Level 5 Module 2
exemplar with comments

2016-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2016

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9 and 11.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

Please stick the barcode label here.

Candidate Number									

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

1. Expand $(5+x)^4$. Hence, find the constant term in the expansion of $(5+x)^4\left(1-\frac{2}{x}\right)^3$. (5 marks)

$(5+x)^4$

$= (5)^4 + C_1^4(5)^3(x) + C_2^4(5)^2(x)^2 + C_3^4(5)(x)^3 + (x)^4$

$= 625 + 500x + 150x^2 + 20x^3 + x^4$

$(5+x)^4(1-\frac{2}{x})^3$

$= (625 + 500x + 150x^2 + 20x^3 + x^4)(1 - \frac{6}{x} + \frac{12}{x^2} - \frac{8}{x^3})$

$= 625 - 6(500) + (150)(12) - (20)(8) + \cdots$

$= -735 + \cdots$

$\therefore$ constant term $= -735$

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2. Prove that $\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}=\frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}$. Hence, find $\frac{d}{dx}\sqrt{\frac{3}{x}}$ from first principles.

(5 marks)

$$\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}$$

$$=\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x}\cdot\sqrt{x+h}}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{(x+h)-x}{\sqrt{x}\cdot\sqrt{x+h}\,(\sqrt{x+h}+\sqrt{x})}$$

$$=\frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}$$

$$\therefore \frac{d}{dx}\sqrt{\frac{3}{x}}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\sqrt{\frac{3}{x+\Delta x}}-\sqrt{\frac{3}{x}}\right]$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\sqrt{\frac{3}{x}}-\sqrt{\frac{3}{x+\Delta x}}\right]$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\cancel{\Delta x}}\left[\frac{\cancel{\Delta x}}{(x+\Delta x)\sqrt{x}+x\sqrt{x+\Delta x}}\right]$$

$$=-\lim_{\Delta x\to 0}\frac{1}{(x+\Delta x)\sqrt{x}+x\sqrt{x+\Delta x}}$$

$$=-\frac{1}{x\sqrt{x}+x\sqrt{x}}$$

$$=-\frac{1}{2x\sqrt{x}}$$

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3. Consider the curve $C: y = 2e^x$ where $x > 0$. It is given that P is a point lying on C. The horizontal line which passes through P cuts the y-axis at the point Q. Let O be the origin. Denote the x-coordinate of P by u.

(a) Express the area of ΔOPQ in terms of u.

(b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of ΔOPQ when $u = 4$.

(5 marks)

a) $\because$ P lies on the curve

Let $P = (u, p)$

$p = 2e^u$

$\therefore$ Area of ΔOPQ

$= \frac{1}{2}[u \cdot 2e^u]$

$= ue^u$

b) Let A be the area of ΔOPQ

$A = ue^u$

$\frac{dA}{dt} = u(e^u)\frac{du}{dt} + e^u(\frac{du}{dt})$

$\frac{dA}{dt} = (4)(e^4)(6) + e^4(6)$

$= 24e^4 + 6e^4$

$= 30e^4$ square units per second

$\therefore$ The rate of change of area of $\Delta OPQ = 30e^4$ sq. units/s

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4. Define $f(x)=\frac{2x^2+x+1}{x-1}$ for all $x \neq 1$. Denote the graph of $y=f(x)$ by G . Find

(a) the asymptote(s) of G ,

(b) the slope of the normal to G at the point $(2,11)$.

(7 marks)

a) Vertical asymptote : $x=1$

$$\begin{array}{r} 2x+3 \\ x-1 \enclose{longdiv}{2x^2+x+1} \\ \underline{2x^2-2x} \\ 3x+1 \\ \underline{3x-3} \\ 4 \end{array}$$

$\therefore\ 2x^2+x+1=(2x+3)(x-1)+4$

$\frac{2x^2+x+1}{x-1}=2x+3+\frac{4}{x-1}$

$\therefore$ oblique asymptote B $y=2x+3$

b) $f(x)=\frac{2x^2+x+1}{x-1}$

$f'(x)=\frac{(x-1)(4x+1)-(2x^2+x+1)}{(x-1)^2}$

$=\frac{4x^2-3x-1-2x^2-x-1}{(x-1)^2}$

$=\frac{2(x^2-2x-1)}{(x-1)^2}$

$\left.\frac{dy}{dx}\right|_{x=2}=\frac{2[(2)^2-2(2)-1]}{(2-1)^2}$

$=-2$

$\therefore$ slope of normal

$=\frac{-1}{-2}$

$=\frac{1}{2}$

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5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}(-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=3}^{333}(-1)^{k+1}k^2$.

