Updated to 06-05-2019

Mathematics
Module 1
HKDSE
2012–2019

Marking Scheme
& Marker’s Report

Content

1 – Binomial

1. Let n be a positive integer.

   (a) Expand $(1+3x)^n$ in ascending powers of x up to the term x^2.

   (b) It is given that the coefficient of x^2 in the expansion of $e^{-2x}(1+3x)^n$ is 62. Find the value of n.

(4 marks)

[HKDSE 2012' Section A#1]

1. (a) $(1+3x)^n = 1 + C_1^n(3x) + C_2^n(3x)^2 + \cdots$		
$= 1 + 3nx + \dfrac{9n(n-1)}{2}x^2 + \cdots$	1A	
(b) $e^{-2x}(1+3x)^n = \left[1+(-2x)+\dfrac{(-2x)^2}{2!}+\cdots\right]\left[1+3nx+\dfrac{9n(n-1)}{2}x^2+\cdots\right]$	1A	For $1+(-2x)+\dfrac{(-2x)^2}{2!}+\cdots$
$= (1-2x+2x^2+\cdots)\left[1+3nx+\dfrac{9n(n-1)}{2}x^2+\cdots\right]$		
$\therefore\ 1\cdot\dfrac{9n(n-1)}{2}+(-2)(3n)+2\cdot 1 = 62$	1M	
$9n^2-21n-120=0$		
$n=5$ or $\dfrac{-8}{3}$ (rejected)	1A	
	(4)	

1	(a)	Very good. A minority of candidates, however, did not simplify the results obtained.
	(b)	Very good. A minority of candidates, however, did not reject the negative root $\dfrac{-8}{3}$.

1. (a) Expand $\left(u+\frac{1}{u}\right)$ in descending powers of u.

(b) Express $(e^{ax}+e^{-ax})^4$ in ascending powers of x up to the term in x^2.

(c) Suppose the coefficient of x^2 in the result of (b) is 2. Find all possible values of a.

(5 marks)

[HKDSE 2013' Section A #1]

1. (a) $\left(u+\frac{1}{u}\right)^4=u^4+4u^3\left(\frac{1}{u}\right)+6u^2\left(\frac{1}{u}\right)^2+4u\left(\frac{1}{u}\right)^3+\left(\frac{1}{u}\right)^4$

$=u^4+4u^2+6+\frac{4}{u^2}+\frac{1}{u^4}$ — 1A

(b) $(e^{ax}+e^{-ax})^4$

$=e^{4ax}+4e^{2ax}+6+4e^{-2ax}+e^{-4ax}$ by (a) — 1M

$=\left[1+\frac{4ax}{1!}+\frac{(4ax)^2}{2!}+\cdots\right]+4\left[1+\frac{2ax}{1!}+\frac{(2ax)^2}{2!}+\cdots\right]+6$

$+4\left[1+\frac{-2ax}{1!}+\frac{(-2ax)^2}{2!}+\cdots\right]+\left[1+\frac{-4ax}{1!}+\frac{(-4ax)^2}{2!}+\cdots\right]$ — 1M

$=1+4ax+8a^2x^2+4+8ax+8a^2x^2+6+4-8ax+8a^2x^2+1-4ax+8a^2x^2+\cdots$

$=16+32a^2x^2+\cdots$ — 1A

(c) $32a^2=2$

$a^2=\frac{1}{16}$

$a=\pm\frac{1}{4}$ — 1A

(5)

1. (a)	Excellent. A few candidates neglected the requirement 'in descending powers of u' when expanding $\left(u+\frac{1}{u}\right)^4$.
(b)	Satisfactory. Some candidates repeated steps in (a) because they did not make use of the fact that $e^{-ax}=\frac{1}{e^{ax}}$. Some candidates were not able to use power series of an exponential function, while some others expressed $(e^{ax}+e^{-ax})^4$ in powers of e^{2ax}.
(c)	Poor. Many candidates were not able to get the correct answer of (b), hence failed to get the answer for this part.

5. (a) Expand e^{-4x} in ascending powers of x as far as the term in x^2.

(b) Find the coefficient of x^2 in the expansion of $\frac{(2+x)^5}{e^{4x}}$.

(5 marks)

[HKDSE 2015' Section A#5]

5. (a) e^{-4x}	
$= 1 + (-4x) + \frac{(-4x)^2}{2!} + \cdots$	1M
$= 1 - 4x + 8x^2 - \cdots$	1A
(b) $(2+x)^5$	
$= 2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$	1M
$= 32 + 80x + 80x^2 + \cdots + x^5$	
The required coefficient	
$= (1)(80) + (-4)(80) + (8)(32)$	1M
$= 16$	1A
	----------(5)

5 (a)	Very good. Most candidates were able to expand e^{-4x} while a few candidates failed to show working steps.
(b)	Very good. Most candidates were able to find the coefficient of x^2 while a few candidates made a carless mistake in expanding $(2+x)^5$ as $2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$.

5. Let k be a constant.

(a) Expand e^{kx} in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x in the expansion of $(1+2x)^7 e^{kx}$ is 8, find the coefficient of x^2.

(5 marks)

[HKDSE 2016' Section A#5]

5. (a) e^{kx}	
$= 1 + kx + \dfrac{(kx)^2}{2!} + \cdots$	
$= 1 + kx + \dfrac{k^2x^2}{2} + \cdots$	1A
(b) $(1+2x)^7 e^{kx}$	
$= \left(1 + C_1^7(2x) + C_2^7(2x)^2 + \cdots + (2x)^7\right)\left(1 + kx + \dfrac{k^2x^2}{2} + \cdots\right)$	1M
$= \left(1 + 14x + 84x^2 + \cdots + (2x)^7\right)\left(1 + kx + \dfrac{k^2x^2}{2} + \cdots\right)$	
$\therefore\ 14 + k = 8$	1M
$k = -6$	
The coefficient of x^2	
$= (1)\left(\dfrac{(-6)^2}{2}\right) + 14(-6) + (84)(1)$	1M
$= 18$	1A
	----------(5)

5 (a)	Very good. A very high proportion of the candidates were able to expand e^{kx} while some candidates were unable to simplify the coefficient of x^2.
(b)	Very good. More than 70% of the candidates were able to find the coefficient of x^2 while a small number of candidates made careless mistakes in expanding $(1+2x)^7$.

5. (a) Expand $(1+e^{3x})^2$ in ascending powers of x as far as the term in x^2 .

(b) Find the coefficient of x^2 in the expansion of $(5-x)^4(1+e^{3x})^2$.

(6 marks)

[HKDSE 2017' Section A#5]

5. (a) $(1+e^{3x})^2$		
$=1+2e^{3x}+e^{6x}$	1M	
$=1+2\left(1+3x+\frac{(3x)^2}{2!}+\cdots\right)+\left(1+6x+\frac{(6x)^2}{2!}+\cdots\right)$	1M	for expanding e^{3x} or e^{6x}
$=4+12x+27x^2+\cdots$	1A	
$(1+e^{3x})^2$		
$=\left(1+1+3x+\frac{(3x)^2}{2!}+\cdots\right)^2$	1M	for expanding e^{3x}
$=(2)(2)+(2)(2)(3x)+(3x)(3x)+(2)(2)\left(\frac{9x^2}{2}\right)+\cdots$	1M	
$=4+12x+27x^2+\cdots$	1A	
(b) $(5-x)^4$		
$=5^4-C_1^4(5^3)x+C_2^4(5^2)x^2-C_3^4(5)x^3+x^4$	1M	
$=625-500x+150x^2-20x^3+x^4$		
The required coefficient		
$=(625)(27)+(-500)(12)+(150)(4)$	1M	withhold 1M if the step is skipped
$=11475$	1A	
	----------(6)	

5 (a)	Very good. Most candidates were able to expand $(1+e^{3x})^2$.
(b)	Very good. Most candidates were able to find the coefficient of x^2 .

6. Let k be a constant.

(a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k.

(6 marks)

[HKDSE 2018' Section A#6]

6. (a) $e^{kx}+e^{2x}$		
$=\left(1+kx+\frac{(kx)^2}{2!}+\cdots\right)+\left(1+2x+\frac{(2x)^2}{2!}+\cdots\right)$	1M	for expanding e^{kx} or e^{2x}
$=2+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$	1A	
(b) $(1-3x)^8$		
$=1+C_1^8(-3x)+C_2^8(-3x)^2+\cdots$	1M	
$=1-24x+252x^2+\cdots$		
$e^{kx}+e^{2x}-1$		
$=1+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$		
$(1)(k+2)+(-24)(1)=(1)\left(\frac{k^2+4}{2}\right)+(-24)(k+2)+(252)(1)$	1M+1M	
$k^2-50k+456=0$		
$k=12$ or $k=38$	1A	
	----------(6)	

6 (a)	Very good. Over 80% of the candidates were able to expand $e^{kx}+e^{2x}$.
(b)	Very good. Most candidates were able to find the value of k by writing down the coefficients of x and x^2 correctly.

6. (a) Expand e^{-18x} in ascending powers of x as far as the term in x^2.

(b) Let n be a positive integer. If the coefficient of x^2 in the expansion of $e^{-18x}(1+4x)^n$ is -38, find n.

(6 marks)

[HKDSE 2019' Section A#6]

6. (a) e^{-18x}	
$=1+(-18x)+\dfrac{(-18x)^2}{2!}+\cdots$	1M
$=1-18x+162x^2+\cdots$	1A
(b) $(1+4x)^n$	
$=1+C_1^n(4x)+C_2^n(4x)^2+\cdots+C_n^n(4x)^n$	1M
$=1+4C_1^n x+16C_2^n x^2+\cdots+4^n x^n$	
$16C_2^n-72C_1^n+162=-38$	1M
$16\left(\dfrac{n(n-1)}{2}\right)-72n+162=-38$	
$n^2-10n+25=0$	1M
$n=5$	1A
	----------(6)

2 – Linear Relationship

3. The population P (in millions) of a city can be modelled by $P = ae^{\frac{kt}{40}} - 5$, where a and k are constants and t is the number of years since the beginning of a certain year. The population of the city is recorded as follows.

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01

(a) Express $\ln(P+5)$ as a linear function of t .

(b) Using the graph paper below, estimate the values of a and k . Correct your answers to the nearest integers.
(5 marks)

[HKDSE 2012' Section A#3]

3. (a) $P = ae^{\frac{kt}{40}} - 5$

$\ln(P+5) = \frac{k}{40}t + \ln a$ 1A

(b)

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01
$\ln(P+5)$	2.00	2.06	2.11	2.15	2.20

1M

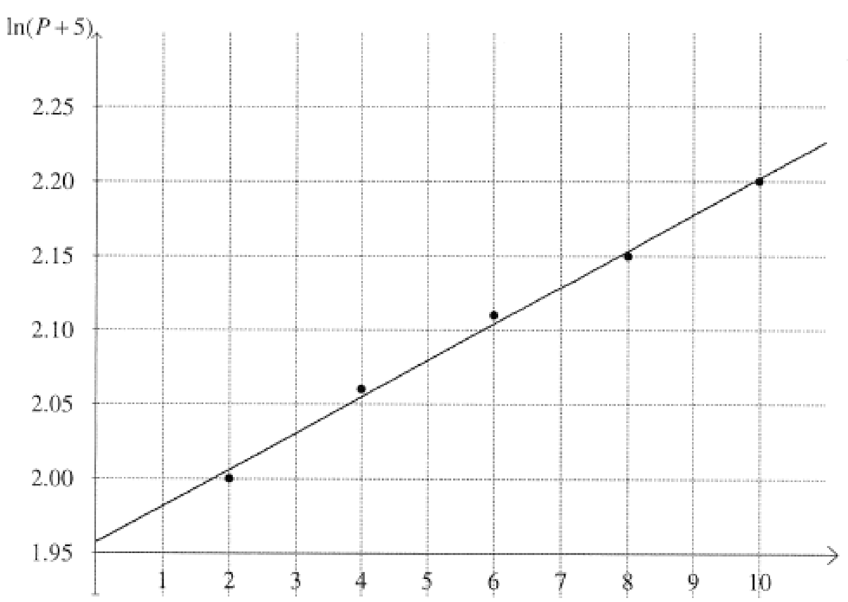
From the graph on the next page, $\ln a \approx 1.96$ 1M

$a \approx 7$

Either one

$\frac{k}{40} \approx \frac{2.21-1.96}{10-0}$

$k \approx 1$ 1A For both a and k



1A

(5)

3	(a) (b)	Very good. Very good. Candidates performed well in plotting graphs, but a small number of them did not use the plotting to estimate the values of a and k .

4. After launching an advertisement for x weeks, the number y (in thousand) of members of a club can be modelled by

$$y = \frac{8(1 - ae^{-bx})}{1 + ae^{-bx}}$$, where a and b are positive integers and $x \geq 0$.

The values of y when $x = 2, 4, 6, 8, 10$ were recorded as follows:

x	2	4	6	8	10
y	5.97	6.26	6.75	7.11	7.37

(a) Let $u = ae^{-bx}$.

(i) Express $\ln u$ as a linear function of x .

(ii) Find u in terms of y .

(b) It is known that one of the values of y in the above table is incorrect.

(i) Using the graph paper on page 9 to determine which value of y is incorrect.

(ii) By removing the incorrect value of y , estimate the values of a and b . Correct your answers to 2 decimal places.

(7 marks)

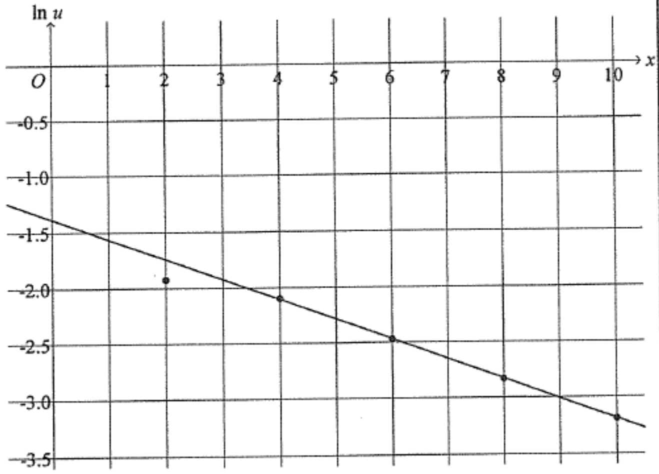
[HKDSE 2013' Section A#4]

4. (a) (i) $u = ae^{-bx}$ $\ln u = \ln a - bx$	1A	
(ii) $y = \dfrac{8(1 - ae^{-bx})}{1 + ae^{-bx}}$ $= \dfrac{8 - 8u}{1 + u}$ $u = \dfrac{8 - y}{8 + y}$	1A	

(b) (i) By (a), $\ln\frac{8-y}{8+y} = \ln a - bx$

x	2	4	6	8	10
$\ln\frac{8-y}{8+y}$	-1.93	-2.10	-2.47	-2.83	-3.19

1A — For any two pairs of values



1A

From the graph, we see that the value $y = 5.97$ is incorrect. 1A

(ii) The y-intercept $= \ln a \approx -1.4$ 1M

$\therefore\ a \approx 0.25$

The slope $= -b \approx \frac{-3.19-(-2.10)}{10-4}$ — For either one

$\therefore\ b \approx 0.18$ 1A — For both a and b

(7)

4.	(a)	(i)	Excellent.
		(ii)	Very good. A few candidates found $\frac{dy}{du}$ or $\ln y$ which was not required.
	(b)	(i)	Satisfactory. Many candidates used values of $\ln u$ with only one decimal place to plot graphs, which made it difficult to determine which value of y should be incorrect. Some others made mistakes in plotting the graph although using more accurate values of $\ln u$.
		(ii)	Fair. Some candidates used correct algebraic method but with values of $\ln u$ not accurate enough. Some others used graphs plotted in (i), but were not able to get the value of the ($\ln u$)-intercept of the straight line accurate enough.

3 –Application of Differentiation (A)

4. Let $y=\sqrt[3]{\dfrac{3x-1}{x-2}}$, where $x>2$.

(a) Use logarithmic differentiation to express $\dfrac{1}{y}\cdot\dfrac{dy}{dx}$ in terms of x .

(b) Using the result of (a), find $\dfrac{d^2y}{dx^2}$ when $x=3$.

(6 marks)

[HKDSE 2012' Section A#4]

4. (a) $y=\sqrt[3]{\dfrac{3x-1}{x-2}}$

$\ln y=\dfrac{1}{3}\ln(3x-1)-\dfrac{1}{3}\ln(x-2)$ — 1A

$\therefore\ \dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}$ — 1A

(b) By (a), $\dfrac{dy}{dx}=\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}$ — 1A

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{1}{3x-1}-\frac{1}{3(x-2)}\right]\sqrt[3]{\frac{3x-1}{x-2}}+\left[\frac{1}{3x-1}-\frac{1}{3(x-2)}\right]\frac{d}{dx}\left(\sqrt[3]{\frac{3x-1}{x-2}}\right)$$

$$=\left[\frac{-3}{(3x-1)^2}+\frac{1}{3(x-2)^2}\right]\sqrt[3]{\frac{3x-1}{x-2}}+\left[\frac{1}{3x-1}-\frac{1}{3(x-2)}\right]^2\sqrt[3]{\frac{3x-1}{x-2}}\quad\text{by (a)}$$ — 1M — For using (a)

When $x=3$, $\dfrac{d^2y}{dx^2}=\left\{\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}+\left[\dfrac{1}{3\cdot3-1}-\dfrac{1}{3(3-2)}\right]^2\right\}\sqrt[3]{\dfrac{3\cdot3-1}{3-2}}$ — 1M

<u>Alternative Solution</u>

When $x=3$, $y=2$ and so $\dfrac{dy}{dx}=\dfrac{-5}{12}$. — 1A — For both y and $\dfrac{dy}{dx}$

By (a), $\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}\cdot\dfrac{dy}{dx}=\dfrac{-3}{(3x-1)^2}+\dfrac{1}{3(x-2)^2}$ — 1M — For chain rule

When $x=3$, $\dfrac{1}{2}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{2^2}\cdot\dfrac{-5}{12}\cdot\dfrac{-5}{12}=\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}$ — 1M

i.e. $\dfrac{d^2y}{dx^2}=\dfrac{95}{144}$ — 1A — OR 0.6597

(6)

4	(a)	Good. Some candidates wrote $\sqrt[3]{\dfrac{3x-1}{x-2}}=\left(\dfrac{3x-1}{x-2}\right)^{3/2}$. Some did not use logarithmic differentiation.
	(b)	Fair. Some candidates did not use the result in (a). Some wrote $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^2$, $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=-y^{-2}\dfrac{d^2y}{dx^2}$ or $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\dfrac{dy}{dx}$.

2. The population p (in million) of a city at time t (in years) can be modelled by

$$p = 8 - \frac{2.1}{\sqrt{t+4}} \quad \text{for} \quad t \geq 0 \ .$$

An environment study indicates that, when the population is p million, the concentration of carbon dioxide in the air is given by

$$C = 2^p \text{ units.}$$

Find the rate of change of the concentration of carbon dioxide in the air at $t = 5$

(4 marks)

[HKDSE 2013' Section A#2]

2. $p = 8 - \frac{2.1}{\sqrt{t+4}}$	
$\frac{dp}{dt} = \frac{2.1}{2(t+4)^{\frac{3}{2}}}$	1A
$C = 2^p$	
$\frac{dC}{dp} = 2^p \ln 2$	1A
$\frac{dC}{dt} = \frac{dC}{dp} \cdot \frac{dp}{dt}$	
$= 2^p \ln 2 \cdot \frac{2.1}{2(t+4)^{\frac{3}{2}}}$	1M
When $t = 5$, $p = 7.3$ and hence	
$\frac{dC}{dt} = 2^{7.3} \ln 2 \cdot \frac{2.1}{2(5+4)^{\frac{3}{2}}}$	
≈ 4.2479	1A
i.e. the rate of change of the concentration of carbon dioxide ≈ 4.2479 units/year.	
	(4)

2.	Satisfactory. Some candidates found $\frac{dp}{dt}$ or $\frac{dC}{dp}$ wrongly (for example, writing $\frac{dC}{dp} = 2^p \ln p$ or $p2^{p-1}$), while some others obtained $\frac{dC}{dt}$ correctly but did not substitute 5 for t .

1. Air is leaking from a spherical balloon at a constant rate of $100\,\text{cm}^3$ per second. Find the rate of change of the radius of the balloon at the instant when the radius is $10\,\text{cm}$.

(3 marks)

[HKDSE 2014' Section A#1]

1. Let V and r be the volume and radius of the spherical balloon respectively.		
$V=\frac{4}{3}\pi r^3$		
$\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$	1M	
$\therefore\ -100=4\pi\cdot 10^2\frac{\mathrm{d}r}{\mathrm{d}t}$	1M	
$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{-1}{4\pi}$	1A	OR -0.0796
Hence the rate of change of the radius is $\frac{-1}{4\pi}\ \text{cm s}^{-1}$.		
	(3)	

1.	Satisfactory. Many candidates set $\frac{\mathrm{d}V}{\mathrm{d}t}$ equal to 100 rather than -100.

2. Let $\mathrm{f}(x)=\frac{x^x}{(2x+13)^6}$, where $x>1$.

(a) By considering $\ln \mathrm{f}(x)$, find $\mathrm{f}'(x)$.

(b) Show that $\mathrm{f}(x)$ is increasing for $x>1$.

(6 marks)

[HKDSE 2014' Section A#2]

2. (a) $\mathrm{f}(x)=\frac{x^x}{(2x+13)^6}$		
$\ln \mathrm{f}(x)=x\ln x-6\ln(2x+13)$	1A	
$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}=\ln x+x\cdot\frac{1}{x}-6\cdot\frac{2}{2x+13}$	1M+1A	
$\mathrm{f}'(x)=\left(\ln x+1-\frac{12}{2x+13}\right)\mathrm{f}(x)$		
$=\left(\ln x+\frac{2x+1}{2x+13}\right)\frac{x^x}{(2x+13)^6}$	1A	Accept $\left(\ln x+\frac{2x+1}{2x+13}\right)\mathrm{f}(x)$
(b) For $x>1$, we have $\ln x>0$, $\frac{2x+1}{2x+13}>0$ and $\frac{x^x}{(2x+13)^6}>0$.	1M	
$\therefore\ \mathrm{f}'(x)>0$		OR $\mathrm{f}'(x)\geq 0$
Hence $\mathrm{f}(x)$ is an increasing function.	1	
	(6)	

2. (a)	Good.
(b)	Very poor. Most candidates failed to show clearly why $\mathrm{f}'(x)>0$.

7. Consider the curve $C: y = x\sqrt{2x^2+1}$.

(a) Find $\frac{dy}{dx}$.

(b) Two of the tangents to C are perpendicular to the straight line $3x+17y=0$. Find the equations of the two tangents.

(7 marks)

[HKDSE 2015' Section A#7]

Solution	Marks	Remarks
7. (a) $y = x\sqrt{2x^2+1}$		
$\frac{dy}{dx} = \sqrt{2x^2+1} + x\left(\frac{1}{2}\right)(2x^2+1)^{\frac{-1}{2}}(4x)$	1M	for chain rule
$\frac{dy}{dx} = \frac{4x^2+1}{\sqrt{2x^2+1}}$	1A	
(b) Note that the slope of the straight line is $\frac{-3}{17}$.		
So, the slope of each tangent is $\frac{17}{3}$.		
$\frac{4x^2+1}{\sqrt{2x^2+1}} = \frac{17}{3}$	1M+1A	1M for using (a)
$3(4x^2+1) = 17\sqrt{2x^2+1}$		
$9(4x^2+1)^2 = 289(2x^2+1)$		
$72x^4 - 253x^2 - 140 = 0$	1M	for $ax^4+bx^2+c=0$
$x=2$ or $x=-2$		
For $x=2$, we have $y=6$.		
The equation of the tangent to C at the point $(2, 6)$ is		
$y-6 = \frac{17}{3}(x-2)$	1M	either one
$17x-3y-16=0$		
For $x=-2$, we have $y=-6$.		
The equation of the tangent to C at the point $(-2, -6)$ is		
$y+6 = \frac{17}{3}(x+2)$		
$17x-3y+16=0$	1A	for both
	----------(7)	

7 (a)	Very good. Most candidates were able to apply chain rule to find $\frac{dy}{dx}$.
(b)	Good. Some candidates made careless mistakes in simplifying the equation involving radical, and some candidates failed to write a quadratic equation in x^2 .

7. Consider the curve $C: y=(2x+8)^{\frac{3}{2}}+3x^2$, where $x>-4$.

(a) Find $\frac{dy}{dx}$.

(b) Someone claims that two of the tangents to C are parallel to the straight line $6x+y+4=0$. Do you agree? Explain your answer.

(7 marks)

[HKDSE 2016' Section A#7]

7. (a) $\frac{dy}{dx}$		
$=\left(\frac{3}{2}\right)(2x+8)^{\frac{1}{2}}(2)+6x$	1M	for chain rule
$=3\sqrt{2x+8}+6x$	1A	
(b) Note that the slope of the straight line $6x+y+4=0$ is -6 .		
So, the slope of the tangent is -6 .		
$3\sqrt{2x+8}+6x=-6$	1M+1A	1M for using (a)
$\sqrt{2x+8}=-2(x+1)$		
$2x+8=4(x+1)^2$		
$2x^2+3x-2=0$	1M	for $ax^2+bx+c=0$
$x=-2$ or $x=\frac{1}{2}$ (rejected)	1A	for ' $x=-2$ or $x=\frac{1}{2}$ '
Hence, there is only one tangent to C parallel to the straight line $6x+y+4=0$.		
Thus, the claim is disagreed.	1A	f.t.
	----------(7)	

7 (a)	Very good. Nearly all of the candidates were able to apply chain rule to find $\frac{dy}{dx}=3\sqrt{2x+8}+6x$.
(b)	Good. Some candidates were unable to solve the equation involving radical $3\sqrt{2x+8}+6x=-6$, and many candidates were unable to reject the inappropriate root $x=\frac{1}{2}$.

6. Let $f(x) = 4x^3 + mx^2 + nx + 615$, where m and n are constants. It is given that $(6, -33)$ is a turning point of the graph of $y = f(x)$. Find

(a) m and n ,

(b) the minimum value(s) and the maximum value(s) of $f(x)$.

(6 marks)

[HKDSE 2017' Section A#6]

6. (a) $f(6) = -33$		
$4(6^3) + m(6^2) + n(6) + 615 = -33$	1M	
$6m + n = -252$		
$f'(x) = 12x^2 + 2mx + n$		
$f'(6) = 0$		
$12(6^2) + 2m(6) + n = 0$	1M	
$12m + n = -432$		
Solving, we have $m = -30$ and $n = -72$.	1A	for both correct
(b) $f'(x) = 12x^2 - 60x - 72$		
$f'(x) = 0$ when $x = -1$ or $x = 6$.	1M	
(table below)	1M	for testing
Thus, the minimum value is -33 and the maximum value is 653 .	1A	for both correct
	----------(6)	

x	$(-\infty, -1)$	-1	$(-1, 6)$	6	$(6, \infty)$
$f'(x)$	+	0	−	0	+
$f(x)$	↗	653	↘	−33	↗

6 (a)	Very good. Most candidates were able to find the values of m and n .
(b)	Very good. Many candidates were able to find the maximum value and the minimum value.

7. Consider the curve $C: y=\dfrac{x}{\sqrt{x-2}}$, where $x>2$.

(a) Find $\dfrac{dy}{dx}$.

(b) A tangent to C passes through the point $(9, 0)$. Find the slope of this tangent.

(7 marks)

[HKDSE 2017' Section A#7]

Solution	Marks	Remarks
7. (a) $y=\dfrac{x}{\sqrt{x-2}}$		
$\dfrac{dy}{dx}=\dfrac{\sqrt{x-2}-x\left(\frac{1}{2}\right)(x-2)^{\frac{-1}{2}}}{x-2}$	1M	for quotient rule
$\dfrac{dy}{dx}=\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$	1A	
(b) Let (h, k) be the coordinates of the point of contact. So, the slope of this tangent is $\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$.		
$\dfrac{k-0}{h-9}=\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$	1M+1A	1M for using (a)
$\dfrac{h}{\sqrt{h-2}}2(h-2)^{\frac{3}{2}}=(h-4)(h-9)$		
$h^2+9h-36=0$ $h=3$ or $h=-12$ (rejected)	1M	
The slope of this tangent $=\dfrac{3-4}{2(3-2)^{\frac{3}{2}}}$	1M	
$=\dfrac{-1}{2}$	1A	
	----------(7)	

7 (a)	Good. Many candidates were able to find $\dfrac{dy}{dx}$ but some candidates did not simplify the answer.
(b)	Fair. Many candidates wrongly thought that $(9, 0)$ was the point of contact.

7. Let h be a constant. Consider the curve $C: y = x^2\sqrt{h-x}$, where $0 < x < h$. It is given that $\frac{dy}{dx} = 30$ when $x = 4$.

(a) Prove that $h = 20$.

(b) Find the maximum point(s) of C.

(c) Write down the equation(s) of the horizontal tangent(s) to C.

