Mock Exam 8

Section A

1. (a) $\because \quad AB = AC$

$$\therefore \quad \sin\frac{\angle BAC}{2} = \frac{\frac{BC}{2}}{AB}$$

$$\sin\theta = \frac{\frac{1}{2}}{x}$$

$$x = \frac{1}{2}\csc\theta \qquad 1$$

(b) When AC is the angle bisector of $\angle BAD$, $\angle CAD = 2\theta$.

Since $CD = AC$, we have $\angle CDA = \angle CAD = 2\theta$. *(base ∠s, isos. Δ)*

$$\angle ACB = 2\theta + 2\theta \quad \textit{(ext. ∠ of Δ)}$$
$$= 4\theta$$

Since $AB = AC$, we have $\angle ABC = \angle ACB = 4\theta$. *(base ∠s, isos. Δ)*

In ΔABC,

$$2\theta + 4\theta + 4\theta = \pi \quad \textit{(∠ sum of Δ)}$$

$$\theta = \frac{\pi}{10} \qquad 1A$$

$$x = \frac{1}{2}\csc\theta$$

$$\frac{dx}{dt} = -\frac{1}{2}\csc\theta\cot\theta\frac{d\theta}{dt} \qquad 1M$$

$$\frac{d\theta}{dt} = -2\sin\theta\tan\theta\frac{dx}{dt}$$

Let S cm^2 be the area of ΔABC.

$$S = \frac{1}{2} \times BC \times \frac{\frac{BC}{2}}{\tan\frac{\angle BAC}{2}}$$

$$= \frac{1}{4}\cot\theta$$

$$\frac{dS}{dt} = -\frac{1}{4}\csc^2\theta\frac{d\theta}{dt}$$

$$= \left(-\frac{1}{4}\csc^2\theta\right)\left(-2\sin\theta\tan\theta\frac{dx}{dt}\right) \qquad 1M$$

$$= \frac{1}{2}\sec\theta\frac{dx}{dt}$$

When $\theta = \frac{\pi}{10}$, $\frac{dx}{dt} = \cos\frac{\pi}{10}$.

$$\therefore \quad \left.\frac{dS}{dt}\right|_{\theta=\frac{\pi}{10}} = \frac{1}{2}\sec\frac{\pi}{10}\cos\frac{\pi}{10} = \frac{1}{2}$$

$\therefore$ The rate of change of the area of ΔABC is $\frac{1}{2}$ cm^{2}s^{-1}. 1A

(5)

2. *Reference: HKDSE Math M2 2014 Q1*

(a) $(1+2x)^n(1-x)^2$

$= [1 + C_1^n(2x) + C_2^n(2x)^2 + \cdots](1 - 2x + x^2)$

$= \left[1 + 2nx + \dfrac{n(n-1)}{2}(4x^2) + \cdots\right](1 - 2x + x^2)$ 1M

$= 1 + (2n-2)x + (2n^2 - 6n + 1)x^2 + \cdots$

Coefficients of $x^2 = 9$ 1M

$$2n^2 - 6n + 1 = 9$$

$$2n^2 - 6n - 8 = 0$$

$$2(n-4)(n+1) = 0$$

$$n = \underline{\underline{4}} \text{ or } -1 \text{ (rejected)}$$ 1A

(b) Coefficient of $x = 2(4) - 2$ 1M

$= \underline{\underline{6}}$ 1A

(5)

3. *Reference: HKDSE Math M2 2015 Q3*

(a) $\int \tan x\, dx = \int \dfrac{\sin x}{\cos x} dx$

$= -\int \dfrac{d(\cos x)}{\cos x}$ 1M

$= -\ln|\cos x| + C$

$= \underline{\underline{\ln|\sec x| + C}}$, where C is a constant 1A

(b) Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}} dx$. 1M

When $x = \dfrac{\pi^2}{16}$, $u = \dfrac{\pi}{4}$; when $x = \dfrac{\pi^2}{9}$, $u = \dfrac{\pi}{3}$.

