Mock Exam 1

Section A

1. *Reference: HKDSE Math M2 2016 Q1*

$(4 - x)^3$

$= 4^3 + C_1^3(4)^2(-x) + C_2^3(4)(-x)^2 + (-x)^3$ 1M

$= \underline{\underline{64 - 48x + 12x^2 - x^3}}$ 1A

$$(4 - x)^3\left(1 + \frac{6}{x}\right)^4$$

$$= (64 - 48x + 12x^2 - x^3)\left[1 + C_1^4\left(\frac{6}{x}\right) + C_2^4\left(\frac{6}{x}\right)^2 + C_3^4\left(\frac{6}{x}\right)^3 + \left(\frac{6}{x}\right)^4\right]$$

$$= (64 - 48x + 12x^2 - x^3)\left(1 + \frac{24}{x} + \frac{216}{x^2} + \frac{864}{x^3} + \frac{1296}{x^4}\right)$$ 1M

$\therefore$ Constant term $= (64)(1) + (-48)(24) + (12)(216) + (-1)(864)$ 1M

$= \underline{\underline{640}}$ 1A

(5)

2. *Reference: HKDSE Math M2 2013 Q1*

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x + h) - \sin x}{h}$$ 1M

$$= \lim_{h\to 0}\frac{2\cos\frac{2x + h}{2}\sin\frac{h}{2}}{h}$$ 1M

$$= \lim_{h\to 0}\left(\cos\frac{2x + h}{2} \times \frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$ 1M

$= (\cos x)(1)$

$= \underline{\underline{\cos x}}$ 1A

(4)

3. *Reference: HKDSE Math M2 2016 Q3*

(a) Area $= \frac{1}{2}(u)(4\ln u)$ square units

$= \underline{\underline{2u \ln u \text{ square units}}}$ 1A

(b) Let v units be the length of PQ and A square units be the area of ΔOPQ.

$v = 4 \ln u$

$$\frac{dv}{dt} = \frac{4}{u} \times \frac{du}{dt}$$

When $u = e$ and $\frac{dv}{dt} = -4$,

$$-4 = \frac{4}{e} \times \frac{du}{dt}$$ 1M

$$\frac{du}{dt} = -e$$ 1A

$$\frac{dA}{dt} = \left(2u \times \frac{1}{u} + 2\ln u\right) \times \frac{du}{dt}$$ 1M

$= (2 + 2\ln e)(-e)$

$= -4e$

$\therefore$ The rate of change of the area of ΔOPQ is $-4e$ square units per second. 1A

(5)

4. *Reference: HKDSE Math M2 2014 Q3*

When $x = 0$, $y = \pm 3$. 1A

Differentiate both sides of the equation of the curve with respect to x,

$$x\left(\frac{2y}{y^2}\times\frac{dy}{dx}\right)+\ln y^2+2y\frac{dy}{dx}=0 \qquad \text{1M + 1M}$$

$$\frac{dy}{dx}=-\frac{\ln y^2}{\frac{2x}{y}+2y}$$

$$=-\frac{y\ln y^2}{2x+2y^2} \qquad \text{1A}$$

$$\left.\frac{dy}{dx}\right|_{(0,\ 3)}=-\frac{3\ln 9}{18}=-\frac{\ln 3}{3}$$

$$\left.\frac{dy}{dx}\right|_{(0,\ -3)}=-\frac{(-3)\ln 9}{18}=\frac{\ln 3}{3} \qquad \text{1M + 1A}$$

$\therefore$ At (0, 3), the equation of the tangent is $y=-\frac{x\ln 3}{3}+3$.

