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HKDSE
Mathematics
Extended Part
Module 2

Past Papers (by topic)
2012 – 2019
Solution

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Mathematical Induction

1.

3. For $n=1$,

L.H.S. $=1\times 2=2$ and R.H.S. $=1^2(1+1)=2$

$\therefore$ L.H.S. = R.H.S. and the statement is true for $n=1$. — 1

Assume $1\times 2+2\times 5+3\times 8+\cdots+k(3k-1)=k^2(k+1)$, where k is a positive integer. — 1

$1\times 2+2\times 5+3\times 8+\cdots+k(3k-1)+(k+1)[3(k+1)-1]$

$=k^2(k+1)+(k+1)(3k+2)$ by the assumption — 1

$=(k+1)(k^2+3k+2)$

$=(k+1)^2(k+1+1)$ — 1

Hence the statement is true for $n=k+1$.

By the principle of mathematical induction, the statement is true for all positive integers n. — 1 Follow through

(5)

[HKDSE 2012 M2 #3]

2.

3. For $n=1$,

L.H.S. $1+\dfrac{1}{1\times 4}=\dfrac{5}{4}$ and R.H.S. $=\dfrac{4(1)+1}{3(1)+1}=\dfrac{5}{4}$

$\therefore$ L.H.S. = R.H.S. and the statement is true for $n=1$. — 1

Assume $1+\dfrac{1}{1\times 4}+\dfrac{1}{4\times 7}+\dfrac{1}{7\times 10}+\cdots+\dfrac{1}{(3k-2)(3k+1)}=\dfrac{4k+1}{3k+1}$, where k is a positive integer. — 1

$1+\dfrac{1}{1\times 4}+\dfrac{1}{4\times 7}+\dfrac{1}{7\times 10}+\cdots+\dfrac{1}{(3k-2)(3k+1)}+\dfrac{1}{[3(k+1)-2][3(k+1)+1]}$

$=\dfrac{4k+1}{3k+1}+\dfrac{1}{(3k+1)(3k+4)}$ by the assumption — 1

$=\dfrac{(12k^2+19k+4)+1}{(3k+1)(3k+4)}$

$=\dfrac{(3k+1)(4k+5)}{(3k+1)(3k+4)}$

$=\dfrac{4(k+1)+1}{3(k+1)+1}$ — 1

Hence the statement is true for $n=k+1$.

By the principle of mathematical induction, the statement is true for all positive integers n. — 1 Follow through

(5)

[HKDSE 2013 M2 #3]

3.

8. (a)	Note that $\sin\frac{x}{2}\cos\frac{(1+1)x}{2}=\sin\frac{x}{2}\cos x$.	1
	So, the statement is true for $n=1$.	
	Assume that $\sin\frac{x}{2}\sum_{k=1}^{m}\cos kx=\sin\frac{mx}{2}\cos\frac{(m+1)x}{2}$ for some positive integer m .	1M
	$\sin\frac{x}{2}\sum_{k=1}^{m+1}\cos kx$	
	$=\sin\frac{x}{2}\sum_{k=1}^{m}\cos kx+\sin\frac{x}{2}\cos(m+1)x$	
	$=\sin\frac{mx}{2}\cos\frac{(m+1)x}{2}+\sin\frac{x}{2}\cos(m+1)x$ (by induction assumption)	1M
	$=\frac{1}{2}\left(\sin\frac{(2m+1)x}{2}-\sin\frac{x}{2}\right)+\frac{1}{2}\left(\sin\frac{(2m+3)x}{2}-\sin\frac{(2m+1)x}{2}\right)$	1M
	$=\frac{1}{2}\left(\sin\frac{(2m+3)x}{2}-\sin\frac{x}{2}\right)$	
	$=\cos\frac{(2m+4)x}{4}\sin\frac{(2m+2)x}{4}$	1M
	$=\sin\frac{(m+1)x}{2}\cos\frac{(m+2)x}{2}$	
	So, the statement is true for $n=m+1$ if it is true for $n=m$.	
	By mathematical induction, we have $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx=\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$ for all positive integers n .	1
(b)	Putting $x=\frac{\pi}{7}$ and $n=567$ in (a), we have	
	$\sum_{k=1}^{567}\cos\frac{k\pi}{7}$	
	$=\dfrac{\sin\frac{(567)(\pi)}{(2)(7)}\cos\frac{(568)(\pi)}{(2)(7)}}{\sin\frac{\pi}{(2)(7)}}$	1M
	$=\dfrac{\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{14}\right)}{\sin\frac{\pi}{14}}$	
	$=\dfrac{-\sin\frac{\pi}{14}}{\sin\frac{\pi}{14}}$	
	$=-1$	1A
		-----------(8)

[HKDSE 2015 M2 #8]

4.

5. (a) Note that $(-1)^1(1^2)=-1=\frac{(-1)^1(1)(2)}{2}$.		
So, the statement is true for $n=1$.	1	
Assume that $\sum_{k=1}^{m}(-1)^k k^2=\frac{(-1)^m m(m+1)}{2}$ for some positive integer m .	1M	
$\sum_{k=1}^{m+1}(-1)^k k^2$		
$=\sum_{k=1}^{m}(-1)^k k^2+(-1)^{m+1}(m+1)^2$		
$=\frac{(-1)^m m(m+1)}{2}+(-1)^{m+1}(m+1)^2$ (by induction assumption)	1M	for using induction assumption
$=\frac{(-1)^m m(m+1)+(-1)^{m+1}(m+1)(2m+2)}{2}$		
$=\frac{(-1)^m (m+1)(m-2m-2)}{2}$		
$=\frac{(-1)^{m+1}(m+1)(m+2)}{2}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(b) Putting $n=333$ in (a), we have $\sum_{k=1}^{333}(-1)^k k^2=\frac{(-1)^{333}(333)(334)}{2}$.	1M	
So, we have $-1+4+\sum_{k=3}^{333}(-1)^k k^2=\frac{(-1)^{333}(333)(334)}{2}$.		
Thus, we have $\sum_{k=3}^{333}(-1)^{k+1}k^2=55\,614$.	1A	
	----------(6)	

[HKDSE 2016 M2 #5]

5.

6. (a) Note that $(1)(1+4)=5=\frac{(1)(2)(15)}{6}$.		
So, the statement is true for $n=1$.	1	
Assume that $\sum_{k=1}^{m} k(k+4)=\frac{m(m+1)(2m+13)}{6}$ for some positive integer m .	1M	
$\sum_{k=1}^{m+1} k(k+4)$		
$=\sum_{k=1}^{m} k(k+4)+(m+1)(m+5)$		
$=\frac{m(m+1)(2m+13)}{6}+(m+1)(m+5)$ (by induction assumption)	1M	for using induction assumption
$=\frac{(m+1)(2m^2+13m+6m+30)}{6}$		
$=\frac{(m+1)(2m^2+19m+30)}{6}$		
$=\frac{(m+1)(m+2)(2m+15)}{6}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(b) Putting $n=555$ in (a), we have		
$\sum_{k=1}^{555} k(k+4)=\frac{(555)(556)(1123)}{6}=57\ 755\ 890$.	1M	either one
Putting $n=332$ in (a), we have		
$\sum_{k=1}^{332} k(k+4)=\frac{(332)(333)(677)}{6}=12\ 474\ 402$.		
$\sum_{k=333}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$		
$=\left(\frac{1}{112}\right)\left(\frac{1}{223}\right)\sum_{k=333}^{555} k(k+4)$	1M	either one
$=\frac{1}{(112)(223)}\left(\sum_{k=1}^{555} k(k+4)-\sum_{k=1}^{332} k(k+4)\right)$		
$=\frac{1}{24\ 976}(57\ 755\ 890-12\ 474\ 402)$		
$=1\ 813$	1A	
	(7)	

[HKDSE 2018 M2 #6]

6.

5. (a) Note that $\frac{1}{(1)(2)}+\frac{1}{(2)(3)}=\frac{2}{3}=\frac{1+1}{(1)(2+1)}$.

So, the statement is true for $n=1$. — 1

Assume that $\sum_{k=m}^{2m}\frac{1}{k(k+1)}=\frac{m+1}{m(2m+1)}$, where m is a positive integer. — 1M

$\sum_{k=m+1}^{2m+2}\frac{1}{k(k+1)}$

$=\sum_{k=m}^{2m}\frac{1}{k(k+1)}-\frac{1}{m(m+1)}+\frac{1}{(2m+1)(2m+2)}+\frac{1}{(2m+2)(2m+3)}$ — 1M — can be absorbed

$=\frac{m+1}{m(2m+1)}-\frac{1}{m(m+1)}+\frac{1}{(2m+1)(2m+2)}+\frac{1}{(2m+2)(2m+3)}$ — 1M — for using induction assumption

$=\frac{(m+1)^2-(2m+1)}{m(m+1)(2m+1)}+\frac{(2m+3)+(2m+1)}{(2m+1)(2m+2)(2m+3)}$

$=\frac{m}{(m+1)(2m+1)}+\frac{2}{(2m+1)(2m+3)}$

$=\frac{(2m+1)(m+2)}{(m+1)(2m+1)(2m+3)}$

$=\frac{m+2}{(m+1)(2m+3)}$

So, the statement is true for $n=m+1$ if it is true for $n=m$.

By mathematical induction, the statement is true for all positive integers n . — 1

(b) Putting $n=50$ in (a), we have $\sum_{k=50}^{100}\frac{1}{k(k+1)}=\frac{51}{(50)(101)}=\frac{51}{5\,050}$. — 1M — either one

Putting $n=100$ in (a), we have $\sum_{k=100}^{200}\frac{1}{k(k+1)}=\frac{101}{(100)(201)}=\frac{101}{20\,100}$.

So, we have $\sum_{k=50}^{200}\frac{1}{k(k+1)}=\frac{51}{5\,050}+\frac{101}{20\,100}-\frac{1}{(100)(101)}$.

Thus, we have $\sum_{k=50}^{200}\frac{1}{k(k+1)}=\frac{151}{10\,050}$. — 1A

----------(7)

[HKDSE 2019 M2 #5]

Binomial Theorem

1.

2. $(1+ax)^n = 1+nax+\frac{n(n-1)}{2}(ax)^2+\cdots$ — 1M+1A — 1A for $\frac{n(n-1)}{2}$

Hence $na=6$ and $\frac{n(n-1)}{2}a^2=16$. — 1M

Solving, $\frac{n(n-1)}{2}\left(\frac{6}{n}\right)^2=16$

$18(n-1)=16n$

$n=9$ — 1A

Therefore, $a=\frac{2}{3}$. — 1A

(5)

[HKDSE 2012 M2 #2]

2.

2. $(1+ax)^n = 1+C_1^n ax + C_2^n (ax)^2+\cdots$ — OR general term $= C_r^n (ax)^r$

$$\begin{cases} na=-20 & \text{------------ (1)} \\ \frac{n(n-1)}{2}a^2=180 & \text{------------ (2)} \end{cases}$$ — 1M

$(2)\div(1)^2$:

$\frac{n-1}{2n}=\frac{180}{400}$ — 1M

$n=10$ — 1A

$\therefore\ a=-2$ — 1A

(4)

[HKDSE 2013 M2 #2]

3.

1. (a) $(1-4x)^2(1+x)^n$

$=(1-8x+16x^2)\left[1+nx+\frac{n(n-1)}{2}x^2+\cdots\right]$ ---------------- (*) — 1M

For binomial expansion of $(1+x)^n$ up to the x^2 term

Coefficient of $x=n-8$

$\therefore\ n-8=1$

i.e. $n=9$ — 1A

(b) $\therefore\ (1-4x)^2(1+x)^9=(1-8x+16x^2)(1+9x+36x^2+\cdots)$

Coefficient of $x^2=36-8\cdot 9+16$ — 1M

Alternative Solution

Coefficient of $x^2=\frac{n(n-1)}{2}-8n+16$ by (*) — 1M

$=-20$ — 1A

(4)

[HKDSE 2014 M2 #1]

4.

1. $(5+x)^4$		
$=5^4+4(5^3)x+6(5^2)x^2+4(5)x^3+x^4$	1M	
$=625+500x+150x^2+20x^3+x^4$	1A	
$\left(1-\frac{2}{x}\right)^3$		
$=1-3\left(\frac{2}{x}\right)+3\left(\frac{2}{x}\right)^2-\left(\frac{2}{x}\right)^3$	1M	
$=1-\frac{6}{x}+\frac{12}{x^2}-\frac{8}{x^3}$		
The constant term		
$=(625)(1)+(500)(-6)+(150)(12)+(20)(-8)$	1M	withhold 1M if the step is skipped
$=-735$	1A	
	----------(5)	

[HKDSE 2016 M2 #1]

5.

2. Note that $(1+ax)^8=1+C_1^8ax+C_2^8(ax)^2+\cdots+(ax)^8$ and	1M	
$(b+x)^9=b^9+C_1^9b^8x+C_2^9b^7x^2+\cdots+C_7^9b^2x^7+C_8^9bx^8+x^9$.	1M	
Also note that that $\lambda_2:\mu_7=7:4$ and $\lambda_1+\mu_8+6=0$.		
Therefore, we have $\frac{C_2^8a^2}{C_7^9b^2}=\frac{7}{4}$ and $8a+9b+6=0$.	1M	for either one
So, we have $4a^2=9b^2$ and $8a+9b+6=0$.		
Hence, we have $4a^2-9\left(\frac{-8a-6}{9}\right)^2=0$.		
Simplifying, we have $7a^2+24a+9=0$.	1M	for $pa^2+qa+r=0$
Thus, we have $a=-3$ or $a=\frac{-3}{7}$.	1A	for both correct
	----------(5)	

[HKDSE 2017 M2 #2]

6.

2. $(x+3)^5$		
$= x^5 + 5(3)x^4 + 10(3^2)x^3 + 10(3^3)x^2 + 5(3^4)x + 3^5$	1M	
$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$	1A	
$\left(x - \dfrac{4}{x}\right)^2$		
$= x^2 - 2x\left(\dfrac{4}{x}\right) + \left(\dfrac{4}{x}\right)^2$	1M	
$= x^2 - 8 + \dfrac{16}{x^2}$		
The coefficient of x^3		
$= (1)(16) + (90)(-8) + (405)(1)$	1M	withhold 1M if this step is skipped
$= -299$	1A	
	----------(5)	

[HKDSE 2018 M2 #2]

7.

2. (a) $\mathrm{P}(x)$		
$= (x+\lambda)(x+\lambda)^2(x+\lambda)^3 - (3)(5)(x+\lambda) + (1)(3)(4) - (2)(4)(x+\lambda)^2$	1M	
$= (x+\lambda)^6 - 8(x+\lambda)^2 - 15(x+\lambda) + 12$		
Note that the coefficient of x^3 in the expansion of $\mathrm{P}(x)$ is 160 .		
So, we have $C_3^6\lambda^3 = 160$.	1M	for $C_3^6\lambda^3$
Thus, we have $\lambda = 2$.	1A	
(b) Note that $\mathrm{P}'(0)$ is the coefficient of x in the expansion of $\mathrm{P}(x)$.	1M	can be absorbed
Also note that the coefficient of x is $6\lambda^5 - 16\lambda - 15$.		
By (a), we have $\mathrm{P}'(0) = 145$.	1A	
	----------(5)	

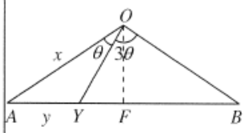
[HKDSE 2019 M2 #2]

Compound Angle Formulae

1.

10. (a) Let F be a point on AB such that $OF \perp AB$. Let OA be x.

$\therefore \quad AF = \frac{1}{2}$ and $\angle AOF = 2\theta$ (properties of isos. Δ)



In ΔOAF, $\sin 2\theta = \dfrac{\frac{1}{2}}{x}$ — 1M

OR

$x^2 = x^2 + 1^2 - 2x\cos(90° - 2\theta)$

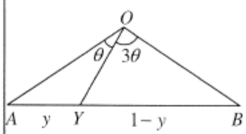
> Alternative Solution
>
> In ΔOAB, $\dfrac{1}{\sin 4\theta} = \dfrac{x}{\sin(90° - 2\theta)}$ — 1M

$x = \dfrac{1}{2\sin 2\theta}$ -- (1)

In ΔOAY, $\dfrac{y}{\sin\theta} = \dfrac{x}{\sin(90° + \theta)}$ --------------------------- (2) — 1M

Substitute (1) into (2) : $\dfrac{y}{\sin\theta} = \dfrac{1}{2\sin 2\theta} \cdot \dfrac{1}{\cos\theta}$ — 1M

> Alternative Solution
>
> In ΔOAY, $\dfrac{y}{\sin\theta} = \dfrac{OY}{\sin\angle OAY}$ — 1M
>
> In ΔOBY, $\dfrac{1-y}{\sin 3\theta} = \dfrac{OY}{\sin\angle OBY}$ — 1M
>
> $\because \quad \angle OAY = \angle OBY$ (base $\angle$s, isos. Δs)
>
> $\therefore \quad \dfrac{y}{\sin\theta} = \dfrac{1-y}{3\sin\theta - 4\sin^3\theta}$ — 1M
>
> $3y - 4y\sin^2\theta = 1 - y$
>
> $4y(1 - \sin^2\theta) = 1$



$y = \dfrac{1}{4}\sec^2\theta$ — 1

(b) $\because \quad 0° < 4\theta < 180°$ — 1M

$\therefore \quad 0° < \theta < 45°$

$\dfrac{1}{4}\sec^2 0° < y < \dfrac{1}{4}\sec^2 45°$ (since $\sec^2\theta$ is an increasing function for $0° < \theta < 45°$)

i.e. $\dfrac{1}{4} < y < \dfrac{1}{2}$ — 1A

Accept using "$\le$" sign

(6)

[HKDSE 2012 M2 #10]

2.

7. (a) R.H.S. $= \dfrac{\sin 2x}{1+\cos 2x}$

$= \dfrac{2\sin x\cos x}{1+2\cos^2 x-1}$ — 1M For either formula

$= \dfrac{\sin x}{\cos x}$

$= \tan x$ — 1

$=$ L.H.S.

(b) R.H.S. $= \dfrac{\sin 8y\cos 4y\cos 2y}{(1+\cos 8y)(1+\cos 4y)(1+\cos 2y)}$

$= \tan 4y \cdot \dfrac{\cos 4y\cos 2y}{(1+\cos 4y)(1+\cos 2y)}$ by (a) — 1M

$= \dfrac{\sin 4y\cos 2y}{(1+\cos 4y)(1+\cos 2y)}$ — 1M

$= \tan 2y \cdot \dfrac{\cos 2y}{1+\cos 2y}$ by (a)

$= \dfrac{\sin 2y}{1+\cos 2y}$

$= \tan y$ by (a) — 1

$=$ L.H.S.

(5)

[HKDSE 2013 M2 #7]

3.

7. (a) $\sin^2 x\cos^2 x$

$= \dfrac{(2\sin x\cos x)^2}{4}$

$= \dfrac{\sin^2 2x}{4}$ — 1M

$= \dfrac{1}{4}\left(\dfrac{1-\cos 4x}{2}\right)$

$= \dfrac{1-\cos 4x}{8}$ — 1

(b) (i) $\mathrm{f}(x)$

$= \sin^4 x+\cos^4 x$

$= (\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x$ — 1M

$= 1^2-2\left(\dfrac{1-\cos 4x}{8}\right)$ (by (a)) — 1M for using (a)

$= \dfrac{1}{4}\cos 4x+\dfrac{3}{4}$ — 1A

(ii) $8\mathrm{f}(x)=7$

$8\left(\dfrac{1}{4}\cos 4x+\dfrac{3}{4}\right)=7$ (by (b)(i)) — 1M for using the result of (b)(i)

$2\cos 4x+6=7$

$\cos 4x=\dfrac{1}{2}$

$4x=\dfrac{\pi}{3}$ or $4x=\dfrac{5\pi}{3}$

$x=\dfrac{\pi}{12}$ or $x=\dfrac{5\pi}{12}$ — 1A for both

----------(7)

[HKDSE 2015 M2 #7]

4.

6. (a) Note that $4(-1)^3+2(-1)^2-3(-1)-1=-4+2+3-1=0$.		
Thus, $x+1$ is a factor of $4x^3+2x^2-3x-1$.	1	
(b) $\cos 3\theta$		
$=\cos(\theta+2\theta)$		
$=\cos\theta\cos 2\theta-\sin\theta\sin 2\theta$	1M	
$=\cos\theta(\cos^2\theta-\sin^2\theta)-\sin\theta(2\sin\theta\cos\theta)$		
$=\cos\theta(\cos^2\theta-(1-\cos^2\theta))-2\cos\theta(1-\cos^2\theta)$		
$=4\cos^3\theta-3\cos\theta$	1A	
(c) Note that $\theta=\frac{3\pi}{5}$ satisfies $\cos 3\theta=\cos(3\pi-2\theta)$.	1M	
Therefore, $\theta=\frac{3\pi}{5}$ satisfies $\cos 3\theta=-\cos 2\theta$.		
By (b), $\theta=\frac{3\pi}{5}$ satisfies $4\cos^3\theta-3\cos\theta=-(2\cos^2\theta-1)$.	1M	for using (b)
Hence, $\cos\frac{3\pi}{5}$ is a root of the equation $4x^3+2x^2-3x-1=0$.		
By (a), we have $4x^3+2x^2-3x-1=(x+1)(4x^2-2x-1)$.		
So, we have $4x^3+2x^2-3x-1=4(x+1)\left(x-\frac{1+\sqrt{5}}{4}\right)\left(x-\frac{1-\sqrt{5}}{4}\right)$.		
Since $\frac{\pi}{2}<\frac{3\pi}{5}<\pi$, we have $-1<\cos\frac{3\pi}{5}<0$.		
Thus, we have $\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$.	1	
	----------(6)	

[HKDSE 2016 M2 #6]

5.

7. (a) $\sin 3x$		
$= \sin(x+2x)$		
$= \sin x \cos 2x + \cos x \sin 2x$	1M	
$= \sin x(\cos^2 x - \sin^2 x) + 2\sin x \cos^2 x$		
$= \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x)$		
$= 3\sin x - 4\sin^3 x$	1	
(b) (i) $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$= \dfrac{\sin\left(3x-\frac{3\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$= \dfrac{\sin 3x \cos\frac{3\pi}{4} - \cos 3x \sin\frac{3\pi}{4}}{\sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4}}$	1M	
$= \dfrac{\frac{-1}{\sqrt{2}}(\sin 3x + \cos 3x)}{\frac{1}{\sqrt{2}}(\sin x - \cos x)}$		
$= \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$	1	
(ii) $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$		
$\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = 2$ (by (b)(i))	1M	for using (b)(i)
Note that $\sin\left(x-\frac{\pi}{4}\right) \neq 0$.		
$3-4\sin^2\left(x-\frac{\pi}{4}\right) = 2$ (by (a))	1M	for using (a)
$1-4\sin^2\left(x-\frac{\pi}{4}\right) = 0$		
$\left(1-2\sin\left(x-\frac{\pi}{4}\right)\right)\left(1+2\sin\left(x-\frac{\pi}{4}\right)\right) = 0$	1M	
Since $\frac{\pi}{4} < x < \frac{\pi}{2}$, we have $\sin\left(x-\frac{\pi}{4}\right) = \frac{1}{2}$.		
Therefore, we have $x - \frac{\pi}{4} = \frac{\pi}{6}$.		
Thus, we have $x = \frac{5\pi}{12}$.	1A	
	----------(8)	

[HKDSE 2017 M2 #7]

6.

3. (a)	$\cot A = 3\cot B$ $\dfrac{\cos A}{\sin A} = \dfrac{3\cos B}{\sin B}$ $3\sin A\cos B = \cos A\sin B$	1M
	$\sin(A+B) - 2\sin(B-A)$ $= (\sin A\cos B + \cos A\sin B) - 2(\sin B\cos A - \cos B\sin A)$ $= 3\sin A\cos B - \cos A\sin B$ $= 0$ Thus, we have $\sin(A+B) = 2\sin(B-A)$.	1
(b)	$\cot\left(x+\dfrac{4\pi}{9}\right) = 3\cot\left(x+\dfrac{5\pi}{18}\right)$ By letting $A = x+\dfrac{4\pi}{9}$ and $B = x+\dfrac{5\pi}{18}$, we have $\cot A = 3\cot B$.	1M
	By (a), we have $\sin(A+B) = 2\sin(B-A)$. With the help of $\sin\left(\dfrac{-\pi}{6}\right) = \dfrac{-1}{2}$, we have $\sin\left(2x+\dfrac{13\pi}{18}\right) = -1$.	1M
	Noting that $0 \le x \le \dfrac{\pi}{2}$, we have $x = \dfrac{7\pi}{18}$.	1A
	Since $\cot\left(\dfrac{7\pi}{18}+\dfrac{4\pi}{9}\right) = -\sqrt{3} = 3\cot\left(\dfrac{7\pi}{18}+\dfrac{5\pi}{18}\right)$, the required solution of the equation $\cot\left(x+\dfrac{4\pi}{9}\right) = 3\cot\left(x+\dfrac{5\pi}{18}\right)$ is $x = \dfrac{7\pi}{18}$.	
		----------(5)

[HKDSE 2018 M2 #3]

Differentiation (First principles)

1.

1. $f(x)=e^{2x}$		
$f'(0)=\lim\limits_{h\to 0}\dfrac{e^{2(0+h)}-e^{2(0)}}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{e^{2h}-1}{2h}\cdot 2$	1M	Accept $\lim\limits_{h\to 0}\dfrac{e^{2h}-1}{h}$
$=2$	1A	
	(3)	

[HKDSE 2012 M2 #1]

2.

