Mock Exam 2

Section A

1. *Reference: HKDSE Math M2 2013 Q2*

$$(1+ax)^n$$
$$= 1 + C_1^n(ax) + C_2^n(ax)^2 + C_3^n(ax)^3 + \cdots$$
$$= 1 + nax + \frac{n(n-1)a^2}{2}x^2 + \frac{n(n-1)(n-2)a^3}{6}x^3 + \cdots \qquad \text{1M}$$

$$\therefore \begin{cases} na = -16 \text{ .... (1)} \\ \dfrac{n(n-1)a^2}{2} + \dfrac{n(n-1)(n-2)a^3}{6} = -336 \text{ .... (2)} \end{cases} \qquad \text{1M}$$

From (1): $a = -\dfrac{16}{n}$ .... (3)

Substituting (3) into (2),

$$\frac{128n(n-1)}{n^2} - \frac{2048n(n-1)(n-2)}{3n^3} = -336$$
$$384n^2(n-1) - 2048n(n-1)(n-2) = -1008n^3$$
$$-656n^3 + 5760n^2 - 4096n = 0$$
$$-16n(n-8)(41n-32) = 0$$

Since n is a positive integer, $n = 8$. 1A

Substituting $n = 8$ into (3),

$$a = -\frac{16}{8}$$
$$= \underline{\underline{-2}} \qquad \text{1A}$$

(4)

2. *Reference: HKDSE Math M2 2016 Q2*

$$\frac{1}{x^{\frac{1}{3}}} - \frac{1}{(x+h)^{\frac{1}{3}}} = \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{1}{3}}(x+h)^{\frac{1}{3}}}$$
$$= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{x^{\frac{1}{3}}(x+h)^{\frac{1}{3}}} \times \frac{(x+h)^{\frac{2}{3}} + x^{\frac{1}{3}}(x+h)^{\frac{1}{3}} + x^{\frac{2}{3}}}{(x+h)^{\frac{2}{3}} + x^{\frac{1}{3}}(x+h)^{\frac{1}{3}} + x^{\frac{2}{3}}} \qquad \text{1M}$$
$$= \frac{(x+h) - x}{x^{\frac{1}{3}}(x+h)^{\frac{1}{3}}[(x+h)^{\frac{2}{3}} + x^{\frac{1}{3}}(x+h)^{\frac{1}{3}} + x^{\frac{2}{3}}]}$$
$$= \frac{h}{x^{\frac{1}{3}}(x+h) + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x(x+h)^{\frac{1}{3}}} \qquad 1$$

Smart Tips

$(a-b)(a^2+ab+b^2)$
$= a^3 - b^3$

$$\frac{d}{dx}\left(\frac{6}{x^{\frac{1}{3}}}\right) = \lim_{h\to 0}\left(\frac{1}{h}\right)\left[\frac{6}{(x+h)^{\frac{1}{3}}} - \frac{6}{x^{\frac{1}{3}}}\right] \qquad \text{1M}$$

$$= \lim_{h\to 0}\frac{-6h}{h[x^{\frac{1}{3}}(x+h) + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x(x+h)^{\frac{1}{3}}]} \quad \text{(by (a))} \qquad \text{1M}$$

$$= \lim_{h\to 0}\frac{-6}{x^{\frac{1}{3}}(x+h) + x^{\frac{2}{3}}(x+h)^{\frac{2}{3}} + x(x+h)^{\frac{1}{3}}}$$

$$= \frac{-6}{x^{\frac{4}{3}} + x^{\frac{4}{3}} + x^{\frac{4}{3}}}$$

$$= -\frac{2}{x^{\frac{4}{3}}} \qquad \text{1A}$$

(5)

3. (a) By long division,

$$\begin{array}{r|l}
 & x^5 - x^3 + x \\ \hline
x^2 + 1 & x^7 \\
 & \underline{x^7 + x^5} \\
 & -x^5 \\
 & \underline{-x^5 - x^3} \\
 & x^3 \\
 & \underline{x^3 + x} \\
 & -x
\end{array}$$

$$\therefore \quad \frac{x^7}{x^2+1} = \frac{(x^5 - x^3 + x)(x^2+1) - x}{x^2+1} \qquad \text{1M}$$

$$= x^5 - x^3 + x - \frac{x}{x^2+1} \qquad \text{1A}$$

Smart Tips
Dividend
= Quotient × Divisor
+ Remainder

(b) $$\int x^5 \ln(x^2+1)\,dx$$

$$= \int \ln(x^2+1)\,d\left(\frac{x^6}{6}\right)$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{6}\int x^6\,d[\ln(x^2+1)] \qquad \text{1M}$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{6}\int x^6 \times \frac{2x}{x^2+1}\,dx$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{3}\int \frac{x^7}{x^2+1}\,dx$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{3}\int\left(x^5 - x^3 + x - \frac{x}{x^2+1}\right)dx \quad \text{(by (a))} \qquad \text{1M}$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{3}\int (x^5 - x^3 + x)\,dx + \frac{1}{3}\int \frac{x}{x^2+1}\,dx$$

