2017-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2017

Please stick the barcode label here.

Candidate Number									

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

Answers written in the margins will not be marked.

1. Find $\dfrac{d}{d\theta}\sec 6\theta$ from first principles. (5 marks)

$$\frac{d}{d\theta}\sec 6\theta = \lim_{h\to 0}\frac{\sec(6(\theta+h)) - \sec 6\theta}{h}$$

$$= \lim_{h\to 0}\frac{\sec(6\theta+6h) - \sec 6\theta}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\cos(6\theta+6h)} - \frac{1}{\cos 6\theta}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\cos 6\theta - \cos(6\theta+6h)}{\cos(6\theta+6h)\cos 6\theta}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot\frac{-2\sin(6\theta+3h)\sin(-3h)}{\frac{1}{2}(\cos 12\theta + \cos 6h)}$$

$$= \lim_{h\to 0}\frac{4}{h}\cdot\frac{\sin(6\theta+3h)\sin 3h}{2\cos(6\theta+3h)\cos(6\theta-3h)}$$

$$= 2\cdot 3\lim_{h\to 0}\frac{\sin 3h}{3h}\lim_{h\to 0}\frac{\sin(6\theta+3h)}{\cos(6\theta+3h)\cos(6\theta-3h)}$$

$$= 6\sec 6\theta\tan 6\theta$$

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2. Let $(1+ax)^8 = \sum_{k=0}^{8} \lambda_k x^k$ and $(b+x)^9 = \sum_{k=0}^{9} \mu_k x^k$, where a and b are constants. It is given that $\lambda_2 : \mu_7 = 7:4$ and $\lambda_1 + \mu_8 + 6 = 0$. Find a. (5 marks)

For $(1+ax)^8$, general term $= C^8_d 1^{8-d}(ax)^d = C^8_d a^d x^d$

$\lambda_2 = C^8_2 a^2 \qquad \lambda_1 = C^8_1 a$

For $(b+x)^9$, general term $= C^9_f b^{9-f} x^f$

$\mu_7 = C^9_7 b^{9-7} = C^9_7 b^2$

$\mu_8 = C^9_8 b^{9-8} = C^9_8 b$

$\frac{\lambda_2}{\mu_7} = \frac{7}{4} \qquad \lambda_1 + \mu_8 + 6 = 0$

$C^8_1 a + C^9_8 b + 6 = 0$

$\frac{C^8_2 a^2}{C^9_7 b^2} = \frac{7}{4} \qquad 8a + 9b + 6 = 0$

$b = \frac{-8a-6}{9}$ —(i)

$\frac{7a^2}{2b^2} = \frac{7}{4} \qquad \frac{a}{\sqrt{2}b} = \frac{1}{2}$

~~$28a^2 = 14b^2$ (ii)~~

~~Put (i) into (ii)~~

~~$28a^2 = 14\left(\frac{-8a-6}{9}\right)^2$~~

~~$28a^2 = 14\left(\frac{64a^2+96a+36}{81}\right)$~~

$a = \frac{1}{2}(\sqrt{2}b)$ —(ii)

Put (i) into (ii)

$a = \frac{\sqrt{2}}{2}\left(\frac{-8a-6}{9}\right)$

$a = \frac{\sqrt{2}}{2}\left(\frac{2(-4a-3)}{9}\right)$

$a = \frac{-\sqrt{2}(4a+3)}{9}$

$9a = -4\sqrt{2}a - 3\sqrt{2}$

$(9+4\sqrt{2})a = -3\sqrt{2}$

$a = \frac{-3\sqrt{2}}{9+4\sqrt{2}}$

$= \frac{-3\sqrt{2}(9-4\sqrt{2})}{49} = \frac{-27\sqrt{2}+24}{49}$

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3. P is a point lying on AB such that $AP : PB = 3 : 2$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}| = 45$, $|\mathbf{b}| = 20$ and $\cos\angle AOB = \frac{1}{4}$. Find

(i) $\mathbf{a} \cdot \mathbf{b}$,

(ii) $|\overrightarrow{OP}|$.

