Mock Exam 4

Section A

1. *Reference: HKDSE Math M2 2015 Q1*

$$\frac{d}{dx}(x^3+2x) = \lim_{h\to 0}\frac{[(x+h)^3+2(x+h)]-(x^3+2x)}{h} \quad \text{1M}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+2x+2h-x^3-2x}{h} \quad \text{1M}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3+2h}{h}$$

$$= \lim_{h\to 0}(3x^2+3xh+h^2+2) \quad \text{1A}$$

$$= \underline{\underline{3x^2+2}} \quad \text{1A}$$

(4)

2. When $x = 0$,

$$e^0 + \ln y + 7(0)(y) = 1$$
$$\ln y = 0$$
$$y = 1 \quad \text{1A}$$

$e^x + \ln y + 7xy = 1$

Differentiate both sides with respect to x,

$$e^x + \frac{1}{y}\bullet\frac{dy}{dx} + 7x\frac{dy}{dx} + 7y = 0 \quad \text{1M}$$

$$\frac{dy}{dx} = \frac{-e^x y - 7y^2}{1+7xy} \quad \text{1A}$$

When $x = 0$ and $y = 1$,

$$\frac{dy}{dx} = \frac{-e^0(1)-7(1)^2}{1+7(0)(1)}$$

$$= \underline{\underline{-8}} \quad \text{1A}$$

(4)

3. $y = \int \frac{-4x}{(x^2+1)^2}\,dx$

Let $u = x^2+1$. Then $du = 2xdx$. 1M

$$\therefore \quad y = \int \frac{-2}{u^2}\,du$$

$$= \frac{2}{u} + C$$

$$= \frac{2}{x^2+1} + C\text{, where } C \text{ is a constant.} \quad \text{1A}$$

When $\frac{dy}{dx} = 0$,

$$\frac{-4x}{(x^2+1)^2} = 0$$

$$x = 0$$

1M

Smart Tips

Since $y = 6$ is a horizontal tangent, $\frac{dy}{dx} = 0$ when $y = 6$.

So, the horizontal tangent $y = 6$ touches the curve at $x = 0$.

$\therefore \quad 6 = \dfrac{2}{0^2 + 1} + C$

$C = 4$

$\therefore$ The equation of the curve is $y = \dfrac{2}{x^2 + 1} + 4$. 1A

(4)

4. *Reference: HKCEE A. Math 2002 Q1*

$(1 - x^2)^n + (1 + 2x)^n$

$= [1 + C_1^n(-x^2) + C_2^n(-x^2)^2 + ...] + [1 + C_1^n(2x) + C_2^n(2x)^2 + ...]$ 1M

$= 2 + 2nx + \left[4 \bullet \dfrac{n(n-1)}{2} - n\right]x^2 + ...$

The coefficient of $x^2 = 104$ 1M

$\therefore \quad 4 \bullet \dfrac{n(n-1)}{2} - n = 104$

$2n^2 - 3n - 104 = 0$ 1A

$(n - 8)(2n + 13) = 0$

$n = \underline{\underline{8}}$ or $-\dfrac{13}{2}$ (rejected) 1A

(4)

Analysis

Since the coefficient of x^2 is given, we should expand the expression as far as the term in x^2. The terms in higher order are omitted and denoted by the symbol '...'.

5. *Reference: HKDSE Math M2 PP Q9*

$\dfrac{dy}{dx} = 3x^2 - 8x + 8$ 1A

Slope of $L = -\dfrac{6}{-2} = 3$

When $\dfrac{dy}{dx} = 3$,

$3x^2 - 8x + 5 = 0$ 1M

$(3x - 5)(x - 1) = 0$

$x = 1$ or $\dfrac{5}{3}$ 1A

When $x = 1, y = -2$; when $x = \dfrac{5}{3}, y = -\dfrac{4}{27}$.