(6 marks)

a) Let $S(n)$ be the statement.

When $n=1$,

$L.H.S. = (-1)^1(1)^2$

$= -1$

$R.H.S. = \frac{(-1)^1(1)(1+1)}{2}$

$= -1 = L.H.S.$

$\therefore S(1)$ is true.

Assume $S(k)$ is true for some positive integer m.

i.e.: $\sum_{k=1}^{m}(-1)^m(m)^2 = \frac{(-1)^m m(m+1)}{2}$

When $n=m+1$

$RHS. = \frac{(-1)^{(m+1)}(m+1)[(m+1)+1]}{2}$

$= \frac{(-1)^{m+1}(m+1)(m+2)}{2}$

$L.H.S. = \sum_{k=1}^{m+1}(-1)^{m+1}(m+1)^2$

$= \frac{(-1)^m(m)(m+1)}{2} + (-1)^{m+1}(m+1)^2$

$= \frac{(-1)^m(m+1)}{2}[m-2(m+1)]$

$= \frac{(-1)^m(m+1)(-m-2)}{2} = \frac{(-1)^{m+1}(m+1)(m+2)}{2}$ (PTO)

Answers written in the margins will not be marked.

$= R.H.S.$

$\therefore S(m+1)$ is also true

$\therefore$ By MI, $S(n)$ is true for all positive integers n.

5b. sub. $n=333$

$$\sum_{k=1}^{333}(-1)^{k+1}(k)^2 = (-1)\frac{[(-1)^{333}(333)(334)]}{2}$$

$$= 55611$$

sub. $n=2$

$$\sum_{k=1}^{2}(-1)^{k+1}(k)^2 = (-1)\frac{(-1)^2(\cancel{2})(3)}{\cancel{2}}$$

$$= -3$$

$$\therefore \sum_{k=3}^{333}(-1)^{k+1}(k)^2 = 55611-(-3)$$

$$= 55614$$

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6. (a) Prove that $x+1$ is a factor of $4x^3+2x^2-3x-1$.

(b) Express $\cos 3\theta$ in terms of $\cos\theta$.

(c) Using the results of (a) and (b), prove that $\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$.

(6 marks)

a) Let $f(x)=4x^3+2x^2-3x-1$

$f(-1)=4(-1)^3+2(-1)^2-3(-1)-1$

$=0$

$\therefore$ $x+1$ is a factor of $4x^3+2x^2-3x-1$.

b) $\cos 3\theta = \cos(2\theta+\theta)$

$=\cos 2\theta\cdot\cos\theta - \sin 2\theta\sin\theta$

$=(\cos^2\theta-\sin^2\theta)\cdot\cos\theta-(2\sin\theta\cos\theta)(\sin\theta)$

$=\cos^3\theta-\sin^2\theta\cos\theta-2\sin^2\theta\cos\theta$

$=\cos^3\theta-(1-\cos^2\theta)\cos\theta-2(1-\cos^2\theta)(\cos\theta)$

$=\cos^3\theta-\cos\theta+\cos^3\theta-2\cos\theta(1-\cos^2\theta)$

$=\cos^3\theta-\cos\theta+\cos^3\theta-2\cos\theta+2\cos^3\theta$

$=4\cos^3\theta-3\cos\theta$

c) Let $\theta=\frac{\pi}{5}$

$\cos\frac{3\pi}{5}=4\cos^3(\frac{\pi}{5})-3\cos(\frac{\pi}{5})$

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7. (a) Using integration by substitution, find $\int (1+\sqrt{t+1})^2 \, dt$.

(b) Consider the curve Γ: $y = 4x^2 - 4x$, where $1 \le x \le 4$. Let R be the region bounded by Γ , the straight line $y = 48$ and the two axes. Find the volume of the solid of revolution generated by revolving R about the y-axis.