1. $\dfrac{d}{dx}(\sin 2x)=\lim\limits_{h\to 0}\dfrac{\sin 2(x+h)-\sin 2x}{h}$	1M
$=\lim\limits_{h\to 0}\left(\dfrac{2}{h}\cos\dfrac{2x+2h+2x}{2}\sin\dfrac{2x+2h-2x}{2}\right)$	1M
$=\lim\limits_{h\to 0}\left[2\cos(2x+h)\dfrac{\sin h}{h}\right]$	
$=2\lim\limits_{h\to 0}\cos(2x+h)\cdot\lim\limits_{h\to 0}\dfrac{\sin h}{h}$	1M
$=2\cos 2x$	1A
	(4)

[HKDSE 2013 M2 #1]

3.

2. (a) $y=x^3-3x$		
$\dfrac{dy}{dx}=\lim\limits_{h\to 0}\dfrac{[(x+h)^3-3(x+h)]-(x^3-3x)}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{x^3+3x^2h+3xh^2+h^3-3x-3h-x^3+3x}{h}$	1M	OR $\dfrac{h[(x+h)^2+(x+h)x+x^2]-3h}{h}$
$=\lim\limits_{h\to 0}(3x^2+3xh+h^2-3)$		
$=3x^2-3$	1A	
(b) When C is decreasing, $\dfrac{dy}{dx}\le 0$.		
$3x^2-3\le 0$	1M	Accept $3x^2-3<0$
$(x+1)(x-1)\le 0$		
$-1\le x\le 1$	1A	Accept $-1<x<1$
	(5)	

[HKDSE 2014 M2 #2]

4.

1. $\frac{d}{dx}(x^5+4)$

$= \lim_{h\to 0} \frac{((x+h)^5+4)-(x^5+4)}{h}$	1M	
$= \lim_{h\to 0} \frac{x^5+5hx^4+10h^2x^3+10h^3x^2+5h^4x+h^5+4-x^5-4}{h}$	1M	for binomial expansion
$= \lim_{h\to 0} (5x^4+10hx^3+10h^2x^2+5h^3x+h^4)$	1M	withhold 1M if the step is skipped
$= 5x^4$	1A	
	----------(4)	

[HKDSE 2015 M2 #1]

5.

2. $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+h}}$

$= \frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x(x+h)}}$		
$= \left(\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x(x+h)}}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$	1M	
$= \frac{x+h-x}{(x+h)\sqrt{x}+x\sqrt{x+h}}$		
$= \frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}$	1	
$\frac{d}{dx}\sqrt{\frac{3}{x}}$		
$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{3}}{\sqrt{x+h}} - \frac{\sqrt{3}}{\sqrt{x}}\right)$	1M	
$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h\sqrt{3}}{(x+h)\sqrt{x}+x\sqrt{x+h}}\right)$		
$= \lim_{h\to 0} \frac{-\sqrt{3}}{(x+h)\sqrt{x}+x\sqrt{x+h}}$	1M	withhold 1M if the step is omitted
$= \frac{-\sqrt{3}}{2x\sqrt{x}}$	1A	
	----------(5)	

[HKDSE 2016 M2 #2]

6.

1. $\frac{d}{d\theta}\sec 6\theta$

$= \lim_{h\to 0} \frac{\sec 6(\theta+h)-\sec 6\theta}{h}$	1M	
$= \lim_{h\to 0} \frac{\cos 6\theta - \cos 6(\theta+h)}{h\cos 6(\theta+h)\cos 6\theta}$		
$= \lim_{h\to 0} \frac{2\sin(6\theta+3h)\sin 3h}{h\cos 6(\theta+h)\cos 6\theta}$	1M	
$= 6\left(\lim_{h\to 0}\frac{\sin 3h}{3h}\right)\left(\lim_{h\to 0}\frac{\sin(6\theta+3h)}{\cos 6(\theta+h)\cos 6\theta}\right)$	1M	
$= 6(1)\left(\frac{\sin 6\theta}{\cos^2 6\theta}\right)$	1M	withhold 1M if the step is skipped
$= 6\sec 6\theta\tan 6\theta$	1A	
	----------(5)	

[HKDSE 2017 M2 #1]

7.

1. $\mathrm{f}(1+h)$		
$=((1+h)^2-1)e^{1+h}$		
$=(2h+h^2)e^{1+h}$	1A	
$\mathrm{f}'(1)$		
$=\lim\limits_{h\to 0}\dfrac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{(2h+h^2)e^{1+h}-0}{h}$		
$=\lim\limits_{h\to 0}(2+h)e^{1+h}$	1M	withhold 1M if this step is skipped
$=2e$	1A	
	----------(4)	

[HKDSE 2018 M2 #1]

8.

1. $\mathrm{f}(1+h)-\mathrm{f}(1)$		
$=\dfrac{10(1+h)}{7+3(1+h)^2}-\dfrac{10}{7+3}$		
$=\dfrac{10+10h}{10+6h+3h^2}-1$		
$=\dfrac{10+10h-10-6h-3h^2}{10+6h+3h^2}$		
$=\dfrac{4h-3h^2}{10+6h+3h^2}$	1	
$\mathrm{f}'(1)$		
$=\lim\limits_{h\to 0}\dfrac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{4h-3h^2}{10+6h+3h^2}\right)$		
$=\lim\limits_{h\to 0}\dfrac{4-3h}{10+6h+3h^2}$	1M	withhold 1M if the step is skipped
$=\dfrac{4-3(0)}{10+6(0)+3(0)^2}$		
$=\dfrac{2}{5}$	1A	0.4
	----------(4)	

[HKDSE 2019 M2 #1]

Applications of Differentiation

1.

5. $y = \dfrac{x^2+x+1}{x+1}$

$= x + \dfrac{1}{x+1}$ — 1M

$\dfrac{dy}{dx} = 1 - \dfrac{1}{(x+1)^2}$ — 1A — OR $\dfrac{x^2+2x}{(x+1)^2}$

$\dfrac{dy}{dx} = 0$ when $x = -2$ or 0

x	$x<-2$	-2	$-2<x<-1$	-1	$-1<x<0$	0	$x>0$
$\dfrac{dy}{dx}$	>0	0	<0	undefined	<0	0	>0

1M

Alternative Solution

$\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+1)^3}$

When $x=0$, $\dfrac{d^2y}{dx^2} = 2 > 0$, when $x=-2$, $\dfrac{d^2y}{dx^2} = -2 < 0$. — 1M

Hence $(0,1)$ is a minimum point. — 1A

The vertical asymptote is $x=-1$. — 1A

$\because \lim\limits_{x\to\pm\infty} \dfrac{1}{x+1} = 0$ — Accept $\lim\limits_{x\to\infty} \dfrac{1}{x+1} = 0$

$\therefore$ the oblique asymptote is $y = x$ — 1A

(6)

[HKDSE 2012 M2 #5]

2.

6. (a) Let the radius of the water surface be a cm.

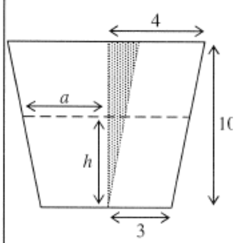
By considering similar triangles, $\dfrac{a-3}{4-3} = \dfrac{h}{10}$. — 1M

i.e. $a = \dfrac{h+30}{10}$

$\therefore \quad V = \dfrac{\pi}{3}h\left[3^2 + 3\left(\dfrac{h+30}{10}\right) + \left(\dfrac{h+30}{10}\right)^2\right]$ — 1M

$= \dfrac{\pi}{300}h[900 + 30(h+30) + (h^2+60h+900)]$

$= \dfrac{\pi}{300}(h^3 + 90h^2 + 2700h)$ — 1



(b) $\dfrac{dV}{dt} = \dfrac{\pi}{300}(3h^2 + 180h + 2700)\dfrac{dh}{dt}$ — 1M+1A

$\therefore \quad 7\pi = \dfrac{\pi}{300}[3(5)^2 + 180(5) + 2700]\dfrac{dh}{dt}$

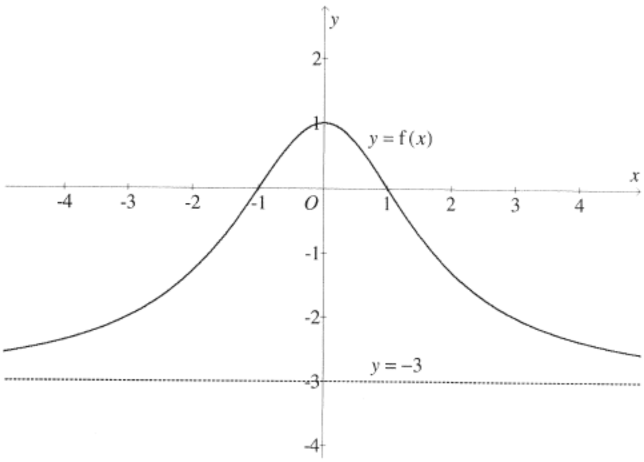
$\dfrac{dh}{dt} = \dfrac{4}{7}$ — 1A

i.e. the rate of increase of depth of water is $\dfrac{4}{7}$ cm s^{-1}.

(6)

[HKDSE 2012 M2 #6]

3.

5. (a) $f(x)=\dfrac{3-3x^2}{3+x^2}$		
$\therefore$ $f(0)=1$, $f(1)=0$ and $f(-1)=0$		
$\therefore$ maximum point is $(0,1)$,	1A	
and points of inflexion are $(1,0)$ and $(-1,0)$.	1A	For both
(b) Since $x^2+3>0$, there is no vertical asymptote.		
$f(x)=-3+\dfrac{12}{x^2+3}$	1M	OR $f(x)=\dfrac{\frac{3}{x^2}-3}{\frac{3}{x^2}+1}$
When $x\to\pm\infty$, $y\to -3$. Hence $y=-3$ is a horizontal asymptote.	1A	
(c) 	1A	For shape of $y=f(x)$
	1A	For all correct
	(6)	

[HKDSE 2013 M2 #5]

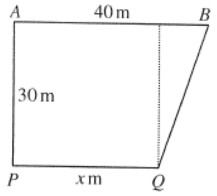
4.

12. (a) (i) $T=\frac{PQ}{7}+\frac{QB}{1.4}$ — 1M

$=\frac{x}{7}+\frac{5\sqrt{30^2+(40-x)^2}}{7}$

$=\frac{x+5\sqrt{x^2-80x+2500}}{7}$ — 1A — OR $\frac{x}{7}+\frac{\sqrt{x^2-80x+2500}}{1.4}$

(ii) When T is minimum, $\frac{dT}{dx}=0$.



$\frac{1}{7}\left[1+\frac{5(2x-80)}{2\sqrt{x^2-80x+2500}}\right]=0$ — 1M

$5(x-40)=-\sqrt{x^2-80x+2500}$

$25x^2-2000x+40000=x^2-80x+2500$

$2x^2-160x+3125=0$ — 1

$\therefore\ x=40-\frac{5\sqrt{6}}{2}$ or $40+\frac{5\sqrt{6}}{2}$ (rejected by checking)

x	$0<x<40-\frac{5\sqrt{6}}{2}$	$x=40-\frac{5\sqrt{6}}{2}$	$x>40-\frac{5\sqrt{6}}{2}$
$\frac{dT}{dx}$	$-$	0	$+$

So, when T is minimum, $x=40-\frac{5\sqrt{6}}{2}$. — 1A

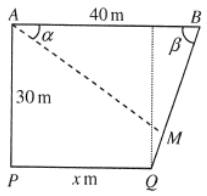
$QB=\sqrt{30^2+\left[40-\left(40-\frac{5\sqrt{6}}{2}\right)\right]^2}$

$=\frac{25\sqrt{6}}{2}$ m — 1

(6)

(b) (i) $\sin\beta=\frac{30}{\frac{25\sqrt{6}}{2}}=\frac{2\sqrt{6}}{5}$ — 1A

$\cos\beta=\frac{40-\left(40-\frac{5\sqrt{6}}{2}\right)}{\frac{25\sqrt{6}}{2}}=\frac{1}{5}$ — 1A



In ΔMAB , $\frac{MB}{\sin\alpha}=\frac{AB}{\sin(\pi-\alpha-\beta)}$. — 1M

$MB=\frac{40\sin\alpha}{\sin(\alpha+\beta)}$

$=\frac{40\sin\alpha}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$ — 1M

$=\frac{40\sin\alpha}{\frac{1}{5}\sin\alpha+\frac{2\sqrt{6}}{5}\cos\alpha}$

$=\frac{200\tan\alpha}{\tan\alpha+2\sqrt{6}}$ — 1

(ii) $\frac{dMB}{dt}=200\cdot\frac{(\tan\alpha+2\sqrt{6})\sec^2\alpha-\tan\alpha\sec^2\alpha}{(\tan\alpha+2\sqrt{6})^2}\cdot\frac{d\alpha}{dt}$ — 1M

$=\frac{400\sqrt{6}\sec^2\alpha}{(\tan\alpha+2\sqrt{6})^2}\cdot\frac{d\alpha}{dt}$

$\therefore\ -1.4=\frac{400\sqrt{6}\sec^2 0.2}{(\tan 0.2+2\sqrt{6})^2}\cdot\frac{d\alpha}{dt}$

$\frac{d\alpha}{dt}\approx-0.0357\ \text{rad s}^{-1}$ — 1A

(7)

[HKDSE 2013 M2 #12]

5.

3. $x\ln y + y = 2$		
$\ln y + x\cdot\frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 0$	1M+1M	1M for product rule 1M for chain rule
$\frac{dy}{dx} = \frac{-y\ln y}{x+y}$		
Alternative Solution		
$x = \frac{2-y}{\ln y}$		
$\frac{dx}{dy} = \frac{\ln y\cdot(-1)-(2-y)\frac{1}{y}}{(\ln y)^2}$	1M	For quotient rule
$\therefore\ \frac{dy}{dx} = \frac{y(\ln y)^2}{y-2-y\ln y}$	1M	For $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$
When the curve cuts the y-axis, $x=0$. $\therefore\ y=2$	1A	

[HKDSE 2014 M2 #3]

6.

4. $x = 2y + \sin y$		
$\frac{dx}{dy} = 2 + \cos y$	1M	
$\frac{dy}{dx} = \frac{1}{2+\cos y}$		
$\frac{d^2y}{dx^2} = -1\cdot(2+\cos y)^{-2}(-\sin y)\frac{dy}{dx}$	1M	OR $\frac{0-1(-\sin y)\frac{dy}{dx}}{(2+\cos y)^2}$
Alternative Solution		
$1 = 2\frac{dy}{dx} + \cos y\cdot\frac{dy}{dx}$	1M	
$0 = 2\frac{d^2y}{dx^2} + \left[\cos y\frac{d^2y}{dx^2} + (-\sin y)\left(\frac{dy}{dx}\right)^2\right]$	1M	
$\sin y\left(\frac{1}{2+\cos y}\right)^2 = (2+\cos y)\frac{d^2y}{dx^2}$		
$\frac{d^2y}{dx^2} = \frac{\sin y}{(2+\cos y)^3}$	1A	
	(3)	

[HKDSE 2014 M2 #4]

7.

10. (a) $HK = HB + BK$

$= \left(\dfrac{24}{\cos\theta} + \dfrac{192}{\sin\theta}\right)$ cm — 1A — OR $(24\sec\theta + 192\csc\theta)$ cm

(1)

(b) $\dfrac{dHK}{d\theta} = -24(\cos\theta)^{-2}(-\sin\theta) - 192(\sin\theta)^{-2}\cos\theta$ — 1A — OR $24\sec\theta\tan\theta - 192\csc\theta\cot\theta$

$\dfrac{dHK}{d\theta} = 0$ when $\dfrac{24\sin\theta}{\cos^2\theta} = \dfrac{192\cos\theta}{\sin^2\theta}$ — 1M

$\tan^3\theta = 8$

$\tan\theta = 2$

$\theta = \tan^{-1}2$

θ	$0 < \theta < \tan^{-1}2$	$\theta = \tan^{-1}2$	$\tan^{-1}2 < \theta < \dfrac{\pi}{2}$
$\dfrac{dHK}{d\theta}$	-ve	0	+ve

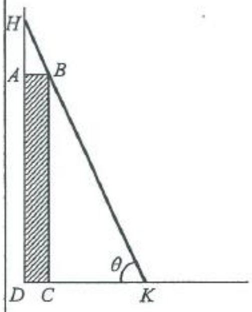
1M

When $\theta = \tan^{-1}2$, HK is minimum. — 1A

By (a), the shortest length of the ladder $= 24\cdot\dfrac{\sqrt{5}}{1} + 192\cdot\dfrac{\sqrt{5}}{2}$

$= 120\sqrt{5}$ cm — 1

(5)



(c) (i) $x + HK\cos\theta = AB + CK$

$x = -270\cos\theta + 24 + 192\cot\theta$ — 1M

$\dfrac{dx}{dt} = 270\sin\theta\cdot\dfrac{d\theta}{dt} - 192\csc^2\theta\cdot\dfrac{d\theta}{dt}$ ---------------------------------- (*) — 1M

When $CK = 160$ cm, $\tan\theta = \dfrac{192}{160} = \dfrac{6}{5}$

$\therefore\ \sin\theta = \dfrac{6}{\sqrt{61}}$ — OR $\theta = 0.87605805$

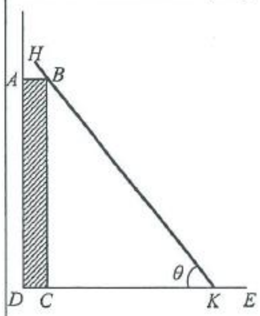
$\dfrac{dx}{dt} = 270\left(\dfrac{6}{\sqrt{61}}\right)(-0.1) - 192\left(\dfrac{\sqrt{61}}{6}\right)^2(-0.1)$ — OR $270\sin 0.87605805\cdot(-0.1) - 192\csc^2 0.87605805\cdot(-0.1)$

≈ 11.79 — 1A

i.e. the rate of change of x is 11.79 cm s^{-1}.

(ii) $y - x = 270\cos\theta$ — 1M

$\dfrac{dy}{dt} - \dfrac{dx}{dt} = -270\sin\theta\cdot\dfrac{d\theta}{dt}$



Alternative Solution

$y = 24 + 192\cot\theta$ — 1M

$\dfrac{dy}{dt} = -192\csc^2\theta\cdot\dfrac{d\theta}{dt}$

By (*), $\dfrac{dx}{dt} - \dfrac{dy}{dt} = 270\sin\theta\cdot\dfrac{d\theta}{dt}$

$\because\ \sin\theta > 0$ and $\dfrac{d\theta}{dt} < 0$

$\therefore\ \dfrac{dy}{dt} > \dfrac{dx}{dt}$ — 1A

Hence K is moving towards E at a speed faster than the horizontal speed H is leaving the wall and Thomas is agreed with. — 1

(6)

[HKDSE 2014 M2 #10]

8.

2. (a) $\dfrac{dy}{dx}$		
$= x\cos x + \sin x - \sin x$	1M	for product rule
$= x\cos x$	1A	
$\dfrac{d^2y}{dx^2}$		
$= -x\sin x + \cos x$	1A	
(b) $x\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + xy$		
$= x(-x\sin x + \cos x) + kx\cos x + x(x\sin x + \cos x)$ (by (a))	1M	for using the results of (a)
$= (2+k)x\cos x$		
Since $x\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + xy = 0$ for all real values of x, we have $k = -2$.	1A	
	----------(5)	

[HKDSE 2015 M2 #2]

9.

9. (a) $f'(x)$		
$=\dfrac{(x-2)(2x)-(x^2+12)}{(x-2)^2}$	1M	for quotient rule
$=\dfrac{x^2-4x-12}{(x-2)^2}$	1A	
	----------(2)	
(b) Note that $f'(x)=\dfrac{(x+2)(x-6)}{(x-2)^2}$.		
So, we have $f'(x)=0 \Leftrightarrow x=-2$ or $x=6$.	1A	
(table below)	1M	
Thus, the maximum value and the minimum value of $f(x)$ are -4	1	
and 12 respectively.	1	
Note that $f'(x)=\dfrac{(x+2)(x-6)}{(x-2)^2}$ and $f''(x)=\dfrac{32}{(x-2)^3}$.		
So, we have $f'(x)=0 \Leftrightarrow x=-2$ or $x=6$.	1A	
Further note that $f''(-2)=\dfrac{-1}{2}<0$ and $f''(6)=\dfrac{1}{2}>0$.	1M	
Thus, the maximum value and the minimum value of $f(x)$ are -4	1	
and 12 respectively.	1	
	----------(4)	
(c) The equation of the vertical asymptote is $x-2=0$.	1A	
Note that $f(x)=x+2+\dfrac{16}{x-2}$.	1M	
Thus, the equation of the oblique asymptote is $y=x+2$.	1A	
	----------(3)	
(d) $\dfrac{x^2+12}{x-2}=14$		
$x^2-14x+40=0$		
$x=4$ or $x=10$	1A	can be absorbed
The required area		
$=\displaystyle\int_4^{10}\left(14-\frac{x^2+12}{x-2}\right)dx$	1M	
$=\displaystyle\int_4^{10}\left(12-x-\frac{16}{x-2}\right)dx$		
$=\left[12x-\dfrac{x^2}{2}-16\ln(x-2)\right]_4^{10}$	1M	
$=30-32\ln 2$	1A	
	----------(4)	

Table for (b):

x	$(-\infty,-2)$	-2	$(-2,2)$	$(2,6)$	6	$(6,\infty)$
$f'(x)$	+	0	−	−	0	+
$f(x)$	↗	−4	↘	↘	12	↗

[HKDSE 2015 M2 #9]

10.

4. (a) Note that the equation of the vertical asymptote is $x=1$.	1A	
$\mathrm{f}(x)$ $=\dfrac{2x^2+x+1}{x-1}$		
$=2x+3+\dfrac{4}{x-1}$	1M	
Thus, the equation of the oblique asymptote is $y=2x+3$.	1A	
(b) $\mathrm{f}'(x)$ $=\dfrac{(x-1)(4x+1)-(2x^2+x+1)}{(x-1)^2}$	1M	
$=\dfrac{2(x^2-2x-1)}{(x-1)^2}$		
$\mathrm{f}'(2)$ $=\dfrac{2(2^2-2(2)-1)}{(2-1)^2}$	1M	
$=-2$		
The slope of the normal to G at the point $(2,11)$ $=\dfrac{-1}{\mathrm{f}'(2)}$	1M	
$=\dfrac{1}{2}$	1A	
$\mathrm{f}'(x)$ $=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2x+3+\dfrac{4}{x-1}\right)$ $=2-\dfrac{4}{(x-1)^2}$	1M	
$\mathrm{f}'(2)$ $=2-\dfrac{4}{(2-1)^2}$	1M	
$=-2$		
The slope of the normal to G at the point $(2,11)$ $=\dfrac{-1}{\mathrm{f}'(2)}$	1M	
$=\dfrac{1}{2}$	1A	
	----------(7)	

[HKDSE 2016 M2 #4]

11.

9. (a) Since $f(x) = x^3 + ax^2 + bx + 5$, we have $f'(x) = 3x^2 + 2ax + b$. — 1A

Note that $P(-1, 10)$ is a turning point of C .

So, we have $-1 + a - b + 5 = 10$ and $3 - 2a + b = 0$. — 1M for either

Therefore, we have $a - b - 6 = 0$ and $-2a + b + 3 = 0$.

Solving, we have $a = -3$ and $b = -9$. — 1A for both

----------(3)

(b) Note that $f''(x) = 6x - 6$.

$f''(-1)$

$= -12$

< 0 — 1M

Thus, P is a maximum point of C . — 1A f.t.

Note that $f'(x) = 3(x+1)(x-3)$.

x	$(-\infty, -1)$	-1	$(-1, 3)$
$f'(x)$	+	0	–
$f(x)$	↗	10	↘

— 1M

Thus, P is a maximum point of C . — 1A f.t.

----------(2)

(c) Note that $f'(x) = 3(x+1)(x-3)$.

So, we have $f'(3) = 0$. — 1M

Also note that $f''(x) = 6x - 6$.

$f''(3)$

$= 12$

> 0

Further note that $f(3) = -22$.

Thus, the minimum value of $f(x)$ is -22 . — 1A

Note that $f'(x) = 3(x+1)(x-3)$.

So, we have $f'(3) = 0$. — 1M

x	$(-1, 3)$	3	$(3, \infty)$
$f'(x)$	–	0	+
$f(x)$	↘	-22	↗

Thus, the minimum value of $f(x)$ is -22 . — 1A

----------(2)

(d) Note that $f''(x) = 6(x-1)$.

Therefore, we have $f''(x) = 0$ when $x = 1$.

x	$(-\infty, 1)$	1	$(1, \infty)$
$f''(x)$	–	0	+

— 1M

Thus, the point of inflexion of C is $(1, -6)$. — 1A

----------(2)

(e) Note that the equation of L is $y = 10$.

$x^3 - 3x^2 - 9x + 5 = 10$ — 1M

$x^3 - 3x^2 - 9x - 5 = 0$

$(x+1)^2(x-5) = 0$

$x = -1$ or $x = 5$

The required area

$= \int_{-1}^{5} (10 - (x^3 - 3x^2 - 9x + 5))\,dx$ — 1M

$= \int_{-1}^{5} (-x^3 + 3x^2 + 9x + 5)\,dx$

$= \left[\frac{-x^4}{4} + x^3 + \frac{9x^2}{2} + 5x\right]_{-1}^{5}$ — 1M

$= 108$ — 1A

----------(4)

[HKDSE 2016 M2 #9]

12.

