Marking Scheme

Module 2 (Algebra and Calculus)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with $\boxed{\text{rectangles}}$. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

Solution	Marks	Remarks
1. $\mathrm{f}(1+h)$		
$=((1+h)^2-1)e^{1+h}$		
$=(2h+h^2)e^{1+h}$	1A	
$\mathrm{f}'(1)$		
$=\lim_{h\to 0}\dfrac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}$	1M	
$=\lim_{h\to 0}\dfrac{(2h+h^2)e^{1+h}-0}{h}$		
$=\lim_{h\to 0}(2+h)e^{1+h}$	1M	withhold 1M if this step is skipped
$=2e$	1A	
	----------(4)	
2. $(x+3)^5$		
$=x^5+5(3)x^4+10(3^2)x^3+10(3^3)x^2+5(3^4)x+3^5$	1M	
$=x^5+15x^4+90x^3+270x^2+405x+243$	1A	
$\left(x-\dfrac{4}{x}\right)^2$		
$=x^2-2x\left(\dfrac{4}{x}\right)+\left(\dfrac{4}{x}\right)^2$	1M	
$=x^2-8+\dfrac{16}{x^2}$		
The coefficient of x^3		
$=(1)(16)+(90)(-8)+(405)(1)$	1M	withhold 1M if this step is skipped
$=-299$	1A	
	----------(5)	

Solution	Marks	Remarks
3. (a) $\cot A = 3\cot B$		
$\dfrac{\cos A}{\sin A} = \dfrac{3\cos B}{\sin B}$		
$3\sin A\cos B = \cos A\sin B$	1M	
$\sin(A+B) - 2\sin(B-A)$		
$= (\sin A\cos B + \cos A\sin B) - 2(\sin B\cos A - \cos B\sin A)$		
$= 3\sin A\cos B - \cos A\sin B$		
$= 0$		
Thus, we have $\sin(A+B) = 2\sin(B-A)$.	1	
(b) $\cot\left(x+\dfrac{4\pi}{9}\right) = 3\cot\left(x+\dfrac{5\pi}{18}\right)$		
By letting $A = x+\dfrac{4\pi}{9}$ and $B = x+\dfrac{5\pi}{18}$, we have $\cot A = 3\cot B$.	1M	
By (a), we have $\sin(A+B) = 2\sin(B-A)$.		
With the help of $\sin\left(\dfrac{-\pi}{6}\right) = \dfrac{-1}{2}$, we have $\sin\left(2x+\dfrac{13\pi}{18}\right) = -1$.	1M	
Noting that $0 \le x \le \dfrac{\pi}{2}$, we have $x = \dfrac{7\pi}{18}$.	1A	
Since $\cot\left(\dfrac{7\pi}{18}+\dfrac{4\pi}{9}\right) = -\sqrt{3} = 3\cot\left(\dfrac{7\pi}{18}+\dfrac{5\pi}{18}\right)$, the required solution of the equation $\cot\left(x+\dfrac{4\pi}{9}\right) = 3\cot\left(x+\dfrac{5\pi}{18}\right)$ is $x = \dfrac{7\pi}{18}$.		
	----------(5)	
4. (a) $\displaystyle\int u(5^u)\,\mathrm{d}u$		
$= \dfrac{1}{\ln 5}\left(u(5^u) - \displaystyle\int 5^u\,\mathrm{d}u\right)$	1M	
$= \dfrac{1}{\ln 5}\left(u(5^u) - \dfrac{5^u}{\ln 5}\right) + \text{constant}$	1A	
$= \dfrac{5^u(u\ln 5 - 1)}{(\ln 5)^2} + \text{constant}$		
(b) The required area		
$= \displaystyle\int_0^1 x(5^{2x})\,\mathrm{d}x$	1M	
$= \dfrac{1}{4}\displaystyle\int_0^2 u(5^u)\,\mathrm{d}u$ (by letting $u = 2x$)	1M	
$= \dfrac{1}{4(\ln 5)^2}\left[5^u(u\ln 5 - 1)\right]_0^2$ (by (a))	1M	for using the result of (a)
$= \dfrac{50\ln 5 - 24}{4(\ln 5)^2}$		
$= \dfrac{25\ln 5 - 12}{2(\ln 5)^2}$	1A	
	----------(6)	