(8 marks)

a) $\int (1+\sqrt{t+1})^2 dt$

$= \int (1+2\sqrt{t+1} + t+1) dt$

$= \int (2+t+2\sqrt{t+1}) dt$

Let $u = \sqrt{t+1}$

$du = \frac{1}{2\sqrt{t+1}} dt$

$\therefore \int (2+t+2\sqrt{t+1}) dt$

$= \int [2+(u^2-1)+2u] \cdot 2u \, du$

$= \int (2+u^2-1+2u) \cdot 2u \, du$

$= \int (u^2+2u+1) \cdot 2u \, du$

$= \int (2u^3+4u^2+2u) du$

$= [\frac{1}{2}u^4 + \frac{4}{3}u^3 + u^2] + C$

$= \frac{1}{2}(t+1)^2 + \frac{4}{3}(t+1)^{\frac{3}{2}} + (t+1) + C$

b) When $x=1$, $y = 4(1)^2 - 4(1) = 0$

$y = 4x^2 - 4x$

$x = 1 + \sqrt{y+1}$

$x^2 = (1+\sqrt{y+1})^2$

$\therefore$ The volume of solid

$= \int_0^{48} (1+\sqrt{y+1})^2 dy$

$= [\frac{1}{2}(y+1)^2 + \frac{4}{3}(y+1)^{\frac{3}{2}} + (y+1)]_0^{48}$ (from (a))

$= \frac{2401}{2} + \frac{1372}{3} + 49$

$= \frac{10241}{6}$

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8. Let n be a positive integer.

(a) Define $A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. Evaluate

(i) A^2 ,

(ii) A^n ,

(iii) $(A^{-1})^n$.

(b) Evaluate

(i) $\sum_{k=0}^{n-1} 2^k$,

(ii) $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}^n$.

(8 marks)

a)(i) $A^2=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$

(ii) $A^3=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$

$\therefore A^n=\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$

aiii) $(A^{-1})^n=(A^n)^{-1}$

$=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$

Answers written in the margins will not be marked.

b i) $\sum_{k=0}^{n-1} 2^k = 1+2+4+\cdots+2^{n-1}$

$= \frac{(1)[2^{n-1}-1]}{2-1}$

$= 2^{n-1}-1$

b ii) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n =$

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SECTION B (50 marks)

9. Let a and b be constants. Define $f(x) = x^3 + ax^2 + bx + 5$ for all real numbers x. Denote the curve $y = f(x)$ by C. It is given that $P(-1, 10)$ is a turning point of C.

(a) Find a and b. (3 marks)

(b) Is P a maximum point of C? Explain your answer. (2 marks)

(c) Find the minimum value(s) of $f(x)$. (2 marks)

(d) Find the point(s) of inflexion of C. (2 marks)

(e) Let L be the tangent to C at P. Find the area of the region bounded by C and L. (4 marks)

a) $f(x) = x^3 + ax^2 + bx + 5$

$f'(x) = 3x^2 + 2ax + b$

$\because$ $P(-1, 10)$ is a turning point of C

$\therefore 3(-1)^2 + 2a(-1) + b = 0$

$-3 - 2a + b = 0$

$b = 2a - 3$ — (1)

$\because$ P lies on C

$\therefore 10 = (-1)^3 + a(-1)^2 + b(-1) + 5$

$10 = -1 + a - b + 5$

$a - b = 6$ — (2)

By solving (1) and (2), $(a, b) = (-3, -9)$

b)

x	~~$x < -1$~~	~~$x = -1$~~	~~$x > -1$~~
$f'(x)$	~~+~~	~~0~~	~~−~~

~~$\therefore$ P is a max. point of C as slope~~

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

$f''(-1) = 6(-1) - 6$

$= -12 < 0$

$\therefore$ P is a max. point of C.

Answers written in the margins will not be marked.

c) Let $f'(x)=0$

$3x^2-6x-9=0$

$x=3$ or $x=-1$

When $x=3$

$f''(x)=6(3)-6=12>0$

$\therefore$ min. at $x=3$

When $x=3$, $f(3)=(3)^3+(-3)(3)^2+(-9)(3)+5$

$=-22$

$\therefore$ min. value $=-22$

d) Let $f''(x)=0$

$6x-6=0$

$x=1$

When $x=1$,

$f(1)=(1)^3+(-3)(1)^2+(-9)(1)+5$

$=-6$

$\therefore$ point of inflexion $=(1,-6)$

e) Equation of tangent is $y=10$.

