L13: Binomial Coefficients

- **Binomial Theorem**
- Pascal's Identity and Triangle
- Some Other Identities

- Reading: Rosen 6.4

Binomial Theorem

- **Definition:** The number of k-combinations of a set with n elements, denoted by $C(n,k)$ or $\binom{n}{k}$, is also called a **binomial coefficient** because it occurs as a coefficient in the expansion of the power of a binomial expression such as $(x+y)^n$.
- **Theorem (Binomial theorem)**

Let x and y be variables and n be a nonnegative integer. Then

$$(x+y)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example: $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

$$(x+y)(x+y)(x+y)$$

$$= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.$$

- Coefficient of xy^2

  = number of length-3 lists having y in **2** places. $\binom{3}{2}$

Proof

We give a combinatorial proof of the theorem here. When we expand the product $(x+y)^n$, the terms are of the form $x^{n-k}y^k$ for $k = 0, 1, \ldots, n$. To count the number of terms of the form $x^{n-k}y^k$, note that to obtain such a term it is necessary to choose k y's from the n sums so that the other $n-k$ terms in the product are x's. Therefore, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$. This proves the theorem.

Example

- **Example 1**

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution:

By the Binomial theorem the coefficient is $\binom{25}{13} = \frac{25!}{13!12!} =$ 5 200 300

Example

- **Example 2**

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution:

By the Binomial theorem the term involving $x^{12}y^{13}$ is $\binom{25}{12}(2x)^{12}(-3y)^{13}$. Therefore the coefficient of $x^{12}y^{13}$ is

$$-\binom{25}{12}2^{12}3^{13}$$

Example

- **Example 3**

What is the constant term in the expansion of $(x + \frac{1}{x^3})^{12}$?

Solution:

We need the Binomial expansion term $\binom{12}{a} x^a (\frac{1}{x^3})^{12-a}$ to produce a constant. In other words, we need the power of x to be zero in the constant term, i.e.,

$$a - 3(12 - a) = 0$$

So $a = 9$. Therefore the constant is $\binom{12}{9} = 220$.

Corollary

- **Corollary**

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- **Proof**

Using the binomial theorem with $x = y = 1$, we can get

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Alternative: Combinatorial Proof

A set with n elements has a total of 2^n different subsets. Each subset has zero elements, one element, two elements, …, or n elements in it. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, …, and $\binom{n}{n}$ subsets with *n* elements. Therefore,

$$\sum_{k=0}^{n} \binom{n}{k}$$

counts the total number of subsets of a set with n elements, which is 2^n.

Corollary

Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- **Proof**

Using the binomial theorem with $x = 1$ and $y = -1$, we can get

$$0 = (1 + (-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

This proves the corollary.

Remark

The corollary implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Corollary

- **Corollary**

Let n be a positive integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

- **Proof**

Using the binomial theorem with $x = 1$ and $y = 2$, we can get

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

This proves the corollary.

Outline

- Binomial Theorem
- **Pascal's Identity and Triangle**
- Trinomial Theorem
- Some Other Identities

Pascal's Identity

The binomial coefficients satisfy many different identities. One of the most important identities is discussed below.

- **Theorem (Pascal's identity)**

Let n and k be integers with $0 < k < n$. Then,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Example:

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$$

Let X = {A, B, C, D, E}

S = 2-combinations of X

= {those containing E} U {those without E}

= S_1 U S_2

S_1 = { {A,E}, {B,E}, {C,E}, {D,E} }

i.e., 1-combinations from {A, B,C,D}

$|S_1| = \binom{4}{1}$

S_2 = {{A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}}

i.e., 2-combinations from {A, B, C, D}

$|S_2| = \binom{4}{2}$

$$\boxed{\binom{5}{2} = |S| = |S_1| + |S_2| = \binom{4}{1} + \binom{4}{2}}$$

Proof

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

#k-combinations of {1,2,3,…,n}

#(k-1)-combinations of {1,2,3, …, n-1}

#k-combinations of {1,2,3,…, n-1}

We are using the Sum Principle

Proof

- **Combinatorial proof of Pascal's identity**

Suppose S is a set containing n elements. Let a be an element of S and $T = S - \{a\}$. Note that there are $\binom{n}{k}$ subsets of S containing k elements. However, a subset of S with k elements either contains a together with $k - 1$ elements of T, or contains k elements of T and does not contain a. Because there are $\binom{n-1}{k-1}$ subsets of $k - 1$ elements of T, there are $\binom{n-1}{k-1}$ subsets of k elements of S that contain a. Also, because there are $\binom{n-1}{k}$ subsets of k elements of T , there are $\binom{n-1}{k}$ subsets of k elements of S that do not contain a. Consequently,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal’s Triangle (1641)

