Level 1 Module 2
exemplar with comments

2019-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2019

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

1. Let $f(x) = \frac{10x}{7+3x^2}$. Prove that $f(1+h) - f(1) = \frac{4h-3h^2}{10+6h+3h^2}$. Hence, find $f'(1)$ from first principles. (4 marks)

$f(x) = \frac{10x}{7+3x^2}$

$f(1+h) - f(1) = \frac{10(1+h)}{7+3(1+h)^2} - \frac{10(1)}{7+3(1)^2}$

$= \frac{10+10h}{7+3(1+h)^2} - 1$

$= \frac{10+10h-(7+3(1+h)^2)}{7+3(1+h)^2}$

$= \frac{10+10h-(7+3+6h+h^2)}{10+6h+3h^2}$

$= \frac{4h-3h^2}{10+6h+3h^2}$

$f'(1) = \frac{4(1)-3(1)^2}{10+6(1)+3(1)^2}$

$= \frac{1}{19}$

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2. Let $P(x)=\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $P(x)$ is 160. Find

(a) λ,

(b) $P'(0)$.

(5 marks)

a) det $P(x) = (x+\lambda)(x+\lambda)^2(x+\lambda)^3 + 12 - 8(x+\lambda)^2 - 15(x+\lambda)$

$= (x+\lambda)^6 - 8(x+\lambda)^2 - 15(x+\lambda) + 12$

$(x+\lambda)^6$

$= {}^6C_3 (x)^3 (\lambda)^3 + \cdots$

${}^6C_3 (\lambda)^3 = 160$

$\lambda^3 = 8$ $\quad\therefore \lambda = 2$.

$\lambda = 2$

b) $P'(0) = (0+2)^6 - 8(0+2)^2 - 15(0+2) + 12$

$= 14$

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3. A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm^3 of liquid X. The researcher finds that during the experiment, $\frac{dV}{dt} = -2t$, where $V\text{ cm}^3$ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

(a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

(b) It is given that $V = h^2 + 24h$, where h cm is the depth of liquid X in the vessel. Find the value of $\frac{dh}{dt}$ when $t = 18$.

(6 marks)

b) $V = h^2 + 24h$

$\frac{dv}{dh} = 2h + 24$

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4. Define $g(x)=\frac{\ln x}{\sqrt{x}}$ for all $x \in (0, 99)$. Denote the graph of $y=g(x)$ by G.

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G, the x-axis and the vertical line passing through the maximum point of G. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

a) $g(x) = \frac{\ln x}{\sqrt{x}}$

$g'(x) = \frac{\sqrt{x}\frac{d}{dx}\ln x - \ln x\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$

$= \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{x}$

$g''(x) = -$

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5. (a) Using mathematical induction, prove that $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(7 marks)

a) $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$

Let $P(n)$ be the statement

When $n=1$,

$L.H.S = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} = \frac{2}{3}$ $\quad$ $R.H.S = \frac{1+1}{1(2(1)+1)} = \frac{2}{3}$

The statement is true when $n=1$.

Assume the statement is true for some positive integer m.

$\sum_{k=m}^{2m} \frac{1}{m(m+1)} = \frac{m+1}{m(2m+1)}$

When $n=m+1$,

$L.H.S = \sum_{k=m}^{2n} \frac{1}{(m+1)(m+2)}$

$= \frac{m+1}{m(2m+1)} + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)}$

$= \frac{m^3+8m^2+11m+4}{m(m+1)(m+2)(2m+1)}$

$= \frac{\cancel{(m+1)}(m^2+7m+4)}{m\cancel{(m+1)}(m+2)\cancel{(2m+1)}}$

$= \frac{m+2}{(m+1)(2m+3)}$.

$\therefore$ The statement is also true for $n=m+1$.

$\therefore$ By Mathematical Induction, the statement is true for all positive integer n.

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b) $\sum_{k=50}^{200} \frac{1}{k(k+1)}$

$= \sum_{k=1}^{200} \frac{1}{k(k+1)} - \sum_{k=1}^{49} \frac{1}{k(k+1)}$

$= \frac{100+1}{100(2(100)+1)} - \frac{\frac{49}{2}+1}{(\frac{49}{2})(49+1)}$

$= \frac{101}{20100} - \frac{\frac{51}{2}}{1225}$

$= \frac{101}{20100} - \frac{51}{2450}$

$=$

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6. Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x - 2y - 2z = \beta \\ 5x + \alpha y + \alpha z = 5\beta \\ 7x + (\alpha-3)y + (2\alpha+1)z = 8\beta \end{cases}, \text{ where } \alpha, \beta \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α.

(ii) Express y in terms of α and β.

(b) Assume that $\alpha = -4$. If (E) is inconsistent, find the range of values of β.

