1. Limits

$\lim\limits_{x \to a} f(x) = L \xrightarrow{\text{means}}$ As x tends to a and $\boxed{x \neq a}$, the value of $f(x)$ approaches L.

gets closer and closer to

can be arbitrary close to L

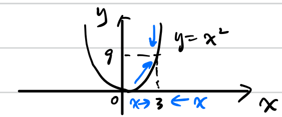
Notice : 1. $|f(x) - L|$ can be arbitrary small if x is close enough to a.

2. We never consider $x = a$ when we find $\lim\limits_{x \to a} f(x)$.

Sometimes, $a \notin$ domain of f.

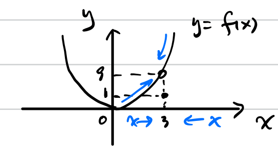
Example 1. $y = f(x) = x^2$.

$\lim\limits_{x \to 3} f(x) = 9$



Example 2. $f(x) = \begin{cases} x^2 & x \neq 3 \\ 1 & x = 3. \end{cases}$

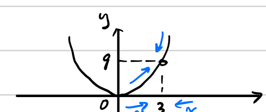
$\lim\limits_{x \to 3} f(x) = 9$.



Example 3. $f(x) = \dfrac{x^3 - 3x^2}{x - 3}$.

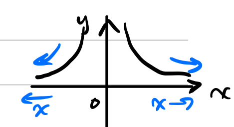
$\lim\limits_{x \to 3} f(x) = 9$

(Observe that $f(x) = x^2$ when $x \neq 3$)



Example 4 : $f(x) = \dfrac{1}{x^2}$

$\lim\limits_{x \to +\infty} f(x) = 0$ $\lim\limits_{x \to -\infty} f(x) = 0$



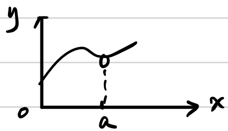
When $x \to \pm\infty$, $\frac{1}{x^2}$ can be arbitrary small. $\xRightarrow{\text{means}}$ $\frac{1}{x^2}$ approaches 0.

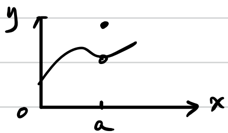
In general, if $\lim_{x\to a} f(x)$ exists, then we have three situations:

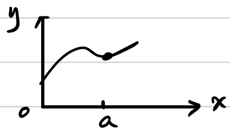
① $f(a)$ is not defined.

② $f(a)$ is defined but $\lim_{x\to a} f(x) \neq f(a)$

③ $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$







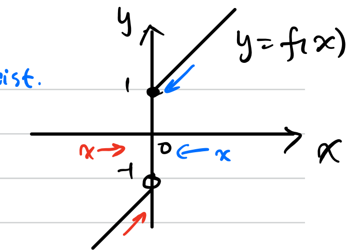
Sometimes, $\lim_{x\to a} f(x)$ does not exist.

Example 1: $f(x) = \frac{1}{x^2}$. $\lim_{x\to 0} f(x)$ does not exist.

When $x \to 0$, $\frac{1}{x^2}$ can be arbitrary large.

$\Rightarrow$ When $x \to 0$, $\frac{1}{x^2}$ does not approach any real number.

Example 2. $f(x) = \begin{cases} x+1 & x \geq 0 \\ x-1 & x < 0. \end{cases}$ $\qquad \lim_{x\to 0} f(x) = ?$ does not exist.



We observe that:

If $x<0$ and $x\to 0$, then $f(x)\to -1$ $\qquad \lim_{x\to 0^-} f(x) = -1$

If $x>0$ and $x\to 0$, then $f(x)\to 1$. $\qquad \lim_{x\to 0^+} f(x) = 1$

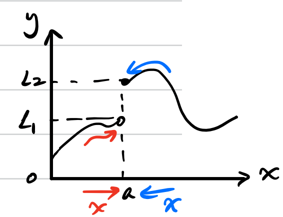
→ As x tends to 0, $f(x)$ does not approach any number.

($f(x)$ can not be arbitrary close to 1 if x tends to 0 from the left.
$f(x)$ can not be arbitrary close to -1 if x tends to 0 from the right.)

2. One-sided Limits

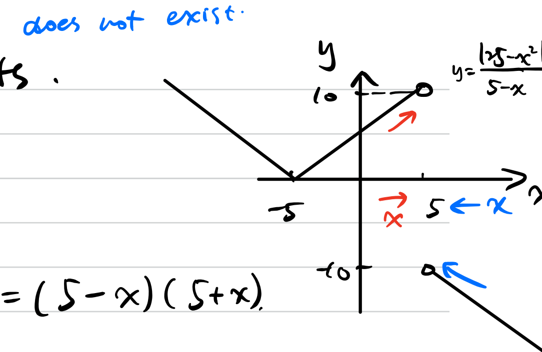
left-hand limit: $\lim_{x\to a^-} f(x) = L_1 \xrightarrow{\text{means}}$ As x approaches a and $x<a$, $f(x)$ approaches L_1.

right-hand limit: $\lim_{x\to a^+} f(x) = L_2 \xrightarrow{\text{means}}$ As x approaches a and $x>a$, $f(x)$ approaches L_2.



