MATHEMATICS EXTENDED PART MODULE 2 (ALGEBRA AND CALCULUS) MARKING SCHEME

General Marking Instructions

1. It is very important that all teachers should adhere as closely as possible to the marking scheme. In many cases, however, students will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Teachers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of teachers, the marking scheme was written as detailed as possible. However, it is still likely that students would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, teachers should exercise their discretion in marking students' work. In general, marks for a certain step should be awarded if students' solution indicated that the relevant concept/technique had been used.

4. In marking students' work, the benefit of doubt should be given in the students' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are **shaded** whereas alternative answers are enclosed with $\boxed{\text{rectangles}}$.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

	Solution	Marks	Remarks
1.	$(1+ax)^n = 1 + C_1^n(ax) + C_2^n(ax)^2 + \cdots$		
	$= 1 + nax + \dfrac{n(n-1)a^2}{2}x^2 + \cdots$	1M	
	$na = 4\sqrt{3}$ ……(1) and $\dfrac{n(n-1)a^2}{2} = 18$ ……(2)	1M	
	From (1), $a = \dfrac{4\sqrt{3}}{n}$ …….(3)		
	Put (3) into (2), $\dfrac{n(n-1)48}{2\cdot n^2} = 18$		
	$4(n-1) = 3n$		
	$n = 4$	1A	
	$4a = 4\sqrt{3}$		
	$a = \sqrt{3}$	1A ------ (4)	
2.	$\dfrac{d}{dx}\left(\sqrt{1+x^3}\right)$		
	$= \lim\limits_{h\to 0}\dfrac{\sqrt{1+(x+h)^3}-\sqrt{1+x^3}}{h}$	1M	
	$= \lim\limits_{h\to 0}\dfrac{\left(\sqrt{1+(x+h)^3}-\sqrt{1+x^3}\right)\left(\sqrt{1+(x+h)^3}+\sqrt{1+x^3}\right)}{h\left(\sqrt{1+(x+h)^3}+\sqrt{1+x^3}\right)}$	1M	
	$= \lim\limits_{h\to 0}\dfrac{1+x^3+3hx^2+3h^2x+h^3-(1+x^3)}{h\left(\sqrt{1+(x+h)^3}+\sqrt{1+x^3}\right)}$	1M	
	$= \lim\limits_{h\to 0}\dfrac{h(3x^2+3hx+h^2)}{h\left(\sqrt{1+(x+h)^3}+\sqrt{1+x^3}\right)}$		
	$= \lim\limits_{h\to 0}\dfrac{3x^2+3hx+h^2}{\sqrt{1+(x+h)^3}+\sqrt{1+x^3}}$	1M	Withhold 1M if the step is skipped.
	$= \dfrac{3x^2}{2\sqrt{1+x^3}}$	1A ------ (5)	

Solution	Marks	Remarks
3. (a) $\tan\left(\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}\right)$		
$=\dfrac{\tan\left(\tan^{-1}\frac{1}{n}\right)-\tan\left(\tan^{-1}\frac{1}{n+1}\right)}{1+\tan\left(\tan^{-1}\frac{1}{n}\right)\tan\left(\tan^{-1}\frac{1}{n+1}\right)}$	1M	
$=\dfrac{\frac{1}{n}-\frac{1}{n+1}}{1+\frac{1}{n(n+1)}}$	1	
$=\dfrac{n+1-n}{n(n+1)+1}$		
$=\dfrac{1}{n^2+n+1}$		
$\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}=\tan^{-1}\frac{1}{n^2+n+1}$	1	f.t.
(b) $\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{3}$		
$\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{7}$		
$\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{1}{13}$		
…		
$\tan^{-1}\frac{1}{m}-\tan^{-1}\frac{1}{m+1}=\tan^{-1}\frac{1}{m^2+m+1}$	1M	
Summing them up,		
$\tan^{-1}1-\tan^{-1}\frac{1}{m+1}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\ldots+\tan^{-1}\frac{1}{m^2+m+1}$	1M	
$\frac{\pi}{4}-\tan^{-1}\frac{1}{m+1}=$ $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\cdots+\tan^{-1}\frac{1}{m^2+m+1}$	1	f.t.
	------ (6)	

