1. (a) From the definition of derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4(x+h)-1}-\sqrt{4x-1}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4(x+h)-1}-\sqrt{4x-1}}{h}\cdot\frac{\sqrt{4(x+h)-1}+\sqrt{4x-1}}{\sqrt{4(x+h)-1}+\sqrt{4x-1}}\\
&= \lim_{h\to 0}\frac{\left(\sqrt{4(x+h)-1}\right)^2-\left(\sqrt{4x-1}\right)^2}{h\left[\sqrt{4(x+h)-1}+\sqrt{4x-1}\right]}\\
&= \lim_{h\to 0}\frac{4(x+h)-1-(4x-1)}{h\left[\sqrt{4(x+h)-1}+\sqrt{4x-1}\right]}\\
&= \lim_{h\to 0}\frac{4}{\sqrt{4(x+h)-1}+\sqrt{4x-1}}\\
&= \boxed{\frac{2}{\sqrt{4x-1}}}.
\end{aligned}$$

(b) From the definition of derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{2+(x+h)^2}-\sqrt{2+x^2}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{2+(x+h)^2}-\sqrt{2+x^2}}{h}\cdot\frac{\sqrt{2+(x+h)^2}+\sqrt{2+x^2}}{\sqrt{2+(x+h)^2}+\sqrt{2+x^2}}\\
&= \lim_{h\to 0}\frac{\left[\sqrt{2+(x+h)^2}\right]^2-\left(\sqrt{2+x^2}\right)^2}{h\left[\sqrt{2+(x+h)^2}+\sqrt{2+x^2}\right]}\\
&= \lim_{h\to 0}\frac{2+(x+h)^2-\left(2+x^2\right)}{h\left[\sqrt{2+(x+h)^2}+\sqrt{2+x^2}\right]}\\
&= \lim_{h\to 0}\frac{h(2x+h)}{h\left[\sqrt{2+(x+h)^2}+\sqrt{2+x^2}\right]}\\
&= \lim_{h\to 0}\frac{2x+h}{\sqrt{2+(x+h)^2}+\sqrt{2+x^2}}\\
&= \boxed{\frac{x}{\sqrt{2+x^2}}}.
\end{aligned}$$

(c) From the definition of derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2(x+h)+1}-\frac{1}{2x+1}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{2x+1-(2(x+h)+1)}{(2(x+h)+1)(2x+1)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{(2x+2h+1)(2x+1)}\right] \\
&= \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)} \\
&= \boxed{-\frac{2}{(2x+1)^2}}.
\end{aligned}
$$

(d) From the definition of derivative and the formula $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-2\sin\frac{x+h+x}{2}\sin\frac{x+h-x}{2}}{h} \\
&= -\lim_{h\to 0}\frac{\sin\frac{2x+h}{2}\sin\frac{h}{2}}{\frac{h}{2}} \\
&= -\left(\lim_{h\to 0}\sin\frac{2x+h}{2}\right)\left(\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \\
&= -\sin x\cdot 1 \\
&= \boxed{-\sin x}.
\end{aligned}
$$

(e) From the definition of derivative and the formula $\sin(A-B) = \sin A\cos B - \sin B\cos A$,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \\
&= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{h\cos(x+h)\cos x} \\
&= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \\
&= \left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left[\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right] \\
&= 1\cdot\frac{1}{(\cos x)^2} \\
&= \boxed{\sec^2 x}.
\end{aligned}
$$

(f) From the definition of derivative,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) \\
&= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} \\
&= \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right].
\end{aligned}
$$

Since $\lim_{h\to 0} \frac{h}{x} = 0$, it follows from the fact $\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$ that

$$\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} = \left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} = e^{\frac{1}{x}}.$$

Hence

$$f'(x) = \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right] = \ln e^{\frac{1}{x}} = \boxed{\frac{1}{x}}.$$

2. (Note: You are required to show working steps, if applicable, in assignments, test and examination)