$\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{9}} \dfrac{\tan\sqrt{x}}{\sqrt{x}} dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan u\, du$ 1M + 1A

$= 2\Big[\ln|\sec u|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ (by (a)) 1M

$= 2(\ln 2 - \ln\sqrt{2})$

$= 2\left(\ln 2 - \dfrac{1}{2}\ln 2\right)$

$= \underline{\underline{\ln 2}}$ 1A

(7)

4. *Reference: HKDSE Math M2 PP Q9*

$x^2 + 2xy - y^2 = 1$

Differentiating both sides with respect to x,

$$2x + 2x\frac{dy}{dx} + 2y - 2y\frac{dy}{dx} = 0$$

$\dfrac{dy}{dx} = \dfrac{x+y}{y-x}$ 1M

Let $P(a, b)$ be the point of contact.

$$\left.\frac{dy}{dx}\right|_{(a,\,b)} = \text{Slope of } PA$$

$$\frac{a+b}{b-a} = \frac{b+1}{a} \qquad \text{1M}$$

$$a^2 + ab = b^2 - ab + b - a$$

$$a^2 + 2ab - b^2 = b - a \text{.....................(1)}$$

Since $P(a, b)$ is a point on C, we have

$$a^2 + 2ab - b^2 = 1 \text{ ....(2)}$$

(1) – (2):

$$0 = b - a - 1$$

$$b = a + 1 \text{ ....(3)} \qquad \text{1M}$$

Substituting (3) into (2),

$$a^2 + 2a(a + 1) - (a + 1)^2 = 1$$

$$2a^2 = 2$$

$$a = 1 \text{ or } -1 \qquad \text{1A}$$

When $a = 1$, $b = 1 + 1 = 2$.

The equation of the tangent is

$$y - 2 = \frac{2+1}{1}(x-1)$$

$$3x - y - 1 = 0 \qquad \text{1A}$$

When $a = -1$, $b = -1 + 1 = 0$.

The equation of the tangent is

$$y - 0 = \frac{0+1}{-1}[x - (-1)]$$

$$x + y + 1 = 0 \qquad \text{1A}$$

(6)

5. *Reference: HKDSE Math M2 PP Q2*

(a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ k & 5 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 5+k & -1-k & 0 \end{array}\right) \quad \begin{array}{l}(R_2 - 2R_1 \to R_2, \\ R_3 - kR_1 \to R_3)\end{array} \qquad \text{1M}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & -4k & 0 \end{array}\right) \quad (5R_3 - (5+k)R_2 \to R_3) \qquad \text{1M}$$

$\therefore$ When (E) has non-trivial solutions, $k = 0$. 1A

For $k = 0$, the augmented matrix is $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Let $z = t$, where t is any real number. Then $y = \frac{t}{5}$ and $x = -\frac{4t}{5}$.

$\therefore$ The solution is $x = -\frac{4t}{5}$, $y = \frac{t}{5}$ and $z = t$, where t is any real number. 1A

(b) $25\left(-\frac{4}{5}t\right)^2 - 175\left(\frac{1}{5}t\right)^2 + (t-p)^2 = 10$ 1M

$16t^2 - 7t^2 + t^2 - 2pt + p^2 - 10 = 0$

$10t^2 - 2pt + p^2 - 10 = 0$

Since t is real, we have

$(-2p)^2 - 4(10)(p^2 - 10) \geq 0$ 1M

$-36p^2 + 400 \geq 0$

$9p^2 - 100 \leq 0$

$(3p+10)(3p-10) \leq 0$

$\therefore \quad -\frac{10}{3} \leq p \leq \frac{10}{3}$ 1A

(7)

6. *Reference: HKDSE Math M2 2013 Q8*

(a) $|M| = (1+x)(1-x) - x(-x)$

$= 1 - x^2 + x^2$

$= 1$

$\neq 0$

$\therefore$ M is invertible 1

$M^{-1} = \frac{1}{1}\begin{pmatrix} 1-x & -x \\ x & 1+x \end{pmatrix}^T$ 1M

$= \underline{\underline{\begin{pmatrix} 1-x & x \\ -x & 1+x \end{pmatrix}}}$ 1A

Smart Tips

M is invertible if and only if $|M| \neq 0$.

