§1.2 Row Reduction and Echelon forms

/ɛʃəlaːn/ (step like)

Def: A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1) All nonzero rows are above any rows of all zeros
2) Each leading entry of a row is in a column to the right of the leading entry o the row above it.
3) All entries in a column below a leading entry are zeros

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

4) The leading entry in each nonzero row is 1.
5) Each leading 1 is the only nonzero entry in its column.

Def: a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Example:

$$\begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

echelon form $\qquad$ reduced echelon form

$$\begin{pmatrix} 3 & -1 & 0 & 8 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form — reduced echelon form

Thm: Uniqueness of the Reduced echelon

Each matrix is row equivalent to one and only one reduced echelon matrix.

If $A \xrightarrow{\text{row operations}} U$,

A: a matrix; U: echelon matrix

U is called an echelon form of A.

* Pivot Positions

Def: A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Example: Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

Solution: $\begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix} \xrightarrow{①\leftrightarrow④} \begin{pmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$ pivot

$\xrightarrow[③+2①]{①+②} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$ pivot $\xrightarrow[③+\frac{3}{2}②]{③-\frac{5}{2}②} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}$

$\xrightarrow{③\leftrightarrow④} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ pivot

pivot position

$A = \begin{pmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}$

pivot columns

* The Row Reduction Algorithm

Example: Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

Solution: Step 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$
\begin{array}{c}
\text{pivot position} \rightarrow
\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}
\end{array}
$$

$\uparrow$ pivot column

Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$
\xrightarrow{①\leftrightarrow③}
\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}
$$

Step 3: Use row replacement operations to create zeros in all positions below the pivot.

$$
\xrightarrow{②-①}
\begin{pmatrix} \cancel{3} & \cancel{-9} & \cancel{12} & \cancel{-9} & \cancel{6} & \cancel{15} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}
$$

New pivot

Step 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

$$\xrightarrow{③-\frac{3}{2}②}\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \leftarrow \text{echelon form}$$

pivot

step 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\xrightarrow[②-2\times③]{①-6\times③}\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}\times②}\begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{①+9\times②}\begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}\times①}\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \quad \text{reduced echlon form}$$

Remark: step 1 – 4 are called forward phase of the row reduction algorithm
step 5 is called backward phase.

* Solutions of Linear Systems

Example: augmented matrix of a linear system

$\Downarrow$ Row Reduction Algorithm

$$\begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system is equivalent to

$$\begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases} \iff \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \end{cases}$$

x_1, x_2 : basic variables

x_3 : free variables

free to choose any value

Example: Find the general solution of the linear system whose augmented matrix is

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[②+③]{①+2\times③} \begin{pmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}\times ②} \begin{pmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{① - 2\times ②} \begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

The system is $\begin{cases} x_1 + 6x_2 \quad + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7 \end{cases}$

The general solution is $\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$

Theorem (Existence and Uniqueness Theorem)

1) A linear system is consistent $\Longleftrightarrow$ the rightmost column of the augmented matrix is not a pivot column.

$\Longleftrightarrow$ an echolon form of the augmented matrix has no row of the form $[0, \cdots, 0 \quad b]$ with b nonzero

2) If a linear system is consistent, then the solution set contains either

i) a unique solution (no free variables)

(All columns of coefficient matrix are pivot columns)

ii) infinitely many solutions (at least 1 free variable)

* Using Row Reduction to solve a linear system

1) Write the augmented matrix of the system
2) Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3) Continue row reduction to obtain the reduced echelon form.
4) Write the system of equations corresponding to the matrix obtained in step 3.
5) Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Exercise: Determine the values of h of the following matrix such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix}$$

Solution: $\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix} \xrightarrow{②-5①} \begin{pmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{pmatrix}$

If $h+15 \neq 0$, i.e. $h \neq -15$, the above matrix is the augmented matrix of a consistent linear system.