- **Corollary**

Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers *n*, can be used to give a geometric arrangement of the binomial coefficients in a triangle, called Pascal's triangle, as follows:

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\cdots
\end{array}
$$

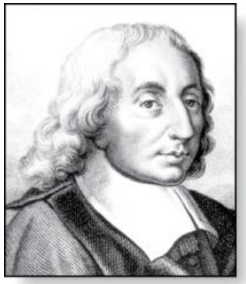
(a)

By Pascal's identity:

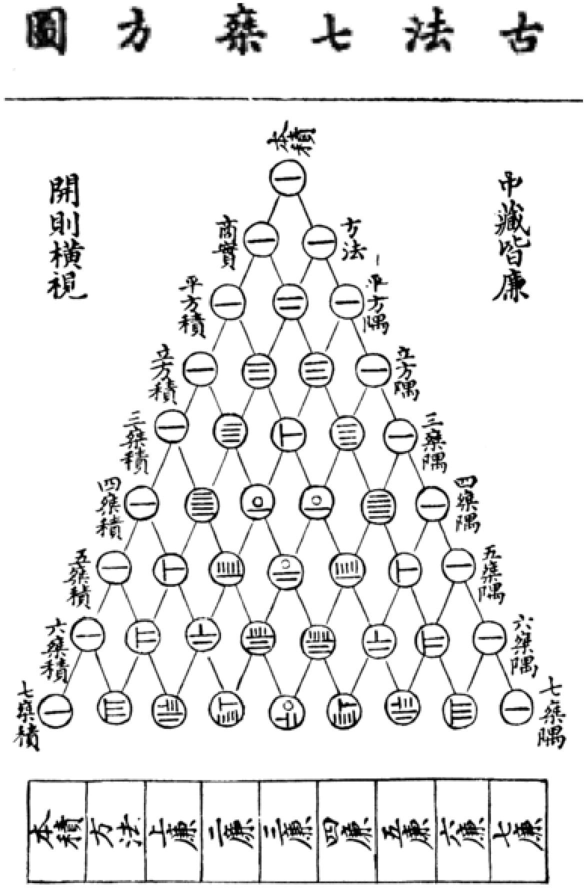
$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)



Pascal’s Triangle



Yang Hui triangle (12??)

	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

The numbers of distinct paths to each square by a rook, when only rightward and downward movements are considered.

Outline

- Binomial Theorem
- Pascal's Identity and Triangle
- **Some Other Identities**

Vandermonde's Identity

- **Theorem (Vandermonde's identity, 1772)**

Let m, n, and r be integers with $0 \leq r \leq m$ and $0 \leq r \leq n$. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}.$$

- Discovered by Zhu Shijie in 1303.

Combinatorial Proof

Suppose that there are m elements in one set and n elements in a second set. Then the total number of ways to choose r elements from the union of the two sets is $\binom{m+n}{r}$. Another way to choose r elements from the union is to choose $r - k$ elements from the first set and then k elements from the second set, where k is an integer with $0 \leq k \leq r$. By the product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Consequently,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Corollary

- **Corollary**

Let n be a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

- **Proof**

Using Vandermonde's identity with $m = n = r$, we can get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Counting bit strings

We can prove combinatorial identities by counting bit strings with different properties.

Recall: Number of bit strings of length *n* containing *r* ones = number of ways to choose *r* positions out of *n* positions = number of *r*-combinations of *n* objects

- **Theorem**

Let *n* and *r* be integers with $0 \leq r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof

Solution 1: The left-hand side counts the number of bit strings of length $n+1$ containing $(r+1)$ $1'$s.

The right-hand side counts the same objects in an alternative way by considering the possible locations of the last 1 in a string with $(r+1)$ 1's: the last 1 must occur at location $r+1, r+2, \ldots,$ or $n+1$.

With the last 1 at the kth bit $(r+1 \leq k \leq n+1)$, there must be r 1s among the first $k-1$ locations. Consequently, there are $\binom{k-1}{r}$ such bit strings.

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Because both sides of the identity count the same objects, they must be equal. This completes the proof.

Proof

Solution 2: The left-hand side counts the number of ways of selecting $(r+1)$ objects from $n+1$ objects labeled 1 to n+1.

The right-hand side considers the possible highest label among the selected $(r+1)$ objects: the highest label must be $r+1, r+2, \ldots,$ or $n+1$.

If the highest label is r+1, the other *r* objects must be selected from the objects labelled 1 to r => **C(r,r)**

If the highest label is r+2, the other *r* objects must be selected from the objects labelled 1 to r+1 => **C(r+1, r)**

…

If the highest label is n+1, the other r objects must be selected from the objects labelled 1 to n => **C(n, r)**

Therefore, the total number of ways is $\sum_{j=r}^{n} \binom{j}{r}$

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