$10=x^3-3x^2-9x+5$

$x=5$ or $x=-1$ (rej.)

$\therefore$ The area bounded by C and L

$=\int_{-1}^{5}[10-(x^3-3x^2-9x+5)]\,dx$

$=\int_{-1}^{5}(-x^3+3x^2+9x+5)\,dx$

$=\left[-\frac{1}{4}x^4+x^3+\frac{9}{2}x^2+5x\right]_{-1}^{5}$

$=\frac{425}{4}+\frac{7}{4}$

$=108$

Answers written in the margins will not be marked.

10. (a) Let $f(x)$ be a continuous function defined on the interval $[0, a]$, where a is a positive constant. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. (3 marks)

(b) Prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$. (3 marks)

(c) Using (b), prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8}$. (3 marks)

(d) Using integration by parts, evaluate $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx$. (3 marks)

a) Let $u = a - x$

$du = -dx$

When $x = a,\ u = 0$

When $x = 0,\ u = a$

$RHS = \int_0^a f(a-x)\,dx$

$= \int_a^0 f(u)(-du)$

$= -\int_a^0 f(u)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

$= L.H.S.$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

b) Let $a = \frac{\pi}{4}$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$

$= \int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$= \int_0^{\frac{\pi}{4}} \ln\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$

$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx = \int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1+\tan x}\right]dx$

Answers written in the margins will not be marked.

c) $2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} \left[\ln(1+\tan x) + \ln\left(\frac{2}{1+\tan x}\right)\right] dx$

$= \int_0^{\frac{\pi}{4}} \ln[2]\,dx$

$= \left[(\ln 2)x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi \ln 2}{4}$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8}$

d) $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{x}{1+\tan x}\,d(\tan x)$

$= \left[\frac{x\tan x}{1+\tan x}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} [\tan x]\,d\left(\frac{x}{1+\tan x}\right)$

$= \frac{\pi}{8} - \int_0^{\frac{\pi}{4}} \tan x \cdot \frac{\cancel{(1+\tan x)} - x(\sec^2 x)}{(1+\cancel{\tan x})^2}\,dx$

$= \frac{\pi}{8} - \int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\tan x} - \frac{x\sec^2 x}{(1+\tan x)^2}\,dx$

$=$

Answers written in the margins will not be marked.

11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + y - z = 3 \\ 4x + 6y + az = b \\ 5x + (1-a)y + (3a-1)z = b-1 \end{cases}$$, where a and b are real numbers.

(i) Assume that (E) has a unique solution.

(1) Prove that $a \neq -2$ and $a \neq -12$.

(2) Solve (E) .

(ii) Assume that $a = -2$ and (E) is consistent.

(1) Find b .

(2) Solve (E) .

(9 marks)

(b) Is there a real solution of the system of linear equations

$$\begin{cases} x + y - z = 3 \\ 2x + 3y - z = 7 \\ 5x + 3y - 7z = 13 \end{cases}$$

satisfying $x^2 + y^2 - 6z^2 > 14$? Explain your answer. (3 marks)

11a i)(1) $\because$ (E) has a unique solution.

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & -1 \\ 4 & 6 & a \\ 5 & 1-a & 3a-1 \end{vmatrix} \neq 0$$

$\Delta = 18a - 6 + 5a - 4 + 4a + 30 - a + \cancel{a^2} - 12a + 4$

$= a^2 + 14a + 24$

$\Delta \neq 0$

$(a+2)(a+12) \neq 0$

$\therefore a \neq -2$ and $a \neq -12$

11a(i)(2) $\Delta = a^2 + 14a + 24$ (from (ai)(1))

$$\Delta x = \begin{vmatrix} 3 & 1 & -1 \\ b & 6 & a \\ b-1 & 1-a & 3a-1 \end{vmatrix}$$

$= 18(3a-1) + a(b-1) - b(1-a) + 6(b-1) - 3a(1-a) - b(3a-1)$

Answers written in the margins will not be marked.