(7 marks)

[HKDSE 2018' Section A#7]

7. (a) $\frac{dy}{dx}$		
$= 2x\sqrt{h-x} + x^2\left(\frac{1}{2}\right)(h-x)^{\frac{-1}{2}}(-1)$	1M	
$= \frac{4hx-5x^2}{2\sqrt{h-x}}$		
$\frac{4h(4)-5(4)^2}{2\sqrt{h-4}} = 30$		
$16h-80 = 60\sqrt{h-4}$		
$(16h-80)^2 = (60\sqrt{h-4})^2$	1M	
$16h^2 - 385h + 1300 = 0$		
$h = 20$ or $h = 4.0625$		
Since $16(4.0625) - 80 = -15 < 0$		
Thus, we have $h = 20$.	1	
(b) For $\frac{dy}{dx} = 0$, we have $\frac{80x-5x^2}{2\sqrt{20-x}} = 0$.		
So, we have $x = 16$ or $x = 0$ (rejected)	1M	
(table below)	1M	for testing
Thus, the maximum point of C is $(16, 512)$.	1A	

x	$(0, 16)$	16	$(16, 20)$
$\frac{dy}{dx}$	$+$	0	$-$
y	↗	512	↘

For $\frac{dy}{dx} = 0$, we have $\frac{80x-5x^2}{2\sqrt{20-x}} = 0$.		
So, we have $x = 16$ or $x = 0$ (rejected)	1M	
$\frac{d^2y}{dx^2}$		
$= \frac{15x^2-480x+3200}{4\sqrt{(20-x)^3}}$		
$\left.\frac{d^2y}{dx^2}\right\|_{x=16}$		
$= -20$		
< 0	1M	for testing
Thus, the maximum point of C is $(16, 512)$.	1A	

(c) $y = 512$	1M	
	----------(7)	

7 (a)	Good. Many candidates were able to find $\frac{dy}{dx}$ but some candidates did not explain clearly why $h = 4.0625$ is not a possible answer.
(b)	Good. Many candidates were able to find the maximum value of y and the corresponding value of x, but some candidates were unable to write down the coordinates of the maximum point.
(c)	Good. Many candidates were able to write down the equation of horizontal tangent by using the result of (b).

7. Consider the curve $C : y=(x-2)\sqrt{3x+6}-8x$, where $x>-2$.

(a) Find $\dfrac{dy}{dx}$.

(b) Someone claims that two of the tangents to C are horizontal lines. Do you agree? Explain your answer.

(6 marks)

[HKDSE 2019' Section A#7]

7. (a) $\dfrac{dy}{dx}$		
$=(3x+6)^{\frac{1}{2}}+\frac{1}{2}(3)(3x+6)^{\frac{-1}{2}}(x-2)-8$	1M	
$=\sqrt{3x+6}+\dfrac{3(x-2)}{2\sqrt{3x+6}}-8$		
$=\dfrac{9x+6}{2\sqrt{3x+6}}-8$	1A	
(b) Note that the slope of each tangent is 0 .		
$\dfrac{9x+6}{2\sqrt{3x+6}}-8=0$	1M	
$9x+6=16\sqrt{3x+6}$		
$(9x+6)^2=256(3x+6)$	1M	
$27x^2-220x-500=0$		
$x=10$ or $x=\dfrac{-50}{27}$		
$\left.\dfrac{dy}{dx}\right\vert_{x=10}=\dfrac{9(10)+6}{2\sqrt{3(10)+6}}-8=0$	1M	for testing
$\left.\dfrac{dy}{dx}\right\vert_{x=\frac{-50}{27}}=\dfrac{9(\frac{-50}{27})+6}{2\sqrt{3(\frac{-50}{27})+6}}-8=-16\neq 0$		
So, we have $x=10$ only. Hence, only one tangent to C is a horizontal line. Thus, the claim is disagreed.	1A	f.t.
	----------(6)	

4 –Application of Differentiation (B)

11. Let y be the amount (in suitable units) of suspended particulate in a laboratory. It is given that

$$(E): \quad y=\frac{340}{2+e^{-t}-2e^{-2t}} \qquad (t \geq 0),$$

where t is the time (in hours) which has elapsed since an experiment started.

(a) Will the value of y exceed 171 in the long run? Justify your answer.

(2 marks)

(b) Find the greatest value and least value of y.

(6 marks)

(c) (i) Rewrite (E) as a quadratic equation in e^{-t}.

(ii) It is known that the amounts of suspended particulate are the same at the time $t=\alpha$ and $t=3-\alpha$. Given that $0 \leq \alpha < 3-\alpha$, find α.

(4 marks)

[HKDSE 2014' Section B#11]

11. (a) $\lim\limits_{t\to\infty}\frac{340}{2+e^{-t}-2e^{-2t}}=\frac{340}{2+0-2\cdot 0}$ 1M

$=170$

Hence the value of y will not exceed 171 in the long run. 1A

(2)

(b) $\frac{dy}{dt}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$ 1A

$=\frac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$

$\therefore \frac{dy}{dt}=0$ when $e^{-t}-4e^{-2t}=0$ 1M For $\frac{dy}{dt}=0$

i.e. $t=\ln 4$ 1A

t	$0 \leq t < \ln 4$	$t=\ln 4$	$t > \ln 4$
$\frac{dy}{dt}$	-ve	0	+ve

1M

Hence y is minimum when $t=\ln 4$.

The minimum value of $y=\frac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$

$=160$ 1A

When $t=0$, $y=\frac{340}{2+e^{0}-2e^{0}}$

$=340$

As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160. 1A

(6)

(c) (i) $y=\dfrac{340}{2+e^{-t}-2e^{-2t}}$

$2y+ye^{-t}-2ye^{-2t}=340$

$2y(e^{-t})^2-ye^{-t}+340-2y=0$ — 1A

(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),

$\dfrac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$ — 1M

$340-2y=2ye^{-3}$

Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$

i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$ — 1A

$\therefore\ e^{-\alpha}=\dfrac{1+\sqrt{1-16e^{-3}}}{4}$ or $\dfrac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)

i.e. $\alpha=-\ln\dfrac{1+\sqrt{1-16e^{-3}}}{4}$ — 1A — OR $\ln\dfrac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140

(4)

11. (a)	Good. Some candidates thought that $\lim_{t\to+\infty} e^{-t}=1$.
(b)	Fair. Quite a lot of candidates failed to consider both the value of y at $t=0$ and the limit found in (a).
(c)	Very poor. Most candidates wrote wrongly the equation required in (i).

12. In an experiment, the temperature (in °C) of a certain liquid can be modelled by

$$S = \frac{200}{1 + a2^{bt}} ,$$

where a and b are constants and t is the number of hours elapsed since the start of the experiment.

(a) Express $\ln\left(\frac{200}{S} - 1\right)$ as a linear function of t. (2 marks)

(b) It is found that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function obtained in (a) are $\ln 4$ and 4 respectively.

(i) Find a and b.

(ii) Find $\frac{dS}{dt}$ and $\frac{d^2S}{dt^2}$.

(iii) Describe how S and $\frac{dS}{dt}$ vary during the first 48 hours after the start of the experiment. Explain your answer.

(11 marks)

[HKDSE 2015' Section B#12]

12. (a) $S = \frac{200}{1 + a2^{bt}}$		
$\frac{200}{S} - 1 = a2^{bt}$	1M	
$\ln\left(\frac{200}{S} - 1\right) = (b\ln 2)t + \ln a$	1A	
	----------(2)	
(b) (i) $\ln a = \ln 4$ $a = 4$	1A	
$b\ln 2 = \frac{0 - \ln 4}{4 - 0}$ $b = -0.5$	1A	
(ii) $\frac{dS}{dt}$		
$= \frac{-200(4)2^{-0.5t}(-0.5)\ln 2}{(1 + 4(2^{-0.5t}))^2}$	1M	for $\frac{d}{dt}2^{bt}$
$= \frac{(400\ln 2)2^{-0.5t}}{(1 + 4(2^{-0.5t}))^2}$	1A	
$\frac{d^2S}{dt^2}$		
$= \frac{-200(\ln 2)^2(1 + 4(2^{-0.5t}))^2 2^{-0.5t} + 1600(\ln 2)^2(1 + 4(2^{-0.5t}))2^{-t}}{(1 + 4(2^{-0.5t}))^4}$	1M	for quotient rule
$= \frac{-200(\ln 2)^2 2^{-0.5t}(1 - 4(2^{-0.5t}))}{(1 + 4(2^{-0.5t}))^3}$	1A	

(iii) Note that $\frac{dS}{dt} > 0$ for $0 \le t \le 48$.	1M	
Therefore, S increases for $0 \le t \le 48$.	1A	f.t.
For $\frac{d}{dt}\left(\frac{dS}{dt}\right) = 0$, we have $4(2^{-0.5t}) = 1$.		
Hence, we have $\frac{d}{dt}\left(\frac{dS}{dt}\right) = 0$ when $t = 4$.		
(table below)	1M+1A	
Thus, $\frac{dS}{dt}$ increases for $0 \le t \le 4$ and $\frac{dS}{dt}$ decreases for $4 \le t \le 48$.	1A	f.t.
	--------(11)	

t	$0 \le t < 4$	$t = 4$	$4 < t \le 48$
$\frac{d}{dt}\left(\frac{dS}{dt}\right)$	+	0	−

12 (a)	Very good. Most candidates were able to express $\ln\left(\frac{200}{S} - 1\right)$ as a linear function of t .
(b) (i)	Very good. A few candidates failed to use the slope of the linear function as a means to find the value of the unknown b .
(ii)	Fair. Many candidates failed to differentiate $2^{-0.5t}$ with respect to t correctly when finding the required derivatives $\frac{dS}{dt}$ and $\frac{d^2S}{dt^2}$.
(iii)	Poor. Only a few candidates were able to make use of the sign of $\frac{dS}{dt}$ to discuss the behaviour of S . Most candidates failed to determine the change of the sign of $\frac{d^2S}{dt^2}$ correctly.

12. The chickens in a farm are infected by a certain bird flu. The number of chickens (in thousand) in the farm is modelled by

$$N = \frac{27}{2+\alpha t e^{\beta t}} ,$$

where $t\ (\geq 0)$ is the number of days elapsed since the start of the spread of the bird flu and α and β are constants.

(a) Express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t. (2 marks)

(b) It is given that the slope and the intercept on the horizontal axis of the graph of the linear function obtained in (a) are -0.1 and $10\ln 0.03$ respectively.

(i) Find α and β.

(ii) Will the number of chickens in the farm be less than 12 thousand on a certain day after the start of the spread of the bird flu? Explain your answer.

(iii) Describe how the rate of change of the number of chickens in the farm varies during the first 20 days after the start of the spread of the bird flu. Explain your answer.

(10 marks)

[HKDSE 2016' Section B#12]

12. (a) $N = \frac{27}{2+\alpha t e^{\beta t}}$

$\frac{27-2N}{Nt} = \alpha e^{\beta t}$ 1M

$\ln\left(\frac{27-2N}{Nt}\right) = \ln\alpha + \beta t$ 1A

----------(2)

(b) (i) $\beta = -0.1$ 1A

$0 = -0.1(10\ln 0.03) + \ln\alpha$

$\ln\alpha = \ln 0.03$

$\alpha = 0.03$ 1A

(ii) $\frac{dN}{dt}$

$= -27(2+0.03te^{-0.1t})^{-2}(0.03)(e^{-0.1t} - 0.1te^{-0.1t})$ 1M for $\frac{d}{dt}e^{\beta t} = \beta e^{\beta t}$

$= \frac{0.081(t-10)e^{-0.1t}}{(2+0.03te^{-0.1t})^2}$ 1A

For $\frac{dN}{dt} = 0$, we have $t = 10$.

t	$0 \leq t < 10$	$t = 10$	$t > 10$
$\frac{dN}{dt}$	$-$	0	$+$

1M

So, N attains its least value when $t = 10$. 1A

The least value of $N = \frac{27}{2+0.3e^{-1}} \approx 12.79400243 > 12$.

Thus, the number of chickens will not be less than 12 thousand on a certain day after the start of the bird flu. 1A f.t.

(iii) $\dfrac{d^2N}{dt^2}$		
$=\dfrac{d}{dt}\left(\dfrac{dN}{dt}\right)$		
$=\dfrac{0.081(2+0.03te^{-0.1t})^2(e^{-0.1t}-0.1(t-10)e^{-0.1t})}{(2+0.03te^{-0.1t})^4}$		
$-\dfrac{0.081(t-10)e^{-0.1t}(2)(2+0.03te^{-0.1t})(0.03)(e^{-0.1t}-0.1te^{-0.1t})}{(2+0.03te^{-0.1t})^4}$	1M	for quotient rule
$=0.0081\left(\dfrac{(2+0.03te^{-0.1t})(20-t)e^{-0.1t}+0.06(t-10)^2e^{-0.2t}}{(2+0.03te^{-0.1t})^3}\right)$	1A	
Hence, we have $\dfrac{d^2N}{dt^2}>0$ for $0\le t\le 20$.		
So, $\dfrac{dN}{dt}$ increases for $0\le t\le 20$.	1A	f.t.
Thus, the rate of change of the number of chickens increases.	---------(10)	

12 (a)	Very good. More than 70% of the candidates were able to express $\ln\left(\dfrac{27-2N}{Nt}\right)$ as a linear function of t .
(b) (i)	Good. Many candidates were able to use the slope of the linear function to find β , while a few candidates wrongly took the given horizontal intercept as the vertical intercept to find α .
(ii)	Fair. Many candidates wrongly gave the limiting value of N instead of the least value of N as the answer. Some candidates were unable to evaluate $\dfrac{d}{dt}te^{-0.1t}$ when finding $\dfrac{dN}{dt}$.
(iii)	Poor. Most candidates were unable to find the derivative of $\dfrac{dN}{dt}$ to describe how the rate of change of the number of chickens varies. Only a very small number of candidates were able to determine the sign of $\dfrac{d^2N}{dt^2}$ for $0\le t\le 20$.

12. A researcher, Peter, models the number of crocodiles in a lake by

$$x = 4 + \frac{3k}{2^{\lambda t} - k} ,$$

where λ and k are positive constants, x is the number in thousands of crocodiles in the lake and t (≥ 0) is the number of years elapsed since the start of the research.

(a) (i) Express $(x-4)(x-1)$ in terms of λ, k and t.

(ii) Peter claims that the number of crocodiles in the lake does not lie between 1 thousand and 4 thousand. Is the claim correct? Explain your answer.

(3 marks)

(b) Peter finds that $\frac{dx}{dt} = \frac{-\ln 2}{24}(x-4)(x-1)$.

(i) Prove that $\lambda = \frac{1}{8}$.

(ii) For each of the following conditions (1) and (2), find k . Also determine whether the crocodiles in the lake will eventually become extinct or not. If your answer is 'yes', find the time it will take for the crocodiles to become extinct; if your answer is 'no', estimate the number of crocodiles in the lake after a very long time.

(1) When $t = 0$, $x = 0.8$.

(2) When $t = 0$, $x = 7$.

(9 marks)

[HKDSE 2017' Section B#12]

12. (a) (i) $x - 4 = \frac{3k}{2^{\lambda t} - k}$		
$x - 1 = \frac{3(2^{\lambda t})}{2^{\lambda t} - k}$		
$(x-4)(x-1) = \frac{9k2^{\lambda t}}{(2^{\lambda t} - k)^2}$	1A	
(ii) $\frac{9k2^{\lambda t}}{(2^{\lambda t} - k)^2} > 0$ (as $k > 0$)		
$(x-4)(x-1) > 0$ (by (a)(i))	1M	
$x > 4$ or $x < 1$		
Thus, the claim is correct.	1A	f.t.
	----------(3)	

(b) (i) $\dfrac{dx}{dt}=\dfrac{-3(\ln 2)k\lambda 2^{\lambda t}}{(2^{\lambda t}-k)^2}$ 1A

$\dfrac{-\ln 2}{24}(x-4)(x-1)=\dfrac{-3(\ln 2)k2^{\lambda t}}{8(2^{\lambda t}-k)^2}$

$\lambda=\dfrac{1}{8}$ 1

(ii) (1) When $t=0$, $x=0.8$.

$-3.2=\dfrac{3k}{1-k}$

$k=16$ 1A

When $x=0$, we have $4+\dfrac{48}{2^{\frac{t}{8}}-16}=0$. 1M either one

So, we have $2^{\frac{t}{8}}=4$. 1M either one

Solving, we have $t=16$.

Thus, the crocodiles in the lake will eventually become extinct in 16 years. 1A

(2) When $t=0$, $x=7$.

$3=\dfrac{3k}{1-k}$

$k=0.5$ 1A

When $x=0$, we have $4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}=0$.

So, we have $2^{\frac{t}{8}}=0.125$.

It is impossible as $2^{\frac{t}{8}}>1$ for $t>0$.

Thus, the crocodiles in the lake will never become extinct. 1A f.t.

Note that $\lim\limits_{t\to\infty}x=\lim\limits_{t\to\infty}\left(4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}\right)=4$.

After a very long time, the estimated number of crocodiles in the lake is 4 000 . 1A

----------(9)

12 (a) (i)	Poor. Only a few candidates were able to express $(x-4)(x-1)$ in terms of λ, k and t.
(ii)	Poor. Only a few candidates were able to use the result in (a)(i) to finish the argument.
(b) (i)	Fair. Many candidates were unable to find $\dfrac{dx}{dt}$.
(ii) (1)	Fair. Only some candidates were able to find the value of k .
(2)	Fair. Many candidates estimated the number of crocodiles in the lake after a very long time without first determining that the crocodiles in the lake will not become extinct eventually.

12. In an experiment, the number of certain bacteria in a room under controlled conditions is recorded. The temperature Q (in °C) in the room can be modelled by the following linear function

$$Q = \ln r + (s \ln 3)t \ ,$$

where r and s are constants and t $(0 \le t \le 20)$ is the number of hours elapsed since the start of the experiment. It is given that the slope and the intercept on the vertical axis of the graph of this linear function of t are $-0.1\ln 9$ and $\ln 9$ respectively.

(a) Find r and s. (2 marks)

(b) It is given that

$$Q = \ln\left(\frac{120-3N}{N}\right) ,$$

where N is the number in millions of bacteria.

(i) Prove that $N = \dfrac{40}{3^{1-0.2t}+1}$.

(ii) Is it possible that there are 4 million bacteria in the room during the experiment? Explain your answer.

(iii) Find $\dfrac{dN}{dt}$ and $\dfrac{d^2N}{dt^2}$.

(iv) Describe how $\dfrac{dN}{dt}$ varies during the experiment. Explain your answer.

(11 marks)

[HKDSE 2018' Section B#12]

12. (a) $r = 9$	1A
$s\ln 3 = -0.1\ln 9$	
$s\ln 3 = -0.2\ln 3$	
$s = -0.2$	1A
	----------(2)
(b) (i) $\ln\left(\dfrac{120-3N}{N}\right) = \ln 9 - (0.2\ln 3)t$	
$\ln\left(\dfrac{120-3N}{N}\right) = \ln 9 + \ln 3^{-0.2t}$	
$\dfrac{120-3N}{N} = 3^{2-0.2t}$	1M
$120-3N = N(3^{2-0.2t})$	
$N = \dfrac{120}{3^{2-0.2t}+3}$	
$N = \dfrac{40}{3^{1-0.2t}+1}$	1

(ii) $4 = \dfrac{40}{3^{1-0.2t}+1}$

$3^{1-0.2t} = 9$

$t = -5$ 1M

Note that $0 \le t \le 20$.

Thus, it is not possible that there are 4 million bacteria in the room during the experiment. 1A f.t.

Note that $N = 10$ when $t = 0$.

$\dfrac{\mathrm{d}N}{\mathrm{d}t}$

$= -40(3^{1-0.2t}+1)^{-2}(3^{1-0.2t})(\ln 3)(-0.2)$

$= \dfrac{8(\ln 3)(3^{1-0.2t})}{(3^{1-0.2t}+1)^2}$

> 0 1M

Note that N is increasing and the least value is 10 for $0 \le t \le 20$.

Thus, it is not possible that there are 4 million bacteria in the room during the experiment. 1A f.t.

(iii) $\dfrac{\mathrm{d}N}{\mathrm{d}t}$

$= -40(3^{1-0.2t}+1)^{-2}(3^{1-0.2t})(\ln 3)(-0.2)$ 1M for $3^{1-0.2t}(\ln 3)(-0.2)$

$= \dfrac{8(\ln 3)(3^{1-0.2t})}{(3^{1-0.2t}+1)^2}$ 1A

$\dfrac{\mathrm{d}^2N}{\mathrm{d}t^2}$

$= \dfrac{8(\ln 3)((3^{1-0.2t}+1)^2(3^{1-0.2t})(\ln 3)(-0.2) - (3^{1-0.2t})(2)(3^{1-0.2t}+1)(3^{1-0.2t})(\ln 3)(-0.2))}{(3^{1-0.2t}+1)^4}$ 1M for quotient rule

$= \dfrac{8(\ln 3)^2 3^{1-0.2t}(3^{1-0.2t}-1)}{5(3^{1-0.2t}+1)^3}$ 1A

(iv) For $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}N}{\mathrm{d}t}\right) = 0$, we have $3^{1-0.2t} = 1$.

Hence, we have $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}N}{\mathrm{d}t}\right) = 0$ when $t = 5$. 1M

t	$[0, 5)$	5	$(5, 20]$
$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}N}{\mathrm{d}t}\right)$	+	0	−

1M for testing

Thus, $\dfrac{\mathrm{d}N}{\mathrm{d}t}$ increases for $0 \le t \le 5$ and

$\dfrac{\mathrm{d}N}{\mathrm{d}t}$ decreases for $5 \le t \le 20$. 1A f.t.

--------(11)

12 (a)	Very good. About 85% of the candidates were able to find r and s .
(b) (i)	Fair. Many candidates were unable to write $e^{\ln 3^{-0.2t}}$ as $3^{-0.2t}$ to complete the proof.
(ii)	Good. Many candidates were able to draw a conclusion by solving the equation $4 = \dfrac{40}{3^{1-0.2t}+1}$.
(iii)	Fair. Many candidates were unable to find $\dfrac{\mathrm{d}}{\mathrm{d}t}3^{1-0.2t}$, hence they were unable to find $\dfrac{\mathrm{d}N}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}^2N}{\mathrm{d}t^2}$ correctly.
(iv)	Poor. Only a small number of candidates were able to describe correctly how $\dfrac{\mathrm{d}N}{\mathrm{d}t}$ varies.

12. A tank contains some water. Water is now leaking from the tank. Let V m^3 be the volume of water in the tank. It is given that

$$V = \frac{64}{he^{kt} + 4} ,$$

where $t\ (\geq 0)$ is the number of hours elapsed since the leaking begins and h and k are constants.

(a) Express $\ln\left(\frac{64}{V} - 4\right)$ as a linear function of t. (1 mark)

(b) It is given that the graph of the linear function obtained in (a) passes through the origin and the point $(2, 1)$. Find

(i) h and k,

(ii) $\frac{dV}{dt}$,

(iii) the value of V when $\frac{dV}{dt}$ attains its least value.

(8 marks)

(c) The owner of the tank finds that $S = V^{\frac{2}{3}}$, where S m^2 is the wet total surface area of the tank.

(i) Find the value of $\frac{dS}{dt}$ when $\frac{dV}{dt}$ attains its least value.

(ii) The owner claims that $\frac{dS}{dt}$ attains its least value when $\frac{dV}{dt}$ attains its least value. Is the claim correct? Explain your answer.

(4 marks)

[HKDSE 2019' Section B#12]

12. (a) $V = \dfrac{64}{he^{kt}+4}$

$\dfrac{64}{V} - 4 = he^{kt}$

$\ln\left(\dfrac{64}{V} - 4\right) = kt + \ln h$ 1A

----------(1)

(b) (i) $\ln h = 0$

$h = 1$ 1A

$k = \dfrac{1-0}{2-0}$

$k = 0.5$ 1A

(ii) $V = \dfrac{64}{e^{0.5t}+4}$

$\dfrac{\mathrm{d}V}{\mathrm{d}t}$

$= -64(e^{0.5t}+4)^{-2}(0.5)e^{0.5t}$ 1M

$= \dfrac{-32e^{0.5t}}{(e^{0.5t}+4)^2}$ 1A

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}V}{\mathrm{d}t}\right)$

$= \dfrac{-32((e^{0.5t}+4)^2(0.5e^{0.5t}) - (e^{0.5t})2(e^{0.5t}+4)(0.5e^{0.5t}))}{(e^{0.5t}+4)^4}$

$= \dfrac{16e^{0.5t}(e^{0.5t}-4)}{(e^{0.5t}+4)^3}$ 1A

For $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}V}{\mathrm{d}t}\right) = 0$, we have $\dfrac{16e^{0.5t}(e^{0.5t}-4)}{(e^{0.5t}+4)^3} = 0$. 1M

So, we have $t = 4\ln 2$.

t	$0 \le t < 4\ln 2$	$t = 4\ln 2$	$t > 4\ln 2$
$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}V}{\mathrm{d}t}\right)$	$-$	0	$+$

1M for testing

Therefore, $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ attains its least value when $t = 4\ln 2$.

The required value of V

$= \dfrac{64}{4+4}$

$= 8$ 1A

----------(8)

(c) (i) $\dfrac{dS}{dt}=\dfrac{2}{3}V^{\frac{-1}{3}}\dfrac{dV}{dt}$	1M	
At $t=4\ln 2$, $\dfrac{dS}{dt}$ $=\dfrac{2}{3}(8)^{\frac{-1}{3}}\left(\dfrac{-32(4)}{(4+4)^2}\right)$ $=\dfrac{-2}{3}$	1A	
$S=16(e^{0.5t}+4)^{\frac{-2}{3}}$ $\dfrac{dS}{dt}$ $=16\left(\dfrac{-2}{3}(e^{0.5t}+4)^{\frac{-5}{3}}(0.5e^{0.5t})\right)$	1M	
$=\dfrac{-16e^{0.5t}}{3(e^{0.5t}+4)^{\frac{5}{3}}}$ At $t=4\ln 2$, we have $\dfrac{dS}{dt}=\dfrac{-2}{3}$.	1A	
(ii) $\dfrac{dS}{dt}=\dfrac{2}{3}V^{\frac{-1}{3}}\dfrac{dV}{dt}$ $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)=\dfrac{2}{3}V^{\frac{-1}{3}}\dfrac{d}{dt}\left(\dfrac{dV}{dt}\right)-\dfrac{2}{9}V^{\frac{-4}{3}}\left(\dfrac{dV}{dt}\right)^2$	1M	
At $t=4\ln 2$, $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)$ $=\dfrac{2}{3}(8)^{\frac{-1}{3}}(0)-\dfrac{2}{9}(8)^{\frac{-4}{3}}(-2)^2$ $=\dfrac{-1}{18}$ $\neq 0$ Thus, the claim is not correct.	1A	f.t.
$\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)=\dfrac{16e^{0.5t}(e^{0.5t}-6)}{9(e^{0.5t}+4)^{\frac{8}{3}}}$	1M	
For $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)=0$, we have $t=2\ln 6\neq 4\ln 2$. Thus, the claim is not correct.	1A	f.t.
	----------(4)	

5 –Application of Integration (A)

2. The rate of change of the value V (in million dollars) of a flat is given by $\dfrac{dV}{dt} = \dfrac{t}{\sqrt{4t+1}}$, where t is the number of years since the beginning of 2012. The value of the flat is 3 million dollars at the beginning of 2012. Find the percentage change in the value of the flat from the beginning of 2012 to the beginning of 2014.

(5 marks)

[HKDSE 2012' Section A#2]

2. Let $u = 4t+1$. $du = 4dt$ When $t=0$, $u=1$; when $t=2$, $u=9$.	1M	
The change in the value of the flat $= \int_0^2 \frac{t}{\sqrt{4t+1}} dt$	1M	
$= \int_1^9 \frac{1}{\sqrt{u}} \cdot \frac{u-1}{4} \frac{du}{4}$ $= \frac{1}{16} \int_1^9 \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) du$		
$= \frac{1}{16} \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^9$	1A	For $\frac{1}{16} \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]$
$= \frac{5}{6}$	1A	
Hence the percentage change $= \dfrac{\frac{5}{6}}{3} \times 100\%$		
$= 27\frac{7}{9}\%$	1A	OR 27.7778%
	(5)	

2	Fair. Many candidates failed to find a suitable substitution or did wrong calculation in substitution. Some found the value of the flat at the beginning of 2014 instead of the percentage change.

5. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x}$. Let L be the tangent to S at the point $A(0,1)$ on S .

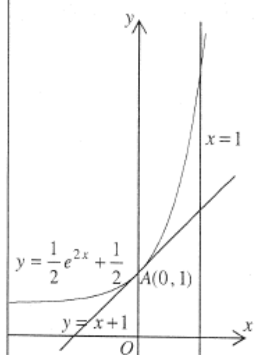
(a) Find the equation of S .

(b) Find the equation of L .

(c) Find the area of the region bounded by S , L and the line $x = 1$.