$$= \frac{x^6\ln(x^2+1)}{6} - \frac{1}{3}\int (x^5 - x^3 + x)\,dx + \frac{1}{6}\int \frac{d(x^2+1)}{x^2+1} \qquad \text{1M}$$

$$= \frac{x^6\ln(x^2+1)}{6} + \frac{\ln(x^2+1)}{6} - \frac{x^6}{18} + \frac{x^4}{12} - \frac{x^2}{6} + C\text{, where } C \text{ is a constant} \qquad \text{1A}$$

(6)

4. (a) Let $u = \pi - x$. Then $du = -dx$. 1M

When $x = 0$, $u = \pi$; when $x = \pi$, $u = 0$.

$$\begin{aligned}\int_0^\pi f(x)\,dx &= -\int_\pi^0 f(\pi - u)\,du \\ &= \int_0^\pi f(\pi - u)\,du \\ &= \int_0^\pi f(\pi - x)\,dx \qquad 1\end{aligned}$$

(b) Let $f(x) = 1 + \dfrac{\cos^3 x}{1+\sin x}$. By (a), we have

$$\int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx = \int_0^\pi \left[1 + \frac{\cos^3(\pi - x)}{1+\sin(\pi - x)}\right]dx$$

$$\int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx = \int_0^\pi \left(1 - \frac{\cos^3 x}{1+\sin x}\right)dx \qquad \text{1M}$$

$$2\int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx = \int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx + \int_0^\pi \left(1 - \frac{\cos^3 x}{1+\sin x}\right)dx \qquad \text{1M}$$

$$2\int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx = 2\int_0^\pi dx \qquad \text{1M}$$

$$\int_0^\pi \left(1 + \frac{\cos^3 x}{1+\sin x}\right)dx = \underline{\underline{\pi}} \qquad \text{1A}$$

(6)

5. *Reference: HKALE P. Math 1997 Paper 1 Q7*

(a)
$$\begin{aligned}\det A \times \det(A^{-1} - xI) &= \det[A(A^{-1} - xI)] \\ &= \det(I - xA) \qquad \text{1M} \\ &= \det[-x(A - x^{-1}I)] \\ &= (-x)^3 \det(A - x^{-1}I) \\ &= -x^3 \det(A - x^{-1}I) \qquad 1\end{aligned}$$

Smart Tips

For $n \times n$ matrix P and real number λ, $\det(\lambda P) = \lambda^n \det P$.

(b) (i)
$$\begin{aligned}\det(A - xI) &= \det\begin{pmatrix} 4-x & -11 & 12 \\ 0 & -x & 4 \\ -2 & 7 & -2-x \end{pmatrix} \\ &= x(4-x)(x+2) + 88 - 24x - 28(4-x) \\ &= -x^3 + 2x^2 + 12x - 24 \text{ ......(*)}\end{aligned}$$

When $x = 2$, $\det(A - xI) = -2^3 + 2(2)^2 + 12(2) - 24 = 0$. 1M

$\therefore$ $x - 2$ is a factor of (*).

$\therefore$ 2 is a root of $\det(A - xI) = 0$. 1

$\det(A - xI) = (x-2)(-x^2 + 12)$

$$\therefore \text{ Other roots } = \pm\sqrt{12} = \underline{\underline{\pm 2\sqrt{3}}} \qquad \text{1A}$$

(ii) $\det A = 88 - 112 = -24 \neq 0$

By (a), $\det(A^{-1} - xI) = 0$ if and only if $\det(A - x^{-1}I) = 0$. 1M

By (b)(i), $x^{-1} = 2$ or $\pm 2\sqrt{3}$

$$\therefore \underline{\underline{x = \frac{1}{2} \text{ or } \pm\frac{\sqrt{3}}{6}}} \qquad \text{1A}$$

(7)

