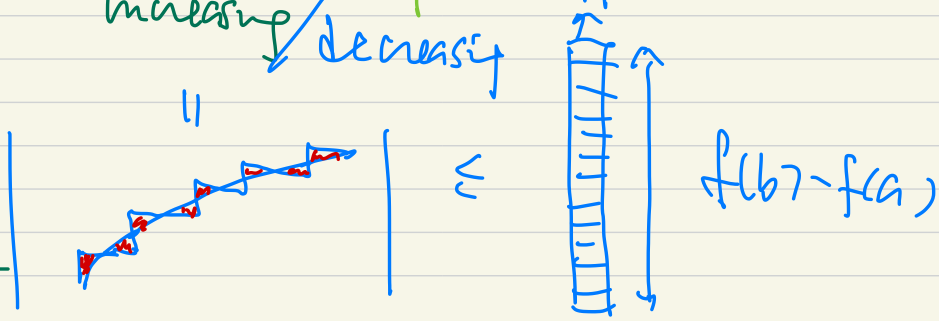
f increasing/decreasing

$$|E_{L_n}| \le \frac{b-a}{n}|f(b)-f(a)|$$

$$|E_{R_n}| \le \frac{b-a}{n}|f(b)-f(a)|$$

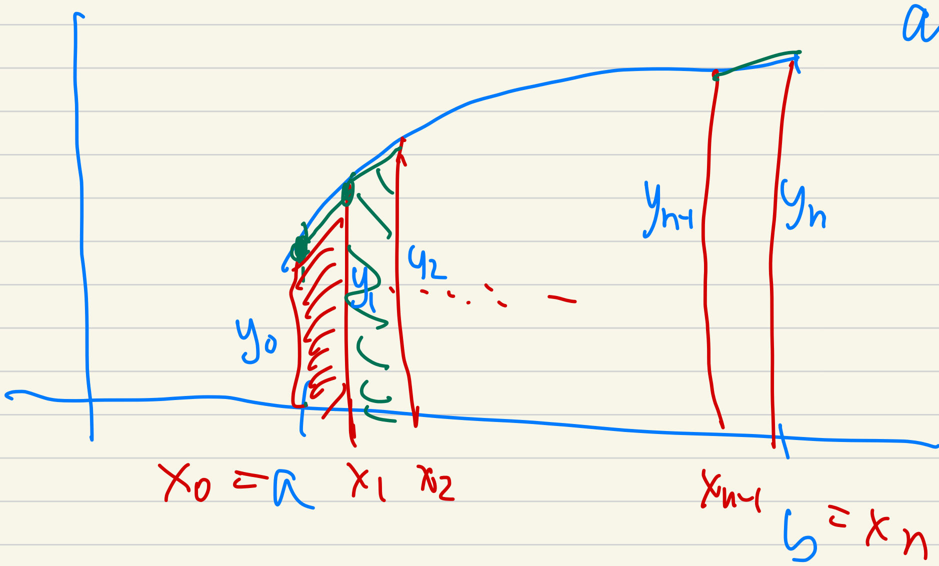
(III) $|E_{M_n}| \longrightarrow$ $$\left|\int_a^b f(x)\,dx - M_n\right| \le \frac{b-a}{2n}|f(b)-f(a)|$$

increasing/decreasing



Trapezoidal Rule $\longleftrightarrow$ Use Trapezoids to estimate areas!

T_n



$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx \approx T_n = \frac{b-a}{2n}\Big[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \vdots + f(x_{n-1}) + f(x_n)\Big]$$

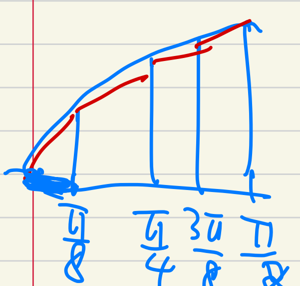
area $= \dfrac{m+M}{2}h$

$$T_n = \frac{b-a}{2n}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right]$$

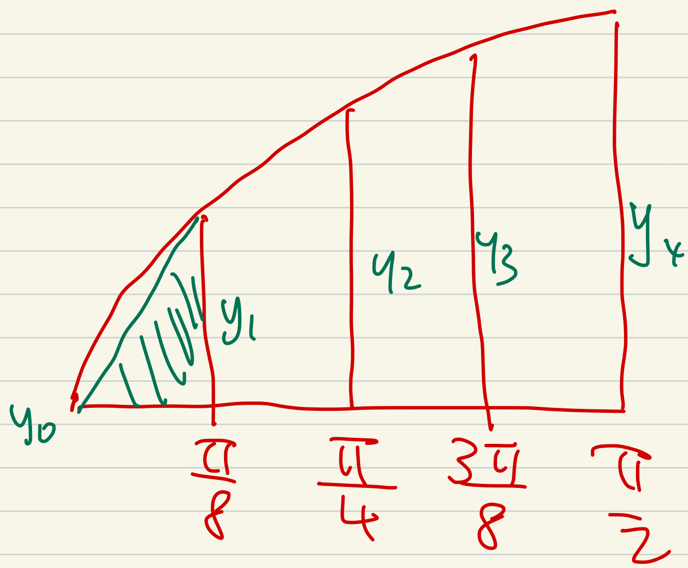
where $y_k = f(x_k)$.

$h = \frac{\pi}{8}$

$$\int_0^{\pi/2} \sin x\,dx \approx T_4 = \frac{\pi}{2\cdot 8}\left[\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + \sin\frac{\pi}{2}\right]$$



Same question:
Can we estimate $|E_{T_n}| = \left|\int_a^b f(x)\,dx - T_n\right|$?



$$\frac{1}{2}\cdot\frac{\pi}{8}\cdot\left[y_0+y_1\right]+\frac{1}{2}\cdot\frac{\pi}{8}\cdot\left[y_1+y_2\right]$$

$$+\frac{1}{2}\frac{\pi}{8}\left[y_2+y_3\right]+\frac{1}{2}\cdot\frac{\pi}{8}\cdot\left[y_3+y_4\right]$$