(7 marks)

[HKDSE 2012' Section A#5]

5. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x}$		
$y = \frac{1}{2}e^{2x} + C$	1A	
Since $A(0,1)$ lies on S , we have $1 = \frac{1}{2}e^{2(0)} + C$.	1M	
i.e. $C = \frac{1}{2}$		
Hence the equation of S is $y = \frac{1}{2}e^{2x} + \frac{1}{2}$.	1A	
(b) At $A(0,1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{2(0)} = 1$.		
Hence the equation of L is $y - 1 = 1(x - 0)$.	1M	
i.e. $y = x + 1$	1A	
(c) The area of the region bounded by S , L and the line $x = 1$		
$= \int_0^1 \left[\left(\frac{1}{2}e^{2x} + \frac{1}{2} \right) - (x+1) \right] \mathrm{d}x$	1M	1M for $A = \int_0^1 (y_1 - y_2)\mathrm{d}x$
$= \left[\frac{1}{4}e^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x \right]_0^1$		
$= \frac{e^2 - 5}{4}$	1A	OR 0.5973
	(7)	

5	(a)	Satisfactory. Some candidates omitted the constant of integration or wrote $\int e^{2x}\,\mathrm{d}x = 2e^{2x} + C$ while others mixed S with L .
	(b)	Satisfactory. Some candidates treated e^{2x} as the slope of L and wrote $y = e^{2x}x + 1$ as the equation of L .
	(c)	Poor. Some candidates regarded $y = e^{2x}$ as the equation of S .

3. Consider the curve C: $y = x(x-2)^{\frac{1}{3}}$ and the straight line L that passes through the origin and is parallel to the tangent to C at $x = 3$.

(a) Find the equation of L .

(b) Find the x-coordinates of the two intersecting points of C and L .

(c) Find the area bounded by C and L .

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(8 marks)

[HKDSE 2013' Section A#3]

3. (a) $y = x(x-2)^{\frac{1}{3}}$

$\frac{dy}{dx} = (x-2)^{\frac{1}{3}} + \frac{1}{3}(x-2)^{\frac{-2}{3}}x$ — 1M — For product rule

When $x = 3$, $\frac{dy}{dx} = 2$.

Hence the equation of L is $y = 2x$. — 1A

(b) Solving C and L :

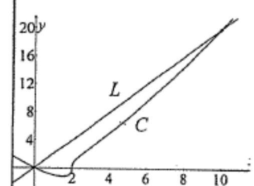
$x(x-2)^{\frac{1}{3}} = 2x$ — 1M

$x\left[(x-2)^{\frac{1}{3}} - 2\right] = 0$

$x = 0$ or 10 — 1A

(c) The area bounded by L and C

$= \int_0^{10}\left[2x - x(x-2)^{\frac{1}{3}}\right]dx$ — 1M



$= \int_0^{10} 2x\,dx - \int_0^{10} x(x-2)^{\frac{1}{3}}dx$

Let $u = x-2$ and so $du = dx$. — 1M

When $x = 0$, $u = -2$; when $x = 10$, $u = 8$.

$\therefore$ the area bounded by L and C

$= \int_0^{10} 2x\,dx - \int_{-2}^{8}(u+2)u^{\frac{1}{3}}du$

$= [x^2]_0^{10} - \int_{-2}^{8}\left(u^{\frac{4}{3}} + 2u^{\frac{1}{3}}\right)du$

$= 100 - \left[\frac{3}{7}u^{\frac{7}{3}} + \frac{3}{2}u^{\frac{4}{3}}\right]_{-2}^{8}$ — 1M — For the primitive function

$= \frac{148 + 9\sqrt[3]{2}}{7}$ — 1A — OR 22.7628

(8)

3.	(a)	Good. Some candidates found the equation of the tangent to C at $x = 3$ instead of the equation of L .
	(b)	Good. Some candidates did not know how to solve equations with fraction exponents or missed out the root $x = 0$ by dividing both sides of an equation by x .
	(c)	Fair. Most candidates made mistakes in finding correct primitive functions or calculating the final answer.

5. (a) Find $\dfrac{d}{dx}(x\ln x)$.

(b) Use (a) to evaluate $\displaystyle\int_1^e \ln x\,dx$.

(4 marks)

[HKDSE 2013' Section A#5]

5. (a) $\dfrac{d}{dx}(x\ln x)=(1)\ln x+x\left(\dfrac{1}{x}\right)$		
$=\ln x+1$	1A	
(b) $\ln x=\dfrac{d}{dx}(x\ln x)-1$		
$\displaystyle\int_1^e \ln x\,dx=[x\ln x]_1^e-\int_1^e 1\,dx$	1M	
$=e\ln e-\ln 1-[x]_1^e$	1A	For x
$=1$	1A	
	(4)	

5. (a)	Excellent.
(b)	Satisfactory. Some candidates failed to use the result of (a), while some others wrote $x\ln x$ instead of $[x\ln x]_1^e$.

3. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{dy}{dx}=\left(2x-\frac{1}{x}\right)^3$, where $x>0$.

A point $P(1, 5)$ lies on S.

(a) Find the equation of the tangent to S at P.

(b) (i) Expand $\left(2x-\frac{1}{x}\right)^3$.

(ii) Find the equation of S for $x>0$.

(7 marks)

[HKDSE 2014' Section A#3]

3. (a) $\left.\frac{dy}{dx}\right\|_{(1,5)}=\left(2\cdot 1-\frac{1}{1}\right)^3$	
$=1$	1A
Hence the equation of tangent is $y-5=1(x-1)$.	
i.e. $x-y+4=0$	1A
(b) (i) $\left(2x-\frac{1}{x}\right)^3=(2x)^3-3(2x)^2\left(\frac{1}{x}\right)+3(2x)\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^3$	1M
$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$	1A
(ii) $y=\int\left(2x-\frac{1}{x}\right)^3 dx$	
$=\int\left(8x^3-12x+\frac{6}{x}-\frac{1}{x^3}\right)dx$ by (i)	1M
$=2x^4-6x^2+6\ln\|x\|+\frac{1}{2x^2}+C$	
Since $P(1, 5)$ lies on S, $5=2(1)^4-6(1)^2+6\ln\|1\|+\frac{1}{2(1)^2}+C$.	1M
i.e. $C=\frac{17}{2}$	
Hence the equation of S is $y=2x^4-6x^2+6\ln x+\frac{1}{2x^2}+\frac{17}{2}$ for $x>0$.	1A
	(7)

3. (a)	Very good.
(b) (i)	Excellent.
(ii)	Satisfactory. Some candidates did not know $\int\frac{1}{x}dx=\ln\|x\|+C$, or wrote $\int\frac{1}{x^3}dx=-\frac{2}{x^2}$ or $\frac{1}{2x^2}$.

4. Evaluate the following definite integrals:

(a) $\displaystyle\int_1^3 \frac{t+2}{t^2+4t+11}\,dt$,

(b) $\displaystyle\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}\,dt$.

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(6 marks)

[HKDSE 2014' Section A#4]

4. (a) Let $u=t^2+4t+11$. $du=(2t+4)dt$ When $t=1, u=16$; when $t=3, u=32$.	1M	
$\displaystyle\int_1^3 \frac{t+2}{t^2+4t+11}\,dt=\int_{16}^{32}\frac{1}{u}\cdot\frac{du}{2}$	1A	OR $\displaystyle\frac{1}{2}\int_{t=1}^{3}\frac{d(t^2+4t+11)}{t^2+4t+11}$
$=\frac{1}{2}\left[\ln\lvert u\rvert\right]_{16}^{32}$	1A	
$=\frac{\ln 32-\ln 16}{2}$		
$=\frac{\ln 2}{2}$	1A	OR 0.3466
(b) $\displaystyle\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}\,dt=\int_1^3\left(1-\frac{t+2}{t^2+4t+11}\right)dt$	1M	
$=[t]_1^3-\displaystyle\int_1^3\frac{t+2}{t^2+4t+11}\,dt$		
$=2-\frac{\ln 2}{2}$	1A	OR 1.6534
	(6)	

4. (a)	Very good.
(b)	Poor. Many candidates seemed to have no idea about how to solve the problem.

5. The government of a country is going to announce a new immigration policy which will last for 3 years. At the moment of the announcement, the population of the country is 8 million. After the announcement, the rate of change of the population can be modelled by

$$\frac{dx}{dt} = \frac{t\sqrt{9-t^2}}{3} \qquad (0 \le t \le 3),$$

where x is the population (in million) of the country and t is the time (in years) which has elapsed since the announcement. Find x in terms of t.

(5 marks)

[HKDSE 2014' Section A#5]

Solution	Marks	Remarks
5. $\frac{dx}{dt} = \frac{t\sqrt{9-t^2}}{3}$		
Let $u = 9-t^2$. $du = -2t\,dt$	1M	
$x = \int \frac{t\sqrt{9-t^2}}{3}\,dt$		
$= \int \frac{u^{\frac{1}{2}}}{3}\frac{du}{-2}$	1A	OR $\int \frac{(9-t^2)^{\frac{1}{2}}}{3}\frac{d(9-t^2)}{-2}$
$= \frac{-1}{6}\cdot\frac{2u^{\frac{3}{2}}}{3} + C$	1A	
$= \frac{-1}{9}(9-t^2)^{\frac{3}{2}} + C$		
When $t=0$, $x=8$.		
$\therefore\ 8 = \frac{-1}{9}(9-0)^{\frac{3}{2}} + C$	1M	
$C = 11$		
i.e. $x = \frac{-1}{9}(9-t^2)^{\frac{3}{2}} + 11$	1A	
	(5)	

5.	Satisfactory. Some candidates substituted $t=3$ and $x=8$ to determine the value of the constant of integration.

6. Consider the curves $C_1 : y = e^{2x} + e^4$ and $C_2 : y = e^{x+3} + e^{x+1}$.

(a) Find the x-coordinates of the two points of intersection of C_1 and C_2 .

(b) Express, in terms of e , the area of the region bounded by C_1 and C_2 .

(6 marks)

[HKDSE 2015' Section A#6]

6. (a) $e^{2x} + e^4 = e^{x+3} + e^{x+1}$	
$(e^x)^2 - (e^3 + e)e^x + e^4 = 0$	1M
$(e^x - e)(e^x - e^3) = 0$	
$e^x = e$ or $e^x = e^3$	
$x = 1$ or $x = 3$	1A
Thus, the x-coordinates are 1 and 3 .	
(b) The area of the region bounded by C_1 and C_2	
$= \int_1^3 (e^{x+3} + e^{x+1} - (e^{2x} + e^4))\, dx$	1M+1A
$= \left[e^{x+3} + e^{x+1} - \frac{e^{2x}}{2} - e^4 x \right]_1^3$	1M
$= \frac{e^6}{2} - 2e^4 - \frac{e^2}{2}$	1A
	----------(6)

6 (a)	Good. Many candidates were able to find the x-coordinates of the two points of intersection of C_1 and C_2 , while some candidates failed to write a quadratic equation in e^x .
(b)	Good. Some candidates failed to give a simplified answer and left an absolute value sign in the answer, and some candidates got a wrong answer $-\frac{e^6}{2} + 2e^4 + \frac{e^2}{2}$ instead of $\frac{e^6}{2} - 2e^4 - \frac{e^2}{2}$.

8. (a) Express $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ in the form $f(x)+g(x)\ln(x^2+1)$, where $f(x)$ and $g(x)$ are polynomials.

(b) Find $\int x^5 \ln(x^2+1)\,dx$.

(7 marks)

[HKDSE 2015' Section A#8]

8. (a) $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$		
$=(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M+1A	1M for product rule
$=(x^2+1)(x^4-x^2+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M	
$=2x^5-2x^3+2x+6x^5\ln(x^2+1)$	1A	
(b) $(x^6+1)\ln(x^2+1)=2\int(x^5-x^3+x)\,dx+6\int x^5\ln(x^2+1)\,dx$	1M	
Note that $\int(x^5-x^3+x)\,dx=\frac{x^6}{6}-\frac{x^4}{4}+\frac{x^2}{2}+\text{constant}$.	1A	
Thus, we have $\int x^5\ln(x^2+1)\,dx=\frac{1}{6}(x^6+1)\ln(x^2+1)-\frac{x^6}{18}+\frac{x^4}{12}-\frac{x^2}{6}+\text{constant}$.	1A	
	----------(7)	

8 (a)	Good. Many candidates were able to apply product rule to find $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ while some candidates did not understand the definition of polynomial and simply left $(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$ as the final answer instead of $(2x^5-2x^3+2x)+6x^5\ln(x^2+1)$.
(b)	Fair. Many candidates employed a wrong substitution in finding $\int(x^6+1)\frac{2x}{x^2+1}dx$, and many candidates made careless mistakes in calculating the integration.

6. Let $f(x)=3^{2x}-10(3^x)+9$.

(a) Find $\int f(x)\,dx$.

(b) The equation of the curve C is $y=f(x)$. Find

(i) the two x-intercepts of C ,

(ii) the exact value of the area of the region bounded by C and the x-axis.

(6 marks)

[HKDSE 2016' Section A#6]

6. (a) $\int f(x)\,dx$		
$=\int (3^{2x}-10(3^x)+9)\,dx$		
$=\dfrac{3^{2x}}{2\ln 3}-\dfrac{10(3^x)}{\ln 3}+9x+\text{constant}$	1M+1A	1M for $\int a^x dx=\dfrac{a^x}{\ln a}+\text{constant}$
(b) (i) $3^{2x}-10(3^x)+9=0$		
$(3^x)^2-10(3^x)+9=0$	1M	
$3^x=1$ or $3^x=9$		
$x=0$ or $x=2$	1A	for both
Thus, the x-intercepts are 0 and 2 .		
(ii) The area of the region bounded by C and the x-axis		
$=-\int_0^2 f(x)\,dx$	1M	
$=-\left[\dfrac{3^{2x}}{2\ln 3}-\dfrac{10(3^x)}{\ln 3}+9x\right]_0^2$ (by (a))		
$=-\left(\dfrac{81}{2\ln 3}-\dfrac{90}{\ln 3}+18\right)+\left(\dfrac{1}{2\ln 3}-\dfrac{10}{\ln 3}\right)$		
$=\dfrac{40}{\ln 3}-18$	1A	
	----------(6)	

6 (a)	Fair. Some candidates wrongly evaluated the indefinite integral $\int 3^x dx$ as $\ln 3(3^x)+\text{constant}$ instead of $\dfrac{3^x}{\ln 3}+\text{constant}$.
(b) (i)	Very good. More than 70% of the candidates were able to find the two x-intercepts of C , while a small number of candidates were unable to write a quadratic equation in 3^x .
(ii)	Fair. Although many candidates were able to use the results of (a) and (b)(i) to find the area of the required region, they were unable to give the answer in exact value.

8. Define $f(x) = \frac{(\ln x)^2}{x}$ for all $x > 0$. Let α and β be the two roots of the equation $f'(x) = 0$, where $\alpha > \beta$.

(a) Express α in terms of e . Also find β.

(b) Using integration by substitution, evaluate $\int_{\beta}^{\alpha} f(x)\,dx$.

(7 marks)

[HKDSE 2016' Section A#8]

8. (a) $f'(x)$		
$= \frac{x\left(2(\ln x)\frac{1}{x}\right) - (\ln x)^2}{x^2}$	1M	for quotient rule
$= \frac{2\ln x - (\ln x)^2}{x^2}$		
$= \frac{(2 - \ln x)(\ln x)}{x^2}$		
$f'(x) = 0$		
$\ln x = 2$ or $\ln x = 0$		
$x = e^2$ or $x = 1$		
$\alpha = e^2$ and $\beta = 1$	1A+1A	
(b) Let $u = \ln x$.	1M	
Then, we have $\frac{du}{dx} = \frac{1}{x}$.		
$\int_{\beta}^{\alpha} f(x)\,dx$		
$= \int_{1}^{e^2} \frac{(\ln x)^2}{x}\,dx$		
$= \int_{0}^{2} u^2\,du$	1A	
$= \left[\frac{u^3}{3}\right]_0^2$	1M	
$= \frac{8}{3}$	1A	r.t. 2.6667
≈ 2.666666667		
≈ 2.6667		
	----------(7)	

8 (a)	Very good. More than 60% of the candidates were able to apply quotient rule or product rule to find $f'(x)$ and hence find the values of α and β by solving the equation $f'(x) = 0$, while some candidates wrongly wrote the value of β as 0 instead of 1 .
(b)	Good. Many candidates employed a suitable substitution in evaluating the definite integral $\int_{1}^{e^2} \frac{(\ln x)^2}{x}\,dx$.

8. Define $g(x)=\frac{1}{x}\ln\left(\frac{e}{x}\right)$ for all $x>0$.

(a) Using integration by substitution, find $\int g(x)\,dx$.

(b) Denote the curve $y=g(x)$ by Γ .

(i) Write down the x-intercept(s) of Γ .

(ii) Find the area of the region bounded by Γ , the x-axis and the straight lines $x=1$ and $x=e^2$.

(7 marks)

[HKDSE 2017' Section A#8]

8. (a) Let $u=\ln x$.	1M
So, we have $\frac{du}{dx}=\frac{1}{x}$.	
$\int g(x)dx$	
$=\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$	
$=\int\left(\frac{1}{x}(1-\ln x)\right)dx$	
$=\int(1-u)du$	1M
$=u-\frac{1}{2}u^2+\text{constant}$	
$=\ln x-\frac{1}{2}(\ln x)^2+\text{constant}$	1A

Let $u=\ln\left(\frac{e}{x}\right)$.	1M	
Then, we have $\frac{du}{dx}=\frac{-1}{x}$.		
$\int g(x)\,dx$		
$=\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$		
$=\int -u\,du$	1M	
$=\frac{-1}{2}u^2+\text{constant}$		
$=\frac{-1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2+\text{constant}$	1A	

(b) (i) e	1A	
(ii) The required area		
$=\int_1^e g(x)\,dx+\int_e^{e^2}-g(x)\,dx$	1M	
$=\left[\ln x-\frac{1}{2}(\ln x)^2\right]_1^e+\left[-\ln x+\frac{1}{2}(\ln x)^2\right]_e^{e^2}$ (by (a))	1M	for using (a)
$=\frac{1}{2}+\frac{1}{2}$		
$=1$	1A	
The required area		
$=\int_1^e g(x)\,dx+\int_e^{e^2}-g(x)\,dx$	1M	
$=\left[\frac{-1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_1^e+\left[\frac{1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_e^{e^2}$ (by (a))	1M	for using (a)
$=\frac{1}{2}+\frac{1}{2}$		
$=1$	1A	
	----------(7)	

8 (a)	Very good. Most candidates were able to use a correct substitution in finding $\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$.
(b) (i)	Very good. Many candidates were able to write down the x-intercept of Γ . However, some candidates wrongly gave $(e, 0)$ instead of e as the answer.
(ii)	Fair. Many candidates were unable to note that part of Γ lies above the x-axis while part of Γ lies below the x-axis.

5. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x .

(a) If $f(x)$ attains its minimum value at $x=\alpha$, find α .

(b) It is given that the extreme value of $f(x)$ is 5 . Find

(i) $f(x)$,

(ii) $\lim\limits_{x\to\infty} f(x)$.

(6 marks)

[HKDSE 2018' Section A#5]

5. Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.	
(a) $f'(x)=0$	
$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$	1M
$x=4$	
So, $f(x)$ attains its minimum value at $x=4$.	
Thus, we have $\alpha=4$.	1A

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

$f'(x)=0$	
$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$	1M
$x=4$	
$f''(x)$ $=\dfrac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$	
$f''(4)$ $=12$ >0	
So, $f(x)$ attains its minimum value at $x=4$.	
Thus, we have $\alpha=4$.	1A

(b) (i) Let $v = 3x^2 - 24x + 49$. Then, we have $\frac{dv}{dx} = 6x - 24$.	
$\mathrm{f}(x)$	
$= \int \frac{12x - 48}{(3x^2 - 24x + 49)^2} dx$	
$= \int \frac{2}{v^2} dv$	1M
$= \frac{-2}{v} + C$	
$= \frac{-2}{3x^2 - 24x + 49} + C$	
Since $\mathrm{f}(x)$ has only one extreme value, we have $\mathrm{f}(4) = 5$.	1M
$\frac{-2}{3(4)^2 - 24(4) + 49} + C = 5$	
$C = 7$	
Thus, we have $\mathrm{f}(x) = \frac{-2}{3x^2 - 24x + 49} + 7$.	1A
(ii) $\lim\limits_{x \to \infty} \mathrm{f}(x)$	
$= 7$	1A
	----------(6)

5 (a)	Very good. Over 85% of the candidates were able to find the value of α .
(b) (i)	Good. Many candidates were able to find $\mathrm{f}(x)$ by indefinite integral but some candidates were unable to use a suitable substitution.
(ii)	Fair. Only some candidates were able to find the constant of integration in (b)(i), and thus the required limit.

8. (a) By considering $\dfrac{d}{dx}(x\ln x)$, find $\displaystyle\int \ln x\,dx$.

(b) Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

(c) Let C be the curve $y=\dfrac{(x-1)(\ln x-1)}{x}$, where $x>0$. Express, in terms of e , the area of the region bounded by C and the x-axis.

(7 marks)

[HKDSE 2018' Section A#8]

8. (a) $\dfrac{d}{dx}(x\ln x)$		
$=x\left(\dfrac{1}{x}\right)+\ln x$	1M	
$=1+\ln x$		
So, we have $\ln x=\dfrac{d}{dx}(x\ln x)-1$.		
$\displaystyle\int \ln x\,dx$		
$=x\ln x-x+\text{constant}$	1A	
(b) $\displaystyle\int \frac{\ln x}{x}\,dx$		
$=\dfrac{(\ln x)^2}{2}+\text{constant}$	1A	
(c) $y=0$		
$\dfrac{(x-1)(\ln x-1)}{x}=0$		
$x=1$ or $x=e$	1A	can be absorbed
The required area		
$=-\displaystyle\int_1^e \frac{(x-1)(\ln x-1)}{x}\,dx$	1M	
$=-\displaystyle\int_1^e \frac{x\ln x-x-\ln x+1}{x}\,dx$		
$=-\displaystyle\int_1^e \left(\ln x-1-\frac{\ln x}{x}+\frac{1}{x}\right)dx$		
$=-\left[x\ln x-x-x-\dfrac{(\ln x)^2}{2}+\ln x\right]_1^e$	1M	
$=e-\dfrac{5}{2}$	1A	
	----------(7)	

8 (a)	Very good. Most candidates were able to use the result of $\dfrac{d}{dx}(x\ln x)$ to find $\displaystyle\int \ln x\,dx$.
(b)	Good. Many candidates were able to find $\displaystyle\int \frac{\ln x}{x}\,dx$ by using integration by substitution.
(c)	Good. Many candidates were able to use the results of (a) and (b) to find $\displaystyle\int \frac{(x-1)(\ln x-1)}{x}\,dx$ but some candidates were unable to find the correct upper and lower limits of the required definite integral.

5. Define $f(x)=\dfrac{6-x}{x+3}$ for all $x>-3$.

(a) Prove that $f(x)$ is decreasing.

(b) Find $\lim\limits_{x\to\infty} f(x)$.

(c) Find the exact value of the area of the region bounded by the graph of $y=f(x)$, the x-axis and the y-axis.

(6 marks)

[HKDSE 2019' Section A#5]

5. (a) For all $x>-3$, $f'(x)$ $=\dfrac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}$ $=\dfrac{-9}{(x+3)^2}$ <0 Thus, $f(x)$ is decreasing.	1	
Note that $f(x)=\dfrac{9}{x+3}-1$ for all $x>-3$. Thus, $f(x)$ is decreasing.	1	
(b) $\lim\limits_{x\to\infty} f(x)$ $=\lim\limits_{x\to\infty}\dfrac{\frac{6}{x}-1}{1+\frac{3}{x}}$ $=-1$	1A	
$\lim\limits_{x\to\infty} f(x)$ $=\lim\limits_{x\to\infty}\left(\dfrac{9}{x+3}-1\right)$ $=-1$	1A	
(c) For $y=0$, we have $x=6$. The required area		
$=\int_0^6 f(x)\mathrm{d}x$	1M	
$=\int_0^6 \dfrac{6-x}{x+3}\mathrm{d}x$		
$=\int_0^6\left(\dfrac{9}{x+3}-1\right)\mathrm{d}x$	1M	
$=\left[9\ln(x+3)-x\right]_0^6$	1M	
$=9\ln 3-6$	1A	
For $y=0$, we have $x=6$. The required area		
$=\int_0^6 f(x)\mathrm{d}x$	1M	
$=\int_0^6 \dfrac{6-x}{x+3}\mathrm{d}x$		
$=\int_3^9 \dfrac{6-(u-3)}{u}\mathrm{d}u$ (by letting $u=x+3$)	1M	
$=\int_3^9\left(\dfrac{9}{u}-1\right)\mathrm{d}u$		
$=\left[9\ln u-u\right]_3^9$	1M	
$=9\ln 3-6$	1A	
	----------(6)	

6 – Application of Integration (B)

10. Let $I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,dt$.

(a) (i) Use the trapezoidal rule with 6 sub-intervals to estimate I.

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Justify your answer.

(7 marks)

(b) Using a suitable substitution, show that $I=2\int_1^2 e^{\frac{-x^2}{2}}\,dx$.

(3 marks)

(c) Using the above results and the Standard Normal Distribution Table on page 14, show that $\pi<3.25$.

(3 marks)

[HKDSE 2012' Section B#10]

10. (a) (i)

$$I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,dt$$

$$=\frac{1}{2}\cdot\frac{4-1}{6}\left[\frac{1}{\sqrt{1}}e^{\frac{-1}{2}}+\frac{1}{\sqrt{4}}e^{\frac{-4}{2}}+2\left(\frac{1}{\sqrt{1.5}}e^{\frac{-1.5}{2}}+\frac{1}{\sqrt{2}}e^{\frac{-2}{2}}+\frac{1}{\sqrt{2.5}}e^{\frac{-2.5}{2}}+\frac{1}{\sqrt{3}}e^{\frac{-3}{2}}+\frac{1}{\sqrt{3.5}}e^{\frac{-3.5}{2}}\right)\right]$$ 1M

≈ 0.692913377

≈ 0.6929 1A

(ii)

$$\frac{d}{dt}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}t^{\frac{-3}{2}}e^{\frac{-t}{2}}+t^{\frac{-1}{2}}\cdot\frac{-1}{2}e^{\frac{-t}{2}}$$ 1M+1A

$$=\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$$

$$\frac{d^2}{dt^2}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}\left[e^{\frac{-t}{2}}\left(\frac{-3}{2}t^{\frac{-5}{2}}+\frac{-1}{2}t^{\frac{-3}{2}}\right)+\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)\right]$$ 1M+1A

$$=\frac{1}{4}e^{\frac{-t}{2}}\left(3t^{\frac{-5}{2}}+2t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$$

>0 for $1\le t\le 4$.

Hence the estimation in (i) is an over-estimate. 1

(7)

(b) Let $t = x^2$. 1M

$dt = 2x dx$

When $t = 1$, $x = 1$; when $t = 4$, $x = 2$. $\Big\}$ 1A

$$I = \int_1^4 \frac{1}{\sqrt{t}} e^{\frac{-t}{2}} dt$$

$$= \int_1^2 \frac{1}{x} e^{\frac{-x^2}{2}} 2x dx$$

$$= 2\int_1^2 e^{\frac{-x^2}{2}} dx$$ 1

(3)

(c) $2\int_1^2 e^{\frac{-x^2}{2}} dx < 0.692913377$ 1M

$$2\sqrt{2\pi}\int_1^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx < 0.692913377$$

$2\sqrt{2\pi}(0.4772 - 0.3413) < 0.692913377$ 1A For 0.4772 and 0.3413

$\pi < 3.249593152$

$\therefore \quad \pi < 3.25$ 1

(3)

10	(a)	(i)	Good. Many candidates applied the trapezoidal rule correctly.
		(ii)	Poor. Many candidates used $\frac{d}{dt}\left(t^{\frac{-1}{2}} e^{\frac{-t}{2}}\right)$ instead of $\frac{d^2}{dt^2}\left(t^{\frac{-1}{2}} e^{\frac{-t}{2}}\right)$ to determine whether the estimate in (i) is an over-estimate or under-estimate.
	(b)		Fair. Many candidates used wrong substitutions.
	(c)		Very poor. Only a few candidates attempted this part. Among them, some wrote $I \approx 0.692913377$ instead of $I < 0.692913377$.

11. In a research of the radiation intensity of a city, an expert modelled the rate of change of the radiation intensity R (in suitable units) by

$$\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{a(30-t)+10}{(t-35)^2+b}$$

where t $(0 \le t \le T)$ is the number of days elapsed since the start of the research, a, b and T are positive constants.

It is known that the intensity increased to the greatest value of 6 units at $t=35$, and then decreased to the level as at the start of the research at $t=T$. Moreover, the decrease of the intensity from $t=40$ to $t=41$ is $\ln\frac{61}{50}$ units.

(a) Find the value of a.

(2 marks)

(b) Find the value of T.

(4 marks)

(c) Express R in terms of t.

(4 marks)

(d) For $0 \le t \le 35$, when would the rate of change of the radiation intensity attain its greatest value?