(a) $-40x^4 + 3\sqrt{3}x^2 + 4\pi x$

(b) $\left(x^{100} + 2x^{50} - 3\right)\left(56x^7 + 20\right) + \left(100x^{99} + 100x^{49}\right)\left(7x^8 + 20x + 5\right)$

(c) $\dfrac{2x^7 + 35x^4 + 2x^3 - 6x^2 - 7}{(x^3+7)^2}$

(d) $4(x^3 - 2x^2 + 7x - 3)^3\left(3x^2 - 4x + 7\right)$

(e) $\dfrac{-24x}{(3x^2+5)^5}$

(f) $\dfrac{1}{\sqrt{2x+7}}$

(g) $\dfrac{-15(x+2)^2}{(x-3)^4}$

(h) $-\sqrt{x}\left(6x^2+1\right)\sin\left(2x^3+x-10\right) + \dfrac{\cos\left(2x^3+x-10\right)}{2\sqrt{x}}$

(i) $-\dfrac{1}{\sqrt{1-x^2}\cos^{-1}x}$

(j) $\dfrac{5}{x\ln x}$

(k) $15\sin^2(5x+4)\cos(5x+4)$

(l) $\dfrac{3\tan^2(\ln x)\sec^2(\ln x)}{x}$

(m) $\dfrac{\cos(2x) + 4x\sin(2x)}{\cos^3(2x)}$

(n) $8\sin(4x)\cos(4x) + 8x\sin\left(x^2-1\right) + \cos(x\ln x)(1+\ln x)$

(o) $e^{3x^2+5x-2}(6x+5)$

(p) $\dfrac{e^x+2}{e^x+2x+1} - \dfrac{e^x-3}{e^x-3x-1}$

(q) xe^{4x}

(r) $\dfrac{2}{(2x+3)\ln x} - \dfrac{\ln(2x+3)}{x(\ln x)^2}$

(s) $-\dfrac{x}{|x|\sqrt{1-x^2}}$

(t) $e^{x-x^4}\left(1 - 12x^2 - 8x^3 + 16x^6\right)$

3. (Note: You are required to show working steps, if applicable, in assignments, test and examination)

(a) $\dfrac{2-x}{3+y}$

(b) $\dfrac{-y}{x+2y}$

(c) $\dfrac{y-1}{\cos y - x}$

(d) $\dfrac{y\sec^2(xy)}{1-x\sec^2(xy)}$

(e) $\dfrac{2\tan(x+y)\sec^2(x+y)}{1-2\tan(x+y)\sec^2(x+y)}$

(f) $\dfrac{y-y^3+4x}{3xy^2-x}$

(g) $\dfrac{2x\cos\left(x^2+y\right)-3y^2}{6xy+2y-\cos\left(x^2+y\right)}$

(h) $-\dfrac{3x+2y+2y\sqrt{x+y}}{x+2x\sqrt{x+y}}$

(i) $\dfrac{2x(x-y)^2+2x^2(x-y)-2x}{2x^2(x-y)-2y}$

(j) $\dfrac{1}{y(x+1)^2}$

(k) $\dfrac{2x^3-xy^2}{x^2y+\frac{1}{y}}$

(l) $\dfrac{\cos\left(y^3\right)-x}{y+3xy^2\sin\left(y^3\right)}$

4. (Note: You are required to show working steps, if applicable, in assignments, test and examination)

(a) $\dfrac{1}{5}\sqrt[5]{\dfrac{x-1}{x+1}}\left(\dfrac{1}{x-1}-\dfrac{1}{x+1}\right)$

(b) $\dfrac{x^2\sqrt[3]{7x-14}}{\left(1+x^2\right)^4}\left(\dfrac{2}{x}+\dfrac{1}{3x-6}-\dfrac{8x}{1+x^2}\right)$

(c) $\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\dfrac{3}{4x}+\dfrac{x}{x^2+1}-\dfrac{15}{3x+2}\right)$

(d) $\dfrac{x^3}{1-x}\sqrt[3]{\dfrac{3-x}{(3+x)^2}}\left(\dfrac{3}{x}+\dfrac{1}{1-x}-\dfrac{1}{9-3x}-\dfrac{2}{9+3x}\right)$