Notice: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.

Example 2. Determine if $\lim\limits_{x\to 5} \dfrac{|25-x^2|}{5-x}$ exists.



We observe that:

1) when $-5 \le x < 5$. $|25-x^2| = 25-x^2 = (5-x)(5+x)$.

$$\lim_{x\to 5^-} \frac{|25-x^2|}{5-x} = \lim_{x\to 5^-} \frac{25-x^2}{5-x} = \lim_{x\to 5^-} 5+x = 10$$

2) when $x > 5$ or $x \le -5$. $|25-x^2| = x^2-25 = (x+5)(x-5)$.

$$\lim_{x\to 5^+} \frac{|25-x^2|}{5-x} = \lim_{x\to 5^+} \frac{x^2-25}{5-x} = \lim_{x\to 5^+} -(x+5) = -10$$

We obtain $\lim\limits_{x\to 5^-} \dfrac{|25-x^2|}{5-x} \neq \lim\limits_{x\to 5^+} \dfrac{|25-x^2|}{5-x}$.

$\Rightarrow \lim\limits_{x\to 5} \dfrac{|25-x^2|}{5-x}$ does not exist.

3. How to compute the limits

1). limit laws: Suppose $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist.

Then we have:

① $\lim\limits_{x\to a} f(x)+g(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

② $\lim\limits_{x\to a} f(x)-g(x) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

③. $\lim\limits_{x\to a} f(x)\cdot g(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$

④ $\lim\limits_{x\to a} c\cdot f(x) = c\cdot \lim\limits_{x\to a} f(x)$ (c is a constant)

⑤ $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ $\boxed{\text{if } \lim\limits_{x\to a} g(x) \neq 0.}$

Example: $\lim\limits_{x\to 2} \dfrac{x^2+2x}{x+3} \overset{⑤}{=} \dfrac{\lim\limits_{x\to 2}(x^2+2x)}{\lim\limits_{x\to 2}(x+3)} \overset{①}{=} \dfrac{\lim\limits_{x\to 2}(x^2)+\lim\limits_{x\to 2}(2x)}{\lim\limits_{x\to 2} x + 3} = \dfrac{4+4}{5} = \dfrac{8}{5}.$

can not directly use limit laws.

2). Some algebraic tricks in limit computation. $\left(\underbrace{\frac{0}{0}, \frac{\infty}{\infty}}, \infty-\infty\right)$.

① $\frac{0}{0}$ type: $\lim\limits_{x\to 0^-} \frac{6x+5|x|}{7x-3|x|}$. $\quad \lim\limits_{x\to 2} \frac{\sqrt{x+7}-3}{x-2}$.

trick: 1) cancel the common factors in the numerator and denominator.

$$\lim_{x\to 0^-} \frac{6x+5|x|}{7x-3|x|} = \lim_{x\to 0^-} \frac{6x-5x}{7x+3x} = \lim_{x\to 0^-} \frac{x}{10x} = \lim_{x\to 0^-} \frac{1}{10} = \frac{1}{10}.$$

Recall $|x| = -x$ when $x<0$.

2) rationalize the numerator or the denominator.

$$\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x-2} = \lim_{x\to 2} \frac{(\sqrt{x+7}-3)(\sqrt{x+7}+3)}{(x-2)(\sqrt{x+7}+3)} = \lim_{x\to 2} \frac{x+7-9}{(x-2)(\sqrt{x+7}+3)}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x+7}+3)} = \lim_{x\to 2} \frac{1}{\sqrt{x+7}+3}$$

$$= \frac{1}{\lim\limits_{x\to 2}(\sqrt{x+7}+3)} = \frac{1}{\left(\lim\limits_{x\to 2}\sqrt{x+7}\right)+3} = \frac{1}{6}.$$

② $\frac{\infty}{\infty}$ type: $\lim\limits_{x\to+\infty} \frac{3x^3+100x^2+1}{2x^3+2}$. $\lim\limits_{x\to+\infty} \frac{\sqrt{4x^4+1}+1}{x^2+1}$.

trick: reduce the highest power of x in the numerator or denominator.

