Math2001 Answer to Homework 8

Exercise 3.37

Given $r \in X$ and $s \notin X$, obviously we have $r \neq s$. If $r < s$, according to the definition of Dedekind cuts, $r < s$ and $r \in X$ imply $s \in X$, contradiction! So we have $r > s$.

Given $r \notin X$ and $s < r$, if $s \in X$, according to the definition of Dedekind cuts, $s < r$ and $s \in X$ imply $r \in X$, contradiction! So we have $s \notin X$.

Exercise 3.38

Consider $\epsilon' = \frac{\epsilon}{2}$, according to Lemma 3.6.2., there are $r \in X$ and $s \notin X$, such that $r - s = \epsilon' = \frac{\epsilon}{2} < \epsilon$.

Exercise 3.40

Given $Y \subseteq \mathbb{Q}$ with lower bound $l \in \mathbb{Q}$, i.e. $l < t$ for all $t \in Y$. First condition: for any $s \in Y$, we have $s > t$ for some $t \in Y$, then $s > l$, i.e. l is the lower bound of Y; second condition: if $r \in Y$ and $s > r$, $r \in Y$ implies $r > t$ for some $t \in Y$, then $s > t$ for same t, hence $s \in Y$; third condition: for any $s \in Y$, i.e. $s > t$ for some $t \in Y$, consider $r = \frac{s+t}{2} < s$, then $r > t$ implies $r \in Y$.

Exercise 3.41

$-\sqrt{2} := \{r \in \mathbb{Q} : r < 0, r^2 < 2\} \cup \{r \in \mathbb{Q} : r \geq 0\}$, $1 - \sqrt{2} := \{r \in \mathbb{Q} : r < 1, (r-1)^2 < 2\} \cup \{r \in \mathbb{Q} : r \geq 1\}$.

Exercise 3.43

Suppose A is a set of real numbers(Dedekind cuts) with lower bound L, $L < X$ for all $X \in A$, i.e. for all $X \in A$, we have $X \subset L$ and $X \neq L$. Consider $Z = \cup_{X \in A} X$ union of all Dedekind cuts in A, then $Z \subset L$ obviously. We will prove that Z is also a Dedekind cut and $Z = \inf A$.

Z is a Dedekind cut: first condition: L is a Dedekind cut, so there is $l \in \mathbb{Q}$ such that $l < r$ for all $r \in L$, by $Z \subset L$, $l < r$ for all $r \in Z$; second condition: if $r \in Z$ and $r < s$, $r \in Z$ implies that $r \in X$ for some $X \in A$, then $r \in X$ and $r < s$ implies $r \in X \subset Z$ for that X is a Dedekind cut; third condition: for any $r \in Z$, then $r \in X$ for some $X \in A$, there exists $s \in X \subset Z$ such that $s < r$ for that X is a Dedekind cut.

$Z = \inf A$: firstly $X \subset Z$ means $Z \leq X$ for all $X \in A$, which means Z is a lower bound of A; secondly for any $Y > Z$, i.e. $Y \subset Z$ and $Y \neq Z$, so there exists $r_0 \in Z$ but $r_0 \notin Y$; $r_0 \in Z$ implies $r_0 \in X_0$ for some $X_0 \in A$, then $X_0 \not\subset Y$ obviously, i.e. $Y \not< X_0$, so Y is not a lower bound.

Exercise 3.42

Given X is a Dedekind cut and $A = \{r \in \mathbb{Q} : r > X\} = \{h(r) \in \mathbb{R} : h(r) > X\}$, where $h : \mathbb{Q} \to \mathbb{R}$ is a map which maps r to $h(r) = \{a \in \mathbb{Q} : a > r\}$.

According to previous Ex3.43, we have $\inf A = \cup_{h(r) \in A} h(r) = \cup_{h(r) > X} h(r)$. $h(r) > X$ implies $h(r) \subset X$, so we have $\inf A = \cup_{h(r) \subset X} h(r) \subset X$. Conversely, for any $a \in X$, by definition of Dedekind cut, there is $r \in X$ such that $r < a$, then $a \in h(r) \subset \inf A$; so we have $X \subset \inf A$. Above all, $X = \inf A$.

Exercise 3.45

For $x, y, z \leq 0$, then $-(x+y), -x, -y, -z \geq 0$, then

$$[-(x+y)](-z) = [(-x) + (-y)](-z) = (-x)(-z) + (-y)(-z)$$

also we have $(x+y)z = [-(x+y)](-z)$, $(-x)(-z) = xz$ and $(-y)(-z) = yz$, so $(x+y)z = xz + yz$.