Solution	Marks	Remarks
4. Slope of tangents $=-\frac{1}{60}$		
$6x+5(x\frac{dy}{dx}+y)+16y\frac{dy}{dx}=0$		
$(5x+16y)\frac{dy}{dx}=-(6x+5y)$		
$\frac{dy}{dx}=-\frac{6x+5y}{5x+16y}$	1A	
Let the point(s) of contact be (x_0, y_0).		
$-\frac{6x_0+5y_0}{5x_0+16y_0}=-\frac{1}{60}$	1M	
$360x_0+300y_0=5x_0+16y_0$		
$355x_0=-284y_0$		
$y_0=-\frac{5x_0}{4}$		
(x_0, y_0) is on the curve,		
$3x_0^2+5x_0y_0+8y_0^2=148$		
$3x_0^2+5x_0\left(-\frac{5x_0}{4}\right)+8\left(-\frac{5x_0}{4}\right)^2=148$	1M	
$37x_0^2=592$		
$x_0=\pm4$		
When $x_0=4$, $y_0=-5$. When $x_0=-4$, $y_0=5$.	1A	
The equations of the tangents are		
$y-(-5)=-\frac{1}{60}(x-4)$ and $y-5=-\frac{1}{60}[x-(-4)]$	1M	
$x+60y+296=0$ and $x+60y-296=0$	1A	
	------ (6)	

Solution	Marks	Remarks
5. (a) Note that $\frac{1}{(4)(9)(14)}=\frac{1}{504}=\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(9)(14)}\right)$.		
So, the statement is true for $n=1$.	1	
Assume that $\sum_{k=1}^{m}\frac{1}{(5k-1)(5k+4)(5k+9)}=\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(5m+4)(5m+9)}\right)$ for some is a positive integer m.	1M	
$\sum_{k=1}^{m+1}\frac{1}{(5k-1)(5k+4)(5k+9)}$		
$=\sum_{k=1}^{m}\frac{1}{(5k-1)(5k+4)(5k+9)}+\frac{1}{(5m+4)(5m+9)(5m+14)}$		
$=\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(5m+4)(5m+9)}\right)+\frac{1}{(5m+4)(5m+9)(5m+14)}$ (by induction assumption)	1M	for using induction assumption
$=\frac{1}{10}\left(\frac{1}{36}-\frac{5m+14-10}{(5m+4)(5m+9)(5m+14)}\right)$		
$=\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(5m+9)(5m+14)}\right)$		
So, the statement is true for $n=m+1$ if it is true for $n=m$. By mathematical induction, the statement is true for all positive integers n.	1	
	------ (4)	
(b) $\sum_{k=13}^{36}\frac{5\,520}{(5k-1)(5k+4)(5k+9)}$		
$=5\,520\left[\sum_{k=1}^{36}\frac{1}{(5k-1)(5k+4)(5k+9)}-\sum_{k=1}^{12}\frac{1}{(5k-1)(5k+4)(5k+9)}\right]$	1M	
$=5\,520\left[\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(184)(189)}\right)-\frac{1}{10}\left(\frac{1}{36}-\frac{1}{(64)(69)}\right)\right]$	1M	
$=\frac{55}{504}$	1A	
	------ (3)	

Solution	Marks	Remarks
6. (a) Let $u=\sqrt{x}$.		
$u^2=x$		
$2u\mathrm{d}u=\mathrm{d}x$	1M	
$\int \sqrt{x}e^{\sqrt{x}}\mathrm{d}x$		
$=\int ue^u\cdot 2u\mathrm{d}u$	1	
$=2\int u^2e^u\mathrm{d}u$		
$=2\left(u^2e^u-\int e^u\cdot 2u\mathrm{d}u\right)$	1M	
$=2u^2e^u-4\left(ue^u-\int e^u\mathrm{d}u\right)$		
$=2u^2e^u-4ue^u+4e^u+C$		
$=2e^{\sqrt{x}}\left(x-2\sqrt{x}+2\right)+C$	1A	
(b) When $y=2e^2$,		
$2e^2=\sqrt{x}e^{\sqrt{x}}$		
$\sqrt{x}=2$		
$x=4$		
The required area		
$=(4)(2e^2)-\int_0^4\sqrt{x}e^{\sqrt{x}}\mathrm{d}x$	1A	
$=8e^2-\left[2e^{\sqrt{x}}\left(x-2\sqrt{x}+2\right)\right]_0^4$	1M	
$=8e^2-(4e^2-4)$		
$=4(e^2+1)$	1A	