(e) $\dfrac{1}{3}\sqrt[3]{\dfrac{(x+2)(3x-1)^4}{(2-x)^5}}\left(\dfrac{1}{x+2}+\dfrac{12}{3x-1}+\dfrac{5}{2-x}\right)$

(f) $2^{\sin x}\cos x\ln 2$

(g) $-\dfrac{\ln 3}{x^2}3^{\tan\frac{1}{x}}\sec^2\dfrac{1}{x}$

(h) $\dfrac{2x^{\ln x}\ln x}{x}$

(i) $(\sin x)^x\left[x\cot x+\ln(\sin x)\right]$

(j) $(1+x)^{\frac{1}{x}}\left[\dfrac{1}{x(1+x)}-\dfrac{\ln(1+x)}{x^2}\right]$

(k) $x^{\cos^{-1}x}\left(\dfrac{\cos^{-1}x}{x}-\dfrac{\ln x}{\sqrt{1-x^2}}\right)$

(l) $\left(\tan^{-1}x\right)^{\sqrt{x}}\left[\dfrac{\sqrt{x}}{\left(1+x^2\right)\tan^{-1}x}+\dfrac{\ln\left(\tan^{-1}x\right)}{2\sqrt{x}}\right]$

(m) $\left(\sin^{-1}x\right)^{x^2}\left[\dfrac{x^2}{\sqrt{1-x^2}\sin^{-1}x}+2x\ln\left(\sin^{-1}x\right)\right]$

(n) $x^{\cos x}\left[\dfrac{\cos x}{x}-(\sin x)(\ln x)\right]\cos\left(x^{\cos x}\right)$

5. (a) Using the chain rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \frac{d}{du}\left(\frac{u^2-1}{u^2+1}\right)\cdot\frac{d}{dx}\left[\left(x^2+2\right)^{\frac{1}{3}}\right] \\
&= \frac{\left(u^2+1\right)\frac{d}{du}\left(u^2-1\right)-\left(u^2-1\right)\frac{d}{du}\left(u^2+1\right)}{\left(u^2+1\right)^2}\cdot\frac{1}{3}\left(x^2+2\right)^{-\frac{2}{3}}\cdot 2x \\
&= \frac{4u}{\left(u^2+1\right)^2}\cdot\frac{2x}{3\left(x^2+2\right)^{\frac{2}{3}}} \\
&= \frac{8\left(x^2+2\right)^{\frac{1}{3}}x}{3\left[\left(x^2+2\right)^{\frac{2}{3}}+1\right]^2\left(x^2+2\right)^{\frac{2}{3}}} \\
&= \boxed{\frac{8x}{3\left[\left(x^2+2\right)^{\frac{2}{3}}+1\right]^2\left(x^2+2\right)^{\frac{1}{3}}}}.
\end{aligned}$$

(b) Using the chain rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \frac{d}{du}\left[\left(3u^2+4\right)^{-\frac{1}{2}}\right]\cdot\frac{d}{dx}\left(e^{-x}\right) \\
&= -\frac{1}{2}\left(3u^2+4\right)^{-\frac{3}{2}}\cdot 6u\left(-e^{-x}\right) \\
&= 3\left[3\left(e^{-x}\right)^2+4\right]^{-\frac{3}{2}}\left(e^{-x}\right)^2 \\
&= \boxed{3e^{-2x}\left(3e^{-2x}+4\right)^{-\frac{3}{2}}}.
\end{aligned}$$

(c) Since $x=\dfrac{u+1}{2}$, it follows from the chain rule that

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}=\frac{d}{dx}\left(2x^2+1\right)\cdot\frac{d}{du}\left(\frac{u+1}{2}\right)=4x\cdot\frac{1}{2}=2x=\boxed{u+1}.$$