6. (a) Let r cm be the radius of the water surface in the container.	
Since $\frac{r}{h}=\frac{15}{20}$, we have $\frac{r}{h}=\frac{3}{4}$.	1M
So, we have $r=\frac{3h}{4}$.	
A	
$=\pi\left(\frac{3h}{4}\right)\sqrt{h^2+\left(\frac{3h}{4}\right)^2}$	1M
$=\pi\left(\frac{3h}{4}\right)\sqrt{\frac{25h^2}{16}}$	
$=\frac{15}{16}\pi h^2$	1
(b) Let d cm be the depth of water when the volume of water in the container is $96\pi\ \text{cm}^3$.	
Note that $\frac{\pi d}{3}\left(\frac{3d}{4}\right)^2=96\pi$.	1M
So, we have $d=8$.	1A
By (a), we have $A=\frac{15}{16}\pi h^2$.	
At time t s , we have $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{15}{8}\pi h\frac{\mathrm{d}h}{\mathrm{d}t}$.	1M
Also note that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{3}{\pi}$.	
Therefore, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=\frac{15}{8}\pi(8)\left(\frac{3}{\pi}\right)$.	
Hence, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=45$.	1A
Thus, the required rate of change is $45\ \text{cm}^2/\text{s}$.	
	----------(7)

[HKDSE 2017 M2 #6]

13.

9. (a) The equation of the vertical asymptote is $x+4=0$.	1A
Note that $f(x)=x-9+\dfrac{36}{x+4}$.	1M
Thus, the equation of the oblique asymptote is $y=x-9$.	1A
	----------(3)
(b) $f'(x)$	
$=\dfrac{d}{dx}\left(x-9+\dfrac{36}{x+4}\right)$	
$=1+36(-1)(x+4)^{-2}$	1M
$=1-\dfrac{36}{(x+4)^2}$	1A
$f'(x)$	
$=\dfrac{d}{dx}\left(\dfrac{x^2-5x}{x+4}\right)$	
$=\dfrac{(x+4)(2x-5)-(x^2-5x)}{(x+4)^2}$	1M
$=\dfrac{x^2+8x-20}{(x+4)^2}$	1A
	----------(2)
(c) Note that $f'(x)=\dfrac{(x+10)(x-2)}{(x+4)^2}$.	
So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$.	1A

x	$(-\infty,-10)$	-10	$(-10,-4)$	$(-4,2)$	2	$(2,\infty)$
$f'(x)$	+	0	–	–	0	+
$f(x)$	↗	-25	↘	↘	-1	↗

(1M)

Thus, the maximum point and the minimum point of G are $(-10,-25)$	1A
and $(2,-1)$ respectively.	1A
Note that $f'(x)=\dfrac{(x+10)(x-2)}{(x+4)^2}$ and $f''(x)=\dfrac{72}{(x+4)^3}$.	
So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$.	1A
Also note that $f''(-10)=\dfrac{-1}{3}<0$ and $f''(2)=\dfrac{1}{3}>0$.	1M
Further note that $f(-10)=-25$ and $f(2)=-1$.	
Thus, the maximum point and the minimum point of G are $(-10,-25)$	1A
and $(2,-1)$ respectively.	1A
	----------(4)

(d) The required volume		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x + 4} \right)^2 \mathrm{d}x$	1M	
$= \pi \int_0^5 \left(x - 9 + \frac{36}{x+4} \right)^2 \mathrm{d}x$		
$= \pi \int_0^5 \left(x^2 - 18x + 81 + \frac{72(x-9)}{x+4} + \frac{1\,296}{(x+4)^2} \right) \mathrm{d}x$		
$= \pi \int_0^5 \left(x^2 - 18x + 153 - \frac{936}{x+4} + \frac{1\,296}{(x+4)^2} \right) \mathrm{d}x$	1M	
$= \pi \left[\frac{x^3}{3} - 9x^2 + 153x - 936 \ln\lvert x+4 \rvert - \frac{1\,296}{x+4} \right]_0^5$	1M	
$= \left(\frac{2\,285}{3} - 1\,872 \ln\left(\frac{3}{2} \right) \right) \pi$	1A	
The required volume		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x + 4} \right)^2 \mathrm{d}x$	1M	
$= \pi \int_4^9 \frac{(x-4)^2 (x-9)^2}{x^2} \mathrm{d}x$		
$= \pi \int_4^9 \left(\frac{x^4 - 26x^3 + 241x^2 - 936x + 1\,296}{x^2} \right) \mathrm{d}x$	1M	
$= \pi \int_4^9 \left(x^2 - 26x + 241 - \frac{936}{x} + \frac{1\,296}{x^2} \right) \mathrm{d}x$		
$= \pi \left[\frac{x^3}{3} - 13x^2 + 241x - 936 \ln\lvert x \rvert - \frac{1\,296}{x} \right]_4^9$	1M	
$= \left(\frac{2\,285}{3} - 1\,872 \ln\left(\frac{3}{2} \right) \right) \pi$	1A	
	----------(4)	

[HKDSE 2017 M2 #9]

14.

8. (a) Note that $A \neq 0$.		
$\mathrm{f}'(x)$		
$= \dfrac{-A(2x-4)}{(x^2-4x+7)^2}$	1M	
So, we have $\mathrm{f}'(x)=0 \Leftrightarrow x=2$.	1M	
Since the equation $\mathrm{f}'(x)=0$ has only one solution $x=2$ and the extreme value of $\mathrm{f}(x)$ is 4 , we have $\mathrm{f}(2)=4$.		
Hence, we have $\dfrac{A}{2^2-4(2)+7}=4$.		
Therefore, we have $A=12$.		
Thus, we have $\mathrm{f}'(x)=\dfrac{24(2-x)}{(x^2-4x+7)^2}$.	1A	
(b) Note that $x^2-4x+7=(x-2)^2+3>0$ for all real values of x .		
So, there are no vertical asymptotes of the graph of $y=\mathrm{f}(x)$.		
Also note that $\mathrm{f}(x)=\dfrac{12}{x^2-4x+7}$.		
Therefore, $y=0$ is the only asymptote of the graph of $y=\mathrm{f}(x)$.	1M	
Hence, there is only one asymptote of the graph of $y=\mathrm{f}(x)$.		
Thus, the claim is disagreed.	1A	f.t.
(c) $\mathrm{f}''(x)$		
$= \dfrac{(x^2-4x+7)^2(-24)-(-24x+48)(2)(x^2-4x+7)(2x-4)}{(x^2-4x+7)^4}$	1M	
$= \dfrac{72(x-3)(x-1)}{(x^2-4x+7)^3}$		
So, we have $\mathrm{f}''(x)=0 \Leftrightarrow x=1$ or $x=3$.		

x	$(-\infty,1)$	1	$(1,3)$	3	$(3,\infty)$
$\mathrm{f}''(x)$	+	0	–	0	+

1M for testing

Thus, the points of inflexion are $(1,3)$ and $(3,3)$. 1A for both correct

----------(8)

[HKDSE 2018 M2 #8]

15.

9. (a) $y=\dfrac{\ln x}{2}$		
$\dfrac{dy}{dx}=\dfrac{1}{2x}$	1M	
The slope of the tangent at P is $\dfrac{1}{2r}$.		
The slope of the normal at P is $-2r$.		
Let a be the x-coordinate of Q .		
$\dfrac{0-\ln\sqrt{r}}{a-r}=-2r$	1M	
$\dfrac{-1}{2}\ln r=2r^2-2ar$		
$2ar=2r^2+\dfrac{1}{2}\ln r$		
$a=\dfrac{4r^2+\ln r}{4r}$	1	
Thus, the x-coordinate of Q is $\dfrac{4r^2+\ln r}{4r}$.		
	----------(3)	
(b) Let A square units be the area of ΔPQR .		
A		
$=\dfrac{1}{2}\left(\dfrac{4r^2+\ln r}{4r}-r\right)\ln\sqrt{r}$	1M	
$=\dfrac{(\ln r)^2}{16r}$	1A	
$\dfrac{dA}{dr}$		
$=\dfrac{r(2\ln r)\frac{1}{r}-(\ln r)^2}{16r^2}$	1M	
$=\dfrac{2\ln r-(\ln r)^2}{16r^2}$		
$=\dfrac{(2-\ln r)\ln r}{16r^2}$		
So, we have $\dfrac{dA}{dr}=0 \Leftrightarrow \ln r=2$ or $\ln r=0$ (rejected) .		
Hence, we have $\dfrac{dA}{dr}=0 \Leftrightarrow r=e^2$.		
(table below)	1M	for testing
Therefore, A attains its greatest value when $r=e^2$.		
Thus, the greatest area of ΔPQR is $\dfrac{1}{4e^2}$ square units.	1A	
	----------(5)	

r	$(1, e^2)$	e^2	(e^2, ∞)
$\dfrac{dA}{dr}$	+	0	–
A	↗	$\dfrac{1}{4e^2}$	↘

(c) OP

$= \sqrt{r^2 + (\ln\sqrt{r})^2}$ 1M

$= \dfrac{1}{2}\sqrt{4r^2 + (\ln r)^2}$

$\dfrac{\mathrm{d}OP}{\mathrm{d}t}$

$= \left(\dfrac{4r^2 + \ln r}{2r\sqrt{4r^2 + (\ln r)^2}}\right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t}\right)$ 1M

$\dfrac{\mathrm{d}A}{\mathrm{d}t}$

$= \left(\dfrac{\mathrm{d}A}{\mathrm{d}r}\right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t}\right)$

$= \left(\dfrac{(2-\ln r)\ln r}{16r^2}\right)\left(\dfrac{\mathrm{d}r}{\mathrm{d}t}\right)$ (by (b))

$= \left(\dfrac{(2-\ln r)\ln r}{16r^2}\right)\left(\dfrac{2r\sqrt{4r^2 + (\ln r)^2}}{4r^2 + \ln r}\right)\left(\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right)$

$= \dfrac{(2-\ln r)(\ln r)\sqrt{4r^2 + (\ln r)^2}}{8r(4r^2 + \ln r)}\left(\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right)$

$\left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e}$

$= \dfrac{(2-\ln e)(\ln e)\sqrt{4e^2 + (\ln e)^2}}{8e(4e^2 + \ln e)}\left(\left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right|_{r=e}\right)$ 1M

$= \dfrac{\sqrt{4e^2+1}}{8e(4e^2+1)}\left(\left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right|_{r=e}\right)$

Since $0 \le \left.\dfrac{\mathrm{d}OP}{\mathrm{d}t}\right|_{r=e} \le 32e^2$, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e} \le \dfrac{32e^2\sqrt{4e^2+1}}{8e(4e^2+1)}$.

So, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e} \le \dfrac{4e}{\sqrt{4e^2+1}}$.

Therefore, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e} < \dfrac{4e}{\sqrt{4e^2}}$.

Hence, we have $0 \le \left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e} < 2$.

Thus, the claim is correct. 1A f.t.

----------(4)

[HKDSE 2018 M2 #9]

16.

3. (a) V		
$= \int -2t \, \mathrm{d}t$	1M	
$= -t^2 + C$, where C is a constant		
Since $V = 580$ when $t = 0$, we have $C = 580$.	1M	
So, we have $V = 580 - t^2$.		
When $t = 24$, we have $V = 4 > 0$.		
Thus, the claim is correct.	1A	f.t.
(b) Let p cm be the depth of liquid X in the vessel when $t = 18$.		
Since $V\big\|_{t=18} = 580 - 18^2 = 256$, we have $p^2 + 24p = 256$.	1M	
So, we have $p^2 + 24p - 256 = 0$.		
Solving, we have $p = 8$ or $p = -32$ (rejected) .		
Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = (2h + 24)\frac{\mathrm{d}h}{\mathrm{d}t}$.	1M	
Since $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right\|_{t=18} = -36$, we have $-36 = (2(8) + 24)\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\|_{t=18}$.		
Thus, we have $\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\|_{t=18} = \frac{-9}{10}$.	1A	-0.9
	----------(6)	

[HKDSE 2019 M2 #3]

Technique of Integration

1.

4. (a) $\int \frac{x+1}{x}\,dx = \int\left(1+\frac{1}{x}\right)dx$	1M	
$= x+\ln\lvert x\rvert + C$	1A	
(b) Let $u = x^2-1$. $du = 2x\,dx$ $\int \frac{x^3}{x^2-1}\,dx = \int \frac{u+1}{u}\cdot\frac{du}{2}$	1A	
$= \frac{1}{2}u+\frac{1}{2}\ln\lvert u\rvert + C$ by (a)	1M	
$= \frac{1}{2}(x^2-1)+\frac{1}{2}\ln\lvert x^2-1\rvert + C$	1A	OR $\frac{1}{2}x^2+\frac{1}{2}\ln\lvert x^2-1\rvert + C$
	(5)	

[HKDSE 2012 M2 #4]

2.

5. (a) $\int \frac{dx}{\sqrt{9-x}} = \int -(9-x)^{\frac{-1}{2}}\,d(9-x)$	1M+1A
Alternative Solution Let $u = 9-x$. $du = -dx$	1M
$\int \frac{dx}{\sqrt{9-x}} = \int -u^{\frac{-1}{2}}\,du$	1A
$= -2u^{\frac{1}{2}} + C$	
$= -2\sqrt{9-x}+C$	1A
(b) Let $x = 3\sin\theta$. $dx = 3\cos\theta\,d\theta$ $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$	1M
$= \int d\theta$	1A
$= \theta + C$ $= \sin^{-1}\frac{x}{3}+C$	1A
	(6)

[HKDSE 2014 M2 #5]

3.

3. (a) $\displaystyle\int \frac{1}{e^{2u}}\,\mathrm{d}u$		
$\displaystyle = \int e^{-2u}\,\mathrm{d}u$	1M	
$\displaystyle = \frac{-1}{2}e^{-2u} + \text{constant}$	1A	
(b) Let $u=\sqrt{x}$. Then, we have $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}$.	1M	
$\displaystyle\int_1^9 \frac{1}{\sqrt{x}\,e^{2\sqrt{x}}}\,\mathrm{d}x$		
$\displaystyle = \int_1^3 \frac{2}{e^{2u}}\,\mathrm{d}u$	1M+1A	
$\displaystyle = 2\int_1^3 \frac{1}{e^{2u}}\,\mathrm{d}u$		
$\displaystyle = 2\left[\frac{-1}{2}e^{-2u}\right]_1^3$ (by (a))	1M	for using the result of (a)
$\displaystyle = \frac{1}{e^2}-\frac{1}{e^6}$	1A	
	----------(7)	

[HKDSE 2015 M2 #3]

4.

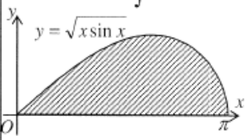
7. (a) $\int e^x \sin \pi x \, dx$		
$= \int \sin \pi x \, de^x$		
$= e^x \sin \pi x - \pi \int e^x \cos \pi x \, dx$	1M	
$= e^x \sin \pi x - \pi \int \cos \pi x \, de^x$		
$= e^x \sin \pi x - \pi \left(e^x \cos \pi x - \pi \int -e^x \sin \pi x \, dx \right)$	1M	
$\int e^x \sin \pi x \, dx = e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x \, dx$		
$\pi^2 \int e^x \sin \pi x \, dx + \int e^x \sin \pi x \, dx = e^x \sin \pi x - \pi e^x \cos \pi x + \text{constant}$	1M	
$(\pi^2+1) \int e^x \sin \pi x \, dx = e^x (\sin \pi x - \pi \cos \pi x) + \text{constant}$		
$\int e^x \sin \pi x \, dx = \frac{1}{\pi^2+1} (e^x (\sin \pi x - \pi \cos \pi x)) + \text{constant}$	1A	
(b) $\int_0^3 e^{3-x} \sin \pi x \, dx$		
$= \int_3^0 -e^u \sin \pi (3-u) \, du$ (by letting $u = 3 - x$)	1M	
$= \int_0^3 e^u \sin \pi u \, du$		
$= \int_0^3 e^x \sin \pi x \, dx$		
$= \frac{1}{\pi^2+1} \left[e^x (\sin \pi x - \pi \cos \pi x) \right]_0^3$ (by (a))	1M	for using the result of (a)
$= \frac{\pi(e^3+1)}{\pi^2+1}$	1A	
	----------(7)	

[HKDSE 2019 M2 #7]

Applications of Integration

1.

9. (a) $\int x\sin x\,dx = x(-\cos x) - \int(-\cos x)dx$	1M	
$= -x\cos x + \sin x + C$	1A	
(b) The volume $= \pi\int_0^{\pi} x\sin x\,dx$	1M	1M for $V = \pi\int y^2 dx$
$= \pi[-x\cos x + \sin x]_0^{\pi}$		
$= \pi^2$	1A	
	(4)	



[HKDSE 2012 M2 #9]

2.

13. (a) (i) $\tan u = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$		
$= \dfrac{-1+1-2\sin^2\frac{\pi}{5}}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$	1M	
$= -\tan\frac{\pi}{5}$		
$= \tan\frac{-\pi}{5}$		
$\therefore\ u = \frac{-\pi}{5}$ for $\frac{-\pi}{2} < u < \frac{\pi}{2}$	1	
(ii) $\tan v = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$		
$= \dfrac{1+2\cos^2\frac{\pi}{5}-1}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$	1M	
$= \cot\frac{\pi}{5}$		
$= \tan\left(\frac{\pi}{2}-\frac{\pi}{5}\right)$		
$\therefore\ v = \frac{3\pi}{10}$ for $\frac{-\pi}{2} < v < \frac{\pi}{2}$	1A	
	(4)	
(b) (i) $x^2+2x\cos\frac{2\pi}{5}+1$		
$x^2+2x\cos\frac{2\pi}{5}+\cos^2\frac{2\pi}{5}+\sin^2\frac{2\pi}{5}$		
$=(x+\cos\frac{2\pi}{5})^2+\sin^2\frac{2\pi}{5}$	1A	
(ii) $\displaystyle\int_{-1}^{1}\frac{\sin\frac{2\pi}{5}}{x^2+2x\cos\frac{2\pi}{5}+1}\,dx = \int_{-1}^{1}\frac{\sin\frac{2\pi}{5}}{(x+\cos\frac{2\pi}{5})^2+\sin^2\frac{2\pi}{5}}\,dx$		
Let $x+\cos\frac{2\pi}{5}=\sin\frac{2\pi}{5}\tan\theta$	1M	
$\therefore\ dx=\sin\frac{2\pi}{5}\sec^2\theta\, d\theta$	1A	
When $x=-1$, $\tan\theta = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$ which gives $\theta=\frac{-\pi}{5}$ (by (a)(i))	1M	For using (a)
When $x=1$, $\tan\theta = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$ which gives $\theta=\frac{3\pi}{10}$ (by (a)(ii))		

$$\therefore \int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2+2x\cos\frac{2\pi}{5}+1}\,\mathrm{d}x = \int_{\frac{-\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin^2\frac{2\pi}{5}\sec^2\theta}{\sin^2\frac{2\pi}{5}(\tan^2\theta+1)}\,\mathrm{d}\theta$$ 1A For integrand

$$= [\theta]_{\frac{-\pi}{5}}^{\frac{3\pi}{10}}$$

$$= \frac{\pi}{2}$$ 1A

(6)

(c) $$\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,\mathrm{d}x = \int_{-1}^{1} \frac{-\sin\frac{2\pi}{5}}{x^2-2x\cos\frac{2\pi}{5}+1}\,\mathrm{d}x$$ 1A

Let $y=-x$. 1M

$\mathrm{d}y=-\mathrm{d}x$

When $x=-1$, $y=1$; when $x=1$, $y=-1$.

$$\therefore \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,\mathrm{d}x = \int_{1}^{-1} \frac{-\sin\frac{2\pi}{5}}{y^2+2y\cos\frac{2\pi}{5}+1}\cdot -\mathrm{d}y$$

Alternative Solution

$$\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,\mathrm{d}x = \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{(x+\cos\frac{7\pi}{5})^2+\sin^2\frac{7\pi}{5}}\,\mathrm{d}x$$

Let $x+\cos\frac{7\pi}{5}=\sin\frac{7\pi}{5}\tan\theta$ 1M

$\therefore\ \mathrm{d}x=\sin\frac{7\pi}{5}\sec^2\theta\,\mathrm{d}\theta$

When $x=-1$, $\theta=\frac{3\pi}{10}$; when $x=1$, $\theta=\frac{-\pi}{5}$. 1A

$$\therefore \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,\mathrm{d}x = \int_{\frac{3\pi}{10}}^{\frac{-\pi}{5}} \frac{\sin^2\frac{7\pi}{5}\sec^2\theta}{\sin^2\frac{7\pi}{5}(\tan^2\theta+1)}\,\mathrm{d}\theta$$

$$= \frac{-\pi}{2} \quad \text{by (b)(ii)}$$ 1A

(3)

[HKDSE 2012 M2 #13]

3.

14. (a) $y = kx^p$

$\frac{dy}{dx} = kpx^{p-1}$ — 1A

The slope of the tangent to Γ at A is kpa^{p-1}.

$\therefore \quad \frac{ka^p - 0}{a-(-a)} = kpa^{p-1}$ — 1M

$p = \frac{1}{2}$ — 1

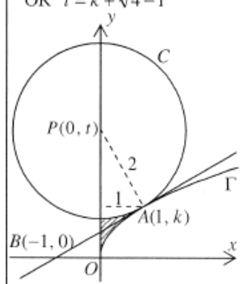
(3)

(b) (i) Let $P(0, t)$ be the centre of C.

$\because \quad AP = 2$

$\therefore \quad (k-t)^2 + (1-0)^2 = 2^2$ — 1M — OR $t = k + \sqrt{4-1}$

$k - t = -\sqrt{3}$ $\boxed{\text{or } \sqrt{3} \text{ (rejected)}}$ ---------------------------- (1)



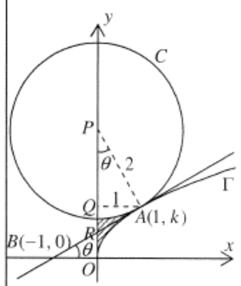
Slope of $AP = \frac{k-t}{1-0}$

Slope of $AB = \frac{k}{2}$ (by (a)) — 1M

$\therefore \quad (k-t)\frac{k}{2} = -1$ -- (2) — 1A

Substitute (1) into (2): $(-\sqrt{3})\frac{k}{2} = -1$

> Alternative Solution
>
> According to the figure, $\angle ROB = \angle RAP = \frac{\pi}{2}$.
>
> $\angle ORB = \angle ARP$ (vert. opp. $\angle$s)
>
> $\therefore \quad \angle RBO = \angle RPA$ and let the angles be θ. — 1M
>
> Since $PA = 2$ and $QA = 1$, $\theta = \frac{\pi}{6}$ — 1A
>
> Slope of $AB = \frac{k}{2}$ (by (a)) — 1M
>
> $\therefore \quad \tan\frac{\pi}{6} = \frac{k}{2}$



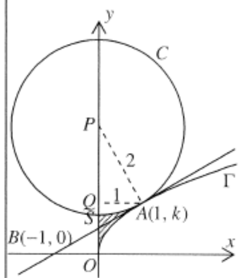
$k = \frac{2\sqrt{3}}{3}$ — 1

(ii) The shaded area

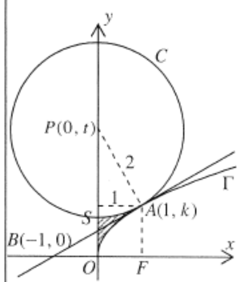
$= $ area of ΔPQA + area on the left of Γ from O to A − area of sector PAS — 1M

$= \frac{1}{2}\cdot 1 \cdot \sqrt{4-1} + \int_0^{\frac{2\sqrt{3}}{3}} \left(\frac{y}{\frac{2\sqrt{3}}{3}}\right)^2 dy - \frac{1}{2}(2)^2\frac{\pi}{6}$ — 1M+1A

$= \frac{\sqrt{3}}{2} + \frac{3}{4}\left[\frac{y^3}{3}\right]_0^{\frac{2\sqrt{3}}{3}} - \frac{\pi}{3}$ — 1M



> Alternative Solution 1
>
> $t = k + \sqrt{3}$
>
> $= \frac{5\sqrt{3}}{3}$
>
> The shaded area
>
> $= $ area of trapezium $OFAP$ − area of sector PAS − area under Γ from O to A — 1M
>
> $= \frac{1}{2}\left(\frac{2\sqrt{3}}{3} + \frac{5\sqrt{3}}{3}\right)(1) - \frac{1}{2}(2)^2\frac{\pi}{6} - \int_0^1 \frac{2\sqrt{3}}{3}x^{\frac{1}{2}}dx$ — 1M+1A
>
> $= \frac{7\sqrt{3}}{6} - \frac{\pi}{3} - \frac{4\sqrt{3}}{9}\left[x^{\frac{3}{2}}\right]_0^1$ — 1M



Alternative Solution 2

$t = k + \sqrt{3}$

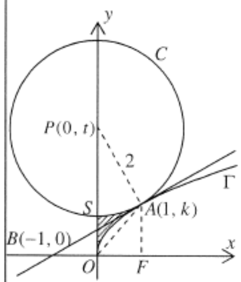
$= \dfrac{5\sqrt{3}}{3}$

The shaded area

= area of ΔOAP + area of ΔOAF − area of sector PAS − area under Γ from O to A — 1M

$= \dfrac{1}{2}\left(\dfrac{5\sqrt{3}}{3}\right)(1) + \dfrac{1}{2}(1)\left(\dfrac{2\sqrt{3}}{3}\right) - \dfrac{1}{2}(2)^2\,\dfrac{\pi}{6} - \int_0^1 \dfrac{2\sqrt{3}}{3}x^{\frac{1}{2}}\,\mathrm{d}x$ — 1M+1A

$= \dfrac{5\sqrt{3}}{6} + \dfrac{\sqrt{3}}{3} - \dfrac{\pi}{3} - \dfrac{4\sqrt{3}}{9}\left[x^{\frac{3}{2}}\right]_0^1$ — 1M



Alternative Solution 3

$t = k + \sqrt{3}$

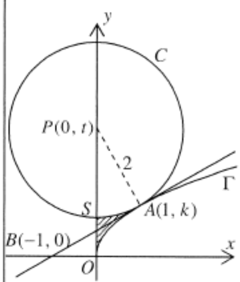
$= \dfrac{5\sqrt{3}}{3}$

The equation of C is $x^2 + \left(y - \dfrac{5\sqrt{3}}{3}\right)^2 = 4$. — 1A

Hence, the equation of $\widehat{AS}$ is $y = \dfrac{5\sqrt{3}}{3} - \sqrt{4 - x^2}$.