$= 3a^2 - ab + 50a + 5b - 17$

$\Delta y = \begin{vmatrix} 1 & 3 & -1 \\ 4 & b & a \\ 5 & b-1 & 3a-1 \end{vmatrix}$

$= b(3a-1) + 15a - 4(b-1) + 5b - a(b-1) - 12(3a-1)$

$= 3ab - b + 15a - 4b + 4 + 5b - ab + a - 36a + 12$

$= 2(ab - 10a + 8)$

$\Delta z = \begin{vmatrix} 1 & 1 & 3 \\ 4 & 6 & b \\ 5 & 1-a & b-1 \end{vmatrix}$

$= ab - 12a + 6b - 80$

$x = \dfrac{\Delta x}{\Delta} = \dfrac{3a^2 - ab + 50a + 5b - 17}{a^2 + 14a + 24}$

$y = \dfrac{\Delta y}{\Delta} = \dfrac{2(ab - 10a + 8)}{a^2 + 14a + 24}$

$z = \dfrac{\Delta z}{\Delta} = \dfrac{ab - 12a + 6b - 80}{a^2 + 14a + 24}$

aii)(1) when $a = -2$, we have

$\begin{cases} x + y - z = 3 \\ 4x + 6y - 2z = b \\ 5x + 3y - 7z = b - 1 \end{cases}$

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 4 & 6 & -2 & b \\ 5 & 3 & -7 & b-1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 2 & b-12 \\ 0 & -2 & -2 & b-16 \end{array}\right)$ (PTO)

Answers written in the margins will not be marked.

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 2 & b-12 \\ 0 & 0 & 0 & 2b-28 \end{array}\right)$$

$\because$ (E) is consistent

$\therefore 2b-28=0$

$b=14$

(ii)(2) Let $z=t$

$2y+2t=2$

$2y=2-2t$

$y=1-t$

$x+(1-t)-t=3$

$x+1-2t=3$

$x=2+2t$

$\therefore (x,y,z)=(2+2t,1-t,t)$, where $t\in R$.

b) sub. $(x,y,z)=(2+2t,1-t,t)$ into $x^2+y^2-6z^2=14$

$(2+2t)^2+(1-t)^2-6t^2=14$

$(\cancel{4}+8t+\cancel{4t^2})+(\cancel{1}-2t+\cancel{t^2})-\cancel{6t^2}=\cancel{14}$

$-t^2+6t-9=0$

$\Delta=(6)^2-4(-1)(-9)$

$=36-36$

$=0$

$\because \Delta=0$

$\therefore$ NO, there isn't a real solution satisfying $x^2+y^2-6z^2>14$.

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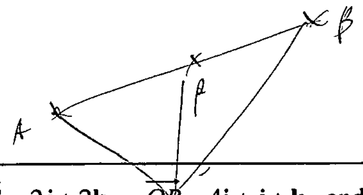

12. Let $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OP} = \mathbf{i} + t\mathbf{j}$, where t is a constant and O is the origin. It is given that P is equidistant from A and B .

(a) Find t . (3 marks)

(b) Let $\overrightarrow{OC} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$. Denote the plane which contains A , B and C by Π .

(i) Find a unit vector which is perpendicular to Π .

(ii) Find the angle between CD and Π .

(iii) It is given that E is a point lying on Π such that $\overrightarrow{DE}$ is perpendicular to Π . Let F be a point such that $\overrightarrow{PF} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$. Describe the geometric relationship between D, E and F. Explain your answer.

(10 marks)

a) $\cancel{\overrightarrow{OP} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}}$

$\cancel{\vec{i} + t\vec{j} = \dfrac{(2\vec{j} + 2\vec{k}) + (4\vec{i} + \vec{j} + \vec{k})}{2}}$