$\therefore$ The equations of the two tangents are $y + 2 = 3(x - 1)$ and $y + \dfrac{4}{27} = 3\left(x - \dfrac{5}{3}\right)$,

i.e., $3x - y - 5 = 0$ and $81x - 27y - 139 = 0$. 1A

(4)

6. (a) $\sin^2 x \cos^2 x = \left(\dfrac{1}{2}\sin 2x\right)^2$ 1M

$= \dfrac{1}{4}\left(\dfrac{1 - \cos 4x}{2}\right)$

$= \dfrac{1 - \cos 4x}{8}$ 1

Smart Tips

1. $\sin 2\theta = 2 \sin\theta \cos\theta$
2. $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

(b) $\int (\sin^4 x + \cos^4 x)\,dx$

$= \int [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]\,dx$ 1M

$= \int \left[1 - 2\left(\dfrac{1-\cos 4x}{8}\right)\right] dx$ (by (a)) 1M

$= \dfrac{1}{4}\int (3 + \cos 4x)\,dx$

$= \dfrac{3x}{4} + \dfrac{\sin 4x}{16} + C$, where C is a constant. 1A

(5)

7. *Reference: HKDSE Math M2 SP Q7*

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -4 & 3 & 9 \\ 5 & 1 & -6 & 3 \\ 3 & 2 & -5 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & 9 \\ 0 & 21 & -21 & -42 \\ 0 & 14 & -14 & -28 \end{array}\right) \quad \begin{array}{l}(R_2 - 5R_1 \to R_2, \\ R_3 - 3R_1 \to R_3)\end{array}$$ 1M

$$\sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & 9 \\ 0 & 21 & -21 & -42 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(R_3 - \frac{2}{3}R_2 \to R_3\right)$$ 1A

$$\sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & 9 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (R_2 \div 21 \to R_2)$$ 1M

We have $\begin{cases} x - 4y + 3z = 9 \\ y - z = -2 \end{cases}$. 1A

Let $z = t$, where t is any real number, then we have $y = t - 2, x = t + 1$.

The required solution is $x = t + 1, y = t - 2, z = t$, where t is any real number. 1A

(5)

8. *Reference: HKDSE Math M2 2013 Q6*

(a) Area $= \int_0^2 [1 - (x-1)^2]\,dx + \int_2^3 [(x-1)^2 - 1]\,dx$ 1M

$= \int_0^2 (2x - x^2)\,dx + \int_2^3 (x^2 - 2x)\,dx$

$= \left[x^2 - \dfrac{x^3}{3}\right]_0^2 + \left[\dfrac{x^3}{3} - x^2\right]_2^3$ 1M

$= \dfrac{8}{3}$ 1A

(b) Volume $= \pi\int_0^3 [(x-1)^2 - 1]^2\,dx$ 1M + 1A

$= \pi\int_0^3 (x^2 - 2x)^2\,dx$

$= \pi\int_0^3 (x^4 - 4x^3 + 4x^2)\,dx$

$= \pi\left[\dfrac{x^5}{5} - x^4 + \dfrac{4x^3}{3}\right]_0^3$

$= \dfrac{18\pi}{5}$ 1A

(6)

9. *Reference: HKDSE Math M2 SP Q9*

(a) The area of the parallelogram $ABEC$

$= \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= |(\mathbf{i} + 3\mathbf{j}) \times (3\mathbf{i} - 4\mathbf{k})|$ 1M

$= |-12\mathbf{i} + 4\mathbf{j} - 9\mathbf{k}|$

$= \underline{\underline{\sqrt{241}}}$ 1A

(b) The volume of the tetrahedron $ABCD$

$= \dfrac{1}{6}\left|(\overrightarrow{AB} \times \overrightarrow{AC}) \bullet \overrightarrow{AD}\right|$

$= \dfrac{1}{6}|(-12\mathbf{i} + 4\mathbf{j} - 9\mathbf{k}) \bullet (4\mathbf{i} - 3\mathbf{j} + \mathbf{k})|$ 1M

$= \underline{\underline{\dfrac{23}{2}}}$ 1A

(c) D' can be any point on the plane $FDGH$ except D. Take $D' = F$.