(b) $M^T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 28 \\ -22 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = (M^T)^{-1}\begin{pmatrix} 28 \\ -22 \end{pmatrix}$

$= (M^{-1})^T\begin{pmatrix} 28 \\ -22 \end{pmatrix}$

$= \begin{pmatrix} 1-x & -x \\ x & 1+x \end{pmatrix}\begin{pmatrix} 28 \\ -22 \end{pmatrix}$ 1M

$= \begin{pmatrix} -6x+28 \\ 6x-22 \end{pmatrix}$ 1M

$\therefore \quad x = -6x + 28$

$7x = 28$

$x = \underline{\underline{4}}$ 1A

$y = 6(4) - 22 = \underline{\underline{2}}$

(6)

7. *Reference: HKDSE Math M2 2014 Q8*

(a) Volume $= \frac{1}{6}\left|\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})\right|$

$= \frac{1}{6}\begin{vmatrix} 3 & -2 & 3 \\ 3 & 1 & -3 \\ 2 & 0 & 1 \end{vmatrix}$ 1M

$= \underline{\underline{\frac{5}{2}}}$ 1A

(b) $OC = \sqrt{2^2 + 0^2 + 1^2}$

$= \sqrt{5}$ 1A

Area of $\Delta OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 3 \\ 3 & 1 & -3 \end{matrix}\right\|$

$= \frac{1}{2}\left|3\mathbf{i} + 18\mathbf{j} + 9\mathbf{k}\right|$ 1M

$= \frac{1}{2}\sqrt{3^2 + 18^2 + 9^2}$

$= \frac{3}{2}\sqrt{46}$ 1A

Let P be the point of intersection of L and the plane OAB.

$\frac{1}{3} \times \frac{3}{2}\sqrt{46} \times CP = \frac{5}{2}$ 1M

$CP = \frac{5}{\sqrt{46}}$

$\cos\theta = \frac{CP}{OC}$

$= \frac{\frac{5}{\sqrt{46}}}{\sqrt{5}}$

$= \underline{\underline{\frac{\sqrt{230}}{46}}}$ 1A

(7)

8. (a) For $n = 1$,

L.H.S. $= 1 \times 2 = 2$

R.H.S. $= (1 - 1) \times 2^{1+1} + 2 = 2 =$ L.H.S.

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = k$, where k is a positive integer, that is,

$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \dots + k \times 2^k = (k - 1) \times 2^{k+1} + 2$, 1

when $n = k + 1$,

L.H.S.

$= 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \dots + k \times 2^k + (k+1) \times 2^{k+1}$

$= (k-1) \times 2^{k+1} + 2 + (k+1) \times 2^{k+1}$ (by the assumption) 1

$= 2k \times 2^{k+1} + 2$

$= k \times 2^{k+2} + 2$ 1

$= [(k+1) - 1] \times 2^{(k+1)+1} + 2 =$ R.H.S.

$\therefore$ The proposition is also true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(b) $\sum_{r=1}^{n}(r+1)\times 2^r$

$= 2\times 2 + 3\times 2^2 + 4\times 2^3 + ... + (n+1)\times 2^n$

$= (1+1)\times 2 + (2+1)\times 2^2 + (3+1)\times 2^3 + ... + (n+1)\times 2^n$

$= (1\times 2 + 2\times 2^2 + 3\times 2^3 + ... + n\times 2^n) + (2 + 2^2 + 2^3 + ... + 2^n)$

$= (n-1)\times 2^{n+1} + 2 + \dfrac{2(2^n-1)}{2-1}$ 1M

$= (n-1)\times 2^{n+1} + 2 + 2(2^n-1)$

$= (n-1)\times 2^{n+1} + 2 + 2^{n+1} - 2$

$= \underline{\underline{n\times 2^{n+1}}}$ 1A

(7)

Smart Tips

For a geometric sequence, the sum of the first n terms is $\dfrac{a(r^n-1)}{r-1}$.