(5 marks)

3a) $\vec{OP} = \frac{3\vec{b} + 2\vec{a}}{3+2} = \frac{2}{5}\vec{a} + \frac{3}{5}\vec{b}$

bi) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\angle AOB$
$\vec{a}\cdot\vec{b} = (45)(20)(\frac{1}{4})$
$\vec{a}\cdot\vec{b} = 225$

bii) $|\vec{OP}|^2 = (\frac{2}{5}\vec{a} + \frac{3}{5}\vec{b})\cdot(\frac{2}{5}\vec{a} + \frac{3}{5}\vec{b})$
$= \frac{4}{25}|\vec{a}|^2 + \frac{6}{25}\vec{a}\cdot\vec{b} + \frac{6}{25}\vec{b}\cdot\vec{a} + \frac{9}{25}|\vec{b}|^2$
$= \frac{4}{25}(45)^2 + 2(\frac{6}{25})(225) + \frac{9}{25}(20)^2$
$= 576$
$|\vec{OP}| = \sqrt{576}$
$= 24$

Answers written in the margins will not be marked.

4. (a) Using integration by parts, find $\int x^2 e^{-x}\, dx$.

(b) Find the area of the region bounded by the graph of $y = x^2 e^{-x}$, the x-axis and the straight line $x = 6$.

(6 marks)

a) $\int x^2 e^{-x} dx$

$= -\int x^2 de^{-x}$

$= -x^2 e^{-x} + \int e^{-x} dx^2$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

$= -x^2 e^{-x} - 2\int x de^{-x}$

$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$

b) When $y = 0$, $x^2 e^{-x} = 0$

$x^2 = 0$ or $e^{-x} = 0$ (rej.)

$x = 0$

$\therefore$ Required area

$= \int_0^6 x^2 e^{-x} dx$

$= \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^6$

$= -36e^{-6} - 12e^{-6} - 2e^{-6} + 2$

$= -\frac{50}{e^6} + 2$

Answers written in the margins will not be marked.

5. Consider the following system of linear equations in real variables x, y, z

$$(E):\begin{cases} x + 2y - z = 11 \\ 3x + 8y - 11z = 49 \\ 2x + 3y + hz = k \end{cases}, \text{ where } h, k \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h.

(ii) Express z in terms of h and k.

(b) Assume that (E) has infinitely many solutions. Solve (E).

(6 marks)

5a i)

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & -11 \\ 2 & 3 & h \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 0 & 1 & -2-h \end{vmatrix}$$

$= (-2)(-2-h) - 8 = 4 + 2h - 8$

$= 2h - 4$

$\because$ (E) has a unique solution

$\therefore \Delta \neq 0$

$2h - 4 \neq 0$

$h \neq 2$

a ii)

$$\Delta_x = \begin{vmatrix} 11 & 2 & -1 \\ 49 & 8 & -11 \\ k & 3 & h \end{vmatrix} = \begin{vmatrix} -5 & 0 & 7 \\ 33-2k & 0 & -3-2h \\ k & 3 & h \end{vmatrix}$$

$= -3[(-5)(-3-2h) - 7(33-2k)]$

$= -3(15 + 10h - 231 + 14k)$

$= -3(10h + 14k - 216)$

$= -6(5h + 7k - 108)$

$$\Delta_y = \begin{vmatrix} 1 & 11 & -1 \\ 3 & 49 & -11 \\ 2 & k & h \end{vmatrix} = \begin{vmatrix} 1 & 11 & -1 \\ 0 & -16 & -22 \\ 0 & 22-k & -2-h \end{vmatrix}$$

$= (-16)(-2-h) + 22(22-k)$

Answers written in the margins will not be marked.

$\Delta y = 32 + 16h + 484 - 22k$

$= 16h - 22k + 516$

$= 2(8h - 11k + 258)$

$\Delta_z = \begin{vmatrix} 1 & 2 & 11 \\ 3 & 8 & 49 \\ 2 & 3 & k \end{vmatrix} = \begin{vmatrix} 1 & 2 & 11 \\ 0 & -2 & -16 \\ 0 & 1 & 22-k \end{vmatrix}$

$= (-2)(22-k) + 16$

$= -44 + 2k + 16$

$= 2k - 28$

$= 2(k-14)$ $\quad \therefore z = \dfrac{k-14}{h-2}$

$(x, y, z) = \left(\dfrac{-3(5h+7k-108)}{h-2}, \dfrac{8h-11k+258}{h-2}, \dfrac{k-14}{h-2} \right)$

b) $\begin{pmatrix} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & h & k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 11 \\ 0 & -2 & 8 & -16 \\ 0 & 1 & -2-h & 22-k \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 1 & -2h & 22-k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & -2+h & -14+k \end{pmatrix}$

$\because$ (E) has infinitely many solutions

$\therefore -2+h = 0$ and $-14+k = 0$

$h = 2$ and $k = 14$

$\begin{pmatrix} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & 2 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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6. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

(a) Let $A\ \text{cm}^2$ be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A = \frac{15}{16}\pi h^2$.

