Level 1 Module 2
exemplar with comments

2020-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2020

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

Answers written in the margins will not be marked.

1. (a) Expand $(1-x)^4$.

(b) Find the constant k such that the coefficient of x^2 in the expansion of $(1+kx)^9(1-x)^4$ is -3.

(4 marks)

a) $(1-x)^4$

$= C^4_0(1)^4 + C^4_1(1)^3(-x)^1 + C^4_2(1)^2(-x)^2 + C^4_3(1)(-x)^3 + C^4_4(-x)^4$

$= 1-4x+6x^2-4x^3+x^4$ //

b) $(1+kx)^9$

$= C^9_0(1)^9 + C^9_1(1)^8(kx)^1 + C^9_2(1)^7(kx)^2 + \cdots\cdots + C^9_9(kx)^9$

The x^2 terms in the expansion of $(1+kx)^9(1-x)^4$

$= (1)(6x^2) + (9kx)(-4x) + (36k^2x^2)(1)$

$(1)(6x^2)+(9kx)(-4x)+(36k^2x^2)(1) = -3x^2$

$6x^2 - 36kx^2 + 36k^2x^2 = -3x^2$

$6-36k+36k^2 = -3$

$36k^2-36k+9 = 0$

$k = \frac{1}{2}$ (repeat) //

$\therefore$ the value of constant k is $\frac{1}{2}$.

Answers written in the margins will not be marked.

2. Define $f(x) = \frac{x}{\sqrt{2+x}}$ for all $x > -2$. Find $f'(2)$ from first principles. (4 marks)

$$f(x) = \frac{d}{dx}\frac{x}{\sqrt{2+x}}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x}{\sqrt{2+(x+\Delta x)}} - \frac{x}{\sqrt{2+x}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{x+\Delta x}{\sqrt{2+(x+\Delta x)}} - \frac{x}{\sqrt{2+x}}\right) \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{(\sqrt{2+x})(x+\Delta x) - x(\sqrt{2+(x+\Delta x)})}{(\sqrt{2+(x+\Delta x)})(\sqrt{2+x})}\right) \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(\sqrt{2+x})(\sqrt{2+(x+\Delta x)}) - x(\sqrt{2+(x+\Delta x)})(\sqrt{2+(x+\Delta x)})}{(\sqrt{2+x})(\sqrt{2+(x+\Delta x)})(\sqrt{2+(x+\Delta x)})} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\quad}{(\sqrt{2+x})(2+x+\Delta x)\Delta x}$$

3. (a) Let x be an angle which is not a multiple of $30°$. Prove that

(i) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$,

(ii) $\tan x \tan(60° - x)\tan(60° + x) = \tan 3x$.

(b) Using (a)(ii), prove that $\tan 55° \tan 65° \tan 75° = \tan 85°$.

(6 marks)

ai) L.H.S $= \tan 3x$

$= \tan(2x + x)$

$= \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

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4. (a) Find $\int \sin^2\theta \, d\theta$.

(b) Define $f(x) = 4x(1-x^2)^{\frac{1}{4}}$ for all $x \in [0,1]$. Denote the graph of $y = f(x)$ by G . Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

a) $\int \sin^2\theta d\theta$

$= -\int \sin\theta \, d(\cos\theta)$

$= -\sin\theta\cos\theta - \int \sin\theta d\theta$

$= -\sin\theta\cos\theta - \cos\theta + C$

$= -\cos\theta(\sin\theta + 1) + C$ //

Answers written in the margins will not be marked.

5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$.

(7 marks)

a) when $n=1$

$L.H.S = \frac{1}{1(1+1)(1+2)}$

$= \frac{1}{6}$

$R.H.S = \frac{(1)(1+3)}{4(1+1)(1+2)}$

$= \frac{1}{6}$

$\therefore n=1$ is proved.

Assume k be all positive integers.

$$\sum_{k=1}^{k} \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

When $k=k+1$

$L.H.S = \frac{1}{(k+1)(k+1+1)(k+1+2)} + \frac{k(k+3)}{4(k+1)(k+2)}$

$= \frac{1}{(k+1)(k+2)(k+3)} + \frac{k(k+3)}{4(k+1)(k+2)}$

$= \frac{4 + k(k+3)(k+3)}{4(k+1)(k+2)(k+3)}$

$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$R.H.S = \frac{(k+1)(k+1+3)}{4(k+1+1)(k+1+2)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

Answers written in the margins will not be marked.

a) $= L.H.S$

$\therefore$ K+1 is proved

$\therefore$ By MI, $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all positive integers n.

b) $\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$

$= \frac{123(123+3)(50)}{4(123+1)(123+2)} - \frac{120(120+3)(50)}{4(120+1)(120+2)}$

$= \left(\frac{7749}{31000} - \frac{1845}{7381}\right)50$

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6. Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the origin by O . Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

(a) Define $S = u^2 + v^2$. Does S increase at a constant rate? Explain your answer.