Section B

9. (a) (i) $\int x\ln x\,dx = \int \ln x\,d\left(\dfrac{x^2}{2}\right)$

$= \dfrac{x^2\ln x}{2} - \int \dfrac{x^2}{2}d(\ln x)$

$= \dfrac{x^2\ln x}{2} - \int \dfrac{x^2}{2}\times\dfrac{1}{x}dx$ 1M

$= \dfrac{x^2\ln x}{2} - \dfrac{1}{2}\int x\,dx$ 1A

$= \underline{\underline{\dfrac{x^2\ln x}{2} - \dfrac{x^2}{4}}} + C$, where C is a constant 1A

(ii) Note that when $y=0$, $x=1$.

Volume $= \pi\int_1^h(\sqrt{x\ln x})^2\,dx$ cubic units

$= \pi\int_1^h x\ln x\,dx$ cubic units 1M

$= \pi\left[\dfrac{x^2\ln x}{2} - \dfrac{x^2}{4}\right]_1^h$ cubic units (by (a)) 1M

$= \pi\left(\dfrac{h^2\ln h}{2} - \dfrac{h^2}{4} - 0 + \dfrac{1}{4}\right)$ cubic units

$= \dfrac{\pi}{4}(2h^2\ln h - h^2 + 1)$ cubic units 1

(6)

(b) For solid X, take $h - 1 = e^2 - 1$, i.e. $h = e^2$.

Volume of solid $X = \dfrac{\pi}{4}(2e^4\ln e^2 - e^4 + 1)$ cubic units (by (a)(ii))

$= \underline{\underline{\dfrac{\pi}{4}(3e^4+1)\text{ cubic units}}}$ 1A

For solid Y, take $h - 1 = e - 1$, i.e. $h = e$.

Volume of solid $Y = \dfrac{\pi}{4}(2e^2\ln e - e^2 + 1)$ cubic units (by (a)(ii))

$= \underline{\underline{\dfrac{\pi}{4}(e^2+1)\text{ cubic units}}}$ 1A

(2)

(c) Let V cubic units be the volume of the solid in (a)(ii).

$$V = \frac{\pi}{4}(2h^2 \ln h - h^2 + 1)$$

$$\frac{dV}{dt} = \frac{\pi}{4}\left[2h^2\left(\frac{1}{h}\right)\frac{dh}{dt} + 4h\ln h\frac{dh}{dt} - 2h\frac{dh}{dt}\right] \quad \text{1M}$$

$$= \pi\, h \ln h \frac{dh}{dt} \quad \text{1A}$$

Let V_M cubic units and V_N cubic units be the volumes of the water in vessels M and N respectively.

Note that $\frac{dV_N}{dt} = -\frac{dV_M}{dt}$. 1M

Let H_M units and H_N units be the heights of water in vessels M and N respectively.
Let h_M and h_N be the corresponding values of h for the volumes of the water in vessels M and N respectively.

Note that $H_M = h_M - 1$ and $H_N = h_N - 1$, so $\frac{dH_M}{dt} = \frac{dh_M}{dt}$ and $\frac{dH_N}{dt} = \frac{dh_N}{dt}$.

At the required moment,

$h_M = e^2$ and $h_N = e$ (by (b))

$$\frac{dh_M}{dt} = \frac{dH_M}{dt} = -1$$

$$\frac{dV_N}{dt} = -\frac{dV_M}{dt}$$

$$\pi\, h_N \ln h_N \frac{dh_N}{dt} = -\pi\, h_M \ln h_M \frac{dh_M}{dt}$$

$$e(\ln e)\frac{dh_N}{dt} = -e^2(\ln e^2)(-1) \quad \text{1M}$$

$$\frac{dh_N}{dt} = 2e$$

$\therefore$ The rate of increase of water level in vessel N is $2e$ units per second. 1A

(5)

