# Trigonometric Integrals

— Exercises in using substitution rule with trigonometric identities.

$$\begin{cases} \sin^2\theta + \cos^2\theta = 1 \\ 1 + \tan^2\theta = \sec^2\theta \end{cases} \longleftrightarrow \begin{cases} u = \sin\theta, & 1-u^2 = \cos^2\theta \\ u = \cos\theta, & 1-u^2 = \sin^2\theta \end{cases}$$

**Example**

$$\int \underbrace{\sin^3\theta}_{\sin^2\theta\,\sin\theta}\, d\theta$$

Let $u = \cos\theta$

$$\frac{du}{d\theta} = -\sin\theta$$

$$-du = \sin\theta\, d\theta$$

$$= \int -(1-u^2)\, du$$ integral of a polynomial

$$= \int (-1+u^2)\, du$$

$$= -u + \frac{u^3}{3} + C$$

$$= -\cos\theta + \frac{\cos^3\theta}{3} + C$$

**Example**

$$\int \underbrace{\sin^5\theta}_{\sin^4\theta\,\sin\theta}\, \boxed{\cos^2\theta}\, d\theta$$

$u = \cos\theta$

$-du = \sin\theta\, d\theta$

$1-u^2 = \sin^2\theta$

$$= \int -(1-u^2)^2\, u^2\, du$$

$$= -\int (1-2u^2+u^4)\, u^2\, du = -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= -\frac{\cos^3\theta}{3} + \frac{2}{5}\cos^5\theta - \frac{\cos^7\theta}{7} + C$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

In general,

① $\boxed{\int \underbrace{\sin^{n}\theta}_{\sin^{n-1}\theta\,\sin\theta} \cos^{m}\theta\, d\theta}$ where $n$ (odd) is a positive odd integer

$\underline{u = \cos\theta}$ $\quad \int (\text{a polynomial in } u)\, du \leftarrow \text{easy!}$

$1-u^2 = \sin^2\theta$

$(n-1 = \text{even})$

e.g. $\int \underbrace{\sin^7\theta}_{\sin^6\theta\,\sin\theta} \cos^4\theta\, d\theta$ $\qquad u = \cos\theta,\ 1-u^2 = \sin^2\theta$

$-du = \sin\theta\, d\theta$

$$= \int -(1-u^2)^3 u^4 \cdot du$$

② $\int \sin^n\theta \underbrace{\cos^m\theta}_{\cos^{m-1}\theta\,\cos\theta} \boxed{d\theta}$, where $m$ is a +ve odd number.

$u = \sin\theta,\ 1-u^2 = \cos^2\theta$

$du = \cos\theta\, d\theta$

$$= \underbrace{\int u^5 (\underbrace{1-u^2}_{\cos^2\theta})^{\frac{m-1}{2}} du}_{\text{a polynomial in } u.}$$

e.g. $\int \sin^4\theta \underbrace{\cos^3\theta}_{\cos^2\theta\,\cos\theta}\, d\theta$

$$= \int u^4 (1-u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5\theta}{5} - \frac{\sin^7\theta}{7} + C$$

$$\boxed{1+\tan^2\theta = \sec^2\theta}$$

$u = \sec\theta$

$u = \tan\theta$

Example

$$\int \tan\theta \sec^4\theta \, d\theta$$

$\sec^2\theta \sec^2\theta$

$u = \tan\theta$, $1+u^2 = \sec^2\theta$

$\frac{du}{d\theta} = \sec^2\theta$

$du = \sec^2\theta \, d\theta$

$$= \int u\cdot(1+u^2)\,du$$

$$= \int (u+u^3)\,du = \frac{u^2}{2} + \frac{u^4}{4} + C$$

$$= \frac{\tan^2\theta}{2} + \frac{\tan^4\theta}{4} + C$$

① $\int \tan^n\theta \sec^m\theta \, d\theta$ (even) $\xrightarrow[du = \sec^2\theta \, d\theta]{\text{Let } u = \tan\theta}$ $\int (\text{a polynomial in } u)\,du$

② $\int \tan^n\theta \sec^m\theta \, d\theta$ (+ve odd number) $= \int \tan^{\boxed{n-1}}\theta \sec^{m-1}\theta \, \underbrace{\boxed{\sec\theta\tan\theta \, d\theta}}_{du}$ (if even)

$u = \sec\theta$

$$\boxed{\begin{aligned} du &= \sec\theta\tan\theta\,d\theta \\ \tan^2\theta &= u^2-1 \end{aligned}}$$

$$\int \underbrace{(u^2-1)^{\frac{n-1}{2}} u^{m-1}}_{\text{a polynomial in } u} du$$

e.g. $\int \tan^3\theta \sec^5\theta \, d\theta = \int \underbrace{(u^2-1)}_{\tan^2\theta} \underbrace{u^4}_{\sec^4\theta} \underbrace{du}_{\sec\theta\tan\theta\,d\theta}$

$u = \sec\theta$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\sec^7\theta}{7} - \frac{\sec^5\theta}{5} + C$$

Example.

