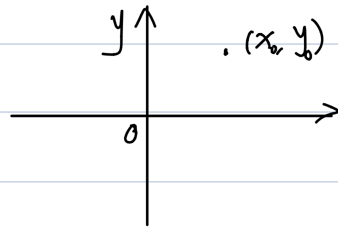
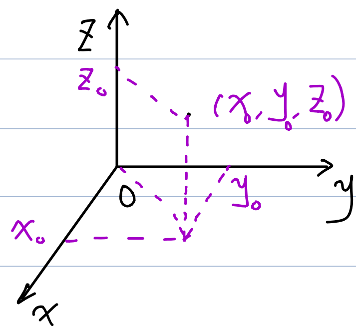
§4.4 Coordinate Systems

Example: $\mathbb{R}^2$



$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathbb{R}^3$



$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

* The existence of coordinate system for vector space V.

Thm: The Unique Representation Theorem

Let $B = \{\vec{b}_1, \cdots, \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1, \cdots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n. \quad \cdots\cdots \quad (*)$$

Proof: Since B spans V, there exist scalars such that $(*)$ holds. Suppose $\vec{x}$ also has the representation $\vec{x} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$ for scalars $d_1, \cdots, d_n$. Then, subtracting, we have

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n \cdots \quad (**)$$

Since B is linearly independent, the weights in $(**)$ must all be zero. That is, $c_j = d_j$ for $1 \le j \le n$.

Def: Suppose $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ is a basis for V and $\vec{x}$ is in V. The coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\vec{x}$) are the weights $c_1, \cdots, c_n$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$.

If $c_1, \cdots c_n$ are the $\mathcal{B}$-coordinates of $\vec{x}$, then the vector in $\mathbb{R}^n$

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is the coordinate vector of $\vec{x}$ (relative to $\mathcal{B}$), or the $\mathcal{B}$-coordinate vector of $\vec{x}$. The mapping

$$x \mapsto [x]_{\mathcal{B}}$$

is the coordinate mapping (determined by $\mathcal{B}$).

Ex: Consider a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$, where $\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Suppose an $\vec{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Find $\vec{x}$.

Solution: $\vec{x} = (-2)\vec{b}_1 + 3\vec{b}_2$

$$= (-2)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

Ex: The entries in the vector $\vec{x} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ are the coordinates of $\vec{x}$ relative to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$, since

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot \vec{e}_1 + 6 \cdot \vec{e}_2$$

If $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$, then $[\vec{x}]_{\mathcal{E}} = \vec{x}$

Ex: $P_1 = \{$Polynomials of degree ≤ 1 in one variable $t\}$

$B = \{P_1(t) = 1, P_2(t) = t\}$ is a basis of P_1

$p(t) = 2 + 3t = 2P_1(t) + 3P_2(t)$ has B-coordinates $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Ex: $M(2,2) = \{2 \times 2$ matrices$\}$

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$B = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of $M(2 \times 2)$.

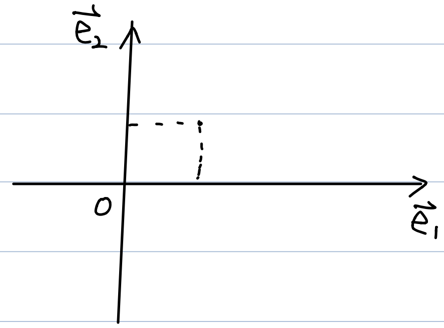
$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ has B-coordinates $\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$.

$A = 2e_{11} + 3e_{12} + 4e_{21} + 5e_{22}$.

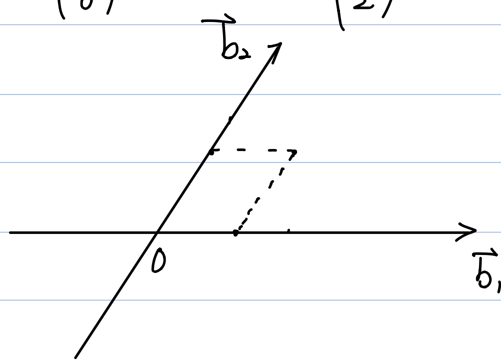
* A Graphical Interpretation of Coordinates.

A coordinate system on a set consists of a one-to-one mapping of the points in the set into $\mathbb{R}^n$.

1) standard basis for $\mathbb{R}^2$.



2) $\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



* Coordinates in $\mathbb{R}^n$.

Ex: Let $\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $B = \{\vec{b}_1, \vec{b}_2\}$.