Solution	Marks	Remarks
7. (a) $P^{-1} = \frac{1}{9-8}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$		
$= \begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$	1A	
$P^{-1}AP = \begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}\begin{pmatrix}26 & -18\\ 36 & -25\end{pmatrix}\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}$		
$= \begin{pmatrix}6 & -4\\ 4 & -3\end{pmatrix}\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}$		
$= \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}$	1A	
(b)(i) $(P^{-1}AP)^n = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}^n$		
$P^{-1}A^nP = \begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}$	1M	
$A^n = P\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}P^{-1}$		
$= \begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$		
$= \begin{pmatrix}3\cdot 2^n & 2(-1)^n\\ 4\cdot 2^n & 3(-1)^n\end{pmatrix}\begin{pmatrix}3 & -2\\ -4 & 3\end{pmatrix}$		
$= \begin{pmatrix}9\cdot 2^n - 8(-1)^n & -6\cdot 2^n + 6(-1)^n\\ 12\cdot 2^n - 12(-1)^n & -8\cdot 2^n + 9(-1)^n\end{pmatrix}$	1A	
(b)(ii) $A^n = 2^n\begin{pmatrix}9 & -6\\ 12 & -8\end{pmatrix} + (-1)^n\begin{pmatrix}-8 & 6\\ -12 & 9\end{pmatrix}$		
$A^n - (-1)^nI = 2^n\begin{pmatrix}9 & -6\\ 12 & -8\end{pmatrix} + (-1)^n\begin{pmatrix}-9 & 6\\ -12 & 8\end{pmatrix}$		
$= [2^n - (-1)^n]\begin{pmatrix}9 & -6\\ 12 & -8\end{pmatrix}$	1M	
For any positive integer n, 2^n is even and $(-1)^n$ is odd,		
$\therefore$ $m = 2^n - (-1)^n$ is an odd integer.	1M	
The claim is justified.	1	f.t.
	------ (7)	

Solution	Marks	Remarks
8. (a) $f'(x) = \frac{(x+6)(2x+a) - (x^2+ax+10)}{(x+6)^2}$	1M	
$= \frac{x^2+12x+(6a-10)}{(x+6)^2}$		
$f'(-10) = 0$		
$\therefore \frac{(-10)^2+12(-10)+(6a-10)}{(-10+6)^2} = 0$	1M	
$6a - 30 = 0$		
$a = 5$		
$f'(x) = \frac{x^2+12x+20}{(x+6)^2}$	1A	
(b) $x + 6 = 0$ is a vertical asymptote.	1A	
$f(x) = x - 1 + \frac{16}{x+6}$	1M	
$y = x - 1$ is a vertical asymptote	1A	
(c) $f''(x) = \frac{(x+6)^2(2x+12) - (x^2+12x+20)(2)(x+6)}{(x+6)^4}$		
$= \frac{2(x^2+12x+36) - 2(x^2+12x+20)}{(x+6)^3}$		
$= \frac{32}{(x+6)^3}$	1A	
As $x \neq -6$, $f''(x)$ does not change sign.		
$\therefore$ G has no points of inflexion.		
The claim is justified.	1	f.t.
	------ (8)	

Solution	Marks	Remarks
9. (a)(i) Let $x = a\sin\theta$.		
$dx = a\cos\theta\, d\theta$		
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$		
$= \int \frac{1}{a\cos\theta}\cdot a\cos\theta\, d\theta$	1M	
$= \int \theta\, d\theta$		
$= \theta + C$		
$= \sin^{-1}\left(\frac{x}{a}\right) + C$	1A	
(a)(ii) $\int \sqrt{a^2-x^2}\,dx$		
$= \left(\sqrt{a^2-x^2}\right)(x) - \int x\cdot\frac{1(-2x)}{2\sqrt{a^2-x^2}}\,dx$		
$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$		
$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$	1M	
$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\sin^{-1}\left(\frac{x}{a}\right)$		
$\therefore\ 2\int\sqrt{a^2-x^2}\,dx = a^2\sin^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2} + C$		
$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$	1	f.t.
	------ (4)	
(b)(i) Let $y = -x$.		
When $x = -a$, $y = a$. When $x = 0$, $y = 0$.		
$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$		
$= \int_{a}^{0} f(-y)(-dy) + \int_{0}^{a} f(x)\,dx$	1M	
$= -\int_{a}^{0} -f(y)\,dy + \int_{0}^{a} f(x)\,dx$		
$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$		
$= 0$	1	f.t.
(b)(ii) $\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-y)(-dy) + \int_{0}^{a} f(x)\,dx$		
$= -\int_{a}^{0} f(y)\,dy + \int_{0}^{a} f(x)\,dx$		
$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$		
$= 2\int_{0}^{a} f(x)\,dx$	1	f.t.
	------ (3)	