6. (a) By the chain rule,

$$F'(5)=f'\left(g(5)\right)g'(5)=f'(-2)g'(5)=4\cdot 6=\boxed{24}.$$

(b) (i) $f'(x)=\dfrac{d\left[(g(x))^2\right]}{d\left(g(x)\right)}\cdot\dfrac{d\left(g(x)\right)}{dx}=\boxed{2g(x)g'(x)}$

(ii) $f'(x)=\dfrac{d\left[\sin\left(g(x)\right)\right]}{d\left(g(x)\right)}\cdot\dfrac{d\left(g(x)\right)}{dx}=\boxed{g'(x)\cos\left(g(x)\right)}$

(iii) $f'(x)=\dfrac{d\left[g(\sin x)\right]}{d\left(\sin x\right)}\cdot\dfrac{d\left(\sin x\right)}{dx}=\boxed{g'(\sin x)\cos x}$

(c) Since $f\left(g(x)\right)=x$, we have

$$\frac{d}{dx}\left(f\left(g(x)\right)\right)=\frac{dx}{dx},$$

i.e.

$$f'\left(g(x)\right)g'(x)=1.$$

With $f'(x)=1+\left(f(x)\right)^2$, it follows that

$$g'(x)=\frac{1}{f'\left(g(x)\right)}=\frac{1}{1+\left(f\left(g(x)\right)\right)^2}=\frac{1}{1+x^2}.$$

7. (a) Since

$$\frac{dy}{dx} = 2x + \sin x,$$

the slope of the tangent line at $x = \pi$ is

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 2\pi + \sin\pi = 2\pi.$$

At $x = \pi$, $y = \pi^2 - \cos\pi - 1 = \pi^2$. Hence the equation of the tangent line is

$$y - \pi^2 = 2\pi(x - \pi),$$

i.e.

$$\boxed{2\pi x - y - \pi^2 = 0}.$$

(b) The slope of a horizontal (tangent) line is zero. Then,

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 6x^2 + 6x - 12 &= 0 \\ 6(x+2)(x-1) &= 0 \\ x &= -2 \text{ or } 1. \end{aligned}$$

When $x = -2$ (resp. $x = 1$), we have $y = 21$ (resp. $y = -6$). The points at which the tangent lines are horizontal are $\boxed{(-2, 21)}$ and $\boxed{(1, -6)}$.

(c) The required condition is

$$\begin{aligned} \frac{dy}{dx} &= 12 \\ 3x^2 &= 12 \\ x &= -2 \text{ or } 2. \end{aligned}$$

When $x = -2$ (resp. $x = 2$), we have $y = -7$ (resp. $y = 9$). The corresponding tangent lines are given by

$$y - (-7) = 12\,[x - (-2)] \quad \text{and} \quad y - 9 = 12(x - 2),$$

i.e.

$$\boxed{12x - y + 17 = 0} \quad \text{and} \quad \boxed{12x - y - 15 = 0}$$

respectively.

(d) The required condition is

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \\ \frac{2}{(x+1)^2} &= \frac{1}{2} \\ x &= -3 \text{ or } 1. \end{aligned}$$

When $x = -3$ (resp. $x = 1$), we have $y = 2$ (resp. $y = 0$). The corresponding tangent lines are given by

$$y - 2 = \frac{1}{2}\,[x - (-3)] \quad \text{and} \quad y - 0 = \frac{1}{2}(x - 1),$$

i.e.

$$\boxed{x - 2y + 7 = 0} \quad \text{and} \quad \boxed{x - 2y - 1 = 0}$$

respectively.

8. (a) If the line $2x + y = b$ is tangent to the curve $y = ax^2$ at $x = 2$, then

$$\frac{d}{dx}\left(ax^2\right)\Big|_{x=2} = -2,$$

i.e.

$$4a = -2.$$

Thus,

$$\boxed{a = -\frac{1}{2}}.$$

When $x = 2$, we have $y = -\frac{1}{2} \cdot 2^2 = -2$. Since the line $2x + y = b$ passes through the point $(2, -2)$, it follows that $b = 2 \cdot 2 + (-2) = \boxed{2}$.