The shaded area $= \int_0^1 \left(\dfrac{5\sqrt{3}}{3} - \sqrt{4 - x^2} - \dfrac{2\sqrt{3}}{3}x^{\frac{1}{2}}\right)\mathrm{d}x$ — 1M

For $A = \int y\,\mathrm{d}x$



For $\int_0^1 \sqrt{4 - x^2}\,\mathrm{d}x$, let $x = 2\sin\phi$.

$\therefore\ \mathrm{d}x = 2\cos\phi\,\mathrm{d}\phi$

When $x = 1$, $\phi = \dfrac{\pi}{6}$; when $x = 0$, $\phi = 0$.

$\int_0^1 \sqrt{4 - x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{4 - 4\sin^2\phi}\;2\cos\phi\,\mathrm{d}\phi$ — 1M

$= \int_0^{\frac{\pi}{6}} 2(1 + \cos 2\phi)\,\mathrm{d}\phi$

$= [2\phi + \sin 2\phi]_0^{\frac{\pi}{6}}$

$= \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

Hence, the shaded area

$= \left[\dfrac{5\sqrt{3}}{3}x - \dfrac{4\sqrt{3}}{9}x^{\frac{3}{2}}\right]_0^1 - \left(\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}\right)$ — 1M

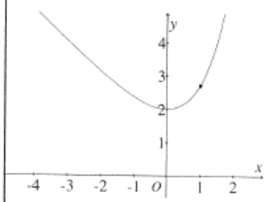
$= \dfrac{13\sqrt{3}}{18} - \dfrac{\pi}{3}$ — 1A

(9)

[HKDSE 2012 M2 #14]

4.

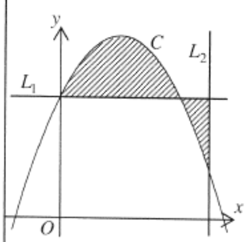
4. (a) $\frac{dy}{dx} = e^x - 1$		
$y = \int (e^x - 1)dx$		
$= e^x - x + C$	1A	
Since the curve passes through the point $(1, e)$, $e = e^1 - 1 + C$.	1M	
i.e. $C = 1$		
$\therefore \quad y = e^x - x + 1$	1A	
(b) The curve cuts the y-axis at $(0, 2)$.		
When $x = 0$, $\frac{dy}{dx} = 0$.	1M	
Hence the equation of tangent to the curve at $(0, 2)$ is		
$y - 2 = 0(x - 0)$		
$y = 2$	1A	
	(5)	



[HKDSE 2013 M2 #4]

5.

6. (a) Area $= \int_0^4 \left[\left(\frac{-x^2}{2} + 2x + 4\right) - 4\right] dx + \int_4^5 \left[4 - \left(\frac{-x^2}{2} + 2x + 4\right)\right] dx$	1M	
$= \int_0^4 \left(\frac{-x^2}{2} + 2x\right) dx + \int_4^5 \left(\frac{x^2}{2} - 2x\right) dx$		
$= \left[\frac{-x^3}{6} + x^2\right]_0^4 + \left[\frac{x^3}{6} - x^2\right]_4^5$	1M	
$= \frac{13}{2}$	1A	
(b) Volume $= \pi \int_0^5 \left(\frac{-x^2}{2} + 2x + 4 - 4\right)^2 dx$	1M+1A	
$= \pi \int_0^5 \left(\frac{x^4}{4} - 2x^3 + 4x^2\right) dx$		
$= \pi \left[\frac{x^5}{20} - \frac{x^4}{2} + \frac{4x^3}{3}\right]_0^5$		
$= \frac{125\pi}{12}$	1A	
	(6)	



[HKDSE 2013 M2 #6]

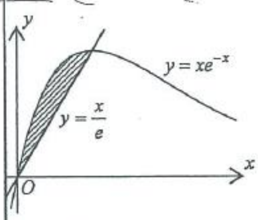
6.

11. (a) $\frac{d}{d\theta}\ln(\sec\theta+\tan\theta)=\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}$	1M
$=\sec\theta$	
Hence $\int\sec\theta\,d\theta=\int\frac{d}{d\theta}\ln(\sec\theta+\tan\theta)\,d\theta$	
$=\ln(\sec\theta+\tan\theta)+C$	1
Alternative Solution	
$\int\sec\theta\,d\theta=\int\frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\,d\theta$	1M
Let $u=\sec\theta+\tan\theta$ which gives $du=(\sec\theta\tan\theta+\sec^2\theta)d\theta$.	
$\therefore$ $\int\sec\theta\,d\theta=\int\frac{du}{u}$	
$=\ln\lvert u\rvert+C$	
$=\ln(\sec\theta+\tan\theta)+C$ since $\sec\theta+\tan\theta>0$ for $0<\theta<\frac{\pi}{2}$	1
	(2)
(b) (i) Let $u=\sec\theta$, where $0<\theta<\frac{\pi}{2}$.	1M
$\therefore$ $du=\sec\theta\tan\theta\,d\theta$	
$\int\frac{du}{\sqrt{u^2-1}}=\int\frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}}$	
$=\int\sec\theta\,d\theta$ since $\tan\theta>0$ for $0<\theta<\frac{\pi}{2}$	
$=\ln(\sec\theta+\tan\theta)+C$ by (a)	
$=\ln(\sec\theta+\sqrt{\sec^2\theta-1})+C$ since $\tan\theta>0$ for $0<\theta<\frac{\pi}{2}$	
$=\ln(u+\sqrt{u^2-1})+C$	1

Solution	Marks	Remarks
(ii) $\displaystyle\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\,dx = \int_0^1 \frac{2x}{\sqrt{(x^2+2)^2-1}}\,dx$		
Let $u = x^2+2$ which gives $du = 2x\,dx$.	1M	
When $x=0$, $u=2$; when $x=1$, $u=3$.		
$\displaystyle\therefore \int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\,dx = \int_2^3 \frac{du}{\sqrt{u^2-1}}$		
$\displaystyle= \left[\ln(u+\sqrt{u^2-1})\right]_2^3$ by (i)	1M	
$= \ln(3+\sqrt{8}) - \ln(2+\sqrt{3})$		
$\displaystyle= \ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)$		
$= \ln(6+4\sqrt{2}-3\sqrt{3}-2\sqrt{6})$	1	
	(5)	
(c) $t = \tan\phi$		
$\displaystyle\frac{dt}{d\phi} = \sec^2\phi$		
$= 1+\tan^2\phi$		
$\displaystyle\therefore \frac{d\phi}{dt} = \frac{1}{1+t^2}$	1	
$\displaystyle\cos^2\phi = \frac{1}{\sec^2\phi}$		
$\displaystyle= \frac{1}{1+t^2}$	1A	
$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1+2\cos^2\phi}}\,d\phi = \int_0^1 \frac{t}{\sqrt{1+\frac{2}{1+t^2}}}\cdot\frac{1}{1+t^2}\,dt$ where $t=\tan\phi$	1M	
$\displaystyle= \int_0^1 \frac{t}{\sqrt{(3+t^2)(1+t^2)}}\,dt$		
$\displaystyle= \frac{1}{2}\int_0^1 \frac{2t}{\sqrt{t^4+4t^2+3}}\,dt$	1M	
$\displaystyle= \frac{1}{2}\ln(6+4\sqrt{2}-3\sqrt{3}-2\sqrt{6})$	1A	OR $\ln\sqrt{6+4\sqrt{2}-3\sqrt{3}-2\sqrt{6}}$
	(5)	

[HKDSE 2013 M2 #11]

7.

6. (a) $\int xe^{-x}dx = x(-e^{-x}) - \int(-e^{-x})\,dx$	1M	
$= -xe^{-x} - e^{-x} + C$	1A	
(b) $\begin{cases} y = xe^{-x} \\ y = \dfrac{x}{e} \end{cases}$		
$xe^{-x} = \dfrac{x}{e}$		
$x\left(e^{-x} - \dfrac{1}{e}\right) = 0$		
$x = 0$ or 1	1A	For $x = 1$
$\therefore$ the area $= \int_0^1\left(xe^{-x} - \dfrac{x}{e}\right)dx$	1M	For $\int_a^b (y_1 - y_2)\,dx$
$= \left[-xe^{-x} - e^{-x} - \dfrac{x^2}{2e}\right]_0^1$	1M	For using (a)
$= \left(-e^{-1} - e^{-1} - \dfrac{1}{2e}\right) - (-1)$		
$= 1 - \dfrac{5}{2e}$	1A	
	(6)	

[HKDSE 2014 M2 #6]

8.

13. (a) $1 - \cos 4\theta - 2\cos 2\theta \sin^2 2\theta$		
$= 2\sin^2 2\theta - 2\cos 2\theta \sin^2 2\theta$	1M	For $1 - \cos 4\theta = 2\sin^2 2\theta$ OR $1 - \cos 2\theta = 2\sin^2\theta$
$= 2\sin^2 2\theta(1 - \cos 2\theta)$		
$= 2(2\sin\theta\cos\theta)^2(2\sin^2\theta)$		
$= 16\cos^2\theta\sin^4\theta$	1	
	(2)	
(b) $\int_0^{n\pi} \cos^2 x \sin^4 x\,dx$		
$= \int_0^{n\pi} \dfrac{1 - \cos 4x - 2\cos 2x \sin^2 2x}{16}\,dx$ by (a)		
$= \dfrac{1}{16}\int_0^{n\pi}(1 - \cos 4x)\,dx - \dfrac{1}{16}\int_0^{n\pi} \sin^2 2x \cdot 2\cos 2x\,dx$	1M	
$= \dfrac{1}{16}\left[x - \dfrac{\sin 4x}{4}\right]_0^{n\pi} - \dfrac{1}{16}\int_{x=0}^{n\pi} \sin^2 2x\,d\sin 2x$	1M	For $d\sin 2x$
$= \dfrac{1}{16}\left[\left(n\pi - \dfrac{\sin 4n\pi}{4}\right) - 0\right] - \dfrac{1}{16}\left[\dfrac{\sin^3 2x}{3}\right]_0^{n\pi}$	1A	For $\dfrac{\sin^3 2x}{3}$
Alternative Solution		
$= \dfrac{1}{16}\int_0^{n\pi}(1 - \cos 4x)\,dx - \dfrac{1}{16}\int_0^{n\pi} \sin 4x \sin 2x\,dx$		
$= \dfrac{1}{16}\left[x - \dfrac{\sin 4x}{4}\right]_0^{n\pi} - \dfrac{1}{16}\int_0^{n\pi} \dfrac{\cos 2x - \cos 6x}{2}\,dx$	1M	For $\dfrac{\cos 2x - \cos 6x}{2}$
$= \dfrac{1}{16}\left[\left(n\pi - \dfrac{\sin 4n\pi}{4}\right) - 0\right] - \dfrac{1}{32}\left[\dfrac{\sin 2x}{2} - \dfrac{\sin 6x}{6}\right]_0^{n\pi}$	1A	For $\dfrac{\sin 2x}{2} - \dfrac{\sin 6x}{6}$
$= \dfrac{n\pi}{16}$	1	
	(4)	

(c) Let $x = k - u$. 1M

$\therefore \quad dx = -du$

When $x = 0, u = k$; when $x = k, u = 0$.

$$\int_0^k x f(x)\,dx = \int_k^0 (k-u) f(k-u)(-du)$$

$$= \int_0^k (k-u) f(u)\,du$$

1M+1M — 1M for reversing the limits; 1M for $f(k-u) = f(u)$

$$= k\int_0^k f(u)\,du - \int_0^k u f(u)\,du$$

$$= k\int_0^k f(x)\,dx - \int_0^k x f(x)\,dx$$

$$\therefore \quad 2\int_0^k x f(x)\,dx = k\int_0^k f(x)\,dx$$

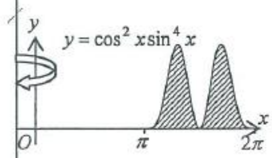
i.e. $\displaystyle\int_0^k x f(x)\,dx = \frac{k}{2}\int_0^k f(x)\,dx$ 1

(4)

(d) Let $f(x) = \cos^2 x \sin^4 x$

$$f(\pi - x) = \cos^2(\pi - x)\sin^4(\pi - x)$$
$$= (-\cos x)^2 (\sin x)^4$$
$$= \cos^2 x \sin^4 x$$
$$= f(x)$$
$$f(2\pi - x) = \cos^2(2\pi - x)\sin^4(2\pi - x)$$
$$= (\cos x)^2(-\sin x)^4$$
$$= f(x)$$

1M



The volume of the solid of revolution

$$= 2\pi\int_\pi^{2\pi} x\cos^2 x \sin^4 x\,dx$$ 1M

$$= 2\pi\left(\int_0^{2\pi} x\cos^2 x\sin^4 x\,dx - \int_0^{\pi} x\cos^2 x \sin^4 x\,dx\right)$$

$$= 2\pi\cdot\frac{2\pi}{2}\int_0^{2\pi}\cos^2 x\sin^4 x\,dx - 2\pi\cdot\frac{\pi}{2}\int_0^{\pi}\cos^2 x\sin^4 x\,dx \quad \text{by (c)}$$ 1M

$$= 2\pi^2\left(\frac{2\pi}{16}\right) - \pi^2\left(\frac{\pi}{16}\right) \quad \text{by (b)}$$

$$= \frac{3\pi^3}{16}$$ 1A

(4)

[HKDSE 2014 M2 #13]

9.

4. (a) $\int x^2 \ln x \,dx$		
$= \frac{1}{3}\int \ln x \,dx^3$		
$= \frac{1}{3}\left(x^3 \ln x - \int x^3 \,d\ln x\right)$	1M	for integration by parts
$= \frac{1}{3}\left(x^3 \ln x - \int x^2 \,dx\right)$	1A	
$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + \text{constant}$	1A	
(b) y		
$= \int 9x^2 \ln x \,dx$	1M	
$= 9\left(\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right) + C$ (by (a))	1M	for using the result of (a)
$= 3x^3 \ln x - x^3 + C$, where C is a constant		
Since Γ passes through the point $(1, 4)$, we have $4 = 3\ln 1 - 1 + C$.	1M	
Solving, we have $C = 5$.		
Thus, the equation of Γ is $y = 3x^3 \ln x - x^3 + 5$.	1A	
	----------(7)	

[HKDSE 2015 M2 #4]

10.

12. (a) (i) Solving $\begin{cases} 3x+y-9=0 \\ x^2-4y+8=0 \end{cases}$, we have $x^2+12x-28=0$.	1M	
So, we have $x=2$ or $x=-14$ (rejected) .		
Thus, the coordinates of B are $(2,3)$.	1A	
(ii) The required capacity		
$=\int_0^3 \pi\left(\frac{9-y}{3}\right)^2 \mathrm{d}y+\int_3^h \pi(4y-8)\,\mathrm{d}y$	1M+1M+1A	
$=\pi\int_0^3\left(9-2y+\frac{y^2}{9}\right)\mathrm{d}y+\pi\int_3^h(4y-8)\,\mathrm{d}y$		
$=\pi\left[9y-y^2+\frac{y^3}{27}\right]_0^3+\pi\left[2y^2-8y\right]_3^h$	1M	either one
$=\pi(2h^2-8h+25)$	1	
	----------(7)	
(b) (i) Putting $x=6$ in $x^2-4y+8=0$, we have $y=11$.	1M	
The required capacity		
$=\pi(2(11)^2-8(11)+25)$ (by (a)(ii))		
$=179\pi\text{ cm}^3$	1A	
(ii) Let h cm be the depth of water in the cup at time t s .		
Also let p cm be the depth of water when the volume of water in the cup is $35\pi\text{ cm}^3$.		
Note that the volume of the frustum is $19\pi\text{ cm}^3$.		
Since $35\pi>19\pi$, we have $p>3$.	1M	withhold 1M if checking is omitted
By (a)(ii), we have $\pi(2p^2-8p+25)=35\pi$.		
Simplifying, we have $p^2-4p-5=0$.	1M	for $k_1p^2+k_2p+k_3=0$
Solving, we have $p=5$ or $p=-1$ (rejected as $3<p\le 11$) .		
Hence, we have $p=5$.		
Let $V\text{ cm}^3$ be the volume of water in the cup at time t s .		
For $3<h\le 11$, we have $V=\pi(2h^2-8h+25)$ (by (a)(ii)) .		
So, we have $\frac{\mathrm{d}V}{\mathrm{d}t}=\pi(4h-8)\frac{\mathrm{d}h}{\mathrm{d}t}$ for $3<h\le 11$.	1M	
Since $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right\|_{h=5}=24\pi$, we have $24\pi=\pi(4(5)-8)\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\|_{h=5}$.		
Therefore, we have $\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\|_{h=5}=2$.	1A	
Thus, the required rate of change is 2 cm/s .		
	----------(6)	

[HKDSE 2015 M2 #12]

11.

3. (a) Note that the coordinates of Q are $(0, 2e^u)$, where $u > 0$.		
The area of ΔOPQ $= \dfrac{u(2e^u)}{2}$ $= ue^u$	1A	
(b) Let v be the y-coordinate of P. Since $v = 2e^u$, we have $\dfrac{dv}{dt} = 2e^u \dfrac{du}{dt}$.	1M	
Therefore, we have $\dfrac{du}{dt} = \dfrac{1}{2e^u}\left(\dfrac{dv}{dt}\right)$. Let A square units be the area of ΔOPQ. By (a), we have $A = ue^u$.		
$\dfrac{dA}{dt}$ $= (e^u + ue^u)\dfrac{du}{dt}$	1M+1M	
$= \left(\dfrac{e^u + ue^u}{2e^u}\right)\dfrac{dv}{dt}$ $= \left(\dfrac{1+u}{2}\right)\dfrac{dv}{dt}$		
So, we have $\left.\dfrac{dA}{dt}\right\rvert_{u=4} = \left(\dfrac{1+4}{2}\right)(6) = 15$.	1A	
Thus, the required rate of change is 15 square units per second.		
Let v be the y-coordinate of P. Since $v = 2e^u$, we have $u = \ln\left(\dfrac{v}{2}\right)$.	1M	
Let A square units be the area of ΔOPQ. By (a), we have $A = \dfrac{v}{2}\ln\left(\dfrac{v}{2}\right)$.		
$\dfrac{dA}{dt}$ $= \left(\left(\dfrac{v}{2}\right)\left(\dfrac{2}{v}\right)\left(\dfrac{1}{2}\right) + \dfrac{1}{2}\ln\left(\dfrac{v}{2}\right)\right)\dfrac{dv}{dt}$	1M+1M	
$= \left(\dfrac{1}{2} + \dfrac{1}{2}\ln\left(\dfrac{v}{2}\right)\right)\dfrac{dv}{dt}$ When $u = 4$, we have $v = 2e^4$.		
$\left.\dfrac{dA}{dt}\right\rvert_{u=4}$ $= \left(\dfrac{1}{2} + \dfrac{1}{2}\ln\left(\dfrac{2e^4}{2}\right)\right)(6)$		
$= 15$	1A	
Thus, the required rate of change is 15 square units per second.		
	----------(5)	

[HKDSE 2016 M2 #3]

12.

7. (a) Let $u=\sqrt{t+1}$.	1M	
So, we have $2u\frac{du}{dt}=1$.		
$\int(1+\sqrt{t+1})^2\,dt$		
$=\int(1+u)^2\,2u\,du$		
$=\int(2u+4u^2+2u^3)\,du$	1M	
$=u^2+\frac{4}{3}u^3+\frac{1}{2}u^4+\text{constant}$		
$=(t+1)+\frac{4}{3}(t+1)^{\frac{3}{2}}+\frac{1}{2}(t+1)^2+\text{constant}$	1A	
$=2t+\frac{t^2}{2}+\frac{4}{3}(t+1)^{\frac{3}{2}}+\text{constant}$		
$\int(1+\sqrt{t+1})^2\,dt$		
$=\int(1+2\sqrt{t+1}+t+1)\,dt$		
$=\int(2+t+2\sqrt{t+1})\,dt$	1M	
$=2t+\frac{t^2}{2}+\int 2\sqrt{t+1}\,dt$		
$=2t+\frac{t^2}{2}+\int 2\sqrt{u}\,du$ (by letting $u=t+1$)	1M	
$=2t+\frac{t^2}{2}+\frac{4}{3}(t+1)^{\frac{3}{2}}+\text{constant}$	1A	
(b) Note that $y=4x^2-4x$, where $1\le x\le 4$.		
So, we have $x=\frac{1}{2}(1+\sqrt{y+1})$.	1M	
The required volume		
$=\int_0^{48}\pi\left(\frac{1}{2}(1+\sqrt{y+1})\right)^2dy$	1M	
$=\frac{\pi}{4}\int_0^{48}(1+\sqrt{y+1})^2\,dy$		
$=\frac{\pi}{4}\int_0^{48}(1+\sqrt{t+1})^2\,dt$	1M	
$=\frac{\pi}{4}\left[2t+\frac{t^2}{2}+\frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^{48}$ (by (a))	1M	for using (a)
$=426\pi$	1A	
	----------(8)	

[HKDSE 2016 M2 #7]

13.

10. (a) Let $x = a - y$.	1M	
So, we have $\frac{dx}{dy} = -1$.		
$\int_0^a f(x)\,dx$		
$= -\int_a^0 f(a-y)\,dy$		
$= \int_0^a f(a-y)\,dy$	1M	
$= \int_0^a f(a-x)\,dx$	1	
	----------(3)	
(b) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$		
$= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$ (by (a) with $a = \frac{\pi}{4} > 0$)	1M	for using (a)
$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$	1M	
$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$	1	
	----------(3)	
(c) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$ (by (b))		
$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} (\ln 2 - \ln(1+\tan x))\,dx$	1M	
$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$		
$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\ dx$	1M	
$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\ dx$		
$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8}$	1	
	----------(3)	
(d) $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx$		
$= \left[x\ln(1+\tan x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$	1M	
$= \left(\frac{\pi \ln 2}{4} - 0\right) - \frac{\pi \ln 2}{8}$ (by (c))	1M	for using (c)
$= \frac{\pi \ln 2}{8}$	1A	
	----------(3)	

[HKDSE 2016 M2 #10]

14.

4. (a) $\int x^2 e^{-x} \mathrm{d}x$		
$= -\int x^2 \mathrm{d}e^{-x}$		
$= -x^2 e^{-x} + \int e^{-x} \mathrm{d}x^2$	1M	for integration by parts
$= -x^2 e^{-x} + 2\int x e^{-x} \mathrm{d}x$	1A	
$= -x^2 e^{-x} - 2\int x \mathrm{d}e^{-x}$		
$= -x^2 e^{-x} - 2\left(x e^{-x} - \int e^{-x} \mathrm{d}x\right)$		
$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + \text{constant}$	1A	
$= -e^{-x}(x^2 + 2x + 2) + \text{constant}$		
(b) The required area		
$= \int_0^6 x^2 e^{-x} \mathrm{d}x$	1M	
$= \left[-e^{-x}(x^2 + 2x + 2)\right]_0^6$ (by (a))	1M	for using the result of (a)
$= 2 - \dfrac{50}{e^6}$	1A	
	----------(6)	

[HKDSE 2017 M2 #4]

15.

8. (a) The slope of the tangent to Γ at P

$= f'(e^3)$

$= \frac{1}{e^3}\ln(e^3)^2$ — 1M

$= \frac{6}{e^3}$

The equation of the tangent to Γ at P is

$y-7=\frac{6}{e^3}(x-e^3)$

$6x-e^3y+e^3=0$ — 1A

(b) $f(x)$

$=\int \frac{1}{x}\ln x^2\,dx$

$=2\int \ln x\,d\ln x$ — 1M

$=(\ln x)^2+C$

Since Γ passes through P, we have $7=(\ln e^3)^2+C$. — 1M

Solving, we have $C=-2$.

Thus, the equation of Γ is $y=(\ln x)^2-2$. — 1A

(c) Note that $f''(x)=\frac{2-2\ln x}{x^2}$. — 1A

Therefore, we have $f''(x)=0 \Leftrightarrow x=e$.

x	$(0,e)$	e	(e,∞)
$f''(x)$	+	0	−

1M

Thus, the point of inflexion of Γ is $(e,-1)$. — 1A

----------(8)

[HKDSE 2017 M2 #8]

16.