10. (a) (i) $\overrightarrow{AD} = k\mathbf{b}$

$$\overrightarrow{OD} = \mathbf{a} + k\mathbf{b}$$

$$|\overrightarrow{OD}|^2 = (\mathbf{a} + k\mathbf{b}) \cdot (\mathbf{a} + k\mathbf{b}) \quad \text{1M}$$

$$= |\mathbf{a}|^2 + 2k\mathbf{a} \cdot \mathbf{b} + k^2|\mathbf{b}|^2$$

$$= |2\mathbf{b}|^2 + k^2|\mathbf{b}|^2$$

$$= (k^2 + 4)|\mathbf{b}|^2$$

$$|\overrightarrow{OD}| = \sqrt{k^2 + 4}\,|\mathbf{b}| \quad \text{1}$$

(ii) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$|\overrightarrow{AB}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

$$= |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2$$

$$= |\mathbf{b}|^2 + |2\mathbf{b}|^2$$

$$= 5|\mathbf{b}|^2$$

$$|\overrightarrow{AB}| = \sqrt{5}\,|\mathbf{b}| \quad \text{1A}$$

Smart Tips

Since $OACB$ is a rectangle, $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b} = 0$.

Analysis

Since θ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{OD}$, $\cos\theta$ can be found from the dot product of $\overrightarrow{AB}$ and $\overrightarrow{OD}$.

$$\overrightarrow{AB} \cdot \overrightarrow{OD} = |\overrightarrow{AB}||\overrightarrow{OD}|\cos\theta$$

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{a} + k\mathbf{b}) = (\sqrt{5}|\mathbf{b}|)(\sqrt{k^2 + 4}|\mathbf{b}|)\cos\theta \qquad \text{1M}$$

$$\sqrt{5(k^2 + 4)}\cos\theta|\mathbf{b}|^2 = \mathbf{a} \cdot \mathbf{b} - |\mathbf{a}|^2 + k|\mathbf{b}|^2 - k\mathbf{a} \cdot \mathbf{b}$$

$$= -|2\mathbf{b}|^2 + k|\mathbf{b}|^2$$

$$\sqrt{5(k^2 + 4)}\cos\theta = k - 4 \qquad \text{1}$$

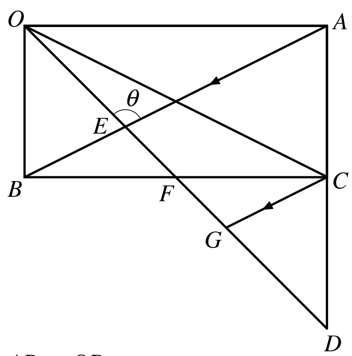
$$\cos\theta = \frac{k - 4}{\sqrt{5(k^2 + 4)}}$$

(5)

Smart Tips

Since $|\mathbf{b}|$ is a scalar, we can divide both sides by $|\mathbf{b}|^2$ directly.

(b) (i)



$\because$ $AB \perp OD$

$\therefore$ $\cos\theta = \cos 90° = 0$

$\therefore$ $\dfrac{k - 4}{\sqrt{5(k^2 + 4)}} = 0$ 1M

$k - 4 = 0$

$k = 4$

$\therefore$ $\overrightarrow{OD} = \mathbf{a} + 4\mathbf{b}$

Let $\overrightarrow{OG} = s\mathbf{a} + 4s\mathbf{b}$, where s is a real number.

$\overrightarrow{CG} = \overrightarrow{OG} - \overrightarrow{OC}$

$= \overrightarrow{OG} - (\overrightarrow{OA} + \overrightarrow{AC})$

$= (s\mathbf{a} + 4s\mathbf{b}) - (\mathbf{a} + \mathbf{b})$

$= (s - 1)\mathbf{a} + (4s - 1)\mathbf{b}$

Since CG // AE // AB, we have

$\dfrac{s - 1}{-1} = \dfrac{4s - 1}{1}$ 1M

$s - 1 = 1 - 4s$

$s = \dfrac{2}{5}$ 1A

$\therefore$ $\overrightarrow{CG} = \left(\dfrac{2}{5} - 1\right)\mathbf{a} + \left(4 \times \dfrac{2}{5} - 1\right)\mathbf{b}$

$= \underline{\underline{-\dfrac{3}{5}\mathbf{a} + \dfrac{3}{5}\mathbf{b}}}$ 1A

(ii) $\overrightarrow{OG} = \frac{2}{5}\mathbf{a} + \frac{8}{5}\mathbf{b}$

Let $BE : EA = r : 1$, where r is a real number.