Find $[\vec{v}]_B$

Solution: $\vec{v} = c_1 \vec{b}_1 + c_2 \vec{b}_2$,

$$c_1\begin{pmatrix}2\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$$

$$\begin{pmatrix}2&-1\\1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}2&-1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}=\frac{1}{\det\begin{pmatrix}2&-1\\1&1\end{pmatrix}}\begin{pmatrix}1&1\\-1&2\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}$$

$$=\frac{1}{2+1}\begin{pmatrix}1&1\\-1&2\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}=\frac{1}{3}\begin{pmatrix}4+5\\-4+10\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$$[\vec{v}]_B=\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

* Change of coordinates matrix

Let $B=\{\vec{b}_1,\cdots,\vec{b}_n\}$ be a basis of $\mathbb{R}^n$. Let $P_B=[\vec{b}_1,\cdots,\vec{b}_n]$

$\vec{v}=c_1\vec{b}_1+c_2\vec{b}_2+\cdots+c_n\vec{b}_n$ if and only if

$$\vec{v}=[\vec{b}_1,\vec{b}_2,\cdots,\vec{b}_n]\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}=P_B[\vec{v}]_B$$

P_B is called the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^n$.

Since $\{\vec{b}_1,\cdots,\vec{b}_n\}$ is linearly independent, P_B is invertible. Thus

$$[\vec{v}]_B=P_B^{-1}\vec{v}.$$

* The coordinate Mapping

Thm: Let $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof: Let $\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$

$\vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n$

$$a\vec{u} + b\vec{v} = a(c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n) + b(d_1\vec{b}_1 + \cdots + d_n\vec{b}_n)$$
$$= (ac_1 + bd_1)\vec{b}_1 + \cdots + (ac_n + bd_n)\vec{b}_n$$

$$[a\vec{u} + b\vec{v}]_{\mathcal{B}} = \begin{pmatrix} ac_1 + bd_1 \\ \vdots \\ ac_n + bd_n \end{pmatrix} = a\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + b\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$
$$= a[\vec{u}]_{\mathcal{B}} + b[\vec{v}]_{\mathcal{B}}$$

Def: A one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism.

Ex: $\mathbb{P}_3 = \{$ polynomials in the variable t with degree $\leq 3\}$

$\mathcal{B} = \{1, t, t^2, t^3\}$

$\mathbb{P}_3 \mapsto p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$$[p(t)]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$p(t) \mapsto [p(t)]_B$ is an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$.

Ex: Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, and $3+2t$ are linearly dependent in $\mathbb{P}_2$.

Solution: $B=\{1, t, t^2\}$ is a basis of $\mathbb{P}_2$.

$$\begin{aligned} \mathbb{P}_2 &\mapsto \mathbb{R}^3 \\ p(t) &\mapsto [p(t)]_B \end{aligned} \quad \text{is an isomorphism}$$

$$p_1(t) = 1+2t^2 \mapsto \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \equiv \vec{v}_1$$

$$p_2(t) = 4+t+5t^2 \mapsto \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \equiv \vec{v}_2$$

$$p_3(t) = 3+2t \mapsto \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \equiv \vec{v}_3$$

Thus $p_1(t)$, $p_2(t)$, $p_3(t)$ are linearly dependent iff $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

$$A = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$A\vec{x} = \vec{0}$ has nontrivial solutions.

So $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Hence $P_1(t)$, $P_2(t)$ and $P_3(t)$ are linearly dependent.

Ex: Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$

and $B = \{\vec{v}_1, \vec{v}_2\}$. Then B is a basis for $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Determine if $\vec{x}$ is in H, and if it is, find the coordinate vector of $\vec{x}$ relative to B.

Solution: If $\vec{x}$ is in H, then the following vector equation is consistent:

$$c_1 \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$$

The scalars c_1 and c_2, if they exist, are the B-coordinates of $\vec{x}$. Using row operations, we obtain

$$\begin{pmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $c_1 = 2$, $c_2 = 3$ and $[x]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Ex: The set $B = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$.
Find the coordinate vector of $p(t) = 6+3t-t^2$ relative to B.

Solution: $c_1(1+t) + c_2(1+t^2) + c_3(t+t^2) = 6+3t-t^2$

$(c_1+c_2) + (c_1+c_3)t + (c_2+c_3)t^2 = 6+3t-t^2$

$$\begin{cases} c_1 + c_2 = 6 \\ c_1 + c_3 = 3 \\ c_2 + c_3 = -1 \end{cases}$$

Solving the equation, we get $\begin{cases} c_1 = 5 \\ c_2 = 1 \\ c_3 = -2 \end{cases}$

So $[p(t)]_B = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$