11. (a) $\int_0^1 \frac{1}{x^2+2x+3}\,dx$

$= \int_0^1 \frac{1}{(x+1)^2+2}\,dx$ 1M

$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^1$ 1M

$= \frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2} - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ 1A

----------(3)

(b) (i) $\frac{2\tan\theta}{1+\tan^2\theta}$

$= \frac{\frac{2\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$

$= 2\sin\theta\cos\theta$

$= \sin 2\theta$ 1

$\frac{1-\tan^2\theta}{1+\tan^2\theta}$

$= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$

$= \cos^2\theta - \sin^2\theta$

$= \cos 2\theta$ 1

(ii) Let $t = \tan\theta$. Then, we have $\frac{d\theta}{dt} = \frac{1}{1+t^2}$. 1M

Note that $\frac{1}{\sin 2\theta + \cos 2\theta + 2} = \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2} = \frac{1+t^2}{t^2+2t+3}$.

$\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$

$= \int_0^1 \frac{1+t^2}{t^2+2t+3}\left(\frac{1}{1+t^2}\right)dt$ 1M

$= \int_0^1 \frac{1}{t^2+2t+3}\,dt$

$= \int_0^1 \frac{1}{x^2+2x+3}\,dx$

$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (a)) 1M (a)

----------(5)

(c) Let $y=\frac{\pi}{4}-\theta$. Then, we have $\frac{d\theta}{dy}=-1$.	1M	
$\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=-\int_{\frac{\pi}{4}}^{0} \frac{\sin\left(\frac{\pi}{2}-2y\right)+1}{\sin\left(\frac{\pi}{2}-2y\right)+\cos\left(\frac{\pi}{2}-2y\right)+2}\,dy$		
$=\int_0^{\frac{\pi}{4}} \frac{\cos 2y+1}{\cos 2y+\sin 2y+2}\,dy$		
$=\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$	1	
	----------(2)	
(d) $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\int_0^{\frac{\pi}{4}} \frac{4(\sin 2\theta+1)+4(\sin 2\theta+1)+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=4\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+4\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=4\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+4\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$ (by (c))	1M	for using (c)
$=4\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+\cos 2\theta+2}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$	1M	
$=4\int_0^{\frac{\pi}{4}} d\theta+\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	$\pi+$(b)(ii)
Let $I=\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$ and $J=\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$.		
Note that $I+J=\int_0^{\frac{\pi}{4}} d\theta=\frac{\pi}{4}$.	1M	
By (c), we have $I=J=\frac{\pi}{8}$.	1M	for using (c)
$\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=8I+\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	$\pi+$(b)(ii)
	----------(3)	

[HKDSE 2017 M2 #11]

17.

4. (a) $\int u(5^u)\,\mathrm{d}u$			
$=\frac{1}{\ln 5}\left(u(5^u)-\int 5^u\,\mathrm{d}u\right)$		1M	
$=\frac{1}{\ln 5}\left(u(5^u)-\frac{5^u}{\ln 5}\right)+\text{constant}$		1A	
$=\frac{5^u(u\ln 5-1)}{(\ln 5)^2}+\text{constant}$			
(b) The required area			
$=\int_0^1 x(5^{2x})\,\mathrm{d}x$		1M	
$=\frac{1}{4}\int_0^2 u(5^u)\,\mathrm{d}u$	(by letting $u=2x$)	1M	
$=\frac{1}{4(\ln 5)^2}\left[5^u(u\ln 5-1)\right]_0^2$	(by (a))	1M	for using the result of (a)
$=\frac{50\ln 5-24}{4(\ln 5)^2}$			
$=\frac{25\ln 5-12}{2(\ln 5)^2}$		1A	
		----------(6)	

[HKDSE 2018 M2 #4]

18.

5. (a) Let $u = 1 + x^2$.	1M	
Then, we have $\frac{du}{dx} = 2x$.		
$\int x^3\sqrt{1+x^2}\,dx$		
$= \int \frac{1}{2}(u-1)u^{\frac{1}{2}}\,du$	1M	
$= \frac{1}{2}\left(\int u^{\frac{3}{2}}\,du - \int u^{\frac{1}{2}}\,du\right)$		
$= \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + \text{constant}$	1A	
Let $x = \tan\theta$.	1M	
Then, we have $\frac{dx}{d\theta} = \sec^2\theta$.		
$\int x^3\sqrt{1+x^2}\,dx$		
$= \int \tan^3\theta\sec\theta(\sec^2\theta)\,d\theta$	1M	
$= \int \tan^3\theta\sec^3\theta\,d\theta$		
$= \int (\sec^2\theta - 1)\sec^2\theta\,d\sec\theta$		
$= \int \sec^4\theta\,d\sec\theta - \int \sec^2\theta\,d\sec\theta$		
$= \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + \text{constant}$		
$= \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + \text{constant}$	1A	
(b) y		
$= \int 15x^3\sqrt{1+x^2}\,dx$	1M	
$= 15\int x^3\sqrt{1+x^2}\,dx$		
$= 15\left(\frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3\right) + C$ (by (a))	1M	for using the result of (a)
$= 3\left(\sqrt{1+x^2}\right)^5 - 5\left(\sqrt{1+x^2}\right)^3 + C$, where C is a constant		
Since the y-intercept of Γ is 2 , we have $3 - 5 + C = 2$. Solving, we have $C = 4$.	1M	
Thus, the equation of Γ is $y = 3\left(\sqrt{1+x^2}\right)^5 - 5\left(\sqrt{1+x^2}\right)^3 + 4$.	1A	
	----------(7)	

[HKDSE 2018 M2 #5]

19.

10. (a) (i)

$$\int \sin^4 x \, dx$$

$$= -\cos x \sin^3 x + \int \cos x (3\sin^2 x \cos x) \, dx \qquad \text{1M}$$

$$= -\cos x \sin^3 x + 3\int (1-\sin^2 x)(\sin^2 x) \, dx$$

So, we have $\int \sin^4 x \, dx = -\cos x \sin^3 x + 3\int \sin^2 x \, dx - 3\int \sin^4 x \, dx$.

Hence, we have $4\int \sin^4 x \, dx = -\cos x \sin^3 x + 3\int \sin^2 x \, dx$.

Thus, we have $\int \sin^4 x \, dx = \dfrac{-\cos x \sin^3 x}{4} + \dfrac{3}{4}\int \sin^2 x \, dx$. 1

(ii)

$$\int \sin^4 x \, dx$$

$$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx \qquad \text{(by (a)(i))}$$

$$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \frac{1-\cos 2x}{2} \, dx \qquad \text{1M}$$

$$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) + \text{constant} \qquad \text{1M}$$

$$= \frac{3x}{8} - \frac{\cos x \sin^3 x}{4} - \frac{3\sin 2x}{16} + \text{constant}$$

$$\int_0^{\pi} \sin^4 x \, dx$$

$$= \left[\frac{3x}{8} - \frac{\cos x \sin^3 x}{4} - \frac{3\sin 2x}{16}\right]_0^{\pi}$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

$$\int_0^{\pi} \sin^4 x \, dx$$

$$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)^2 dx \qquad \text{1M}$$

$$= \frac{1}{4}\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int_0^{\pi} \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int_0^{\pi} (3 - 4\cos 2x + \cos 4x) \, dx$$

$$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi} \qquad \text{1M}$$

$$= \frac{3\pi}{8} \qquad \text{1A}$$

----------(5)

(b) (i) Let $x=\beta-u$. Then, we have $\frac{dx}{du}=-1$.		
$\int_0^{\beta} x\,f(x)\,dx$		
$=\int_{\beta}^{0} -(\beta-u)\,f(\beta-u)\,du$		
$=\int_0^{\beta} (\beta f(\beta-u)-u\,f(\beta-u))\,du$	1M	
$=\int_0^{\beta} \beta f(x)\,dx-\int_0^{\beta} x\,f(x)\,dx$		
So, we have $2\int_0^{\beta} x\,f(x)\,dx=\beta\int_0^{\beta} f(x)\,dx$.		
Thus, we have $\int_0^{\beta} x\,f(x)\,dx=\frac{\beta}{2}\int_0^{\beta} f(x)\,dx$.	1	
(ii) Note that $\sin^4(\pi-x)=\sin^4 x$ for all real numbers x.	1M	withhold 1M if checking is skipped
$\int_0^{\pi} x\sin^4 x\,dx$		
$=\frac{\pi}{2}\int_0^{\pi}\sin^4 x\,dx$ (by (b)(i))	1M	for using the result of (b)(i)
$=\frac{\pi}{2}\left(\frac{3\pi}{8}\right)$ (by (a)(ii))		
$=\frac{3\pi^2}{16}$	1M	for $\frac{\pi}{2}$(a)(ii)
	----------(5)	
(c) The required volume		
$=\int_{\pi}^{2\pi}\pi(\sqrt{x}\sin^2 x)^2\,dx$	1M	
$=\pi\int_{\pi}^{2\pi} x\sin^4 x\,dx$		
$=\pi\int_0^{\pi}(\pi+y)\sin^4(\pi+y)\,dy$ (by letting $x=\pi+y$)	1M	accept $x=2\pi-y$
$=\pi\int_0^{\pi}(\pi\sin^4 y+y\sin^4 y)\,dy$		
$=\pi\int_0^{\pi}(\pi\sin^4 x+x\sin^4 x)\,dx$		
$=\pi\left(\pi\int_0^{\pi}\sin^4 x\,dx+\int_0^{\pi}x\sin^4 x\,dx\right)$		
$=\pi\left(\pi\left(\frac{3\pi}{8}\right)+\frac{3\pi^2}{16}\right)$ (by (a)(ii) and (b)(ii))		
$=\frac{9\pi^3}{16}$	1A	
	----------(3)	

[HKDSE 2018 M2 #10]

20.

4. (a) $g'(x)$

$$= \frac{\sqrt{x}\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{2\sqrt{x}}\right)}{x}$$ — 1M — for quotient rule

$$= \frac{2 - \ln x}{2\sqrt{x^3}}$$

So, we have $g'(x) = 0 \Leftrightarrow x = e^2$.

x	$(0, e^2)$	e^2	$(e^2, 99)$
$g'(x)$	$+$	0	$-$

— 1M

Therefore, G attains its maximum value only at $x = e^2$.
Thus, G has only one maximum point. — 1

(b) Note that $g(x) < 0$ for all $x \in (0, 1)$ and $g(x) > 0$ for all $x \in (1, 99)$.
So, we have $g(x) = 0 \Leftrightarrow x = 1$.

The required volume

$$= \int_1^{e^2} \pi \left(\frac{\ln x}{\sqrt{x}}\right)^2 \mathrm{d}x$$ — 1M

$$= \pi \int_0^2 u^2 \, \mathrm{d}u \qquad \text{(by letting } u = \ln x \text{)}$$ — 1M

$$= \pi \left[\frac{u^3}{3}\right]_0^2$$

$$= \frac{8\pi}{3}$$ — 1A

----------(6)

[HKDSE 2019 M2 #4]

21.

8. (a) For all $x>0$,

$\mathrm{h}'(x)$

$=\frac{2}{x}\left(\left(x^2-\frac{7}{2}x+\left(\frac{7}{4}\right)^2-\left(\frac{7}{4}\right)^2\right)+4\right)$ — 1M

$=\frac{2}{x}\left(\left(x-\frac{7}{4}\right)^2+\frac{15}{16}\right)$

>0

Thus, $\mathrm{h}(x)$ is an increasing function. — 1A f.t.

(b) (i) y

$=\int\left(2x-7+\frac{8}{x}\right)\mathrm{d}x$ — 1M

$=x^2-7x+8\ln x+C$, where C is a constant

Note that $y=3$ when $x=1$.

So, we have $1^2-7(1)+8\ln 1+C=3$. — 1M

Therefore, we have $C=9$.

Thus, the equation of H is $y=x^2-7x+8\ln x+9$. — 1A

(ii) $\mathrm{h}''(x)$

$=\frac{x(4x-7)-(2x^2-7x+8)}{x^2}$ — 1M

$=\frac{2x^2-8}{x^2}$

$=\frac{2(x-2)(x+2)}{x^2}$

Since $x>0$, we have $\mathrm{h}''(x)=0 \Leftrightarrow x=2$.

x	$(0,2)$	2	$(2,\infty)$
$\mathrm{h}''(x)$	$-$	0	$+$

— 1M

Thus, the point of inflexion of H is $(2,8\ln 2-1)$. — 1A

----------(8)

[HKDSE 2019 M2 #8]

22.

9. (a) $\dfrac{dy}{dx}$		
$=\left(\dfrac{1}{3}\right)\left(\dfrac{1}{2}\right)(12-x^2)^{\frac{-1}{2}}(-2x)$	1M	for chain rule
$=\dfrac{-x}{3\sqrt{12-x^2}}$		
So, we have $\left.\dfrac{dy}{dx}\right\|_{x=3}=\dfrac{-1}{\sqrt{3}}$.		
The equation of L is		
$y-\dfrac{\sqrt{3}}{3}=\dfrac{-1}{\sqrt{3}}(x-3)$	1M	
$x+\sqrt{3}y-4=0$	1A	
	----------(3)	
(b) (i) Putting $y=\dfrac{1}{\sqrt{3}}(4-x)$ in $y=\sqrt{4-x^2}$, we have	1M	
$\dfrac{1}{\sqrt{3}}(4-x)=\sqrt{4-x^2}$		
$x^2-2x+1=0$		
So, we have $x=1$ and $y=\sqrt{3}$.		
Thus, the point of contact of L and C is $(1,\sqrt{3})$.	1A	
(ii) When $\sqrt{4-x^2}=\dfrac{1}{3}\sqrt{12-x^2}$, we have $36-9x^2=12-x^2$.	1M	
So, we have $8x^2-24=0$.		
Since $0<x<2$, we have $x=\sqrt{3}$.		
Thus, the point of intersection of C and Γ is $(\sqrt{3},1)$.	1A	
(iii) The required area		
$=\displaystyle\int_1^{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(4-x)-\sqrt{4-x^2}\right)dx+\int_{\sqrt{3}}^{3}\left(\frac{1}{\sqrt{3}}(4-x)-\frac{1}{3}\sqrt{12-x^2}\right)dx$	1M+1A	1M for either integral
$=\displaystyle\int_1^3\frac{1}{\sqrt{3}}(4-x)\,dx-\int_1^{\sqrt{3}}\sqrt{4-u^2}\,du-\int_{\sqrt{3}}^{3}\frac{1}{3}\sqrt{12-v^2}\,dv$		
$=\dfrac{4\sqrt{3}}{3}-\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}4\cos^2\alpha\,d\alpha-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{12\cos^2\beta}{3}d\beta$ (by letting $u=2\sin\alpha$ and $v=\sqrt{12}\sin\beta$)	1M	for either substitution
$=\dfrac{4\sqrt{3}}{3}-8\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos 2\theta+1}{2}d\theta$	1M	
$=\dfrac{4\sqrt{3}}{3}-8\left[\dfrac{\sin 2\theta}{4}+\dfrac{\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$		
$=\dfrac{4\sqrt{3}}{3}-\dfrac{2\pi}{3}$	1A	
	----------(9)	

[HKDSE 2019 M2 #9]

23.

10. (a) $\dfrac{1}{2+\cos 2x}$		
$=\dfrac{1}{2+2\cos^2 x-1}$		
$=\dfrac{1}{2\cos^2 x+1}$		
$=\dfrac{\sec^2 x}{2+\sec^2 x}$	1	
	----------(1)	
(b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,dx$		
$=\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{2+\sec^2 x}\,dx$ (by (a))	1M	for using (a)
$=\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3+\tan^2 x}\,dx$		
$=\displaystyle\int_0^1 \frac{1}{3+t^2}\,dt$ (by letting $t=\tan x$)	1M	
$=\left[\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{t}{\sqrt{3}}\right)\right]_0^1$		
$=\dfrac{\sqrt{3}\pi}{18}$	1A	
	----------(3)	
(c) By letting $u=-x$, we have $\displaystyle\int_{-a}^0 f(x)\ln(1+e^x)\,dx=\int_a^0 -f(-u)\ln(1+e^{-u})\,du$.	1M	
So, we have $\displaystyle\int_{-a}^0 f(x)\ln(1+e^x)\,dx=\int_0^a f(-x)\ln(1+e^{-x})\,dx$.		
$\displaystyle\int_{-a}^a f(x)\ln(1+e^x)\,dx$		
$=\displaystyle\int_{-a}^0 f(x)\ln(1+e^x)\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$	1M	
$=\displaystyle\int_0^a f(-x)\ln(1+e^{-x})\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$=-\displaystyle\int_0^a f(x)\ln(1+e^{-x})\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$	1M	
$=-\displaystyle\int_0^a f(x)\ln\left(\frac{e^x+1}{e^x}\right)dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$=-\displaystyle\int_0^a f(x)\ln(e^x+1)\,dx+\int_0^a x f(x)\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$=\displaystyle\int_0^a x f(x)\,dx$	1	
	----------(4)	

(d) Note that $\dfrac{\sin 2(-x)}{(2+\cos 2(-x))^2}=\dfrac{-\sin 2x}{(2+\cos 2x)^2}$ for all $x \in \mathbf{R}$.	1M	withhold 1M if checking is omitted
By (c), we have $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,\mathrm{d}x=\int_0^{\frac{\pi}{4}}\frac{x\sin 2x}{(2+\cos 2x)^2}\,\mathrm{d}x$.	1M	for using (c)
Also note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2(2+\cos 2x)}\right)=\dfrac{\sin 2x}{(2+\cos 2x)^2}$.		
$\displaystyle\int_0^{\frac{\pi}{4}}\frac{x\sin 2x}{(2+\cos 2x)^2}\,\mathrm{d}x$		
$\displaystyle=\left[\frac{x}{2(2+\cos 2x)}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{2(2+\cos 2x)}\,\mathrm{d}x$	1M	
$=\dfrac{1}{2}\left(\dfrac{\pi}{4}\right)\left(\dfrac{1}{2+0}\right)-\dfrac{1}{2}\left(\dfrac{\sqrt{3}\pi}{18}\right)$ (by (b))	1M	for using the result of (b)
$=\dfrac{(9-4\sqrt{3})\pi}{144}$		
Thus, we have $\displaystyle\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,\mathrm{d}x=\frac{(9-4\sqrt{3})\pi}{144}$.	1A	
	----------(5)	

[HKDSE 2019 M2 #10]

Matrices

1.

11. (a) $\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0$

$(1-x)(3-x)-2\cdot 4=0$ — 1M

$x^2-4x-5=0$

$x=-1$ or 5 — 1A

(2)

(b) (i) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1\cdot\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} a+4b=-a \\ 2a+3b=-b \end{cases}$ — 1M

$a+2b=0$ -- (1) — 1A

$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = 5\begin{pmatrix} c \\ 1 \end{pmatrix}$

Either one

Either one

$\begin{cases} c+4=5c \\ 2c+3=5 \end{cases}$

$c=1$ -- (2)

$\begin{vmatrix} a & c \\ b & 1 \end{vmatrix} = 1$

By (2), $a-b=1$ -- (3) — 1M — For $a-bc=1$

Solving (1) and (3), we have $a=\dfrac{2}{3}$ and $b=\dfrac{-1}{3}$.

$\therefore\ P=\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$ — 1A

(ii) $P^{-1}=\dfrac{1}{\frac{2}{3}+\frac{1}{3}}\begin{pmatrix} 1 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}^t$

$=\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ — 1M

$P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P=\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$

$=\begin{pmatrix} -1 & 1 \\ \frac{5}{3} & \frac{10}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$ — 1M — OR $\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{-2}{3} & 5 \\ \frac{1}{3} & 5 \end{pmatrix}$

$=\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ — 1A

Solution	Marks	Remarks
(iii) $P^{-1}\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}P=\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}$		
$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}=P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}P^{-1}$	1M	
$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}^{12}=\underbrace{P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}P^{-1}P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}P^{-1}\cdots P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}P^{-1}}_{12\text{ times}}$		
$=P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}^{12}P^{-1}$	1M	
$=\begin{pmatrix}\frac{2}{3} & 1\\ \frac{-1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^{12}\end{pmatrix}\begin{pmatrix}1 & -1\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$	1M	For $\begin{pmatrix}1 & 0\\ 0 & 5^{12}\end{pmatrix}$
$=\begin{pmatrix}\frac{2}{3} & 5^{12}\\ \frac{-1}{3} & 5^{12}\end{pmatrix}\begin{pmatrix}1 & -1\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$		OR $\begin{pmatrix}\frac{2}{3} & 1\\ \frac{-1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ \frac{5^{12}}{3} & \frac{2\cdot 5^{12}}{3}\end{pmatrix}$
$=\begin{pmatrix}\frac{5^{12}+2}{3} & \frac{2\cdot 5^{12}-2}{3}\\ \frac{5^{12}-1}{3} & \frac{2\cdot 5^{12}+1}{3}\end{pmatrix}$	1A	OR $\begin{pmatrix}81380209 & 162760416\\ 81380208 & 162760417\end{pmatrix}$
	(11)	

[HKDSE 2012 M2 #11]

2.

8. (a) $\begin{pmatrix}1 & k & 0\\ 0 & 1 & 1\\ k & 0 & 0\end{pmatrix}^{-1} = \dfrac{1}{\begin{vmatrix}1 & k & 0\\ 0 & 1 & 1\\ k & 0 & 0\end{vmatrix}}\begin{pmatrix}0 & k & -k\\ 0 & 0 & k^2\\ k & -1 & 1\end{pmatrix}^{-1}$ 1M+1A

$= \dfrac{1}{k^2}\begin{pmatrix}0 & 0 & k\\ k & 0 & -1\\ -k & k^2 & 1\end{pmatrix}$ 1A

(b) $\begin{pmatrix}1 & k & 0\\ 0 & 1 & 1\\ k & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ 1\\ z\end{pmatrix} = \begin{pmatrix}2\\ 2\\ 1\end{pmatrix}$

$\begin{pmatrix}x\\ 1\\ z\end{pmatrix} = \dfrac{1}{k^2}\begin{pmatrix}0 & 0 & k\\ k & 0 & -1\\ -k & k^2 & 1\end{pmatrix}\begin{pmatrix}2\\ 2\\ 1\end{pmatrix}$ by (a) 1M

$= \dfrac{1}{k^2}\begin{pmatrix}k\\ 2k-1\\ 2k^2-2k+1\end{pmatrix}$

From the second row, we have $\dfrac{2k-1}{k^2} = 1$.

> Alternative Solution
>
> $\begin{pmatrix}x+k\\ 1+z\\ kx\end{pmatrix} = \begin{pmatrix}2\\ 2\\ 1\end{pmatrix}$ 1M
>
> From the first and third rows, we have $x+k=2$ and $x = \dfrac{1}{k}$.
>
> $\therefore \ \dfrac{1}{k} + k = 2$.

i.e. $k^2 - 2k + 1 = 0$

$k = 1$ 1A

(5)

[HKDSE 2013 M2 #8]

3.

13. (a) (i) $MN=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix}$

$=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}$

$\mathrm{tr}(MN)=ae+bg+cf+dh$ — 1A

$NM=\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$=\begin{pmatrix}ea+fc & eb+fd\\ ga+hc & gb+hd\end{pmatrix}$ — Either one

$\mathrm{tr}(MN)=ea+fc+gb+hd$

$\therefore\ \mathrm{tr}(MN)=\mathrm{tr}(NM)$ — 1

(ii) $\mathrm{tr}(BAB^{-1})=1+3$

$\mathrm{tr}(AB^{-1}B)=4$ by (a)(i) — 1M

$\mathrm{tr}(A)=4$ — 1

(iii) $BAB^{-1}=\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$\left|BAB^{-1}\right|=\begin{vmatrix}1 & 0\\ 0 & 3\end{vmatrix}$

$|B|\cdot|A|\cdot\left|B^{-1}\right|=1\cdot 3-0\cdot 0$ — 1M

$|B|\cdot|A|\cdot|B|^{-1}=3$

$|A|=3$ — 1A

(6)

(b) (i) $C\begin{pmatrix}x\\ y\end{pmatrix}=\lambda_1\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}px+qy\\ rx+sy\end{pmatrix}=\begin{pmatrix}\lambda_1 x\\ \lambda_1 y\end{pmatrix}$

$\begin{cases}(p-\lambda_1)x+qy=0\\ rx+(s-\lambda_1)y=0\end{cases}$ — 1A

Since this system of equations has non-zero solutions $\begin{pmatrix}x\\ y\end{pmatrix}$, $\begin{vmatrix}p-\lambda_1 & q\\ r & s-\lambda_1\end{vmatrix}=0$. — 1

Similarly, $C\begin{pmatrix}x\\ y\end{pmatrix}=\lambda_2\begin{pmatrix}x\\ y\end{pmatrix}$ gives $\begin{vmatrix}p-\lambda_2 & q\\ r & s-\lambda_2\end{vmatrix}=0$. — 1

(ii) By (b)(i), λ_1 and λ_2 are the roots of the equation

$\begin{vmatrix}p-\lambda & q\\ r & s-\lambda\end{vmatrix}=0$ — 1M

$(p-\lambda)(s-\lambda)-qr=0$

$\lambda^2-(p+s)\lambda+ps-qr=0$

$\lambda^2-\mathrm{tr}(C)\cdot\lambda+|C|=0$ — 1

(5)

(c) $A\begin{pmatrix}x\\ y\end{pmatrix}=\lambda\begin{pmatrix}x\\ y\end{pmatrix}$ for some non-zero matrices $\begin{pmatrix}x\\ y\end{pmatrix}$

$\lambda^2-\mathrm{tr}(A)\cdot\lambda+|A|=0$ by (b)(ii)

$\lambda^2-4\lambda+3=0$ by (a)(ii) & (a)(iii) — 1M For either one

$\lambda=1$ or 3 — 1A

(2)

[HKDSE 2013 M2 #13]

4.

