MATH 2111: Tutorial 2 Echelon Form and Linear Combinations

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Review

- Existence and Uniqueness Theorem (important!!!!!)
- Geometric visualization of linear equation
- Vector equation (sum & scalar multiple & some other algebraic properties) $---->$ Linear combinations
- The subset spanned by vector $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_p\}$.

Suppose $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & a & 3 \\ 1 & 1 & b & 4 \end{pmatrix}$ is an augumented matrix. Determine a and b such that the linear system

(1) is inconsistent,

(2) has a unique solution,

(3) has infinite many solutions.

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 2 & 2 & a & 3 \\ 1 & 1 & b & 4 \end{array}\right)\begin{array}{c} ① \\ ② \\ ③ \end{array} \xrightarrow[③-1\cdot①]{②-2\cdot①} \left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & b-1 & 4 \end{array}\right)$$

① when $a=2$ or $b=1$

it is inconsistent

② when $a\neq 2$, and $b\neq 1$

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & b-1 & 4 \end{array}\right)\begin{array}{c} ①' \\ ②' \\ ③ \end{array} \xrightarrow{③'-②'\cdot\frac{b-1}{a-2}} \left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & a-2 & 3 \\ 0 & 0 & 0 & 4-\frac{b-1}{a-2}\cdot 3 \end{array}\right).$$

2.1). when $4-\frac{b-1}{a-2}\cdot 3\neq 0$ it is inconsistent.

$\downarrow$

namely, $4a-8-3b+3\neq 0$, $\Rightarrow$ $4a-3b\neq 5$

2.2) when $4a-3b=5$, it has infinite many solutions.

In conclusion, when $a=2$ or $b=1$ or $4a-3b\neq 5$, it's inconsistent,

when $4a-3b=5$ & $a\neq 2$, $b\neq 1$,

it has infinite many solutions.

Suppose $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & a & 3 & 3 \\ 1 & 2 & b & 4 \end{pmatrix}$ is an augumented matrix. Determine a and b such that the linear system

(1) is inconsistent,

(2) has a unique solution,

(3) has infinite many solutions.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & a & 3 & 3 \\ 1 & 2 & b & 4 \end{pmatrix}\begin{matrix} ① \\ ② \\ ③ \end{matrix} \xrightarrow[③-①]{②-2\cdot①} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & a-2 & 1 & 3 \\ 0 & 1 & b-1 & 4 \end{pmatrix}\begin{matrix} ①' \\ ②' \\ ③' \end{matrix}$$

$$\xrightarrow{②' \leftrightarrow ③'} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & b-1 & 4 \\ 0 & a-2 & 1 & 3 \end{pmatrix}\begin{matrix} ①'' \\ ②'' \\ ③'' \end{matrix}$$

$$\xrightarrow{③''-(a-2)\cdot②''} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & b-1 & 4 \\ 0 & 0 & 1-(a-2)(b-1) & 3-4(a-2) \end{pmatrix}$$

① when $a \neq \frac{11}{4}$ & $1 = (a-2)(b-1)$, it is inconsistent.

② when $(a-2)(b-1) \neq 1$, it has unique solution

③ when $a = \frac{11}{4}$ & $(a-2)(b-1) = 1$, it has infinitely many solutions.

namely, $a = \frac{11}{4}$, $b = \frac{7}{3}$

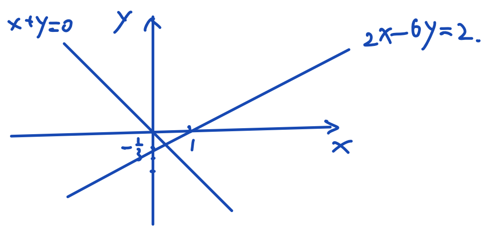
Plot the following linear systems:

(1) Two variables: $\begin{cases} x + y = 0, \\ 2x - 6y = 2. \end{cases}$

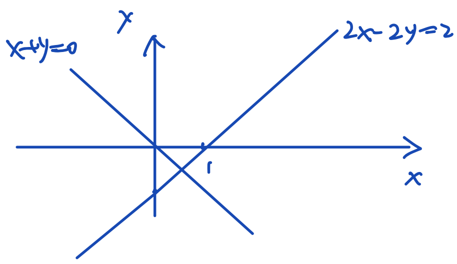
(2) Two variables: $\begin{cases} x + y = 0, \\ 2x - 2y = 2. \end{cases}$