(b) The depth of water in the container increases at a constant rate of $\frac{3}{\pi}$ cm/s . Find the rate of change of the wet curved surface area of the container when the volume of water in the container is $96\pi\ \text{cm}^3$.

(7 marks)

a) Let a cm be the radius of water surface

$\frac{r}{h} = \frac{15}{20}$

$r = \frac{3}{4}h$

A

$= \pi\left(\frac{3}{4}h\right)\sqrt{h^2+\left(\frac{3h}{4}\right)^2}$

$= \pi\left(\frac{3}{4}h\right)\sqrt{\frac{25h^2}{16}}$

$= \pi\left(\frac{3}{4}h\right)\left(\frac{5h}{4}\right)$

$= \frac{15}{16}\pi h^2$

b) Let V be the volume of water

$V = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h$

$= \frac{1}{3}\pi\left(\frac{9}{16}h^2\right)h$

$= \frac{9}{48}h^3\pi$

When $V = 96\pi$, $96\pi = \frac{9}{48}h^3\pi$

$h^3 = 512$

$h = 8$

Answers written in the margins will not be marked.

$A = \frac{15}{16}\pi h^2$

$\frac{dA}{dt} = \left(\frac{15}{16}\right)(2h)(\pi)\frac{dh}{dt}$

$\frac{dA}{dt} = \left(\frac{15}{8}\right)(8)(\pi)\left(\frac{3}{\pi}\right)$

$\frac{dA}{dt} = 45$

$\therefore$ Required rate of change is

$45 \text{ cm}^2/\text{s}$

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7. (a) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

(b) Let $\frac{\pi}{4} < x < \frac{\pi}{2}$.

(i) Prove that $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$.

(ii) Solve the equation $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$.

(8 marks)

7(a) $\sin 3x = \sin(2x+x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos^2 x + (1-2\sin^2 x)\sin x$

$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

b i) $\dfrac{\sin 3(x-\frac{\pi}{4})}{\sin(x-\frac{\pi}{4})}$

$= \dfrac{\sin 3x \cos\frac{3\pi}{4} - \cos 3x \sin\frac{3\pi}{4}}{\sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4}}$

$= \dfrac{-\frac{\sqrt{2}}{2}\sin 3x - \frac{\sqrt{2}}{2}\cos 3x}{\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x}$

$= \dfrac{-\frac{\sqrt{2}}{2}(\cos 3x + \sin 3x)}{-\frac{\sqrt{2}}{2}(\cos x - \sin x)}$

$= \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$

Answers written in the margins will not be marked.

7b ii) $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$

$\dfrac{\sin 3(x - \frac{\pi}{4})}{\sin (x - \frac{\pi}{4})} = 2$

$\sin 3(x - \frac{\pi}{4}) = 2\sin (x - \frac{\pi}{4})$

$3\sin (x - \frac{\pi}{4}) - 4\sin^3 (x - \frac{\pi}{4}) = 2\sin (x - \frac{\pi}{4})$

$\sin (x - \frac{\pi}{4}) = 0$ or $3 - 4\sin^2 (x - \frac{\pi}{4}) = 2$

$\sin (x - \frac{\pi}{4}) = 0$ or $4\sin^2 (x - \frac{\pi}{4}) = 1$

(rej.) $\sin (x - \frac{\pi}{4}) = 0$ or $\sin (x - \frac{\pi}{4}) = \frac{1}{2}$

$x - \frac{\pi}{4} = \frac{\pi}{6}$ $\quad$ ($\sin (x - \frac{\pi}{4}) = -\frac{1}{2}$ is rej.)

$x = \frac{5\pi}{12}$

8. Let $f(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $P(e^3, 7)$ and $f'(x) = \frac{1}{x}\ln x^2$ for all $x > 0$. Find

(a) the equation of the tangent to Γ at P,

(b) the equation of Γ,

(c) the point(s) of inflexion of Γ.