(7 marks)

a i) $\because$ (E) has a unique solution

$\therefore |E| \neq 0$

$$\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix} \neq 0$$

$\alpha(2\alpha+1) - \cancel{14\alpha} - 10(\alpha-3) + \cancel{14\alpha} + 10(2\alpha+1) - \alpha(\alpha-3) \neq 0$

$2\alpha^2 + \alpha - 10\alpha + 30 + 20\alpha + 10 - \alpha^2 + 3\alpha \neq 0$

$\alpha^2 + 14\alpha + 40 \neq 0$

$(\alpha-4)(\alpha-10) \neq 0$

$\alpha \neq 4$ or $\alpha \neq 10$,

$\therefore \alpha \neq 4$ or $\alpha \neq 10$

ii) $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & \alpha & \alpha & 5\beta \\ 7 & \alpha-3 & 2\alpha+1 & 8\beta \end{array}\right)$$

$\therefore |E| = \alpha^2 + 14\alpha + 40$

By using Cramer's rule,

$$|\Delta y| = \begin{vmatrix} 1 & \beta & -2 \\ 5 & 5\beta & \alpha \\ 7 & 8\beta & 2\alpha+1 \end{vmatrix}$$

$= 5\beta(2\alpha+1) + 7\alpha\beta - 80\beta + 70\beta - 8\alpha\beta - 5\beta(2\alpha+1)$

$= \cancel{10\alpha\beta} + \cancel{5\beta} + 7\alpha\beta - 80\beta + 70\beta - 8\alpha\beta - \cancel{10\alpha\beta} - \cancel{5\beta}$

$= -\alpha\beta - 10\beta$

$\therefore y = \dfrac{|\Delta y|}{|E|}$

(P. T. O).

Answers written in the margins will not be marked.

$y = \dfrac{-\alpha\beta - 6\beta}{\alpha^2 + 14\alpha + 46}$,

b) Sub $\alpha = -4$ into the equations.

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & -4 & -4 & 5\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 3 & 0 & 0 & 3\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 3 & 0 & 0 & 3\beta \\ \frac{7}{2} & 0 & 0 & \frac{9}{2}\beta \end{array}\right)$$

$3x = 3\beta$

$\beta = x$

$\frac{7}{2}x = \frac{9}{2}\beta$

$\beta = \frac{7}{9}x$

$\therefore$ (E) is inconsistent

$\therefore$ Range of values of β is between $\frac{7}{9}x$ and x.

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7. (a) Using integration by parts, find $\int e^x \sin \pi x \, dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x \, dx$.

(7 marks)

a) $\int e^x \sin \pi x \, dx$

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8. Let $h(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \frac{2x^2 - 7x + 8}{x}$ for all $x > 0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = h(x)$ by H. It is given that H passes through the point $(1, 3)$. Find

(i) the equation of H,

(ii) the point(s) of inflexion of H.

(8 marks)

a) $h'(x) = \frac{2x^2-7x+8}{x} = 2x - 7 + \frac{8}{x}$

$h(x) = \int 2x - 7 + \frac{8}{x}\,dx$

$= x^2 - 7x - \frac{4}{x^2} + C$.

$\therefore$ h(x) is an increasing function.

bi) $h(x) = x^2 - 7x - \frac{4}{x^2} + C$, $y = h(x)$

Sub the point (1,3) into the equation

$3 = (1)^2 - 7(1) - \frac{4}{(1)^2} + C$

$3 = 1 - 7 - 4 + C$

$C = 13$

$\therefore$ The equation of H is $y = x^2 - 7x - \frac{4}{x^2} + 13$

ii) $h'(x) = \frac{2x^2-7x+8}{x}$

$h''(x) = \frac{x(4x-7) - (2x^2-7x+8)}{x^2}$

$= \frac{4x^2 - 7x - 2x^2 + 7x - 8}{x^2}$.

$= \frac{2x^2 - 8}{x^2} = \frac{2(x+2)(x-2)}{x^2}$

When $h''(x) = 0$

$2(x+2)(x-2) = 0$

$x = -2$ or $x = 2$

Answers written in the margins will not be marked.

When $x = 2$,

$y = (2)^2 - 7(2) - \frac{4}{(2)^2} + 13$

$= 4 - 14 - 1 + 13$

$= 2$

When $x = -2$,

$y = (-2)^2 - 7(-2) - \frac{4}{(-2)^2} + 13$

$= 4 + 14 - 1 + 13$

$= 30$

$\therefore$ The points of inflexion of H are $(2, 2)$ and $(-2, 30)$.

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SECTION B (50 marks)

9. Consider the curve Γ: $y=\frac{1}{3}\sqrt{12-x^2}$, where $0<x<2\sqrt{3}$. Denote the tangent to Γ at $x=3$ by L .

(a) Find the equation of L . (3 marks)

(b) Let C be the curve $y=\sqrt{4-x^2}$, where $0<x<2$. It is given that L is a tangent to C . Find

(i) the point(s) of contact of L and C ;

(ii) the point(s) of intersection of C and Γ ;

(iii) the area of the region bounded by L , C and Γ .