At (0, –3), the equation of the tangent is $y=\frac{x\ln 3}{3}-3$. 1A

(7)

5. *Reference: HKDSE Math M2 2016 Q5*

(a) When $n = 1$,

L.H.S. = (–1)(1) = –1

$$\text{R.H.S.}=\frac{(-1)[2(1)+1]-1}{4}=\frac{-3-1}{4}=-1$$

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = m$, where m is a positive integer, that is,

$$\sum_{k=1}^{m}(-1)^k k=\frac{(-1)^m(2m+1)-1}{4}, \qquad 1$$

when $n = m + 1$,

$$\text{L.H.S.}=\sum_{k=1}^{m+1}(-1)^k k$$

$$=\sum_{k=1}^{m}(-1)^k k+(-1)^{m+1}(m+1)$$

$$=\frac{(-1)^m(2m+1)-1}{4}+(-1)^{m+1}(m+1) \quad \text{(by the assumption)} \qquad 1$$

$$=\frac{(-1)^m(2m+1)-1+4(-1)^{m+1}(m+1)}{4}$$

$$=\frac{(-1)^{m+1}(-2m-1+4m+4)-1}{4}$$

$$=\frac{(-1)^{m+1}[2(m+1)+1]-1}{4}$$

$$=\text{R.H.S.}$$

$\therefore$ The proposition is true for $n = m + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(b) $\sum_{k=3}^{2017}(-1)^k(k+1) = \sum_{k=3}^{2017}(-1)^k k + \sum_{k=3}^{2017}(-1)^k$

$= \sum_{k=1}^{2017}(-1)^k k - (-1)(1) - (-1)^2(2) - 1$

$= \frac{(-1)^{2017}[2(2017)+1]-1}{4} + 1 - 2 - 1$ 1M

$= \underline{\underline{-1011}}$ 1A

(6)

6. (a) $\sin 3x = \sin(x + 2x)$

$= \sin x \cos 2x + \cos x \sin 2x$ 1M

$= \sin x\,(1 - 2\sin^2 x) + \cos x\,(2\sin x \cos x)$ 1M

$= \sin x - 2\sin^3 x + 2\sin x\cos^2 x$

$= \sin x - 2\sin^3 x + 2\sin x\,(1 - \sin^2 x)$

$= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$ 1

(b) $\sin 108° = \sin 72°$

$\sin 3(36°) = \sin 2(36°)$

$3\sin 36° - 4\sin^3 36° = 2\sin 36°\cos 36°$ (by (a)) 1M

$3 - 4\sin^2 36° = 2\cos 36°$ $(\sin 36° \neq 0)$

$3 - 4(1 - \cos^2 36°) = 2\cos 36°$ 1M

$4\cos^2 36° - 2\cos 36° - 1 = 0$

$\therefore$ $\cos 36°$ is a root of the equation $4x^2 - 2x - 1 = 0$. 1

$\therefore$ $\cos 36° = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$

$= \underline{\underline{\frac{1+\sqrt{5}}{4}}}$ or $\frac{1-\sqrt{5}}{4}$ (rejected) 1A

(7)

7. *Reference: HKALE P. Math 2011 Paper 2 Q5*

(a) Let $x = 3 + 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$.

$\because$ $\sin\theta = \frac{x-3}{3}$

$\therefore$ $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{x-3}{3}\right)^2} = \frac{\sqrt{6x - x^2}}{3}$

$\therefore$ $3\cos\theta = \sqrt{x(6-x)}$

$\int \sqrt{x(6-x)}\, dx = \int (3\cos\theta)(3\cos\theta)\, d\theta$ 1M

$= 9\int \cos^2\theta\, d\theta$

$= \frac{9}{2}\int (1 + \cos 2\theta)\, d\theta$ 1A

$= \frac{9}{2}\theta + \frac{9\sin 2\theta}{4} + C$ 1A

$= \frac{9}{2}\theta + \frac{9\sin\theta\cos\theta}{2} + C$

$= \frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{9}{2}\left(\frac{x-3}{3}\right)\left(\frac{\sqrt{6x-x^2}}{3}\right) + C$

$= \underline{\underline{\frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{x-3}{2}\sqrt{6x-x^2} + C}}$, where C is a constant. 1A

(b) The equation of the curve can be rewritten as $y = \pm\sqrt[4]{x(6-x)}$.

