MATH 2111: Tutorial 1 Linear System and Echelon Form

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Review

- Linear equation & Linear systems
- Matrix & Augumented matrix
- Elemantary Row Operations & Row Equivalent
- Echelon Form & Reduced Echelon Form

Example 1

Can a linear system has finite many solutions, like 2 solutions, or 100 solutions?

Proof: Denote the linear system as

$$\begin{cases} a_{11}x_1 + \dots + a_{1m}x_m = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nm}x_m = b_n. \end{cases}$$

Suppose $x_1 = y_1^{(1)}, \dots, x_m = y_m^{(1)}$

and

$x_1 = y_1^{(2)}, \dots, x_m = y_m^{(2)}$

are two different solutions.

Then for any $\alpha \in \mathbb{R}$

$$x_1 = \alpha y_1^{(1)} + (1-\alpha) y_1^{(2)}, \quad \dots, \quad x_m = \alpha y_m^{(1)} + (1-\alpha) y_m^{(2)}$$

is solution to the linear system.

$\Rightarrow$ then the linear system has infinite many solutions.

Example 2

Solve the following linear system with Echelon form

$$\begin{cases} x_1 - x_2 + x_3 = 2, \\ x_1 + 2x_2 = 1, \\ x_1 - x_3 = 4. \end{cases} \tag{1}$$

② − ① & ③ − ①

$$\begin{cases} x_1 - x_2 + x_3 = 2 & ①' \\ 3x_2 - x_3 = -4 & ②' \\ x_2 - 2x_3 = 2 & ③' \end{cases} \quad \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & -1 & -4 \\ 0 & 1 & -2 & 2 \end{pmatrix}$$

②' − 3×③'.

$$\begin{cases} x_1 - x_2 + x_3 = 2 & ①'' \\ 5x_3 = -7 & ②'' \\ x_2 - 2x_3 = 2 & ③'' \end{cases} \quad \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 1 & -2 & 2 \end{pmatrix}$$

①'' + ③'' & ③'' + $\frac{2}{5}$②'' & interchange ②'' and ③''

$$\begin{cases} x_1 - x_3 = 4 & ①''' \\ x_2 = \frac{4}{5} & ②''' \\ 5x_3 = -7 & ③''' \end{cases} \quad \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 5 & -7 \end{pmatrix}$$

Finaly →

$$\begin{pmatrix} 1 & 0 & 0 & \frac{13}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & -\frac{7}{5} \end{pmatrix}$$

$x_1 = \frac{13}{5}$, $x_2 = -\frac{4}{5}$, $x_3 = -\frac{7}{5}$

Solve the following linear system with Echelon form

$$\begin{cases} x + y + z = 0, \\ 2x - 6y + 6z = 2, \\ 4x + 8y + 2z = 4. \end{cases} \tag{2}$$

Augumented Matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -6 & 6 & 2 \\ 4 & 8 & 2 & 4 \end{pmatrix} \begin{matrix} \to ① \\ \to ② \\ \to ③ \end{matrix} \quad \begin{matrix} ② - 2 \cdot ① \\ ③ - 4 \cdot ① \end{matrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -8 & 4 & 2 \\ 0 & 4 & -2 & 4 \end{pmatrix} \begin{matrix} \to ①' \\ \to ②' \\ \to ③' \end{matrix}$$

$$③' + \frac{1}{2}②' \Rightarrow \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -8 & 4 & 2 \\ 0 & 0 & 0 & 5 \end{pmatrix} \to \text{inconsistent.}$$

Solve the following linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2, \end{cases} \qquad (3)$$

where $a_{11} \neq 0$.

Hint

Need to discuss different cases: inconsisten case, only one solution and infinite many solutions case.

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} \begin{array}{c} ① \\ ② \end{array}$$

$$② - ① \cdot \frac{a_{21}}{a_{11}} \Longrightarrow \begin{pmatrix} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & b_2 - b_1 \cdot \frac{a_{21}}{a_{11}} \end{pmatrix}$$

(a) $a_{22} - a_{12}\dfrac{a_{21}}{a_{11}} \neq 0.$ only one solution

(b) $a_{22} - a_{12} \cdot \dfrac{a_{21}}{a_{11}} = 0$ & $b_2 - b_1 \cdot \dfrac{a_{21}}{a_{11}} \neq 0$ $\Rightarrow$ inconsistent

(c) $a_{22} - a_{12} \cdot \dfrac{a_{21}}{a_{11}} = 0$ & $b_2 - b_1 \cdot \dfrac{a_{21}}{a_{11}} = 0$ $\Rightarrow$ infinitely many solutions.

$a_{11}\, a_{22} = a_{21} \cdot a_{12}$. ??

Suppose $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 3 \\ 1 & a & b & 4 \end{pmatrix}\begin{array}{l} ① \\ ② \\ ③ \end{array}$ is an augumented matrix. Determine a and b such that the linear system

(1) is inconsistent,

(2) has a unique solution,

(3) has infinite many solutions.

$$\begin{array}{l} ② - ① \cdot 2 \\ ③ - ① \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & a-1 & b-1 & 4 \end{pmatrix}\begin{array}{l} ①' \\ ②' \\ ③' \end{array}$$

$$\begin{array}{l} ①' + ②'/3 \\ ②' \cdot \left(-\frac{1}{3}\right) \end{array} \Rightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & a-1 & b-1 & 4 \end{pmatrix} \end{array} \begin{array}{l} ①'' \\ ②'' \\ ③'' \end{array}$$

$$③'' - (a-1) \cdot ②'' \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & b-a & \underline{4+(a-1)} \end{pmatrix}$$

ⓐ $b-a \neq 0$ $\Rightarrow$ unique solution

ⓑ $b-a = 0$, & $4+(a-1) = 0$ $\Rightarrow$ infinitely many

ⓒ $b-a = 0$ & $4+(a-1) \neq 0$ $\Rightarrow$ inconsistent.