Solution	Marks	Remarks
9. (c) Note that $\sqrt{4-(-x)^2} = \sqrt{4-x^2}$ for $-\sqrt{3} \le x \le \sqrt{3}$.	1M	Withhold 1M if the step is skipped.
The volume of the required solid		
$= \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left[(4-x^2)^{\frac{1}{4}}\right]^2 dx$	1A	
$= \pi\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\,dx$		
$= 2\pi\int_{0}^{\sqrt{3}} \sqrt{4-x^2}\,dx$ (by (b)(ii))		
$= 2\pi\left[\frac{2^2}{2}\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\right]_0^{\sqrt{3}}$ (by (a)(ii))	1M+1A	1M for using the result of (a)(ii)
$= 2\pi\left(2\sin^{-1}\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\sqrt{4-3}\right)$		
$= \frac{(4\pi + 3\sqrt{3})\pi}{3}$ cubic units	1A	
	------ (5)	

Solution	Marks	Remarks
10. (a) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$		
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2a}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$	1A	
LHS $=a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$		
$=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$		
RHS $=\sqrt{1+\left[\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)\right]^2}$		
$=\frac{1}{2}\sqrt{4+\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)^2}$		
$=\frac{1}{2}\sqrt{\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2}=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$		
$\therefore$ it satisfies the condition.	1	f.t.
P(0, 1) is on the curve.		
$1=\frac{a}{2}\left(e^0+e^0\right)$		
$a=1$	1A ------ (3)	
(b)(i) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(e^x-e^{-x}\right)$		
Let $N=(0,\alpha)$.		
$\frac{\alpha-y}{0-x}\cdot\frac{1}{2}\left(e^x-e^{-x}\right)=-1$	1M	
$\alpha-y=\frac{2x}{e^x-e^{-x}}$		
R		
$=NM$		
$=\sqrt{(0-x)^2+(\alpha-y)^2}$	1M	
$=\sqrt{x^2+\frac{4x^2}{(e^x-e^{-x})^2}}$		
$=\frac{x}{e^x-e^{-x}}\sqrt{(e^x-e^{-x})^2+4}$		
$=\frac{x(e^x+e^{-x})}{e^x-e^{-x}}$		
$=\frac{x(e^{2x}+1)}{e^{2x}-1}$	1	f.t.

Solution	Marks	Remarks
10. (b)(ii)(1) $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{2}\left(e^x-e^{-x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$		
$2=\left(\frac{e^{2x}-1}{2e^x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$	1M	
$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{4e^x}{e^{2x}-1}$	1M	
$\frac{\mathrm{d}A}{\mathrm{d}t}$		
$=2\pi R\frac{\mathrm{d}R}{\mathrm{d}t}$		
$=2\pi R\cdot\frac{(e^{2x}-1)[e^{2x}+1+x(2e^{2x})]-x(e^{2x}+1)(2e^{2x})}{(e^{2x}-1)^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$	1M	
$=2\pi\cdot\frac{x(e^{2x}+1)}{e^{2x}-1}\cdot\frac{e^{4x}-1-4xe^{2x}}{(e^{2x}-1)^2}\cdot\frac{4e^x}{e^{2x}-1}$		
$=\frac{8\pi xe^x(e^{2x}+1)(e^{4x}-4xe^{2x}-1)}{(e^{2x}-1)^4}$	1	f.t.
(b)(ii)(2) $t=0.125$		
$y=1+0.125\times 2=1.25$		
$\frac{1}{2}\left(e^x+e^{-x}\right)=1.25$	1M	
$e^{2x}-2.5e^x+1=0$		
$e^x=2$ or 0.5		
$x=\ln 2$ or $-\ln 2$ (rejected)	1A	
$\frac{\mathrm{d}A}{\mathrm{d}t}$		
$=\frac{8\pi(\ln 2)(2)(4+1)(16-4(\ln 2)(4)-1)}{(4-1)^4}$		
$=\frac{80\pi(\ln 2)(15-16\ln 2)}{81}$ sq. units/s	1A ------ (10)	

Solution	Marks	Remarks

11. (a) For unique solution,

$$\begin{vmatrix} 4 & a & 3 \\ 1 & -1 & 2 \\ 2 & 3 & -a \end{vmatrix} \neq 0$$ — 1M

$4a + 4a + 9 - (-6 - a^2 + 24) \neq 0$ — 1A

$a^2 + 8a - 9 \neq 0$

$a \neq 1$ and $a \neq -9$

(E) has a unique solution for all real numbers except 1 and -9. — 1A

------ (3)

(b)(i) For $a = 1$,

$$\left(\begin{array}{ccc|c} 4 & 1 & 3 & k \\ 1 & -1 & 2 & 2 \\ 2 & 3 & -1 & k-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 4 & 1 & 3 & k \\ 2 & 3 & -1 & k-4 \end{array}\right)$$ — 1M

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 5 & -5 & k-8 \\ 0 & 5 & -5 & k-8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 5 & -5 & k-8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(E) is consistent for all values of k. — 1 — f.t.