Solution	Marks	Remarks
5. (a) Let $u=1+x^2$.	1M	
Then, we have $\frac{\mathrm{d}u}{\mathrm{d}x}=2x$.		
$\int x^3\sqrt{1+x^2}\ \mathrm{d}x$		
$=\int \frac{1}{2}(u-1)u^{\frac{1}{2}}\ \mathrm{d}u$	1M	
$=\frac{1}{2}\left(\int u^{\frac{3}{2}}\ \mathrm{d}u-\int u^{\frac{1}{2}}\ \mathrm{d}u\right)$		
$=\frac{1}{5}\left(\sqrt{1+x^2}\right)^5-\frac{1}{3}\left(\sqrt{1+x^2}\right)^3+\text{constant}$	1A	
Let $x=\tan\theta$.	1M	
Then, we have $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$.		
$\int x^3\sqrt{1+x^2}\ \mathrm{d}x$		
$=\int \tan^3\theta\sec\theta(\sec^2\theta)\ \mathrm{d}\theta$	1M	
$=\int \tan^3\theta\sec^3\theta\ \mathrm{d}\theta$		
$=\int (\sec^2\theta-1)\sec^2\theta\ \mathrm{d}\sec\theta$		
$=\int \sec^4\theta\ \mathrm{d}\sec\theta-\int\sec^2\theta\ \mathrm{d}\sec\theta$		
$=\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}+\text{constant}$		
$=\frac{1}{5}\left(\sqrt{1+x^2}\right)^5-\frac{1}{3}\left(\sqrt{1+x^2}\right)^3+\text{constant}$	1A	
(b) y		
$=\int 15x^3\sqrt{1+x^2}\ \mathrm{d}x$	1M	
$=15\int x^3\sqrt{1+x^2}\ \mathrm{d}x$		
$=15\left(\frac{1}{5}\left(\sqrt{1+x^2}\right)^5-\frac{1}{3}\left(\sqrt{1+x^2}\right)^3\right)+C$ (by (a))	1M	for using the result of (a)
$=3\left(\sqrt{1+x^2}\right)^5-5\left(\sqrt{1+x^2}\right)^3+C$, where C is a constant		
Since the y-intercept of Γ is 2 , we have $3-5+C=2$. Solving, we have $C=4$.	1M	
Thus, the equation of Γ is $y=3\left(\sqrt{1+x^2}\right)^5-5\left(\sqrt{1+x^2}\right)^3+4$.	1A	
	----------(7)	

Solution	Marks	Remarks
6. (a) Note that $(1)(1+4)=5=\dfrac{(1)(2)(15)}{6}$.		
So, the statement is true for $n=1$.	1	
Assume that $\displaystyle\sum_{k=1}^{m} k(k+4)=\frac{m(m+1)(2m+13)}{6}$ for some positive integer m .	1M	
$\displaystyle\sum_{k=1}^{m+1} k(k+4)$		
$\displaystyle=\sum_{k=1}^{m} k(k+4)+(m+1)(m+5)$		
$=\dfrac{m(m+1)(2m+13)}{6}+(m+1)(m+5)$ (by induction assumption)	1M	for using induction assumption
$=\dfrac{(m+1)(2m^2+13m+6m+30)}{6}$		
$=\dfrac{(m+1)(2m^2+19m+30)}{6}$		
$=\dfrac{(m+1)(m+2)(2m+15)}{6}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(b) Putting $n=555$ in (a), we have		
$\displaystyle\sum_{k=1}^{555} k(k+4)=\frac{(555)(556)(1\,123)}{6}=57\,755\,890$.	1M	either one
Putting $n=332$ in (a), we have		
$\displaystyle\sum_{k=1}^{332} k(k+4)=\frac{(332)(333)(677)}{6}=12\,474\,402$.		
$\displaystyle\sum_{k=333}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$		
$\displaystyle=\left(\frac{1}{112}\right)\left(\frac{1}{223}\right)\sum_{k=333}^{555} k(k+4)$	1M	either one
$\displaystyle=\frac{1}{(112)(223)}\left(\sum_{k=1}^{555} k(k+4)-\sum_{k=1}^{332} k(k+4)\right)$		
$=\dfrac{1}{24\,976}(57\,755\,890-12\,474\,402)$		
$=1813$	1A	
	----------(7)	