$\overrightarrow{AD'} = \overrightarrow{AF}$

$= \overrightarrow{AC} + \overrightarrow{CF}$

$= \overrightarrow{AC} + \overrightarrow{AD}$ 1M

$= (3\mathbf{i} - 4\mathbf{k}) + (4\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

$= \underline{\underline{7\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}}}$ (or other reasonable answers) 1A

(6)

10. (a) For $n = 1$,

L.H.S. $= M = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} =$ R.H.S.

$\therefore$ The proposition is true for $n = 1$.

Next, assume the proposition is true for $n = k$, where k is a positive integer, i.e.,

$M^k = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}$. 1

When $n = k + 1$,

L.H.S. $= M^{k+1}$

$= MM^k$

$= \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}\begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}$

$= \begin{pmatrix} \cos x\cos kx - \sin x\sin kx & -\cos x\sin kx - \sin x\cos kx \\ \sin x\cos kx + \cos x\sin kx & -\sin x\sin kx + \cos x\cos kx \end{pmatrix}$

$= \begin{pmatrix} \cos(k+1)x & -\sin(k+1)x \\ \sin(k+1)x & \cos(k+1)x \end{pmatrix} =$ R.H.S. 1

$\therefore$ The proposition is also true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(b) (i) Since $A = r\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$,

we have $\begin{cases} r\cos x = -\sqrt{3} \text{.......(1)} \\ r\sin x = -1 \text{..........(2)} \end{cases}$ 1M

(2) ÷ (1),

$\tan x = \dfrac{1}{\sqrt{3}}$

$x = \underline{\underline{210°}}$

Substituting $x = 210°$ into (1),

$r\cos 210° = -\sqrt{3}$

$r = \underline{\underline{2}}$ 1A

(ii) $A^n = 2^n\begin{pmatrix} \cos 210° & -\sin 210° \\ \sin 210° & \cos 210° \end{pmatrix}^n$

$= 2^n\begin{pmatrix} \cos(210n)° & -\sin(210n)° \\ \sin(210n)° & \cos(210n)° \end{pmatrix}$

A^n is a 2×2 diagonal matrix when $2^n\sin(210n)° = 0$. 1M

When $\sin(210n)° = 0$,

$210n = 180m$, where m is an integer.

$n = \dfrac{6}{7}m$

For the least positive integer n,

$n = \dfrac{6}{7}(7)$

$= \underline{\underline{6}}$ 1A

When $n = 6$, we have

$A^6 = 2^6\begin{pmatrix} \cos 1260° & 0 \\ 0 & \cos 1260° \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} -64 & 0 \\ 0 & -64 \end{pmatrix}}}$ 1A

(8)

Smart Tips

Since $\cos x < 0$ and $\sin x < 0$, x should be in quadrant III.

Smart Tips

If m is a multiple of 7, then n is an integer. To make n the least positive integer, we take $m = 7$.

Section B

11. *Reference: HKALE P. Math 2011 Paper 1 Q8*

(a) $|P| = -y - x = -(x + y)$ 1M

Since $xy > 0$, x and y are both positive or both negative,

so $x + y \neq 0$ and $|P| \neq 0$.

Therefore, P is invertible. 1

(2)

Smart Tips

A matrix A is invertible if and only if $|A| \neq 0$.

(b) $P^{-1} = \dfrac{-1}{x+y}\begin{pmatrix} -1 & -1 \\ -x & y \end{pmatrix} = \dfrac{1}{x+y}\begin{pmatrix} 1 & 1 \\ x & -y \end{pmatrix}$ 1A

$$P^{-1}MP = \frac{1}{x+y}\begin{pmatrix} 1 & 1 \\ x & -y \end{pmatrix}\begin{pmatrix} 3+x & -y \\ -x & 3+y \end{pmatrix}\begin{pmatrix} y & 1 \\ x & -1 \end{pmatrix}$$

$$= \frac{1}{x+y}\begin{pmatrix} 3 & 3 \\ x(3+x+y) & -y(3+x+y) \end{pmatrix}\begin{pmatrix} y & 1 \\ x & -1 \end{pmatrix}$$ 1A

$$= \frac{1}{x+y}\begin{pmatrix} 3(x+y) & 0 \\ 0 & (x+y)(3+x+y) \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 3 & 0 \\ 0 & 3+x+y \end{pmatrix}}}$$ 1A

$$P^{-1}M^nP = \overbrace{(P^{-1}MP)(P^{-1}MP)\dots(P^{-1}MP)}^{n \text{ times}}$$

$$= (P^{-1}MP)^n$$ 1M

$$= \underline{\underline{\begin{pmatrix} 3^n & 0 \\ 0 & (3+x+y)^n \end{pmatrix}}}$$ 1A

