Lecture 9

§2.3 Characterization of Invertible Matrices

Thm: Let A be an $n\times n$ matrix. Then the following statements are equivalent.

(a) A is an invertible matrix

(b) A is row equivalent to the $n\times n$ identity matrix.

(c) A has n pivot positions

(d) The equation $A\vec{x}=\vec{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\vec{x}\mapsto A\vec{x}$ is one-to-one.

(g) The equation $A\vec{x}=\vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.

(h) The columns span $\mathbb{R}^n$.

(i) The linear transformation $\vec{x}\mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n\times n$ matrix C such that $CA=I$.

(k) There is an $n\times n$ matrix D such that $AD=I$.

(l) A^T is an invertible matrix.

Proof: (a) $\Rightarrow$ (j) with $C=A^{-1}$.

(j) $\Rightarrow$ (d) $CA\vec{x}=C\vec{0}=\vec{0} \Rightarrow \vec{x}=\vec{0}$

(d) $\Rightarrow$ (c) because A has n columns, no free variables

(c) $\Rightarrow$ (b) $\Rightarrow$ (a)

So we have (a) $\to$ (j) $\to$ (d) $\to$ (c) $\to$ (b) $\to$ (a)

(a) $\Rightarrow$ (k) since we take $D = A^{-1}$.

(k) $\Rightarrow$ (g) $AD\vec{b} = I \cdot \vec{b}$ So $D \cdot \vec{b}$ is a solution of $A\vec{x} = \vec{b}$.

(g) $\Rightarrow$ (d) since any row contain a pivot. So A has n pivot.

(d) $\Leftrightarrow$ (a) $\qquad$ (a) $\to$ (k) $\to$ (g) $\to$ (d) $\to$ (a)

(g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (i) $\qquad$ (g) $\to$ (h) $\to$ (i) $\to$ (g)

(d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\qquad$ (d) $\to$ (e) $\to$ (f) $\to$ (d)

(a) $\Leftrightarrow$ (l)

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Ex: Is $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$ invertible?

Solution: Use row operations

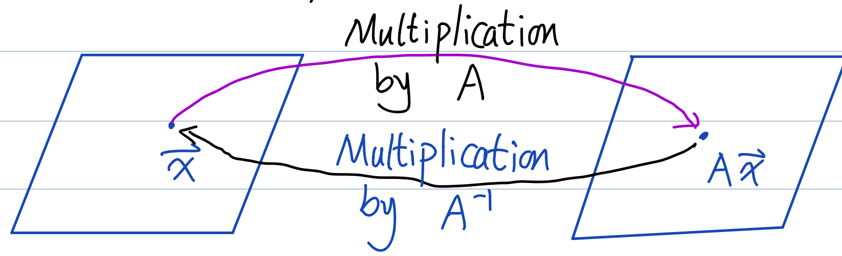
$$A \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

Since A has three pivots, A is invertible.

Invertible Linear Transformations

A linear transformation $T: \mathbb{R}^n \mapsto \mathbb{R}^n$ is invertible

if there exists a map $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that
$S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.
and $T(S(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.



Thm: Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T, Then T is invertible if and only if A is an invertible matrix. In this case, the linear transformation S given by $S(\vec{x}) = A^{-1}\vec{x}$ is the unique function satisfying

$$S(T(\vec{x})) = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n \quad (1)$$
$$T(S(\vec{x})) = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n \quad (2)$$

Proof: $\Rightarrow$ If T is invertible, then T is onto $\mathbb{R}^n$.
In fact, if $\vec{b} \in \mathbb{R}^n$ and $\vec{x} = S(\vec{b})$, then
$T(\vec{x}) = T(S(\vec{b})) = \vec{b}$, so each $\vec{b}$ is in the range of T.
Thus A is invertible, by the Invertible Matrix Theorem.

$\Leftarrow$ Suppose A is invertible, and let $S(\vec{x}) = A^{-1}\vec{x}$. Then, S is a linear transformation, and S obviously satisfies (1) and (2). $S(T(\vec{x})) = S(A\vec{x}) = A^{-1}(A\vec{x}) = \vec{x}$. Thus T invertible.

§3.1 Introduction to Determinants

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$ $\quad A_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$ $\quad A_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

In general, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $n \times n$ matrix

A_{ij} is the matrix obtained from deleting ith row and jth column of A. A_{ij} has size $(n-1) \times (n-1)$.

$\det A = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{2+1} a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$

$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$

Ex: Compute the determinant of $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

Solution: $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$= 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$

$$= -2 - 5\cdot 0 + 0 = -2$$

Def: The (i,j)-cofactor of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$

Thm: The determinant of an $n\times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Ex: Use a cofactor expansion across the third row to compute $\det(A)$, where

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

Solution: $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$$= (-1)^{3+1}\cdot 0\cdot C_{31} + (-1)^{3+2}\cdot(-2)\,C_{32} + (-1)^{3+3}\,0\cdot C_{33}$$

$$= (-1)^{3+2}\cdot(-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 0 + 2\cdot(-1) + 0 = -2$$

Ex: Compute $\det A$, where $A = \begin{pmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$

Solution: $\det A = (-1)^{1+1} a_{11} \det A_{11} + (-1)^{1+2} a_{21} \det A_{21} + (-1)^{1+3} a_{31} \det A_{31}$
$+ (-1)^{1+4} a_{41} \det A_{41} + (-1)^{1+5} a_{51} \det A_{51}$

$$= 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

$$= 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 3 \cdot 2 \cdot (-1)^{3+2} \cdot (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 12 \cdot (-1-0) = -12$$

Thm: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

$$A = \begin{pmatrix} * & * & * & & * \\ 0 & * & * & & * \\ 0 & 0 & * & & * \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & \cdots & * \end{pmatrix}$$

Exercise: Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$

Solution: $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}\, 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$

$$= 2\cdot(-1)^{1+2}\cdot(-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

$$= 10\cdot(3\cdot(-6) - 5\times(-4))$$

$$= 10(-18+20)$$

$$= 20$$

§3.2 Properties of Determinants

Theorem: Row Operations

Let A be a square matrix.

a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

i.e. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

then $\det B = \det A$

b) If two rows of A are interchanged to produce B, then $\det B = -\det A$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

c) If one row of A is multiplied by k to produce B, then $\det B = k\cdot\det A$.

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Example: Compute $\det A$, where $A=\begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$

Solution: $\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = (-1)^{2+1}\cdot 2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$

$$= -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2 \cdot (-1) \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2 \cdot 1 \cdot (-3) \cdot 5$$

$$= -30$$