$\overrightarrow{OE} = \frac{r\mathbf{a} + \mathbf{b}}{r+1}$ 1M

$\because$ O, E and G lie on the same straight line.

$$\therefore \quad \frac{\frac{r}{r+1}}{\frac{2}{5}} = \frac{\frac{1}{r+1}}{\frac{8}{5}} \qquad \text{1M}$$

$$r = \frac{1}{4}$$

$$\therefore \quad \overrightarrow{OE} = \frac{1}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$$
$$= \frac{1}{2}\left(\frac{2}{5}\mathbf{a} + \frac{8}{5}\mathbf{b}\right)$$
$$= \frac{1}{2}\overrightarrow{OG}$$

$\therefore$ E is the mid-point of OG.

$\therefore$ The claim is agreed. 1A

(7)

11. *Reference: HKDSE Math M2 2015 Q11*

(a) (i)
$$P = \frac{1}{\alpha - \beta + 4}\left[\begin{pmatrix} \alpha & 2 \\ \alpha - \beta + 2 & \beta \end{pmatrix} - \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$
$$= \frac{1}{\alpha - \beta + 4}\begin{pmatrix} \alpha - \beta + 2 & 2 \\ \alpha - \beta + 2 & 2 \end{pmatrix}$$

$$Q = \frac{1}{\alpha - \beta + 4}\left[\begin{pmatrix} \alpha & 2 \\ \alpha - \beta + 2 & \beta \end{pmatrix} - \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$
$$= \frac{1}{\alpha - \beta + 4}\begin{pmatrix} -2 & 2 \\ \alpha - \beta + 2 & \beta - \alpha - 2 \end{pmatrix}$$

$$PQ = \frac{1}{(\alpha - \beta + 4)^2}\begin{pmatrix} \alpha - \beta + 2 & 2 \\ \alpha - \beta + 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ \alpha - \beta + 2 & \beta - \alpha - 2 \end{pmatrix} \qquad \text{1M}$$
$$= \underline{\underline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}} \qquad \text{1A}$$

$$QP = \frac{1}{(\alpha - \beta + 4)^2}\begin{pmatrix} -2 & 2 \\ \alpha - \beta + 2 & \beta - \alpha - 2 \end{pmatrix}\begin{pmatrix} \alpha - \beta + 2 & 2 \\ \alpha - \beta + 2 & 2 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}}$$

$$P - Q = \frac{1}{\alpha - \beta + 4}[(M - \beta I + 2I) - (M - \alpha I - 2I)]$$
$$= \frac{\alpha - \beta + 4}{\alpha - \beta + 4}I$$
$$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}} \qquad \text{1A}$$

(ii) $P^2 = P(I + Q)$ 1M

$= P + PQ$

$= P$ 1

$Q^2 = Q(P - I)$

$= QP - Q$

$= -Q$

(iii) For $n = 1$,

L.H.S. $= M^1 = \begin{pmatrix} \alpha & 2 \\ \alpha - \beta + 2 & \beta \end{pmatrix}$

R.H.S. $= (\alpha + 2)^1 P - (\beta - 2)^1 Q$

$= (\alpha + 2)\left(\dfrac{1}{\alpha - \beta + 4}\right)\begin{pmatrix} \alpha - \beta + 2 & 2 \\ \alpha - \beta + 2 & 2 \end{pmatrix}$

$-(\beta - 2)\left(\dfrac{1}{\alpha - \beta + 4}\right)\begin{pmatrix} -2 & 2 \\ \alpha - \beta + 2 & \beta - \alpha - 2 \end{pmatrix}$

$= \dfrac{1}{\alpha - \beta + 4}\begin{pmatrix} \alpha^2 + 4\alpha - \alpha\beta - 2\beta + 4 & 2\alpha + 4 \\ \alpha^2 + 4\alpha - \alpha\beta - 2\beta + 4 & 2\alpha + 4 \end{pmatrix}$