(5)

(c) Note that $A = \begin{pmatrix} 3+(-1) & -(-3) \\ -(-1) & 3+(-3) \end{pmatrix}$ and $(-1)(-3) > 0$. 1M

Analysis

Consider the values of x and y in order to apply the result of (b).

Putting $M = A$ in (b),

$$\left.\begin{aligned} P^{-1}A^{2n}P &= \begin{pmatrix} 3^{2n} & 0 \\ 0 & (3-1-3)^{2n} \end{pmatrix}, \text{ where } P = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{2n} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}\right\}$$ 1M

$$A^{2n} = P\begin{pmatrix} 3^{2n} & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$ 1M

$$= \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3^{2n} & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{1}{4}\begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}\right]$$ 1M

$$= \frac{1}{4}\begin{pmatrix} -3^{2n+1} & 1 \\ -3^{2n} & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{4}\begin{pmatrix} 3^{2n+1}+1 & 3^{2n+1}-3 \\ 3^{2n}-1 & 3^{2n}+3 \end{pmatrix}}}$$ 1A

(5)

12. (a) Since C_1 passes through (0, 0), $0 = \dfrac{p(0)+q}{2(0)+r}$,
i.e., $q = 0$. 1M

When $2x + r = 0, x = -\dfrac{r}{2}$.
The equation of the vertical asymptote of C_1 is $x = -\dfrac{r}{2}$.
Since it is given that the vertical asymptote of C_1 is $x = -\dfrac{3}{2}$,
$$-\frac{r}{2} = -\frac{3}{2}$$
$\therefore \quad r = 3$ 1M

$$y = \frac{px}{2x+3}$$
$$= \frac{\frac{p}{2}(2x+3) - \frac{3p}{2}}{2x+3}$$
$$= \frac{p}{2} - \frac{\frac{3p}{2}}{2x+3}$$

$\therefore$ The horizontal asymptote is $y = \dfrac{p}{2}$.
$\therefore \quad \dfrac{p}{2} = -1$
$p = -2$ 1M

The equation of C_1 is $y = \dfrac{-2x}{2x+3}$. 1A
(4)

(b) The equation C_2 is $y = \dfrac{2x+3}{-2x}$.
$$\frac{-2x}{2x+3} = \frac{2x+3}{-2x}$$
$$4x^2 + 12x + 9 = 4x^2$$
$$4x + 3 = 0$$
$$x = -\frac{3}{4}$$ 1M

When $x = -\dfrac{3}{4}$, $y = 1$.

$\therefore$ The coordinates of the point of intersection are $\left(-\dfrac{3}{4}, 1\right)$. 1A
(2)

(c) $\frac{d}{dx}\left(\frac{-2x}{2x+3}\right) = \frac{(2x+3)(-2)-(-2x)(2)}{(2x+3)^2}$ 1M

$= -\frac{6}{(2x+3)^2} < 0$ for $x \neq -\frac{3}{2}$ 1

$\frac{d}{dx}\left(\frac{2x+3}{-2x}\right) = \frac{(-2x)(2)-(2x+3)(-2)}{(-2x)^2}$

$= \frac{6}{4x^2} > 0$ for $x \neq 0$ 1

(3)