Exercise 3.46

For $X, Y > 0, X > Y$, we have $X \subset Y$ and $X \neq Y$. The construction of X^{-1} and Y^{-1} is as follows: consider

$$\begin{aligned}\bar{X}^{-1} &= \{t \in \mathbb{Q} : tr > 1 \text{ for all } r \in X\}\\ \bar{Y}^{-1} &= \{t \in \mathbb{Q} : tr > 1 \text{ for all } r \in Y\}\end{aligned}$$

then we can set

$$\begin{aligned}X^{-1} &= \{s \in \mathbb{Q} : s > t \text{ for some } t \in \bar{X}^{-1}\}\\ Y^{-1} &= \{s \in \mathbb{Q} : s > t \text{ for some } t \in \bar{Y}^{-1}\}\end{aligned}$$

by the construction of the multiplicative inverse, we heve $X^{-1}, Y^{-1} > 0$. Then $X^{-1} \neq Y^{-1}$, otherwise $X^{-1} = Y^{-1}$ implies $1 = XX^{-1} > YX^{-1} = 1$, which is a contradiction.

Then to prove $X^{-1} < Y^{-1}$, we only need to prove that $Y^{-1} \subset X^{-1}$: given $s \in Y^{-1}$, we have $s > t_0$ for some t_0 satisfies $t_0 r > 1$ for all $r \in Y$; by $X \subset Y$, $t_0 r > 1$ for all $r \in X$, which means $s \in X^{-1}$, hence $Y^{-1} \subset X^{-1}$.

Exercise 3.53

The equation can be written as $z^2 + z + 1 = (z + \frac{1}{2})^2 + \frac{3}{4} = 0$, i.e. $(z + \frac{1}{2})^2 = -\frac{3}{4}$, the solutions will be $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Note that $z^3 - 1 = (z - 1)(z^2 + z + 1)$, so $z^3 = 1$ have solutions $z_0 = 1$, $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Exercise 3.54(2)

Suppose $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, for $r_1, r_2 \geq 0$. Let $\gamma = \theta_2 - \theta_1$, then

$$\begin{aligned}&|z_1 + z_2| = |e^{i\theta_1}||r_1 + r_2 e^{i\gamma}|\\ =&1 \cdot \sqrt{(r_1 + r_2\cos\gamma)^2 + r_2^2 \sin^2\gamma}\\ =&\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\gamma}\\ \leq&\sqrt{r_1^2 + r_2^2 + 2r_1 r_2}\\ =&r_1 + r_2 = |z_1| + |z_2|.\end{aligned}$$

Exercise 3.54(5)

Note that $z\bar{z} = |z|^2$, then

$$\begin{aligned}&|z_1 + z_2|^2 + |z_1 - z_2|^2\\ =&(z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})\\ =&(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)\\ =&2z_1\bar{z}_1 + 2z_2\bar{z}_2 = 2(|z_1|^2 + |z_2|^2).\end{aligned}$$

Exercise 3.55

The equation can be written as $(z+1)^4 = -i(z+2)^4$, if $z + 2 = 0$, then $(z+1)^4 = 0$ which means $z + 1 = 0$, contradiction with $z + 2 = 0$, so we have $z + 2 \neq 0$. So we will have

$$\left(\frac{z+1}{z+2}\right)^4 = -i = e^{-i\frac{\pi}{2}}$$

then we have

$$1 + \frac{1}{z+1} = \frac{z+2}{z+1} = e^{i\alpha}$$

for $\alpha = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. Then $z = \frac{1}{e^{i\alpha}-1} - 1$ for $\alpha = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$.

Exercise 3.56

Suppose $z = re^{i\theta}$ for $r \geq 0$, then $z^2 = r^2 e^{2i\theta}$ and $\bar{z} = re^{-i\theta}$, then $z^2 = \bar{z}$ can be written as

$$r^2 e^{2i\theta} = re^{-i\theta}$$

take modulus, we get $r^2 = r$, so $r = 0$ or $r = 1$. $r = 0$ implies $z = 0$. If $r = 1$, then $e^{2i\theta} = e^{-i\theta}$, i.e. $z^3 = e^{3i\theta} = 1$, then $z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, or $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ by previous exercise.

Above all, $z = 0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Exercise 3.58

Let $z = 1 + \sqrt{3}i = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2e^{\frac{\pi}{3}i}$, then $z^{10} = 2^{10} e^{\frac{10\pi}{3}i} = 2^{10} e^{\frac{4\pi}{3}i} = 2^{10}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = -2^9(1 + \sqrt{3}i)$.

Exercise 3.59

For $|z| = 1$, we have $z\bar{z} = |z|^2 = 1$, so $\frac{1}{z} = z^{-1} = \bar{z}$. Then

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = |\overline{z_1 + z_2 + z_3}| = |z_1 + z_2 + z_3| = 1.$$

Exercise 3.60

Firstly, we have $|z+3| = |(z+1)+2| \leq |z+1| + 2 \leq 4$. Let $z = 1$, then $|z+1| = |1+1| = 2$, and $|z+3| = |1+3| = 4$. So the maximum of $|z+3|$ is 4.