(8 marks)

8a) $f'(e^3) = \frac{1}{e^3}\ln e^{3(2)} = \frac{1}{e^3}\ln e^6 = \frac{6}{e^3}$

Equation of the tangent to Γ at P:

$y - 7 = \frac{6}{e^3}(x - e^3)$

$ye^3 - 7e^3 = 6x - 6e^3$

$ye^3 = 6x + e^3$

$y = \frac{6}{e^3}x + 1$

b) $f(x) = \int \frac{1}{x}\ln x^2\,dx$

$= 2\int \frac{1}{x}\ln x\,dx$

$= 2\int \ln x\,d\ln x$

$= 2\left(\frac{1}{2}\right)(\ln x)^2 + C$

$= (\ln x)^2 + C$

Put $P(e^3, 7)$ into $y = f(x)$,

$7 = (\ln e^3)^2 + C$

$7 = 9 + C$

$C = -2$

$\therefore$ Equation of Γ:

$f(x) = (\ln x)^2 - 2$

Answers written in the margins will not be marked.

8c) $f'(x)=\frac{1}{x}\ln x^2$

$f''(x)=-\frac{1}{x^2}\ln x^2+\left(\frac{1}{x}\right)\left(\frac{2x}{x^2}\right)$

$=-\frac{1}{x^2}\ln x^2+\frac{2}{x^2}$

$f''(x)=0$

$-\frac{1}{x^2}\ln x^2+\frac{2}{x^2}=0$

$-\ln x^2+2=0$

$\ln x^2=2$

$\ln x^2=\ln e^2$

$x=e$

x	$x<0$	$x=0$	$0<x<e$	$x=e$	$x>e$
$f''(x)$	$-$	$\times$	$+$	0	$-$

$\therefore$ Point of inflexion

$=(e,-1)$

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SECTION B (50 marks)

9. Define $f(x) = \frac{x^2 - 5x}{x + 4}$ for all $x \neq -4$. Denote the graph of $y = f(x)$ by G.

(a) Find the asymptote(s) of G. (3 marks)

(b) Find $f'(x)$. (2 marks)

(c) Find the maximum point(s) and the minimum point(s) of G. (4 marks)

(d) Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis. (4 marks)

9a) $f(x) = \frac{x^2-5x}{x+4}$

$= x - 9 + \frac{36}{x+4}$

Oblique asymptote: $y = x - 9$

Vertical asymptote: $x = -4$

b) $f'(x) = 1 - \frac{36}{(x+4)^2}$

$\left(f(x) = x - 9 + \frac{36}{x+4}\right)$

c) $f'(x) = 0$

$1 - \frac{36}{(x+4)^2} = 0$

$(x+4)^2 - 36 = 0$

$(x+4)^2 = 36$

$x + 4 = 6$ or $x + 4 = -6$

$x = 2$ or $x = -10$

x	$x<-10$	$x=-10$	$-10<x<-4$	$x=-4$	$-4<x<2$	$x=2$	$x>2$
$f'(x)$	$+$	0	$-$	$\times$	$-$	0	$+$

$\therefore$ Maximum point $= (-10, -25)$

Minimum point $= (2, -1)$

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9d) When $y=0$,

$\frac{x^2-5x}{x+4}=0$

$x^2-5x=0$

$x(x-5)=0$

$x=0$ or 5

Required volume

$=\pi\int_0^5 (x-9+\frac{36}{x+4})^2 dx$

$=\pi\int_0^5 (x^2-18x+81+\frac{72(x-9)}{x+4}+\frac{1296}{(x+4)^2})dx$

$=\pi\int_0^5 (x^2-18x+81+\frac{72x}{x+4}-\frac{648}{x+4}+\frac{1296}{(x+4)^2})dx$

$=\pi\int_0^5 (x^2-18x+81+72-\frac{936}{x+4}+\frac{1296}{(x+4)^2})dx$

$=\pi\left[\frac{1}{3}x^3-6x^3+153x-936\ln|x+4|-\frac{1296}{x+4}\right]_0^5$

$=\pi(\frac{125}{3}-750+765-936\ln 9-144+936\ln 4+324)$

$=\pi(\frac{710}{3}+936\ln\frac{4}{9})$

$=\pi(\frac{710}{3}+1872\ln\frac{2}{3})$

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10. ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE:EC=1:r$. AB produced and DE produced meet at the point F. It is given that $DE:EF=1:10$. Let $\overrightarrow{OA}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\overrightarrow{OB}=4\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\overrightarrow{OC}=8\mathbf{i}-3\mathbf{j}-2\mathbf{k}$, where O is the origin.

(a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r. (4 marks)

(b) (i) Find $\overrightarrow{AD}\cdot\overrightarrow{DE}$.

(ii) Are B, D, C and F concyclic? Explain your answer.