Solution	Marks	Remarks
7. (a) $MX = XM$		
$\begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} a & 6a \\ b & c \end{pmatrix} = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}\begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$		
$\begin{pmatrix} 7a+3b & 42a+3c \\ -a+5b & -6a+5c \end{pmatrix} = \begin{pmatrix} a & 33a \\ 7b-c & 3b+5c \end{pmatrix}$	1M	
$\begin{cases} 7a+3b=a \\ -a+5b=7b-c \\ 42a+3c=33a \\ -6a+5c=3b+5c \end{cases}$	1M	
$\begin{cases} b=-2a \\ c=-3a \end{cases}$	1A	for both correct
(b) $\lvert X \rvert$		
$=\begin{vmatrix} a & 6a \\ -2a & -3a \end{vmatrix}$	1M	for considering the determinant
$=(a)(-3a)-(6a)(-2a)$		
$=9a^2$		
Note that X is a non-zero real matrix. By (a), a is a non-zero real number. So, we have $\lvert X \rvert > 0$. Therefore, we have $\lvert X \rvert \neq 0$.		
Thus, X is a non-singular matrix.	1	
(c) $(X^T)^{-1}$		
$=(X^{-1})^T$	1M	
$=\left(\dfrac{1}{\lvert X \rvert}\begin{pmatrix} -3a & -6a \\ 2a & a \end{pmatrix}\right)^T$	1M	
$=\left(\dfrac{1}{9a}\begin{pmatrix} -3 & -6 \\ 2 & 1 \end{pmatrix}\right)^T$		
$=\dfrac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$	1A	
$(X^T)^{-1}$		
$=\begin{pmatrix} a & -2a \\ 6a & -3a \end{pmatrix}^{-1}$		
$=\dfrac{1}{\lvert X^T \rvert}\begin{pmatrix} -3a & 2a \\ -6a & a \end{pmatrix}$	1M	
$=\dfrac{1}{\lvert X \rvert}\begin{pmatrix} -3a & 2a \\ -6a & a \end{pmatrix}$	1M	
$=\dfrac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$	1A	
	----------(8)	

Solution	Marks	Remarks
8. (a) Note that $A \neq 0$.		
$\mathrm{f}'(x)$		
$= \dfrac{-A(2x-4)}{(x^2-4x+7)^2}$	1M	
So, we have $\mathrm{f}'(x)=0 \Leftrightarrow x=2$.	1M	
Since the equation $\mathrm{f}'(x)=0$ has only one solution $x=2$ and the extreme value of $\mathrm{f}(x)$ is 4 , we have $\mathrm{f}(2)=4$.		
Hence, we have $\dfrac{A}{2^2-4(2)+7}=4$.		
Therefore, we have $A=12$.		
Thus, we have $\mathrm{f}'(x)=\dfrac{24(2-x)}{(x^2-4x+7)^2}$.	1A	
(b) Note that $x^2-4x+7=(x-2)^2+3>0$ for all real values of x .		
So, there are no vertical asymptotes of the graph of $y=\mathrm{f}(x)$.		
Also note that $\mathrm{f}(x)=\dfrac{12}{x^2-4x+7}$.		
Therefore, $y=0$ is the only asymptote of the graph of $y=\mathrm{f}(x)$.	1M	
Hence, there is only one asymptote of the graph of $y=\mathrm{f}(x)$.		
Thus, the claim is disagreed.	1A	f.t.
(c) $\mathrm{f}''(x)$		
$= \dfrac{(x^2-4x+7)^2(-24)-(-24x+48)(2)(x^2-4x+7)(2x-4)}{(x^2-4x+7)^4}$	1M	
$= \dfrac{72(x-3)(x-1)}{(x^2-4x+7)^3}$		
So, we have $\mathrm{f}''(x)=0 \Leftrightarrow x=1$ or $x=3$.		

x	$(-\infty,1)$	1	$(1,3)$	3	$(3,\infty)$
$\mathrm{f}''(x)$	$+$	0	$-$	0	$+$

1M for testing

Thus, the points of inflexion are $(1,3)$ and $(3,3)$.	1A	for both correct
	----------(8)	