(b) Suppose that the graph of $y = \dfrac{c}{x+1}$ touches the given line at $\left(a, \dfrac{c}{a+1}\right)$, where the value(s) of a is/are to be determined. At this point,

$$\frac{dy}{dx}\Big|_{x=a} = \frac{-2-3}{5-0},$$

i.e.

$$\frac{c}{(a+1)^2} = 1.$$

Thus,

$$c = (a+1)^2.$$

Since this point is also on the given line whose equation is $y = -x + 3$, we have

$$\frac{c}{a+1} = -a + 3.$$

It follows that

$$a + 1 = -a + 3,$$

or

$$a = 1.$$

Hence

$$c = (1+1)^2 = \boxed{4}.$$

9. (a) (i) The given condition can be expressed as

$$\begin{aligned} \frac{dy}{dx} &= 4 \\ 3x^2 - 4x &= 4 \\ (3x+2)(x-2) &= 0 \\ x &= -\frac{2}{3} \text{ or } 2. \end{aligned}$$

When $x = -\frac{2}{3}$ (resp. $x = 2$), we have $y = -\frac{140}{27}$ (resp. $y = -4$). Thus, the required points are

$$\boxed{\left(-\frac{2}{3}, -\frac{140}{27}\right)} \text{ and } \boxed{(2, -4)}.$$

(ii) A point on C is of the form $\left(a, a^3 - 2a^2 - 4\right)$. At this point,

$$\left.\frac{dy}{dx}\right|_{x=a} = 3a^2 - 4a$$

and the equation of the tangent line to C is given by

$$y - \left(a^3 - 2a^2 - 4\right) = \left(3a^2 - 4a\right)(x - a).$$

If the above tangent line passes through the point $(-1, 0)$, then

$$\begin{aligned} 0 - \left(a^3 - 2a^2 - 4\right) &= \left(3a^2 - 4a\right)(-1 - a) \\ 2a^3 + a^2 - 4a + 4 &= 0 \\ (a + 2)\left(2a^2 - 3a + 2\right) &= 0 \\ a &= -2 \end{aligned}$$

(the equation $2a^2 - 3a + 2 = 0$ has *no* real roots, since $(-3)^2 - 4 \cdot 2 \cdot 2 = -7 < 0$). When $a = -2$, $(-2)^3 - 2(-2)^2 - 4 = -20$. Thus, the required point is $\boxed{(-2, -20)}$.

(b) (i) The equations of the corresponding tangent lines are

$$y - \left(-\frac{140}{27}\right) = 4\left[x - \left(-\frac{2}{3}\right)\right] \quad \text{and} \quad y - (-4) = 4(x - 2),$$

i.e.

$$\boxed{4x - y - \frac{68}{27} = 0} \text{ and } \boxed{4x - y - 12 = 0}.$$

(ii) The equation of the corresponding tangent line is

$$\frac{y - 0}{x - (-1)} = \frac{-20 - 0}{-2 - (-1)},$$

or

$$\boxed{20x - y + 20 = 0}.$$

10. (a) Let $y = f(x)$. Since

$$\frac{dy}{dx} = 6x^2 + 10x,$$

the slope of the tangent line at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 6 \cdot 1^2 + 10 \cdot 1 = 16.$$

At $x = 1$, $y = 2 \cdot 1^3 + 5 \cdot 1^2 - 12 = -5$. Hence the equation of the tangent line is

$$y - (-5) = 16(x - 1),$$

i.e.

$$\boxed{16x - y - 21 = 0}.$$

(b) Suppose that the tangent line $y = mx$ touches the graph of $y = f(x)$ at (a, ma), where the value(s) of a is/are to be determined. Since this point is also on the graph of $y = f(x)$, we have

$$ma = 2a^3 + 5a^2 - 12.$$

At this point,

$$\left.\frac{dy}{dx}\right|_{x=a} = m,$$

i.e.

