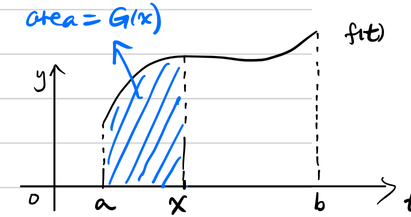
Review: Fundamental Theorem of Calculus (FTC)



1st version of FTC: Define $G(x) = \int_a^x f(t)\,dt$ (called "area function")

Then $\boxed{G'(x) = f(x)}$.

2nd version of FTC: If $F(x)$ is an antiderivatie of $f(x)$.

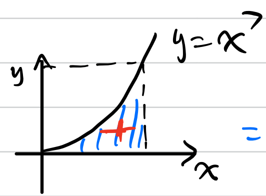
then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

If we use FTC to calculate $\int_a^b f(x)\,dx$, we take 2 steps:

Step 1: Find an antiderivative $F(x)$ of $f(x)$. (Find $\int f(x)\,dx$)

step 2: Calculate $F(b) - F(a)$

Example: Find $\int_0^1 x^7\,dx =$



$= \frac{1}{8}$

Step 1: Find $\int x^7\,dx$: $\int x^7\,dx = \underbrace{\frac{1}{8}x^8}_{= F(x)} + C$.

Step 2: Find $F(1) - F(0)$: $\int_0^1 x^7\,dx \overset{FTC}{=} F(1) - F(0) = \frac{1}{8}$

idea: We make a substitution to replace a complicated integral by a simple integral

Let's combine FTC with the Substitution Rule.

Recall: If $u=g(x)$, then $\int \underline{f(g(x))} \cdot g'(x)\,dx = \int f(u)\,du$

Example 1: $\int_0^{\frac{\pi}{2}} \cos x \cdot (\sin x)^2 \cdot dx = \frac{1}{3}$

Step 1: Find $\int \cos x\, (\sin x)^2\, dx$. (use Substitution Rule).

Let $u=g(x)=\sin x$. Let $f(x)=x^2$.

$$\int \cos x \cdot (\sin x)^2\, dx = \int g'(x)\cdot (g(x))^2\, dx = \int g'(x)\cdot f(g(x))\, dx$$

$$\overset{\text{Substitution Rule}}{=} \int f(u)\,du = \int u^2\,du = \underbrace{\frac{1}{3}u^3 + C}_{\text{easy to find.}}$$

$$\overset{\text{return to } x}{=} \underbrace{\frac{1}{3}(\sin x)^3}_{=F(x)} + C$$

Check: $\frac{d}{dx}\left(\frac{1}{3}\cdot(\underset{u}{\sin x})^3\right) \overset{\text{chain rule}}{=} \frac{d}{du}\left(\frac{1}{3}u^3\right)\cdot\frac{du}{dx} = u^2\cdot\cos x = (\sin x)^2\cdot\cos x$

Step 2: Calculate $F(\frac{\pi}{2}) - F(0)$

$$\int_0^{\frac{\pi}{2}} \cos x\cdot(\sin x)^2\, dx \overset{\text{FTC}}{=} \frac{1}{3}\left(\sin\frac{\pi}{2}\right)^3 - \frac{1}{3}(\sin 0)^3 = \frac{1}{3}$$

Example 2. $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \, e^{\sin x + 3} \, dx = e^4 - e^3$

step 1: Find $\displaystyle\int \cos x \, e^{\sin x + 3} \, dx$. (use Substitution Rule).

Let $u = g(x) = \sin x + 3$. Let $f(x) = e^x$.