In the first quadrant, $y = \sqrt[4]{x(6-x)}$. 1M

When $y = 0$,

$$\sqrt[4]{x(6-x)} = 0$$
$$x(6-x) = 0$$
$$x = 0 \text{ or } 6$$

The required volume $= \pi\int_0^6 [\sqrt[4]{x(6-x)}]^2\,dx$

$= \pi\int_0^6 \sqrt{x(6-x)}\,dx$ 1M

$= \pi\left[\frac{9}{2}\sin^{-1}\frac{x-3}{3} + \frac{x-3}{2}\sqrt{6x-x^2}\right]_0^6$ (by (a)) 1M

$= \frac{9\pi}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= \underline{\underline{\frac{9\pi^2}{2}}}$ 1A

(8)

8. *Reference: HKDSE Math M2 2016 Q8*

(a) (i) $M^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2 \\ 0 & 1 \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}}}$ 1A

(ii) $M^3 = \begin{pmatrix} 1 & 2+2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2+2 \\ 0 & 1 \end{pmatrix}$ 1M

Similarly, we have $M^n = \underline{\underline{\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}}}$. 1A

(iii) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = (M^T)^n$

$= (M^n)^T$ 1M

$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}}}$ 1A

(b) (i) $\sum_{k=1}^{n} 2^k = 2 + 2^2 + 2^3 + \cdots + 2^n$

$= \frac{2(2^n - 1)}{2-1}$

$= \underline{\underline{2^{n+1} - 2}}$ 1A

Smart Tips

It is the sum of a geometric sequence with first term 2 and common ratio 2.

(ii) $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2^2 \\ 0 & 2^2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2+2^2 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2^2+2^3 \\ 0 & 2^3 \end{pmatrix}$ 1M

Similarly,

$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2+2^2+2^3+\cdots+2^n \\ 0 & 2^n \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} 1 & 2^{n+1}-2 \\ 0 & 2^n \end{pmatrix}}}$ 1A

(8)

Section B

9. *Reference: HKDSE Math M2 2016 Q9*

(a) Since $P(-2, 9)$ is a point on C, we have

$(-2)^3 + p(-2)^2 + q(-2) + 1 = 9$

$4p - 2q = 16$ ...... (1) 1M

$\frac{dy}{dx} = 3x^2 + 2px + q$ 1A

Since $P(-2, 9)$ is a stationary point on C, we have

$\left.\frac{dy}{dx}\right|_{x=-2} = 0$

$3(-2)^2 + 2p(-2) + q = 0$

$4p - q = 12$ ...... (2)

(2) − (1): $q = \underline{\underline{-4}}$ 1A

Substituting $q = -4$ into (2),

$4p - (-4) = 12$

$p = \underline{\underline{2}}$

(3)

(b) From (a),

$\frac{dy}{dx} = 3x^2 + 4x - 4$

$\frac{d^2y}{dx^2} = 6x + 4$

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 6(-2) + 4 = -8 < 0$ 1M

$\therefore$ P is a maximum point of C.

i.e., P is not a minimum point of C. 1A

(2)

(c) When $\frac{dy}{dx} = 0$,

$$3x^2 + 4x - 4 = 0$$
$$(3x - 2)(x + 2) = 0$$
$$x = \frac{2}{3} \text{ or } -2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{2}{3}} = 6\left(\frac{2}{3}\right) + 4 = 8 > 0$$ 1M

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 1 = -\frac{13}{27}$$

∴ (–2, 9) is a maximum point. $\left(\frac{2}{3}, -\frac{13}{27}\right)$ is a minimum point. 1A

(2)

(d) When $\frac{d^2y}{dx^2} = 0$, $x = -\frac{2}{3}$.

x	$x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$x > -\frac{2}{3}$
$\frac{d^2y}{dx^2}$	−	0	+

1M

$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 1 = \frac{115}{27}$$

∴ $\left(-\frac{2}{3}, \frac{115}{27}\right)$ is the point of inflexion. 1A

(2)

(e) The equation of L is $y = 9$.

