Example:

$$\int \frac{x+4}{(x+1)(x^2+1)}\,dx$$

← no real roots!

$$\frac{d}{dx}\ln|x^2+1| = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}$$

Same idea

$$\int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right) dx$$

for some constants A, B, C

$$= A\int \frac{dx}{x+1} + \frac{B}{2}\int \frac{2x}{x^2+1}\,dx + C\int \frac{1}{x^2+1}\,dx$$

Let $x = \tan\theta$

$$= \underbrace{\boxed{A}}_{?}^{3/2}\ln|x+1| + \underbrace{\boxed{\frac{B}{2}}}_{?}^{-3/4}\ln|x^2+1| + \underbrace{\boxed{C}}_{?}^{5/2}\tan^{-1}x + C_1$$

Back to an algebra problem!

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$x+4 = A(x^2+1) + (Bx+C)(x+1)$$

① $A+B=0$ since there is no x^2 term!

i.e. $A=-B$

② Let $x=-1$.

$$-1+4 = A(1+1) + 0$$

$A = 3/2$, $B = -3/2$

③ $x+4 = \frac{3}{2}(x^2+1) + (-\frac{3}{2}x + C)(x+1)$

Let $x=0$, $4 = \frac{3}{2} + C$, $C = 4-\frac{3}{2} = \frac{5}{2}$

Why can't we have

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x^2+1} = \frac{Ax^2+A+Bx+B}{(x+1)(x^2+1)}$$

?

no x^2-term!

$$\frac{d}{dx}\ln|x^2+a| = \frac{2x}{x^2+a}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}\cdot\frac{1}{a} = \frac{1}{a^2+x^2}$$

$$\longrightarrow \frac{d}{dx}\ln|x^2+a| = \frac{2x}{x^2+a}, \quad \frac{d}{dx}\frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{1}{a^2+x^2}$$

Example.

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx = \int \left(\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}\right) dx$$

$$\longrightarrow = \frac{A}{2}\int \frac{2x}{x^2+1}\,dx + B\int \frac{1}{x^2+1}\,dx + \frac{C}{2}\int \frac{2x}{x^2+4}\,dx + D\int \frac{1}{x^2+4}\,dx$$

$$= \underbrace{\boxed{\frac{A}{2}}}_{0}\ln|x^2+1| + \underbrace{\boxed{B}}_{\frac{1}{3}}\tan^{-1}x\,dx + \underbrace{\boxed{\frac{C}{2}}}_{\shortparallel\ 0}\ln|x^2+4| + \underbrace{\boxed{D\cdot\frac{1}{2}}}_{-\frac{1}{6}}\tan^{-1}\frac{x}{2} + C_1$$

Length Algebra:

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)}$$

$$1 = Ax^3 + 4Ax + \boxed{Bx^2} + 4B + Cx^3 + Cx + \boxed{Dx^2} + D$$

$$1 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B+D$$

$x+1$

$$\begin{cases} A+C=0 & -(1) \\ B+D=0 & -(2) \\ 4A+C=0 & -(3) \\ 4B+D=1 & -(4) \end{cases}$$

$(4)-(2)$: $3B = 1$, $B=\frac{1}{3}$ and $D=-B=-\frac{1}{3}$

$(3)-(1)$: $3A=0$, $A=C=0$

How about $\displaystyle\int \frac{x+1}{(x^2+1)(x^2+4)}\,dx =$ Same calculation

$$x+1 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B+D$$

$$\begin{cases} A+C=0 & -(1) \\ B+D=0 & -(2) \\ 4A+C=1 & -(3) \\ 4B+D=1 & -(4) \end{cases}$$

$(3)-(1) \Rightarrow 3A=1$

$A=1/3$, $C=-A=-1/3$

$(4)-(2) \Rightarrow$ $B=1/3$, $D=-1/3$

$$= \frac{1}{6}\ln|x^2+1| + \frac{1}{3}\tan^{-1}x - \frac{1}{6}\ln|x^2+4| - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Consider a proper rational function.

$$\frac{f(x)}{g(x)}, \quad \text{where degree of } p(x) < \text{degree of } g(x)$$

polynomials

$\|$

a sum of terms like $\frac{A}{x-a}$, $\frac{Bx+C}{x^2+a^2}$ $\Longrightarrow$ can be done by standard integration formulas.