(5 marks)

(c) Let $\overrightarrow{OP}=3\mathbf{i}+10\mathbf{j}-4\mathbf{k}$. Denote the circumcentre of ΔBCF by Q. Find the volume of the tetrahedron $ABPQ$. (3 marks)

10a) $\vec{AE}=\dfrac{\vec{AC}+r\vec{AB}}{1+r}$

$=\dfrac{(8\vec{i}-3\vec{j}-2\vec{k})-(2\vec{i}+3\vec{j}-2\vec{k})+r\left((4\vec{i}+4\vec{j}-\vec{k})-(2\vec{i}+3\vec{j}-2\vec{k})\right)}{1+r}$

$=\dfrac{6\vec{i}-6\vec{j}+r(2\vec{i}+\vec{j}+\vec{k})}{1+r}$

$=\dfrac{2(r+3)}{1+r}\vec{i}+\dfrac{2(r-3)}{1+r}\vec{j}+\dfrac{r}{1+r}\vec{k}$

$\vec{AF}=$

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10(c) $\overrightarrow{AB} \times \overrightarrow{AP}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 7 & -2 \end{vmatrix} = -9\vec{i} + 11\vec{j} + 13\vec{k}$$

$\overrightarrow{AQ} \cdot \overrightarrow{AB} \times \overrightarrow{AP}$

$=$

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11. (a) Using $\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$, evaluate $\int_0^1 \frac{1}{x^2+2x+3}\,dx$. (3 marks)

(b) (i) Let $0\le\theta\le\frac{\pi}{4}$. Prove that $\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$.

(ii) Using the substitution $t=\tan\theta$, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$.

(5 marks)

(c) Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta=\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$. (2 marks)

(d) Evaluate $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$. (3 marks)

11a) $\int_0 \frac{1}{x^2+2x+3}\,dx$

$=\int_0^1 \frac{1}{(x+1)^2+2}\,dx$

$=\int_{\tan^{-1}(\frac{\sqrt{2}}{2})}^{\tan^{-1}(\sqrt{2})} \frac{1}{(\sqrt{2}\tan\theta)^2+2}\,d(\sqrt{2}\tan\theta-1)$

$=\sqrt{2}\int_{\tan^{-1}(\frac{\sqrt{2}}{2})}^{\tan^{-1}(\sqrt{2})} \frac{\sec^2\theta}{2\tan^2\theta+2}\,d\theta$

$=2\sqrt{2}\int_{\tan^{-1}(\frac{\sqrt{2}}{2})}^{\tan^{-1}(\sqrt{2})} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$

$=2\sqrt{2}\int_{\tan^{-1}(\frac{\sqrt{2}}{2})}^{\tan^{-1}(\sqrt{2})} d\theta$

$=2\sqrt{2}\left[\theta\right]_{\tan^{-1}(\frac{\sqrt{2}}{2})}^{\tan^{-1}(\sqrt{2})}$

$=2\sqrt{2}\left(\tan^{-1}(\sqrt{2})-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

$=2\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

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11(b)(i) $\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta}$

$= \frac{2\sin\theta\cdot\cos^2\theta}{\cos\theta}$

$= 2\sin\theta\cos\theta$

$= \sin 2\theta$

$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(\sec^2\theta-1)}{\sec^2\theta}$

$= \frac{2-\sec^2\theta}{\sec^2\theta}$

$= 2\cos^2\theta-1$

$= \cos 2\theta$

b)ii) $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\frac{2\tan\theta}{1+\tan^2\theta}+\frac{1-\tan^2\theta}{1+\tan^2\theta}+2}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1+\tan^2\theta}{2\tan\theta+1-\tan^2\theta+2(1+\tan^2\theta)}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{2\tan\theta+1-\tan^2\theta+2+2\tan^2\theta}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2\theta+2\tan\theta+3}\,d\tan\theta$

$= \int_0^1 \frac{1}{t^2+2t+3}\,dt$

$= 2\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

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11(c) $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2} d\theta+\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2} d\theta$

$=\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+\cos 2\theta+2}{\sin 2\theta+\cos 2\theta+2} d\theta$

$\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2} d\theta$

d) $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2} d\theta$

$=\int_0^{\frac{\pi}{4}}\left(8-\frac{8\cos 2\theta+7}{\sin 2\theta+\cos 2\theta+2}+\frac{9}{\sin 2\theta+\cos 2\theta+2}\right) d\theta$

$=8[\theta]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} \frac{8\cos 2\theta+7}{\sin 2\theta+\cos 2\theta+2}+9(2\sqrt{2})\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

$=2\pi+18\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)-\int_0^{\frac{\pi}{4}} \frac{8\cos 2\theta+7}{\sin 2\theta+\cos 2\theta+2} d\theta$

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12. Let $A=\begin{pmatrix}3&1\\0&3\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Using mathematical induction, prove that $A^n=3^nI+3^{n-1}n\begin{pmatrix}0&1\\0&0\end{pmatrix}$ for all positive integers n.

