COMP 2711 Discrete Math Tools for Computer Science

2022 Fall Semester - Homework 1

Question 1: Let $Z(x)$, $D(x)$, $F(x)$ and $C(x)$ be the following predicates:

$Z(x)$: "x attended every COMP2711 tutorial classes".

$D(x)$: "x gets F in COMP2711".

$F(x)$: "x cheated in the exams".

$C(x)$: "x has not done any tutorial question".

$K(x)$: "x asked some questions in the telegram group".

Express the following statements using quantifiers, logical connectives, and the predicates above, where the domain consists of all students in COMP2711.

(a) A student gets F in COMP2711 if and only if he/she hasn't done any tutorial question and cheated in the exams.

(b) Some students did some tutorial questions but he/she either absent from some of the tutorial classes or cheated in the exams.

(c) If a student attended every tutorial classes but gets F, then he/she must have cheated in the exams.

(d) Any student who asked some questions in the telegram group and didn't cheat in the exams won't get F.

Answer: (a) $\forall x(D(x) \leftrightarrow C(x) \land F(x))$.

(b) $\exists x(\neg C(x) \land (\neg Z(x) \lor F(x)))$.

(c) $\forall x(Z(x) \land D(x) \to F(x))$.

(d) $\forall x(K(x) \land \neg F(x) \to \neg D(x))$.

Question 2: Show that the following two propositions are logically equivalent by developing series of logical equivalences.

(i) $(((p \to q) \leftrightarrow (\neg q \vee r)) \wedge (p \to \neg r)) \to \neg((s \vee r) \leftarrow (\neg r \wedge p))$,

(ii) $(r \vee (\neg q \wedge (s \vee \neg p))) \to (p \wedge (\neg q \vee r))$

Answer:

$$
\begin{aligned}
&(((p \to q) \leftrightarrow (\neg q \vee r)) \wedge (p \to \neg r)) \to \neg((s \vee r) \leftarrow (\neg r \wedge p))\\
\equiv &((((p \to q) \wedge (\neg q \vee r)) \vee (\neg(p \to q) \wedge \neg(\neg q \vee r))) \wedge (p \to \neg r)) \to \neg((s \vee r) \leftarrow (\neg r \wedge p))\\
\equiv &((((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg(\neg p \vee q) \wedge \neg(\neg q \vee r))) \wedge (\neg p \vee \neg r)) \to \neg((s \vee r) \vee \neg(\neg r \wedge p))\\
\equiv &((((\neg p \vee q) \wedge (\neg q \vee r)) \vee ((p \wedge \neg q) \wedge (q \wedge \neg r))) \wedge (\neg p \vee \neg r)) \to \neg((s \vee r) \vee (r \vee \neg p))\\
\equiv &((((\neg p \vee q) \wedge (\neg q \vee r)) \vee (p \wedge (\neg q \wedge q) \wedge \neg r)) \wedge (\neg p \vee \neg r)) \to \neg(s \vee (r \vee r) \vee \neg p)\\
\equiv &((((\neg p \vee q) \wedge (\neg q \vee r)) \vee (p \wedge F \wedge \neg r)) \wedge (\neg p \vee \neg r)) \to \neg(s \vee r \vee \neg p)\\
\equiv &((((\neg p \vee q) \wedge (\neg q \vee r)) \vee F) \wedge (\neg p \vee \neg r)) \to \neg(s \vee r \vee \neg p)\\
\equiv &(((\neg p \vee q) \wedge (\neg q \vee r)) \wedge (\neg p \vee \neg r)) \to \neg(s \vee r \vee \neg p)\\
\equiv &\neg(((\neg p \vee q) \wedge (\neg q \vee r)) \wedge (\neg p \vee \neg r)) \vee \neg(s \vee r \vee \neg p)\\
\equiv &(p \wedge \neg q) \vee (q \wedge \neg r) \vee (p \wedge r) \vee (\neg s \wedge \neg r \wedge p)
\end{aligned}
$$

$$
\begin{aligned}
&(r \vee (\neg q \wedge (s \vee \neg p))) \to (p \wedge (\neg q \vee r))\\
\equiv &\neg(r \vee (\neg q \wedge (s \vee \neg p))) \vee (p \wedge (\neg q \vee r))\\
\equiv &(\neg r \wedge (q \vee (\neg s \wedge p))) \vee (p \wedge (\neg q \vee r))\\
\equiv &((\neg r \wedge q) \vee (\neg r \wedge \neg s \wedge p)) \vee (p \wedge \neg q) \wedge (p \vee r))\\
\equiv &(p \wedge \neg q) \vee (q \wedge \neg r) \vee (p \wedge r) \vee (\neg s \wedge \neg r \wedge p)
\end{aligned}
$$

Question 3: Determine the truth value of each of these statements if the domain for all variables consists of all real numbers.

(a) $\forall x \exists y (y > 2711x)$

(b) $\exists x \forall y (x \leq y^2)$

(c) $\exists x \exists y \forall z (x^2 + y^2 = z^3)$

(d) $\forall x((x > 2) \to (\log_2 x < x - 1) \leftrightarrow \neg \exists x((x > 2) \land (\log_2 x \geq x - 1))$

Answer: (a) True. We could always take $y = 2711x + 1$.

(b) True. We can take $x = 0$, obviously $\forall y(0 \leq y^2)$.

(c) False. For every x and y, there exists a z that $z^3 \neq x^2 + y^2$.

(d) True.

$$\begin{aligned}
&\neg \exists x((x > 2) \land (\log_2 x \geq x - 1)) \\
\equiv\ &\forall x \neg((x > 2) \land (\log_2 x \geq x - 1)) \\
\equiv\ &\forall x((x \leq 2) \lor (\log_2 x < x - 1)) \\
\equiv\ &\forall x((x > 2) \to (\log_2 x < x - 1))
\end{aligned}$$

Question 4: Prove the following statement by contradiction for any integers a, b, c.

"If $a^2 + b^2 = c^2$, then a or b is even"

Answer: We assume $a^2 + b^2 = c^2$ and both a and b are odd. Let $a = 2k_a + 1$ and $b = 2k_b + 1$ for some integers k_a, k_b. We have

$$\begin{aligned} a^2 + b^2 &= 4k_a^2 + 4k_a + 1 + 4k_b^2 + 4k_b + 1 \\ &= 4(k_a^2 + k_a + k_b^2 + k_b) + 2 \end{aligned}$$

which is even but not a multiple of 4. For any integer c, c^2 is even if and only if c is even. The square of an even number must be multiple of 4. This contradicts the assumption that $a^2 + b^2 = c^2$, completing the proof.