Other identities?

$\boxed{\text{Double Angle Formula}}$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(1) $\displaystyle\int \cos^2\theta \, d\theta$

$= \displaystyle\int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4} + C$

(2) $\displaystyle\int \cos^4\theta \, d\theta \longleftarrow$ $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$= \displaystyle\int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

corrected version after lecture! (I forgot the "2")

$= \dfrac{1}{4}\displaystyle\int \left(\underbrace{1 + 2\cos 2\theta}_{\text{easy to handle}} + \cos^2 2\theta\right) d\theta$

use double angle formula again

$= \dfrac{1}{4}\left[\theta + \sin 2\theta\right] + \dfrac{1}{4}\displaystyle\int \frac{1+\cos 4\theta}{2} \, d\theta$

$= \dfrac{\theta}{4} + \dfrac{\sin 2\theta}{4} + \dfrac{\theta}{8} + \dfrac{\sin 4\theta}{32} + C$

$3\theta/8$

$\left(\displaystyle\int \cos 4\theta \, d\theta = \frac{\sin 4\theta}{4} + C\right)$

Fact: $\int \cos^{2m}\theta\, d\theta$ can be handle by double angle formula and the substitution "$u = \sin k\theta$".

Same with $\int \sin^{2m}\theta\, d\theta$

Example $\int \cos^6\theta\, d\theta$

$(\cos^2\theta)^3 = \left(\frac{1+\cos 2\theta}{2}\right)^3$

$$= \frac{1}{8}\int (1+\cos 2\theta)^3\, d\theta$$

$$= \frac{1}{8}\int \left(1 + 3\cos 2\theta + 3\boxed{\cos^2 2\theta} + \cos^3 2\theta\right) d\theta$$

$3\cos^2 2\theta \to 3\,\frac{1+\cos 4\theta}{2}$

odd: $\boxed{u = \sin 2\theta}$, $1-u^2 = \cos^2 2\theta$

$du = 2\cos 2\theta\, d\theta$

$$= \frac{1}{8}\int \left(\frac{5}{2} + 3\cos 2\theta + \frac{3}{2}\cos 4\theta\right) d\theta + \frac{1}{8}\int \cos^2 2\theta\, \underbrace{\cos 2\theta\, d\theta}_{\frac{1}{2}du}$$

$$= \frac{1}{8}\left[\frac{5\theta}{2} + \frac{3\sin 2\theta}{2} + \frac{3\sin 4\theta}{8}\right] + \frac{1}{16}\underbrace{\int (1-u^2)\, du}_{u - u^3/3 + C}$$

$$= \frac{5\theta}{16} + \frac{3\sin 2\theta}{16} + \frac{3\sin 4\theta}{64} + \frac{\sin 2\theta}{16} - \frac{\sin^3 2\theta}{48} + C.$$

Example. $\int \sin^4\theta \, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$\boxed{(1-\cos 2\theta)^2 = 1 - 2\cos 2\theta + \cos^2 2\theta}$

$$= \frac{1}{4}\int \left(1 - \cancel{2\cos 2\theta} + \underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}}\right) d\theta$$

$$= \frac{1}{4}\left[\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{4}\right] + C$$

How about $\int \tan^4\theta \, d\theta$?

$1 + \tan^2\theta = \sec^2\theta$

$\longrightarrow \boxed{\int \tan^n\theta \, d\theta}$

$$= \int \tan^2\theta \, (\sec^2\theta - 1) \, d\theta$$

$$= \int \tan^2\theta \boxed{\sec^2\theta \, d\theta} - \underbrace{\int \frac{\tan^2\theta}{\sec^2\theta - 1} d\theta}_{-\tan\theta + \theta + C}$$

$u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$= \int u^2 \, du$$

$$= \frac{\tan^3\theta}{3} - \tan\theta + \theta + C$$

$$\int \tan^n\theta \, d\theta$$

power is reduced to $n-2$

$$= \int \tan^{n-2}\theta \,(\underset{\tan^2\theta}{\sec^2\theta - 1})\, d\theta$$

$$= \int \tan^{n-2}\theta \,\underbrace{\sec^2\theta \, d\theta}_{\text{Let } u=\tan\theta} - \int \tan^{n-2}\theta \, d\theta$$

$$\int u^{n-2}\, du \qquad\qquad \int \tan^{n-4}\theta \, d\theta$$

$$\vdots$$

reduction formula

$$\boxed{\int \tan^n\theta \, d\theta = \frac{\tan^{n-1}\theta}{n-1} - \int \tan^{n-2}\theta \, d\theta}$$

$$\int \tan\theta \, d\theta = \ln|\sec\theta| + C \qquad \Bigg| \qquad \int 1 \, d\theta = \theta + C$$