(b)(ii) For $k = 3$,

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 5 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$(x, y, z) = (1 - t, t - 1, t)$ where $t \in \mathbf{R}$ — 1A

$(1 - t)^2 + (t - 1)^2 + t^2 \leq 9$ — 1M

$3t^2 - 4t - 7 \leq 0$

$-1 \leq t \leq \dfrac{7}{3}$ — 1A

For x, y, z being integers, t must be an integer.

$\therefore t = -1, 0, 1$ or 2 — 1M

$(x, y, z) = (2, -2, -1), (1, -1, 0), (0, 0, 1)$ or $(-1, 1, 2)$ — 1A

------ (7)

(c)(i)

$$\left(\begin{array}{ccc|c} 4 & -9 & 3 & k \\ 1 & -1 & 2 & 2 \\ 2 & 3 & 9 & k-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 4 & -9 & 3 & k \\ 2 & 3 & 9 & k-4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & -5 & -5 & k-8 \\ 0 & 5 & 5 & k-8 \end{array}\right)$$ — 1M

(E) is consistent

$\therefore k - 8 = -(k - 8)$

$k = 8$ — 1A

(c)(ii)

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & -5 & -5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$(x, y, z) = (2 - 3t, -t, t)$ where $t \in \mathbf{R}$ — 1A

------ (3)

Solution	Marks	Remarks

12. (a) $\dfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right) = \dfrac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OD}\right)$

$(a + 17)\mathbf{i} + 8\mathbf{j} + (c - 3)\mathbf{k} = 23\mathbf{i} + (b + 5)\mathbf{j} - 12\mathbf{k}$ — 1M

$a + 17 = 23$, $8 = b + 5$, $c - 3 = -12$

$a = 6$, $b = 3$, $c = -9$ — 1A

------ (2)

(b) $\overrightarrow{OA} \cdot \overrightarrow{AD}$

$= (6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j})$

$= 0$ — 1A

------ (1)

(c) $\overrightarrow{AB} \times \overrightarrow{AD}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & -3 & -6 \\ 1 & -1 & 0 \end{vmatrix}$$

$= -6\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}$ — 1A

------ (1)

(d)(i) The volume of the pyramid $OABCD$

$= \dfrac{1}{3}\left|\overrightarrow{OA} \cdot (\overrightarrow{AB} \times \overrightarrow{AD})\right|$ — 1M

$= \dfrac{1}{3}\left|(6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) \cdot (-6\mathbf{i} - 6\mathbf{j} - 7\mathbf{k})\right|$

$= 17$ cubic units — 1A

------ (4)

(b)(ii) Area of $ABCD$ $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

$= |-6\mathbf{i} - 6\mathbf{j} - 7\mathbf{k}|$

$= \sqrt{(-6)^2 + (-6)^2 + (-7)^2}$

$= 11$ sq. units — 1M

The shortest distance from O to the plane $ABCD$

$= \dfrac{3 \times 17}{11}$

$= \dfrac{51}{11}$ — 1A

$OA = |6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}|$

$= \sqrt{6^2 + 6^2 + (-3)^2}$

$= 9$ — 1M

Let θ be the angle between OA and the plane $ABCD$.

$\sin\theta = \dfrac{51}{11} \div 9 = \dfrac{51}{99}$

$\theta = \sin^{-1}\dfrac{51}{99}$

The angle between OA and the plane $ABCD$ is $\sin^{-1}\dfrac{51}{99}$. — 1A

	Solution	Marks	Remarks
12.	(d)(iii) $\overrightarrow{OA} \cdot \overrightarrow{AD} = 0$		
	$\therefore \angle OAD = 90°$	1M	
	$\therefore \angle EAD = 90°$		
	The angle between ΔOAD and the plane $ABCD$ is $\angle OAE$.		
	The claim is justified.	1A	
		------ (8)	