$$6a^2 + 10a = m.$$

It follows that

$$2a^3 + 5a^2 - 12 = \left(6a^2 + 10a\right) a,$$

or

$$4a^3 + 5a^2 + 12 = 0.$$

Solving for a gives

$$\begin{aligned} (a+2)\left(4a^2 - 3a + 6\right) &= 0 \\ a &= -2 \end{aligned}$$

(the equation $4a^2 - 3a + 6 = 0$ has *no* real roots, since $(-3)^2 - 4 \cdot 4 \cdot 6 = -87 < 0$). Hence

$$m = 6(-2)^2 + 10(-2) = \boxed{4}.$$

11. (a) Differentiate both sides of the given equation with respect to x as follows.

$$\begin{aligned} 2\frac{d}{dx}\left(x^4\right) - 2\frac{d}{dx}\left(x^2y^2\right) - \frac{d}{dx}\left(y^3\right) &= 0 \\ 2\frac{d}{dx}\left(x^4\right) - 2\left[x^2\frac{d\left(y^2\right)}{dy}\frac{dy}{dx} + y^2\frac{d}{dx}\left(x^2\right)\right] - \frac{d\left(y^3\right)}{dy}\frac{dy}{dx} &= 0 \\ 2\left(4x^3\right) - 2\left(2x^2y\frac{dy}{dx} + 2xy^2\right) - 3y^2\frac{dy}{dx} &= 0 \end{aligned}$$

Putting $x = y = 1$ on both sides of the above equality, we have

$$8 - 2\left(2\left.\frac{dy}{dx}\right|_{x=1} + 2\right) - 3\left.\frac{dy}{dx}\right|_{x=1} = 0.$$

Hence

$$\boxed{\left.\frac{dy}{dx}\right|_{x=1} = \frac{4}{7}},$$

which is also the slope of the tangent to the given curve at the point $(1, 1)$.

(b) Differentiate both sides of the given implicit function with respect to x as follows.

$$\begin{aligned} \frac{d}{dx}(x) &= 2\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}\left(y^3\right) \\ 1 &= 2\frac{d\left(y^2\right)}{dy}\frac{dy}{dx} - \frac{d\left(y^3\right)}{dy}\frac{dy}{dx} \\ 1 &= 4y\frac{dy}{dx} - 3y^2\frac{dy}{dx} \\ 1 &= \left(4y - 3y^2\right)\frac{dy}{dx} \end{aligned}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{4y - 3y^2}.$$

The condition that the slope of a tangent line is one can be expressed as

$$\frac{dy}{dx} = 1,$$

or

$$3y^2 - 4y + 1 = 0.$$

Solving for y gives

$$\begin{aligned} (3y-1)(y-1) &= 0 \\ y &= \frac{1}{3} \text{ or } 1. \end{aligned}$$

At $y = \dfrac{1}{3}$ (resp. $y = 1$), $x = 2\left(\dfrac{1}{3}\right)^2 - \left(\dfrac{1}{3}\right)^3 = \dfrac{5}{27}$ (resp. $x = 2 \cdot 1^2 - 1^3 = 1$). Hence the equations of the corresponding tangent lines are

$$y - \frac{1}{3} = x - \frac{5}{27} \quad \text{and} \quad y - 1 = x - 1,$$

i.e.

$$\boxed{27x - 27y + 4 = 0} \text{ and } \boxed{x - y = 0}.$$

12. (a) $\left(f^{-1}\right)'(1) = \dfrac{1}{f'\left(f^{-1}(1)\right)} = \dfrac{1}{f'(2)} = \dfrac{1}{\left.(3x^2+1)\right|_{x=2}} = \boxed{\dfrac{1}{13}}$

(b) $\left(g^{-1}\right)'(1) = \dfrac{1}{g'\left(g^{-1}(1)\right)} = \dfrac{1}{g'(0)} = \dfrac{1}{\left.(2-\sin x)\right|_{x=0}} = \boxed{\dfrac{1}{2}}$