(3) Three variables: $\begin{cases} x + y = 0, \\ y + z = 2. \end{cases}$

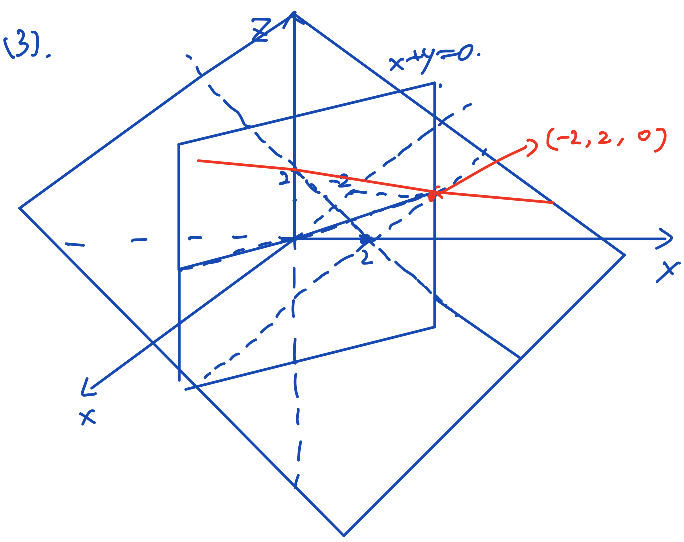
(1).



(2)



(3).



The two planes are not parallel or coincide, their intersection is a line.

Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(2) Determine whether vector $\boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(1). set $\{\vec{x} : \vec{x} = a_1\vec{u} + a_2\vec{v}$ for all $a_1, a_2 \in \mathbb{R}\}$

(2). If $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$ then the exist $a_1 . a_2 \in \mathbb{R}$ such that

$\vec{w} = a_1 \vec{u} + a_2 \vec{v}$

$$\begin{array}{c}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}\end{array}\begin{array}{c} ① \\ ② \\ ③ \end{array} \xrightarrow{②-①} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{array}{c} ①' \\ ②' \\ ③' \end{array}$$

$$\Big\downarrow \begin{array}{l} ①' + ②' \\ ③' - ②' \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Rightarrow$ the system is inconsistent

$\Rightarrow$ $\vec{w}$ is not spanned by $\vec{u}$ and $\vec{v}$

Let $\boldsymbol{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(2) Determine h such that vector $\boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(1). set $\{\vec{x}: \vec{x} = a_1\vec{u} + a_2\vec{v}$ for all $a_1, a_2 \in \mathbb{R}\}$

(2). If $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$ then the exist $a_1, a_2 \in \mathbb{R}$ such that

$\vec{w} = a_1 \vec{u} + a_2 \vec{v}$.

$$\begin{array}{c} \begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & h \end{pmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array} \xrightarrow{② - ① \cdot \frac{1}{3}} \begin{pmatrix} 3 & -2 & 2 \\ 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & h \end{pmatrix} \begin{array}{c} ①' \\ ②' \\ ③' \end{array} \xrightarrow{②' \cdot \frac{3}{2}} \begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & h \end{pmatrix}$$

$$\xrightarrow{③'' - ②''} \begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & h-2 \end{pmatrix}$$

$\Rightarrow$ when $h=2$, $\vec{w}$ is spanned by $\vec{u}$ and $\vec{v}$

Let $\boldsymbol{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

(1) Write down the subset of $\mathbb{R}^3$ spanned by $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

(2) Determine h such that vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

(2) Determine h such that vector $\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ could be spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{w}$.

(1) Set $\{\vec{y} : \vec{y} = a_1\vec{u} + a_2\vec{v} + a_3\vec{w}$ for all $a_1, a_2, a_3 \in \mathbb{R}\}$.

(2). If $\vec{x}$ could be spanned by $\vec{u}$ and $\vec{v}$, then there exist $a_1, a_2 \in \mathbb{R}$ such that

$$\vec{x} = a_1\vec{u} + a_2\vec{v}.$$

namely, augument matrix $\begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & h \end{pmatrix}$ is consistent

$$\begin{array}{c} \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & h \end{pmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix} \xrightarrow{② - ① \cdot \frac{1}{3}} \begin{pmatrix} 3 & -2 & 1 \\ 0 & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & h \end{pmatrix} \begin{matrix} ①' \\ ②' \\ ③' \end{matrix} \end{array}$$

$$\Big\downarrow ②' \cdot \tfrac{3}{2}$$

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 1 & h \end{pmatrix} \begin{matrix} ①'' \\ ②'' \\ ③'' \end{matrix}$$

$$\Big\downarrow ③'' - ②''$$

$$\begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & h-4 \end{pmatrix}$$

$\Rightarrow$ when $h = 4$, it is consistent.

(3). Similarly, consider

$$\begin{pmatrix} 3 & -2 & 2 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & h \end{pmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix} \xrightarrow{② - ① \cdot \frac{1}{3}} \begin{pmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{8}{3} \\ 0 & 1 & 2 & h \end{pmatrix} \begin{matrix} ①' \\ ②' \\ ③' \end{matrix}$$

$$\xrightarrow{2' \cdot \frac{3}{2}} \begin{array}{cc} \begin{pmatrix} 3 & -2 & 2 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & h \end{pmatrix} & \begin{array}{c} 1'' \\ 2'' \\ 3'' \end{array} \end{array} \xrightarrow{3'' - 2''} \begin{pmatrix} 3 & -2 & 2 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & h-4 \end{pmatrix}$$

$\Rightarrow$ when $h=4$, it is consistent.

Question: think about why (2) and (3) have same solution.