# Lecture 13

## §4.2 Null spaces, Column Spaces and Linear Transformation

$\mathrm{Nul}\,A = \{\vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\}$

$\mathrm{Col}\,A = \mathrm{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$

$= \{\vec{b} \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n\}$

* Contrast Between Nul A and Col A for an $m\times n$ Matrix A

| Nul A | Col A |
|---|---|
| 1) Nul A is a subspace of $\mathbb{R}^n$ | 1) Col A is a subspace of $\mathbb{R}^m$. |
| 2) Solve $A\vec{x} = 0$ to get Nul A | 2) $\mathrm{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$ |
| 3) $\vec{v} \in \mathrm{Nul}\,A \Leftrightarrow A\vec{v} = \vec{0}$ | 3) $\vec{v} \in \mathrm{Col}\,A \Leftrightarrow A\vec{x} = \vec{v}$ consistent |

* kernel and Range of a Linear Transformation

Def: A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, such that

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V

(ii) $T(c\vec{u}) = c\,T(\vec{u})$ for all $\vec{u}$ in V and all scalar c.

or $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in V and all scalars a, b.

kernel (or Null space) of $T = \{\vec{u} \in V \mid T(\vec{u}) = \vec{0}\}$

Range of $T = \{\vec{w} \text{ in } W \mid \vec{w} = T(\vec{u}) \text{ for some } \vec{u} \text{ in } V\}$.

## §4.3 Linearly Independent Sets; Bases

**Def:** An indexed set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ in $V$ is said to be *linearly independent* if the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$ has only the trivial solution $c_1 = 0, \cdots, c_p = 0$.

**Remark:** 1) The set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is said to be *linearly dependent* if there exist $c_1, \cdots, c_p$ not all zero such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$.

2) In such a case, $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ is called a linear dependence relation among $\vec{v}_1, \cdots, \vec{v}_p$.

**Thm:** An indexed set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ of two or more vectors, with $\vec{v}_1 \neq \vec{0}$, is linearly dependent if and only if some $\vec{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\vec{v}_1, \cdots, \vec{v}_{j-1}$.

**Proof:** If $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is linearly dependent, then there exist $c_1, \cdots, c_p$ not all zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

If $c_2 = c_3 = \cdots = c_p = 0$, then $c_1\vec{v}_1 = \vec{0}$. Since $\vec{v}_1 \neq 0$, $c_1 = 0$. It is a contradiction. Hence $c_2, \cdots, c_p$ can't be all zero. Let $j$ be the largest in $\{i \mid c_i \neq 0\}$. Then $c_{j+1} = \cdots = c_p = 0$.

Hence we have $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_j\vec{v}_j = \vec{0}$

$$c_j\vec{v}_j = -c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_{j-1}\vec{v}_{j-1}$$

$$\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \cdots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}.$$

Conversely, if $\vec{v}_j$ is a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{j-1}$, then

$$\vec{v}_j = a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1}.$$

$$a_1\vec{v}_1 + \cdots + a_{j-1}\vec{v}_{j-1} - \vec{v}_j = \vec{0}$$

Take $c_1 = a_1, \cdots, c_{j-1} = a_{j-1}, c_j = -1, c_{j+1} = \cdots = c_p = 0.$

Example: Let $p_1(t) = 1$, $p_2(t) = t$, and $p_3(t) = 4 - t$.

$\{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent in $\mathbb{P}$ because

$$4p_1(t) - p_2(t) - p_3(t) = 0$$

Def: Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $B = \{\vec{b}_1, \cdots, \vec{b}_p\}$ in $V$ is a basis for $H$ if

1) $B$ is a linearly independent set, and
2) the subspace spanned by $B$ coincides with $H$.
   That is, $H = \mathrm{Span}\{\vec{b}_1, \cdots, \vec{b}_p\}$

Ex: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

is a basis of $\mathbb{R}^n$.

