Math1014 Calculus II

Week 3-4: Brief Review and Some Practice Problems
Definite Integrals: Work, Arc Length, Surface Area, ...

Just recall that the application of integration can be considered as a process of summing up "pieces of tiny quantities" through limit taking.

- Work W as an integral:

  Moving a single point mass: $\underbrace{\Delta W}_{\text{"tiny work"}} \approx \underbrace{F(x)}_{force} \cdot \underbrace{\Delta x}_{displacement} \xrightarrow{\text{"sum"}} W = \int_a^b F(x)dx$

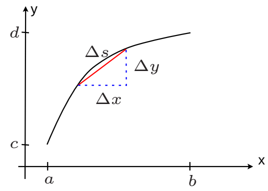
  Lifting a continuum of mass distribution: (An extension of the "mgh" idea.)

$$\Delta W \approx (\text{volume of a thin layer of mass at the same height})\,(\text{density})g \cdot (\text{distance lifted})$$

$$\longrightarrow W = \int_a^b (\text{density})g \underbrace{h(t)}_{(\text{distance lifted})} \underbrace{A(t)}_{(\text{cross section area})} dt$$

- Arc length L as an integral: a matter of approximating "tiny arc length" by short line segment:

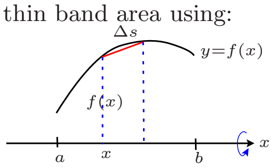
$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \begin{cases} \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\,\Delta x & \rightarrow \quad L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\,dx \\ \sqrt{1 + (\frac{\Delta x}{\Delta y})^2}\,\Delta y & \rightarrow \quad L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}\,dy \end{cases}$$



- Area A of a surface of revolution as an integral: a matter of approximating thin band area using:

$$\Delta A \approx (\text{circular length})(\text{tiny arc length}) = 2\pi f(x)\Delta s$$

$$\longrightarrow \quad A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$



rotate the curve about the x-axis

  Note that there are variations of the "formula" above, according to the axis of rotation chosen: y-axis or other lines.

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1. A swimming pool is 20 m long an 10 m wide, with a bottom that slopes uniformly from a depth of 1 m at one end to a depth of 2 m at the other end. Assuming the pool if full, how much work is required to pump the water to a level 0.2 m above the top of the pool?

2. A water trough has a semicircular cross section with a radius of 0.25m ans a length of 3m. How much work is required to pump the water out of the trough when it is full?

3. Find the arc length of the curve.

   (i) $y^2 = 4(x+4)^3$, $0 \le x \le 2$, $y > 0$. (ii) $x = \frac{y^3}{6} + \frac{1}{2y}$, $\frac{1}{2} \le y \le 1$.

4. Find the area of the surface of revolution obtained by rotating the curve $y = 1 - x^2$, $0 \le x \le 1$, about: (i) the y-axis; (ii) the line $x = -1$. ($\int \sqrt{a^2+u^2}du = \frac{x}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2+u^2}) + C$)

5. Find the surface area of the torus obtained by rotating the circle $x^2 + (y-3)^2 = 1$ about the x-axis.

6. Find the area of the surface of revolution obtained by rotating the part of the curve $y = \frac{1}{2}\ln(2x + \sqrt{4x^2-1})$ between the points $(\frac{1}{2}, 0)$ and $(\frac{17}{16}, \ln 2)$ about the y-axis.

7. Find the mass of a metal plate in the shape of the region bounded by the curves $y = x^3$, $x + y = 2$, $y = 0$, if the value of the density function at the coordinate point (x, y) is $\rho(x, y) = (1 + y)$ kg/m^2. (Hint: Look at a thin horizontal rectangle of across the region, and consider its tiny mass by $\Delta m \approx (\textit{thin area})(\textit{density})$. Then, $\sum \Delta m \longrightarrow \int_?^{??}$ (a suitable function of y)dy.)