(b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q . Find the rate of change of the area of ΔOPQ when $u = 2$.

(7 marks)

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7. Let $f(x)$ be a continuous function defined on $\mathbf{R}$. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $(1, 2)$ and $f'(x) = -2x + 8$ for all $x \in \mathbf{R}$.

(a) Find the equation of Γ.

(b) Let L be a tangent to Γ such that L passes through the point $(5, 14)$ and the slope of L is negative. Denote the point of contact of Γ and L by P. Find

(i) the coordinates of P,

(ii) the equation of the normal to Γ at P.

(8 marks)

a) equation of Γ

$$\frac{y-2}{x-1} = -2x+8$$

$$y-2 = (-2x+8)(x-1)$$

$$y-2 = -2x^2+2x+8x-8$$

$$2x^2+10x+y+6 = 0$$

bi)

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8. Define $P=\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Let $M=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}$ such that $|M|=1$ and $PM=MQ$, where a, b and c are real numbers.

(a) Find a, b and c.

(b) Define $R=\begin{pmatrix}6 & 2\\ -15 & -5\end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b)(i), prove that $(\alpha P+\beta R)^{99}=\alpha^{99}P+\beta^{99}R$ for any real numbers α and β.

(8 marks)

a) $M=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}$

$|M|=1$

$\begin{vmatrix}1 & a\\ b & c\end{vmatrix}=1$

$c-ab=1$

$c=1+ab$ ①

$PM=MQ$

$\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix}\begin{pmatrix}1 & a\\ b & c\end{pmatrix}=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$\begin{pmatrix}-5 & -2a\\ 15b & 6c\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ ← $\begin{vmatrix}-5 & -2a\\ 15b & 6c\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}$

$-30c+30ab=0$

$-30c=-30ab$

$ab=c$ ②

sub ② to ①

$c=1+c$

$2c=1$

$c=\frac{1}{2}$ //

from ①

$\frac{1}{2}=1+ab$

$-\frac{1}{2}=ab$

$a=-\frac{1}{2b}$ — ③

~~1+ab~~

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a)

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SECTION B (50 marks)

9. Let $f(x)=\frac{(x+4)^3}{(x-4)^2}$ for all real numbers $x \neq 4$. Denote the graph of $y=f(x)$ by H.

(a) Find the asymptote(s) of H. (3 marks)

(b) Find $f''(x)$. (2 marks)

(c) Someone claims that there are two turning points of H. Do you agree? Explain your answer. (2 marks)

(d) Find the point(s) of inflexion of H. (2 marks)

(e) Find the area of the region bounded by H, the x-axis and the y-axis. (3 marks)

a) the verticle asymptote of H is $x=4$
the oblique asymptote of H is $y=x+20$

b) $f'(x)=\frac{(x-4)^2\frac{d}{dx}(x+4)^3-(x+4)^3\frac{d}{dx}(x-4)^2}{(x-4)^{2\cdot 2}}$

$=\frac{(x+4)^2(x-20)}{(x-4)^3}$

$f''(x)=\frac{(x-4)^3\frac{d}{dx}((x+4)^2(x-20))-(x+4)^2(x-20)\frac{d}{dx}(x-4)^3}{(x-4)^{3\cdot 2}}$

$=\frac{(x-4)^3(x+4)(3x-36)-(x+4)^2(x-20)3(x-4)^2}{(x-4)^6}$

$=\frac{3(x-4)^2(x+4)(x^2-16x+48-x^2+16x+80)}{(x-4)^6}$

$=\frac{384(x+4)}{(x-4)^4}$ //

Answers written in the margins will not be marked.

c) when $f''(x) = 0$

$x = -4$

	$x<-4$	$x=-4$	$x>-4$
$f(x)$	$-$	0	$+$

$\because$ H only have minimum point, so it only have one turning point.

