MATH 2111: Tutorial 7 Determinants, Vector Spaces and Subspaces

T1A&T1B QUAN Xueyang
T1C&T2A SHEN Yinan
T2B&T2C ZHANG Fa

Department of Mathematics, HKUST

Review

- Cramer's Rule
- Inverse formula
- Area and volume (using determinant)
- Vector spaces and subspaces

Example 1

Use Cramer's rule to solve the following linear system.

$$\begin{cases} x_1 + x_2 = 3 \\ -3x_1 + 2x_3 = 0 \\ x_2 - 2x_3 = 2 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}}$$

Since $|A| = \begin{vmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} = (1)\cdot \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix}$

$= -2 - 6 = -8$

$$|A_1(\vec{b})| = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} = (3)\cdot\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}$$

$= -6 + 4 = -2$

$$|A_2(\vec{b})| = \begin{vmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix} = (1)\cdot\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} + (-3)\cdot\begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix}$$

$= -4 - 18 = -22$

$$|A_3(\vec{b})| = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -(-3)\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$= -3$

By Cramer's Rule,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-2}{-8} \\ \frac{-22}{-8} \\ \frac{-3}{-8} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{3}{8} \end{bmatrix}$$

Example 2

Compute the adjugate of the given matrix, and then use the inverse formula to give A^{-1}.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\det A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{vmatrix} \end{array} = (1)\cdot\begin{vmatrix} 1 & 4 \\ -3 & 4 \end{vmatrix} + (-2)\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 16 - 14 = 2$$

$$C_{11} = \begin{vmatrix} 1 & 4 \\ -3 & 4 \end{vmatrix} = 16 \qquad C_{12} = -\begin{vmatrix} -3 & 4 \\ 2 & 4 \end{vmatrix} = 20 \qquad C_{13} = \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} = 7$$

$$C_{21} = -\begin{vmatrix} 0 & -2 \\ -3 & 4 \end{vmatrix} = 6 \qquad C_{22} = \begin{vmatrix} 1 & -2 \\ 2 & 4 \end{vmatrix} = 8 \qquad C_{23} = -\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} = 2 \qquad C_{33} = \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} = 1$$

$$\therefore \ \operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{2}\begin{bmatrix} 16 & 6 & 2 \\ 20 & 8 & 2 \\ 7 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Example 3

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$

Area of image of S

$= |\det A| \cdot$ Area of S

det

$$= \left| \begin{vmatrix} 6 & -2 \\ -3 & 2 \end{vmatrix} \right| \cdot \left| \begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix} \right|$$

absolute value

$$= |6| \cdot |-4| = 24$$

Example 4

Let R be the triangle with vertices at $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3). Show that

$$\{ \text{ area of triangle } \} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Translate triangle to one having the origin as a vertex.

One can subtract the vertex (x_1, y_1) from 3 vertices :

$$(0,0),\ (x_2-x_1,\ y_2-y_1),\ (x_3-x_1,\ y_3-y_1).$$

① LHS :

$$\{\text{Area of triangle}\} = \frac{1}{2}\{\text{area of parallelogram}\}$$

$$= \frac{1}{2}\det\begin{bmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{bmatrix}$$

② RHS:

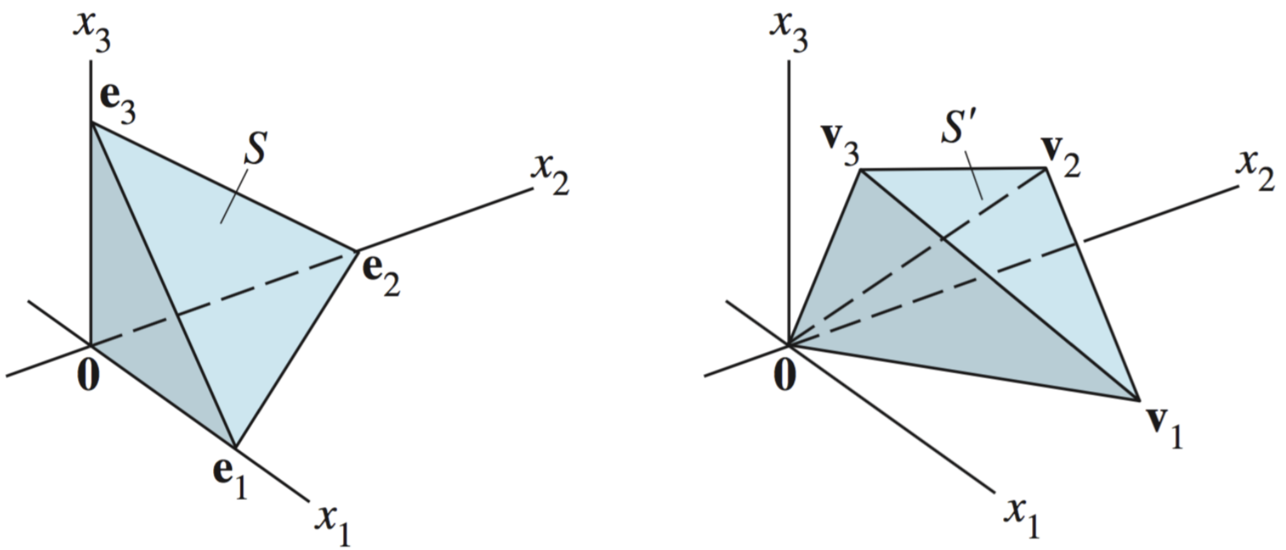
$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}\det \begin{array}{r c} & \begin{array}{ccc} + & - & + \end{array} \\ & \begin{bmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ R_2-R_1\to R_2 \\ \\ R_3-R_1\to R_3 \end{array}$$

$$= \frac{1}{2}\det\begin{bmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{bmatrix}$$

Therefore, $\{\text{Area of triangle}\} = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

Example 5

Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.



a. Describe a linear transformation that maps S onto S'.
b. Find a formula for the volume of the tetrahedron S' using the fact that $\{\text{ volume of } S\} = (1/3)\{\text{ area of base }\} \cdot \{\text{ height }\}$

a. A linear transformation T which maps S onto S' will map $\vec{e_1}$ to $\vec{v_1}$, $\vec{e_2}$ to $\vec{V_2}$, $\vec{e_3}$ to $\vec{V_3}$,

that is,

$T(\vec{e_1}) = \vec{V_1}$, $T(\vec{e_2}) = \vec{V_2}$, $T(\vec{e_3}) = \vec{V_3}$

The standard matrix A will be:

$$A = [\vec{V_1} \ \vec{V_2} \ \vec{V_3}]$$

b. Area of base of S is $1 \times 1 \times \frac{1}{2} = \frac{1}{2}$.

$$\{\text{volume of } S\} = \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}$$

$$\therefore \{\text{volume of } S'\} = \{\text{volume of } T(S)\}$$

$$= |\det A| \cdot \{\text{volume of } S\}$$

$$= \frac{1}{6} |\det A|$$

Example 6

Let S be a set of 2×2 matrices, whose sum of all diagonal entries is zero. Verify S is a subspace of the vector space of all 2×2 matrices.

let $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+d=0$.

① S contains 0 ($0 \in M_{2\times2}$) matrix, i.e.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$

② let $S_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $S_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $S_1, S_2 \in S$

$S_1+S_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

Note that $a_1+a_2+d_1+d_2 = (a_1+d_1)+(a_2+d_2) = 0+0 = 0$

$\therefore$ S is closed under addition.

③ for $t \in \mathbb{R}$, $S_1 \in S$

$tS_1 = t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ta_1 & tb_1 \\ tc_1 & td_1 \end{bmatrix}$

note that $ta_1+td_1 = t(a_1+d_1) = t\cdot 0 = 0$

$\therefore$ S is closed under multiplication.

$\therefore$ S is a subspace of the vector space of all 2x2 matrices.