Then $\displaystyle\int \underset{\sim}{\cos x} \cdot \underline{\underline{e^{\sin x + 3}}} \, dx = \int \underset{\sim}{g'(x)} \cdot \underline{\underline{e^{g(x)}}} \, dx = \int \underset{\sim}{g'(x)} \cdot \underline{\underline{f(g(x))}} \, dx$

$$\overset{\text{Substitution}}{\underset{\text{Rule}}{=\!=}} \int f(u) \, du = \int e^u \, du = e^u + C$$

easy to find.

$$\overset{\text{return to } x}{=\!=} \underbrace{e^{\sin x + 3}}_{= F(u)} + C.$$

step 2: Calculate $F(\frac{\pi}{2}) - F(0)$

$$\int_0^{\frac{\pi}{2}} \cos x \cdot e^{\sin x + 3} \, dx \overset{\text{FTC}}{=\!=} F(\tfrac{\pi}{2}) - F(0)$$

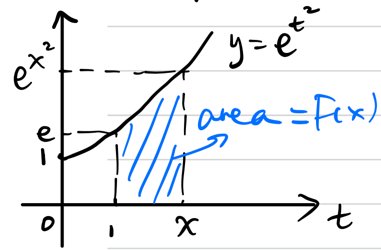
$$= e^{\sin \frac{\pi}{2} + 3} - e^{\sin 0 + 3} = e^4 - e^3$$

Notice: The simpler f is, the easier it is to find $\int f(u) \, du$.

Two $\boxed{\text{important}}$ applications of definite integral :

1st application : We can define some new functions by using definite integral, and their derivatives are easy to find

Example 1: Define $F(x) = \int_1^x e^{t^2}\,dt$ for $x \geq 1$

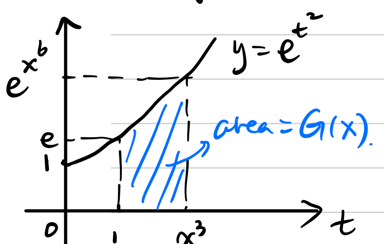


Recall: $F(x)$ is the area function of e^{x^2}.

$F(x)$ is hard to find,

but $\frac{d}{dx}F(x)$ is easy to find: $\frac{d}{dx}F(x) \underset{\text{of FTC}}{\overset{\text{1st version}}{=\!=}} e^{x^2}$.

Example 2: Define $G(x) = \int_1^{x^3} e^{t^2}\,dt$ for $x \geq 1$



Notice that $G(x) = F(x^3)$

so $\frac{d}{dx}G(x)$ is also easy to find:

$$\frac{d}{dx}G(x) = \frac{d}{dx}F(\underbrace{x^3}_{=u}) \underset{\text{rule}}{\overset{\text{chain}}{=\!=}} \frac{d}{du}F(u) \cdot \frac{du}{dx} = e^{u^2} \cdot 3x^2$$

$$= e^{x^6} \cdot 3x^2.$$

In general, if $G(x)=\int_{h(x)}^{f(x)} g(t)\,dt$, we take 2 steps to find $\frac{d}{dx}G(x)$:

$\boxed{\text{Step 1}}$: Let $F(x)$ be an antiderivative of $g(x)$. → Notice: we do not need to find $F(x)$

Then we have $G(x) \overset{FTC}{=} F(f(x)) - F(h(x))$.

$\boxed{\text{Step 2}}$: use $\boxed{\text{the chain rule}}$ to calculate $\frac{d}{dx}G(x)$.

Example 3. Find $\frac{d}{dx}G(x)$ if $G(x)=\int_0^{\cos x}\sin(t^2)\,dt$.

Step 1: Let $F(x)$ be an antiderivative of $\sin(x^2)$ → We do not need to know what $F(x)$ is.

Then $\underline{\frac{d}{dx}F(x)=\sin(x^2)}$

and $G(x)=\int_0^{\cos x}\sin(t^2)\,dt \overset{FTC}{=} F(\cos x)-F(0)$.

Step 2: $\frac{d}{dx}G(x) = \frac{d}{dx}F(\cos x) - \frac{d}{dx}\underline{F(0)}$ → just a constant.