$-\dfrac{1}{\alpha - \beta + 4}\begin{pmatrix} -2\beta + 4 & 2\beta - 4 \\ -2\alpha - \beta^2 + \alpha\beta + 4\beta - 4 & 2\alpha + \beta^2 - \alpha\beta - 4\beta + 4 \end{pmatrix}$

$= \dfrac{1}{\alpha - \beta + 4}\begin{pmatrix} \alpha^2 + 4\alpha - \alpha\beta & 2\alpha - 2\beta + 8 \\ \alpha^2 + 6\alpha + \beta^2 - 2\alpha\beta - 6\beta + 8 & -\beta^2 + \alpha\beta + 4\beta \end{pmatrix}$

$= \dfrac{1}{\alpha - \beta + 4}\begin{pmatrix} \alpha(\alpha - \beta + 4) & 2(\alpha - \beta + 4) \\ (\alpha - \beta + 2)(\alpha - \beta + 4) & \beta(\alpha - \beta + 4) \end{pmatrix}$ 1M

$= \begin{pmatrix} \alpha & 2 \\ \alpha - \beta + 2 & \beta \end{pmatrix}$

$\therefore$ The proposition is true for $n = 1$.

Next, assume the proposition is true for $n = k$, where k is a positive integer, i.e.

$M^k = (\alpha + 2)^k P - (\beta - 2)^k Q$,

when $n = k + 1$,

L.H.S.

$= M^{k+1}$

$= MM^k$

$= [(\alpha + 2)P - (\beta - 2)Q][(\alpha + 2)^k P - (\beta - 2)^k Q]$ (by the assumption)

$= (\alpha + 2)^{k+1}P^2 - (\alpha + 2)(\beta - 2)^k PQ - (\alpha + 2)^k(\beta - 2)QP + (\beta - 2)^{k+1}Q^2$ 1M

$= (\alpha + 2)^{k+1}P^2 - (\alpha + 2)(\beta - 2)^k(0) - (\alpha + 2)^k(\beta - 2)(0) + (\beta - 2)^{k+1}Q^2$

$= (\alpha + 2)^{k+1}P^2 + (\beta - 2)^{k+1}Q^2$

$= (\alpha + 2)^{k+1}P - (\beta - 2)^{k+1}Q$ (by (a)(ii))

$=$ R.H.S.

$\therefore$ The proposition is true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(8)

(b) $\begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}^{2017} = \dfrac{1}{2^{2017}}\begin{pmatrix} 8 & 2 \\ 10 & 0 \end{pmatrix}^{2017}$ 1M

Let $\alpha = 8$, $\beta = 0$ and $n = 2017$.

$\because \quad \beta - \alpha = 0 - 8 = -8 \neq 4$ and $\alpha - \beta + 2 = 8 - 0 + 2 = 10$. 1M

$$\therefore \quad \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}^{2017} = \frac{1}{2^{2017}}\begin{pmatrix} 8 & 2 \\ 10 & 0 \end{pmatrix}^{2017}$$

$$= \frac{1}{2^{2017}}\left[(8+2)^{2017}\left(\frac{1}{8-0+4}\right)\begin{pmatrix} 8-0+2 & 2 \\ 8-0+2 & 2 \end{pmatrix} - (0-2)^{2017}\left(\frac{1}{8-0+4}\right)\begin{pmatrix} -2 & 2 \\ 8-0+2 & 0-8-2 \end{pmatrix}\right] \quad \text{1M}$$

$$= \frac{1}{2^{2017}}\left[(10)^{2017}\left(\frac{1}{12}\right)\begin{pmatrix} 10 & 2 \\ 10 & 2 \end{pmatrix} - (-2)^{2017}\left(\frac{1}{12}\right)\begin{pmatrix} -2 & 2 \\ 10 & -10 \end{pmatrix}\right]$$

$$= \frac{1}{12}\begin{pmatrix} 2\times 5^{2018} - 2 & 2\times 5^{2017} + 2 \\ 2\times 5^{2018} + 10 & 2\times 5^{2017} - 10 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{6}\begin{pmatrix} 5^{2018} - 1 & 5^{2017} + 1 \\ 5^{2018} + 5 & 5^{2017} - 5 \end{pmatrix}}} \quad \text{1A}$$