$\therefore$ I don't agree.

d) when $x=-4$,

$$y = \frac{(-4+4)^3}{(-4-4)^2}$$

$$y = 0$$

$\therefore$ the minimum point is $(-4, 0)$ //

e)

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10. (a) Using integration by substitution, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$.

(3 marks)

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\,dx$. (3 marks)

(c) (i) Using $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot\frac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \dfrac{x\csc^2 x}{\cot x - 1}\,dx = \dfrac{\pi}{8}\ln(2+\sqrt{3})$.

(7 marks)

(i) $\cot\frac{\pi}{12} = \cot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

L.H.S

$= \cot\frac{\pi}{12}$

$= \cot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$= \dfrac{\cot\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{4}\right)+1}{\cot\left(\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{3}\right)}$

$= \dfrac{\left(\frac{\sqrt{3}}{3}\right)(1)+1}{1-\frac{\sqrt{3}}{3}}$

$= \dfrac{\left(\frac{\sqrt{3}}{3}+1\right)}{\left(1-\frac{\sqrt{3}}{3}\right)} \cdot \dfrac{1+\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}$

$= \dfrac{\left(\frac{\sqrt{3}}{3}+1\right)^2}{1-\frac{3}{9}}$

$= \left(\dfrac{3+\sqrt{3}}{3}\right)^2 \times \dfrac{3}{2}$

$= \dfrac{9+6\sqrt{3}+3}{9} \times \dfrac{3}{2}$

$= \dfrac{12+6\sqrt{3}}{9} \times \dfrac{3}{2}$

$= \dfrac{36+18\sqrt{3}}{18}$

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(i) $= 2+\sqrt{3}$

$= R.H.S$

$\therefore \cot\frac{\pi}{12} = 2+\sqrt{3}$

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11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x - y - 2z = 1 \\ x - 2y + hz = k \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h, k \in \mathbf{R} .$$

(i) Assume that (E) has a unique solution.

(1) Prove that $h \neq -3$.

(2) Solve (E) .

(ii) Assume that $h = -3$ and (E) is consistent.

(1) Prove that $k = -2$.

(2) Solve (E) .

(9 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F): \begin{cases} x - y - 2z = 1 \\ x - 2y + hz = -2 \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h \in \mathbf{R} .$$

Someone claims that there are at least two values of h such that (F) has a real solution (x, y, z) satisfying $3x^2 + 4y^2 - 7z^2 = 1$. Do you agree? Explain your answer. (4 marks)

ai)

$$(E) = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -2 & h \\ 4 & h & -7 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -1 & -2 \\ 1 & -2 & h \\ 4 & h & -7 \end{vmatrix}$$

$= (1)(-2)(-7) + (1)(h)(-2) + (-1)(h)(4) - (4)(-2)(-2) - (-1)(1)(-7) - (1)(h)(h)$

$= 14 - 2h - 4h - 8 - 7 - h^2$

$= -h^2 - 6h - 1$

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12. Let $\overrightarrow{OP} = \mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i}-7\mathbf{j}-4\mathbf{k}$, where O is the origin. R is a point lying on PQ such that $PR:RQ = 1:3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$. (2 marks)

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$. (2 marks)

(c) Let N be a point such that $\overrightarrow{ON} = \lambda(\overrightarrow{OP} \times \overrightarrow{OR})$, where λ is a real number.

(i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

(ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i}+\mu\mathbf{j}-10\mathbf{k}$.

(1) Find λ and μ.

(2) Denote the angle between ΔOPQ and ΔNPQ by θ. Find $\tan\theta$.

(8 marks)

a) $\overrightarrow{OR}$

$= \dfrac{(i+j+4k)\times 3+(5i-7j-4k)\times 1}{1+3}$

$= \dfrac{3i+3j+12k+5i-7j-4k}{4}$

$= \dfrac{8i-4j+8k}{4}$

$= 2i-j+2k$

$\overrightarrow{OP} \times \overrightarrow{OR}$

$= (i+j+4k)(2i-j+2k)$

$= 2i^2-ij+2ik+2ij-j^2+2jk+8ik-4jk+8k^2$

$= 2i^2-j^2+8k^2+ij+10ik-2jk$ //

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END OF PAPER

Answers written in the margins will not be marked.

Comments

The candidate communicates and expresses simple ideas using mathematical language and notations.

He/She is able to apply binomial theorem and to find the required constant in Question 1.

It can be concluded that the candidate demonstrates elementary knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by performing straightforward procedures according to direct instructions.