$\int \uparrow$ ln $\int \uparrow$ ln, $\tan^{-1}$

Slightly more complicated if we have repeated factors!

$$\int \frac{1}{(x-a)^k}\,dx = \frac{(x-a)^{-k+1}}{-k+1} + C$$

Example: $\displaystyle\int \frac{(x^3+2x-1)\,dx}{(x-1)(x-2)(x+4)^2}$

$$= \int \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+4} + \frac{D}{(x+4)^2}\,dx$$

$$= A\ln|x-1| + B\ln|x-2| + C\ln|x+4| - D(x-4)^{-1} + C_1$$

$$x^3+2x-1 = A(x-2)(x+4)^2 + B(x-1)(x+4)^2 + C(x-1)\underbrace{(x+4)}_{(x-2)} + D(x-1)(x-2)$$

Let $x=1$, $\quad 1+2-1 = A(-1)(5)^2, \quad A = -\frac{2}{25}$

$x=2 \quad 8+4-1 = B(1)(6)^2, \quad B = \frac{11}{36}$

$$x^3+2x-1 = \boxed{-\frac{2}{25}(x-2)(x+4)^2 + \frac{11}{36}(x-1)(x+4)^2}$$
$$+ C(x-1)(x-2)(x+4) + D(x-1)(x-2)$$

$\to x=-4 \quad -64 - 8 - 1 = D(-4-1)(-4-2) = 30D$

$D = \frac{-73}{30}$

Put $x=0$, $\quad -1 = -\frac{2}{25}(2)(4)^2 + \frac{11}{36}(-1)(4)^2 + C(8) - \frac{73}{30}\cdot 2$

C = an ugly number!

i.e.

$$\int \frac{p(x)}{(x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_n)^{k_n}}\,dx$$

$p(x) \to \deg p(x) < k_1+k_2+\cdots+k_n$

$\Leftrightarrow$ sum of integrals of the form $\int \frac{A}{(x-a)^k}\,dk = \begin{cases} A\ln|x-a|+C & \text{if } k=1 \\ \frac{(x-a)^{-k+1}}{-k+1}+C & \text{if } k\neq 1 \end{cases}$

$$\int \frac{x^3+x+2}{(x-1)(x^2+4)^2}\,dx$$

"partial fraction decomposition"

$$= \int \left[\frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \boxed{\frac{Dx+E}{(x^2+4)^2}}\right] dx$$

$u = x^2+4$

$$\frac{D}{2}\int \frac{2x}{(x^2+4)^2}\,dx + E\int \frac{dx}{(x^2+4)^2} \leftarrow x = 2\tan\theta$$

$$-\frac{D}{2}(x^2+4)^{-1} +$$

$$\parallel$$

$$\int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta\,d\theta}$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$

a well-known integral!

$$\frac{A}{x-1} + \frac{\text{cubic polynomial}}{(x^2+4)^2}$$

Remark

cubic $\to$ $\frac{x^3}{(x^2+4)^2} = \frac{((x^2+4)-4)x}{(x^2+4)^2}$

$$= \frac{x\,\cancel{(x^2+4)}}{(x^2+4)^{\cancel{2}}} + \frac{-4x}{(x^2+4)^2}$$

$$\parallel$$

$$\frac{A}{x^2+4} + \frac{Bx+E}{(x^2+4)^2}$$

$$\int \frac{A\,x}{x^2+1}\,dx \leftarrow \text{or let } u = x^2+1$$

$$\frac{d}{dx} \ln|x^2+1| = \frac{2x}{x^2+1}$$

$$\frac{A}{2}\boxed{\int \frac{2x}{x^2+1}\,dx}$$

$$= \frac{A}{2}\ln|x^2+1| + C.$$