6. *Reference: HKDSE Math M2 2016 Q6*

(a) $\cos 3\theta = \sin 2\theta$

$\cos 3\theta = \cos(90° - 2\theta)$

$3\theta = 90° - 2\theta$

$5\theta = 90°$

$\theta = \underline{\underline{18°}}$ 1A

(b)
$$\begin{aligned}
\cos 3\theta &= \sin 2\theta \\
\cos(2\theta + \theta) &= \sin 2\theta \\
\cos 2\theta \cos \theta - \sin 2\theta \sin \theta &= 2 \sin \theta \cos \theta && \text{1M} \\
(2 \cos^2 \theta - 1)\cos \theta - 2 \sin^2 \theta \cos \theta &= 2 \sin \theta \cos \theta \\
(2 \cos^2 \theta - 1) - 2 \sin^2 \theta &= 2 \sin \theta \quad (\cos \theta \neq 0 \text{ for } 0° < \theta < 45°) \\
2 \cos^2 \theta - 1 - 2(1 - \cos^2 \theta) &= 2 \sin \theta \\
4 \cos^2 \theta - 3 &= 2 \sin \theta \\
16 \cos^4 \theta - 24 \cos^2 \theta + 9 &= 4 \sin^2 \theta && \text{1M} \\
16 \cos^4 \theta - 24 \cos^2 \theta + 9 &= 4 - 4 \cos^2 \theta \\
16 \cos^4 \theta - 20 \cos^2 \theta + 5 &= 0 && \text{1}
\end{aligned}$$

(c) By (a), $\theta = 18°$. By (b), we have

$16 \cos^4 18° - 20 \cos^2 18° + 5 = 0$

$$\cos^2 18° = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(16)(5)}}{2(16)}$$

$$\cos^2 18° = \frac{5 + \sqrt{5}}{8} \quad \text{or} \quad \frac{5 - \sqrt{5}}{8} \quad \text{(rejected)} \qquad \text{1M}$$

$$\begin{aligned}
\sin 54° &= \cos 36° \\
&= \cos 2(18°) \\
&= 2\cos^2 18° - 1 && \text{1M} \\
&= 2\left(\frac{5 + \sqrt{5}}{8}\right) - 1 \\
&= \underline{\underline{\frac{1 + \sqrt{5}}{4}}} && \text{1A}
\end{aligned}$$

(7)

7. *Reference: HKDSE Math M2 2016 Q4*

(a) Note that $x = -2$ is the vertical asymptote. 1A

$$\begin{aligned}
f(x) &= \frac{x^2 + 5x + 1}{x + 2} \\
&= \frac{x(x + 2) + 3x + 1}{x + 2} \\
&= \frac{x(x + 2) + 3(x + 2) - 5}{x + 2} \\
&= x + 3 - \frac{5}{x + 2} && \text{1M}
\end{aligned}$$

Note that $\frac{5}{x + 2} \to 0$ when $x \to \pm\infty$.

$\therefore$ $y = x + 3$ is the oblique asymptote. 1A

(b) $f(0) = \dfrac{1}{2}$

$f'(x) = 1 + \dfrac{5}{(x+2)^2}$ 1M

$f'(0) = 1 + \dfrac{5}{(0+2)^2} = \dfrac{9}{4}$ 1M

Slope of the normal $= -\dfrac{4}{9}$ 1M

The equation of the normal is $y = -\dfrac{4}{9}x + \dfrac{1}{2}$. 1A

(7)

8. *Reference: HKDSE Math M2 2015 Q8*

(a) When $n = 1$,

L.H.S. $= \sin x \cos 2(1)x = \sin x \cos 2x$

R.H.S. $= \sin x \cos (1 + 1)x = \sin x \cos 2x$

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = m$, where m is a positive integer, that is,

$$\sin x \sum_{k=1}^{m} \cos 2kx = \sin mx \cos(m+1)x \ , \quad \text{1M}$$

when $n = m + 1$,

$$\sin x \sum_{k=1}^{m+1} \cos 2kx$$

$$= \sin x \sum_{k=1}^{m} \cos 2kx + \sin x \cos 2(m+1)x$$

$$= \sin mx \cos(m+1)x + \sin x \cos 2(m+1)x \quad \text{(by the assumption)} \quad \text{1M}$$

$$= \frac{\sin(2m+1)x + \sin(-x)}{2} + \frac{\sin(2m+3)x + \sin(-2m-1)x}{2} \quad \text{1M}$$

$$= \frac{\sin(2m+3)x - \sin x}{2} \quad \text{1M}$$

$$= \cos\left(\frac{2m+4}{2}\right)x \sin\left(\frac{2m+2}{2}\right)x$$

$$= \sin(m+1)x \cos[(m+1)+1]x$$

$\therefore$ The proposition is true for $n = m + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(b) Let $x = \dfrac{\pi}{18}$ and $n = 2025$.