$= \frac{d}{dx}F(\underbrace{\cos x}_{u}) \overset{\text{chain rule}}{=} \underline{\frac{d}{du}F(u)} \cdot \underline{\frac{du}{dx}}$

$= \underline{\sin(u^2)} \cdot \underline{(-\sin x)} = -\sin((\cos x)^2)\cdot \sin x$

Summary:

If $G(x)=\int_{h(x)}^{f(x)} g(t)\,dt$, then we can use FTC and chain rule to find $\frac{d}{dx}G(x)$.

A new notation: Suppose that $m \le n$ and $a_m, a_{m+1}, \cdots, a_n$ are real numbers.

$$\underbrace{\sum_{i=m}^{n} a_i}_{} \overset{\text{denote}}{=} a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k = \sum_{t=m}^{n} a_t.$$

→ a new way to rewrite a sum of real numbers

Example 1.
$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5 = \sum_{k=1}^{5} a_k = \sum_{t=1}^{5} a_t$$

$$\sum_{i=2}^{4} a_i = a_2 + a_3 + a_4. \xRightarrow{\text{if } a_i = i} \sum_{i=2}^{4} i = 2+3+4 = 9.$$

Example 2.
$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{1}{2} n(n+1).$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} \cdot n \cdot (n+1)(2n+1)$$

Example 3: Riemann sum $\frac{b-a}{n} f(c_1) + \cdots + \frac{b-a}{n} f(c_n) = \sum_{i=1}^{n} \frac{b-a}{n} \cdot f(c_i)$

2nd application: we can use the definite integral to calculate the limit of a sum of real numbers.

Example 1: Let's find $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{1}{n+i} = \ln 2 \approx 0.6931$

$$\sum_{i=1}^{n} \frac{1}{n+i} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \rightarrow \text{a sum of } n \text{ real numbers.}$$

e.g. If $n=100$, then $\sum\limits_{i=1}^{n} \frac{1}{n+i} = \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \cdots + \frac{1}{200} \approx 0.6906$ a sum of 100 numbers

If $n=1000$, then $\sum\limits_{i=1}^{n} \frac{1}{n+i} = \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots + \frac{1}{2000} \approx 0.6929$ a sum of 1000 numbers

As $n \to +\infty$: ① each term $\frac{1}{n+i} \to 0$

② but more and more numbers are added up.

How to find $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{1}{n+i} = \lim\limits_{n\to+\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right)$?

key idea: consider $\sum_{i=1}^{n} \frac{1}{n+i}$ as a Riemann sum $\Rightarrow$ $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \int_0^1 \frac{1}{1+x}\,dx \overset{FTC}{=} F(1)-F(0)$
$= \ln 2.$

To find this limit, we take the following 2 steps:

$\boxed{\text{Step 1}}$ We find ① a function $f(x)$ ② an interval $[a,b]$

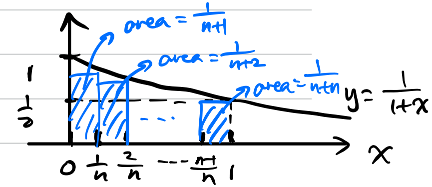
③ n respresentative points $c_1, ---, c_n$

such that $\sum_{i=1}^{n} \frac{1}{n+i} = \sum_{i=1}^{n} \frac{b-a}{n} f(c_i)$ $\rightarrow$ Riemann sum of $f(x)$ on $[a,b]$.

Then we have: $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(c_i) = \int_a^b f(x)\,dx.$

$\boxed{\text{Step 2}}$ We use FTC to calculate $\int_a^b f(x)\,dx$: $\int_a^b f(x)\,dx \overset{FTC}{=} F(b)-F(a).$

$\Rightarrow$ we obtain $\boxed{\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n+i} = F(b)-F(a)}$



Step 1: Notice that $\frac{1}{n+i} = \frac{1}{n} \cdot \frac{1}{1+\frac{i}{n}}$,

so if we let $a=0$, $b=1$, each $c_i = \frac{i}{n}$ and $f(x) = \frac{1}{1+x}$, then $\frac{1}{n+i} = \frac{b-a}{n} f(c_i)$

$\Rightarrow \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \int_0^1 \frac{1}{1+x}\,dx. = \ln 2.$

Step 2: Notice $\int \frac{1}{1+x}\,dx = \underbrace{\ln(1+x)}_{=F(x)} + C$ (if $x>-1$) $\Rightarrow \int_0^1 \frac{1}{1+x}\,dx \overset{FTC}{=} F(1)-F(0) = \ln 2.$

The choices of $f(x)$, $[a,b]$, $c_1 \cdots c_n$ in step 1 are not unique.