(c) $\left(h^{-1}\right)'(2) = \dfrac{1}{h'\left(h^{-1}(2)\right)} = \dfrac{1}{h'(1)} = \dfrac{1}{\left.\frac{4x^4+12x^2}{(x^2+1)^2}\right|_{x=1}} = \boxed{\dfrac{1}{4}}$

13. (a) Since

$$g'(x) = \frac{d}{dx}\left[\left(f^{-1}(x)\right)^{-1}\right] = -\left(f^{-1}(x)\right)^{-2} \cdot \frac{1}{f'\left(f^{-1}(x)\right)},$$

we have

$$g'(7) = -\frac{1}{\left(f^{-1}(7)\right)^2 f'\left(f^{-1}(7)\right)} = -\frac{1}{1^2 f'(1)} = \boxed{-\frac{1}{9}}.$$

(b) Applying the chain rule, we have

$$h'(x) = f'\left(\cos^{-1}\frac{x}{3}\right) \cdot \frac{d}{dx}\left(\cos^{-1}\frac{x}{3}\right) = -\frac{f'\left(\cos^{-1}\frac{x}{3}\right)}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}.$$

Thus,

$$h'(0) = -\frac{f'\left(\cos^{-1} 0\right)}{3\sqrt{1-0^2}} = -\frac{1}{3} f'\left(\frac{\pi}{2}\right) = -\frac{1}{3} \cdot 15 = \boxed{-5}.$$

14. (a) $\dfrac{dy}{dx} = \dfrac{1}{x}$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$ and $\dfrac{d^3y}{dx^3} = \dfrac{2}{x^3}$

(b) $\dfrac{dy}{dx} = -\dfrac{a}{(ax+b)^2}$, $\dfrac{d^2y}{dx^2} = \dfrac{2a^2}{(ax+b)^3}$ and $\dfrac{d^3y}{dx^3} = -\dfrac{6a^3}{(ax+b)^4}$

(c) $\dfrac{dy}{dx} = xe^x$, $\dfrac{d^2y}{dx^2} = (1+x)e^x$ and $\dfrac{d^3y}{dx^3} = (2+x)e^x$

(d) $\dfrac{dy}{dx} = \sin(2x)$, $\dfrac{d^2y}{dx^2} = 2\cos(2x)$ and $\dfrac{d^3y}{dx^3} = -4\sin(2x)$

15. (a) Differentiate both sides of the given implicit function with respect to x as follows.

$$\begin{aligned}
\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(4y^2\right) &= \frac{d}{dx}(4)\\
2x+4\frac{d\left(y^2\right)}{dy}\frac{dy}{dx} &= 0\\
2x+8y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{x}{4y}
\end{aligned}$$

Differentiating both sides of the equality $2x+8y\dfrac{dy}{dx}=0$ with respect to x gives

$$\begin{aligned}
\frac{d}{dx}(2x)+\frac{d}{dx}\left(8y\frac{dy}{dx}\right) &= 0\\
2+8\left[y\frac{d}{dx}\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2\right] &= 0\\
2+8y\frac{d^2y}{dx^2}+8\left(\frac{dy}{dx}\right)^2 &= 0.
\end{aligned}$$

Hence

$$\frac{d^2y}{dx^2}=\frac{-2-8\left(\frac{dy}{dx}\right)^2}{8y}=\frac{-2-8\left(-\frac{x}{4y}\right)^2}{8y}=\frac{\frac{-4y^2-x^2}{2y^2}}{8y}=\boxed{-\frac{1}{4y^3}},$$

since $x^2+4y^2=4$. Moreover,

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)}=-\frac{1}{4\cdot 1^3}=\boxed{-\frac{1}{4}}.$$

(b) Differentiate both sides of the given implicit function with respect to x as follows.

$$\begin{aligned}
\frac{d}{dx}\left(x^3\right)-\frac{d}{dx}(3xy)+\frac{d}{dx}\left(y^3\right) &= \frac{d}{dx}(1)\\
3x^2-3\left[x\frac{dy}{dx}+y\frac{d}{dx}(x)\right]+\frac{d\left(y^3\right)}{dy}\frac{dy}{dx} &= 0\\
3x^2-3x\frac{dy}{dx}-3y+3y^2\frac{dy}{dx} &= 0\\
\left(3y^2-3x\right)\frac{dy}{dx}+3x^2-3y &= 0\\
\frac{dy}{dx} &= \frac{y-x^2}{y^2-x}
\end{aligned}$$

