HKDSE MATH M2 Practice Paper

1. **HKDSE Math M2 Practice Paper Q1**
   Find the coefficient of x^5 in the expansion of $(2-x)^9$.
   (4 marks)

2. **HKDSE Math M2 Practice Paper Q2**
   Consider the following system of linear equation in x, y, z

$$\begin{cases} x & - & 7y & + & 7z & = & 0 \\ x & - & ky & + & 3z & = & 0 \\ 2x & + & y & + & kz & = & 0 \end{cases}, \text{ where } k \text{ is a real number.}$$

   If the system has non-trival solutions, find the two possible values of k.
   (4 marks)

3. **HKDSE Math M2 Practice Paper Q3**
   Prove by mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all positive integers n.
   (5 marks)

4. **HKDSE Math M2 Practice Paper Q4**

   (a) Let $x = \tan\theta$, show that $\dfrac{2x}{1+x^2} = \sin 2\theta$.

   (b) Using (a), find the greatest value of $\dfrac{(1+x)^2}{1+x^2}$, where x is real.

   (5 marks)

5. **HKDSE Math M2 Practice Paper Q5**

   (a) It is given that $\cos(x+1) + \cos(x-1) = k\cos x$ for any real x. Find the value of k.

   (b) Without using a calculator, find the value of $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$.

   (6 marks)

6. **HKDSE Math M2 Practice Paper Q6**
   Find $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$ from first principles.
   (4 marks)

7. **HKDSE Math M2 Practice Paper Q7**
   Let $f(x) = e^x(\sin x + \cos x)$.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Find the value of x such that $f''(x) - f'(x) + f(x) = 0$ for $0 \le x \le \pi$.

(5 marks)

8. **HKDSE Math M2 Practice Paper Q8**

   (a) Using integration by substitution, find $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$.

   (b) Using integration by parts, find $\displaystyle\int \ln x \, dx$.

   (5 marks)

9. **HKDSE Math M2 Practice Paper Q9**

   Find the equations of the two tangents to the curve $x^2 - xy - 2y^2 - 1 = 0$ which are parallel to the straight line $y = 2x + 1$.

   (6 marks)

10. **HKDSE Math M2 Practice Paper Q10**

    (a) Find $\displaystyle\int xe^{-x^2} \, dx$.

    (b) In Figure 1, the shaded region is bounded by the curves $y = \dfrac{x^2}{2}$ and $y = e^{-x^2}$, where $1 \le x \le 2$. Find the volume of the solid generated by revolving the shaded region about the y-axis.

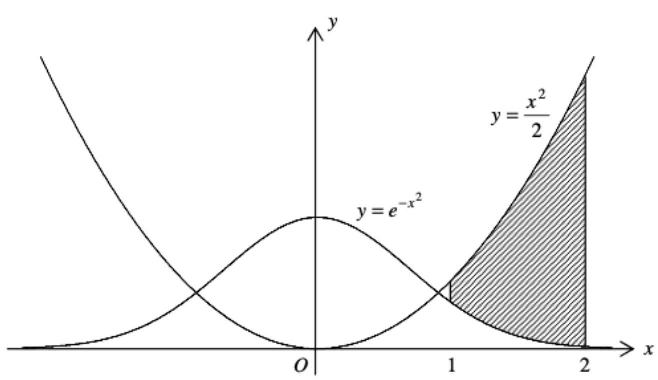

Figure 1

    (6 marks)

11. **HKDSE Math M2 Practice Paper Q11**

    Let $A = \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$ where α and β are distinct real numbers. Let I be the 2×2 identity matrix.

    (a) Show that $A^2 = (\alpha+\beta)A - \alpha\beta I$.
    (2 marks)

    (b) Using (a), or otherwise, show that $(A-\alpha I)^2 = (\beta-\alpha)(A-\alpha I)$ and $(A-\beta I)^2 = (\alpha-\beta)(A-\beta I)$.
    (3 marks)

(c) Let $X = s(A - \alpha I)$ and $Y = t(A - \beta I)$ where s and t are real numbers. Suppose $A = X + Y$.

(i) Find s and t in terms of α and β.

(ii) For any positive integer n, prove that
$X^n = \dfrac{\beta^n}{\beta - \alpha}(A - \alpha I)$ and $Y^n = \dfrac{\alpha^n}{\alpha - \beta}(A - \beta I)$.

(iii) For any positive integer n, express A^n in the form of $pA + qI$, where p and q are real numbers. [Note: It is known that for any 2×2 matrices H and K, if $HK = KH = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $(H + K)^n = H^n + K^n$ for any positive integer n.]

(9 marks)

12. **HKDSE Math M2 Practice Paper Q12**

Let $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (see Figure 2). Let M and N be points on the straight lines AB and OC respectively such that $AM : MB = a : (1 - a)$ and $ON : NC = b : (1 - b)$, where $0 < a < 1$ and $0 < b < 1$. Suppose that MN is perpendicular to both AB and OC.