$$\lim_{x\to+\infty} \frac{3x^3+100x^2+1}{2x^3+2} \overset{\text{divide by } x^3}{=} \lim_{x\to+\infty} \frac{3+\frac{100}{x}+\frac{1}{x^3}}{2+\frac{2}{x^3}} \overset{⑤}{=} \frac{\lim\limits_{x\to+\infty} 3+\frac{100}{x}+\frac{1}{x^3}}{\lim\limits_{x\to+\infty} 2+\frac{2}{x^3}}$$

$$\overset{①}{=} \frac{3+\lim\limits_{x\to+\infty}\frac{100}{x}+\lim\limits_{x\to+\infty}\frac{1}{x^3}}{2+\lim\limits_{x\to+\infty}\frac{2}{x^3}} = \frac{3+0+0}{2+0} = \frac{3}{2}$$

$$\lim_{x\to+\infty} \frac{\sqrt{4x^4+1}+1}{x^2+1} \overset{\text{divide by } x^2}{=} \lim_{x\to+\infty} \frac{\sqrt{4+\frac{1}{x^4}}+\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\lim\limits_{x\to+\infty}\left(\sqrt{4+\frac{1}{x^4}}+\frac{1}{x^2}\right)}{\left(\lim\limits_{x\to+\infty}\frac{1}{x^2}\right)+1}$$

$$= \frac{\sqrt{4+0}+0}{1+0} = 2.$$

③ $\infty - \infty$ type: $\lim_{x\to+\infty} \sqrt{x^2+3x+2} - x$. $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x\sqrt{x+1}}$.

trick: rationalize $f(x)-g(x)$

$$\lim_{x\to+\infty} \sqrt{x^2+3x+2} - x = \lim_{x\to+\infty} \frac{\sqrt{x^2+3x+2} - x}{1} = \lim_{x\to+\infty} \frac{(\sqrt{x^2+3x+2}-x)(\sqrt{x^2+3x+2}+x)}{1\cdot(\sqrt{x^2+3x+2}+x)}.$$

$$= \lim_{x\to+\infty} \frac{x^2+3x+2-x^2}{\sqrt{x^2+3x+2}+x} = \lim_{x\to+\infty} \frac{3x+2}{\sqrt{x^2+3x+2}+x} \to \frac{\infty}{\infty} \text{ type.}$$

divide by x

$$= \lim_{x\to+\infty} \frac{3+\frac{2}{x}}{\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}+1} \overset{⑤}{=} \frac{\lim_{x\to+\infty} 3+\frac{2}{x}}{\lim_{x\to+\infty} \sqrt{1+\frac{3}{x}+\frac{2}{x^2}}+1} = \frac{3}{2}.$$

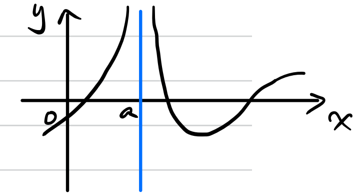
$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x\sqrt{x+1}} = \lim_{x\to 0} \frac{\sqrt{x+1}}{x\cdot\sqrt{x+1}} - \frac{1}{x\sqrt{x+1}}. = \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x\sqrt{x+1}} \to \frac{0}{0} \text{ type}$$

$$= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x\sqrt{x+1}(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{x+1-1}{x\sqrt{x+1}(\sqrt{x+1}+1)}$$

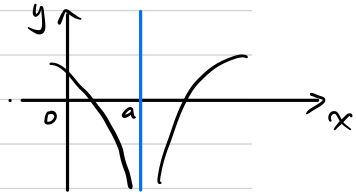
$$= \lim_{x\to 0} \frac{x}{x\sqrt{x+1}(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}(\sqrt{x+1}+1)} = \frac{1}{2}.$$

4. Infinite Limits

$\lim_{x\to a} f(x) = +\infty \xrightarrow{\text{means}}$ As x approaches a, $f(x)$ can be arbitrary large.



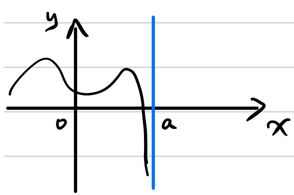
$\lim_{x\to a} f(x) = -\infty \xrightarrow{\text{means}}$ As x approaches a, $f(x)$ can be arbitrary large negative.

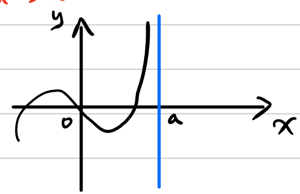


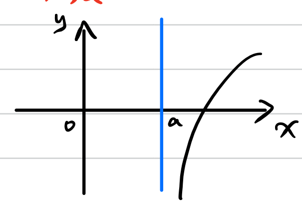
Notice: $\lim_{x\to a} f(x) = \pm\infty \Longrightarrow \lim_{x\to a} f(x)$ does not exist.

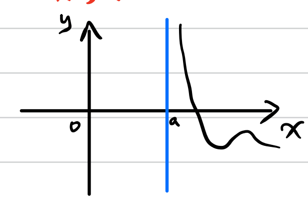
one-sided infinite limits:

$\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = +\infty$, $\lim_{x\to a^+} f(x) = -\infty$. $\lim_{x\to a^+} f(x) = +\infty$.









The line $x=a$ is called a "vertical asymptote" of $y=f(x)$.