When $y = 9$,

$$x^3 + 2x^2 - 4x + 1 = 9$$ 1M
$$x^3 + 2x^2 - 4x - 8 = 0$$
$$(x + 2)^2(x - 2) = 0$$
$$x = 2 \text{ or } -2$$

$$\text{Area} = \int_{-2}^{2}[9 - (x^3 + 2x^2 - 4x + 1)]\,dx$$ 1M
$$= \int_{-2}^{2}(-x^3 - 2x^2 + 4x + 8)\,dx$$
$$= \left[-\frac{x^4}{4} - \frac{2x^3}{3} + 2x^2 + 8x\right]_{-2}^{2}$$ 1M
$$= \underline{\underline{\frac{64}{3}}}$$ 1A

(4)

10. (a) (i) Let $u = k - x$. Then $dx = -du$. 1M

When $x = \frac{\pi}{4}$, $u = k - \frac{\pi}{4}$.

When $x = k - \frac{\pi}{4}$, $u = \frac{\pi}{4}$.

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\sin(k - x)\,dx$$

$$= -\int_{k-\frac{\pi}{4}}^{\frac{\pi}{4}} \ln\sin u\,du \qquad \text{1M}$$

$$= \int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\sin u\,du$$

$$= \int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\sin x\,dx \qquad 1$$

(ii) By (a)(i),

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\sin(k - x)\,dx - \int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\sin x\,dx = 0 \qquad \text{1M}$$

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} [\ln\sin(k - x) - \ln\sin x]\,dx = 0$$

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\left[\frac{\sin(k - x)}{\sin x}\right]dx = 0 \qquad \text{1M}$$

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln\left(\frac{\sin k\cos x - \cos k\sin x}{\sin x}\right)dx = 0$$

$$\int_{\frac{\pi}{4}}^{k-\frac{\pi}{4}} \ln(\sin k\cot x - \cos k)\,dx = 0 \qquad 1$$

(6)

(b) (i) $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\csc^2 x}{\cot x + 1}\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{-x}{\cot x + 1}\,d(\cot x + 1)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-x)d[\ln(\cot x + 1)] \qquad \text{1M}$$

$$= [-x\ln(\cot x + 1)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cot x + 1)\,dx \qquad \text{1M}$$

$$= \frac{\pi\ln 2}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cot x + 1)\,dx \qquad 1$$

(ii) Note that $\frac{\pi}{2} < \frac{3\pi}{4} < \pi$.

Putting $k = \frac{3\pi}{4}$ in (a)(ii), we have

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\sin\frac{3\pi}{4}\cot x - \cos\frac{3\pi}{4}\right) dx = 0$$ 1M

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\frac{1}{\sqrt{2}}\cot x + \frac{1}{\sqrt{2}}\right) dx = 0$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\ln(\cot x + 1) - \ln\sqrt{2}]\, dx = 0$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cot x + 1)\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sqrt{2}\, dx$$

$$= \frac{\pi\ln 2}{8}$$ 1A

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x\csc^2 x}{\cot x + 1}\, dx = \frac{\pi\ln 2}{4} + \frac{\pi\ln 2}{8} \quad \text{(by (b)(i))}$$

$$= \underline{\underline{\frac{3\pi\ln 2}{8}}}$$ 1A

(6)

Analysis

Choose a suitable value of k in (a)(ii) so as to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cot x + 1)\, dx$.

Smart Tips

$$\ln\left(\frac{1}{\sqrt{2}}\cot x + \frac{1}{\sqrt{2}}\right)$$
$$= \ln\left(\frac{\cot x + 1}{\sqrt{2}}\right)$$
$$= \ln(\cot x + 1) - \ln\sqrt{2}$$

11. *Reference: HKDSE Math M2 2016 Q11*

(a) (i) (1) Let A be the coefficient matrix.