(d) Since the equation of C_2 is $y = \frac{2x+3}{-2x}$,

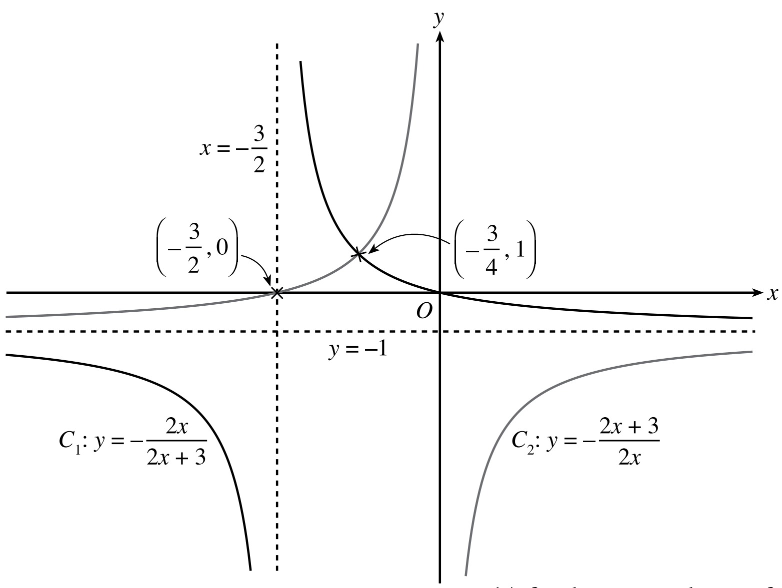
$y = \frac{2x+3}{-2x} = -1 - \frac{3}{2x}$

The vertical asymptote of C_2 is $x = 0$. 1A

The horizontal asymptote of C_2 is $y = -1$. 1A

(2)

(e)



1A for the correct shapes of C_1 and C_2

1A for the correct asymptotes, intercepts and points of intersection

1A for all correct

(3)

13. *Reference: HKALE P. Math 2011 Paper 2 Q8*

(a) Let $\tan x = 2\tan\theta$.

Then $\sec^2 x\, dx = 2\sec^2\theta\, d\theta$. 1M

$$\int \frac{dx}{1+3\cos^2 x} = \int \frac{\sec^2 x dx}{\sec^2 x + 3}$$

$$= \int \frac{\sec^2 x dx}{\tan^2 x + 4} \quad \text{1M}$$

$$= \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta + 4}$$

$$= \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta} \quad \text{1M}$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{\theta}{2} + C \quad \text{1A}$$

$$= \underline{\underline{\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right)}} + C\text{, where } C \text{ is a constant} \quad \text{1A}$$

(5)

(b) Let $y = 2\pi - x$. Then $dy = -dx$.

When $x = \frac{3\pi}{4}$, $y = \frac{5\pi}{4}$.

When $x = \frac{5\pi}{4}$, $y = \frac{3\pi}{4}$.

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln(1+e^{\sin x})\, dx = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} f(2\pi - y)\ln[1+e^{\sin(2\pi-y)}](-1)\, dy$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y)\ln(1+e^{-\sin y})\, dy \quad \text{1M}$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(y)\ln\frac{1+e^{\sin y}}{e^{\sin y}}\, dy$$

$$= -\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)[\ln(1+e^{\sin x}) - \ln e^{\sin x}]\, dx \quad \text{1M}$$

Therefore, $2\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln(1+e^{\sin x})\, dx = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\sin x dx$.

$\therefore \quad \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\ln(1+e^{\sin x})\, dx = \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} f(x)\sin x dx$ 1

(3)

WatchOut

The upper and lower limits should be changed accordingly when substitution takes place.

(c) Let $g(x) = \dfrac{\sin x}{1 + 3\cos^2 x}$.

$g(2\pi - x) = \dfrac{\sin(2\pi - x)}{1 + 3\cos^2(2\pi - x)} = \dfrac{-\sin x}{1 + 3\cos^2 x} = -g(x)$ 1M

Smart Tips

The result of (b) is based on $f(2\pi - x) = -f(x)$. Therefore, we must check that $g(2\pi - x) = -g(x)$ to apply the result.