Ex: A: $n \times n$ matrix, $A = [\vec{a}_1, \cdots, \vec{a}_n]$

If A is invertible, then $\vec{a}_1, \cdots, \vec{a}_n$ are linearly independent. $\vec{b} = A\vec{x}$ always has a solution $\vec{x} = A^{-1}\vec{b}$. Thus $\mathbb{R}^n = \text{span}\{\vec{a}_1, \cdots, \vec{a}_n\}$. So $\{\vec{a}_1, \cdots, \vec{a}_n\}$ is a basis of $\mathbb{R}^n$.

Ex: Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$. Determine if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Solution: $\begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + (-1)^{3+1} \cdot (-6) \begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$

$= 3(5-7) - 6(-4+2)$

$= -6 + 12 = 6 \neq 0$

Thus $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is invertible.

Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.

Ex: Let $S = \{1, t, t^2, \cdots, t^n\}$. Verify that $S$ is a basis for $\mathbb{P}_n$. This basis is called the standard basis for $\mathbb{P}_n$.

Solution: Any $f(t) \in \mathbb{P}_n$, $f(t) = C_0 + C_1 t + \cdots + C_n t^n$

Thus $\mathbb{P}_n = \text{span}\{1, t, t^2, \cdots, t^n\}$

If $a_0 + a_1 t + \cdots + a_n t^n = 0$ as polynomial

$a_0 = 0,\ a_1 = 0,\ \cdots,\ a_n = 0$

Thus $\{1, t, t^2, \cdots, t^n\}$ is linearly independent.

Thus $S$ is a basis of $P_n$.

## * The spanning Set Theorem

Ex: Let $\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$,

and $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Note that $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$, and show that $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Then find a basis for the subspace $H$.

Solution: Since $c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3$,

$\text{Span}\{\vec{v}_1, \vec{v}_2\} \subset H$.

For any $\vec{x} \in H$, $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

Since $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$,

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(5\vec{v}_1 + 3\vec{v}_2)$$

$$= (c_1 + 5c_3)\vec{v}_1 + (c_2 + 3c_3)\vec{v}_2$$

Thus $\vec{x} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

So $H \subset \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Therefore, $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

## Thm: The Spanning Set Theorem

Let $S = \{\vec{v}_1, \cdots, \vec{v}_p\}$ be a set in $V$, and let $H = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$.

a) If one of the vectors in S — say, $\vec{v}_k$ — is a linear combination of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

## Bases of NulA and ColA.

Ex: Find a basis for Col B, where

$$B = [\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5] = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: $\vec{b}_2 = 4\vec{b}_1$, $\vec{b}_4 = -\vec{b}_3 + 2\vec{b}_1$

Thus by the Spanning Set Theorem,

$$\text{col}\,B = \text{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}$$
$$= \text{span}\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$$

$S = \{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is linearly independent.

Thus S is a basis for Col B.

Observation: 1) Each nonpivot column of B is a linear combination of the pivot columns.

2) If A is a matrix whose reduced echelon form is B, then $A\vec{x} = \vec{0}$ iff $B\cdot\vec{x} = \vec{0}$.

i.e. if $A = [\vec{a}_1, \cdots, \vec{a}_n]$, $B = [\vec{b}_1, \cdots, \vec{b}_n]$, then $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$ iff $x_1\vec{b}_1 + \cdots + x_n\vec{b}_n = \vec{0}$.

Therefore, columns of A have exactly the same linear dependence relations as the columns of B.

Ex: $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5] = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}$

is row equivalent to the matrix

$$B = (\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_5) = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for Col A.

Solution: Since $\vec{b}_2 = 4\vec{b}_1$, we must have $\vec{a}_2 = 4\vec{a}_1$.

$\vec{b}_4 = -\vec{b}_3 + 2\vec{b}_1$, we must have $\vec{a}_4 = -\vec{a}_3 + 2\vec{a}_1$

Thus $\text{col} A = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$

$= \text{Span}\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$.