Differentiating both sides of the equality $\left(3y^2-3x\right)\dfrac{dy}{dx}+3x^2-3y=0$ with respect to x gives

$$\begin{aligned}
\left(3y^2-3x\right)\frac{d}{dx}\left(\frac{dy}{dx}\right)+\left[\frac{d}{dx}\left(3y^2\right)-\frac{d}{dx}(3x)\right]\frac{dy}{dx}+\frac{d}{dx}\left(3x^2\right)-\frac{d}{dx}(3y) &= 0\\
\left(3y^2-3x\right)\frac{d^2y}{dx^2}+\left[3\frac{d\left(y^2\right)}{dy}\frac{dy}{dx}-3\right]\frac{dy}{dx}+6x-3\frac{dy}{dx} &= 0\\
\left(3y^2-3x\right)\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2-6\frac{dy}{dx}+6x &= 0\\
\left(3y^2-3x\right)\frac{d^2y}{dx^2}+6y\left(\frac{y-x^2}{y^2-x}\right)^2-6\left(\frac{y-x^2}{y^2-x}\right)+6x &= 0\\
\left(3y^2-3x\right)\frac{d^2y}{dx^2}+6\left[\frac{y\left(y-x^2\right)^2-\left(y-x^2\right)\left(y^2-x\right)+x\left(y^2-x\right)^2}{\left(y^2-x\right)^2}\right] &= 0\\
3\left(y^2-x\right)\frac{d^2y}{dx^2}+\frac{6xy\left(x^3-3xy+y^3\right)}{\left(y^2-x\right)^2} &= 0.
\end{aligned}$$

Since $x^3 - 3xy + y^3 = 1$, it follows that

$$\boxed{\frac{d^2y}{dx^2} = -\frac{2xy}{(y^2 - x)^3}}$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = -\frac{2 \cdot 0 \cdot 1}{(1^2 - 0)^3} = \boxed{0}.$$

16. (a) Since

$$\frac{dy}{dx} = \frac{(x+b)\frac{d}{dx}(x+a) - (x+a)\frac{d}{dx}(x+b)}{(x+b)^2} = \frac{x+b-(x+a)}{(x+b)^2} = \frac{b-a}{(x+b)^2}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[(b-a)(x+b)^{-2}\right] = -\frac{2(b-a)}{(x+b)^3},$$

we have

$$\begin{aligned} 2\left(\frac{dy}{dx}\right)^2 + (1-y)\frac{d^2y}{dx^2} &= 2\left[\frac{b-a}{(x+b)^2}\right]^2 + \left(1 - \frac{x+a}{x+b}\right)\left[-\frac{2(b-a)}{(x+b)^3}\right] \\ &= \frac{2(b-a)^2}{(x+b)^4} - \frac{b-a}{x+b} \cdot \frac{2(b-a)}{(x+b)^3} \\ &= 0. \end{aligned}$$

(b) Write

$$\sqrt{1+x^2}\, y = 1.$$

Differentiating both sides of the above equality with respect to x gives

$$\sqrt{1+x^2}\frac{dy}{dx} + y\frac{d}{dx}\left(\sqrt{1+x^2}\right) = 0,$$

or

$$\sqrt{1+x^2}\frac{dy}{dx} + \frac{xy}{\sqrt{1+x^2}} = 0.$$

Thus,

$$\left(1+x^2\right)\frac{dy}{dx} + xy = 0.$$

Differentiating both sides of this equality with respect to x gives

$$\left(1+x^2\right)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}\left(1+x^2\right) + x\frac{dy}{dx} + y\frac{dx}{dx} = 0,$$

i.e.

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + x\frac{dy}{dx} + y = 0.$$

Hence

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0.$$