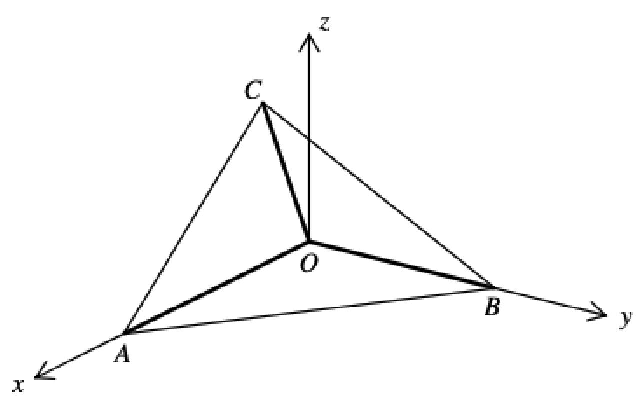

Figure 2

(a) (i) Show that $\overrightarrow{MN} = (a + b - 1)\mathbf{i} + (b - a)\mathbf{j} + b\mathbf{k}$.

(ii) Find the values of a and b.

(iii) Find the shortest distance between straight lines AB and OC.

(8 marks)

(b) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Let G be the projection of O on the plane ABC, find the coordinates of the intersecting point of the two straight lines OG and MN.

(5 marks)

13. **HKDSE Math M2 Practice Paper Q13**

(a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.
Prove that $\displaystyle\int_0^{2p} f(x - p)\, dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where q is a constant.

(4 marks)

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

(2 marks)

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln\left(1+\sqrt{3}\tan x\right) dx$.

(4 marks)

14. **HKDSE Math M2 Practice Paper Q14**

(a) In Figure 3, the shaded region enclosed by the circle $x^2+y^2=25$, the x-axis and the straight line $y=h$ (where $0 \le h \le 5$) is revolved about the y-axis. Show that the volume of the solid of revolution is $\left(25h-\dfrac{h^3}{3}\right)\pi$.

(2 marks)

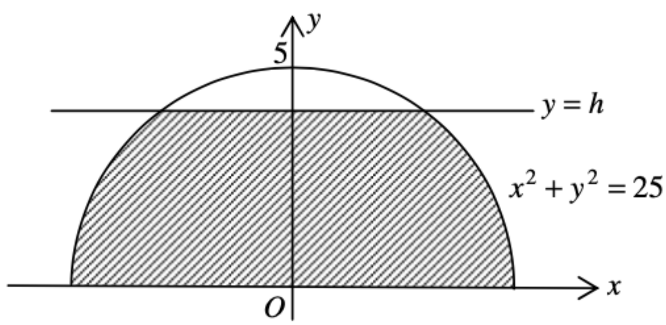

Figure 3

(b) In Figure 4, an empty coffee cup consists of two portions. The lower portion is in the shape of the solid described in (a) with height 4 cm. The upper portion is a frustum of a circular cone. The height of the frustum is 8 cm. The radius of the top of the cup is 6 cm. Hot coffee is poured into the cup to a depth h cm at a rate of 8 cm^3s^{-1}, where $0 \le h \le 12$. Let V cm^3 be the volume of coffee in the cup.

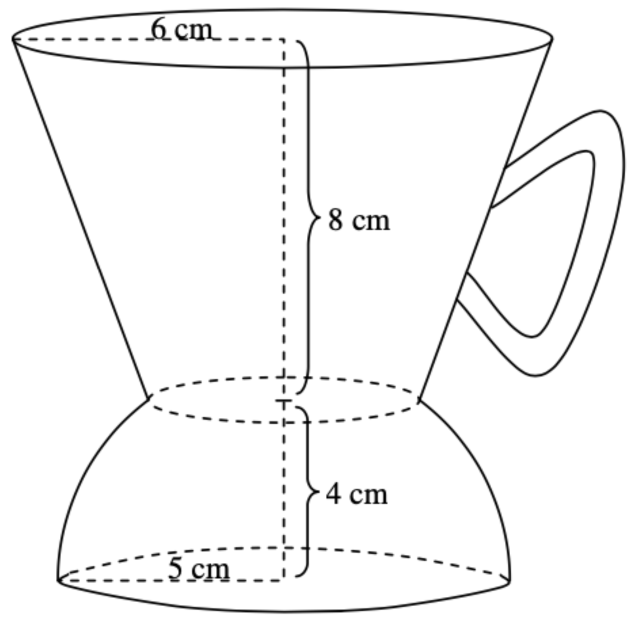

Figure 4

(i) Find the rate of increase of the depth of coffee when the depth is 3 cm.

(ii) Show that $V = \dfrac{164\pi}{3} + \dfrac{3\pi}{64}(h+4)^3$ for $4 \leq h \leq 12$.

(iii) After the cup is fully filled, suddenly it cracks at the bottom. The coffee leaks at a rate of 2 cm^3s^{-1}. Find the rate of decrease of the depth of coffee after 15 seconds of leaking, giving your answer correct to 3 significant figures.

(11 marks)