(4 marks)

[HKDSE 2012' Section B#11]

11. (a) When $t=35$, the intensity increased to a maximum and therefore $\frac{\mathrm{d}R}{\mathrm{d}t}=0$.	
$\frac{a(30-35)+10}{(35-35)^2+b}=0$	1A
$a=2$	1A
	(2)
(b) $\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{2(30-t)+10}{(t-35)^2+b}$	
Let $u=(t-35)^2+b$.	1M
$\mathrm{d}u=2(t-35)\mathrm{d}t$	
$R=\int\frac{-2t+70}{(t-35)^2+b}\,\mathrm{d}t$	
$=\int\frac{-2t+70}{u}\frac{\mathrm{d}u}{2(t-35)}$	
$=-\ln\lvert u\rvert+C$	
$=-\ln[(t-35)^2+b]+C$	1A
$R\vert_{t=T}=R\vert_{t=0}$	
$-\ln[(T-35)^2+b]+C=-\ln[(0-35)^2+b]+C$	1M
$(T-35)^2=35^2$	
$T=70$ or 0 (rejected)	1A
	(4)

(c) $R\|_{t=40} - R\|_{t=41} = \ln\dfrac{61}{50}$	
$-\ln[(40-35)^2+b]+C-\{-\ln[(41-35)^2+b]+C\} = \ln\dfrac{61}{50}$	1M
$-\ln(25+b)+\ln(36+b) = \ln\dfrac{61}{50}$	
$\ln\dfrac{36+b}{25+b} = \ln\dfrac{61}{50}$	
$b=25$	1A
$\therefore\ R = -\ln[(t-35)^2+25]+C$	
$R\|_{t=35} = 6$	
$-\ln[(35-35)^2+25]+C=6$	1M
$C = 6+\ln 25$	
i.e. $R = -\ln[(t-35)^2+25]+6+\ln 25$	1A
	(4)
(d) $\dfrac{dR}{dt} = \dfrac{2(30-t)+10}{(t-35)^2+25}$	
$= \dfrac{70-2t}{t^2-70t+1250}$	
$\dfrac{d^2R}{dt^2} = \dfrac{(t^2-70t+1250)(-2)-(70-2t)(2t-70)}{(t^2-70t+1250)^2}$	1M+1A
$= \dfrac{2t^2-140t+2400}{(t^2-70t+1250)^2}$	
When the rate of change of the radiation intensity attains its greatest value, $\dfrac{d^2R}{dt^2}=0$.	
$2t^2-140t+2400=0$	
$t=30$ or 40 (rejected)	
(see table below)	1M
Hence, the rate of change of the radiation intensity would attain its greatest value when $t=30$.	1A
	(4)

t	$0 \le t < 30$	$t=30$	$30 < t \le 35$
$\dfrac{d^2R}{dt^2}$	+ve	0	-ve

11	A common mistake was to mix up R with $\dfrac{dR}{dt}$.
(a)	Fair. However, many candidates knew that maximum intensity implied $\dfrac{dR}{dt}=0$
(b)	Poor. Some candidates were not able to choose a suitable substitution to solve for R, while others did not go on after substitution or made careless mistakes in further calculations.
(c)	Very poor. A common mistake was $R\|_{t=41} - R\|_{t=40} = \ln\dfrac{61}{50}$.
(d)	Very poor. Only a few candidates attempted this part. Among them, some forgot to square the denominator when applying quotient rule to calculate $\dfrac{d^2R}{dt^2}$.

10. (a) Consider the function $f(x) = \ln(x^2+16) - \ln(3x+20)$ for $x > \frac{-20}{3}$.

(i) Find the range of values of x such that $f(x) < 0$.

(ii) Consider the integral $I = \int_0^4 f(x)\,dx$.

(1) Using the trapezoidal rule with 4 subintervals, find an estimate for I .

(2) Determine whether the estimate in (1) is an over-estimate or under-estimate. Justify your answer.

(8 marks)

(b) A certain species of insects lives in a certain environment. Let $N(t)$ (in thousand) be the number of the insects at time t (in months). Assume that $N(t)$ can be treated as a differentiable function when $N(t) > 0$. The birth rate and death rate of the insects at time t are equal to $10\ln(t^2+16)$ and $10\ln(3t+20)$ respectively when $N(t) > 0$. It is given that $N(0) = 8$.

(i) Express $N'(t)$ in terms of t when $N(t) > 0$.

(ii) Jane claims that the species will not die out until $t = 4$. Do you agree? Justify your answer.

(4 marks)

[HKDSE 2013' Section B#10]

10. (a) (i) $\ln(x^2+16)-\ln(3x+20)<0$		
$\ln(x^2+16)<\ln(3x+20)$		
$x^2+16<3x+20$	1A	
$x^2-3x-4<0$		
$-1<x<4$	1A	
(ii) (1) $I=\int_0^4[\ln(x^2+16)-\ln(3x+20)]\mathrm{d}x$		
$\approx\frac{1}{2}[-0.223143551+0+2(-0.302280871-0.262364264-0.148420005)]$	1M	OR $\frac{1}{2}\{(\ln 16-\ln 20)+0+2[(\ln 17-\ln 23)+(\ln 20-\ln 26)+(\ln 25-\ln 29)]\}$
≈ -0.824636917		
≈ -0.8246	1A	
(2) $f(x)=\ln(x^2+16)-\ln(3x+20)$		
$f'(x)=\frac{2x}{x^2+16}-\frac{3}{3x+20}$	1M+1A	OR $\frac{3x^2+40x-48}{(x^2+16)(3x+20)}$
$f''(x)=2\cdot\frac{(x^2+16)-x(2x)}{(x^2+16)^2}-3\cdot\frac{(-1)\cdot 3}{(3x+20)^2}$	1M	
$=\frac{2(4+x)(4-x)}{(x^2+16)^2}+\frac{9}{(3x+20)^2}$		
>0 for $0\le x\le 4$		
Hence the estimate in (1) is an over-estimate.	1A	Follow through
	(8)	
(b) (i) $N'(t)=10\ln(t^2+16)-10\ln(3t+20)$	1A	
(ii) Assume that Jane's claim is true: the species will not die out until $t=4$, i.e. $N(t)>0$ for $0\le t\le 4$.		
$N(4)-N(0)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$	1A	
$N(4)-8<-8.24636917$ (since the estimate is an over-estimate)	1M	
$N(4)<0$		
Hence Jane's claim is false and cannot be agreed with.	1A	Follow through
	(4)	

10. (a) (i)	Satisfactory. Some candidates were not able to solve the inequality $\ln(x^2+16)-\ln(3x+20)<0$. Some considered $[\ln(x^2+16)-\ln(3x+20)]'<0$ instead.
(ii) (1)	Satisfactory. Some candidates did not apply the formula for trapezoidal rule correctly. Some found the absolute value of I instead.
(2)	Satisfactory. After obtaining $f'(x)$, many candidates got a point $x_0\in[0,4]$ such that $f(x)$ would decrease on $[0,x_0)$ and increase on $(x_0,4]$, and then claimed immediately that the estimate in (1) was an over-estimate. Among those who were able to find $f''(x)$, only few showed that $f''(x)>0$ for all $x\in[0,4]$ correctly.
(b) (i)	Poor. A common mistake was to write $N(t)=10\ln(t^2+16)-10\ln(3t+20)$ and then to differentiate both sides of it.
(ii)	Very poor. Common mistakes included writing $N(t)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$ and failing to apply the result of (a)(ii)(2) in additional to that of (a)(ii)(1).

11. Let $\mathrm{P}(t)$ and $\mathrm{C}(t)$ (in suitable units) be the electric energy produced and consumed respectively in a city during the time period $[0, t]$, where t is in years and $t \geq 0$. It is known that $\mathrm{P}'(t) = 4(4 - e^{\frac{-t}{5}})$ and $\mathrm{C}'(t) = 9(2 - e^{\frac{-t}{10}})$. The redundant electric energy being generated during the time period $[0, t]$ is $\mathrm{R}(t)$, where $\mathrm{R}(t) = \mathrm{P}(t) - \mathrm{C}(t)$ and $t \geq 0$.

(a) Find t such that $\mathrm{R}'(t) = 0$.

(3 marks)

(b) Show that $\mathrm{R}'(t)$ decreases with t.

(3 marks)

(c) Find the total redundant electric energy generated during the period when $\mathrm{R}'(t) > 0$.

(3 marks)

(d) The electric energy production is improved at $t = 5$. Let $\mathrm{Q}(t)$ be the electric energy produced during the period $[5, t]$, where $t \geq 5$, and

$$\mathrm{Q}'(t) = \frac{(t+1)\,[\ln(t^2 + 2t + 3)]^3}{t^2 + 2t + 3} + 9 .$$

Find the total electric energy produced for the first 3 years after the improvement.

(5 marks)

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

[HKDSE 2013' Section B#11]

1. (a) $\mathrm{R}'(t) = 0$

$\mathrm{P}'(t) - \mathrm{C}'(t) = 0$

$4(4 - e^{\frac{-t}{5}}) - 9(2 - e^{\frac{-t}{10}}) = 0$ — 1A

$-4\left(e^{\frac{-t}{10}}\right)^2 + 9e^{\frac{-t}{10}} - 2 = 0$ — 1M — For $e^{\frac{-t}{5}} = \left(e^{\frac{-t}{10}}\right)^2$

$e^{\frac{-t}{10}} = 0.25$ or 2

$t = 20\ln 2$ or $-10\ln 2$ (rejected as $t \geq 0$) — 1A — OR $t \approx 13.8629$

(3)

(b) $\mathrm{R}'(t) = -4e^{\frac{-t}{5}} + 9e^{\frac{-t}{10}} - 2$ — 1A

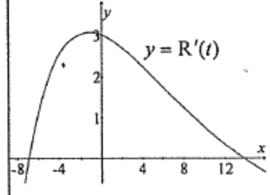
$\mathrm{R}''(t) = \frac{4}{5}e^{\frac{-t}{5}} - \frac{9}{10}e^{\frac{-t}{10}}$

$= \frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}} - 9\right)$ — 1M

< 0 for $t \geq 0$ (since $e^{\frac{-t}{10}} \leq 1$ for $t \geq 0$)

Therefore $\mathrm{R}'(t)$ decreases with t. — 1

(3)



(c) By (a) and (b), $R'(t) > 0$ when $0 \le t < 20\ln 2$.		
The total redundant electric energy generated during the period when $R'(t) > 0$		
$= \int_0^{20\ln 2} \left(-4e^{\frac{-t}{5}} + 9e^{\frac{-t}{10}} - 2 \right) dt$	1M	For lower and upper limits
$= \left[20e^{\frac{-t}{5}} - 90e^{\frac{-t}{10}} - 2t \right]_0^{20\ln 2}$	1A	For primitive function
$= 48.75 - 40\ln 2$	1A	OR 21.0241
	(3)	

(d) Consider $\int_5^8 \frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3} dt$		
Let $u = \ln(t^2+2t+3)$.	1M	
$du = \frac{2t+2}{t^2+2t+3} dt$	1A	
When $t = 5$, $u = \ln 38$; when $t = 8$, $u = \ln 83$.		
$\therefore \int_5^8 \frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3} dt = \int_{\ln 38}^{\ln 83} u^3 \frac{du}{2}$	1A	For $\frac{u^3}{2}$
$= \frac{1}{8}\left[u^4 \right]_{\ln 38}^{\ln 83}$		
$= \frac{1}{8}[(\ln 83)^4 - (\ln 38)^4]$		
Hence the total electric energy produced for the first 3 years after the improvement		
$= \int_5^8 \left[\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3} + 9 \right] dt$	1A	
$= \int_5^8 \frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3} dt + \int_5^8 9\, dt$		
$= \frac{1}{8}[(\ln 83)^4 - (\ln 38)^4] + [9t]_5^8$		
$= \frac{1}{8}[(\ln 83)^4 - (\ln 38)^4] + 27$	1A	OR 52.7730
	(5)	

11. (a)	Fair. Some candidates confused $R(t)$ with $R'(t)$, or found $R(t) = P(t) - C(t)$ by integration first and then obtained the expression for $R'(t) = P'(t) - C'(t)$ by differentiation. Many candidates failed to make use of knowledge about quadratic equations to solve for t . Some got wrong answers such as '$e^{\frac{-t}{5}} = 0.25$ or 2' or did not reject $t = -10\ln 2$.
(b)	Very poor. Many candidates failed to find $R''(t)$ correctly. Among those who were able to find $R''(t)$, only few provided sufficient reasons to conclude that '$R'(t)$ decreases with t'.
(c)	Very poor. Common mistakes included putting wrong values as limits of the definite integral involved and getting wrong primitives of its integrand.
(d)	Poor. Few candidates were able to use correctly the method of substitution for integration. Among them, some put wrong values as limits of definite integrals, while some others missed out the term $\int_5^8 9\, dt$ or wrote $\int_{\ln 38}^{\ln 83} 9\, dt$.

10. (a) (i) Find $\dfrac{\mathrm{d}}{\mathrm{d}v}(ve^{-v})$.

(ii) Using (a)(i), or otherwise, show that $\displaystyle\int ve^{-v}\,\mathrm{d}v = -e^{-v}(1+v)+C$, where C is a constant.

(3 marks)

(b)

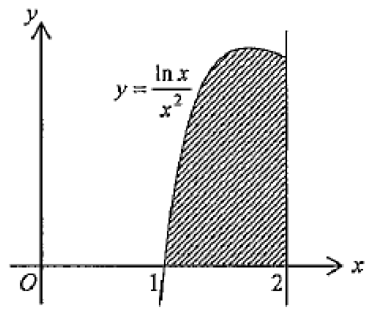


Figure 1

Figure 1 shows a shaded region bounded by the curve $y=\dfrac{\ln x}{x^2}$, the line $x=2$ and the x-axis. Using a suitable substitution and the result of (a), show that the area of the shaded region is $\dfrac{1-\ln 2}{2}$.

(5 marks)

(c) (i) Find $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\dfrac{\ln x}{x^2}\right)$.

(ii) Using (b) and (c)(i), show that

$$\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\frac{\ln 1.3}{1.3^2}+\cdots+\frac{\ln 1.9}{1.9^2}<5-\frac{41}{8}\ln 2 .$$

(6 marks)

[HKDSE 2014' Section B#10]

10. (a) (i) $\dfrac{\mathrm{d}}{\mathrm{d}v}(ve^{-v})=e^{-v}-ve^{-v}$ 1A

(ii) $ve^{-v}=e^{-v}-\dfrac{\mathrm{d}}{\mathrm{d}v}ve^{-v}$

$\displaystyle\int ve^{-v}\mathrm{d}v=\int e^{-v}\mathrm{d}v-ve^{-v}$ 1M

$=-e^{-v}-ve^{-v}+C$

$=-e^{-v}(1+v)+C$ 1

(3)

(b) The area of the shaded region $= \int_1^2 \frac{\ln x}{x^2}\,dx$ — 1A

Let $x = e^u$. — OR $u = \ln x$

$dx = e^u\,du$ — 1M

When $x=1$, $u=0$; when $x=2$, $u=\ln 2$

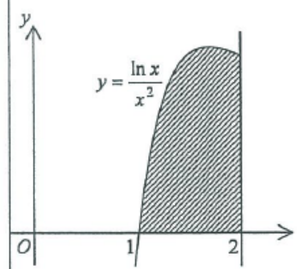
$\therefore$ the area $= \int_0^{\ln 2} \frac{u}{e^{2u}} \cdot e^u\,du$ — 1A

$= \int_0^{\ln 2} u e^{-u}\,du$

$= [-e^{-u}(1+u)]_0^{\ln 2}$ by (a) — 1M

$= \frac{-1}{2}(1+\ln 2)+1$

$= \frac{1-\ln 2}{2}$ — 1



(5)

(c) (i) $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2}$

$= \frac{1-2\ln x}{x^3}$ — 1A

$\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right) = \frac{x^3 \cdot \frac{-2}{x} - (1-2\ln x)3x^2}{x^6}$

$= \frac{6\ln x - 5}{x^4}$ — 1A

(ii) $\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right) < 0$ when $x < e^{\frac{5}{6}} \approx 2.30098$ — 1M — OR when $1 \le x \le 2$

Hence the trapezoidal rule will underestimate $\int_1^2 \frac{\ln x}{x^2}\,dx$. — 1A

Consider the trapezoidal rule with 10 intervals.

$\therefore \frac{1}{2} \cdot \frac{1}{10}\left[\frac{\ln 1}{1^2} + 2\left(\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \cdots + \frac{\ln 1.9}{1.9^2}\right) + \frac{\ln 2}{2^2}\right] < \frac{1-\ln 2}{2}$ — 1M — For L.H.S.

$0 + 2\left(\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \cdots + \frac{\ln 1.9}{1.9^2}\right) + \frac{\ln 2}{4} < 10 - 10\ln 2$

$\frac{\ln 1.1}{1.1^2} + \frac{\ln 1.2}{1.2^2} + \frac{\ln 1.3}{1.3^2} + \cdots + \frac{\ln 1.9}{1.9^2} < 5 - \frac{41}{8}\ln 2$ — 1

(6)

10. (a) (i)	Excellent.
(ii)	Fair. Some candidates wrote nothing but only the expression provided in the question.
(b)	Fair. Some candidates failed to write the correct integral for the area. Some omitted details showing why $[-e^{-u}(1+u)]_0^{\ln 2} = \frac{1-\ln 2}{2}$.
(c) (i)	Satisfactory. Careless mistakes prevented some candidates from obtaining the correct answer.
(ii)	Very poor. Most candidates did not attempt this part.

11. An engineer models the rates of change of the amount of oil produced (in hundred barrels per day) by oil companies X and Y respectively by

$$f(t)=\ln(e^t-t) \quad \text{and} \quad g(t)=\frac{8t}{1+t},$$

where t ($2 \le t \le 12$) is the time measured in days.

(a) Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of oil produced by oil company X from $t=2$ to $t=12$. (3 marks)

(b) Determine whether the estimate in (a) is an over-estimate or an under-estimate. Explain your answer. (3 marks)

(c) Find $\int \frac{t}{1+t}\,dt$. (3 marks)

(d) The engineer claims that the total amount of oil produced by oil company X from $t=2$ to $t=12$ is less than that of oil company Y. Do you agree? Explain your answer. (3 marks)

[HKDSE 2015' Section B#11]

11. (a) The total amount of oil produced by oil company X		
$=\int_2^{12} f(t)\,dt$	1M	
$\approx \frac{1}{2}\left(\frac{12-2}{5}\right)(f(2)+f(12)+2(f(4)+f(6)+f(8)+f(10)))$	1M	
≈ 69.49587529		
≈ 69.4959 hundred barrels	1A	r.t. 69.4959
	(3)	
(b) $\frac{df(t)}{dt}$		
$=\frac{e^t-1}{e^t-t}$	1A	
$\frac{d^2f(t)}{dt^2}$		
$=\frac{(e^t-t)e^t-(e^t-1)(e^t-1)}{(e^t-t)^2}$	1A	
$=\frac{e^t(2-t)-1}{(e^t-t)^2}$		
<0 (since $2 \le t \le 12$)		
Thus, the estimate in (a) is an under-estimate.	1A	f.t.
	(3)	

(c) Let $u=1+t$. Then, we have $\frac{du}{dt}=1$.	1M	
$\int \frac{t}{1+t}dt$ $=\int \frac{u-1}{u}du$ $=\int \left(1-\frac{1}{u}\right)du$	1A	
$=u-\ln u+\text{constant}$ $=t-\ln(1+t)+\text{constant}$	1A	
Note that $\frac{t}{1+t}=1-\frac{1}{1+t}$.	1A	
$\int \frac{t}{1+t}dt$ $=\int \left(1-\frac{1}{1+t}\right)dt$	1M	
$=t-\ln(1+t)+\text{constant}$	1A	
	----------(3)	
(d) The total amount of oil produced by oil company Y $=8\int_{2}^{12}\frac{t}{1+t}dt$		
$=8\left[t-\ln(1+t)\right]_{2}^{12}$ (by (c))	1M	for using the result of (c)
≈ 68.26930345	1A	
<69.49587529 By (b), the claim is disagreed.	1A	f.t.
	----------(3)	

11 (a)	Good. Some candidates did not formulate the required amount as a definite integral, and some candidates did not use the correct number of sub-intervals when applying the trapezoidal rule.
(b)	Fair. Many candidates failed to find $\frac{d^2f(t)}{dt^2}$ correctly, as a result, many candidates failed to determine the nature of the estimate in (a).
(c)	Very good. Most candidates were able to find the indefinite integral by using the method of substitution.
(d)	Good. Many candidates were able to find the correct total amount of oil produced by company Y for comparison. However, some candidates failed to show that the estimate in (a) is an under-estimate and hence could not complete the argument.

11. An investment consultant, Albert, predicts the total profit made by a factory in the coming year. He models the rate of change of profit (in million dollars per month) made by the factory by

$$A(t) = \ln(t^2 - 8t + 95) \ ,$$

where $t \ (0 \le t \le 12)$ is the number of months elapsed since the prediction begins. Let P_1 million dollars be the total profit made by the factory in the coming year under Albert's model.

(a) (i) Using the trapezoidal rule with 4 sub-intervals, estimate P_1 .

(ii) Find $\dfrac{d^2A(t)}{dt^2}$.

(4 marks)

(b) The factory manager, Christine, models the rate of change of profit (in million dollars per month) made by the factory in the coming year by

$$B(t) = \frac{t+8}{\sqrt{t+3}} \ ,$$

where $t \ (0 \le t \le 12)$ is the number of months elapsed since the prediction begins. Let P_2 million dollars be the total profit made by the factory in the coming year under Christine's model.

(i) Find P_2 .

(ii) Albert claims that the difference between P_1 and P_2 does not exceed 2 . Do you agree? Explain your answer.

(9 marks)

[HKDSE 2016' Section B#11]

11. (a) (i) P_1		
$= \int_0^{12} A(t)\,dt$		
$\approx \frac{1}{2}\left(\frac{12-0}{4}\right)(A(0)+A(12)+2(A(3)+A(6)+A(9)))$	1M	
≈ 54.61085671		
≈ 54.6109	1A	r.t. 54.6109
(ii) $\frac{dA(t)}{dt}$		
$= \frac{2t-8}{t^2-8t+95}$	1A	
$\frac{d^2A(t)}{dt^2}$		
$= \frac{2(t^2-8t+95)-(2t-8)^2}{(t^2-8t+95)^2}$		
$= \frac{-2t^2+16t+126}{(t^2-8t+95)^2}$	1A	
$= \frac{-2(t^2-8t-63)}{(t^2-8t+95)^2}$		
	----------(4)	

(b) (i) Let $u=t+3$.	1M	
Then, we have $\dfrac{\mathrm{d}u}{\mathrm{d}t}=1$.		
P_2		
$=\int_0^{12}\mathrm{B}(t)\,\mathrm{d}t$	1M	
$=\int_0^{12}\dfrac{t+8}{\sqrt{t+3}}\,\mathrm{d}t$		
$=\int_3^{15}\dfrac{u-3+8}{\sqrt{u}}\,\mathrm{d}u$	1A	
$=\int_3^{15}(u^{\frac{1}{2}}+5u^{\frac{-1}{2}})\,\mathrm{d}u$		
$=\left[\dfrac{2}{3}u^{\frac{3}{2}}+10u^{\frac{1}{2}}\right]_3^{15}$	1M	
$=20\sqrt{15}-12\sqrt{3}$	1A	r.t. 56.6751
≈ 56.67505723		
≈ 56.6751		

(b) (ii) $\dfrac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}=\dfrac{-2[t-(4-\sqrt{79})][t-(4+\sqrt{79})]}{(t^2-8t+95)^2}$	1M	1M for considering $\dfrac{\mathrm{d}^2A(t)}{\mathrm{d}t^2}$
Note that $4-\sqrt{79}<0$ and $4+\sqrt{79}>12$.		
Therefore, we have $\dfrac{\left(t-(4-\sqrt{79})\right)\left(t-(4+\sqrt{79})\right)}{(t^2-8t+95)^2}<0$		
for $0\le t\le 12$.		
Hence, we have $\dfrac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}>0$ for $0\le t\le 12$.	1A	f.t.
So, the estimate of P_1 is an over-estimate. $P_1<54.61085671$.		
P_2-P_1		
$=20\sqrt{15}-12\sqrt{3}-P_1$		
$>20\sqrt{15}-12\sqrt{3}-54.61085671$	1M	
≈ 2.064200523		
>2		
Thus, the claim is disagreed.	1A ----------(9)	f.t.

11 (a) (i)	Very good. More than 60% of the candidates were able to find the correct answer using trapezoidal rule. However, a small number of candidates were unable to use the correct sub-intervals when applying the trapezoidal rule.
(ii)	Good. Many candidates were able to find $\dfrac{\mathrm{dA}(t)}{\mathrm{d}t}$ by quotient rule, but some candidates were unable to simplify $\dfrac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}$.
(b) (i)	Very good. Most candidates were able to formulate and evaluate the definite integral $\int_0^{12}\dfrac{t+8}{\sqrt{t+3}}\mathrm{d}t$ by using a suitable substitution.
(ii)	Poor. Most candidates just mentioned $\dfrac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}>0$ without proof. They showed difficulties in using inequality to express the relation between P_1 and its over-estimate, hence unable to complete the argument.

11. Let $f(x)=\dfrac{e^{0.1x}}{x}$. Define $I=\int_{0.5}^{1} f(x)\,dx$. In order to estimate the value of I , Ada suggests using trapezoidal rule with 5 sub-intervals while Billy suggests replacing $e^{0.1x}$ with $1+0.1x+0.005x^2$ and then evaluating the integral.

(a) Find the estimates of I according to the suggestions of Ada and Billy respectively. (5 marks)

(b) Determine each of the two estimates in (a) is an over-estimate or an under-estimate. Explain your answer. (6 marks)

(c) Someone claims that the difference of I and 0.746 is less than 0.002 . Do you agree? Explain your answer. (2 marks)

[HKDSE 2017' Section B#11]

11. (a) According to the suggestion by Ada,		
I		
$\approx \frac{1}{2}\left(\frac{1-0.5}{5}\right)(f(0.5)+f(1)+2(f(0.6)+f(0.7)+f(0.8)+f(0.9)))$	1M	
≈ 0.747559672		
≈ 0.7476	1A	r.t. 0.7476
According to the suggestion by Billy,		
I		
$\approx \int_{0.5}^{1}\left(\frac{1}{x}+0.1+0.005x\right)dx$	1M	
$=\left[\ln x+0.1x+0.0025x^2\right]_{0.5}^{1}$	1M	
$=\ln 2+0.051875$	1A	r.t. 0.7450
≈ 0.74502218		
≈ 0.7450		
	----------(5)	

(b) $f(x)$		
$= \dfrac{e^{0.1x}}{x}$		
$f'(x)$		
$= \dfrac{0.1\,e^{0.1x}}{x^2}(x-10)$	1M	
$f''(x)$		
$= \dfrac{0.01\,e^{0.1x}}{x^3}(x^2-20x+200)$	1A	
$= \dfrac{0.01\,e^{0.1x}}{x^3}((x-10)^2+100)$	1M	
>0 for $0.5 \le x \le 1$.		
Thus, the estimate suggested by Ada is an over-estimate.	1A	f.t.
$e^{0.1x} = 1 + 0.1x + \dfrac{(0.1x)^2}{2!} + \dfrac{(0.1x)^3}{3!} + \cdots$	1M	
$e^{0.1x} > 1 + 0.1x + 0.005x^2$ for $0.5 \le x \le 1$		
$I > \displaystyle\int_{0.5}^{1} \left(\frac{1}{x} + 0.1 + 0.005x\right) dx$		
Thus, the estimate suggested by Billy is an under-estimate.	1A	f.t.
	----------(6)	
(c) $0.7450 < I < 0.7476$	1M	
$-0.0010 < I - 0.746 < 0.0016$		
So, we have $-0.002 < I - 0.746 < 0.002$.		
Thus, the claim is agreed.	1A	f.t.
$I - 0.746 < 0.7476 - 0.746 = 0.0016$	1M	
$0.746 - I < 0.746 - 0.7450 = 0.0010$		
So, the difference of I and 0.746 is less than 0.002 .		
Thus, the claim is agreed.	1A	f.t.
	----------(2)	

11 (a)	Very good. Most candidates were able to use correct sub-intervals when applying the trapezoidal rule to find an estimate of I .
(b)	Fair. Many candidates were unable to find $\dfrac{d^2f(t)}{dt^2}$ correctly, hence they were unable to determine the nature of the estimate according to the suggestion of Ada in (a).
(c)	Poor. Most candidates did not prove that one of the estimates in (a) is an over-estimate while the other is an under-estimate, hence they were unable to finish the argument.

11. In a research, the rate of change of the distance (in cm/s) travelled by a particle is given by

$$A(t)=60(1+10t)e^{-2t} \ ,$$

where t is the number of seconds elapsed since the start of the research. Let D cm be the distance travelled by the particle from $t=0.1$ to $t=0.5$. Denote the estimate of D by using the trapezoidal rule with 4 sub-intervals by D_1 .

(a) (i) Find D_1 .

(ii) Is D_1 an over-estimate or an under-estimate? Explain your answer.

(6 marks)

(b) In order to estimate D , a researcher, Mary, models the rate of change of the distance travelled by the particle by

$$B(t)=\frac{50(1+10t)}{1+2t} \ ,$$

where t is the number of seconds elapsed since the start of the research. Let D_2 cm be the distance travelled by the particle from $t=0.1$ to $t=0.5$ under this model.