7. (a) $A^2=\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$	
$=\begin{pmatrix}2&0&2\\0&4&0\\2&0&2\end{pmatrix}$	
$=2A$	1
Hence the statement is true for $n=1$.	
Assume the statement is true for $n=k$, i.e. $A^{k+1}=2^kA$.	1
$A^{k+2}=A^{k+1}A$	
$=(2^kA)A$ by assumption	1
$=2^kA^2$	
$=2^k\cdot 2A$ by the statement for $n=1$	
$=2^{k+1}A$	1
Hence the statement is also true for $n=k+1$.	
By the principle of mathematical induction, the statement is true for all positive integers n .	1
(b) $\lvert A\rvert=0$	1A
Hence A^{-1} does not exist and so Willy arrives at a wrong conclusion by using A^{-1}.	1
	(7)

[HKDSE 2014 M2 #7]

5.

12. (a) (i) $A^{-1}=\begin{pmatrix}1 & p\\ -1 & 1\end{pmatrix}^{-1}$

$=\dfrac{1}{\begin{vmatrix}1 & p\\ -1 & 1\end{vmatrix}}\begin{pmatrix}1 & 1\\ -p & 1\end{pmatrix}^{T}$ 1M

$=\dfrac{1}{1+p}\begin{pmatrix}1 & -p\\ 1 & 1\end{pmatrix}$ 1A

(ii) $A^{-1}MA=\dfrac{1}{1+p}\begin{pmatrix}1 & -p\\ 1 & 1\end{pmatrix}\begin{pmatrix}k-1 & k\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & p\\ -1 & 1\end{pmatrix}$

$=\dfrac{1}{1+p}\begin{pmatrix}k-p-1 & k\\ k & k\end{pmatrix}\begin{pmatrix}1 & p\\ -1 & 1\end{pmatrix}$ 1M+1A OR $\dfrac{1}{1+p}\begin{pmatrix}1 & -p\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & k+kp-p\\ 1 & p\end{pmatrix}$

$=\dfrac{1}{1+p}\begin{pmatrix}-1-p & k+kp-p-p^2\\ 0 & k+kp\end{pmatrix}$

$=\begin{pmatrix}-1 & k-p\\ 0 & k\end{pmatrix}$ 1

(iii) By (ii), $(A^{-1}MA)^n=\begin{pmatrix}-1 & 0\\ 0 & k\end{pmatrix}^n$ for $p=k$

$A^{-1}M^nA=\begin{pmatrix}(-1)^n & 0\\ 0 & k^n\end{pmatrix}$ 1M For either side

$M^n=\begin{pmatrix}1 & k\\ -1 & 1\end{pmatrix}\begin{pmatrix}(-1)^n & 0\\ 0 & k^n\end{pmatrix}\cdot\dfrac{1}{1+k}\begin{pmatrix}1 & -k\\ 1 & 1\end{pmatrix}$ 1M

$=\dfrac{1}{1+k}\begin{pmatrix}(-1)^n & k^{n+1}\\ (-1)^{n+1} & k^n\end{pmatrix}\begin{pmatrix}1 & -k\\ 1 & 1\end{pmatrix}$ OR $\dfrac{1}{1+k}\begin{pmatrix}1 & k\\ -1 & 1\end{pmatrix}\begin{pmatrix}(-1)^n & (-1)^{n+1}k\\ k^n & k^n\end{pmatrix}$

$=\dfrac{1}{1+k}\begin{pmatrix}k^{n+1}+(-1)^n & k^{n+1}+(-1)^{n+1}k\\ k^n+(-1)^{n+1} & k^n+(-1)^nk\end{pmatrix}$ 1A

(8)

(b) $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=M\begin{pmatrix}x_{n-1}\\ x_{n-2}\end{pmatrix}$ where $M=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}$ after substituting $k=2$

$=M^2\begin{pmatrix}x_{n-2}\\ x_{n-3}\end{pmatrix}$ 1M

$=\cdots$

$=M^{n-2}\begin{pmatrix}x_2\\ x_1\end{pmatrix}$ 1A

$=\dfrac{1}{1+2}\begin{pmatrix}2^{n-1}+(-1)^{n-2} & 2^{n-1}+(-1)^{n-1}2\\ 2^{n-2}+(-1)^{n-1} & 2^{n-2}+(-1)^{n-2}2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$ by (a)(iii)

$\therefore\ x_n=\dfrac{2^{n-1}+(-1)^{n-2}}{3}$ 1A OR $\dfrac{2^{n-1}+(-1)^n}{3}$

(3)

[HKDSE 2014 M2 #12]

6.

6. (a) Note that $\lvert M^T \rvert = \lvert M \rvert$ and $\lvert -M \rvert = -\lvert M \rvert$.	1M	either one
Since $\lvert M^T \rvert = \lvert -M \rvert$, we have $\lvert M \rvert = -\lvert M \rvert$.		
So, we have $2\lvert M \rvert = 0$.		
Thus, we have $\lvert M \rvert = 0$.	1	
(b) (i) $A+I = \begin{pmatrix} 0 & a & b \\ -a & 0 & -8 \\ -b & 8 & 0 \end{pmatrix}$		
So, we have $(A+I)^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & 8 \\ b & -8 & 0 \end{pmatrix} = -(A+I)$.	1M	
By (a), we have $\lvert A+I \rvert = 0$.	1	
$A+I = \begin{pmatrix} 0 & a & b \\ -a & 0 & -8 \\ -b & 8 & 0 \end{pmatrix}$ $\lvert A+I \rvert = \begin{vmatrix} 0 & a & b \\ -a & 0 & -8 \\ -b & 8 & 0 \end{vmatrix}$ $= 0 + 8ab - 8ab - 0 - 0 - 0$	1M	
$= 0$	1	
(ii) Note that $A^3 + I = (A+I)(A^2 - A + I)$.	1M	
$\lvert A^3 + I \rvert$ $= \lvert A+I \rvert \lvert A^2 - A + I \rvert$ $= (0)\lvert A^2 - A + I \rvert$ (by (b)(i)) $= 0$		
Therefore, $A^3 + I$ is a singular matrix. Thus, the claim is agreed.	1A ----------(6)	f.t.

[HKDSE 2015 M2 #6]

7.

11. (a) (i) AB

$= \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}$ 1M — either one — for both

$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

BA

$= \frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}$

$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ 1A

$A+B$

$= \frac{1}{\lambda-\mu+2}\left[\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}\right]$

$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ 1A

(ii) A^2

$= A(I-B)$ (by (a)(i)) 1M — either one — for both

$= A - AB$

$= A - 0$ (by (a)(i))

$= A$

B^2

$= B(I-A)$ (by (a)(i))

$= B - BA$

$= B - 0$ (by (a)(i))

$= B$ 1

(iii) $(\lambda+1)A+(\mu-1)B$

$= \frac{\lambda+1}{\lambda-\mu+2}(I-\mu I+M)+\frac{\mu-1}{\lambda-\mu+2}(I+\lambda I-M)$

$= \frac{-\mu\lambda+\lambda-\mu+1}{\lambda-\mu+2}I+\frac{\lambda+1}{\lambda-\mu+2}M+\frac{\mu\lambda-\lambda+\mu-1}{\lambda-\mu+2}I-\frac{\mu-1}{\lambda-\mu+2}M$

$= M$ 1

So, we have $M=(\lambda+1)A+(\mu-1)B$.

M^2

$= ((\lambda+1)A+(\mu-1)B)((\lambda+1)A+(\mu-1)B)$

$= (\lambda+1)^2A^2+(\lambda+1)(\mu-1)AB+(\lambda+1)(\mu-1)BA+(\mu-1)^2B^2$

$= (\lambda+1)^2A+(\mu-1)^2B$ (by (a)(i) and (a)(ii)) 1M for using (a)(i) and (a)(ii)

M^3

$= M^2M$

$= ((\lambda+1)^2A+(\mu-1)^2B)((\lambda+1)A+(\mu-1)B)$

$= (\lambda+1)^3A^2+(\lambda+1)^2(\mu-1)AB+(\lambda+1)(\mu-1)^2BA+(\mu-1)^3B^2$

$= (\lambda+1)^3A+(\mu-1)^3B$ (by (a)(i) and (a)(ii))

Thus, we have $M^n=(\lambda+1)^nA+(\mu-1)^nB$. 1

$(\lambda+1)A+(\mu-1)B$		
$=\frac{\lambda+1}{\lambda-\mu+2}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}+\frac{\mu-1}{\lambda-\mu+2}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}$		
$=\frac{1}{\lambda-\mu+2}\begin{pmatrix}\lambda^2-\lambda\mu+2\lambda & \lambda-\mu+2\\ (\lambda-\mu+1)(\lambda-\mu+2) & \lambda\mu-\mu^2+2\mu\end{pmatrix}$		
$=M$	1	
So, the statement is true for $n=1$.		
Assume that $M^k=(\lambda+1)^kA+(\mu-1)^kB$, where k is a positive integer.		
M^{k+1}		
$=MM^k$		
$=((\lambda+1)A+(\mu-1)B)((\lambda+1)^kA+(\mu-1)^kB)$		
$=(\lambda+1)^{k+1}A^2+(\lambda+1)(\mu-1)^kAB+(\lambda+1)^k(\mu-1)BA+(\mu-1)^{k+1}B^2$		
$=(\lambda+1)^{k+1}A+(\mu-1)^{k+1}B$ (by (a)(i) and (a)(ii))	1M	
So, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, we have $M^n=(\lambda+1)^nA+(\mu-1)^nB$.	1	
	----------(8)	
(b) Note that $\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}=2^{315}\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}^{315}$.	1M	
Also note that $3-2=1\neq 2$ and $2-3+1=0$.	1M	withhold 1M if checking is omitted
Putting $\lambda=2$, $\mu=3$ and $n=315$ in (a)(iii), we have	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}$		
$=\frac{(2^{315})(3^{315})}{1}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}+\frac{(2^{315})(2^{315})}{1}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$		
$=\begin{pmatrix}2^{630} & 6^{315}-2^{630}\\ 0 & 6^{315}\end{pmatrix}$	1A	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}$	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^2=\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4^2 & 6^2-4^2\\ 0 & 6^2\end{pmatrix}$	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^3=\begin{pmatrix}4^2 & 6^2-4^2\\ 0 & 6^2\end{pmatrix}\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4^3 & 6^3-4^3\\ 0 & 6^3\end{pmatrix}$	1M	withhold 1M if the step is skipped
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}=\begin{pmatrix}4^{315} & 6^{315}-4^{315}\\ 0 & 6^{315}\end{pmatrix}=\begin{pmatrix}2^{630} & 6^{315}-2^{630}\\ 0 & 6^{315}\end{pmatrix}$	1A	
	----------(4)	

[HKDSE 2015 M2 #11]

8.

8. (a) (i) A $=\begin{pmatrix}1&0\\1&1\end{pmatrix}$		
A^2 $=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}$		
$=\begin{pmatrix}1&0\\2&1\end{pmatrix}$	1A	
(ii) A^3 $=\begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}$	1M	
$=\begin{pmatrix}1&0\\3&1\end{pmatrix}$		
A^n $=\begin{pmatrix}1&0\\n&1\end{pmatrix}$	1A	
(iii) $\det(A^n)$ $=\begin{vmatrix}1&0\\n&1\end{vmatrix}$ $=1$		
$(A^{-1})^n$ $=(A^n)^{-1}$		
$=\frac{1}{\det(A^n)}\begin{pmatrix}1&0\\-n&1\end{pmatrix}$	1M	
$=\begin{pmatrix}1&0\\-n&1\end{pmatrix}$	1A	
$A^{-1}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$		
$(A^{-1})^2=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\-2&1\end{pmatrix}$		
$(A^{-1})^3=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\-3&1\end{pmatrix}$	1M	
Thus, we have $(A^{-1})^n=\begin{pmatrix}1&0\\-n&1\end{pmatrix}$.	1A	
	----------(5)	

(b) (i) $\sum_{k=0}^{n-1} 2^k$		
$= \frac{2^n - 1}{2-1}$		
$= 2^n - 1$	1A	
(ii) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^0 & 2^1 \end{pmatrix}$		
$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2^0 & 2^1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2^0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^0 + 2^1 & 2^2 \end{pmatrix}$		
$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 2^0 + 2^1 & 2^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2^0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^0 + 2^1 + 2^2 & 2^3 \end{pmatrix}$	1M	
So, we have $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{n-1} 2^k & 2^n \end{pmatrix}$.		
By (b)(i), we have $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$.	1A	
$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^1 - 1 & 2^1 \end{pmatrix}$		
$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2^1 - 1 & 2^1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^2 - 1 & 2^2 \end{pmatrix}$		
$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 2^2 - 1 & 2^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^3 - 1 & 2^3 \end{pmatrix}$	1M	
Thus, we have $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$.	1A	
	----------(3)	

[HKDSE 2016 M2 #8]

9.

12. (a)

A

$=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$

$=3^1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+3^0(1)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

So, the statement is true for $n=1$. — 1

Assume that $A^k=3^kI+3^{k-1}k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, where k is a positive integer. — 1M

A^{k+1}

$=A^kA$

$=\left(3^kI+3^{k-1}k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$ (by induction assumption) — 1M — for using induction assumption

$=\left(3^kI+3^{k-1}k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\left(3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$

$=3^{k+1}I+3^kk\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+3^k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+3^{k-1}k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^2$

$=3^{k+1}I+3^k(k+1)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$=3^{k+1}I+3^k(k+1)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

Therefore, the statement is true for $n=k+1$ if it is true for $n=k$.

By mathematical induction, the statement is true for all positive integers n . — 1

----------(4)

(b) (i) Note that $P^{-1}=\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$. — 1A

$P^{-1}BP$

$=\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}$

$=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$ — 1A

$=A$

(ii) By (b)(i), we have $P^{-1}BP=A$.

So, we have $(P^{-1}BP)^n=A^n$.

Therefore, we have $P^{-1}B^nP=A^n$. — 1M

Hence, we have $B^n=PA^nP^{-1}$.

B^n

$=\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}\left(3^nI+3^{n-1}n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$ — 1M

$=3^nI+3^{n-1}n\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ -2 & -1\end{pmatrix}$

$=3^nI+3^{n-1}n\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}$ — 1

B $=\begin{pmatrix}5 & 1\\-4 & 1\end{pmatrix}$ $=3^1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+3^0(1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$ So, the statement is true for $n=1$. Assume that $B^k=3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$, where k is a positive integer. B^{k+1} $=B^kB$		
$=\left(3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)\begin{pmatrix}5 & 1\\-4 & 1\end{pmatrix}$ (by induction assumption)	1M	for using induction assumption
$=\left(3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)\left(3I+\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)$	1M	
$=3^{k+1}I+3^kk\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+3^k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}^2$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$ Therefore, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(iii) $\lvert A^m-B^m\rvert=4m^2$ $\left\lvert 3^{m-1}m\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}-3^{m-1}m\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right\rvert=4m^2$		
$(3^{m-1})^2m^2\begin{vmatrix}-2 & 0\\4 & 2\end{vmatrix}=4m^2$	1M	
$-4m^2(3^{2(m-1)})=4m^2$		
$3^{2(m-1)}=-1$	1M	
Note that $-1<0<3^{2(m-1)}$.		
Thus, there does not exist a positive integer m such that $\lvert A^m-B^m\rvert=4m^2$.	1A	f.t.
	----------(8)	

[HKDSE 2017 M2 #12]

10.

7. (a) $MX = XM$		
$\begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} a & 6a \\ b & c \end{pmatrix}=\begin{pmatrix} a & 6a \\ b & c \end{pmatrix}\begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$		
$\begin{pmatrix} 7a+3b & 42a+3c \\ -a+5b & -6a+5c \end{pmatrix}=\begin{pmatrix} a & 33a \\ 7b-c & 3b+5c \end{pmatrix}$	1M	
$\begin{cases} 7a+3b=a \\ -a+5b=7b-c \\ 42a+3c=33a \\ -6a+5c=3b+5c \end{cases}$	1M	
$\begin{cases} b=-2a \\ c=-3a \end{cases}$	1A	for both correct
(b) $\lvert X \rvert$		
$=\begin{vmatrix} a & 6a \\ -2a & -3a \end{vmatrix}$	1M	for considering determinant
$=(a)(-3a)-(6a)(-2a)$		
$=9a^2$		
Note that X is a non-zero real matrix.		
By (a), a is a non-zero real number.		
So, we have $\lvert X \rvert > 0$.		
Therefore, we have $\lvert X \rvert \neq 0$.		
Thus, X is a non-singular matrix.	1	
(c) $(X^T)^{-1}$		
$=(X^{-1})^T$	1M	
$=\left(\dfrac{1}{\lvert X \rvert}\begin{pmatrix} -3a & -6a \\ 2a & a \end{pmatrix}\right)^T$	1M	
$=\left(\dfrac{1}{9a}\begin{pmatrix} -3 & -6 \\ 2 & 1 \end{pmatrix}\right)^T$		
$=\dfrac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$	1A	
$(X^T)^{-1}$		
$=\begin{pmatrix} a & -2a \\ 6a & -3a \end{pmatrix}^{-1}$		
$=\dfrac{1}{\lvert X^T \rvert}\begin{pmatrix} -3a & 2a \\ -6a & a \end{pmatrix}$	1M	
$=\dfrac{1}{\lvert X \rvert}\begin{pmatrix} -3a & 2a \\ -6a & a \end{pmatrix}$	1M	
$=\dfrac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$	1A	
	----------(8)	

[HKDSE 2018 M2 #7]

11.

11. (a) M^2 $=\begin{pmatrix}2&7\\-1&-6\end{pmatrix}\begin{pmatrix}2&7\\-1&-6\end{pmatrix}$ $=\begin{pmatrix}-3&-28\\4&29\end{pmatrix}$	1M	
$M^2=aM+bI$ $\begin{pmatrix}-3&-28\\4&29\end{pmatrix}=\begin{pmatrix}2a&7a\\-a&-6a\end{pmatrix}+\begin{pmatrix}b&0\\0&b\end{pmatrix}$ $\begin{cases}-3=2a+b\\-28=7a\\4=-a\\29=-6a+b\end{cases}$	1M	
Thus, we have $a=-4$ and $b=5$.	1A	for both correct
	----------(3)	
(b) Note that $(1-(-5))M+(5+(-5))I=6M$.	1M	
So, the statement is true for $n=1$.		
Assume that $6M^k=(1-(-5)^k)M+(5+(-5)^k)I$, where k is a positive integer.		
$6M^{k+1}$ $=M(6M^k)$		
$=M((1-(-5)^k)M+(5+(-5)^k)I)$	1M	
$=(1-(-5)^k)M^2+(5+(-5)^k)M$		
$=(1-(-5)^k)((1+(-5))M+5I)+(5+(-5)^k)M$	1M	for using the result of (a)
$=(1+(-5)-(-5)^k-(-5)^{k+1})M+(5+(-5)^{k+1})I+(5+(-5)^k)M$		
$=(1-(-5)^{k+1})M+(5+(-5)^{k+1})I$		
So, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, we have $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$.	1	
$6M$ $=(1-(-5))M+(5+(-5))I$	1M	
$6M^2$ $=(1-(-5))M^2$		
$=(1-(-5))((1+(-5))M+5I)$	1M	for using the result of (a)
$=(1-(-5)^2)M+(5+(-5)^2)I$		
$6M^3$ $=M(6M^2)$		
$=M((1-(-5)^2)M+(5+(-5)^2)I)$	1M	
$=(1-(-5)^2)M^2+(5+(-5)^2)M$		
$=(1-(-5)^2)((1+(-5))M+5I)+(5+(-5)^2)M$		
$=(1+(-5)-(-5)^2-(-5)^3)M+(5+(-5)^3)I+(5+(-5)^2)M$		
$=(1-(-5)^3)M+(5+(-5)^3)I$		
Thus, we have $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$.	1	
	----------(4)	

(c) By (b), we have $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ and

$6M^{n+1} = (1-(-5)^{n+1})M + (5+(-5)^{n+1})I$. 1M

$6(1-(-5)^n)M^{n+1} - 6(1-(-5)^{n+1})M^n = ((1-(-5)^n)(5+(-5)^{n+1}) - (1-(-5)^{n+1})(5+(-5)^n))I$ 1M

$$M^n\left(\frac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}\right)=\left(\frac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}\right)M^n = I$$

So, we have $(M^n)^{-1} = \dfrac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}$. 1M

$$(M^n)^{-1}$$
$$=\frac{(-5)^n-1}{6(-5)^n}M+\frac{-(-5)^{n+1}+1}{6(-5)^n}I$$
$$=\left(\frac{1}{6}M+\frac{5}{6}I\right)+\frac{1}{(-5)^n}\left(\frac{-1}{6}M+\frac{1}{6}I\right)$$ 1M

Letting $A=\dfrac{1}{6}M+\dfrac{5}{6}I=\dfrac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix}$ and $B=\dfrac{-1}{6}M+\dfrac{1}{6}I=\dfrac{1}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix}$,

we have $(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$.

Thus, there exists a pair of 2×2 real matrices A and B such that

$(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$ for all positive integers n . 1A f.t.

$6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ (by (b))

$$M^n=\frac{1}{6}(1-(-5)^n)\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}+\frac{1}{6}(5+(-5)^n)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}\dfrac{-(-5)^n+7}{6} & \dfrac{-7(-5)^n+7}{6}\\ \dfrac{(-5)^n-1}{6} & \dfrac{7(-5)^n-1}{6}\end{pmatrix}$$ 1M

$$\det(M^n)=\left(\frac{-(-5)^n+7}{6}\right)\left(\frac{7(-5)^n-1}{6}\right)-\left(\frac{-7(-5)^n+7}{6}\right)\left(\frac{(-5)^n-1}{6}\right)$$ 1M
$$=(-5)^n$$

$$(M^n)^{-1}=\frac{1}{\det(M^n)}\begin{pmatrix}\dfrac{7(-5)^n-1}{6} & \dfrac{7(-5)^n-7}{6}\\ \dfrac{-(-5)^n+1}{6} & \dfrac{-(-5)^n+7}{6}\end{pmatrix}$$ 1M

$$=\begin{pmatrix}\dfrac{7}{6} & \dfrac{7}{6}\\ \dfrac{-1}{6} & \dfrac{-1}{6}\end{pmatrix}+\frac{1}{(-5)^n}\begin{pmatrix}\dfrac{-1}{6} & \dfrac{-7}{6}\\ \dfrac{1}{6} & \dfrac{7}{6}\end{pmatrix}$$ 1M

Letting $A=\begin{pmatrix}\frac{7}{6} & \frac{7}{6}\\ \frac{-1}{6} & \frac{-1}{6}\end{pmatrix}$ and $B=\begin{pmatrix}\frac{-1}{6} & \frac{-7}{6}\\ \frac{1}{6} & \frac{7}{6}\end{pmatrix}$, we have $(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$.

Thus, there exists a pair of 2×2 real matrices A and B such that

$(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$ for all positive integers n . 1A f.t.

----------(5)

[HKDSE 2019 M2 #11]

Systems of Linear Equations

1.

8. (a) The augmented matrix is $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 5 & 6 \end{array}\right)$		
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 6 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 \end{array}\right)$		
Let $z = t$, where t is a real number. Then $y = t - 2$ and $x = 2 - 2t$.	1A	OR Solution Set = $\{(2-2t, t-2, t) : t \in \mathbf{R}\}$
(b) Substitute $(x, y, z) = (2-2t, t-2, t)$ into the last equation:	1M	
$(2-2t)-(t-2)+\lambda(t)=4$		
$(\lambda - 3)t = 0$		
When $\lambda \neq 3$, $t = 0$.		
$\therefore$ $(x, y, z) = (2, -2, 0)$	1A	
When $\lambda = 3$, t can be any real number.		
$\therefore$ $(x, y, z) = (2-2t, t-2, t)$	1A	
Alternative Solution		
The augmented matrix is $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 5 & 6 \\ 1 & -1 & \lambda & 4 \end{array}\right)$		
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 3 & 6 \\ 0 & -2 & \lambda-1 & 4 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \lambda-3 & 0 \end{array}\right)$		
When $\lambda \neq 3$, $z = 0$.		
$\therefore$ $(x, y, z) = (2, -2, 0)$	1A	
When $\lambda = 3$, z can be any real number.		
$\therefore$ $(x, y, z) = (2-2t, t-2, t)$, where t is a real number.	1A	
	(5)	

[HKDSE 2012 M2 #8]

2.

9. (a) The augmented matrix is $\left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 2 & 1-2a & 2-b & a+4 \\ 3 & 1-3a & 3-ab & 4 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 0 & 1 & -b & a \\ 0 & 1 & -ab & -2 \end{array}\right)$ 1M

$\sim \left(\begin{array}{ccc|c} 1 & -a & 1 & 2 \\ 0 & 1 & -b & a \\ 0 & 0 & ab-b & a+2 \end{array}\right)$

Hence the system has infinitely many solutions when

$\begin{cases} b(a-1)=0 \\ a+2=0 \end{cases}$ 1M

i.e. $a=-2$ and $b=0$ 1A For both

(b) The system becomes $\begin{cases} x + 2y + z = 2 \\ 2x + 5y + 2z = 2 \\ 3x + 7y + 3z = 4 \end{cases}$ 1M

i.e. $\begin{cases} x+z=6 \\ y=-2 \end{cases}$

$(x, y, z) = (6-t, -2, t)$ for any real number t 1A OR $(t, -2, 6-t)$

(5)

[HKDSE 2013 M2 #9]

3.