(9 marks)

a) Γ : $y=\frac{1}{3}\sqrt{12-x^2}$

When $x=3$, $y=\frac{1}{3}\sqrt{12-9}$

$=\frac{\sqrt{3}}{3}$.

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10. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2+\cos 2x} = \frac{\sec^2 x}{2+\sec^2 x}$. $\frac{1}{\cos^2\theta}$ (1 mark)

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,dx$. (3 marks)

(c) Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} f(x)\ln(1+e^x)\,dx = \int_0^a x\,f(x)\,dx$ for any $a \in \mathbf{R}$. (4 marks)

(d) Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,dx$. (5 marks)

a) L.H.S $= \frac{1}{2+\cos 2x}$

$= \frac{1}{\quad}$

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11. Let $M=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2=aM+bI$. (3 marks)

(b) Prove that $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$ for all positive integers n. (4 marks)

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$ for all positive integers n? Explain your answer. (5 marks)

a) $M=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$

$M^2=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\begin{pmatrix}2 & 7\\ -1 & 6\end{pmatrix}$

$=\begin{pmatrix}-3 & -56\\ 4 & -43\end{pmatrix}$

$M^2=aM+bI$

$\begin{pmatrix}-3 & -56\\ 4 & -43\end{pmatrix}=a\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}+b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\therefore a=5,\ b=-13$

$2a+b=-3$

$b=-3-2a$ — ①

$-6a+b=-43$ — ②

sub $b=-3-2a$ into ②

$-6a+(-3-2a)=-43$

$-8a=-40$

$a=5$

sub $a=5$ into ① $b=-3-2(5)$

$=-13$

b) $6M^n=(1-(-5)^n)M+(5+(-5)^n M)I$

Let $P(n)$ be the statement

when $n=1$,

$L.H.S=6M^{(1)}=6M$ $R.H.S=(1-(-5)^n)M+(5+(-5)^{(1)}M)I$

$=6M$

$\therefore$ The statement is true when $n=1$

Assume the statement is also true for some positive integer k.

$6M^k=(1-(-5)^k)M+(5+(-5)^k M)I$

when $n=k+1$

$L.H.S=6M^{k+1}$

$=6M^k\cdot M$

$=[(1-(-5)^k)M+(5+(-5)^k M)I]\cdot M$

$=[(1-(-5)^k)M^2+(5+(-5)^k M)I\cdot M]$

$=[(1-(-5)^k)M^2+(5+(-5)^k M)M]$

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$= [(1-(-5)^k)\mu^2 + (5+(-5)^k)\mu^2]$

$= \mu^2 [1-(-5)^k + 5 - (-5)^k]$

$= \mu^2 [6 - \cancel{2}(-5)^k]$

$=$ R.H.S.

$\therefore$ The statement is also true when $n=k+1$.

$\therefore$ By Mathematical Induction, the statement is true for all positive integer n.

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12. Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$.

(a) Find t . (3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (2 marks)

(c) Find the volume of the pyramid $OABC$. (2 marks)

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OR} = r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer.

(6 marks)

a) $\vec{AC} = \vec{OC} - \vec{OA}$

$= (-5\vec{i} - 12\vec{j} + t\vec{k}) - (\vec{i} - 4\vec{j} + 2\vec{k})$

$= -6\vec{i} - 8\vec{j} + (t-2)\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (-5\vec{i} - 12\vec{j} + 4\vec{k}) - (-5\vec{i} - 4\vec{j} + 8\vec{k})$

$= -8\vec{j} - 4\vec{k}$

$\because |\vec{AC}| = |\vec{BC}|$

$\therefore t - 2 = -4$

$t = -2$

b) $\vec{AB} = \vec{OB} - \vec{OA}$

$= (-5\vec{i} - 4\vec{j} + 8\vec{k}) - (\vec{i} - 4\vec{j} + 2\vec{k})$

$= -6\vec{i} + 6\vec{k}$

$\vec{AC} = -6\vec{i} - 8\vec{j} - 4\vec{k}$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 0 & 6 \\ -6 & -8 & -4 \end{vmatrix} = -36\vec{j} + 48\vec{k} + 48\vec{i} - 24\vec{j}$

$= 48\vec{i} - 60\vec{j} + 48\vec{k}$

Answers written in the margins will not be marked.

c) $V=\frac{1}{3}\left|(\vec{AB}\times\vec{AC})\cdot\vec{OA}\right|$

$=\frac{1}{3}\left|(48\vec{i}-60\vec{j}+48\vec{k})\cdot(\vec{i}-4\vec{j}+2\vec{k})\right|$

$=\frac{1}{3}\left|48+240+96\right|$

$=128$ cubic units

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END OF PAPER

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Comments

The candidate communicates and expresses simple ideas using mathematical language and notations.

He/She is able to expand the determinant and to apply binomial theorem to find the unknown in Question 2(a).

It can be concluded that the candidate demonstrates elementary knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by performing straightforward procedures according to direct instructions.