(i) Find D_2 .

(ii) Mary claims that in order to estimate D , D_2 is more accurate than D_1 . Do you agree? Explain your answer.

(6 marks)

[HKDSE 2018' Section B#11]

11. (a) (i)	D_1 $=\frac{1}{2}\left(\frac{0.5-0.1}{4}\right)(\mathrm{A}(0.1)+\mathrm{A}(0.5)+2(\mathrm{A}(0.2)+\mathrm{A}(0.3)+\mathrm{A}(0.4)))$	1M	
	≈ 50.25132348 ≈ 50.2513	1A	r.t. 50.2513
(ii)	$\frac{dA}{dt}$ $=60(e^{-2t}(10)+(1+10t)e^{-2t}(-2))$ $=480e^{-2t}-1200te^{-2t}$	1A	
	For all $t\in[0.1,0.5]$, $\frac{d^2A}{dt^2}$ $=480e^{-2t}(-2)-1200e^{-2t}-1200te^{-2t}(-2)$	1M	
	$=2400te^{-2t}-2160e^{-2t}$	1A	
	$=240e^{-2t}(10t-9)$ <0 Thus, D_1 is an under-estimate of D .	1A	f.t.
		----------(6)	
(b) (i)	Let $u=1+2t$.	1M	
	Then, we have $\frac{du}{dt}=2$. D_2 $=\int_{0.1}^{0.5}\mathrm{B}(t)\,dt$	1M	
	$=25\int_{1.2}^{2}\frac{5u-4}{u}du$ $=25[5u-4\ln u]_{1.2}^{2}$	1M	
	$=100-100\ln\frac{5}{3}$ ≈ 48.91743762 ≈ 48.9174	1A	r.t. 48.9174
	Note that $\frac{1+10t}{1+2t}=\frac{-4}{1+2t}+5$.	1M	
	D_2 $=\int_{0.1}^{0.5}\mathrm{B}(t)\,dt$	1M	
	$=50[-2\ln(1+2t)+5t]_{0.1}^{0.5}$	1M	
	$=100-100\ln\frac{5}{3}$ ≈ 48.91743762 ≈ 48.9174	1A	r.t. 48.9174
(ii)	By (a)(ii), D_1 is an under-estimate of D . Since $D_2<D_1$, we have $D_2<D_1<D$.	1M	
	Thus, the claim is disagreed.	1A	f.t.
		----------(6)	

11 (a) (i)	Very good. Most candidates were able to use correct sub-intervals when applying the trapezoidal rule to find D_1 .
(ii)	Good. Many candidates were able to consider the second derivative $\frac{d^2A}{dt^2}$ to determine whether D_1 is an over-estimate or an under-estimate.
(b) (i)	Good. Many candidates were able to use integration by substitution to find D_2 .
(ii)	Poor. Most candidates were unable to use a compound inequality such as $D_2<D_1<D$ to finish the argument.

11. A steel factory has two machines, P and Q, for producing steel. The two machines start production at the same time. The manager of the factory models the rates of change of the amount of steel produced (in thousand tonnes per month) by P and Q respectively by

$$\mathrm{p}(t)=2t\ln(t^2+4) \quad \text{and} \quad \mathrm{q}(t)=\frac{4\ln(2e^t+1)}{e^{-t}+2} \quad (0\le t\le 4)\ ,$$

where t is the number of months elapsed since the steel production begins. Denote the total amount of steel produced by P in the first 4 months by α thousand tonnes. Let α_1 be the estimate of α by using the trapezoidal rule with 4 sub-intervals.

(a) (i) Find α_1 .

(ii) Is α_1 an over-estimate or an under-estimate? Explain your answer.

(6 marks)

(b) Let β thousand tonnes be the total amount of steel produced by Q in the first 4 months.

(i) Using the substitution $u=\ln(2e^t+1)$, find β .

(ii) The manager claims that the total amount of steel produced by Q in the first 4 months exceeds 30% of the sum of the total amount of steel produced by P and Q in the first 4 months. Do you agree? Explain your answer.

(6 marks)

[HKDSE 2019' Section B#11]

11. (a) (i) α_1		
$=\frac{1}{2}\left(\frac{4-0}{4}\right)(\mathrm{p}(0)+\mathrm{p}(4)+2(\mathrm{p}(1)+\mathrm{p}(2)+\mathrm{p}(3)))$	1M	
$=2\ln 281216000$	1A	r.t. 38.9093
≈ 38.90926723		
≈ 38.9093		
(ii) $\frac{\mathrm{dp}(t)}{\mathrm{d}t}$		
$=\frac{4t^2}{t^2+4}+2\ln(t^2+4)$	1A	
$\frac{\mathrm{d}^2\mathrm{p}(t)}{\mathrm{d}t^2}$		
$=4\left(\frac{(t^2+4)(2t)-(t^2)(2t)}{(t^2+4)^2}\right)+\frac{4t}{t^2+4}$	1M	
$=\frac{32t}{(t^2+4)^2}+\frac{4t}{t^2+4}$	1A	
$=\frac{4t(t^2+12)}{(t^2+4)^2}$		
$\frac{\mathrm{d}^2\mathrm{p}(t)}{\mathrm{d}t^2}=0$ when $t=0$ and $\frac{\mathrm{d}^2\mathrm{p}(t)}{\mathrm{d}t^2}>0$ for $0<t\le 4$.		
Thus, α_1 is an over-estimate.	1A	f.t.
	----------(6)	
(b) (i) Let $u=\ln(2e^t+1)$.		
So, we have $\frac{du}{dt}=\frac{2e^t}{2e^t+1}$.	1A	
β		
$=\int_0^4 \mathrm{q}(t)\,\mathrm{d}t$	1M	
$=\int_0^4 \frac{4e^t\ln(2e^t+1)}{(2+e^{-t})e^t}\mathrm{d}t$		
$=\int_0^4 \frac{4e^t\ln(2e^t+1)}{2e^t+1}\mathrm{d}t$		
$=2\int_{\ln 3}^{\ln(2e^4+1)} u\,\mathrm{d}u$	1M	
$=\left[u^2\right]_{\ln 3}^{\ln(2e^4+1)}$		
$=(\ln(2e^4+1))^2-(\ln 3)^2$	1A	r.t. 20.9043
≈ 20.90433138		
≈ 20.9043		

(b) (ii) By (a)(ii), α_1 is an over-estimate of α . $\alpha_1 > \alpha$ $\alpha_1 + \beta > \alpha + \beta$ $\frac{\beta}{\alpha+\beta} > \frac{\beta}{\alpha_1+\beta}$	1M	
$\frac{\beta}{\alpha_1+\beta}$ $= \frac{(\ln(2e^4+1))^2-(\ln 3)^2}{2\ln 281216000+(\ln(2e^4+1))^2-(\ln 3)^2}$ ≈ 0.349491284 > 0.3 So, we have $\frac{\beta}{\alpha+\beta} > \frac{\beta}{\alpha_1+\beta} > 30\%$. Thus, the claim is agreed.	1A	f.t.
By (a)(ii), α_1 is an over-estimate of α . $\alpha_1 > \alpha$ $0.3(\alpha_1+\beta) > 0.3(\alpha+\beta)$	1M	
$0.3(\alpha_1+\beta)$ $= 0.3\left(2\ln 281216000+(\ln(2e^4+1))^2-(\ln 3)^2\right)$ ≈ 17.94407958 $< \beta$ So, we have $\beta > 0.3(\alpha_1+\beta) > 0.3(\alpha+\beta)$. Thus, the claim is agreed.	1A	f.t.
	----------(6)	

7 – Further Probability

7. Let A and B be two events such that $P(A|B)=0.4$, $P(A\cup B)=0.45$ and $P(B')=0.75$, where B' is the complementary event of B .

   (a) Find $P(A\cap B)$ and $P(A)$.

   (b) Are events A and B independent? Justify your answer.

(6 marks)

[HKDSE 2014' Section A#7]

7. (a)	$P(A\cap B)=P(B)P(A\mid B)$	
	$=(1-0.75)\times 0.4$	1M
	$=0.1$	1A
	$P(A\cup B)=P(A)+P(B)-P(A\cap B)$	
	$0.45=P(A)+(1-0.75)-0.1$	1M
	$P(A)=0.3$	1A
(b)	$P(A\mid B)=0.4$	
	$\neq P(A)$	1M
	Alternative Solution $P(A)P(B)=0.3\times 0.25$ $=0.075$ $\neq P(A\cap B)$	1M
	Hence A and B are not independent events.	1
		(6)

7.	(a) (b)	Excellent. Good. Few candidates wrote $P(A\cap B)=0.1$ for (a) and then $P(A)\cdot P(B)=\cdots\neq 0.1\neq P(A\cap B)$ for (b); while others tried to make conclusion by comparing $P(A\cap B)$ with 0 , or $P(A\mid B)$ with $P(A)\cdot P(B)$.

2. A and B are two events. Suppose that $P(A)=0.3$, $P(B)=0.28$ and $P(B'|A')=0.6$, where A' and B' are the complementary events of A and B respectively.

(a) Find $P(A'\cap B')$ and $P(A'\cap B)$.

(b) Are A and B mutually exclusive? Explain your answer.

(6 marks)

[HKDSE 2015' Section A#2]

2. (a) $P(A'\cap B')$		
$=P(B'\mid A')P(A')$		
$=0.6(1-0.3)$	1M	
$=0.42$	1A	
$P(A'\cap B)$		
$=P(A')-P(A'\cap B')$		
$=1-0.3-0.42$	1M	
$=0.28$	1A	
(b) Note that $P(B)=P(A\cap B)+P(A'\cap B)$.		
Since $P(B)=P(A'\cap B)=0.28$, we have $P(A\cap B)=0$.	1M	
Thus, A and B are mutually exclusive.	1A	f.t.
	----------(6)	

2 (a)	Very good. Most candidates were able to find the value of $P(A'\cap B')$ while a few candidates failed to find the value of $P(A'\cap B)$ properly.
(b)	Fair. Many candidates mixed up mutually exclusive events with independent events. Only some candidates were able to mention $P(A\cap B)=0$ to conclude that A and B are mutually exclusive events.

1. Let X and Y be two events such that $P(X)=0.4$, $P(Y)=0.7$ and $P(Y\mid X)=0.5$.

(a) Are X and Y independent? Explain your answer.

(b) Find $P(X\cup Y)$.

(5 marks)

[HKDSE 2016' Section A#1]

1. (a) $P(Y\mid X)$ $=0.5$ $\neq 0.7$ $=P(Y)$	1M	
Thus, X and Y are not independent.	1A	f.t.
$P(X)P(Y)$ $=(0.4)(0.7)$ $=0.28$ $P(X\cap Y)$ $=P(Y\mid X)P(X)$ $=(0.5)(0.4)$ $=0.2$ $P(X\cap Y)\neq P(X)P(Y)$	1M	
Thus, X and Y are not independent.	1A	f.t.
(b) $P(X\cap Y)$ $=P(Y\mid X)P(X)$ $=(0.5)(0.4)$	1M	
$=0.2$ $P(X\cup Y)$ $=P(X)+P(Y)-P(X\cap Y)$ $=0.4+0.7-0.2$	1M	
$=0.9$	1A	
	----------(5)	

1 (a)	Very good. More than 70% of the candidates were able to mention $P(Y\mid X)\neq P(Y)$ or $P(X\cap Y)\neq P(X)P(Y)$ to conclude that A and B were not independent events. Only some candidates were unable to show their numerical values in comparison.
(b)	Very good. A very high proportion of the candidates were able to use the identity $P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$ to find the value of $P(X\cup Y)$ while a few candidates were unable to find the value of $P(X\cap Y)$.

2. Let A and B be two events. Suppose that $\mathrm{P}(A)=0.2$, $\mathrm{P}(B')=0.7$ and $\mathrm{P}(A|B)=0.6$, where B' is the complementary event of B .

(a) Find $\mathrm{P}(B|A)$.

(b) Are A and B mutually exclusive? Explain your answer.

(c) Are A and B independent? Explain your answer.

(6 marks)

[HKDSE 2017' Section A#2]

2. (a) $\mathrm{P}(B\|A)$		
$=\dfrac{\mathrm{P}(A\|B)\mathrm{P}(B)}{\mathrm{P}(A)}$		
$=\dfrac{\mathrm{P}(A\|B)(1-\mathrm{P}(B'))}{\mathrm{P}(A)}$		
$=\dfrac{0.6(1-0.7)}{0.2}$	1M	
$=0.9$	1A	
(b) $\mathrm{P}(A\cap B)$		
$=\mathrm{P}(A\|B)\mathrm{P}(B)$		
$=\mathrm{P}(A\|B)(1-\mathrm{P}(B'))$		
$=0.6(1-0.7)$		
$=0.18$		
$\neq 0$	1M	
Thus, A and B are not mutually exclusive.	1A	f.t.
(c) Note that $\mathrm{P}(A\|B)=0.6\neq 0.2=\mathrm{P}(A)$.	1M	
Thus, A and B are not independent.	1A	f.t.
Note that $\mathrm{P}(A\cap B)=0.18\neq 0.06=\mathrm{P}(A)\mathrm{P}(B)$.	1M	
Thus, A and B are not independent.	1A	f.t.
	----------(6)	

2 (a)	Very good. Over 90% of the candidates were able to find the value of $\mathrm{P}(B\|A)$ by using Bayes' Theorem.
(b)	Very good. Most candidates were able to conclude that A and B are not mutually exclusive events.
(c)	Very good. About 80% of the candidates were able to conclude that A and B are not independent events.

1. Let A and B be two events. Suppose that $P(A)=0.8$, $P(B|A)=0.45$ and $P(B|A')=0.6$, where A' is the complementary event of A . Find

(a) $P(B)$,

(b) $P(A|B)$,

(c) $P(A \cup B)$.

(5 marks)

[HKDSE 2018' Section A#1]

1. (a) $P(B)$ $=P(B\|A)P(A)+P(B\|A')P(A')$ $=(0.45)(0.8)+(0.6)(1-0.8)$ $=0.48$	 1M 1A	
$P(B\cap A)$ $=P(B\|A)P(A)$ $=(0.45)(0.8)$ $=0.36$ $P(B\cap A')$ $=P(B\|A')P(A')$ $=(0.6)(1-0.8)$ $=0.12$ $P(B)$ $=P(B\cap A)+P(B\cap A')$ $=0.36+0.12$ $=0.48$	 1M 1A	
(b) $P(A\|B)$ $=\dfrac{P(B\|A)P(A)}{P(B)}$ $=\dfrac{(0.45)(0.8)}{0.48}$ $=0.75$	 1M 1A	
(c) $P(B\cap A)$ $=P(B\|A)P(A)$ $=(0.45)(0.8)$ $=0.36$ $P(A\cup B)$ $=P(A)+P(B)-P(A\cap B)$ $=0.8+0.48-0.36$ $=0.92$	 1A ----------(5)	

1 (a)	Very good. Over 80% of the candidates were able to find $P(B)$ by using $P(B)=P(B\|A)P(A)+P(B\|A')P(A')$.
(b)	Very good. About 80% of the candidates were able to find the value of $P(A\|B)$ by using Bayes' Theorem.
(c)	Very good. Most candidates were able to find the value of $P(A\cup B)$.

2. Let A and B be two events. Denote the complementary events of A and B by A' and B' respectively. Suppose that $P(A' \cap B) = 0.12$ and $P(B' \mid A') = 2P(A)$.

(a) By considering $P(A' \cap B')$, or otherwise, find $P(A)$.

(b) If A and B are independent, find $P(B)$.

(6 marks)

[HKDSE 2019' Section A#2]

2. (a) Let $P(A) = a$.		
$P(A' \cap B')$		
$= P(B' \mid A')P(A')$	1M	for conditional probability
$= 2a(1-a)$		
$= 2a - 2a^2$		
$P(A') = P(A' \cap B') + P(A' \cap B)$		
$1 - a = 2a - 2a^2 + 0.12$	1M	
$2a^2 - 3a + 0.88 = 0$		
$a = 0.4$ or $a = 1.1$ (rejected)		
Thus, we have $P(A) = 0.4$.	1A	
(b) $P(A)P(B) = P(A \cap B)$	1M	
$P(A)P(B) = P(B) - P(A' \cap B)$		
$0.4P(B) = P(B) - 0.12$	1M	for using the result of (a)
$0.6P(B) = 0.12$		
$P(B) = 0.2$	1A	
	----------(6)	

8 – Expectation and Variance

8. Let X be a discrete random variable with probability function shown below:

x	1	3	4	6	9	13
$P(X=x)$	0.1	a	0.25	0.15	b	0.05

where a and b are constants. It is known that $E(X)=5.5$.

(a) Find the values of a and b .

(b) Let F be the event that $X \geq 4$ and G be the event that $X < 8$.

(i) Find $P(F \cap G)$.

(ii) Are F and G independent events? Justify your answer.

(6 marks)

[HKDSE 2012' Section A#8]

8. (a) $P(X=1)+P(X=3)+\cdots+P(X=13)=1$

$0.1+a+0.25+0.15+b+0.05=1$ — 1M

$a+b=0.45$ ---------- (1)

$E(X)=5.5$

$1\times0.1+3a+4\times0.25+6\times0.15+9b+13\times0.05=5.5$ — 1M

$a+3b=0.95$ ---------- (2)

Solving (1) and (2), we get $a=0.2$ and $b=0.25$. — 1A For both

(b) (i) $P(F \cap G)=0.25+0.15$

$=0.4$ — 1A

(ii) $P(F)\times P(G)=(0.25+0.15+0.25+0.05)(0.1+0.2+0.25+0.15)$

$=0.49$ — 1A

$\neq P(F \cap G)$

Alternative Solution 1

$P(F \mid G)=\dfrac{P(F\cap G)}{P(G)}$

$=\dfrac{0.4}{0.1+0.2+0.25+0.15}$

≈ 0.571428571 — 1A

$P(F)=0.25+0.15+0.25+0.05$

$=0.7$

$\neq P(F \mid G)$

Alternative Solution 2

$P(G \mid F)=\dfrac{P(F\cap G)}{P(F)}$

$=\dfrac{0.4}{0.25+0.15+0.25+0.05}$

≈ 0.571428571 — 1A

$P(G)=0.1+0.2+0.25+0.15$

$=0.7$

$\neq P(G \mid F)$

Hence, F and G are not independent. — 1

(6)

8	(a)	Excellent.
	(b)	Satisfactory. Quite a number of candidates did not understand the concept of independence — some calculated $P(F \cap G)$ using $P(F)\times P(G)$ and some mixed up independent events with mutually exclusive events.

7. Let X and Y be two independent discrete random variables with their respective probability distributions shown as follows:

x	0	1	3	5	7
$\mathrm{P}(X=x)$	0.2	0.3	0.3	0.1	0.1

y	1	2	4	m
$\mathrm{P}(Y=y)$	0.4	0.3	0.2	0.1

Suppose that $\mathrm{E}(Y)=2.4$.

(a) Find the value of m .

(b) Let A be the event that $X+Y\le 2$ and B be the event that $X=0$.

(i) Find $\mathrm{P}(A)$.

(ii) Are events A and B independent? Justify your answer.

(5 marks)

[HKDSE 2013' Section A#7]

7. (a) $\mathrm{E}(Y)=1\times 0.4+2\times 0.3+4\times 0.2+m\times 0.1=2.4$

$\therefore\ m=6$ — 1A

(b) (i) $\mathrm{P}(A)=\mathrm{P}(X=0,Y=1)+\mathrm{P}(X=0,Y=2)+\mathrm{P}(X=1,Y=1)$

$=0.2\times 0.4+0.2\times 0.3+0.3\times 0.4$ — 1M

$=0.26$ — 1A

(ii) $\mathrm{P}(A\cap B)=\mathrm{P}(X=0,Y=1)+\mathrm{P}(X=0,Y=2)$

$=0.2\times 0.4+0.2\times 0.3$

$=0.14$ — 1A

$\mathrm{P}(A)\mathrm{P}(B)=0.26\times 0.2$

$=0.052$

$\ne \mathrm{P}(A\cap B)$

<u>Alternative Solution</u>

$\mathrm{P}(A\mid B)=\mathrm{P}(Y=1)+\mathrm{P}(Y=2)$

$=0.4+0.3$

$=0.7$ — 1A

$\ne \mathrm{P}(A)$ by (i)

Thus, A and B are not independent. — 1A — Follow through

(5)

7. (a)	Excellent.
(b) (i)	Good. Mistakes were occasionally found in computations.
(ii)	Fair. A lot of candidates thought that the independence of two events A and B could be verified by checking $\mathrm{P}(A\cap B)=0$. Among those who found correct values of related probabilities, some did not mention $\mathrm{P}(A\cap B)\ne \mathrm{P}(A)\cdot \mathrm{P}(B)$ as the reason to make conclusion, while some made a wrong conclusion that 'A and B are independent'.

6. Let X be a discrete random variable with probability function as shown in the following table.

x	k	0	4	6
$P(X=x)$	0.1	0.2	0.3	0.4

It is given that $E(X)=3.4$.

(a) Find the value of k .

(b) Find $\mathrm{Var}(3-4X)$.

(c) Let G be the event that $X<4$ and H be the event that $X \geq -1$. Find $P(G \cap H)$.

(5 marks)

[HKDSE 2014' Section A#6]

6. (a) $0.1k+0.2(0)+0.3(4)+0.4(6)=3.4$

$k=-2$ — 1A

(b) $\mathrm{Var}(3-4X)=16\mathrm{Var}(X)$ — 1M

$=16[E(X^2)-E(X)^2]$

$=16[0.1(-2)^2+0.2(0)^2+0.3(4)^2+0.4(6)^2-3.4^2]$ — 1M

Alternative Solution

x	-2	0	4	6
$3-4x$	11	3	-13	-21
$P(X=x)$	0.1	0.2	0.3	0.4

$E(3-4X)=0.1(11)+0.2(3)+0.3(-13)+0.4(-21)$ — 1M — OR $3-4(3.4)$

$=-10.6$

$\mathrm{Var}(3-4X)=0.1(11+10.6)^2+0.2(3+10.6)^2+0.3(-13+10.6)^2+0.4(-21+10.6)^2$ — 1M

$=128.64$ — 1A

(c) $P(G \cap H)=P(-1 \leq X<4)$

$=P(X=0)$

$=0.2$ — 1A

(5)

6.	(a)	Excellent.
	(b)	Very good. Some candidates equated $\mathrm{Var}(3-4X)$ to $3^2\mathrm{Var}(X)$ or $3-4\mathrm{Var}(X)$.
	(c)	Good.

1. The table below shows the probability distribution of a discrete random variable X, where a and b are constants:

x	2	3	5	7	9
$P(X=x)$	0.08	0.15	a	0.45	b

It is given that $E(X)=5.64$. Find

(a) a and b ,

(b) $E((6-5X)^2)$ and $Var(6-5X)$.

(6 marks)

[HKDSE 2015' Section A#1]

Solution	Marks	Remarks
1. (a) $0.08+0.15+a+0.45+b=1$	1M	either one
$2(0.08)+3(0.15)+5a+7(0.45)+9b=5.64$		
Solving, we have $a=0.25$ and $b=0.07$.	1A	for both
(b) $E((6-5X)^2)$		
$=E(36-60X+25X^2)$		
$=36-60E(X)+25E(X^2)$	1M	
$=36-60(5.64)+25(35.64)$		
$=588.6$	1A	
$Var(6-5X)$		
$=E((6-5X)^2)-(E(6-5X))^2$		
$=E((6-5X)^2)-(6-5E(X))^2$	1M	accept $(-5)^2 Var(X)$
$=588.6-(6-5(5.64))^2$		
$=95.76$	1A	
	----------(6)	

1 (a)	Very good. Most candidates were able to find the values of a and b by setting up two equations involving them.
(b)	Good. Many candidates were able to find the value of $Var(6-5X)$ while some candidates wrongly found the value of $(E(6-5X))^2$ instead of $E((6-5X)^2)$.

1. The table below shows the probability distribution of a discrete random variable X, where k is a constant:

x	0	2	4	5	8	9
$\mathrm{P}(X=x)$	k^2	0.16	0.18	0.3	k	0.12

Find

(a) k,

(b) $\mathrm{E}(X)$,

(c) $\mathrm{Var}(2-3X)$.

(6 marks)

[HKDSE 2017' Section A#1]

1. (a) $k^2+0.16+0.18+0.3+k+0.12=1$	1M	
$k^2+k-0.24=0$		
$k=0.2$ or $k=-1.2$ (rejected)	1A	
Thus, we have $k=0.2$.		
(b) $\mathrm{E}(X)$		
$=0(0.04)+2(0.16)+4(0.18)+5(0.3)+8(0.2)+9(0.12)$	1M	
$=5.22$	1A	
(c) $\mathrm{Var}(2-3X)$		
$=9\mathrm{Var}(X)$		
$=9((0-5.22)^2(0.04)+(2-5.22)^2(0.16)+(4-5.22)^2(0.18)$		
$+(5-5.22)^2(0.3)+(8-5.22)^2(0.2)+(9-5.22)^2(0.12))$	1M	
$=56.6244$	1A	
$\mathrm{Var}(2-3X)$		
$=9\mathrm{Var}(X)$		
$=9(\mathrm{E}(X^2)-(\mathrm{E}(X))^2)$		
$=9(33.54-(5.22)^2)$	1M	
$=56.6244$	1A	
	----------(6)	

1 (a)	Very good. About 98% of the candidates were able to find the value of k by setting up a quadratic equation.
(b)	Very good. Over 90% of the candidates were able to find the value of $\mathrm{E}(X)$.
(c)	Very good. Most candidates were able to find the value of $\mathrm{Var}(2-3X)$.

4. The table below shows the probability distribution of a discrete random variable Y , where m and p are constants:

y	-2	2	m
$P(Y=y)$	p	0.25	0.5

(a) Prove that $\text{Var}(Y)=0.25m^2+2$.

(b) If $\text{Var}(2Y-1)=8\text{E}(2Y-1)$, find m .

(7 marks)

[HKDSE 2018' Section A#4]

4. (a) $p+0.25+0.5=1$ $p=0.25$	1M
$\text{E}(Y)$ $=-2(0.25)+2(0.25)+0.5m$ $=0.5m$	1M
$\text{Var}(Y)$ $=0.25(-2-0.5m)^2+0.25(2-0.5m)^2+0.5(m-0.5m)^2$ $=0.25(4+2m+0.25m^2+4-2m+0.25m^2)+0.125m^2$	1M
$=0.25m^2+2$	1
$0.25+p+0.5=1$ $p=0.25$	1M
$\text{E}(Y)$ $=-2(0.25)+2(0.25)+0.5m$ $=0.5m$	1M
$\text{Var}(Y)$ $=(-2)^2(0.25)+(2)^2(0.25)+m^2(0.5)-(0.5m)^2$	1M
$=0.25m^2+2$	1
(b) Note that $\text{E}(2Y-1)=2\text{E}(Y)-1$ and $\text{Var}(2Y-1)=4\text{Var}(Y)$. $\text{Var}(2Y-1)=8\text{E}(2Y-1)$ $4\text{Var}(Y)=16\text{E}(Y)-8$ $4(0.25m^2+2)=16(0.5m)-8$ $m^2-8m+16=0$	1M+1M
$m=4$	1A
	----------(7)

4 (a)	Very good. About 80% of the candidates were able to complete the proof by expressing $\text{Var}(Y)$ correctly in form of either $\text{E}(Y^2)-[\text{E}(Y)]^2$ or $\text{E}([Y-E(Y)]^2)$.
(b)	Very good. About 70% of the candidates were able to find the value of m by using $8\text{E}(2Y-1)=16\text{E}(Y)-8$ and $\text{Var}(2Y-1)=4\text{Var}(Y)$.

1. The table below shows the probability distribution of a discrete random variable X, where k is a constant:

x	8	11	k	27	32
$\mathrm{P}(X=x)$	0.2	0.1	0.3	0.3	0.1

It is given that $\mathrm{Var}(X)=66$. Find k, $\mathrm{E}(3X+5)$ and $\mathrm{Var}(3X+5)$. (6 marks)

[HKDSE 2019' Section A#1]

1. $\mathrm{E}(X)$		
$=0.2(8)+0.1(11)+0.3(k)+0.3(27)+0.1(32)$	1M	for either one
$=14+0.3k$		
$\mathrm{E}(X^2)$		
$=0.2(8^2)+0.1(11^2)+0.3(k^2)+0.3(27^2)+0.1(32^2)$		
$=346+0.3k^2$		
$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$		
$66=346+0.3k^2-(14+0.3k)^2$	1M	
$0.21k^2-8.4k+84=0$		
$k=20$	1A	
$\mathrm{E}(3X+5)$		
$=3\mathrm{E}(X)+5$	1M	for either one
$=3(14+0.3(20))+5$		
$=65$	1A	
$\mathrm{Var}(3X+5)$		
$=9\mathrm{Var}(X)$		
$=9(66)$		
$=594$	1A	
	----------(6)	

7. The number of goals scored in a randomly selected match by a football team follows a Poisson distribution with mean λ . The probability that the team scores no goals in a match is 0.1653 .