Solution	Marks	Remarks
9. (a) $y=\dfrac{\ln x}{2}$		
$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2x}$	1M	
The slope of the tangent at P is $\dfrac{1}{2r}$.		
The slope of the normal at P is $-2r$.		
Let a be the x-coordinate of Q .		
$\dfrac{0-\ln\sqrt{r}}{a-r}=-2r$	1M	
$\dfrac{-1}{2}\ln r=2r^2-2ar$		
$2ar=2r^2+\dfrac{1}{2}\ln r$		
$a=\dfrac{4r^2+\ln r}{4r}$	1	
Thus, the x-coordinate of Q is $\dfrac{4r^2+\ln r}{4r}$.		
	----------(3)	
(b) Let A square units be the area of ΔPQR .		
A		
$=\dfrac{1}{2}\left(\dfrac{4r^2+\ln r}{4r}-r\right)\ln\sqrt{r}$	1M	
$=\dfrac{(\ln r)^2}{16r}$	1A	
$\dfrac{\mathrm{d}A}{\mathrm{d}r}$		
$=\dfrac{r(2\ln r)\frac{1}{r}-(\ln r)^2}{16r^2}$	1M	
$=\dfrac{2\ln r-(\ln r)^2}{16r^2}$		
$=\dfrac{(2-\ln r)\ln r}{16r^2}$		
So, we have $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0 \iff \ln r=2$ or $\ln r=0$ (rejected) .		
Hence, we have $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0 \iff r=e^2$.		
r: $(1,e^2)$, e^2, (e^2,∞); $\dfrac{\mathrm{d}A}{\mathrm{d}r}$: $+$, 0, $-$; A: ↗, $\dfrac{1}{4e^2}$, ↘	1M	for testing
Therefore, A attains its greatest value when $r=e^2$.		
Thus, the greatest area of ΔPQR is $\dfrac{1}{4e^2}$ square units.	1A	
	----------(5)	

r	$(1,e^2)$	e^2	(e^2,∞)
$\dfrac{\mathrm{d}A}{\mathrm{d}r}$	$+$	0	$-$
A	↗	$\dfrac{1}{4e^2}$	↘

Solution	Marks	Remarks
(c) OP		
$= \sqrt{r^2 + (\ln \sqrt{r})^2}$	1M	
$= \dfrac{1}{2}\sqrt{4r^2 + (\ln r)^2}$		
$\dfrac{\mathrm{d}OP}{\mathrm{d}t}$		
$= \left(\dfrac{4r^2 + \ln r}{2r\sqrt{4r^2 + (\ln r)^2}} \right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t} \right)$	1M	
$\dfrac{\mathrm{d}A}{\mathrm{d}t}$		
$= \left(\dfrac{\mathrm{d}A}{\mathrm{d}r} \right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t} \right)$		
$= \left(\dfrac{(2-\ln r)\ln r}{16r^2} \right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t} \right)$ (by (b))		
$= \left(\dfrac{(2-\ln r)\ln r}{16r^2} \right)\left(\dfrac{2r\sqrt{4r^2 + (\ln r)^2}}{4r^2 + \ln r} \right)\left(\dfrac{\mathrm{d}OP}{\mathrm{d}t} \right)$		
$= \dfrac{(2-\ln r)(\ln r)\sqrt{4r^2 + (\ln r)^2}}{8r(4r^2 + \ln r)}\left(\dfrac{\mathrm{d}OP}{\mathrm{d}t} \right)$		
$\left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right\|_{r=e}$		
$= \dfrac{(2-\ln e)(\ln e)\sqrt{4e^2 + (\ln e)^2}}{8e(4e^2 + \ln e)}\left(\left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right\|_{r=e} \right)$	1M	
$= \dfrac{\sqrt{4e^2+1}}{8e(4e^2+1)}\left(\left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right\|_{r=e} \right)$		
Since $0 \le \left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right\|_{r=e} \le 32e^2$, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right\|_{r=e} \le \dfrac{32e^2\sqrt{4e^2+1}}{8e(4e^2+1)}$.		
So, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right\|_{r=e} \le \dfrac{4e}{\sqrt{4e^2+1}}$.		
Therefore, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right\|_{r=e} < \dfrac{4e}{\sqrt{4e^2}}$.		
Hence, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right\|_{r=e} < 2$.		
Thus, the claim is correct.	1A	f.t.
	----------(4)	

