HKDSE MATH M2 2012

1. **HKDSE Math M2 2012 Q1**
   Let $f(x) = e^{2x}$. Find $f'(0)$ from first principles.
   (3 marks)

2. **HKDSE Math M2 2012 Q2**
   It is given that

   $$(1 + ax)^n = 1 + 6x + 16x^2 + \text{terms involving higher powers of } x,$$

   where n is a positive integer and a is a constant. Find the values of a and n.
   (5 marks)

3. **HKDSE Math M2 2012 Q3**
   Prove, by mathematical induction, that for all positive integers n,
   $1 \times 2 + 2 \times 5 + 3 \times 8 + \cdots + n(3n - 1) = n^2(n + 1)$.
   (5 marks)

4. **HKDSE Math M2 2012 Q4**
   (a) Find $\displaystyle\int \frac{x+1}{x}\,dx$

   (b) Using the substitution $u = x^2 - 1$, find $\displaystyle\int \frac{x^3}{x^2-1}\,dx$.

   (5 marks)

5. **HKDSE Math M2 2012 Q5**
   Find the minimum point(s) and asymptote(s) of the graph of $y = \dfrac{x^2 + x + 1}{x + 1}$.
   (6 marks)

6. **HKDSE Math M2 2012 Q6**
   A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (See Figure 1). It is given that the volume of the frustum is $\dfrac{\pi}{3}H(r^2 + rR + R^2)$.
   An empty glass is in the form of an inverted frustum described above with height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass. Let h cm ($0 \le h \le 10$) be the depth of the water inside the glass at time t s (see Figure 2).

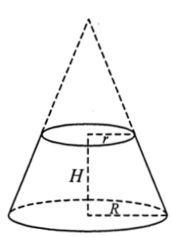

Figure 1

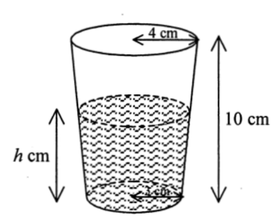

Figure 2

(a) Show that the volume V cm^3 of water inside the glass at time t s is given by $V = \frac{\pi}{300}(h^3 + 90h^2 + 2700h)$.

(b) If the volume of water in the glass is increasing at the rate 7π cm^3 s^{-1}, find the rate of increase of depth of water at the instant when $h = 5$.

(6 marks)

7. **HKDSE Math M2 2012 Q7**

Figure 3 shows a parallelepiped $OADBECFG$. Let $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OC} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

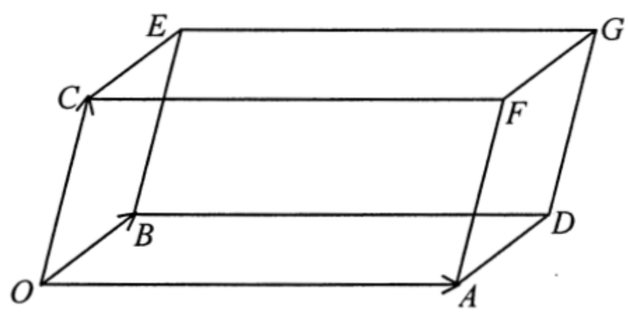

Figure 3

(a) Find the area of the parallelogram $OADB$.

(b) Find the distance between point C and the plane $OADB$.

(5 marks)

8. **HKDSE Math M2 2012 Q8**

(a) Solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ 2x - y + 5z = 6 \end{cases}$$

(b) Using (a), or otherwise, solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ 2x - y + 5z = 6 \\ x - y + \lambda z = 4 \end{cases}, \text{ where } \lambda \text{ is a constant.}$$

(5 marks)

9. **HKDSE Math M2 2012 Q9**

(a) Using integration by parts, find $\int x \sin x \, dx$.

(b) Figure 4 shows the shaded region bounded by the curve $y = \sqrt{x \sin x}$ for $0 \le x \le \pi$ and the x-axis. Find the volume of the solid generated by revolving the region about the x-axis.

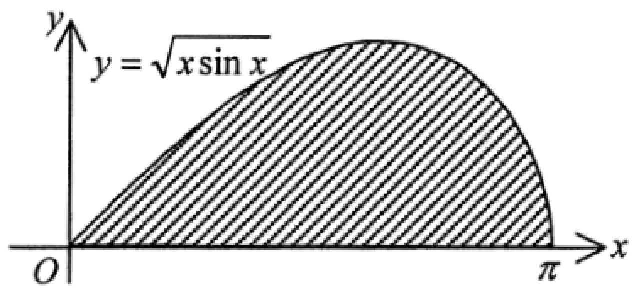

Figure 4

(4 marks)

10. **HKDSE Math M2 2012 Q10**

In Figure 5, OAB is an isosceles triangle with $OA = OB$, $AB = 1$, $AY = y$, $\angle AOY = \theta$ and $\angle BOY = 3\theta$.

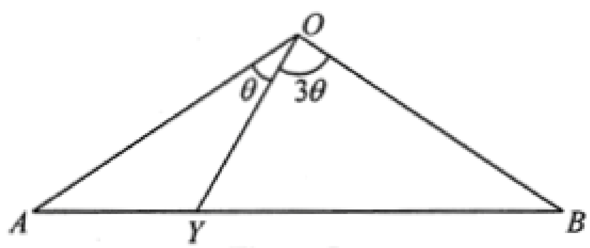

Figure 5

(a) Show that $y = \dfrac{1}{4}\sec^2\theta$.