(a) Find the value of λ correct to 1 decimal place.

(b) Find the probability that the team scores less than 3 goals in a match.

(c) It is known that the numbers of goals scored by the team in any two matches are independent. Find the probability that the team totally scores less than 3 goals in two randomly selected matches.

(5 marks)

[HKDSE 2012' Section A#7]

Solution	Marks
7. (a) $\dfrac{e^{-\lambda}}{0!}=0.1653$ $\lambda=-\ln 0.1653$ ≈ 1.8	1A
(b) $\text{P(no. of goals in a match} < 3) = \dfrac{e^{-1.8}}{0!}+\dfrac{e^{-1.8}(1.8)}{1!}+\dfrac{e^{-1.8}(1.8)^2}{2!}$	1M
≈ 0.7306	1A
(c) The number of goals scored in two matches by the team $\sim \text{Po}(3.6)$. $\therefore$ P(no. of goals in two matches < 3) $=\dfrac{e^{-3.6}}{0!}+\dfrac{e^{-3.6}(3.6)}{1!}+\dfrac{e^{-3.6}(3.6)^2}{2!}$	1M
Alternative Solution P(no. of goals in two matches < 3) $=\text{P}(0,0)+\text{P}(0,1)+\text{P}(1,0)+\text{P}(1,1)+\text{P}(0,2)+\text{P}(2,0)$ $=\left(\dfrac{e^{-1.8}}{0!}\right)^2+2\left(\dfrac{e^{-1.8}}{0!}\right)\left[\dfrac{e^{-1.8}(1.8)}{1!}\right]+\left[\dfrac{e^{-1.8}(1.8)}{1!}\right]^2+2\left(\dfrac{e^{-1.8}}{0!}\right)\left[\dfrac{e^{-1.8}(1.8)^2}{2!}\right]$	1M
≈ 0.3027	1A
	(5)

7	(a)	Excellent.
	(b)	Very good.
	(c)	Poor. A few candidates used the Poisson distribution with mean 2λ . Many failed to consider all the events related to the required probability when using the Poisson distribution with mean λ .

8. In a shooting game, one member from each team will be selected to shoot a target three times. The team will get a prize if the target is hit at least once. Team A consists of Mabel and Owen, with the probability that Mabel is selected to shoot being 0.7. Suppose that the probabilities of Mabel and Owen to hit the target in each shot are 0.6 and 0.5 respectively.

(a) Find the probability that Team A will get a prize if Mabel is selected.

(b) Find the probability that Team A will get a prize.

(c) Given that Team A does not get a prize, find the probability that Owen is selected.

(6 marks)

[HKDSE 2013' Section A#8]

8. (a)	$P(\text{get a prize} \mid \text{Mabel}) = 0.6 + 0.4 \times 0.6 + 0.4^2 \times 0.6$	1M	OR $1-(1-0.6)^3$
	$= 0.936$	1A	OR $(0.6)^3 + C_1^3(0.6)^2(0.4) + C_1^3(0.6)(0.4)^2$
(b)	$P(\text{get a prize} \mid \text{Owen}) = 0.5 + 0.5 \times 0.5 + 0.5^2 \times 0.5$		OR $1-(1-0.5)^3$
	$= 0.875$		
	$\therefore P(\text{win}) = 0.7 \times 0.936 + 0.3 \times 0.875$	1M	
	$= 0.9177$	1A	
(c)	$P(\text{Owen} \mid \text{does not get a prize}) = \dfrac{0.3 \times (1-0.875)}{1-0.9177}$	1M	OR $\dfrac{0.3 \times (1-0.5)^3}{1-0.9177}$
	$= \dfrac{375}{823}$	1A	OR 0.4557
		(6)	

8. (a)	Fair. Many candidates actually found P(Mabel is selected and Team A gets a prize) instead of the required probability.
(b)	Good.
(c)	Satisfactory. Quite a number of candidates were weak in applying Bayes' theorem.

8. A company produces microwave ovens by production lines A and B. It is known that 4% of all microwave ovens fail to function properly and that 2% of microwave ovens produced by line A fail to function properly. Among the microwave ovens which function properly, $\frac{2}{3}$ of them are produced by line B. Suppose a microwave oven is randomly selected.

(a) What is the probability that the microwave oven is produced by line B and functions properly?

(b) What is the probability that the microwave oven is produced by line A?

(c) If the microwave oven is produced by line B, what is the probability that it functions properly?

(5 marks)

[HKDSE 2014' Section A#8]

8. (a) P(the selected microwave oven is produced by line B and functions properly) $= (1-0.04)\times\frac{2}{3}$ $= 0.64$	1A	
(b) P(A) P(functions properly $\mid A$) = P(functions properly) P(A $\mid$ functions properly)		
$\mathrm{P}(A)\ (1-0.02) = (1-0.04)\left(1-\frac{2}{3}\right)$	1M	
$\mathrm{P}(A) = \frac{16}{49}$	1A	OR 0.3265
(c) $\mathrm{P}(\text{functions properly} \mid B) = \dfrac{0.64}{1-\frac{16}{49}}$	1M	
$= \frac{784}{825}$	1A	OR 0.9503
	(5)	

8.	(a)	Very good.
	(b)	Poor. Most candidates did not understand the meanings of the conditional probabilities given.
	(c)	Fair. Many candidates used correct methods but obtained wrong answers, because their answers to (a) or (b) were wrong.

3. A bag contains 2 white balls and 5 yellow balls. In a survey, each interviewee draws a ball randomly from the bag. If a white ball is drawn, then the interviewee considers the question 'Are you a smoker?'. If a yellow ball is drawn, then the interviewee considers the question 'Are you a non-smoker?'. Finally, the interviewee answers either 'Yes' or 'No'. Let p be the probability that a randomly selected interviewee is a smoker.

(a) Express, in terms of p, the probability that a randomly selected interviewee answers 'Yes'.

(b) In this survey, 50 out of 91 interviewees answer 'Yes'.

(i) Find p.

(ii) Given that an interviewee answers 'No', find the probability that the interviewee is a non-smoker.

(6 marks)

[HKDSE 2015' Section A#3]

3. (a) The required probability		
$=\frac{2}{7}p+\frac{5}{7}(1-p)$	1M	for $rs+(1-r)(1-s)$
$=\frac{5-3p}{7}$	1A	
(b) (i) $\frac{5-3p}{7}=\frac{50}{91}$	1M	for using (a)
$p=\frac{5}{13}$	1A	
$p\approx 0.384615384$		r.t. 0.3846
$p\approx 0.3846$		
(ii) The required probability		
$=\frac{\frac{2}{7}\left(1-\frac{5}{13}\right)}{1-\frac{50}{91}}$	1M	for numerator using (b)(i)
$=\frac{16}{41}$	1A	
≈ 0.3902439024		r.t. 0.3902
≈ 0.3902		
	----------(6)	

3 (a)	Good. Some candidates did not simplify the answer and some candidates failed to give an answer as an expression in terms of p.
(b) (i)	Very good. Most candidates were able to set up an equation by using the result of (a).
(ii)	Good. Some candidates failed to use the result of (b)(i) to find the required probability.

4. A manufacturer of brand B biscuits starts a promotion plan by giving one reward points card in each packet of biscuits. It is found that 75% of the packets of brand B biscuits contain 3-point cards and the rest contain 7-point cards. A total of 20 points or more can be exchanged for a gift coupon. John buys 4 packets of brand B biscuits and he opens them one by one.

(a) Find the probability that John gets the first 7-point card when the 4th packet of brand B biscuits has been opened.

(b) Find the probability that John can exchange for a gift coupon.

(c) Given that John can exchange for a gift coupon, find the probability that he gets a 7-point card when the 4th packet of brand B biscuits has been opened.

(7 marks)

[HKDSE 2015' Section A#4]

Solution	Marks	Remarks
4. (a) The required probability		
$=(0.75)^3(1-0.75)$	1M	for $p^3(1-p)$, $0<p<1$
$=\frac{27}{256}$	1A	
$=0.10546875$		
≈ 0.1055		r.t. 0.1055
(b) The required probability		
$=1-\left((0.75)^4+4\left(\frac{27}{256}\right)\right)$	1M+1M	1M for $1-p$ + 1M for using (a)
$=\frac{67}{256}$	1A	
$=0.26171875$		
≈ 0.2617		r.t. 0.2617
The required probability		
$=(1-0.75)^4+C_1^4(1-0.75)^3(0.75)+C_2^4(1-0.75)^2(0.75)^2$	1M+1M	1M for the 3 cases + 1M for binomial probability
$=\frac{67}{256}$	1A	
$=0.26171875$		
≈ 0.2617		r.t. 0.2617
(c) The required probability		
$=\frac{\left(1-(0.75)^3\right)(1-0.75)}{0.26171875}$	1M	for denominator using (b)
$=\frac{37}{67}$	1A	
≈ 0.552238806		
≈ 0.5522		r.t. 0.5522
The required probability		
$=\frac{\left((1-0.75)^3+C_1^3(1-0.75)^2(0.75)+C_2^3(1-0.75)(0.75)^2\right)(1-0.75)}{0.26171875}$	1M	for denominator using (b)
$=\frac{37}{67}$	1A	
≈ 0.552238806		
≈ 0.5522		r.t. 0.5522
	----------(7)	

4 (a)	Very good. Most candidates were able to write a binomial probability while a few candidates wrongly wrote $(0.25)^3(1-0.25)$ instead of $(0.75)^3(1-0.75)$.
(b)	Very good. Most candidates were able to use the result of (a) while a few candidates wrongly wrote $1-\left((0.75)^4+\left(\frac{27}{256}\right)\right)$ instead of $1-\left((0.75)^4+4\left(\frac{27}{256}\right)\right)$.
(c)	Good. Some candidates failed to get the correct answer because they made a mistake in (b).

2. A box contains six cards numbered 1 , 2 , 3 , 4 , 5 and 6 respectively.

(a) Three cards are drawn randomly from the box one by one with replacement. Given that the sum of the numbers drawn is 7 , find the probability that the number 1 is drawn exactly two times.

(b) If the card numbered 6 is taken away before three cards are drawn, will the probability described in (a) change? Explain your answer.

(6 marks)

[HKDSE 2016' Section A#2]

2. (a) The required probability		
$=\dfrac{\left(\frac{1}{6}\right)^3(3)}{\left(\frac{1}{6}\right)^3(3+3!+3+3)}$	1M+1M+1M	1M for $\left(\frac{1}{6}\right)^3$ + 1M for numerator + 1M for denominator
$=\dfrac{1}{5}$	1A	
(b) The required probability		
$=\dfrac{\left(\frac{1}{5}\right)^3(3)}{\left(\frac{1}{5}\right)^3(3+3!+3+3)}$	1M	
$=\dfrac{1}{5}$		
Thus, the required probability will not change.	1A ----------(6)	f.t.

2 (a)	Good. Some candidates were unable to count the correct number of relevant outcomes for the sum of 7 , hence unable to work out the denominator of the required probability properly.
(b)	Fair. Although many candidates guessed correctly that the required conditional probability remains unchanged, they were unable to provide a mathematical argument to justify the guess.

3. A museum opens at 10:00 . The number of visitors entering the museum in a minute follows a Poisson distribution with a mean of 1.8 .

(a) Write down the variance of the number of visitors entering the museum in a minute.

(b) Find the probability that 3 visitors entered the museum in the first two minutes after the museum opens.

(c) At 10:00 , only one gate at the entrance of the museum is opened. If in any two consecutive minutes, there are at least 4 visitors entering the museum in each minute, then a second gate will be opened. Find the probability that the second gate is just opened three minutes after the museum opens.

(7 marks)

[HKDSE 2016' Section A#3]

3. (a) The variance of the number of visitor entering the museum in a minute is 1.8	1A	
(b) The required probability		
$=\frac{e^{-3.6}3.6^3}{3!}$	1M+1M	1M for Poisson probability + 1M using mean 3.6
$=\frac{7.776}{e^{3.6}}$	1A	r.t. 0.2125
≈ 0.212469265		
≈ 0.2125		
The required probability		
$=2\left(\frac{e^{-1.8}1.8^0}{0!}\right)\left(\frac{e^{-1.8}1.8^3}{3!}\right)+2\left(\frac{e^{-1.8}1.8^1}{1!}\right)\left(\frac{e^{-1.8}1.8^2}{2!}\right)$	1M + 1M	1M for 4 cases + 1M for Poisson probability using mean 1.8
$=\frac{7.776}{e^{3.6}}$	1A	r.t. 0.2125
≈ 0.212469265		
≈ 0.2125		
(c) P(at most 3 visitors in a minute)		
$=\frac{e^{-1.8}1.8^0}{0!}+\frac{e^{-1.8}1.8^1}{1!}+\frac{e^{-1.8}1.8^2}{2!}+\frac{e^{-1.8}1.8^3}{3!}$	1M	
≈ 0.891291605		
≈ 0.8913		
The required probability		
$\approx (0.891291605)(1-0.891291605)^2$	1M	
≈ 0.010532851		
≈ 0.0105	1A	r.t. 0.0105
	----------(7)	

3 (a)	Very good. A very high proportion of the candidates were able to write down the required variance.
(b)	Very good. More than 70% of the candidates were able to find the answer using a Poisson probability with a mean of 3.6 instead of a mean of 1.8 .
(c)	Good. Only a number of candidates made careless mistakes in finding the required probability.

4. Susan plays a game. In each trial of the game, her probability of winning a doll is 0.6 . Susan plays the game until she wins a doll.

(a) Find the probability that Susan wins a doll at the 4th trial in the game.

(b) If Susan cannot win a doll in k trials, then the probability that she wins a doll within 10 trials in the game is greater than 0.95 . Find the greatest value of k .

(c) In each trial of the game, Susan has to pay $15 . Find the expected amount of money she has to pay to win a doll in the game.

(7 marks)

[HKDSE 2017' Section A#4]

4. (a) The required probability		
$= (1-0.6)^3(0.6)$	1M	for $(1-p)^3 p$, $0<p<1$
$= 0.0384$	1A	
(b) $1-(1-0.6)^{10-k} > 0.95$	1M	for $1-(1-q)^{10-k}$, $0<q<1$
$0.4^{10-k} < 0.05$		
$\log(0.4^{10-k}) < \log 0.05$	1M	
$k < 6.730587608$		
Thus, the greatest value of k is 6 .	1A	
(c) The expected amount of money		
$=15\left(\frac{1}{0.6}\right)$	1M	for $15\left(\frac{1}{r}\right)$, $0<r<1$
$= \$25$	1A	
	----------(7)	

4 (a)	Very good. Most candidates were able to write down a probability of geometric distribution but a few candidates wrongly wrote down $(0.6)^3(1-0.6)$ instead of $(1-0.6)^3(0.6)$.
(b)	Poor. Less than 10% of the candidates were able to set up the correct inequality $1-(1-0.6)^{10-k} > 0.95$.
(c)	Good. Only some candidates were unable to find the expected amount of money correctly.

3. A lucky draw is held in a shop. In each day, there is a big prize in this lucky draw. When the big prize is won, the lucky draw on that day stops. For each draw, the probability of winning the big prize is 0.2 .

(a) Write down the mean and the variance of the number of draws for winning the big prize in a day.

(b) Within the first 4 draws in a day, are winning the big prize and not winning the big prize of equal chance? Explain your answer.

(c) Find the probability of not winning the big prize within the first 4 draws in each day for 5 days.

(7 marks)

[HKDSE 2019' Section A#3]

3. (a) The mean $= 5$	1A	
The variance $= 20$	1A	
(b) The probability of winning the big prize within the first 4 draws		
$= 0.2 + (1-0.2)(0.2) + (1-0.2)^2(0.2) + (1-0.2)^3(0.2)$	1M	for $(1-p)^k p$
$= 1-(1-0.2)^4$		
$= 0.5904$	1A	
> 0.5		
Thus, winning the big prize and not winning the big prize within the first 4 draws are not of equal chance.	1A	f.t.
The probability of not winning the big prize within the first 4 draws		
$= (1-0.2)^4$	1M	for $(1-p)^k$
$= 0.4096$	1A	
The probability of winning the big prize within the first 4 draws		
$= 1 - 0.4096$		
$= 0.5904$		
$\neq 0.4096$		
Thus, winning the big prize and not winning the big prize within the first 4 draws are not of equal chance.	1A	f.t.
(c) The required probability		
$= (1-0.5904)^5$	1M	1M for q^5
≈ 0.011529215		
≈ 0.0115	1A	r.t. 0.0115
	----------(7)	

4. In each month, the probability that a store offers a discount to its products is 0.35 . If a discount is offered in a certain month, then the probability that the store makes a profit in that month is 0.7 ; otherwise, the probability of making a profit in that month is 0.28 .

(a) Find the probability that the store makes a profit in a certain month.

(b) Given that the store makes a profit in a certain month, find the probability that the store offers a discount in that month.

(c) Find the probability that the store makes a profit in at least 2 months out of 12 months.

(6 marks)

[HKDSE 2019' Section A#4]

4. (a) The required probability		
$=(0.35)(0.7)+(1-0.35)(0.28)$	1M	for $pq+(1-p)r$
$=0.427$	1A	
(b) The required probability		
$=\dfrac{(0.35)(0.7)}{0.427}$	1M	for denominator using (a)
$=\dfrac{35}{61}$	1A	r.t. 0.5738
≈ 0.573770491		
≈ 0.5738		
(c) The required probability		
$=1-(1-0.427)^{12}-C_1^{12}(1-0.427)^{11}(0.427)$	1M	
≈ 0.987544904		
≈ 0.9875	1A	r.t. 0.9875
	----------(6)	

10 – Discrete Probability Distributions (B)

13. Drunk driving is against the law in a city. The police set up an inspection block at the entrance of a certain highway at night in order to arrest drunk drivers. From past experience, the number of drunk drivers arrested follows a Poisson distribution with mean 2.3 per hour.

(a) Find the probability that at least 2 drunk drivers are arrested in a certain hour.

(2 marks)

(b) Given that at least 2 drunk drivers are arrested in a certain hour, find the probability that not more than 4 drunk drivers are arrested.

(3 marks)

(c) In a certain week, the police sets up an inspection block for three nights, all at the same period from 1:00 am to 2:00 am. It is known that the numbers of drunk drivers arrested in different nights are independent.

(i) Find the probability that the third night is the first night to have at least 2 drunk drivers arrested.

(ii) Find the probability that at least 2 drunk drivers are arrested in each of the 3 nights and there are totally 10 drunk drivers arrested.

(5 marks)

[HKDSE 2012'Section B#13]

Solution	Marks	Remarks
13. (a) P(at least 2 drunk drivers are prosecuted)		
$=1-e^{-2.3}-e^{-2.3}(2.3)$	1A	
≈ 0.669145815		
≈ 0.6691	1A	
	(2)	
(b) P(≤ 4 drunk drivers are prosecuted | at least 2 drunk drivers are prosecuted)		
$\approx \dfrac{e^{-2.3}\left(\dfrac{2.3^2}{2!}+\dfrac{2.3^3}{3!}+\dfrac{2.3^4}{4!}\right)}{0.669145815}$	1M+1M	1M for Poisson 1M for conditional prob
≈ 0.8748	1A	
	(3)	
(c) (i) P(the third night was the 1st night to have ≥ 2 drunk drivers prosecuted)		
$\approx (1-0.669145815)^2(0.669145815)$	1M	
≈ 0.0732	1A	
(ii) P(≥ 2 drunk drivers prosecuted in each night and totally 10 prosecuted)		
$=C_2^3\left(e^{-2.3}\dfrac{2.3^2}{2!}\right)^2\left(e^{-2.3}\dfrac{2.3^6}{6!}\right)+3!\left(e^{-2.3}\dfrac{2.3^2}{2!}\right)\left(e^{-2.3}\dfrac{2.3^3}{3!}\right)\left(e^{-2.3}\dfrac{2.3^5}{5!}\right)$ $+C_2^3\left(e^{-2.3}\dfrac{2.3^2}{2!}\right)\left(e^{-2.3}\dfrac{2.3^4}{4!}\right)^2+C_2^3\left(e^{-2.3}\dfrac{2.3^3}{3!}\right)^2\left(e^{-2.3}\dfrac{2.3^4}{4!}\right)$	1M+1M	1M for any one case 1M for all cases
≈ 0.0471	1A	
	(5)	

13 (a)		Excellent. However, a small number of candidates forgot the formula of Poisson probabilities.
(b)		Satisfactory. Some candidates failed to write all the terms needed in the numerator.
(c)	(i)	Satisfactory. Many candidates were able to apply the correct method, although some got wrong numerical answers.
	(ii)	Poor. Most candidates failed to identify all the events related to the probability required and some even used 4.6 instead of 2.3 as the mean of the Poisson distribution.

13. A lift company provides a regular maintenance service for every lift in an estate at the beginning of each month. Assume that the number of breakdowns of a lift in a month follows the Poisson distribution with mean 1.9 . Suppose there are totally 15 lifts in the estate, and the regular maintenance service of a lift in a month is regarded as unacceptable if there are more than 2 breakdowns in that month after the regular maintenance. Assume that the monthly numbers of breakdowns of lifts are independent.

(a) Find the probability that the regular maintenance service of a randomly selected lift in a certain month in the estate is unacceptable.

(2 marks)

(b) For a certain lift, find the probability that June of 2014 is the 3rd month in 2014 such that the regular maintenance service of that lift is unacceptable.

(2 marks)

(c) Find the expected total number of unacceptable regular maintenance services of all lifts in the estate for one year.

(2 marks)

(d) In order to assure the quality of the maintenance service provided by the lift company, the estate management office introduces the following term in the new maintenance contract for the 15 lifts, which will be effective on 1st January 2015.

> For each lift in the estate, if the regular maintenance services is unacceptable for 3 consecutive months in the new contract period, one warning letter will be immediately issued to the lift company, provided that no warning letter has been issued for that lift before.

(i) For a randomly selected lift, find the probability that a warning letter will be issued to the lift company on or before 30th April 2015.

(ii) Find the probability that 3 or more warning letters will be issued to the lift company on or before 30th April 2015.

(6 marks)

[HKDSE 2013'Section B#13]

13. (a) P(the regular maintenance service of a lift in a certain month in the estate is unacceptable)

$$=1-e^{-1.9}\left(1+\frac{1.9^1}{1!}+\frac{1.9^2}{2!}\right)$$ 1M

≈ 0.296279646

≈ 0.2963 1A

(2)

(b) P(the maintenance service of a lift in June of 2014 is the 3rd month unacceptable)

$\approx C_2^5(0.296279646)^2(1-0.296279646)^3\cdot(0.296279646)$ 1M

≈ 0.0906 1A

(2)

(c) The expected total number of unacceptable maintenance services of all lifts for one year

$\approx 15\times 12\times 0.296279646$ 1M

≈ 53.3303 1A

(2)

(d) (i)	P(a warning letter will be issued for a lift on or before 30th April 2015) $\approx (0.296279646)^3 + (1-0.296279646)\cdot(0.296279646)^3$	1M+1M
	≈ 0.044310205	
	≈ 0.0443	1A
(ii)	P(3 or more warning letters will be issued on or before 30th April 2015) $\approx 1-(1-0.044310205)^{15} - C_1^{15}(0.044310205)(1-0.044310205)^{14} - C_2^{15}(0.044310205)^2(1-0.044310205)^{13}$	1M+1M
	≈ 0.0265	1A
		(6)

13. (a)	Good. Some candidates missed out the term $e^{-1.9}\frac{1.9^2}{2!}$ in the expression $1-e^{-1.9}\left(1+\frac{1.9^1}{1!}+\frac{1.9^2}{2!}\right)$, while some others missed out the factor $e^{-1.9}$.
(b)	Satisfactory. Mistakes found were missing the factor C_2^5 or replacing it by C_3^6.
(c)	Poor. Some candidates missed out the factor 15 or 12, while some others used 1.9, the mean of the Poisson distribution given, instead of the probability found in (a).
(d)	Poor. Most candidates were not able to analyse the events correctly to calculate the probabilities.
(i)	Some candidates multiplied factors such as C_3^4, 15 or $\frac{1}{15}$ to the probability $(1-0.296279646)\cdot 0.296279646^3$, some multiplied 2 to 0.296279646^3, while some others wrote $2(1-0.296279646)\cdot 0.296279646^3$ without adding 0.296279646^4 to it.
(ii)	Some candidates used the probability found in (a) instead of that in (d)(i).

13. The number of delays in a day of a railway system follows the Poisson distribution with mean 4.8 . Assume that the daily numbers of delays are independent.

(a) Find the probability that there are not more than 3 delays in a day.

(2 marks)

(b) Find the probability that, in 3 consecutive days, there are at most 2 days with not more than 3 delays in each day.

(2 marks)

(c) A day is called a *bad day* if there are more than 5 delays in that day; otherwise it is called a *good day*.

(i) Suppose today is a *bad day*. Find the mean number of *good days* between today and next *bad day*.

(ii) Find the probability that the last day of a week is the third *bad day* in that week.

(iii) Find the probability that there are at least 4 consecutive *bad days* in a week.

(7 marks)

[HKDSE 2014'Section B#13]

13. (a) P(not more than 3 delays in a day)

$$= e^{-4.8}\left(1+4.8+\frac{4.8^2}{2!}+\frac{4.8^3}{3!}\right)$$ 1M

≈ 0.294229916

≈ 0.2942 1A

(2)

(b) P(at most 2 days with not more than 3 delays in a day in 3 consecutive days)

$\approx 1-0.294229916^3$ 1M

≈ 0.9745 1A

(2)

OR $\sum_{r=0}^{2} C_r^3 p^r (1-p)^{3-r}$,
where $p \approx 0.294229916$

(c) Denote P(*bad day*) by k.

(i) $k=1-e^{-4.8}\left(1+4.8+\frac{4.8^2}{2!}+\frac{4.8^3}{3!}+\frac{4.8^4}{4!}+\frac{4.8^5}{5!}\right)$

≈ 0.348993562 — 1A

$\therefore$ the mean number of *good days* between today and next *bad day*

$=\frac{1}{k}-1$ — 1M

≈ 1.8654 — 1A

(ii) P(the last day in a week is the third *bad day* in that week)

$=C_2^6 k^2(1-k)^4 k$ — 1M

≈ 0.1145 — 1A

(iii) P(there are at least 4 consecutive bad days in a week)

$=k^4\cdot 1^3+(1-k)k^4\cdot 1^2+1(1-k)k^4\cdot 1+1^2(1-k)k^4$ — 1M

Alternative Solution 1

$=[2k^4(1-k)+2k^4(1-k)^2]+[2k^5(1-k)+k^5(1-k)^2]+2k^6(1-k)+k^7$ — 1M

$\boxed{=2(k^4-k^5+k^4-2k^5+k^6)+2k^5-2k^6+k^5-2k^6+k^7+2k^6-2k^7+k^7}$

Alternative Solution 2

$=4k^4(1-k)^3+9k^5(1-k)^2+6k^6(1-k)+k^7$ — 1M

$\boxed{=4(k^4-3k^5+3k^6-k^7)+9(k^5-2k^6+k^7)+6k^6-6k^7+k^7}$

$\boxed{=4k^4-3k^5}$

≈ 0.0438 — 1A

(7)

(c)(iii)

S	M	T	W	T	F	S
B	B	B	B			
G	B	B	B	B		
	G	B	B	B	B	
		G	B	B	B	B

(c)(iii) Alt Sol 1

S	M	T	W	T	F	S
B	B	B	B	G		
G	B	B	B	B	G	
	G	B	B	B	B	G
		G	B	B	B	B
B	B	B	B	B	G	
G	B	B	B	B	B	G
	G	B	B	B	B	B
B	B	B	B	B	B	G
G	B	B	B	B	B	B
B	B	B	B	B	B	B

(c)(iii) Alt Sol 2

S	M	T	W	T	F	S
B	B	B	B	G	G	G
G	B	B	B	B	G	G
G	G	B	B	B	B	G
G	G	G	B	B	B	B
B	B	B	B	B	G	G
G	B	B	B	B	B	G
G	G	B	B	B	B	B
B	B	B	B	G	B	G
B	B	B	B	G	G	B
G	B	B	B	B	G	B
B	G	B	B	B	B	G
B	G	G	B	B	B	B
G	B	G	B	B	B	B
B	B	B	B	B	B	G
B	B	B	B	B	G	B
B	B	B	B	G	B	B
B	B	G	B	B	B	B
B	G	B	B	B	B	B
G	B	B	B	B	B	B
B	B	B	B	B	B	B

13. (a)		Excellent. Some candidates missed the case of 3 delays in a day.
(b)		Good. Some candidates used incorrect expressions such as $1-(1-0.2942)^3$ to find the required probability.
(c)	(i)	Poor. Quite a number of candidates wrongly used $\frac{1}{\text{P(bad day)}}$ to find the required mean number.
	(ii)	Satisfactory. Some candidates used C_2^7 instead of C_2^6 in the calculation.
	(iii)	Very poor. Many candidates were able to write the related terms for the required probability, but assigned wrong coefficients to them.

