# Math1014 Calculus II

## Week 5-6: Brief Review and Some Practice Problems

### Integration Techniques

---

No matter what techniques you use, the main strategy is to reduce an unfamiliar integral $\int f(x)dx$ to a more familiar one.

$$\int f(x)dx \begin{cases} \text{Any nice substitution to simplify the original one?} \\ \text{Any useful trigonometric substitution which may help simply the integral?} \\ \text{Any trigonometric identities helpful for simplifying the integral?} \\ \text{Will integration by parts work?} \\ \text{Can } f(x) \text{ be broken up into simpler pieces? } (\textit{Partial Fractions}) \end{cases}$$

- *Integration by parts*: $\displaystyle\int udv = uv - \int vdu$, which is just the antidifferentiation version of the *product rule* $uv' = (uv)' - vu'$. Roughly speaking $\int vdu$ should not look more difficult than $\int udv$ if you hit the correct choice of $u$, $v$.
- When using a substitution, say $u = u(x)$ or $x = x(u)$ to turn $\int f(x)dx$ into an easier $\int F(u)du$, don't forget to use $du = u'(x)dx$ or $dx = x'(u)du$ to "turn $dx$ into $du$" in a suitable way. Must work out the new $u$-interval endpoints too if working with definite integral.
- Trigonometric substitutions: essentially a matter of using the identities

  $$\sin^2\theta + \cos^2\theta = 1$$

  $$1 + \tan^2\theta = \sec^2\theta$$

  to take care of terms like $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, or power expressions of those.

  (If you are familiar with hyperbolic functions, may also try to use the identity $\cosh^2 t - \sinh^2 t = 1$.)
- Trigonometric integrals: often used identities are

  $$\begin{aligned} &\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \\ &\sin 2\theta = 2\sin\theta\cos\theta \\ &\sin A\cos B = \tfrac{1}{2}\big[\sin(A-B) + \sin(A+B)\big] \\ &\sin A\sin B = \tfrac{1}{2}\big[\cos(A-B) - \cos(A+B)\big] \\ &\cos A\cos B = \tfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big] \end{aligned}$$

  Very useful for bring down the powers of certain trigonometric terms (by the double angle formula), or breaking up products of trigonometric functions.
- Partial fractions: next exercise-example sheet.

---

1. Practise integration by parts by evaluating some of the following integrals.

   (i) $\displaystyle\int t\sin 2tdt$ (ii) $\displaystyle\int p^5 \ln pdp$ (iii) $\displaystyle\int e^{-\theta}\cos 2\theta d\theta$,

   (iv) $\displaystyle\int (x^2+1)e^{-x}dx$ (v) $\displaystyle\int_1^{\sqrt{3}} \tan^{-1}\frac{1}{x}dx$ (vi) $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}dx$

   (vii) $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr$ (viii) $\displaystyle\int x^2\sin 2xdx$
2. Use integration by parts to work out the reduction formula:

$$\int \sec^n xdx = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}dx\ .$$

3. If $f(0) = g(0) = 0$ and $f''$ and $g''$ are continuous, show that

$$\int_0^a f(x)g''(x)dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

4. Use suitable trigonometric identities and substitutions to evaluate the following integrals.

(i) $\displaystyle\int_0^{\pi/2} \sin^2(2\theta)d\theta$ (ii) $\displaystyle\int \frac{\sin^2(\sqrt{x})}{\sqrt{x}}dx$ (iii) $\displaystyle\int_0^{\pi} \sin^2 t \cos^4 t dt$

(iv) $\displaystyle\int \cos^2 x \sin 2x dx$ (v) $\displaystyle\int \tan^2(2x)\sec^5(2x)dx$ (vi) $\displaystyle\int_0^{\pi/3} \tan^5 x \sec^6 x dx$

5. Find the volume obtained by rotating the region bounded by the curves $y = \sec x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{3}$ about the line $y = -1$.

6. Use suitable trigonometric identities to help show that:

(i) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ for any integers $m, n$.

(ii) A *finite Fourier series* is given by the sum

$$f(x) = \sum_{i=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin(2x) + \cdots + a_N \sin(Nx)\ .$$

Show that the $m$-th coefficient $a_m$ is given by the formula $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin mx dx$.

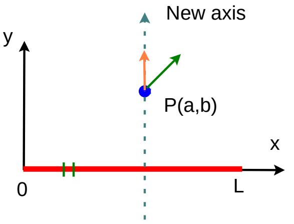
7. Evaluate the following integrals by suitable trigonometric substitutions.

(i) $\displaystyle\int_0^2 x^2\sqrt{x^2+4}dx$ (ii) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}dx$, where $a > 0$ is a constant. (iii) $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$,

(iv) $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx$ (v) $\displaystyle\int \frac{x^2}{(x^2+a^2)^{3/2}}dx$ by the substitution $x = a\sinh t$.

8. A charge rod of length $L$ produces an electric field at a point $P(a,b)$ *which has a vertical component* given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\epsilon_0(x^2+b^2)^{3/2}}dx$$




where $\lambda$ is the charge density per unit length on the rod and $\epsilon_0$ is the free space permittivity. Evaluate the integral. [Recall that by Coulomb's Law, the **magnitude** of the force on a test charge (of 1 coulomb) at a distance of $r$ away from another charge $q$ is given by $\dfrac{q}{4\pi\epsilon_0 r^2}$. So, consider a tiny piece of the charge rod and the resulting electrostatic force on the test charge at $P$.]

9. (Integration by parts.) Suppose that $f$ is a positive function such that $f'$ is continuous.

(i) How is the graph of $y = f(x)\sin nx$ related to the graph of $y = f(x)$? What happens as $n \longrightarrow \infty$? (Try $f(x) = x^2$, and $n = 2, 3, 4$ as starting examples.)

(ii) Make a guess as to the value of the limit $\displaystyle\lim_{n\to\infty}\int_0^1 f(x)\sin nx dx$ based on graphs of the integrand.

(iii) Using integration by parts, confirm the guess you made in part (b). [Use the fact that, since $f'$ is continuous, there is a constant $M$ such that $|f'(x)| \leq M$ for $0 \leq x \leq 1$.]