(4 marks)

(b) Let $B=\begin{pmatrix}5&1\\-4&1\end{pmatrix}$.

(i) Define $P=\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n=3^nI+3^{n-1}n\begin{pmatrix}2&1\\-4&-2\end{pmatrix}$ for any positive integer n.

(iii) Does there exist a positive integer m such that $|A^m-B^m|=4m^2$? Explain your answer.

(8 marks)

12 a) Let $P(n)$: $A^n=3^nI+3^{n-1}n\begin{pmatrix}0&1\\0&0\end{pmatrix}$

For $n=1$

LHS $=A^1=\begin{pmatrix}3&1\\0&3\end{pmatrix}$

RHS $=3I+\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}3&1\\0&3\end{pmatrix}$

$\therefore$ $P(1)$ is true

Assume $P(k)$ is true. i.e $A^k=3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}$

For $n=k+1$,

$A^{k+1}=A^kA^1$

$=\left[3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]\begin{pmatrix}3&1\\0&3\end{pmatrix}$

$=\begin{pmatrix}3^k&3^{k-1}k\\0&3^k\end{pmatrix}\begin{pmatrix}3&1\\0&3\end{pmatrix}$

$=\begin{pmatrix}3^{k+1}&3^k+3^kk\\0&3^{k+1}\end{pmatrix}$

$=3^{k+1}I\begin{pmatrix}0&3^{-1}+3^{-1}k\\&1\end{pmatrix}$

Answers written in the margins will not be marked.

$= 3^{k+1}I + 3^{k}(k+1)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore P(k+1)$ is true

$\therefore P(n)$ is true for all positive integers n.

b i) $P^{-1} = \frac{1}{\det P}\begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}^{T}$

$= \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$

$P^{-1}BP$

$= \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} -5 & -1 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

b ii) Let $Q(n): B^{n} = 3^{n}I + 3^{n-1}n\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

For $n=1$,

$LHS = B = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$

$RHS = 3I + \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$

$\therefore Q(1)$ is true.

Assume $Q(k)$ is true. i.e. $B^{k} = 3^{k}I + 3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

For $n = k+1$,

$B^{k+1} = B^{k}B$

$= \left[3^{k}I + 3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right]\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$

Answers written in the margins will not be marked.

$$B^{k+1}=\begin{pmatrix}3^k+2k(3^{k-1}) & 3^{k-1}k\\ (-4k)3^{k-1} & 3^k-2k(3^{k-1})\end{pmatrix}\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}$$

$$=\begin{pmatrix}5(3^k)+10k(3^{k-1})-4(3^{k-1}k) & 3^k+3k(3^{k-1})\\ -20k3^{k-1}-4(3^k)+8k(3^{k-1}) & (-4k)3^{k-1}+3^k-2k(3^{k-1})\end{pmatrix}$$

$$=\begin{pmatrix}5(3^k)+6k(3^{k-1}) & 2(3^k)+k\\ (-12k)3^{k-1}-4(3^k) & (-6k)(3^{k-1})+3^k\end{pmatrix}$$

$$=3^{k+1}I+3^k(k+1)\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}$$

$\therefore Q(k+1)$ is true

$\therefore Q(n)$ is true for any positive integers n

12b iii)

END OF PAPER

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香港考試及評核局
HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
補充答題紙 **(A)** Supplementary answer sheet **(A)**

考生編號 Candidate Number

試題編號 Question No.

1	2	3	4	5	6	7	8	9	10	11	12	
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☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
13	14	15	16	17	18	19	20	21	22	23	24	≥25

1. 每題另起新頁作答。
   Start each question on a new page.
2. 補充答題紙不可撕開使用。
   Do not tear the supplementary answer sheet apart.

5b) $\begin{pmatrix} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\sim$

$(x,y,z) = (-7t-5, 4t+8, t)$ for all $t \in \mathbb{R}$

寫於邊界以外的答案，將不予評閱。
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