MATH 1012 Calculus IA.

$\boxed{\text{Functions}}$

A function f is a rule that defines the relationship between x and y.

$x \xrightarrow{f} y = f(x)$ — y: output, x: input

Notice: essentially same $\begin{cases} y = x+1. \\ g = u+1. \end{cases}$

x: independent variable. y: dependent variable (y is computed by x when f is given).

Notice: For a function f:

1) One input will only give one output.

$y = \pm\sqrt{x}$. → This rule is not a function. $x=4 \rightrightarrows y=2,\ y=-2$.

2). It's possible that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

$y = x^2$ → This is a function. $f(2) = f(-2) = 4$.

3). f is called a one-to-one function if $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$. → different inputs give different outputs.

$y = x^2$. is not one-to-one because $f(2) = f(-2) = 4$

$y = x$ is one-to-one.

$\boxed{\text{1. Domain and range}}$

Given $f(x)$.

Domain of f: the set of real numbers that can be mapped by f.
→ the set where we choose inputs.

Range of f: the set of real numbers to which f maps.
→ the set consisting of all possible outputs.

Notice: $f(x)$ is meaningful only if x is in the domain of f.

Example: $y = \frac{1}{x}$. $(x \neq 0)$.

Domain: $(-\infty, 0) \cup (0, +\infty)$ → always use a round bracket next to "∞".

Range: $(-\infty, 0) \cup (0, +\infty)$.

Example: $y = \sqrt{x}$.

Domain: $[0, +\infty)$.

Range: $[0, +\infty)$.

Example: $y_1 = \frac{x^2-1}{x-1}$. Domain: $(-\infty, 1) \cup (1, +\infty)$.

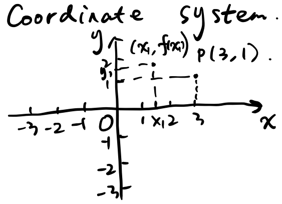
$y_2 = x+1$. Domain: $(-\infty, +\infty)$.

Notice: $y_1 \neq y_2$. (because they have different domains)

$\boxed{\text{2. Graph of function}}$

(the visualization of graph).

Coordinate system.



$x_1 \xrightarrow{f} y_1 = f(x_1)$.

The graph of f is the set $\{(x, f(x)) \mid x \in \underline{\underline{D}}\}$ → a curve formed by points $(x, f(x))$.

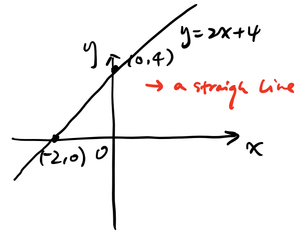
D: the domain of f.

Example : $y = 2x+4$. $\rightarrow$ linear function.

Find $\begin{cases} x\text{-intercept} : (x, f(x)) \text{ with } f(x)=0. \\ y\text{-intercept} : (0, f(0)). \end{cases}$

Table.

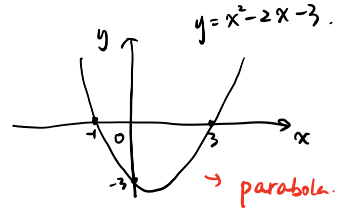
x	-3	-2	-1	0	1
y	-2	0	2	4	6



Example : $y = x^2-2x-3$
$= (x-1)^2-4.$

Table

x	-2	-1	0	1	2	3
y	5	0	-3	-4	-3	0

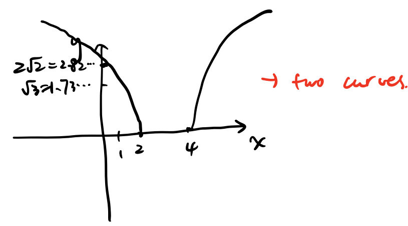


Example. $y = \sqrt{(x-2)(x-4)}$. $\rightarrow (x-2)(x-4) \geq 0$.

Domain : $(-\infty, 2] \cup [4, +\infty)$.

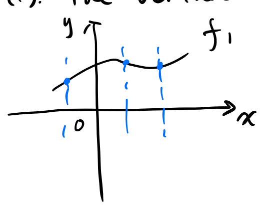
Table

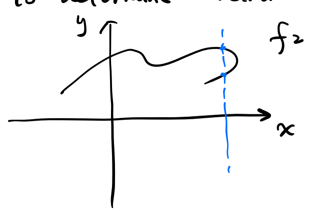
x	0	1	2	4	5
y	$2\sqrt{2}$	$\sqrt{3}$	0	0	$\sqrt{3}$



3. Two tests

(1). The vertical line test: to determine whether a curve is the graph of a function.