(4)

12. *Reference: HKALE P. Math 2006 Paper 2 Q7*

(a) $f(x) = x - \dfrac{x}{x+1}$

$$f'(x) = 1 - \frac{(x+1)(1) - (x)(1)}{(x+1)^2} \quad \text{1M}$$

$$= \underline{\underline{1 - \frac{1}{(x+1)^2}}} \quad \text{1A}$$

$$f''(x) = -\frac{(x+1)^2(0) - (1)(2)(x+1)}{(x+1)^4}$$

$$= \underline{\underline{\frac{2}{(x+1)^3}}}$$

(2)

(b) (i) When $f'(x) = 0$,

$$1 - \frac{1}{(x+1)^2} = 0 \quad \text{1M}$$

$$\frac{1}{(x+1)^2} = 1$$

$$(x+1)^2 = 1$$

$$x + 1 = \pm 1$$

$$x = -2 \text{ or } 0$$

x	$x < -2$	$x = -2$	$-2 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	+	0	–	undefined	–	0	+
$f''(x)$	–	–	–	undefined	+	+	+

1M

$$f(-2) = -2 - \frac{-2}{-2+1} = -4$$

$$f(0) = 0 - \frac{0}{0+1} = 0$$

From the table, the maximum point is $(-2, -4)$ 1A

and the minimum point is $(0, 0)$. 1A

(ii) When $f''(x) = 0$,

$\frac{2}{(x+1)^3} = 0$, which is impossible 1M

$\therefore$ There is no solutions for $f''(x) = 0$.

From the table, the graph does not have any points of inflexion. 1

(6)

(c) Note that the denominator $x + 1$ is 0 when $x = -1$.

$\therefore$ $x = -1$ is a vertical asymptote. 1A

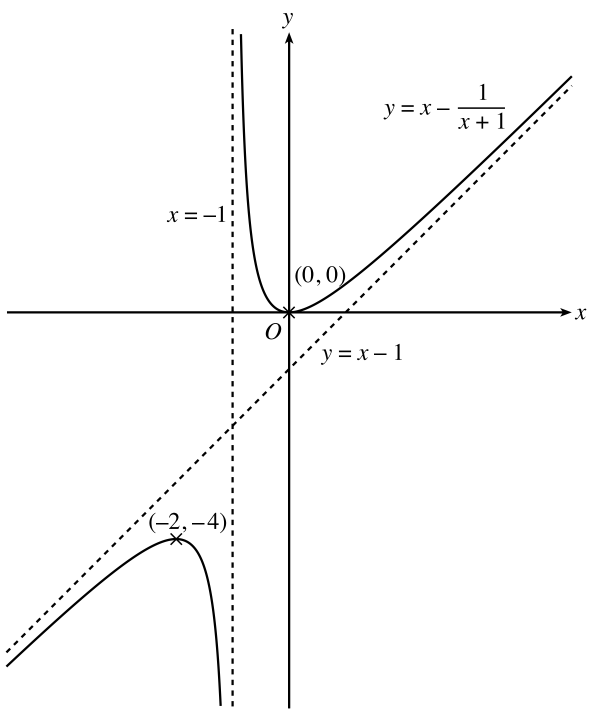
$$\begin{aligned} f(x) &= x - \frac{x}{x+1} \\ &= x - \frac{x+1-1}{x+1} \\ &= x - 1 + \frac{1}{x+1} \end{aligned}$$

When $x \to \pm\infty$, $\frac{1}{x+1} \to 0$.

$\therefore$ $y = x - 1$ is an oblique asymptote. 1A

(2)

(d)



1A for the correct shape

1A for the correct asymptotes and extreme points

1A for all correct

(3)