9. (a) The augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 1 & 6 & 10 & 200 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 5 & 9 & 100 \end{array}\right)$ — 1M

Let $z = t$, where t is a real number.

$\therefore \ y = 20 - \frac{9t}{5}$ and $x = 80 + \frac{4t}{5}$ — 1A+1A

> Alternative Solution
> Let $y = t$, where t is a real number.
> $\therefore \ z = \frac{100-5t}{9}$ and $x = \frac{800-4t}{9}$ — 1A+1A
>
> OR let $x = t$ and so $y = 200 - \frac{9t}{4}$, $z = \frac{5t}{4} - 100$

(b) $\begin{cases} m + n + k = 100 \\ 0.5m + 3n + 5k = 100 \end{cases}$ — 1A

$\therefore \ \begin{cases} m + n + k = 100 \\ m + 6n + 10k = 200 \end{cases}$

By (a), if both $20 - \frac{9t}{5}$ and $80 + \frac{4t}{5}$ are integers, then t is a multiple of 5.

$m \geq 0$ gives $t \geq -100$

$n \geq 0$ gives $t \leq \frac{100}{9}$

$k \geq 0$ gives $t \geq 0$

Combining all the conditions above, we have $t = 0$, 5 or 10. — 1M

> Alternative Solution (1)
> By trying out different values of t, we see that m, n and k are all non-negative when $t = 0$, 5 or 10 (OR any two of these). — 1M

> Alternative Solution (2)
> By trying out different values of t, we see that (m, n, k) can be $(80, 20, 0)$, $(84, 11, 5)$ or $(88, 2, 10)$ (OR any two of these). — 1M

Hence there are more than one set of combination of m, n and k and so Aubrey cannot be agreed with. — 1

(6)

[HKDSE 2014 M2 #9]

4.

5. (a)	The augmented matrix is $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \end{array}\right)$.	1M
	Thus, the solution set is $\{(2-6t, 5t, t) : t \in \mathbf{R}\}$.	1A
(b)	Putting $(x, y, z) = (2-6t, 5t, t)$ in the last equation, we have $3(2-6t)+2(5t)+kt=6$. So, we have $(k-8)t=0$.	1M
	We now consider the cases $k=8$ and $k \neq 8$. Case 1: $k=8$ The system of linear equations in (b) is equivalent to the system of linear equations in (a).	1M
	Thus, the solution set is $\{(2-6t, 5t, t) : t \in \mathbf{R}\}$.	1M
	Case 2: $k \neq 8$ So, we have $t=0$. Thus, the solution is $(2, 0, 0)$.	1A
	The augmented matrix is $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 2 & 3 & -3 & 4 \\ 3 & 2 & k & 6 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & k-8 & 0 \end{array}\right)$.	1M
	We now consider the cases $k=8$ and $k \neq 8$.	1M
	Case 1: $k=8$ In this case, the augmented matrix becomes $\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.	
	Thus, the solution set is $\{(2-6t, 5t, t) : t \in \mathbf{R}\}$.	1M
	Case 2: $k \neq 8$ The solution is $(2, 0, 0)$.	1A
		----------(6)

[HKDSE 2015 M2 #5]

5.

11. (a) (i) (1) Note that $\begin{vmatrix} 1 & 1 & -1 \\ 4 & 6 & a \\ 5 & 1-a & 3a-1 \end{vmatrix}$ $= 6(3a-1)+(a)(5)+(-1)(4)(1-a)-(a)(1-a)-4(3a-1)-(-1)(6)(5)$	1A	
$=(a+2)(a+12)$		
As (E) has a unique solution, we have $\begin{vmatrix} 1 & 1 & -1 \\ 4 & 6 & a \\ 5 & 1-a & 3a-1 \end{vmatrix} \neq 0$.	1M	
So, we have $(a+2)(a+12) \neq 0$.		
Thus, we have $a \neq -2$ and $a \neq -12$.	1	
The augmented matrix of (E) is $\left(\begin{array}{ccc\|c} 1 & 1 & -1 & 3 \\ 4 & 6 & a & b \\ 5 & 1-a & 3a-1 & b-1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & -1 & 3 \\ 0 & 2 & a+4 & b-12 \\ 0 & -a-4 & 3a+4 & b-16 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & -1 & 3 \\ 0 & 2 & a+4 & b-12 \\ 0 & 0 & (a+2)(a+12) & ab-12a+6b-80 \end{array}\right)$	1A	
As (E) has a unique solution, we have $(a+2)(a+12) \neq 0$.		
Thus, we have $a \neq -2$ and $a \neq -12$.	1	
(2) Since (E) has a unique solution, we have		
$x = \dfrac{\begin{vmatrix} 3 & 1 & -1 \\ b & 6 & a \\ b-1 & 1-a & 3a-1 \end{vmatrix}}{(a+2)(a+12)}$	1M	for Cramer's Rule
$= \dfrac{3a^2-ab+50a+6b-24}{(a+2)(a+12)}$		
$y = \dfrac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & b & a \\ 5 & b-1 & 3a-1 \end{vmatrix}}{(a+2)(a+12)}$		
$= \dfrac{2(ab-10a+8)}{(a+2)(a+12)}$		
$z = \dfrac{\begin{vmatrix} 1 & 1 & 3 \\ 4 & 6 & b \\ 5 & 1-a & b-1 \end{vmatrix}}{(a+2)(a+12)}$		
$= \dfrac{ab-12a+6b-80}{(a+2)(a+12)}$	1A+1A	1A for any one + 1A for all

Since (E) has a unique solution, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & 1 & -1 & 3 \\ 0 & 2 & a+4 & b-12 \\ 0 & 0 & (a+2)(a+12) & ab-12a+6b-80 \end{array}\right)$	1M	
$\sim\left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{3a^2-ab+50a+6b-24}{(a+2)(a+12)} \\ 0 & 1 & 0 & \frac{2(ab-10a+8)}{(a+2)(a+12)} \\ 0 & 0 & 1 & \frac{ab-12a+6b-80}{(a+2)(a+12)} \end{array}\right)$		
Thus, we have $\begin{cases} x=\frac{3a^2-ab+50a+6b-24}{(a+2)(a+12)} \\ y=\frac{2(ab-10a+8)}{(a+2)(a+12)} \\ z=\frac{ab-12a+6b-80}{(a+2)(a+12)} \end{cases}$	1A+1A	1A for any one + 1A for all

(ii) (1) When $a=-2$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 4 & 6 & -2 & b \\ 5 & 3 & -7 & b-1 \end{array}\right)\sim\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 2 & b-12 \\ 0 & 0 & 0 & 2b-28 \end{array}\right) \quad \text{1M}$$

Since (E) is consistent, we have $b=14$. 1A

either one

(2) When $a=-2$ and $b=14$, the augmented matrix of (E)

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the solution set of (E) is $\{(2t+2, 1-t, t): t\in\mathbf{R}\}$. 1A

------(9)

(b) Putting $a=-2$ and $b=14$, (E) becomes

$$\begin{cases} x+y-z=3 \\ 2x+3y-z=7 \\ 5x+3y-7z=13 \end{cases}$$

By (b)(ii), the solution set is $\{(2t+2, 1-t, t): t\in\mathbf{R}\}$. 1M

So, we have

$x^2+y^2-6z^2$

$=(2t+2)^2+(1-t)^2-6t^2$

$=-t^2+6t+5$

$=-(t^2-6t+3^2)+3^2+5$

$=-(t-3)^2+14$ 1M

Therefore, the greatest value of $x^2+y^2-6z^2$ is 14.

Thus, there is no real solution of the system of linear equations satisfying $x^2+y^2-6z^2>14$. 1A f.t.

------(3)

[HKDSE 2016 M2 #11]

6.

5. (a) (i) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & -11 \\ 2 & 3 & h \end{vmatrix} \neq 0$	1M
$8h-44-9+16+33-6h \neq 0$ $2h-4 \neq 0$ $h \neq 2$ $h<2$ or $h>2$	1A
(ii) z $= \dfrac{\begin{vmatrix} 1 & 2 & 11 \\ 3 & 8 & 49 \\ 2 & 3 & k \end{vmatrix}}{2h-4}$	1M
$= \dfrac{k-14}{h-2}$	1A
(b) When $h=2$, the augmented matrix of (E) is $\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & 2 & k \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right)$.	1M
Since (E) has infinitely many solutions, we have $h=2$ and $k=14$. Thus, the solution set is $\{(-7t-5, 4t+8, t) : t \in \mathbf{R}\}$.	1A
	----------(6)

[HKDSE 2017 M2 #5]

7.

11. (a) (i) (1) Note that

$$\begin{vmatrix} 1 & a & 4(a+1) \\ 2 & a-1 & 2(a-1) \\ 1 & -1 & -12 \end{vmatrix}$$

$= (a-1)(-12) + a(2)(a-1) + 4(a+1)(2)(-1) - 4(a-1)(a+1) + 2(a-1) - 2a(-12)$ — 1A

$= -2(a-3)(a+1)$

Since (E) has a unique solution, we have $\begin{vmatrix} 1 & a & 4(a+1) \\ 2 & a-1 & 2(a-1) \\ 1 & -1 & -12 \end{vmatrix} \neq 0$. — 1M

So, we have $-2(a-3)(a+1) \neq 0$.

Therefore, we have $a \neq 3$ and $a \neq -1$. — 1A

Thus, we have $a < -1$, $-1 < a < 3$ or $a > 3$.

The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 2 & a-1 & 2(a-1) & 20 \\ 1 & -1 & -12 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & -a-1 & -4a-16 & b-18 \end{array}\right)$$ — 1M

$$\sim \left(\begin{array}{ccc|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & 0 & 2a-6 & b-2 \end{array}\right)$$ — 1A

Since (E) has a unique solution, we have $2a-6 \neq 0$ and $-a-1 \neq 0$.

Therefore, we have $a \neq 3$ and $a \neq -1$. — 1A

Thus, we have $a < -1$, $-1 < a < 3$ or $a > 3$.

---

(2) Since (E) has a unique solution, we have

$$x = \frac{\begin{vmatrix} 18 & a & 4(a+1) \\ 20 & a-1 & 2(a-1) \\ b & -1 & -12 \end{vmatrix}}{-2(a-3)(a+1)}$$ — 1M (for Cramer's Rule)

$$= \frac{a^2b + ab + 10a - 2b - 50}{(a-3)(a+1)}$$

$$y = \frac{\begin{vmatrix} 1 & 18 & 4(a+1) \\ 2 & 20 & 2(a-1) \\ 1 & b & -12 \end{vmatrix}}{-2(a-3)(a+1)}$$

$$= \frac{-3ab + 22a - 5b - 38}{(a-3)(a+1)}$$

$$z = \frac{\begin{vmatrix} 1 & a & 18 \\ 2 & a-1 & 20 \\ 1 & -1 & b \end{vmatrix}}{-2(a-3)(a+1)}$$

$$= \frac{b-2}{2(a-3)}$$ — 1A+1A (1A for any one + 1A for all)

Solution	Marks	Remarks
Since (E) has a unique solution, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & a & 4(a+1) & 18 \\ 0 & -a-1 & -6a-10 & -16 \\ 0 & 0 & 2a-6 & b-2 \end{array}\right)$	1M	
$\sim\left(\begin{array}{ccc\|c} 1 & 0 & 0 & \dfrac{a^2b+ab+10a-2b-50}{(a-3)(a+1)} \\ 0 & 1 & 0 & \dfrac{-3ab+22a-5b-38}{(a-3)(a+1)} \\ 0 & 0 & 1 & \dfrac{b-2}{2(a-3)} \end{array}\right)$		
Thus, we have $\begin{cases} x=\dfrac{a^2b+ab+10a-2b-50}{(a-3)(a+1)} \\ y=\dfrac{-3ab+22a-5b-38}{(a-3)(a+1)} \\ z=\dfrac{b-2}{2(a-3)} \end{cases}$.	1A+1A	1A for any one + 1A for all

(ii) (1) When $a=3$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & 3 & 16 & 18 \\ 2 & 2 & 4 & 20 \\ 1 & -1 & -12 & b \end{array}\right)\sim\left(\begin{array}{ccc|c} 1 & 3 & 16 & 18 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & b-2 \end{array}\right)$$ 1M

Since (E) is consistent, we have $b=2$. 1A

either one

(2) When $a=3$ and $b=2$, the augmented matrix of (E)

$$\sim\left(\begin{array}{ccc|c} 1 & 3 & 16 & 18 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the solution set of (E) is $\{(5u+6, -7u+4, u):\ u\in\mathbf{R}\}$. 1A

----------(9)

(b) When $a=3$ and $b=s$, (E) becomes

$$(G):\begin{cases} x+3y+16z=18 \\ x+y+2z=10 \\ x-y-12z=s \end{cases}.$$

Since (F) is consistent, (G) is consistent.

By (a)(ii), we have $s=2$.

When $s=2$, the solution set of (G) is $\{(5u+6, -7u+4, u):\ u\in\mathbf{R}\}$. 1M

Therefore, we have $2(5u+6)-5(-7u+4)-45u=t$. 1M

Solving, we have $t=-8$. 1A for both correct

Thus, we have $s=2$ and $t=-8$.

----------(3)

[HKDSE 2018 M2 #11]

8.

6. (a) (i) Note that

$$\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix}$$

$= (\alpha)(2\alpha+1)+(-2)(\alpha)(7)+(-2)(5)(\alpha-3)-(\alpha)(\alpha-3)-(-2)(5)(2\alpha+1)-(-2)(\alpha)(7)$ — 1A

$= (\alpha+4)(\alpha+10)$

Since (E) has a unique solution, we have $\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix} \neq 0$. — 1M

Hence, we have $(\alpha+4)(\alpha+10) \neq 0$.

So, we have $\alpha \neq -4$ and $\alpha \neq -10$. — 1A

Thus, we have $\alpha < -10$, $-10 < \alpha < -4$ or $\alpha > -4$.

The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & \alpha & \alpha & 5\beta \\ 7 & \alpha-3 & 2\alpha+1 & 8\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & \alpha+10 & \alpha+10 & 0 \\ 0 & \alpha+11 & 2\alpha+15 & \beta \end{array}\right)$$ — 1M

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & \alpha+5 & \beta \\ 0 & 0 & (\alpha+4)(\alpha+10) & (\alpha+10)\beta \end{array}\right)$$ — 1A

Since (E) has a unique solution, we have $(\alpha+4)(\alpha+10) \neq 0$.

So, we have $\alpha \neq -4$ and $\alpha \neq -10$. — 1A

Thus, we have $\alpha < -10$, $-10 < \alpha < -4$ or $\alpha > -4$.

(ii) Since (E) has a unique solution, we have

$$y = \frac{\begin{vmatrix} 1 & \beta & -2 \\ 5 & 5\beta & \alpha \\ 7 & 8\beta & 2\alpha+1 \end{vmatrix}}{(\alpha+4)(\alpha+10)}$$ — 1M for Cramer's Rule

$$= \frac{-\beta}{\alpha+4}$$ — 1A

(b) When $\alpha = -4$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & -4 & -4 & 5\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{array}\right)$$ — 1M

Since (E) is inconsistent, we have $\beta \neq 0$. — 1A

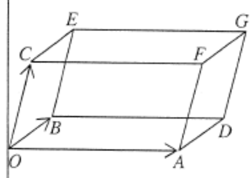
Thus, we have $\beta < 0$ or $\beta > 0$.

----------(7)

[HKDSE 2019 M2 #6]

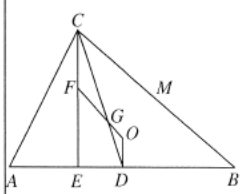
Vectors

1.

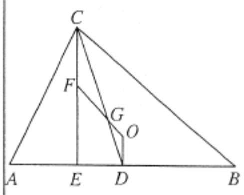
	Marks	Remarks
7. (a) The area of the parallelogram $OADB$ $= \|(6\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+\mathbf{j})\|$	1M	
$= \|\mathbf{i}-2\mathbf{j}+2\mathbf{k}\|$ $=\sqrt{1^2+2^2+2^2}$ $=3$	1A	
(b) The volume of the parallelepiped $OADBECFG$ $=(6\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i}-\mathbf{j}+2\mathbf{k})$	1M	
$=(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\cdot(5\mathbf{i}-\mathbf{j}+2\mathbf{k})$ $=1\cdot 5+(-2)(-1)+2\cdot 2$ $=11$		
Hence, the distance between point C and plane $OADB$ is $\frac{11}{3}$.	1M+1A	1M for height $=\frac{\text{volume}}{\text{base}}$
	(5)	

[HKDSE 2012 M2 #7]

2.

	Marks	Remarks
12. (a) Let $\overrightarrow{AG}=\frac{\overrightarrow{AC}+\lambda\overrightarrow{AD}}{1+\lambda}$. Since $\overrightarrow{AG}$ lies on a median, $\overrightarrow{AG}=k\frac{\overrightarrow{AC}+\overrightarrow{AB}}{2}=k\frac{\overrightarrow{AC}+2\overrightarrow{AD}}{2}$ for some k . Comparing the two expressions of $\overrightarrow{AG}$, we get $\lambda=2$.		
$\overrightarrow{AG}=\frac{\overrightarrow{AC}+2\overrightarrow{AD}}{3}$	1A	
$=\frac{\overrightarrow{AC}+\overrightarrow{AB}}{3}$ $=\frac{(\mathbf{c}-\mathbf{a})+(\mathbf{b}-\mathbf{a})}{3}$	1M	For tip-to-tail method
Alternative Solution 1 Let M be the mid-point of BC . $\overrightarrow{AG}=\frac{2}{3}\overrightarrow{AM}$	1A	
$=\frac{2}{3}\cdot\frac{\overrightarrow{AC}+\overrightarrow{AB}}{2}$ $=\frac{(\mathbf{c}-\mathbf{a})+(\mathbf{b}-\mathbf{a})}{3}$	1M	For tip-to-tail method
Alternative Solution 2 $\overrightarrow{AG}=\overrightarrow{OG}-\overrightarrow{OA}$	1M	
$=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}-\mathbf{a}$	1A	For $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$
$=\frac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3}$	1A	
	(3)	

Solution	Marks	Remarks
(b) (i) Since O is the circumcentre of the ΔABC , $OD \perp AB$.		
$\therefore \quad OD // CE$	1	
$\angle DOG = \angle CFG$ (alt. $\angle$s, $OD // CF$)		
$\angle ODG = \angle FCG$ (alt. $\angle$s, $OD // CF$)		
$\angle OGD = \angle FGC$ (vert. opp. $\angle$s)		
$\therefore \quad \Delta DOG \sim \Delta CFG$ (A.A.A.)	1	
$FG : GO = CG : GD$ (corr. sides, $\sim \Delta$s)		
$= 2:1$	1A	
(ii) $\overrightarrow{AG} = \dfrac{\overrightarrow{AF} + 2\overrightarrow{AO}}{3}$	1M	For using (b)(i)
$\overrightarrow{AF} = 3\overrightarrow{AG} - 2\overrightarrow{AO}$		
$= 3 \cdot \dfrac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3} - 2(-\mathbf{a})$	1M	For using (a)
Alternative Solution 1		
$\overrightarrow{AF} = \overrightarrow{AG} + \overrightarrow{GF}$		
$= \overrightarrow{AG} + 2\overrightarrow{OG}$	1M	For using (b)(i)
$= \dfrac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3} + 2 \cdot \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$	1M	For using (a)
Alternative Solution 2		
$\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF}$		
$= \overrightarrow{AC} + 2\overrightarrow{OD}$	1M	
$= \overrightarrow{AC} + \overrightarrow{OA} + \overrightarrow{OB}$	1M	
$= (\mathbf{c}-\mathbf{a}) + \mathbf{a} + \mathbf{b}$		
Alternative Solution 3		
$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA}$		
$= 3\overrightarrow{OG} - \overrightarrow{OA}$	1M	For using (b)(i)
$= 3 \cdot \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3} - \mathbf{a}$	1M	
$= \mathbf{b} + \mathbf{c}$	1	
$\overrightarrow{AF} \cdot \overrightarrow{BC} = (\mathbf{b}+\mathbf{c}) \cdot (\mathbf{c}-\mathbf{b})$	1M	
$= \lvert\mathbf{c}\rvert^2 - \lvert\mathbf{b}\rvert^2$		
$= 0$ ($\because$ O is the circumcentre)	1A	
$\therefore \quad AF \perp BC$		
$\therefore \quad AF$ is another altitude of ΔABC .		
Alternative Solution		
$\overrightarrow{BF} \cdot \overrightarrow{AC} = (\overrightarrow{BA} + \overrightarrow{AF}) \cdot \overrightarrow{AC}$	1M	
$= (\mathbf{a}-\mathbf{b}+\mathbf{b}+\mathbf{c}) \cdot (\mathbf{c}-\mathbf{a})$		
$= \lvert\mathbf{c}\rvert^2 - \lvert\mathbf{a}\rvert^2$		
$= 0$ ($\because$ O is the circumcentre)	1A	
$\therefore \quad BF \perp AC$		
$\therefore \quad BF$ is another altitude of ΔABC .		
$\therefore \quad F$ is the orthocentre of ΔABC .	1	
	(9)	



[HKDSE 2012 M2 #12]

3.

10. (a) $\overrightarrow{ON}=\dfrac{k\overrightarrow{OA}+\overrightarrow{OB}}{k+1}$ 1M

$=\dfrac{k(2\mathbf{i})+(\mathbf{i}+2\mathbf{j})}{k+1}$

$=\dfrac{(2k+1)\mathbf{i}+2\mathbf{j}}{k+1}$ 1A

(b) $\because$ $\overrightarrow{MB}=2\mathbf{j}$, $\therefore$ $BM\perp OA$ 1M

Since A , N , P and M are concyclic, $ON\perp AB$.

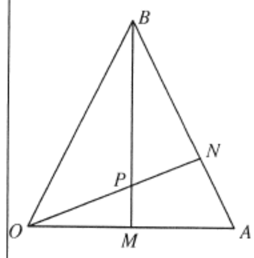
$\therefore$ $\overrightarrow{ON}\cdot\overrightarrow{AB}=0$ 1M

$\dfrac{(2k+1)\mathbf{i}+2\mathbf{j}}{k+1}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{i})=0$

$-(2k+1)+2\cdot 2=0$

$k=\dfrac{3}{2}$ 1A

(5)



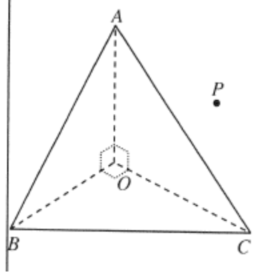
[HKDSE 2013 M2 #10]

4.

14. (a) (i) $\overrightarrow{AP}\cdot\overrightarrow{BP}=(\mathbf{p}-\mathbf{a})\cdot(\mathbf{p}-\mathbf{b})$ 1M For "tail minus head"

$=\mathbf{p}\cdot\mathbf{p}-\mathbf{a}\cdot\mathbf{p}-\mathbf{b}\cdot\mathbf{p}+\mathbf{a}\cdot\mathbf{b}$

$=\mathbf{p}\cdot\mathbf{p}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{p}$ $(\because\ \mathbf{a}\cdot\mathbf{b}=0)$ 1



(ii) $\overrightarrow{AP}\cdot\overrightarrow{BP}+\overrightarrow{BP}\cdot\overrightarrow{CP}+\overrightarrow{CP}\cdot\overrightarrow{AP}=0$

By (i) and some similar results, we have

$\mathbf{p}\cdot\mathbf{p}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{p}-(\mathbf{b}+\mathbf{c})\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{p}-(\mathbf{c}+\mathbf{a})\cdot\mathbf{p}=0$ 1M

$3\mathbf{p}\cdot\mathbf{p}-2(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{p}=0$

$3\mathbf{p}\cdot\mathbf{p}-2(3\mathbf{d})\cdot\mathbf{p}=0$ 1M

$\mathbf{p}\cdot\mathbf{p}=2\mathbf{p}\cdot\mathbf{d}$ -------------------------------- (*) 1

(iii) $|\mathbf{p}-\mathbf{d}|^2=(\mathbf{p}-\mathbf{d})\cdot(\mathbf{p}-\mathbf{d})$ 1M

$=\mathbf{p}\cdot\mathbf{p}-2\mathbf{p}\cdot\mathbf{d}+\mathbf{d}\cdot\mathbf{d}$

$=\mathbf{d}\cdot\mathbf{d}$ by (*)

$=|\mathbf{d}|^2$

Hence $|\mathbf{p}-\mathbf{d}|=|\mathbf{d}|$. 1

$\therefore$ $|\overrightarrow{DP}|=|\overrightarrow{OD}|$

$\therefore$ $PD=OD$ ------------------------------ (**)

Thus, the distance between P and D is a constant and therefore P lies on the sphere centred at D with fixed radius. 1

(8)

(b) (i) Yes. Since O satisfies (**), O lies on the sphere mentioned in (i). 1A

(ii) Yes. Since $\overrightarrow{DP_1}\times\overrightarrow{DP_2}=\overrightarrow{DP_2}\times\overrightarrow{DP_3}$, the normal vector of the plane containing D, P_1 and P_2 equals the normal vector of the plane containing D, P_2 and P_3. 1M

Thus the plane containing D , P_1 and P_2 parallel to the plane containing D , P_2 and P_3 .