$\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is linearly independent since

if $x_1\vec{a}_1 + x\,\vec{a}_3 + x_5\vec{a}_5 = \vec{0}$, we must have

$x_1\vec{b}_1 + x_3\vec{b}_3 + x_5\vec{b}_5 = \vec{0}$, we must have

$x_1 = 0$, $x_3 = 0$, $x_5 = 0$.

**Thm:** The pivot columns of a matrix $A$ form a basis for $\text{Col}A$.

**Proof:** Let $B$ be the reduced echelon matrix of $A$.

$B = [\vec{b}_1, \cdots, \vec{b}_n]$, $A = [\vec{a}_1, \cdots, \vec{a}_n]$.

Suppose $\vec{b}_{i_1}, \cdots, \vec{b}_{i_l}$ be the pivot columns of $B$, then for any $\vec{b}_j$ with $j \neq i_1, \cdots, i_l$ is a linear combination of $\vec{b}_{i_1}, \cdots, \vec{b}_{i_l}$,

$$\vec{b}_{i_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \vec{b}_{i_l} = \begin{array}{r} \\ \\ l\text{th} \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vec{b}_j = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_l \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \vec{b}_j = \beta_1 \vec{b}_{i_1} + \cdots + \beta_l \vec{b}_{i_l}$$

Clearly $\{\vec{b}_{i_1}, \cdots, \vec{b}_{i_l}\}$ is linearly independent.
Thus $\vec{a}_j$ for $j \neq i_1, \cdots, i_l$ is a linearly combination of $a_{i_1}, \cdots, \vec{a}_{i_l}$ and $\{\vec{a}_{i_1}, \cdots, \vec{a}_{i_l}\}$ is linearly independent.

**Basis:** $S = \{\vec{v}_1, \cdots, \vec{v}_p\}$ is a basis of vector space $V$. If we delete one $\vec{v}_i$ from $S$, $T = \{\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, v_p\}$ is still linearly independent, but $T$ cannot generate $V$.

if we add a vector $\vec{v}$ in $V$ to $S$ to get $R = \{\vec{v}_1, \cdots, \vec{v}_p, v\}$. Clearly $R$ generate $V$, but $R$ is linearly dependent since $S$ generate $V \Rightarrow \vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$

Example: $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix} \right\}$ linearly independent but does not span $\mathbb{R}^3$.

$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix} \right\}$ A basis for $\mathbb{R}^3$.

$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \right\}$

Spans $\mathbb{R}^3$ but is linearly dependent

Example: Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

Solution: $x = -2y - z$.

$y, z$ are free variables.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2y-z\\y\\z\end{pmatrix} = \begin{pmatrix}-2y\\y\\0\end{pmatrix} + \begin{pmatrix}-z\\0\\z\end{pmatrix}$$

$$= y\begin{pmatrix}-2\\1\\0\end{pmatrix} + z\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Since $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linear independent

and all the solutions is a linear combination of $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$,

$B = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the

plane $x + 2y + z = 0$.

Ex: Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix}$. Determine if $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\vec{v}_1, \vec{v}_2\}$ a basis for $\mathbb{R}^2$?

Solution:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$$

Not every row of $A$ contains a pivot position. So the columns of $A$ do not span $\mathbb{R}^3$. Hence $\{\vec{v}_1, \vec{v}_2\}$ is not a basis for $\mathbb{R}^3$. Since $\vec{v}_1$ and $\vec{v}_2$ are not in $\mathbb{R}^2$, they cannot possibly be a basis for $\mathbb{R}^2$.

Ex: Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $\vec{v}_4 = \begin{pmatrix} -4 \\ -8 \\ 9 \end{pmatrix}$.

Find a basis for the subspace $W$ spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

Solution:

$$A = \begin{pmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first two columns of $A$ are the pivot columns and hence form a basis of $\mathrm{Col} A = W$. Hence $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $W$.