Solution	Marks	Remarks
10. (a) (i) $\int \sin^4 x \, dx$		
$= -\cos x \sin^3 x + \int \cos x (3\sin^2 x \cos x) \, dx$	1M	
$= -\cos x \sin^3 x + 3\int (1-\sin^2 x)(\sin^2 x) \, dx$		
So, we have $\int \sin^4 x \, dx = -\cos x \sin^3 x + 3\int \sin^2 x \, dx - 3\int \sin^4 x \, dx$.		
Hence, we have $4\int \sin^4 x \, dx = -\cos x \sin^3 x + 3\int \sin^2 x \, dx$.		
Thus, we have $\int \sin^4 x \, dx = \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$.	1	
(ii) $\int \sin^4 x \, dx$		
$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$ (by (a)(i))		
$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \frac{1-\cos 2x}{2} \, dx$	1M	
$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) + \text{constant}$	1M	
$= \frac{3x}{8} - \frac{\cos x \sin^3 x}{4} - \frac{3\sin 2x}{16} + \text{constant}$		
$\int_0^{\pi} \sin^4 x \, dx$		
$= \left[\frac{3x}{8} - \frac{\cos x \sin^3 x}{4} - \frac{3\sin 2x}{16}\right]_0^{\pi}$		
$= \frac{3\pi}{8}$	1A	
$\int_0^{\pi} \sin^4 x \, dx$		
$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)^2 dx$	1M	
$= \frac{1}{4}\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) \, dx$		
$= \frac{1}{4}\int_0^{\pi} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$		
$= \frac{1}{8}\int_0^{\pi} (3 - 4\cos 2x + \cos 4x) \, dx$		
$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi}$	1M	
$= \frac{3\pi}{8}$	1A	
	----------(5)	

Solution	Marks	Remarks
(b) (i) Let $x=\beta-u$. Then, we have $\frac{dx}{du}=-1$.		
$\int_0^{\beta} x\,f(x)\,dx$		
$=\int_{\beta}^{0} -(\beta-u)\,f(\beta-u)\,du$		
$=\int_0^{\beta}(\beta f(\beta-u)-u\,f(\beta-u))\,du$	1M	
$=\int_0^{\beta}\beta f(x)\,dx-\int_0^{\beta}x\,f(x)\,dx$		
So, we have $2\int_0^{\beta}x\,f(x)\,dx=\beta\int_0^{\beta}f(x)\,dx$.		
Thus, we have $\int_0^{\beta}x\,f(x)\,dx=\frac{\beta}{2}\int_0^{\beta}f(x)\,dx$.	1	
(ii) Note that $\sin^4(\pi-x)=\sin^4 x$ for all real numbers x .	1M	withhold 1M if checking is skipped
$\int_0^{\pi}x\sin^4 x\,dx$		
$=\frac{\pi}{2}\int_0^{\pi}\sin^4 x\,dx$ (by (b)(i))	1M	for using the result of (b)(i)
$=\frac{\pi}{2}\left(\frac{3\pi}{8}\right)$ (by (a)(ii))		
$=\frac{3\pi^2}{16}$	1M	for $\frac{\pi}{2}$(a)(ii)
	----------(5)	
(c) The required volume		
$=\int_{\pi}^{2\pi}\pi(\sqrt{x}\sin^2 x)^2\,dx$	1M	
$=\pi\int_{\pi}^{2\pi}x\sin^4 x\,dx$		
$=\pi\int_0^{\pi}(\pi+y)\sin^4(\pi+y)\,dy$ (by letting $x=\pi+y$)	1M	accept $x=2\pi-y$
$=\pi\int_0^{\pi}(\pi\sin^4 y+y\sin^4 y)\,dy$		
$=\pi\int_0^{\pi}(\pi\sin^4 x+x\sin^4 x)\,dx$		
$=\pi\left(\pi\int_0^{\pi}\sin^4 x\,dx+\int_0^{\pi}x\sin^4 x\,dx\right)$		
$=\pi\left(\pi\left(\frac{3\pi}{8}\right)+\frac{3\pi^2}{16}\right)$ (by (a)(ii) and (b)(ii))		
$=\frac{9\pi^3}{16}$	1A	
	----------(3)	