10. The number of customers buying tickets at cinema A in a minute can be modelled by a Poisson distribution with a mean of 3.2. The probability distribution of the number of tickets bought by a customer at cinema A is shown in the following table:

Number of tickets bought	1	2	3	4	5	6	≥ 7
Probability	0.12	0.7	0.08	0.04	0.03	0.02	0.01

(a) Find the probability that fewer than 4 customers buy tickets at cinema A in a certain minute. (3 marks)

(b) Find the probability that the 8th customer buying tickets at cinema A is the 3rd customer who buys 2 tickets. (2 marks)

(c) Find the probability that exactly 3 customers buy tickets at cinema A in a certain minute and each of them buys 2 tickets. (2 marks)

(d) Find the probability that exactly 3 customers buy tickets at cinema A in a certain minute and they buy a total of 6 tickets. (3 marks)

(e) Given that fewer than 4 customers buy tickets at cinema A in a certain minute, find the probability that they buy a total of 6 tickets. (3 marks)

[HKDSE 2015'Section B#10]

Solution	Marks	Remarks
10. (a) The required probability		
$=\frac{3.2^0e^{-3.2}}{0!}+\frac{3.2^1e^{-3.2}}{1!}+\frac{3.2^2e^{-3.2}}{2!}+\frac{3.2^3e^{-3.2}}{3!}$	1M+1M	1M for the 4 cases + 1M for Poisson probability
≈ 0.602519724		
≈ 0.6025	1A	r.t. 0.6025
	----------(3)	
(b) The required probability		
$=C_2^7(0.7)^2(1-0.7)^5(0.7)$	1M	for binomial probability
≈ 0.01750329		
≈ 0.0175	1A	r.t. 0.0175
	----------(2)	
(c) The required probability		
$=\frac{3.2^3e^{-3.2}}{3!}(0.7)^3$	1M	
≈ 0.076357282		
≈ 0.0764	1A	r.t. 0.0764
	----------(2)	
(d) The required probability		
$\approx 0.076357282+\frac{3.2^3e^{-3.2}}{3!}\left(3(0.12)^2(0.04)+3!(0.12)(0.7)(0.08)\right)$	1M+1A	1M for using (c) + 1A for any one correct
≈ 0.085717839		
≈ 0.0857	1A	r.t. 0.0857
	----------(3)	
(e) The required probability		
$\approx \frac{\left(\frac{3.2e^{-3.2}}{1!}\right)(0.02)+\left(\frac{3.2^2e^{-3.2}}{2!}\right)\left(2(0.12)(0.03)+2(0.7)(0.04)+(0.08)^2\right)+0.085717839}{0.602519724}$	1M+1M	1M for numerator using (d) +1M for denominator using (a)
≈ 0.170703644		
≈ 0.1707	1A	r.t. 0.1707
	----------(3)	

10 (a)	Very good. A few candidates missed the first case in the required sum of the Poisson probabilities.
(b)	Very good. A few candidates unnecessarily multiplied the Poisson probability to the required probability form.
(c)	Very good. A few candidates wrongly used $\frac{3.2^3 e^{-3.2}}{3!}(0.7)^2$ instead of $\frac{3.2^3 e^{-3.2}}{3!}(0.7)^3$ in the calculation.
(d)	Good. Some candidates failed to count the number of cases correctly, such as they wrongly multiplied 3 instead of $3!$ to the term $(0.12)(0.7)(0.08)$.
(e)	Good. Some candidates did not realize that a conditional probability is considered here. Some candidates did not consider the Poisson probabilities as a part of the joint probability in the numerator of the required conditional probability.

10. Tom arrives at the bus stop at 7:10 . A bus arrives at 7:20 and another bus arrives at 7:30 . The probability that Tom can take the bus is 0.9 each time. If Tom takes the bus at 7:20 , the probability for him to be late is 0.1 . If Tom takes the bus at 7:30 , the probability for him to be late is 0.4 . Tom will be late if he cannot take these two buses.

(a) Find the probability that Tom takes a bus on or before 7:30 on a certain day. (2 marks)

(b) Find the probability that Tom is late on a certain day. (2 marks)

(c) Find the probability that Tom is late 2 times in 6 days. (2 marks)

(d) There are 7 persons, including Tom, waiting for a lift at the lobby. If Tom is late, he will go to the second floor; otherwise he will go to the third floor. The probabilities for each of the other 6 persons to go to the second and third floor are 0.7 and 0.3 respectively. When an empty lift arrives, the 7 persons enter the lift. No person enters the lift afterwards.

(i) Find the probability that the 7 persons are going to the same floor.

(ii) Find the probability that exactly 3 persons are going to the third floor.

(iii) Given that exactly 3 persons are going to the third floor, find the probability that Tom is late.

(7 marks)

[HKDSE 2016'Section B#10]

10. (a) The required probability		
$=0.9+(1-0.9)(0.9)$	1M	
$=0.99$	1A	
	----------(2)	
(b) The required probability		
$=(0.9)(0.1)+(1-0.9)(0.9)(0.4)+(1-0.9)^2(1)$	1M	
$=0.136$	1A	
	----------(2)	
(c) The required probability		
$=C_2^6(1-0.136)^4(0.136)^2$	1M	
≈ 0.154605181		
≈ 0.1546	1A	r.t. 0.1546
	----------(2)	
(d) (i) The required probability		
$=(0.136)(0.7)^6+(1-0.136)(0.3)^6$	1M	
≈ 0.01663012		
≈ 0.0166	1A	r.t. 0.0166
(ii) The required probability		
$=(0.136)C_3^6(0.7)^3(0.3)^3+(1-0.136)C_2^6(0.7)^4(0.3)^2$	1M+1M	
≈ 0.30524256		
≈ 0.3052	1A	r.t. 0.3052
(iii) The required probability		
$=\dfrac{(0.136)C_3^6(0.7)^3(0.3)^3}{(0.136)C_3^6(0.7)^3(0.3)^3+(1-0.136)C_2^6(0.7)^4(0.3)^2}$	1M	1M for denominator using (d)(ii)
≈ 0.082524271		
≈ 0.0825	1A	r.t. 0.0825
	----------(7)	

10 (a)	Very good. More than 70% of the candidates were able to find the required probability.
(b)	Good. Some candidates missed the term $(1-0.9)^2(1)$ when finding the required probability.
(c)	Very good. Most candidates were able to formulate the required probability using binomial distribution.
(d) (i)	Good. About half of the candidates were able to find the required probability by using the result of (b). However, some candidates wrongly used 0.7 and 0.3 instead of $(0.7)^6$ and $(0.3)^6$ respectively in the required probability.
(ii)	Good. Many candidates were able to formulate the required probability by using an appropriate binomial probability
(iii)	Good. Many candidates were able to formulate the required conditional probability by using the result in (d)(ii).

10. A department store issues a cash coupon to a customer spending at least $\$500$ in a transaction. The details are given in the following table:

Transaction amount ($\$x$)	Cash coupon
$500 \leq x < 1\,000$	$\$50$
$1\,000 \leq x < 2\,000$	$\$100$
$x \geq 2\,000$	$\$200$

At the department store, 45% , 20% and 10% of the customers each gets one cash coupon of $\$50$, $\$100$ and $\$200$ respectively in a transaction. Assume that the number of transactions per minute follows a Poisson distribution with a mean of 2 .

(a) Find the probability that there are at most 4 transactions at the department store in a certain minute. (3 marks)

(b) Find the probability that there are exactly 3 transactions at the department store in a certain minute and cash coupons of total value $\$200$ are issued. (3 marks)

(c) If there are exactly 4 transactions at the department store in a certain minute, find the probability that cash coupons of total value $\$200$ are issued by the department store in this minute. (3 marks)

(d) Given that there are at most 4 transactions at the department store in a certain minute, find the probability that cash coupons of total value $\$200$ are issued by the department store in this minute. (3 marks)

[HKDSE 2017'Section B#10]

Solution	Marks	Remarks
10. (a) The required probability $=\frac{2^0e^{-2}}{0!}+\frac{2^1e^{-2}}{1!}+\frac{2^2e^{-2}}{2!}+\frac{2^3e^{-2}}{3!}+\frac{2^4e^{-2}}{4!}$	1M+1M	1M for the 5 cases + 1M for Poisson probability
≈ 0.947346982 ≈ 0.9473	1A ----------(3)	r.t. 0.9473
(b) The required probability $=\frac{2^3e^{-2}}{3!}(3(0.25)^2(0.1)+3(0.25)(0.2)^2+3(0.45)^2(0.2))$	1M+1M	1M for Poisson probability + 1M for any one correct
≈ 0.030721109 ≈ 0.0307	1A ----------(3)	r.t. 0.0307
(c) The required probability $=4(0.25)^3(0.1)+6(0.25)^2(0.2)^2+(4)(3)(0.45)^2(0.2)(0.25)+(0.45)^4$	1M+1M	1M for any one correct + 1M for any three correct
$= 0.18375625$ ≈ 0.1838	1A ----------(3)	r.t. 0.1838
(d) The required probability $\approx\frac{\left(\frac{2e^{-2}}{1!}\right)(0.1)+\left(\frac{2^2e^{-2}}{2!}\right)(2(0.25)(0.1)+(0.2)^2)+0.030721109+\left(\frac{2^4e^{-2}}{4!}\right)(0.18375625)}{0.947346982}$	1M+1M	1M for numerator using (b) or (c) +1M for denominator using (a)
≈ 0.10421488 ≈ 0.1042	1A ----------(3)	r.t. 0.1042

10 (a)	Very good. Over 85% of the candidates were able to write down all the five Poisson probabilities.
(b)	Very good. A few candidates were unable to use correct combinations in counting.
(c)	Good. Some candidates wrongly multiplied the Poisson probability to the required probability.
(d)	Good. Only some candidates were unable to consider all the possible cases that cash coupons of total value $\$200$ are issued in a minute.

10. A company records the numbers of lateness of its staff monthly. The performance of a staff member in a month is regarded as *good* if the staff member is late for fewer than 2 times in that month. Albert is a staff member of the company. The number of lateness of Albert in a month follows a Poisson distribution with a mean of 1.8 .

(a) Find the probability that Albert's performance in a certain month is *good*. (2 marks)

(b) To improve the performance of the staff, the company launches a bonus scheme on staff performance in the coming four months. Two suggestions for the bonus scheme are listed below:

Suggestion I

Number of month with *good* performance	4	3	2	1	0
Bonus	\$5000	\$2500	\$1500	\$600	\$0

Suggestion II

Total number of lateness in these four months	Fewer than 5	Otherwise
Bonus	\$8000	\$0

Which one of the above suggestions is more favourable to Albert? Explain your answer. (6 marks)

(c) The company also records the numbers of early leaves of its staff monthly. The number of early leaves of Albert in a month follows a Poisson distribution with a mean of λ . It is assumed that whether Albert is late and whether he leaves early are independent events.

(i) Express, in terms of e and λ , the probability that Albert is late for 2 times and does not leave early in a certain month.

(ii) Given that the sum of the number of lateness and the number of early leaves of Albert in a certain month is 2 , the probability that Albert is late for 2 times and does not leave early in that month is 0.36 . Find λ .

(5 marks)

[HKDSE 2018'Section B#10]

10. (a) The required probability

$= e^{-1.8}\left(\frac{1.8^0}{0!}+\frac{1.8}{1!}\right)$ 1M

$= 2.8e^{-1.8}$ 1A r.t. 0.4628

≈ 0.462836887

≈ 0.4628

----------(2)

(b) Let $p = 2.8e^{-1.8}$.

The expected bonus according to Suggestion I

$= 5000p^4 + 2500C_1^4 p^3(1-p) + 1500C_2^4 p^2(1-p)^2 + 600C_3^4 p(1-p)^3$ 1M+1M

$\approx \$1490.505464$

$\approx \$1490.5055$

The probability that Albert is late for fewer than 5 times in four months

$= e^{-7.2}\left(\frac{7.2^0}{0!}+\frac{7.2^1}{1!}+\frac{7.2^2}{2!}+\frac{7.2^3}{3!}+\frac{7.2^4}{4!}\right)$ 1M+1M

$= 208.3024e^{-7.2}$

≈ 0.155515616

The expected bonus according to Suggestion II

$= (8000)(208.3024e^{-7.2})$ 1M

$= \$1666419.2e^{-7.2}$

$\approx \$1244.1249$

$< \$1490.5055$

Thus, Suggstion I is more favourable to Albert. 1A f.t.

----------(6)

(c) (i) The required probability

$= \left(\frac{1.8^2}{2!}e^{-1.8}\right)\left(\frac{\lambda^0}{0!}e^{-\lambda}\right)$ 1M

$= 1.62e^{-1.8-\lambda}$ 1A

(ii) $\dfrac{1.62e^{-1.8-\lambda}}{\left(\frac{1.8^2}{2!}e^{-1.8}\right)\left(\frac{\lambda^0}{0!}e^{-\lambda}\right)+\left(\frac{1.8}{1!}e^{-1.8}\right)\left(\frac{\lambda^1}{1!}e^{-\lambda}\right)+\left(\frac{1.8^0}{0!}e^{-1.8}\right)\left(\frac{\lambda^2}{2!}e^{-\lambda}\right)} = 0.36$ 1M+1M 1M for using (c)(i) in numerator+1M for denominator

$\dfrac{1.62}{1.62+1.8\lambda+0.5\lambda^2} = 0.36$

$\lambda^2 + 3.6\lambda - 5.76 = 0$

$\lambda = 1.2$ or $\lambda = -4.8$ (rejected)

Thus, we have $\lambda = 1.2$. 1A

----------(5)

10 (a)	Very good. Over 85% of the candidates were able to write down the two Poisson probabilities.
(b)	Good. In finding the expected bonus, many candidates were able to use Poisson distribution with 7.2 in Suggestion II, but some candidates missed the binomial coefficients in Suggestion I.
(c) (i)	Good. Many candidates were able to express the answer in terms of e and λ .
(ii)	Good. Many candidates were able to use conditional probability to find the value of λ .

9. The number of matches won by a basketball team in a season follows a Poisson distribution with a mean of 3 matches per season. The points scored by the team in a match follows a normal distribution with a mean of 66 points and a standard deviation of 10 points.

(a) Find the probability that the team wins fewer than 6 matches in a certain season. (3 marks)

(b) Find the probability that the team scores higher than 70 points in a certain match. (2 marks)

(c) The team receives a certificate if the team wins a match and scores more than 70 points in that match. The team is awarded a bonus in a certain season if the team receives more than 2 certificates in that season.

(i) Find the probability that the team wins exactly 3 matches in a certain season and is awarded a bonus in that season.

(ii) If the team wins exactly 4 matches in a certain season, find the probability that the team is awarded a bonus in that season.

(iii) Given that the team wins fewer than 6 matches in a certain season, find the probability that the team is awarded a bonus in that season.

(7 marks)

[HKDSE 2019' Section B#9]

Solution	Marks	Remarks
9. (a) The required probability		
$=\frac{3^0e^{-3}}{0!}+\frac{3^1e^{-3}}{1!}+\frac{3^2e^{-3}}{2!}+\frac{3^3e^{-3}}{3!}+\frac{3^4e^{-3}}{4!}+\frac{3^5e^{-3}}{5!}$	1M+1M	1M for the 6 cases + 1M for Poisson probability
$=18.4e^{-3}$	1A	r.t. 0.9161
≈ 0.916082058		
≈ 0.9161		
	----------(3)	
(b) The required probability		
$=\mathrm{P}\left(Z>\frac{70-66}{10}\right)$	1M	
$=\mathrm{P}(Z>0.4)$		
$=0.5-0.1554$		
$=0.3446$	1A	
	----------(2)	
(c) (i) The required probability		
$=(0.3446)^3\left(\frac{3^3e^{-3}}{3!}\right)$	1M	
≈ 0.009168006		
≈ 0.0092	1A	r.t. 0.0092
(ii) The required probability		
$=C_3^4(0.3446)^3(1-0.3446)+(0.3446)^4$	1M	
≈ 0.121379753		
≈ 0.1214	1A	r.t. 0.1214
(iii) The probability that the team is awarded a bonus in a certain season if the team wins exactly 5 matches in that season		
$=C_3^5(0.3446)^3(1-0.3446)^2+C_4^5(0.3446)^4(1-0.3446)+(0.3446)^5$		
≈ 0.226845138		
The required probability		
$\approx\frac{0.009168006+(0.121379753)\left(\frac{3^4e^{-3}}{4!}\right)+(0.226845138)\left(\frac{3^5e^{-3}}{5!}\right)}{18.4e^{-3}}$	1M+1M	1M for numerator + 1M for denominator
≈ 0.057237086		
≈ 0.0572	1A	r.t. 0.0572
	----------(7)	

9. Among the students sitting for a Mathematics test, 73% of them had revised before the test. For those who had revised, their scores are real numbers which can be modelled by $N(59, 10^2)$; and for those who had not revised, their scores are real numbers which can be modelled by $N(35.2, 12^2)$. Students who scored at least 43 passed the test.

   (a) Find the probability that a randomly selected student passed the test.

   (b) Given that a randomly selected student passed the test, find the probability that he had not revised before the test.

   (c) Ten students are randomly selected among those who passed the test. Find the probability that exactly four of them had not revised before the test.

(7 marks)

[HKDSE 2012'Section A#9]

9. (a) Let X be the score of a student who had revised. $P(X \ge 43) = P\left(Z \ge \frac{43-59}{10}\right)$ $= P(Z \ge -1.6)$ ≈ 0.9452	1A	Either one
Let Y be the score of a student who had not revised. $P(Y \ge 43) = P\left(Z \ge \frac{43-35.2}{12}\right)$ $= P(Z \ge 0.65)$ ≈ 0.2578		
$\therefore$ P(pass the test) $\approx 0.73 \times 0.9452 + 0.27 \times 0.2578$	1M	
$= 0.759602$	1A	OR 0.7596
(b) P(a student had not revised for the test | he passed the test) $= \frac{0.27 \times 0.2578}{0.759602}$	1M	
≈ 0.091634829 ≈ 0.0916	1A	
(c) P(4 students had not revised for the test among 10 passed students) $\approx C_6^{10}(0.091634829)^4(1-0.091634829)^6$	1M	
≈ 0.0083	1A	
	(7)	

9	(a)	Good. Nevertheless, some candidates did not figure out that the required probability was $0.73\,P(X \ge 43) + 0.27\,P(Y \ge 43)$, and some failed to use the standard normal distribution table.
	(b)	Satisfactory. Many candidates were able to apply the correct method, although some got wrong numerical answers.
	(c)	Fair. Some candidates wrote a binomial probability but did not use the result of (b).

9. The lifetime of a randomly selected LED bulb produced by a manufacturer is assumed to be normally distributed with mean μ hours and standard deviation 5000 hours. It is known that 96.41% of the bulbs will have a lifetime shorter than 39000 hours.

(a) Find the value of μ .

(b) Suppose a random sample of 100 bulbs is drawn. Find the probability that the mean lifetime of the sample lies between 30200 hours and 30800 hours.

(c) The manufacturer wants to select another random sample of n bulbs such that the probability that the mean lifetime of the sample exceeding 28500 hours is at least 0.985 . Find the least value of n .

(7 marks)

[HKDSE 2013'Section A#9]

9. (a) P (lifetime of a bulb < 39000) = 0.9641		
$P\left(Z<\frac{39000-\mu}{5000}\right)=0.9641$	1M	
$\frac{39000-\mu}{5000}\approx 1.8$		
$\mu \approx 30000$	1A	
(b) P (30200 < sample mean < 30800)		Can use '≤' sign
$=P\left(\frac{30200-30000}{\frac{5000}{\sqrt{100}}}<Z<\frac{30800-30000}{\frac{5000}{\sqrt{100}}}\right)$	1M	
$=P(0.4<Z<1.6)$		
$\approx 0.4452-0.1554$		
$=0.2898$	1A	
(c) P (sample mean > 28500) ≥ 0.985		
$P\left(Z>\frac{28500-30000}{\frac{5000}{\sqrt{n}}}\right)\geq 0.985$	1M	
$-0.3\sqrt{n}\leq -2.17$	1A	
$n\geq 52.32111111$		
Thus, the least value of n is 53 .	1A	
	(7)	

9. (a)	Very good. Some candidates showed poor presentation such as '$P\left(Z<\frac{39000-\mu}{5000}\right)=0.9641$', '$P\left(\frac{39000-\mu}{5000}\right)=0.9641$', '$P\left(Z<\frac{39000-\mu}{5000}\right)=0.4641=0.9641$' or '$0.9641=0.18$'.
(b)	Good. Some candidates used '0.4452 + 0.1554' to find the probability required.
(c)	Fair. Some candidates used same symbols for both random variables before and after standardization. Many did not show enough ability to solve inequalities, such as writing a negative number greater than or equal to a positive number.

3. In a large farm, the weights of chickens follow a normal distribution with a mean of μ kg and a standard deviation of σ kg . It is given that the percentage of chickens being lighter than 1.83 kg is the same as the percentage of those being heavier than 3.43 kg . Moreover, 89.04% chickens weigh between 1.83 kg and 3.43 kg .

(a) Find μ and σ .

(b) If 9 chickens are selected randomly from the farm, find the probability that the mean of their weights lies between 2.5 kg and 3.1 kg .

(5 marks)

[HKDSE 2017'Section A#3]

3. (a)

μ	
$= \dfrac{1.83+3.43}{2}$	
$= 2.63$	1A
$P\left(\dfrac{1.83-2.63}{\sigma} < Z < \dfrac{3.43-2.63}{\sigma}\right) = 0.8904$	1M
$P\left(\dfrac{-0.8}{\sigma} < Z < \dfrac{0.8}{\sigma}\right) = 0.8904$	
$P\left(0 < Z < \dfrac{0.8}{\sigma}\right) = 0.4452$	
$\dfrac{0.8}{\sigma} = 1.6$	
$\sigma = 0.5$	1A
(b) The required probability	
$= P\left(\dfrac{2.5-2.63}{\frac{0.5}{\sqrt{9}}} < Z < \dfrac{3.1-2.63}{\frac{0.5}{\sqrt{9}}}\right)$	1M
$= P(-0.78 < Z < 2.82)$	
$= 0.2823 + 0.4976$	
$= 0.7799$	1A
	----------(5)

3 (a)	Very good. Most candidates were able to find the required mean μ and standard deviation σ .
(b)	Good. Some candidates mistook σ as the standard deviation of the sample mean.

3. A factory manufactures a batch of marbles. The diameters of the marbles follow a normal distribution with a mean of 9 mm and a standard deviation of 0.125 mm . A marble is classified as *oversized* if its diameter is more than 9.16 mm .

(a) Find the probability that a randomly selected marble from the batch is *oversized.*

(b) The diameters of the marbles are measured one by one. Let X be the random variable representing the number of measurements taken when the first *oversized* marble is found. Find

(i) $P(X \leq 3)$,

(ii) $E(X)$.

(6 marks)

[HKDSE 2018'Section A#3]

3. (a) The required probability		
$= P\left(Z > \dfrac{9.16-9}{0.125}\right)$	1M	
$= P(Z > 1.28)$		
$= 0.1003$	1A	
(b) (i) $P(X \leq 3)$		
$= 0.1003 + (1-0.1003)(0.1003) + (1-0.1003)^2(0.1003)$	1M	
≈ 0.271728757		
≈ 0.2717	1A	r.t. 0.2717
(ii) $E(X)$		
$= \dfrac{1}{0.1003}$	1M	
≈ 9.970089731		
≈ 9.9701	1A	r.t. 9.9701
	----------(6)	

3 (a)	Very good. About 90% of the candidates were able to perform standardization and find the required probability.
(b) (i)	Very good. About 75% of the candidates were able to find the value of $P(X \leq 3)$.
(ii)	Good. Many candidates were able to write down $E(X) = \dfrac{1}{p}$, where p is the probability of getting an *oversized* marble, but some candidates wrongly thought that $E(X) = \dfrac{1}{p} - 1$.

12 – Sampling and Confidence Interval

6. The weights (in kg) of the students in a school can be modelled by the normal distribution with mean 67 and standard deviation 15. A random sample of 36 students is taken.

   (a) Find the probability that the mean weight of the 36 students is over 70 kg.

   (b) It is found that 9 students in the sample like French fries. Find an approximate 95% confidence interval for the proportion of students in the school who like French fries.

(5 marks)

[HKDSE 2012'Section A#6]

6. (a) Let X be the weight of a student. The sample mean $\overline{X} \sim N\left(67, \frac{15^2}{36}\right)$.	
$P(\overline{X} > 70) = P\left(Z > \frac{70-67}{\frac{15}{6}}\right)$	1M
$= P(Z > 1.2)$ ≈ 0.1151	1A
(b) The sample proportion is $\frac{9}{36} = 0.25$.	1A
An approximate 95% confidence interval for the proportion $\approx \left(0.25 - 1.96 \times \sqrt{\frac{0.25 \times 0.75}{36}}, 0.25 + 1.96 \times \sqrt{\frac{0.25 \times 0.75}{36}}\right)$	1M
$\approx (0.1085, 0.3915)$	1A
	(5)

6	(a)	Good. Some candidates failed to perform the standardisation related to the distribution of a sample mean correctly.
	(b)	Satisfactory. Many candidates found the sample proportion but failed to find the confidence interval required.

6. In a random sample of 120 swimmers in a certain beach, 75 of them are not satisfied with the water quality of the beach. Let p be the population proportion of the swimmers in this beach who are not satisfied with the water quality of the beach. Find an approximate 90% confidence interval for p.

(4 marks)

[HKDSE 2013'Section A#6]

5. An estimate for p is $\frac{75}{120}=0.625$.	1A	OR $\frac{5}{8}$
An approximate 90% confidence interval for p $\approx\left(0.625-1.645\sqrt{\frac{0.625(1-0.625)}{120}},\ 0.625+1.645\sqrt{\frac{0.625(1-0.625)}{120}}\right)$	1M+1M	
$\approx(0.5523,\ 0.6977)$	1A	
	(4)	

6.	Good. Some candidates treated 75 as the sample size. Some wrote wrong expressions for the approximate standard deviation of the sample proportion.

9. The manager of a fitness centre wants to promote aerobic classes.

(a) The manager randomly selected 200 Hong Kong residents and found out that 80 of them had taken part in aerobic classes. Let p be the proportion of Hong Kong residents who had taken part in aerobic classes. Find an approximate 95% confidence interval for p.

(b) The manager wants to randomly select n Hong Kong residents and invite them to take part in a free aerobic class. The probability that an invited resident will show up is 0.85. Let X be the proportion of the n invited residents who will show up. Assume that X can be modelled by a normal distribution with mean 0.85 and variance $\frac{0.85(1-0.85)}{n}$. Find the maximum number of n such that the probability that more than 100 invited residents will show up is less than 0.05.

(7 marks)

[HKDSE 2014'Section A#9]

9. (a) An estimate of $p = \frac{80}{200}$	
$= 0.4$	1A
An approximate 95% confidence interval for p	
$= \left(0.4-1.96\sqrt{\frac{0.4\times 0.6}{200}},\ 0.4+1.96\sqrt{\frac{0.4\times 0.6}{200}}\right)$	1M
$\approx (0.3321, 0.4679)$	1A
(b) $X \sim \mathrm{N}\left(0.85, \frac{0.85(1-0.85)}{n}\right)$	
$\mathrm{P}\left(X > \frac{100}{n}\right) < 0.05$	1A
$\mathrm{P}\left(Z > \frac{\frac{100}{n}-0.85}{\sqrt{\frac{0.85(0.15)}{n}}}\right) < 0.05$	1M
$\frac{100-0.85n}{n}\sqrt{\frac{n}{0.1275}} > 1.645$	
$0.85n + 1.645\sqrt{0.1275n} - 100 < 0$	
$-11.19754391 < \sqrt{n} < 10.50650569$	1M
$0 < n < 110.3866618$	
Hence the maximum number of n is 110.	1A
	(7)

9. (a)	Very good.
(b)	Poor. Some candidates considered $\mathrm{P}(X > 100)$ rather than $\mathrm{P}(X > 100/n)$. Some candidates used wrong means such as 85 and $0.85n$, or wrong standard deviations such as $\frac{0.85(1-0.85)}{n}$, for standardisation. Others got inequalities in $\sqrt{n}$ with incorrect direction of sign.

4. There are many packs of seeds and each pack contains 100 seeds. Let p be the population proportion of seeds that germinate in a pack.

(a) A pack of seeds is randomly selected, 64 seeds germinate. Find an approximate 95% confidence interval for p.

(b) It is given that the proportion of seeds that germinate in these packs of seeds follows a normal distribution with a mean of p and a standard deviation of 0.05. Find the least sample size to be taken such that the width of a 90% confidence interval for p is less than 0.04.