Recall : One input will only give one output.

Test : A curve is the graph of a function. if and only if each vertice line intersects this curve at most once.

Example : f_1 passes the test $\rightarrow f_1$ is the graph of a function.

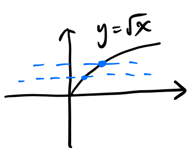
f_2 does not pass the test. $\rightarrow$ f_2 is not the graph of a function.

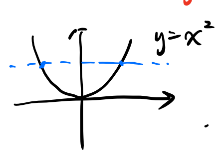
(2) The horizontal line test : to verify whether a function is a one-to-one function.

Recall : A one-to-one function satisfies $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.

Test : A function is one-to-one if and only if each horizontal line intersects the graph of f at most once.

Example :





$y=\sqrt{x}$ is a one-to-one function.

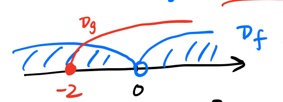
$y = x^2$ is not a one-to-one function.

4. Operations on functions

$f(x)$ $\quad g(x)$ $\quad D_f$: domain of f. $\quad D_g$: domain of g.

Operations	Domain of new function.
Sum: $(f+g)(\overset{\text{input}}{x}) = f(x)+g(x)$.	$D_{f+g} = \{x \mid x\in D_f.\ x\in D_g\} = D_f \cap D_g$.
Difference: $(f-g)(x) = f(x)-g(x)$.	$D_{f-g} = \{x \mid x\in D_f.\ x\in D_g\}$
Product: $(f\cdot g)(x) = f(x)\cdot g(x)$.	$D_{f\cdot g} = \{x \mid x\in D_f.\ x\in D_g\}$
Quotient: $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.	$D_{\frac{f}{g}} = \{x \mid x\in D_f.\ x\in D_g.\ \underline{g(x)\neq 0}\}$.
Composition: $(f\circ g)(x) = f(g(x))$.	$D_{f\circ g} = \{x \mid x\in D_g,\ \underline{g(x)\in D_f}\}$.
$\neq$ $(g\circ f)(x) = g(f(x))$.	$D_{g\circ f} = \{x \mid x\in D_f,\ \underline{f(x)\in D_g}\}$.

Example: $f(x) = \frac{1}{x^2}$. $g(x) = \sqrt{x+2}$.



D_f : $x\neq 0 \rightarrow (-\infty,0)\cup(0,+\infty)$. $\quad D_g$: $x+2\geq 0 \rightarrow [-2,+\infty)$.

$(f+g)(x) = \frac{1}{x^2} + \sqrt{x+2}$. $\qquad D_{f+g} = D_f \cap D_g = [-2,0)\cup(0,+\infty)$.

$(f-g)(x) = \frac{1}{x^2} - \sqrt{x+2}$. $\qquad D_{f-g} = D_f \cap D_g = [-2,0)\cup(0,+\infty)$.

$(f\cdot g)(x) = \frac{\sqrt{x+2}}{x^2}$ $\qquad D_{f\cdot g} = D_f \cap D_g = [-2,0)\cup(0,+\infty)$.

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{1}{x^2\cdot\sqrt{x+2}}$. $\qquad D_{\frac{f}{g}} = D_f \cap D_g \cap \underline{\{x \mid g(x)\neq 0\}} \rightarrow x>-2$.

$= (-2,0)\cup(0,+\infty)$.

$(f\circ g)(x) = f(g(x))$ $\qquad D_{f\circ g} = D_g \cap \underline{\{x \mid g(x)\in D_f\}} \rightarrow \sqrt{x+2}\neq 0 \rightarrow x>-2$.

$= \frac{1}{(g(x))^2} = \frac{1}{x+2}$. $\qquad = (-2,+\infty)$.

$(g\circ f)(x) = g(f(x))$ $\qquad D_{g\circ f} = D_f \cap \underline{\{x \mid f(x)\in D_g\}} \rightarrow \frac{1}{x^2}+2\geq 0$.

$= \sqrt{\frac{1}{x^2}+2}$. $\qquad = (-\infty,0)\cup(0,+\infty)$.

Notice: (1). $h(x) = \frac{1}{x+2}$. Domain of h : $x\neq -2. \rightarrow (-\infty,-2)\cup(-2,+\infty)$.

$f\circ g \neq h. \rightarrow$ They have different domains.

(2). In general, $f\circ g \neq g\circ f$.