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & p \\ 4 & 2-p & 4p-5 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 1-p \\ 2 & -1 & p \\ 4 & 2-p & 4p-5 \end{vmatrix} \quad (R_1 - R_2 \to R_1)$$

$$= 4p - 5 + 2(1-p)(2-p) + 4(1-p) + p(2-p)$$ 1A

$$= p^2 - 4p + 3$$

$$= (p-1)(p-3)$$

(E) has a unique solution if and only if $|A| \neq 0$. 1M

$\therefore$ $p \neq 1$ and $p \neq 3$. 1

(2) $\begin{vmatrix} 2 & -1 & 1 \\ q & -1 & p \\ q-8 & 2-p & 4p-5 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ q & -1 & p-1 \\ q-8 & 2-p & 3(p-1) \end{vmatrix} (C_3 + C_2 \to C_3)$

$= \begin{vmatrix} 0 & -1 & 0 \\ q-2 & -1 & p-1 \\ -2p+q-4 & 2-p & 3(p-1) \end{vmatrix} (C_1 + 2C_2 \to C_1)$

$= 3(p-1)(q-2) - (p-1)(-2p+q-4)$

$= 2(p-1)(p+q-1)$

$\begin{vmatrix} 1 & 2 & 1 \\ 2 & q & p \\ 4 & q-8 & 4p-5 \end{vmatrix} = q(4p-5) + 8p + 2(q-8) - 4q - 4(4p-5) - p(q-8)$

$= 3pq - 7q + 4$

$\begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & q \\ 4 & 2-p & q-8 \end{vmatrix} = -(q-8) - 4q + 4(2-p) + 8 + 2(q-8) - q(2-p)$

$= pq - 4p - 5q + 8$

1M

By the Cramer's rule, the solution is

$x = \dfrac{2(p-1)(p+q-1)}{(p-1)(p-3)} = \dfrac{2(p+q-1)}{p-3}$

$y = \dfrac{3pq-7q+4}{(p-1)(p-3)}$

$z = \dfrac{pq-4p-5q+8}{p-3}$

1A + 1A

(ii) (1) The augmented matrix is

$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & q \\ 4 & -1 & 7 & q-8 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & q-4 \\ 0 & 3 & 3 & q-16 \end{array}\right)$ $(R_2 - 2R_1 \to R_2;$ $R_3 - 4R_1 \to R_3)$

$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & q-4 \\ 0 & 0 & 0 & -2q-4 \end{array}\right)$ $(R_3 - 3R_2 \to R_3)$ 1M

Since (E) is consistent, $-2q - 4 = 0$, i.e., $q = -2$. 1A

(2) By (a)(ii)(1), the augmented matrix is

$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $(R_1 + R_2 \to R_1)$

Let $z = t$, where t is any real number.

Then $y = -6 - t$ and $x = -4 - 2t$.

$\therefore$ The solution is $x = -4 - 2t$, $y = -6 - t$ and $z = t$, where t is any real number.

1A

(9)

(b) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 8 & -4 & 12 & -8 \\ 8 & -2 & 14 & -20 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & -2 \\ 4 & -1 & 7 & -10 \end{array}\right) \qquad \left(\frac{R_2}{4} \to R_2,\ \frac{R_3}{2} \to R_3\right)$$

$\therefore$ The system of linear equations is equivalent to (E) for $p = 3$ and $q = -2$.

By (a)(ii)(2), $x = -4 - 2t$, $y = -6 - t$ and $z = t$, where t is any real number. 1M

If $xy = 3z$, then

$(-4 - 2t)(-6 - t) = 3t$

$24 + 16t + 2t^2 = 3t$

$2t^2 + 13t + 24 = 0$ .... (*)

Note that $13^2 - 4(2)(24) = -23 < 0$. 1M

$\therefore$ (*) does not have real solution.