Trick: We should first consider the case: $a=0$, $b=1$, each $c_i = \frac{i}{n}$

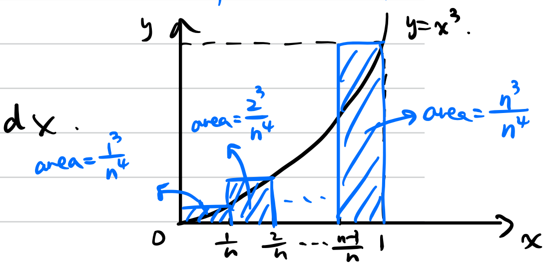
Example 2: Find $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{i^3}{n^4} = \lim\limits_{n\to+\infty} \frac{1^3+2^3+3^3+\cdots+n^3}{n^4} = \frac{1}{4}$.

Step 1: We aim to rewrite $\sum\limits_{i=1}^{n} \frac{i^3}{n^4}$ as $\sum\limits_{i=1}^{n} \frac{b-a}{n} f(c_i)$.

Let $a=0$, $b=1$, each $c_i = \frac{i}{n}$. Let $f(x) = x^3$.

Then $\frac{i^3}{n^4} = \frac{1}{n} \cdot \left(\frac{i}{n}\right)^3 = \frac{b-a}{n} \cdot c_i^3 = \frac{b-a}{n} \cdot f(c_i) \Rightarrow \sum\limits_{i=1}^{n} \frac{i^3}{n^4} = \sum\limits_{i=1}^{n} \frac{b-a}{n} f(c_i)$

$\Rightarrow \lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{i^3}{n^4} = \int_0^1 x^3\,dx$.



Step 2: We use FTC to calculate $\int_0^1 x^3\,dx$.

Notice: $\int x^3\,dx = \underbrace{\frac{1}{4}x^4}_{=F(x)} + C$.

$\int_0^1 x^3\,dx \overset{FTC}{=} F(1) - F(0) = \frac{1}{4}$.

Similarly, we can calculate $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{i^4}{n^5}$, $\lim\limits_{n\to+\infty} \sum\limits_{i=1}^{n} \frac{i^5}{n^6}$ $\cdots$

Summary: We have two different ways to calculate $\int_a^b f(x)dx$:

Method 1: $\int_a^b f(x)dx$ is the limit of a Riemann sum: $\int_a^b f(x)\,dx = \lim_{n\to+\infty} \underbrace{\sum_{i=1}^{n} \frac{b-a}{n} f(c_i)}_{\text{a sum of } n \text{ real numbers}}$.

Method 2: If we use FTC, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

An application: Calculate the limit of a sum of real numbers ($\lim_{n\to+\infty} \sum_{i=1}^{n} a_i$)

To find $\lim_{n\to+\infty} \sum_{i=1}^{n} a_i$, we take 2 steps:

Step 1: relate $\sum_{i=1}^{n} a_i$ to a Riemann sum $\sum_{i=1}^{n} \frac{b-a}{n} f(c_i)$

Then $\lim_{n\to+\infty} \sum_{i=1}^{n} a_i = \int_a^b f(x)\,dx$.

Step 2: use FTC to find $\int_a^b f(x)dx$.

Examples: $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n+i}$, $\lim_{n\to+\infty} \sum_{i=1}^{2n} \frac{1}{n+i}$, $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$, $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$, ...