Solution	Marks	Remarks
11. (a) (i) (1) Note that		
$\begin{vmatrix} 1 & a & 4(a+1) \\ 2 & a-1 & 2(a-1) \\ 1 & -1 & -12 \end{vmatrix}$		
$=(a-1)(-12)+a(2)(a-1)+4(a+1)(2)(-1)-4(a-1)(a+1)+2(a-1)-2a(-12)$	1A	
$=-2(a-3)(a+1)$		
Since (E) has a unique solution, we have $\begin{vmatrix} 1 & a & 4(a+1) \\ 2 & a-1 & 2(a-1) \\ 1 & -1 & -12 \end{vmatrix} \neq 0$.	1M	
So, we have $-2(a-3)(a+1) \neq 0$.		
Therefore, we have $a \neq 3$ and $a \neq -1$.	1A	
Thus, we have $a < -1$, $-1 < a < 3$ or $a > 3$.		
The augmented matrix of (E) is		
$\left(\begin{array}{ccc\|c} 1 & a & 4(a+1) & 18 \\ 2 & a-1 & 2(a-1) & 20 \\ 1 & -1 & -12 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & -a-1 & -4a-16 & b-18 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & 0 & 2a-6 & b-2 \end{array}\right)$	1A	
Since (E) has a unique solution, we have $2a-6 \neq 0$ and $-a-1 \neq 0$.		
Therefore, we have $a \neq 3$ and $a \neq -1$.	1A	
Thus, we have $a < -1$, $-1 < a < 3$ or $a > 3$.		
(2) Since (E) has a unique solution, we have		
$x = \dfrac{\begin{vmatrix} 18 & a & 4(a+1) \\ 20 & a-1 & 2(a-1) \\ b & -1 & -12 \end{vmatrix}}{-2(a-3)(a+1)}$	1M	for Cramer's Rule
$=\dfrac{a^2b+ab+10a-2b-50}{(a-3)(a+1)}$		
$y = \dfrac{\begin{vmatrix} 1 & 18 & 4(a+1) \\ 2 & 20 & 2(a-1) \\ 1 & b & -12 \end{vmatrix}}{-2(a-3)(a+1)}$		
$=\dfrac{-3ab+22a-5b-38}{(a-3)(a+1)}$		
$z = \dfrac{\begin{vmatrix} 1 & a & 18 \\ 2 & a-1 & 20 \\ 1 & -1 & b \end{vmatrix}}{-2(a-3)(a+1)}$		
$=\dfrac{b-2}{2(a-3)}$	1A+1A	1A for any one + 1A for all

Solution	Marks	Remarks
Since (E) has a unique solution, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & 0 & 2a-6 & b-2 \end{array}\right)$	1M	
$\sim\left(\begin{array}{ccc\|c} 1 & 0 & 0 & \dfrac{a^2b+ab+10a-2b-50}{(a-3)(a+1)} \\ 0 & 1 & 0 & \dfrac{-3ab+22a-5b-38}{(a-3)(a+1)} \\ 0 & 0 & 1 & \dfrac{b-2}{2(a-3)} \end{array}\right)$		
Thus, we have $\begin{cases} x=\dfrac{a^2b+ab+10a-2b-50}{(a-3)(a+1)} \\ y=\dfrac{-3ab+22a-5b-38}{(a-3)(a+1)} \\ z=\dfrac{b-2}{2(a-3)} \end{cases}$.	1A+1A	1A for any one + 1A for all
(ii) (1) When $a=3$, the augmented matrix of (E) is $\left(\begin{array}{ccc\|c} 1 & 3 & 16 & 18 \\ 2 & 2 & 4 & 20 \\ 1 & -1 & -12 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 3 & 16 & 18 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & b-2 \end{array}\right)$	1M	either one
Since (E) is consistent, we have $b=2$.	1A	
(2) When $a=3$ and $b=2$, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & 3 & 16 & 18 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$		
Thus, the solution set of (E) is $\{(5u+6, -7u+4, u):\ u\in\mathbf{R}\}$.	1A	
	----------(9)	
(b) When $a=3$ and $b=s$, (E) becomes $(G):\begin{cases} x+3y+16z=18 \\ x+y+2z=10 \\ x-y-12z=s \end{cases}$.		
Since (F) is consistent, (G) is consistent.		
By (a)(ii), we have $s=2$.		
When $s=2$, the solution set of (G) is $\{(5u+6, -7u+4, u):\ u\in\mathbf{R}\}$.	1M	
Therefore, we have $2(5u+6)-5(-7u+4)-45u=t$.	1M	
Solving, we have $t=-8$.	1A	for both correct
Thus, we have $s=2$ and $t=-8$.		
	----------(3)	