(7 marks)

[HKDSE 2016'Section A#4]

4. (a) The point estimate of p is $\frac{64}{100}=0.64$.	1A	
An approximate 95% confidence interval for p		
$=\left(\frac{64}{100}-1.96\sqrt{\frac{(0.64)(0.36)}{100}},\frac{64}{100}+1.96\sqrt{\frac{(0.64)(0.36)}{100}}\right)$	1M+1A	1A for 1.96
$=(0.54592, 0.73408)$	1A	
$\approx(0.5459, 0.7341)$		
(b) Let n be the number of packs in the sample.		
The width of a 90% confidence interval for p is $(2)(1.645)\left(\frac{0.05}{\sqrt{n}}\right)$.		
$(2)(1.645)\left(\frac{0.05}{\sqrt{n}}\right)<0.04$	1M+1A	
$\sqrt{n}>4.1125$		
$n>16.91265625$		
Thus, the least sample size is 17.	1A	
	----------(7)	

4 (a)	Very good. More than 60% of the candidates were able to evaluate the approximate 95% confidence interval for the population proportion p. However, some candidates wrongly used $n=64$ instead of $n=100$ in evaluating the approximate confidence interval $\left(\frac{64}{100}-1.96\sqrt{\frac{(0.64)(0.36)}{n}},\frac{64}{100}+1.96\sqrt{\frac{(0.64)(0.36)}{n}}\right)$.
(b)	Good. Some candidates were unable to distinguish the concept between the confidence interval for the population mean and the approximate confidence interval for the population proportion.

2. In an estate, Peter wants to study the proportion p of households who keep pets. He conducts a survey of a random sample of 64 households and finds that an approximate $\beta\%$ confidence interval for p is $(0.0915, 0.3085)$.

(a) Find

(i) the sample proportion of households who keep pets,

(ii) β.

(b) Using the sample proportion obtained in (a)(i), find the least number of households such that the probability of at least 1 of these households who keeps pets is greater than 0.999.

(6 marks)

[HKDSE 2018'Section A#2]

2. (a) (i) The sample proportion	
$=\dfrac{0.0915+0.3085}{2}$	
$=0.2$	1A
(ii) z	
$=\dfrac{0.3085-0.0915}{2\sqrt{\dfrac{0.2(1-0.2)}{64}}}$	1M
$=2.17$	
Thus, we have $\beta=97$.	1A
(b) Let n be the number of households.	
$1-(1-0.2)^n>0.999$	1M
$0.001>0.8^n$	
$\log 0.001>\log(0.8^n)$	1M
$\log 0.001>n\log 0.8$	
$n>\dfrac{-3}{\log 0.8}$	
$n>30.95655348$	
Thus, the least number of households is 31.	1A
	----------(6)

2 (a) (i)	Good. Many candidates were able to find the required sample proportion of households.
(ii)	Good. Many candidates were able to find the value of β but some candidates wrongly thought that $\beta=97\%$.
(b)	Fair. Many candidates were unable to set up the correct inequality $1-(1-0.2)^n>0.999$.

13 – Normal Distribution (B)

12. A company provides cable-car service for tourists. Tourists complain that the waiting time for the cable-car is too long. From past experience, the waiting time (in minutes) of a randomly selected tourist follows a normal distribution with mean μ and standard deviation 9 .

(a) The customer service manager of the company conducts a survey on the waiting time to estimate μ .

(i) A random sample of 16 tourists is taken and their waiting times are recorded as below:

56	36	48	63	57	41	50	43
56	55	62	46	55	69	38	50

Construct a 90% confidence interval for μ .

(ii) Find the least sample size to be taken such that the width of the 90% confidence interval for μ is less than 6 minutes.

(7 marks)

(b) Suppose that $\mu = 51.5$. The customer service manager of the company interviews tourists and will give a coupon to a tourist whose waiting time is more than 65 minutes.

(i) Find the probability that he gives less than 2 coupons to the first 10 tourists interviewed.

(ii) Find the probability that the 5th coupon is given to the 20th tourist interviewed.

(6 marks)

[HKDSE 2012'Section B#12]

12. (a) (i) The sample mean $= \dfrac{56+\cdots+50}{16}$

$= 51.5625$ — 1A

A 90% confidence interval

$\approx \left(51.5625 - 1.645 \times \dfrac{9}{\sqrt{16}}, 51.5625 + 1.645 \times \dfrac{9}{\sqrt{16}}\right)$ — 1M+1A

$= (47.86125, 55.26375)$ — 1A — OR $(47.8613, 55.2638)$

(ii) Let n be the sample size.

$\therefore \quad 2\left(1.645 \cdot \dfrac{9}{\sqrt{n}}\right) < 6$ — 1M

$n > 24.354225$ — 1A

Hence, the least sample size is 25 . — 1A

(7)

(b) (i) P(a tourist waits for more than 65 minutes)	
$=P\left(Z>\dfrac{65-51.5}{9}\right)$	1M
$=P(Z>1.5)$	
≈ 0.0668	1A
P(less than 2 coupons are sent to the first 10 tourists interviewed)	
$\approx (1-0.0668)^{10}+C_1^{10}(1-0.0668)^9(0.0668)$	1M
≈ 0.8594	1A
(ii) P(the 5th coupon is sent to the 20th tourist interviewed)	
$\approx C_4^{19}(1-0.0668)^{15}(0.0668)^4\cdot 0.0668$	1M
≈ 0.0018	1A
	(6)

12	(a)	(i)	Good. However, some candidates used the standard deviation of the sample instead of the population, used values other than 1.645 , or interchanged the upper and lower confidence limits.
		(ii)	Fair. Besides mistakes similar to (i), many candidates did not write the width of the confidence interval correctly or failed to solve inequalities.
	(b)	(i)	Good. Most candidates were able to express the probability of the mentioned event, but some failed in the standardisation of normal distributions.
		(ii)	Satisfactory. Binomial coefficients were omitted or written wrongly by some candidates.

12. The cholesterol levels (in suitable units) of the adults in a city are assumed to be normally distributed with mean μ and variance σ^2. From a random sample of 49 adults, a 95% confidence interval for μ is found to be $(4.596, 5.044)$.

(a) (i) Find the value of σ.

(ii) Find the mean of the sample.

(3 marks)

(b) Another sample of 15 adults is randomly selected and their cholesterol levels are recorded as follows:

3.6	3.8	3.9	4.3	4.3	4.5	4.8	5.0
5.1	5.2	5.3	5.5	5.8	6.0	6.4	

The two samples are then combined. Construct a 99% confidence interval for μ using the combined sample.

(4 marks)

(c) A health organisation classifies the cholesterol level of an adult to be low, medium and high if his/her cholesterol value is respectively at most 5.2, between 5.2 and 6.2, and at least 6.2. Suppose $\mu = 4.8$.

(i) Find the probability that the cholesterol level of a randomly selected adult in the city is low.

(ii) A sample of 20 adults is randomly selected in the city. Find the probability that there are more than 17 adults with low cholesterol level and at least 1 adult with medium cholesterol level in this sample.

(5 marks)

[HKDSE 2013'Section B#12]

12. (a) (i) $2 \times 1.96 \times \frac{\sigma}{\sqrt{49}} = 5.044 - 4.596$ — 1M

$\sigma = 0.8$ — 1A

(ii) The mean of the sample $= \frac{4.596 + 5.044}{2}$

$= 4.82$ — 1A

(3)

(b) The combined sample mean $= \frac{4.82 \times 49 + 3.6 + 3.8 + \cdots + 6.4}{49 + 15}$ — 1M

$= 4.83875$ — 1A — OR 4.8388

A 99% confidence interval for μ

$\approx \left(4.83875 - 2.575 \times \frac{0.8}{\sqrt{64}}, 4.83875 + 2.575 \times \frac{0.8}{\sqrt{64}} \right)$ — 1M

$= (4.58125, 5.09625)$ — 1A — OR $(4.5813, 5.0963)$

(4)

(c) Let X be the cholesterol level of a randomly selected adult.

(i) $\text{P(low)} = \text{P}(X \le 5.2)$
$$= \text{P}\left(Z \le \frac{5.2-4.8}{0.8}\right)$$
$$= \text{P}(Z \le 0.5)$$
$$\approx 0.6915$$ 1A

(ii) $\text{P(high)} = \text{P}(X \ge 6.2)$
$$= \text{P}\left(Z \ge \frac{6.2-4.8}{0.8}\right)$$
$$= \text{P}(Z \ge 1.75)$$
$$\approx 0.0401$$ 1A

$\text{P(medium)} \approx 1 - 0.6915 - 0.0401$
$$= 0.2684$$ 1A

P(more than 17 adults with low level and at least 1 adult with medium level)
$$\approx C_{18}^{20}(0.6915)^{18}[C_1^2(0.2684)(0.0401)+(0.2684)^2]+C_{19}^{20}(0.6915)^{19}(0.2684)$$ 1M
$$\approx 0.0281$$ 1A

(5)

12. (a)	Good. Some wrote $\frac{\sigma}{49}$ or $\frac{\sigma^2}{\sqrt{49}}$ instead of $\frac{\sigma}{\sqrt{49}}$ as the standard deviation of the sample mean.
(b)	Poor. Many candidates tried to use the standard deviation of the combined sample instead of that of the population. Some thought that the mean of the combined sample would be $\frac{4.82+4.9}{2}$.
(c) (i)	Fair. Some candidates assigned wrong values to μ and σ.
(ii)	Poor. Some candidates missed out the factor C_1^2 in calculating the probability required.

12. The delivery time X (in minutes) of an order received by a pizza restaurant follows a normal distribution with mean μ and standard deviation σ . It is known that 27.43% of the delivery times are longer than 25 minutes and 51.60% of the delivery times fall within 3.5 minutes of μ .

(a) Find μ and σ .

(4 marks)

(b) If the delivery time of an order is longer than k minutes, then a coupon will be given as a compensation to the customer who has made the order. Suppose that a total of 200 orders are received in a day. Assuming independence among delivery times of different orders, find the minimum integral value of k such that the expected number of coupons given out is at most 5 in that day.

(3 marks)

(c) The employees of the pizza restaurant recently received training to improve their efficiency. After training, the delivery time Y (in minutes) of an order follows a normal distribution with mean θ and standard deviation 4.7 .

(i) Manager A draws a random sample of 12 orders and the delivery times (in minutes) are recorded as follows:

22	15	18	21	22	31
20	16	21	19	23	24

Construct a 90% confidence interval for θ .

(ii) Manager B is going to draw another random sample of n orders. He requires that the probability that the mean delivery time of the n orders falls within 3 minutes of θ be greater than 0.99 . Find the minimum value of n to meet his requirement.

(6 marks)

[HKDSE 2014'Section B#12]

12. (a) $P(\mu-3.5 \le X \le \mu+3.5) = 0.5160$	1M
$P\left(0 \le Z \le \frac{3.5}{\sigma}\right) = 0.2580$	
$\frac{3.5}{\sigma} = 0.7$	
$\sigma = 5$	1A
$P(X > 25) = 0.2743$	1M
$P\left(0 < Z < \frac{25-\mu}{\sigma}\right) = 0.2257$	
$\frac{25-\mu}{5} = 0.6$	
$\mu = 22$	1A
	(4)

(b) $P(X > k) \le \frac{5}{200}$ — 1A

$P\left(0 < Z < \frac{k-22}{5}\right) \ge 0.475$

$\frac{k-22}{5} \ge 1.96$ — 1M

$k \ge 31.8$

Hence the minimum integral value of k is 32. — 1A

(3)

(c) (i) Sample mean $= \frac{22+15+\cdots+24}{12}$

$= 21$ — 1A

A 90% confidence interval

$\approx \left(21 - 1.645 \times \frac{4.7}{\sqrt{12}}, 21 + 1.645 \times \frac{4.7}{\sqrt{12}}\right)$ — 1M

$\approx (18.7681, 23.2319)$ — 1A

(ii) Let $\overline{Y}$ be the mean delivery time of the n orders.

$P(\theta - 3 \le \overline{Y} \le \theta + 3) > 0.99$

$P\left(\frac{-3}{\frac{4.7}{\sqrt{n}}} \le Z \le \frac{3}{\frac{4.7}{\sqrt{n}}}\right) > 0.99$

$\frac{3}{\frac{4.7}{\sqrt{n}}} > 2.575$ — 1M

$n > 16.27450069$ — 1A

Hence the minimum value of n is 17. — 1A

(6)

Alternative Solution

$\left(\overline{Y} - 2.575 \times \frac{4.7}{\sqrt{n}}, \overline{Y} + 2.575 \times \frac{4.7}{\sqrt{n}}\right)$

$\subsetneq (\overline{Y} - 3, \overline{Y} - 3)$

$\therefore 2.575 \times \frac{4.7}{\sqrt{n}} < 3$

12. (a)		Satisfactory. Some candidates found the value of $P(\mu - 1.75 \le X \le \mu + 1.75)$ instead of $P(\mu - 3.5 \le X \le \mu + 3.5)$.
(b)		Fair. Some candidates got inequalities in k with incorrect direction of sign.
(c)	(i)	Good. Few candidates used the sample standard deviation in the calculation.
	(ii)	Poor. Some candidates wrongly treated 21 as the mean of $\overline{Y}$. Some candidates thought that the length of the confidence interval was 3.

9. The speeds of cars passing a checkpoint on a highway follow a normal distribution with a mean of μ km/h and a standard deviation of 16 km/h .

(a) A survey on the speeds of cars to estimate μ is conducted.

(i) A random sample of 25 cars is taken and the stem-and-leaf diagram below shows the distribution of their speeds (in km/h) :

Stem (tens)	Leaf (units)
6	0 0 1 1 1 2 2 3 4 4 5 5 6 6 7
7	1 1 2 3 5 5 6
8	3 6 7

Find a 95% confidence interval for μ .

(ii) Find the least sample size to be taken such that the width of a 97.5% confidence interval for μ is less than 9 .

(7 marks)

(b) Suppose that $\mu = 66$. If the speed of a car passing the checkpoint exceeds 90 km/h , a penalty ticket will be issued.

(i) If a car passes the checkpoint, find the probability that a penalty ticket will be issued.

(ii) If 12 cars pass the checkpoint, find the probability that more than 2 penalty tickets will be issued.

(5 marks)

[HKDSE 2015'Section B#9]

9. (a) (i) The sample mean		
$= 68.64$ km/h	1A	
A 95% confidence interval for μ		
$= \left(68.64 - 1.96\left(\frac{16}{\sqrt{25}}\right), 68.64 + 1.96\left(\frac{16}{\sqrt{25}}\right)\right)$	1M+1A	1A for 1.96
$= (62.368, 74.912)$	1A	
(ii) Let n be the sample size.		
$2(2.24)\left(\frac{16}{\sqrt{n}}\right) < 9$	1M+1A	1A for 2.24
$n > 63.43237531$		
Thus, the least sample size is 64 .	1A	
	----------(7)	
(b) (i) The required probability		
$= \mathrm{P}\left(Z > \frac{90-66}{16}\right)$	1M	
$= \mathrm{P}(Z > 1.5)$		
$= 0.5 - 0.4332$		
$= 0.0668$	1A	
(ii) The required probability		
$= 1-(1-0.0668)^{12} - C_1^{12}(1-0.0668)^{11}(0.0668) - C_2^{12}(1-0.0668)^{10}(0.0668)^2$	1M+1M	1M for using (b)(i) + 1M for binomial probability
≈ 0.041574551		
≈ 0.0416	1A	r.t. 0.0416
	----------(5)	

9 (a) (i)	Very good. Most candidates were able to use the correct formula to find the confidence interval while a few candidates treated 16 as the variance rather than the standard deviation of the given distribution.
(ii)	Very good. A few candidates wrongly used the sample mean in (a)(i) to find the width of the interval concerned.
(b) (i)	Very good. Most candidates were able to perform standardization and find the required probability.
(ii)	Very good. Most candidates were able to formulate the required probability form while a few candidates used wrong probabilities in substitution.

9. X and Y are two schools with the same number of students. The daily reading times (in minutes) of the students in each school are assumed to be normally distributed. In school X, 0.6% of the students read less than 40 minutes daily while 1.5% read more than 70 minutes. In school Y, 1.5% of the students read less than 48 minutes daily while 1.7% read more than 72 minutes.

(a) Which school has less students reading more than 60 minutes daily? Explain your answer. (6 marks)

(b) For the school that has less students reading more than 60 minutes daily, find the probability that the 4th randomly selected student is the 2nd one who reads more than 60 minutes daily. (2 marks)

(c) Students reading T minutes or more daily will be awarded. What should the least value of T be so that no more than 10% of students are awarded in each school? Give your answer in integral minutes. (4 marks)

[HKDSE 2016'Section B#9]

Solution	Marks	Remarks
9. Let J minutes and K minutes be the random variables representing the daily reading times of the students in schools X and Y respectively.		
(a) Let μ_1 and σ_1 be the mean and the standard deviation of the daily reading time of the students in school X respectively, while μ_2 and σ_2 be the mean and the standard deviation of the daily reading time of the students in schools Y respectively.		
$\begin{cases} \dfrac{40-\mu_1}{\sigma_1} = -2.51 \\ \dfrac{70-\mu_1}{\sigma_1} = 2.17 \end{cases}$	1M+1A	either one
$\begin{cases} \dfrac{48-\mu_2}{\sigma_2} = -2.17 \\ \dfrac{72-\mu_2}{\sigma_2} = 2.12 \end{cases}$		
Solving, we have		
$\mu_1 = \dfrac{4375}{78}, \sigma_1 = \dfrac{250}{39}$	1A	(for both)
$\mu_1 \approx 56.08974359, \sigma_1 \approx 6.41025641$		r.t. $\mu_1 \approx 56.0897, \sigma_1 \approx 6.4103$
$\mu_1 \approx 56.0897, \sigma_1 \approx 6.4103$		
$\mu_2 = \dfrac{8600}{143}, \sigma_2 = \dfrac{800}{143}$	1A	(for both)
$\mu_2 \approx 60.13986014, \sigma_2 \approx 5.594405594$		r.t. $\mu_2 \approx 60.1399, \sigma_2 \approx 5.5944$
$\mu_2 \approx 60.1399, \sigma_2 \approx 5.5944$		

P(students reading more than 60 minutes in school X)
$= \mathrm{P}(J > 60)$
$= \mathrm{P}\left(Z > \dfrac{60 - \frac{4375}{78}}{\frac{250}{39}} \right)$ — 1M — either one
$= \mathrm{P}(Z > 0.61)$
$= 0.2709$

P(students reading more than 60 minutes in school Y)
$= \mathrm{P}(K > 60)$
$= \mathrm{P}\left(Z > \dfrac{60 - \frac{8600}{143}}{\frac{800}{143}} \right)$
$= \mathrm{P}(Z > \frac{-1}{40})$
$> \mathrm{P}(Z > 0)$
$= 0.5$
> 0.2709

Thus, there are less students reading more than 60 minutes daily in school X. — 1A — f.t.
----------(6)

(b) The required probability
$= C_1^3(0.2709)(1-0.2709)^2(0.2709)$ — 1M
≈ 0.11703438
≈ 0.1170 — 1A — r.t. 0.1170
----------(2)

(c) For school X,
$\mathrm{P}(J \geq T) \leq 0.1$
$\dfrac{T - \frac{4375}{78}}{\frac{250}{39}} \geq 1.29$ — 1M+1A — 1A for 1.29 — either one
$T \geq \frac{2510}{39}$
$T \geq 64.35897436$
$T \geq 65$ — 1A — accept $T = 65$ — either one

For school Y,
$\mathrm{P}(K \geq T) \leq 0.1$
$\dfrac{T - \frac{8600}{143}}{\frac{800}{143}} \geq 1.29$
$T \geq \frac{9632}{143}$
$T \geq 67.35664336$
$T \geq 68$

Thus, the least value of T is 68. — 1A — f.t.
----------(4)

9 (a)	Good. Many candidates were able to formulate the corresponding equations in means and standard deviations, but some candidates were unable to give the numerical answers either in an exact fraction or correct to 4 decimal places.
(b)	Good. Many candidates were able to apply the result of (a).
(c)	Fair. About half of the candidates were unable to use inequality to formulate the problem. Besides, many candidates used 1.28 instead of 1.29 in the inequality.

9. The daily times spent on homework of the students in a school follow a normal distribution with a mean of μ hours and a standard deviation of 0.4 hour.

(a) A survey is conducted in the school to estimate μ .

(i) A sample of 40 students in the school is randomly selected and their daily times spent on homework are recorded below:

Daily time spent (x hours)	Number of students
$0.5 < x \le 1.0$	11
$1.0 < x \le 1.5$	13
$1.5 < x \le 2.0$	8
$2.0 < x \le 2.5$	5
$2.5 < x \le 3.0$	3

Find a 90% confidence interval for μ .

(ii) Find the least sample size to be taken such that the width of a 97% confidence interval for μ is at most 0.3 .

(7 marks)

(b) Suppose that $\mu = 1.48$. If the daily time spent on homework of a student exceeds 2 hours, then the student has to attend homework guidance class.

(i) If a student is randomly selected from the school, find the probability that the student has to attend homework guidance class.

(ii) A sample of 15 students is now randomly drawn from the school and their daily times spent on homework are examined one by one. Given that more than 1 student in the sample have to attend homework guidance class, find the probability that the 10th student is the 2nd student who has to attend homework guidance class.

(6 marks)

[HKDSE 2017'Section B#9]

9. (a) (i) The sample mean		
$= \dfrac{(0.75)(11)+(1.25)(13)+(1.75)(8)+(2.25)(5)+(2.75)(3)}{40}$		
$= 1.45$ hours	1A	
A 90% confidence interval for μ		
$= \left(1.45-1.645\left(\dfrac{0.4}{\sqrt{40}}\right), 1.45+1.645\left(\dfrac{0.4}{\sqrt{40}}\right)\right)$	1M+1A	1A for 1.645
$\approx (1.3460, 1.5540)$	1A	r.t. (1.3460, 1.5540)
(ii) Let n be the sample size.		
$2(2.17)\left(\dfrac{0.4}{\sqrt{n}}\right) \le 0.3$	1M+1A	1A for 2.17
$n \ge 33.48551111$		
Thus, the least sample size is 34 .	1A ----------(7)	
(b) (i) The required probability		
$= \mathrm{P}\left(Z > \dfrac{2-1.48}{0.4}\right)$	1M	
$= \mathrm{P}(Z > 1.3)$		
$= 0.5 - 0.4032$		
$= 0.0968$	1A	
(ii) The required probability		
$= \dfrac{C_1^9(1-0.0968)^8(0.0968)^2}{1-(1-0.0968)^{15}-C_1^{15}(1-0.0968)^{14}(0.0968)}$	1M+1M+1M	1M for using (b)(i) + 1M for numerator + 1M for denominator
≈ 0.086102962		
≈ 0.0861	1A ----------(6)	r.t. 0.0861

9 (a) (i)	Very good. Most candidates were able to find the confidence interval correctly.
(ii)	Very good. A few candidates wrongly used the sample mean obtained in (a)(i) to find the width of the interval concerned.
(b) (i)	Very good. About 80% of the candidates were able to find the required probability.
(ii)	Good. Many candidates were able to find the required conditional probability.

9. A fruit wholesaler, John, grades a batch of apples according to their weights. The following table shows the classification of the apples, where a is a constant.

Weight of an apple (W g)	$W \leq a$	$a < W \leq 260$	$W > 260$
Classification	*small*	*medium*	*large*

The weights of the apples follow a normal distribution with a mean of μ g and a standard deviation of 16 g. It is known that 10.56% and 73.57% of the apples are *large* and *medium* respectively. Every 8 of the apples are packed in a box. A box of apples is regarded as *regular* if there are at least 6 *medium* apples in the box.

(a) Find μ and a. (3 marks)

(b) Find the probability that a randomly chosen box of apples is *regular*. (2 marks)

(c) John randomly chooses 3 boxes of apples.

(i) Find the probability that these 3 boxes of apples are *regular* and there are totally 21 *medium* apples and 3 *small* apples.

(ii) Given that these 3 boxes of apples are *regular*, find the probability that there are totally 21 *medium* apples and 3 *small* apples.

(iii) Given that there are totally 21 *medium* apples and 3 *small* apples in these 3 boxes of apples, find the probability that these 3 boxes of apples are *regular*.

(7 marks)

[HKDSE 2018'Section B#9]

9. (a) $P(W > 260) = 0.1056$

$P(\mu < W \le 260) = 0.5 - 0.1056$

$P\left(0 < Z \le \frac{260-\mu}{16}\right) = 0.3944$

$\frac{260-\mu}{16} = 1.25$ — 1M — for either one

$\mu = 240$ — 1A

$P(a < W \le 240) = 0.7357 - 0.3944$

$P\left(\frac{a-240}{16} < Z \le 0\right) = 0.3413$

$\frac{a-240}{16} = -1$

$a = 224$ — 1A

----------(3)

(b) The required probability

$= C_6^8(0.7357)^6(1-0.7357)^2 + C_7^8(0.7357)^7(1-0.7357) + C_8^8(0.7357)^8$ — 1M

≈ 0.642619261

≈ 0.6426 — 1A — r.t. 0.6426

----------(2)

(c) (i) The required probability

$= (C_7^8(0.7357)^7(0.1587))^3 + 6(C_6^8(0.7357)^6(0.1587)^2)(C_7^8(0.7357)^7(0.1587))(0.7357)^8$ — 1M

$= 1856(0.7357)^{21}(0.1587)^3$

≈ 0.011776727

≈ 0.0118 — 1A — r.t. 0.0118

(ii) The required probability

$\approx \frac{1856(0.7357)^{21}(0.1587)^3}{(0.642619261)^3}$ — 1M + 1M

≈ 0.044377559

≈ 0.0444 — 1A — r.t. 0.0444

(iii) The required probability

$= \frac{1856(0.7357)^{21}(0.1587)^3}{c_{21}^{24}(0.7357)^{21}(0.1587)^3}$ — 1M

$= \frac{232}{253}$ — 1A — r.t. 0.9170

≈ 0.916996047

≈ 0.9170

The required probability

$= \frac{1856(0.7357)^{21}(0.1587)^3}{1856(0.7357)^{21}(0.1587)^3 + 3(C_5^8(0.7357)^5(0.1587)^3)((0.7357)^8)^2}$ — 1M

$= \frac{232}{253}$ — 1A — r.t. 0.9170

≈ 0.916996047

≈ 0.9170

----------(7)

9 (a)	Very good. Many candidates were able to find μ and a by standardizing the normal variable correctly.
(b)	Very good. About 75% of the candidates were able to find the required probability.
(c) (i)	Fair. Many candidates missed the binomial coefficients in finding the probability.
(ii)	Fair. Some candidates were able to use the results of (b) and (c)(i) to find the required conditional probability.
(iii)	Fair. Many candidates were unable to count correctly the number of cases under the given condition, hence they were unable to find the required conditional probability.

10. In city H, the water consumption (in m^3) of each family in a certain month follows a normal distribution with a mean of $\mu\ \text{m}^3$ and a standard deviation of $4\ \text{m}^3$.

(a) A survey is conducted to estimate μ.

(i) A random sample of 16 families is selected and their water consumptions (in m^3) in that month are recorded as follows:

17 17 18 19 19 20 20 21 21 21 22 23 23 23 24 24

Find a 95% confidence interval for μ.

(ii) Find the least sample size to be taken such that the width of a 99.5% confidence interval for μ is less than 3.

(7 marks)

(b) Suppose that $\mu = 20$. If the water consumption of a family in that month lies between $18\ \text{m}^3$ and $23\ \text{m}^3$, the family is regarded as *ordinary*.

(i) Find the percentage of *ordinary* families in city H.

(ii) The families in city H are randomly selected one by one and their water consumptions in that month are recorded. The recording stops when 3 *ordinary* families are found. Given that more than 6 families are selected in this recording process, find the probability that the water consumptions of exactly 9 families are recorded.

(6 marks)

[HKDSE 2019' Section B#10]

10. (a) (i) The sample mean		
$=\dfrac{17+17+18+19+19+20+20+21+21+21+22+23+23+23+24+24}{16}$		
$= 20.75\ \text{m}^3$	1A	
A 95% confidence interval for μ		
$=\left(20.75-1.96\left(\dfrac{4}{\sqrt{16}}\right),\ 20.75+1.96\left(\dfrac{4}{\sqrt{16}}\right)\right)$	1M+1A	1A for 1.96
$=(18.79,\ 22.71)$	1A	
(ii) Let n be the sample size.		
$2(2.81)\left(\dfrac{4}{\sqrt{n}}\right)<3$	1M+1A	1A for 2.81
$n>56.15004444$		
Thus, the least sample size is 57.	1A	
	----------(7)	
(b) (i) The required percentage		
$=\text{P}\left(\dfrac{18-20}{4}<Z<\dfrac{23-20}{4}\right)\times 100\%$	1M	
$=\text{P}(-0.5<Z<0.75)\times 100\%$		
$=(0.1915+0.2734)\times 100\%$		
$=0.4649\times 100\%$		
$=46.49\%$	1A	
(ii) Take $p=0.4649$.		
The required probability		
$=\dfrac{C_2^8(1-p)^6p^3}{1-p^3-C_1^3(1-p)p^3-C_2^4(1-p)^2p^3-C_3^5(1-p)^3p^3}$	1M+1M+1M	1M for using (b)(i) + 1M for numerator +1M for denominator
≈ 0.160443919		
≈ 0.1604	1A	r.t. 0.1604
The required probability		
$=\dfrac{C_2^8(1-0.4649)^6(0.4649)^3}{(1-0.4649)^6+C_1^6(1-0.4649)^5(0.4649)+C_2^6(1-0.4649)^4(0.4649)^2}$	1M+1M+1M	1M for using (b)(i) + 1M for numerator+1M for denominator
≈ 0.160443919		
≈ 0.1604	1A	r.t. 0.1604
	----------(6)	