$\therefore$ There is no solution of the system of linear equations satisfying $xy = 3z$. 1A

(3)

12. *Reference: HKCEE A. Math 2008 Q15*

(a) $\overrightarrow{OM} = 2\mathbf{a}$

$\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC} = (2 - r)\mathbf{a} - s\mathbf{b}$

Since C is the circumcentre of ΔOPQ,

$\overrightarrow{OP} \cdot \overrightarrow{CM} = 0 \quad (\because \quad OP \perp CM)$

$4\mathbf{a} \cdot [(2 - r)\mathbf{a} - s\mathbf{b}] = 0$

$(2 - r)\mathbf{a} \cdot \mathbf{a} - s\mathbf{a} \cdot \mathbf{b} = 0$ 1M

$(2 - r) - s\left(\frac{1}{3}\right) = 0$ 1M

$6 - 3r - s = 0$

$3r + s = 6$ ...... (1) 1

(3)

(b) Let R be the mid-point of OQ.

$\overrightarrow{OR} = \mathbf{b}$

$\overrightarrow{CR} = \overrightarrow{OR} - \overrightarrow{OC} = -r\mathbf{a} + (1 - s)\mathbf{b}$

$\overrightarrow{OQ} \cdot \overrightarrow{CR} = 0 \quad (\because \quad OQ \perp CR)$

$2\mathbf{b} \cdot [-r\mathbf{a} + (1 - s)\mathbf{b}] = 0$ 1M

$-r\mathbf{b} \cdot \mathbf{a} + (1 - s)\mathbf{b} \cdot \mathbf{b} = 0$

$-r\left(\frac{1}{3}\right) + 1 - s = 0$

$-r + 3 - 3s = 0$

$r + 3s = 3$ ..... (2) 1M

$3 \times (2)$: $3r + 9s = 9$ .... (3)

$(3) - (1)$: $8s = 3$

$s = \frac{3}{8}$

Analysis

Consider $\overrightarrow{OQ} \cdot \overrightarrow{CR}$, where R is the mid-point of OQ.

Substituting $s = \frac{3}{8}$ into (2),

$$r + 3\left(\frac{3}{8}\right) = 3$$

$$r = \frac{15}{8}$$

$\therefore \quad r = \underline{\underline{\frac{15}{8}}}$ and $s = \underline{\underline{\frac{3}{8}}}$ 1A

(3)

(c) $\overrightarrow{ON} = \frac{1}{2}(4\mathbf{a} + 2\mathbf{b}) = 2\mathbf{a} + \mathbf{b}$

$\overrightarrow{CN} = \overrightarrow{ON} - \overrightarrow{OC}$ 1M

$= 2\mathbf{a} + \mathbf{b} - \left(\frac{15}{8}\mathbf{a} + \frac{3}{8}\mathbf{b}\right)$

$= \frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}$ 1A

Smart Tips

N is the mid-point of PQ.

Since H is the orthocentre, $OH \perp PQ$.
Since C is the circumcentre, $CN \perp PQ$.
$\therefore \quad OH \parallel CN$
Since $OH \parallel CN$ and $OH : CN = k : 1$,

$\overrightarrow{OH} = k\left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}\right)$ 1M

$\overrightarrow{QH} = \overrightarrow{OH} - \overrightarrow{OQ}$

$= \frac{k}{8}\mathbf{a} + \left(\frac{5k}{8} - 2\right)\mathbf{b}$ 1A

$$\overrightarrow{QH} \cdot \overrightarrow{OP} = 0$$

$$\left[\frac{k}{8}\mathbf{a} + \left(\frac{5k}{8} - 2\right)\mathbf{b}\right] \cdot (4\mathbf{a}) = 0$$

$\frac{k}{8}\mathbf{a} \cdot \mathbf{a} + \left(\frac{5k}{8} - 2\right)\mathbf{b} \cdot \mathbf{a} = 0$ 1M

$$\frac{k}{8} + \left(\frac{5k}{8} - 2\right)\left(\frac{1}{3}\right) = 0$$

$$3k + (5k - 16) = 0$$

$k = 2$ 1A

$\therefore \quad \overrightarrow{OH} = 2\left(\frac{1}{8}\mathbf{a} + \frac{5}{8}\mathbf{b}\right) = \underline{\underline{\frac{1}{4}\mathbf{a} + \frac{5}{4}\mathbf{b}}}$ 1A

(7)