Solution	Marks	Remarks
12. (a) (i) Note that $\overrightarrow{AB}=-5\mathbf{i}+6\mathbf{j}-4\mathbf{k}$ and $\overrightarrow{AC}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$.		
$\overrightarrow{AB}\times\overrightarrow{AC}$		
$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-5&6&-4\\3&2&4\end{vmatrix}$		
$=32\mathbf{i}+8\mathbf{j}-28\mathbf{k}$	1A	
(ii) Note that $\overrightarrow{AD}=-\mathbf{i}+\mathbf{j}-6\mathbf{k}$.		
The required volume		
$=\frac{1}{6}\left\lvert(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right\rvert$	1M	
$=\frac{1}{6}\lvert(32\mathbf{i}+8\mathbf{j}-28\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}-6\mathbf{k})\rvert$		
$=\frac{1}{6}\lvert(32)(-1)+(8)(1)+(-28)(-6)\rvert$		
$=24$	1A	
(iii) $\overrightarrow{DE}$		
$=\left(\overrightarrow{DA}\cdot\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)\left(\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)$	1M	
$=\left((\mathbf{i}-\mathbf{j}+6\mathbf{k})\cdot\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)\left(\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)$		
$=\frac{-32}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}+\frac{28}{13}\mathbf{k}$	1A	
	----------(5)	
(b) (i) Let $\overrightarrow{BF}=t\overrightarrow{BC}$, where $0<t<1$.		
$\overrightarrow{DF}$		
$=(1-t)\overrightarrow{DB}+t\overrightarrow{DC}$	1M	
$=(1-t)(-4\mathbf{i}+5\mathbf{j}+2\mathbf{k})+t(4\mathbf{i}+\mathbf{j}+10\mathbf{k})$		
$=(8t-4)\mathbf{i}+(5-4t)\mathbf{j}+(8t+2)\mathbf{k}$		
Since $DF\perp BC$, we have $\overrightarrow{DF}\cdot\overrightarrow{BC}=0$.	1M	
Note that $\overrightarrow{BC}=8\mathbf{i}-4\mathbf{j}+8\mathbf{k}$.		
Hence, we have $(8t-4)(8)+(5-4t)(-4)+(8t+2)(8)=0$.		
So, we have $144t-36=0$.		
Solving, we have $t=\frac{1}{4}$.		
Thus, we have $\overrightarrow{DF}=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.	1A	

<table>
<tr><th>Solution</th><th>Marks</th><th>Remarks</th></tr>
<tr><td>(ii) $\overrightarrow{EF}$
$= \overrightarrow{DF} - \overrightarrow{DE}$</td><td>1M</td><td></td></tr>
<tr><td>$= -2\mathbf{i}+4\mathbf{j}+4\mathbf{k}-\left(\frac{-32}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}+\frac{28}{13}\mathbf{k}\right)$
$= \frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}$

$\overrightarrow{BC}\cdot\overrightarrow{EF}$
$= (8\mathbf{i}-4\mathbf{j}+8\mathbf{k})\cdot\left(\frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}\right)$
$= 8\left(\frac{6}{13}\right)-4\left(\frac{60}{13}\right)+8\left(\frac{24}{13}\right)$
$= 0$</td><td></td><td></td></tr>
<tr><td>Thus, $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{EF}$.</td><td>1A
----------(5)</td><td>f.t.</td></tr>
<tr><td>(c) Note that the required angle is $\angle DFE$.</td><td>1M</td><td>for identifying the required angle</td></tr>
<tr><td>$\cos\angle DFE$
$= \frac{\overrightarrow{DF}\cdot\overrightarrow{EF}}{\left|\overrightarrow{DF}\right|\left|\overrightarrow{EF}\right|}$</td><td>1M</td><td></td></tr>
<tr><td>$= \frac{(-2\mathbf{i}+4\mathbf{j}+4\mathbf{k})\cdot\left(\frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}\right)}{\sqrt{(-2)^2+4^2+4^2}\sqrt{\left(\frac{6}{13}\right)^2+\left(\frac{60}{13}\right)^2+\left(\frac{24}{13}\right)^2}}$
$= \frac{324}{(6)(18\sqrt{13})}$
$= \frac{3}{\sqrt{13}}$
$= \frac{3\sqrt{13}}{13}$</td><td></td><td></td></tr>
<tr><td>Thus, the required angle is $\cos^{-1}\left(\frac{3\sqrt{13}}{13}\right)$.</td><td>1A
----------(3)</td><td></td></tr>
</table>