$\cancel{= \dfrac{4\vec{i} + 3\vec{j} + 3\vec{k}}{2}}$

$\cancel{= 2\vec{i} + \frac{3}{2}\vec{j} + \frac{3}{2}\vec{k}}$

$|\overrightarrow{AP}| = |\overrightarrow{BP}|$

$\overrightarrow{AP} = (\vec{i} + t\vec{j}) - (2\vec{j} + 2\vec{k})$

$= \vec{i} + (t-2)\vec{j} - 2\vec{k}$

$|\overrightarrow{AP}| = \sqrt{(1)^2 + (t-2)^2 + (-2)^2}$

$= \sqrt{t^2 - 4t + 9}$

$\overrightarrow{BP} = (\vec{i} + t\vec{j}) - (4\vec{i} + \vec{j} + \vec{k})$

$= -3\vec{i} + (t-1)\vec{j} - \vec{k}$

$|\overrightarrow{BP}| = \sqrt{(-3)^2 + (t-1)^2 + (-1)^2}$

$= \sqrt{t^2 - 2t + 11}$

$\therefore \sqrt{t^2 - 4t + 9} = \sqrt{t^2 - 2t + 11}$

$t^2 - 4t + 9 = t^2 - 2t + 11$

$2t = -2$

$t = -1$

b i) $\vec{AB} = (4\vec{i}+\vec{j}+\vec{k}) - (2\vec{j}+2\vec{k})$

$= 4\vec{i} - \vec{j} - \vec{k}$

$\vec{AC} = (2\vec{i}-\vec{j}+4\vec{k}) - (2\vec{j}+2\vec{k})$

$= 2\vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{AB} \times \vec{AC} = (4\vec{i}-\vec{j}-\vec{k}) \times (2\vec{i}-3\vec{j}+2\vec{k})$

$= -5\vec{i} - 10\vec{j} - 10\vec{k}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -1 \\ 2 & -3 & 2 \end{vmatrix}$$

$|\vec{AB} \times \vec{AC}| = \sqrt{(-5)^2 + (-10)^2 + (-10)^2}$

$= 15$

$\therefore$ The req. unit vector $= -\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$

b ii) $\vec{CD} = (3\vec{i}+2\vec{j}+5\vec{k}) - (2\vec{i}-\vec{j}+4\vec{k})$

$= \vec{i} + 3\vec{j} + \vec{k}$

$|\vec{CD}| = \sqrt{(1)^2 + (3)^2 + (1)^2}$

$= \sqrt{11}$

$\because$ Let θ be the angle between CD and Π.

$\vec{CD} \cdot (\vec{AB} \times \vec{AC}) = |\vec{CD}||\vec{AB} \times \vec{AC}| \cos\theta$

$(\vec{i}+3\vec{j}+\vec{k}) \cdot (-5\vec{i}-10\vec{j}-10\vec{k}) = (\sqrt{11})(15)\cos\theta$

$-5 - 30 - 10 = 15\sqrt{11}\cos\theta$

$\theta = 154.7605982°$

b iii) $\vec{PF} = \vec{PA} + \vec{PB} + \vec{PC}$

$= (\cancel{\vec{i}}+\cancel{\vec{j}}+\cancel{2\vec{k}}) + (\cancel{3\vec{i}}+\cancel{2\vec{j}}+\cancel{\vec{k}}) + (\cancel{\vec{i}}+\cancel{4\vec{k}})$

$= 5\vec{i} + 3\vec{j} + 7\vec{k}$

$\vec{PF} = \vec{OF} - \vec{OP}$

$\vec{OF} = \vec{PF} + \vec{OP}$

$= (5\vec{i}+3\vec{j}+7\vec{k}) + (\vec{i}-\vec{j})$

$= 6\vec{i} + 2\vec{j} + 7\vec{k}$

Answers written in the margins will not be marked.

$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$

$= (6\vec{i} + 2\vec{j} + 7\vec{k}) - (3\vec{i} + 2\vec{j} + 5\vec{k})$

$= 3\vec{i} + 2\vec{k}$

$\overrightarrow{DE} = \lambda\left(-\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}\right)$, where λ is a scalar

$\overrightarrow{DE} \cdot \overrightarrow{DF} = (3\vec{i} + 2\vec{k}) \cdot \left(-\frac{\lambda}{3}\vec{i} - \frac{2\lambda}{3}\vec{j} - \frac{2\lambda}{3}\vec{k}\right)$

$= -\lambda - \frac{4}{3}\lambda$

END OF PAPER

Answers written in the margins will not be marked.

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Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations as in Questions 9, 10, 11 and 12.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language, notations and diagrams, such as using binomial theorem in Question 1, limit notations in Question 2, differentiation symbols in Questions 3 and 4, mathematical induction in Question 5, trigonometric formulas in Question 6, integration symbols in Questions 7 and 10, matrix notations in Questions 8 and 11, derivative tests in Question 9 and vector manipulations in Question 12.

He/She also provides complex mathematical proofs in a logical, rigorous and concise manner in Questions 10, 11 and 12.

It can be concluded that the candidate has the ability to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies.