Math2001 Answer to Homework 6

Exercise 3.1

The equality $m+1=1+m$ may be proved by inducting on m. For $m=1$, the equality holds trivially. Next assume $m+1=1+m$. Then

$$\begin{aligned} m^{'}+1 &= (m+1)+1 \text{ (first property)} \\ &= (m+1)^{'} \text{ (first property)} \\ &= (1+m)^{'} \text{ (induction assumption)} \\ &= 1+m^{'} \text{ (second property).} \end{aligned}$$

Exercise 3.2

The equality $m+n=n+m$ may be proved by inducting on n. For $n=1$, the equality is proved in Exercise 3.1. Next assume $m+n=n+m$. Then

$$\begin{aligned} m+n^{'} &= m+(n+1) \text{ (first property)} \\ &= (m+n)+1 \text{ (associativity)} \\ &= 1+(m+n) \text{ (Ex 3.1 or base case)} \\ &= 1+(n+m) \text{ (induction assumption)} \\ &= (1+n)+m \text{ (associativity)} \\ &= (n+1)+m \text{ (Ex 3.1 or base case)} \\ &= n^{'}+m \text{ (first property).} \end{aligned}$$

Exercise 3.5

For integers $a=[m,n], b=[k,l], c=[p,q]$, we have

$$\begin{aligned} a+(b+c) &= [m,n]+([k,l]+[p,q]) \\ &= [m,n]+[k+p,l+q] \\ &= [m+(k+p),n+(l+q)] \\ &= [(m+k)+p,(n+l)+q] \text{ (associativity for } \mathbb{N}) \\ &= [(m+k),(n+l)]+[p,q] \\ &= ([m,n]+[k,l])+[p,q] \\ &= (a+b)+c. \end{aligned}$$

then associativity follows. Also we have

$$\begin{aligned} a+b &= [m,n]+[k,l] \\ &= [m+k,n+l] \\ &= [k+m,l+n] \text{ (commutativity for } \mathbb{N}) \\ &= [k,l]+[m,n] \\ &= b+a \end{aligned}$$

then commutativity follows.

Exercise 3.6

Note that $[m, n] + [1, 1] = [m + 1, n + 1]$, and

$$(m + 1) + n = m + (1 + n) = m + (n + 1)$$

we see that $[m+1, n+1] = [m, n]$. Similarly we can prove that $[1, 1]+[m, n] = [1+m, 1+n] = [m, n]$. Let $0 = [1, 1]$.

If $\bar{0}$ is another integer such that $a + \bar{0} = \bar{0} + a$ for all integer a, then

$$\bar{0} = \bar{0} + 0 = 0$$

which prove the uniqueness of zero.

Exercise 3.7

For any integer $a = [m, n]$, note that $[m, n] + [n, m] = [m + n, n + m]$,

$$(m + n) + 1 = 1 + (m + n) = 1 + (n + m)$$

which means that $[m, n] + [n, m] = [1, 1] = 0$. Similarly we have $[n, m] + [m, n] = 0$ which shows the existence of negative.

If b, c are two integers such that $a + b = 0 = b + a$, $a + c = 0 = c + a$ for given a, then

$$\begin{aligned} b &= b + 0 \text{ definition of zero} \\ &= b + (a + c) \text{ definition of negative} \\ &= (b + a) + c \text{ associativity} \\ &= 0 + c \text{ definition of negative} \\ &= c \text{ definition of zero} \end{aligned}$$

which shows the uniqueness of negative.

Exercise 3.8

From Ex 3.7, we can set $-[n + 1, 1] = [1, n + 1]$, then

$$\begin{aligned} c &= [m + 1, 1] + (-[n + 1, 1]) \\ &= [m + 1, 1] + [1, n + 1] \\ &= [(m + 1) + 1, 1 + (n + 1)] \end{aligned}$$

note that

$$((m+1)+1)+n = (m+(1+1))+n = m+((1+1)+n) = m+(1+(1+n)) = m+(1+(n+1))$$

which means $[(m + 1) + 1, 1 + (n + 1)] = [m, n]$, then $[m, n] = [m + 1, 1] - [n + 1, 1]$.

Exercise 3.9

Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} &(a + b) + ((-a) + (-b)) \\ =&(b + a) + ((-a) + (-b)) \text{ commutativity} \\ =&((b + a) + (-a)) + (-b) \text{ associativity} \\ =&(b + (a + (-a))) + (-b) \text{ associativity} \\ =&(b + 0) + (-b) \text{ definition of negative} \\ =&b + (-b) \text{ definition of zero} \\ =&0 \text{ definition of negative} \end{aligned}$$

Similarly, we have

$$\begin{aligned}
&((-a)+(-b))+(a+b)\\
=&((-a)+(-b))+(b+a) \text{ commutativity}\\
=&(-a)+((-b)+(b+a)) \text{ associativity}\\
=&(-a)+(((-b)+b)+a) \text{ associativity}\\
=&(-a)+(0+a) \text{ definition of negative}\\
=&(-a)+a \text{ definition of zero}\\
=&0 \text{ definition of negative}
\end{aligned}$$

which means $-a-b:=(-a)+(-b)$ is a negative of $a+b$, by the uniquess of negative, we have $-(a+b)=-a-b$.

Exercise 3.11

For any integer a, its negative is $-a$, there exists an integer $-(-a)$ which is the negative of $-a$, satisfying

$$-(-a)+(-a)=0=-a+(-(-a))$$

a also satisfying $a+(-a)=0=-a+a$, which means a is also the negative of $-a$, then $-(-a)=a$ by the uniqueness of negative.

Exercise 3.13

Given two integers a,b, we have

$$\begin{aligned}
-(a-b)&=-(a+(-b)) \text{ definition of subtraction}\\
&=-((-b)+a) \text{ commutativity}\\
&=-(-b)-a \text{ Ex 3.9}\\
&=b-a \text{ Ex 3.11}
\end{aligned}$$

then $-(a-b)=b-a$.

Exercise 3.14

By Ex 3.9, we have $-(b+c)=-b-c$, then

$$\begin{aligned}
&(a+c)-(b+c)\\
=&(a+c)+(-(b+c)) \text{ definition of subtraction}\\
=&(a+c)+((-b)+(-c))\\
=&(a+c)+((-c)+(-b)) \text{ commutativity}\\
=&((a+c)+(-c))+(-b) \text{ associativity}\\
=&(a+(c+(-c)))+(-b) \text{ associativity}\\
=&(a+0)+(-b) \text{ definition of negative}\\
=&a+(-b) \text{ definition of zero}
\end{aligned}$$

then we have $(a+c)-(b+c)=a-b$.

By Ex 3.9, we have $(-b)-(-a)=(-b)+(-(-a))=-(b+(-a))=-(b-a)$; by Ex 3.13, we have $-(b-a)=a-b$, so we have $(-b)-(-a)=a-b$.

Exercise 3.17

If $a>b, c>d$, then for order compatible with addition, we have

$$a+c>b+c \quad c+b>d+b$$

note that $c+b=b+c, d+b=b+d$, then by transitivity, we have $a+c>b+d$.

Exercise 3.18

For Reflexivity, $a=a$ implies $a \geq a$;

For Antisymmetry, given $a \geq b$ and $b \geq a$, then $a \geq b$ implies $a>b$ or $a=b$; $a>b$ means $a-b \in \mathbb{N}$, then $-(a-b)=b-a \neq 0$ and $-(a-b)=b-a \notin \mathbb{N}$ which contradicts with $b \geq a$, then we should have $a=b$;

For Transitivity, given $a \geq b$ and $b \geq c$, then if $a<c$, i.e. $-(a-c)=c-a \in \mathbb{N}$, note that $a \geq b$ implies $a-b=0$ or $a-b \in \mathbb{N}$, then $c-b=(c-a)+(a-b) \in \mathbb{N}$, which contradicts with $b \geq c$! So we have $a \geq c$.