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin x \ln(1 + e^{\sin x})}{1 + 3\cos^2 x}\,dx = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin x}{1 + 3\cos^2 x}\ln(1 + e^{\sin x})\,dx$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\sin^2 x}{1 + 3\cos^2 x}\,dx \quad \text{(by (b))} \qquad \text{1A}$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1 - \cos^2 x}{1 + 3\cos^2 x}\,dx$$

$$= \frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\frac{4}{3} - \frac{1}{3}(1 + 3\cos^2 x)}{1 + 3\cos^2 x}\,dx \qquad \text{1M}$$

$$= \frac{2}{3}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1 + 3\cos^2 x}\,dx - \frac{1}{6}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} dx$$

$$= \frac{2}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right)\right]_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} - \frac{1}{6}\left(\frac{\pi}{2}\right) \quad \text{(by (a))}$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{1}{2}\right)\right] - \frac{\pi}{12}$$

$= \underline{\underline{0.0473}}$ *(cor. to 3 sig. fig.)* 1A

(4)

14. *Reference: HKCEE A. Math 2009 Q14*

(a) $\overrightarrow{PV} = \underline{\underline{\mathbf{a} + \mathbf{b}}}$ 1A

(1)

(b) $\overrightarrow{PR} = \mathbf{a} + \dfrac{3}{2}\mathbf{b}$

$$\overrightarrow{PT} = \frac{r\left(\mathbf{a} + \frac{3}{2}\mathbf{b}\right) + 1(k\mathbf{a})}{r + 1} \qquad \text{1M}$$

$$= \underline{\underline{\frac{r + k}{r + 1}\mathbf{a} + \frac{3r}{2(r + 1)}\mathbf{b}}} \qquad \text{1A}$$

Since $PT \parallel PV$,

$$\frac{\frac{r + k}{r + 1}}{1} = \frac{\frac{3r}{2(r + 1)}}{1} \qquad \text{1M}$$

$2(r + k) = 3r$

$r = \underline{\underline{2k}}$ 1A

(4)

(c) (i) $\overrightarrow{RQ} = k\mathbf{a} - \left(\mathbf{a} + \frac{3}{2}\mathbf{b}\right) = (k-1)\mathbf{a} - \frac{3}{2}\mathbf{b}$

Since $RQ \perp PV$, $\overrightarrow{RQ} \cdot \overrightarrow{PV} = 0$. 1M

$$(k-1)|\mathbf{a}|^2 - \frac{3}{2}|\mathbf{b}|^2 = 0$$ 1M

$$(k-1)|\mathbf{a}|^2 - \frac{3}{2}|\mathbf{a}|^2 = 0$$

$$k - 1 - \frac{3}{2} = 0$$

$$k = \frac{5}{2}$$ 1A

$r = 2\left(\frac{5}{2}\right) = 5$ (by (b))

$$\overrightarrow{PT} = \frac{5 + \frac{5}{2}}{5+1}\mathbf{a} + \frac{3(5)}{2(5+1)}\mathbf{b}$$

$$= \underline{\underline{\frac{5}{4}(\mathbf{a} + \mathbf{b})}}$$ 1A

Smart Tips

Since V is the orthocentre, PVT is the altitude with respect to the base RQ.

(ii) $\overrightarrow{QV} = (\mathbf{a} + \mathbf{b}) - \frac{5}{2}\mathbf{a}$

$$= -\frac{3}{2}\mathbf{a} + \mathbf{b}$$ 1A

Let $\frac{RU}{UP} = s$.

$$\overrightarrow{QU} = \frac{s\overrightarrow{QP} + \overrightarrow{QR}}{s+1}$$

$$= \frac{-\frac{5s}{2}\mathbf{a} + \left(-\frac{3}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}\right)}{s+1}$$ 1M

$$= -\frac{5s+3}{2(s+1)}\mathbf{a} + \frac{3}{2(s+1)}\mathbf{b}$$

Since Q, V and U are collinear, we have

$$\frac{-\frac{5s+3}{2(s+1)}}{-\frac{3}{2}} = \frac{\frac{3}{2(s+1)}}{1}$$

$$5s + 3 = \frac{9}{2}$$

$$s = \frac{3}{10}$$

$\therefore \quad \frac{RU}{UP} = \underline{\underline{\frac{3}{10}}}$ 1A

(7)