(b) Find the range of values of y. [Hint: you may use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.]

(6 marks)

11. **HKDSE Math M2 2012 Q11**

(a) Solve the equation

$$\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0 \text{ ———————(*)}.$$

(2 marks)

(b) Let x_1, x_2 ($x_1 < x_2$) be the roots of (*). Let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$. It is given that

$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix}$ and $|P| = 1$,

where a, b and c are constants.

(i) Find P.

(ii) Evaluate $P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P$.

(iii) Using (b)(ii), evaluate $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$.

(11 marks)

12. **HKDSE Math M2 2012 Q12**

Figure 6 shows an acute angled scalene triangle ABC, where D is the mid-point of AB, G is the centroid and O is the circumcentre. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

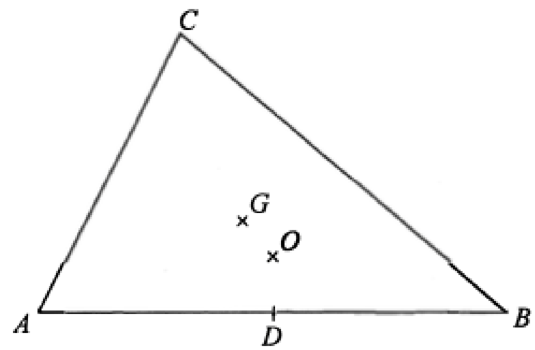

Figure 6

(a) Express $\overrightarrow{AG}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(3 marks)

(b) It is given that E is a point on AB such that CE is an altitude. Extend OG to meet CE at F.

(i) Prove that $\triangle DOG \sim \triangle CFG$.

Hence find $FG : GO$.

(ii) Show that $\overrightarrow{AF} = \mathbf{b} + \mathbf{c}$.

Hence prove that F is the orthocentre of $\triangle ABC$.

(9 marks)

13. **HKDSE Math M2 2012 Q13**

(a) (i) Suppose $\tan u = \dfrac{-1 + \cos \dfrac{2\pi}{5}}{\sin \dfrac{2\pi}{5}}$, where $\dfrac{-\pi}{2} < u < \dfrac{\pi}{2}$.

Show that $u = \dfrac{-\pi}{5}$.

(ii) Suppose $\tan v = \dfrac{1 + \cos \dfrac{2\pi}{5}}{\sin \dfrac{2\pi}{5}}$.

Find v, where $\dfrac{-\pi}{2} < v < \dfrac{\pi}{2}$.

(4 marks)

(b) (i) Express $x^2 + 2x \cos \dfrac{2\pi}{5} + 1$ in the form $(x + a)^2 + b^2$, where a and b are constants.

(ii) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2+2x\cos\frac{2\pi}{5}+1}\,dx.$

(6 marks)

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,dx.$

(3marks)

14. **HKDSE Math M2 2012 Q14**

Consider the curve $\Gamma : y = kx^p$, where $k > 0$, $p > 0$. In Figure 7, the tangent to Γ at $A(a, ka^p)$ cuts the x-axis at $B(-a, 0)$, where $a > 0$.

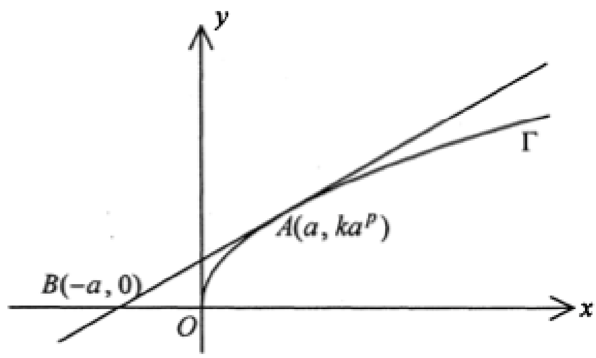

Figure 7

(a) Show that $p = \frac{1}{2}$.
(3 marks)

(b) Suppose that $a = 1$. As shown in Figure 8, the circle C, with radius 2 and centre on the y-axis, touches Γ at point A.

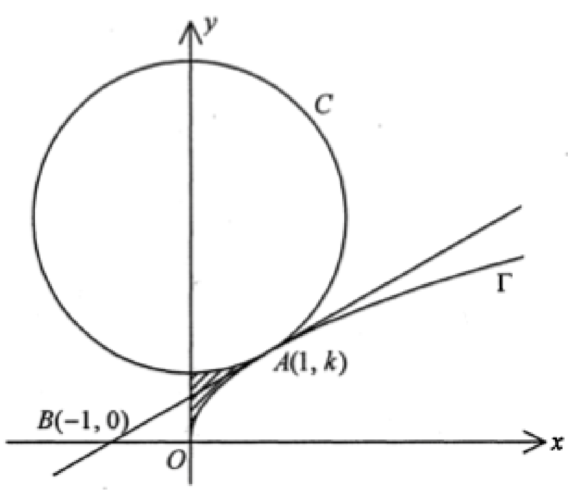

Figure 8

(i) Show that $k = \frac{2\sqrt{3}}{3}$.

(ii) Find the area of the shaded region bounded by Γ, C and the y-axis.

(9 marks)