Since D and P_2 are common points of the planes, D , P_1 , P_2 and P_3 are on the same plane. 1M

Since D is the centre of the sphere and P_1 , P_2 and P_3 lie on the largest circle on the sphere, the radius of the circle equals the radius of the sphere, which is OD . 1A Follow through

(4)

[HKDSE 2013 M2 #14]

5.

8. (a) $\overrightarrow{OP}\times\overrightarrow{OQ}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 2\\ 1 & -1 & 2\end{vmatrix}$

$=6\mathbf{i}+4\mathbf{j}-\mathbf{k}$ — 1A

The volume of tetrahedron $OPQR=\frac{1}{6}\left|(\overrightarrow{OP}\times\overrightarrow{OQ})\cdot\overrightarrow{OR}\right|$

$=\frac{1}{6}\left|(6\mathbf{i}+4\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})\right|$ — 1M

$=1$ — 1A

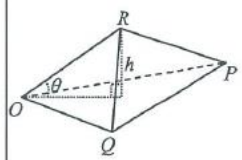
(b) $OR=\sqrt{2^2+(-3)^2+6^2}$

$=7$ — 1A

The area of $\Delta OPQ=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|$

$=\frac{1}{2}\sqrt{6^2+4^2+(-1)^2}$

$=\frac{\sqrt{53}}{2}$ — 1A



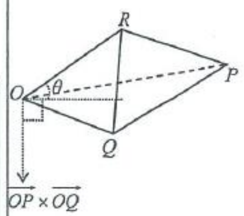
Let h be the height of the tetrahedron with OPQ as base.

$\therefore\ \frac{1}{3}\cdot\frac{\sqrt{53}}{2}h=1$ — 1M

$h=\frac{6}{\sqrt{53}}$

Let θ be the angle between the plane OPQ and the line OR.

$\therefore\ \sin\theta=\frac{\frac{6}{\sqrt{53}}}{7}$ — 1M



Alternative Solution

$\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|=\sqrt{6^2+4^2+(-1)^2}$

$=\sqrt{53}$ — 1A

Let θ be the angle between the plane OPQ and the line OR.

$\overrightarrow{OR}\cdot(\overrightarrow{OP}\times\overrightarrow{OQ})=\left|\overrightarrow{OR}\right|\cdot\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|\cos(\theta+90^\circ)$ — 1M+1M

1M for dot product formula
1M for $\theta+90^\circ$

$\cos(\theta+90^\circ)=\frac{2\cdot6-3\cdot4+6(-1)}{7\sqrt{53}}$

$\theta\approx6.8^\circ$ — 1A

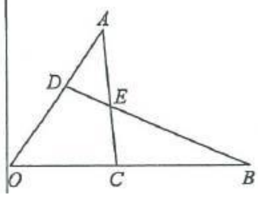
(8)

[HKDSE 2014 M2 #8]

6.

11. (a) (i) $\overrightarrow{OC} = t\mathbf{b}$

$\therefore \ \overrightarrow{OE} = \dfrac{\mathbf{a} + mt\mathbf{b}}{1+m}$ — 1M+1A



(ii) $\overrightarrow{OD} = (1-t)\mathbf{a}$

$\therefore \ \overrightarrow{OE} = \dfrac{n(1-t)\mathbf{a} + \mathbf{b}}{1+n}$ — 1A

(iii) Comparing (i) and (ii), we have

$$\begin{cases} \dfrac{1}{1+m} = \dfrac{n(1-t)}{1+n} & \text{------------------(1)} \\ \dfrac{mt}{1+m} = \dfrac{1}{1+n} & \text{------------------(2)} \end{cases}$$ — 1M

$(2) \div (1)$:

$$mt = \frac{1}{n(1-t)} \text{ ------------------(3)}$$ — 1M

By (1), $\dfrac{1}{1+\dfrac{1}{nt(1-t)}} = \dfrac{n(1-t)}{1+n}$

$t(1+n) = nt(1-t) + 1$

$t = -nt^2 + 1$

$n = \dfrac{1-t}{t^2}$ — 1

By (3), $mt = \dfrac{1}{\dfrac{1-t}{t^2}(1-t)}$

$m = \dfrac{t}{(1-t)^2}$ — 1

(iv) If $m = n$, then $\dfrac{t}{(1-t)^2} = \dfrac{1-t}{t^2}$. — 1A

$t^3 = (1-t)^3$

$t = \dfrac{1}{2}$

Hence C and D are the mid-points of OB and OA respectively.
Therefore, E is the centroid of ΔOAB and Chris is agreed with. — 1A

(9)

(b) $\overrightarrow{AC} = t\mathbf{b} - \mathbf{a}$

$\overrightarrow{AC} \cdot \overrightarrow{OB} = (t\mathbf{b} - \mathbf{a}) \cdot \mathbf{b}$

$= 4t - \mathbf{a} \cdot \mathbf{b}$ — 1A

When $AC \perp OB$, $\overrightarrow{AC} \cdot \overrightarrow{OB} = 0$ which gives $\mathbf{a} \cdot \mathbf{b} = 4t$ ------------------ (4) — 1M

$\overrightarrow{BD} = (1-t)\mathbf{a} - \mathbf{b}$

$\overrightarrow{BD} \cdot \overrightarrow{OA} = [(1-t)\mathbf{a} - \mathbf{b}] \cdot \mathbf{a}$

$= (1-t) - \mathbf{a} \cdot \mathbf{b}$ — 1A

By (4), $\overrightarrow{BD} \cdot \overrightarrow{OA} = 1 - 5t$.

So, $\overrightarrow{BD} \cdot \overrightarrow{OA} \neq 0$ in general.

i.e. BD is not always perpendicular to OA and Francis is not agreed with. — 1A

(4)

[HKDSE 2014 M2 #11]

7.

10. (a) (i) $\overrightarrow{OQ}$		
$=\frac{2}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}$	1M	any one
$\overrightarrow{OC}$		
$=(1-t)\overrightarrow{OP}+t\overrightarrow{OR}$		
$=(1-t)\left(\frac{1}{2}\overrightarrow{OA}\right)+t\left(\frac{3}{4}\overrightarrow{OB}\right)$		
$=\frac{1-t}{2}\overrightarrow{OA}+\frac{3t}{4}\overrightarrow{OB}$	1A	can be absorbed
Note that C lies on OQ.		
So, we have $\frac{1-t}{2}:\frac{2}{3}=\frac{3t}{4}:\frac{1}{3}$.	1M	
Thus, we have $t=\frac{1}{4}$.	1A	
(ii) By (a)(i), we have $\overrightarrow{OC}=\frac{3}{8}\overrightarrow{OA}+\frac{3}{16}\overrightarrow{OB}$.	1M	for using (a)(i)
Therefore, we have $\overrightarrow{OC}=\frac{9}{16}\overrightarrow{OQ}$.	1M	
Hence, we have $\left\lvert\overrightarrow{OC}\right\rvert=\frac{9}{16}\left\lvert\overrightarrow{OQ}\right\rvert$.		
So, we have $OC:OQ=9:16$.		
Thus, we have $CQ:OQ=7:16$.	1A	
	----------(7)	
(b) (i) The area of ΔOAB		
$=\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$	1M	
$=\frac{1}{2}\left\lVert\begin{matrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 20 & -6 & -12\\ 16 & -16 & 0\end{matrix}\right\rVert$		
$=\frac{1}{2}\lvert-192\mathbf{i}-192\mathbf{j}-224\mathbf{k}\rvert$		
$=176$	1A	
(ii) The volume of the tetrahedron $ABCD$		
$=\frac{1}{6}\left\lvert\overrightarrow{CD}\cdot(\overrightarrow{CA}\times\overrightarrow{CB})\right\rvert$	1M	
$=\left(\frac{1}{6}\right)\left(\frac{7}{16}\right)\left\lvert\overrightarrow{OD}\cdot(\overrightarrow{OA}\times\overrightarrow{OB})\right\rvert$ (by (a)(ii))	1M	for using the result of (a)(ii)
$=\frac{7}{96}\lvert(\mathbf{i}+3\mathbf{j}-6\mathbf{k})\cdot(-192\mathbf{i}-192\mathbf{j}-224\mathbf{k})\rvert$ (by (b)(i))		
$=\left(\frac{7}{96}\right)(576)$		
$=42$	1A	

Since $\overrightarrow{OC}=\frac{3}{8}\overrightarrow{OA}+\frac{3}{16}\overrightarrow{OB}$, we have $\overrightarrow{OC}=\frac{21}{2}\mathbf{i}-\frac{21}{4}\mathbf{j}-\frac{9}{2}\mathbf{k}$.	1M	
$\overrightarrow{CA}=\frac{19}{2}\mathbf{i}-\frac{3}{4}\mathbf{j}-\frac{15}{2}\mathbf{k}$		
$\overrightarrow{CB}=\frac{11}{2}\mathbf{i}-\frac{43}{4}\mathbf{j}+\frac{9}{2}\mathbf{k}$		
$\overrightarrow{CD}=\frac{-19}{2}\mathbf{i}+\frac{33}{4}\mathbf{j}-\frac{3}{2}\mathbf{k}$		
The volume of the tetrahedron $ABCD$		
$=\frac{1}{6}\left\|\overrightarrow{CD}\cdot(\overrightarrow{CA}\times\overrightarrow{CB})\right\|$	1M	
$=\frac{1}{6}\left\|\begin{vmatrix}\frac{-19}{2} & \frac{33}{4} & \frac{-3}{2}\\ \frac{19}{2} & \frac{-3}{4} & \frac{-15}{2}\\ \frac{11}{2} & \frac{-43}{4} & \frac{9}{2}\end{vmatrix}\right\|$		
$=\frac{252}{6}$		
$=42$	1A	
	----------(5)	

[HKDSE 2015 M2 #10]

8.

12. (a) $\left\|\overrightarrow{PA}\right\|=\left\|\overrightarrow{PB}\right\|$		
$\left\|\overrightarrow{OA}-\overrightarrow{OP}\right\|=\left\|\overrightarrow{OB}-\overrightarrow{OP}\right\|$	1M	
$\left\|-\mathbf{i}+(2-t)\mathbf{j}+2\mathbf{k}\right\|=\left\|3\mathbf{i}+(1-t)\mathbf{j}+\mathbf{k}\right\|$		
$\sqrt{(-1)^2+(2-t)^2+2^2}=\sqrt{3^2+(1-t)^2+1^2}$	1M	
$t^2-4t+9=t^2-2t+11$		
$t=-1$	1A	
	----------(3)	
(b) (i) $\overrightarrow{CA}\times\overrightarrow{CB}$		
$=(-2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-3\mathbf{k})$		
$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 3 & -2\\ 2 & 2 & -3\end{vmatrix}$		
$=-5\mathbf{i}-10\mathbf{j}-10\mathbf{k}$	1M	
$\left\|\overrightarrow{CA}\times\overrightarrow{CB}\right\|$		
$=\sqrt{(-5)^2+(-10)^2+(-10)^2}$		
$=15$		
A unit vector which is perpendicular to Π		
$=\frac{\overrightarrow{CA}\times\overrightarrow{CB}}{\left\|\overrightarrow{CA}\times\overrightarrow{CB}\right\|}$	1M	
$=\frac{-1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$	1A	accept $\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

(ii) Note that $\overrightarrow{CD} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Let $\mathbf{n} = \frac{-1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

$\overrightarrow{CD} \cdot \mathbf{n}$

$= (\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot \left(\frac{-1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$

$= \frac{-1}{3} - 2 - \frac{2}{3}$ 1M

$= -3$

Let θ be the angle between CD and Π .

Since $\overrightarrow{CD} \cdot \mathbf{n} < 0$, the angle between $\overrightarrow{CD}$ and $\mathbf{n}$ is $\frac{\pi}{2} + \theta$.

$\overrightarrow{CD} \cdot \mathbf{n} = \left|\overrightarrow{CD}\right| |\mathbf{n}| \cos\left(\frac{\pi}{2} + \theta\right)$ 1M

$3 = \sqrt{1^2 + 3^2 + 1^2} \sin\theta$

$\sin\theta = \frac{3\sqrt{11}}{11}$

$\theta = \sin^{-1}\left(\frac{3\sqrt{11}}{11}\right)$ 1A

Thus, the angle between CD and Π is $\sin^{-1}\left(\frac{3\sqrt{11}}{11}\right)$.

(iii) $\overrightarrow{DE}$

$= (\overrightarrow{DC} \cdot \mathbf{n})\mathbf{n}$ 1M

$= (3)\left(\frac{-1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$ (by (b)(ii))

$= -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{PF}$

$= \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$

$= (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + (\mathbf{i} + 4\mathbf{k})$

$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$

$\overrightarrow{FD}$

$= \overrightarrow{OD} - (\overrightarrow{OP} + \overrightarrow{PF})$

$= (3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) - ((\mathbf{i} - \mathbf{j}) + (3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}))$ 1M

$= -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

Therefore, we have $\overrightarrow{DE} = \overrightarrow{FD}$.

So, we have $DE // FD$ and $DE = DF$.

Hence, D , E and F are collinear and $DE = DF$. 1A+1A f.t.

Thus, D is the mid-point of the line segment joining E and F .

---------(10)

[HKDSE 2016 M2 #12]

9.

3. (a) $\overrightarrow{OP}$

$= \frac{2}{2+3}\mathbf{a} + \frac{3}{2+3}\mathbf{b}$

$= \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$ 1A

(b) (i) $\mathbf{a} \cdot \mathbf{b}$

$= |\mathbf{a}||\mathbf{b}|\cos\angle AOB$ 1M

$= (45)(20)\left(\frac{1}{4}\right)$

$= 225$ 1A

(ii) $\left|\overrightarrow{OP}\right|^2$

$= \left(\frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}\right) \cdot \left(\frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}\right)$

$= \frac{4}{25}|\mathbf{a}|^2 + 2\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\mathbf{a} \cdot \mathbf{b} + \frac{9}{25}|\mathbf{b}|^2$

$= 324 + 108 + 144$ 1M for using (b)(i)

$= 576$

$\left|\overrightarrow{OP}\right|$

$= \sqrt{576}$

$= 24$ 1A

----------(5)

[HKDSE 2017 M2 #3]

10.

10. (a) Note that $\overrightarrow{AB}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\overrightarrow{AC}=6\mathbf{i}-6\mathbf{j}$.		
$\overrightarrow{AE}$		
$=\frac{1}{1+r}\overrightarrow{AC}+\frac{r}{1+r}\overrightarrow{AB}$	1M	any one
$=\frac{2r+6}{r+1}\mathbf{i}+\frac{r-6}{r+1}\mathbf{j}+\frac{r}{r+1}\mathbf{k}$		
Also note that $\overrightarrow{AE}=\frac{1}{11}\overrightarrow{AF}+\frac{10}{11}\overrightarrow{AD}$ and $\overrightarrow{AC}=2\overrightarrow{AD}$.		
$\overrightarrow{AF}$		
$=11\overrightarrow{AE}-5\overrightarrow{AC}$		
$=\frac{-8r+36}{r+1}\mathbf{i}+\frac{41r-36}{r+1}\mathbf{j}+\frac{11r}{r+1}\mathbf{k}$	1A	for both
Since A , B and F are collinear, we have $\frac{2}{-8r+36}=\frac{1}{41r-36}=\frac{1}{11r}$.	1M	
Solving, we have $r=\frac{6}{5}$.	1A	1.2
	----------(4)	
(b) (i) Note that $\overrightarrow{AD}=\frac{1}{2}\overrightarrow{AC}=3\mathbf{i}-3\mathbf{j}$.		
By (a), we have $\overrightarrow{AE}=\frac{1}{11}(42\mathbf{i}-24\mathbf{j}+6\mathbf{k})$.	1M	for using (a)
$\overrightarrow{AD}\cdot\overrightarrow{DE}$		
$=\overrightarrow{AD}\cdot(\overrightarrow{AE}-\overrightarrow{AD})$		
$=(3\mathbf{i}-3\mathbf{j})\cdot\left(\frac{1}{11}(9\mathbf{i}+9\mathbf{j}+6\mathbf{k})\right)$		
$=0$	1A	
(ii) $\overrightarrow{AB}\cdot\overrightarrow{BC}$	1M	
$=\overrightarrow{AB}\cdot(\overrightarrow{AC}-\overrightarrow{AB})$		
$=(2\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}-7\mathbf{j}-\mathbf{k})$		
$=0$		
Therefore, we have $\angle ABC=90°=\angle ADE$.	1M	
So, we have $\angle CBF=90°=\angle CDF$.		
Thus, B , D , C and F are concyclic.	1A	f.t.
	----------(5)	
(c) Note that $\overrightarrow{AF}=12\mathbf{i}+6\mathbf{j}+6\mathbf{k}$ and $\overrightarrow{AP}=\mathbf{i}+7\mathbf{j}-2\mathbf{k}$.		
Since $\angle CBF=90°$, Q is the mid-point of CF .		
Therefore, we have $\overrightarrow{AQ}=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{AF})=9\mathbf{i}+3\mathbf{k}$.	1M	
The volume of the tetrahedron $ABPQ$		
$=\frac{1}{6}\left\|\overrightarrow{AQ}\cdot(\overrightarrow{AB}\times\overrightarrow{AP})\right\|$	1M	
$=\frac{1}{6}\left\|\begin{vmatrix}9&0&3\\2&1&1\\1&7&-2\end{vmatrix}\right\|$		
$=7$	1A	
	----------(3)	

[HKDSE 2017 M2 #10]

11.

12. (a) (i) Note that $\overrightarrow{AB}=-5\mathbf{i}+6\mathbf{j}-4\mathbf{k}$ and $\overrightarrow{AC}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$.	
$\overrightarrow{AB}\times\overrightarrow{AC}$ $=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -5 & 6 & -4\\ 3 & 2 & 4\end{vmatrix}$	
$=32\mathbf{i}+8\mathbf{j}-28\mathbf{k}$	1A
(ii) Note that $\overrightarrow{AD}=-\mathbf{i}+\mathbf{j}-6\mathbf{k}$. The required volume	
$=\frac{1}{6}\left\|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right\|$	1M
$=\frac{1}{6}\left\|(32\mathbf{i}+8\mathbf{j}-28\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}-6\mathbf{k})\right\|$	
$=\frac{1}{6}\left\|(32)(-1)+(8)(1)+(-28)(-6)\right\|$	
$=24$	1A
(iii) $\overrightarrow{DE}$	
$=\left(\overrightarrow{DA}\cdot\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)\left(\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)$	1M
$=\left((\mathbf{i}-\mathbf{j}+6\mathbf{k})\cdot\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)\left(\frac{32\mathbf{i}+8\mathbf{j}-28\mathbf{k}}{\sqrt{32^2+8^2+(-28)^2}}\right)$	
$=\frac{-32}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}+\frac{28}{13}\mathbf{k}$	1A
	----------(5)
(b) (i) Let $\overrightarrow{BF}=t\overrightarrow{BC}$, where $0<t<1$.	
$\overrightarrow{DF}$ $=(1-t)\overrightarrow{DB}+t\overrightarrow{DC}$	1M
$=(1-t)(-4\mathbf{i}+5\mathbf{j}+2\mathbf{k})+t(4\mathbf{i}+\mathbf{j}+10\mathbf{k})$	
$=(8t-4)\mathbf{i}+(5-4t)\mathbf{j}+(8t+2)\mathbf{k}$	
Since $DF\perp BC$, we have $\overrightarrow{DF}\cdot\overrightarrow{BC}=0$.	1M
Note that $\overrightarrow{BC}=8\mathbf{i}-4\mathbf{j}+8\mathbf{k}$.	
Hence, we have $(8t-4)(8)+(5-4t)(-4)+(8t+2)(8)=0$.	
So, we have $144t-36=0$.	
Solving, we have $t=\frac{1}{4}$.	
Thus, we have $\overrightarrow{DF}=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.	1A

(ii) $\overrightarrow{EF}$		
$=\overrightarrow{DF}-\overrightarrow{DE}$	1M	
$=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}-\left(\frac{-32}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}+\frac{28}{13}\mathbf{k}\right)$		
$=\frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}$		
$\overrightarrow{BC}\cdot\overrightarrow{EF}$		
$=(8\mathbf{i}-4\mathbf{j}+8\mathbf{k})\cdot\left(\frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}\right)$		
$=8\left(\frac{6}{13}\right)-4\left(\frac{60}{13}\right)+8\left(\frac{24}{13}\right)$		
$=0$		
Thus, $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{EF}$.	1A	f.t.
	----------(5)	
(c) Note that the required angle is $\angle DFE$.	1M	for identifying the required angle
$\cos\angle DFE$		
$=\frac{\overrightarrow{DF}\cdot\overrightarrow{EF}}{\left\lvert\overrightarrow{DF}\right\rvert\left\lvert\overrightarrow{EF}\right\rvert}$	1M	
$=\frac{(-2\mathbf{i}+4\mathbf{j}+4\mathbf{k})\cdot\left(\frac{6}{13}\mathbf{i}+\frac{60}{13}\mathbf{j}+\frac{24}{13}\mathbf{k}\right)}{\sqrt{(-2)^2+4^2+4^2}\sqrt{\left(\frac{6}{13}\right)^2+\left(\frac{60}{13}\right)^2+\left(\frac{24}{13}\right)^2}}$		
$=\frac{324}{(6)(18\sqrt{13})}$		
$=\frac{3}{\sqrt{13}}$		
$=\frac{3\sqrt{13}}{13}$		
Thus, the required angle is $\cos^{-1}\left(\frac{3\sqrt{13}}{13}\right)$.	1A	
	----------(3)	

[HKDSE 2018 M2 #12]

12.

12. (a) $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$

$\left|\overrightarrow{OC} - \overrightarrow{OA}\right| = \left|\overrightarrow{OC} - \overrightarrow{OB}\right|$ 1M

$\left|-6\mathbf{i} - 8\mathbf{j} + (t-2)\mathbf{k}\right| = \left|-8\mathbf{j} + (t-8)\mathbf{k}\right|$

$\sqrt{(-6)^2 + (-8)^2 + (t-2)^2} = \sqrt{(-8)^2 + (t-8)^2}$ 1M

$t^2 - 4t + 104 = t^2 - 16t + 128$

$12t = 24$

$t = 2$ 1A

----------(3)

(b) $\overrightarrow{AB} \times \overrightarrow{AC}$

$= (-6\mathbf{i} + 6\mathbf{k}) \times (-6\mathbf{i} - 8\mathbf{j})$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 0 & 6 \\ -6 & -8 & 0 \end{vmatrix}$

$= ((0)(0) - (6)(-8))\mathbf{i} + ((6)(-6) - (6)(0))\mathbf{j} + ((-6)(-8) - (0)(-6))\mathbf{k}$ 1M

$= 48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}$ 1A

----------(2)

(c) The volume of the pyramid $OABC$

$= \frac{1}{6}\left|\overrightarrow{OA} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right|$ 1M

$= \frac{1}{6}\left|(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k})\right|$

$= \frac{1}{6}\left|(1)(48) + (-4)(-36) + (2)(48)\right|$

$= 48$ 1A

The volume of the pyramid $OABC$

$= \frac{1}{6}\left|\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})\right|$ 1M

$= \frac{1}{6}\left\|\begin{matrix} 1 & -4 & 2 \\ -5 & -4 & 8 \\ -5 & -12 & 2 \end{matrix}\right\|$

$= \frac{1}{6}\left|(1)(-4)(2) + (-4)(8)(-5) + (2)(-5)(-12) - (1)(8)(-12) - (-4)(-5)(2) - (2)(-4)(-5)\right|$

$= 48$ 1A

----------(2)

(d) (i) By (c), the volume of the pyramid $OABC$ is not equal to 0.

So, O does not lie on Π.

Therefore, $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ are non-zero vectors.

Hence, we have $p \neq 0$, $q \neq 0$ and $r \neq 0$.

Thus, we have $pqr \neq 0$. 1

(ii) $\overrightarrow{OD}$

$= \overrightarrow{OA} \cdot \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2} (\overrightarrow{AB} \times \overrightarrow{AC})$ 1M

$= \dfrac{(6)(48)}{48^2 + (-36)^2 + 48^2} (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k})$

$= \dfrac{288}{5\,904} (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k})$

$= \dfrac{96}{41}\mathbf{i} - \dfrac{72}{41}\mathbf{j} + \dfrac{96}{41}\mathbf{k}$ 1A

(iii) $\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ 1M

$((p-1)\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$

$48p - 48 - 144 - 96 = 0$

$p = 6$

$\overrightarrow{AQ} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ any one

$(-\mathbf{i} + (q+4)\mathbf{j} - 2\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$

$-48 - 36q - 144 - 96 = 0$ for all

$q = -8$

$\overrightarrow{AR} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

$(-\mathbf{i} + 4\mathbf{j} + (r-2)\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$

$-48 - 144 + 48r - 96 = 0$

$r = 6$ 1A

So, we have $\overrightarrow{OE} = \dfrac{1}{6}\mathbf{i} - \dfrac{1}{8}\mathbf{j} + \dfrac{1}{6}\mathbf{k}$.

By (b)(ii), we have $\overrightarrow{OE} = \dfrac{41}{576}\overrightarrow{OD}$.

Thus, D , E and O are collinear. 1A f.t.

(6)

[HKDSE 2019 M2 #12]