By (a), $\sin\dfrac{\pi}{18}\sum_{k=1}^{2025}\cos\dfrac{k\pi}{9} = \sin\dfrac{2025\pi}{18}\cos\dfrac{2026\pi}{18}$

$$\sum_{k=1}^{2025}\cos\frac{k\pi}{9} = \frac{\sin\left(112\pi+\frac{\pi}{2}\right)\cos\left(112\pi+\frac{5\pi}{9}\right)}{\sin\frac{\pi}{18}} \qquad \text{1M}$$

$$= \frac{\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{18}\right)}{\sin\frac{\pi}{18}}$$

$$= \frac{-\sin\frac{\pi}{18}}{\sin\frac{\pi}{18}}$$

$$= \underline{\underline{-1}} \qquad \text{1A}$$

(8)

Section B

9. *Reference: HKDSE Math M2 2016 Q9*

(a) Since $(0, -1)$ is a point on C,

$$f(0) = -1 \qquad \text{1M}$$

$$e^0 + \frac{a}{e^0} + b(0) = -1$$

$$1 + a = -1$$

$$a = \underline{\underline{-2}} \qquad \text{1A}$$

$$f'(x) = e^x - \frac{a}{e^x} + b = e^x + \frac{2}{e^x} + b \qquad \text{1M}$$

Since $(0, -1)$ is a stationary point of C,

$$f'(0) = 0$$

$$e^0 + \frac{2}{e^0} + b = 0$$

$$b = \underline{\underline{-3}}$$

(3)

(b) By (a), $f'(x) = e^x + \dfrac{2}{e^x} - 3$

$$f''(x) = e^x - \frac{2}{e^x}$$

$$f''(0) = e^0 - \frac{2}{e^0} = 1 - 2 = -1 < 0 \qquad \text{1M}$$

$\therefore$ P is a maximum point of C.

$\therefore$ The claim is agreed. 1A

(2)

(c) When $f'(x) = 0$,

$$e^x + \frac{2}{e^x} - 3 = 0$$
$$(e^x)^2 - 3e^x + 2 = 0$$
$$(e^x - 1)(e^x - 2) = 0$$
$$e^x = 1 \text{ or } 2 \qquad \text{1M}$$
$$x = 0 \text{ or } \ln 2$$

$f''(\ln 2) = 2 - \frac{2}{2} = 1 > 0$ 1M

$\therefore$ C has a minimum point at $x = \ln 2$.

$f(\ln 2) = 2 - \frac{2}{2} - 3\ln 2 = 1 - 3\ln 2$

$\therefore$ The coordinates of Q are $(\ln 2,\ 1 - 3\ln 2)$. 1A

(3)

(d) When $f''(x) = 0$,

$$e^x - \frac{2}{e^x} = 0$$
$$e^x = \frac{2}{e^x}$$
$$e^{2x} = 2$$
$$x = \frac{\ln 2}{2}$$

x	$x < \frac{\ln 2}{2}$	$x = \frac{\ln 2}{2}$	$x > \frac{\ln 2}{2}$
$f''(x)$	$-$	0	$+$

1M

$f\left(\frac{\ln 2}{2}\right) = \sqrt{2} - \frac{2}{\sqrt{2}} - \frac{3\ln 2}{2} = -\frac{3\ln 2}{2}$

$\therefore$ $\left(\frac{\ln 2}{2},\ -\frac{3\ln 2}{2}\right)$ is the point of inflexion. 1A

(2)

(e) Note that the equation of L is $y = 1 - 3\ln 2$.

$$\text{Area} = \int_0^{\ln 2}\left[\left(e^x - \frac{2}{e^x} - 3x\right) - (1 - 3\ln 2)\right]dx \qquad \text{1M}$$
$$= \left[e^x + \frac{2}{e^x} - \frac{3x^2}{2} - (1 - 3\ln 2)x\right]_0^{\ln 2} \qquad \text{1M}$$
$$= \underline{\underline{\frac{3(\ln 2)^2}{2} - \ln 2}} \qquad \text{1A}$$

(3)

10. *Reference: HKCEE A. Math 1992 Paper 2 Q11*

(a)
$$PA + PB = 10$$
$$\sqrt{(x+4)^2 + (y-0)^2} + \sqrt{(x-4)^2 + (y-0)^2} = 10$$
$$\sqrt{x^2 + 8x + 16 + y^2} + \sqrt{x^2 - 8x + 16 + y^2} = 10$$
$$\sqrt{x^2 + 8x + 16 + y^2} = 10 - \sqrt{x^2 - 8x + 16 + y^2} \quad \text{1M}$$
$$x^2 + 8x + 16 + y^2 = 100 - 20\sqrt{x^2 - 8x + 16 + y^2} + x^2 - 8x + 16 + y^2$$
$$20\sqrt{x^2 - 8x + 16 + y^2} = 100 - 16x$$
$$5\sqrt{x^2 - 8x + 16 + y^2} = 25 - 4x \quad \text{1M}$$
$$25(x^2 - 8x + 16 + y^2) = 625 - 200x + 16x^2$$
$$25x^2 - 200x + 400 + 25y^2 = 625 - 200x + 16x^2$$
$$9x^2 + 25y^2 = 225$$
$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \text{1}$$

(3)

(b) (i) Let V cubic units be the volume of water.

$$V = \pi\int_{-3}^{-3+h} 25\left(1 - \frac{y^2}{9}\right)dy \quad \text{1M}$$
$$= \frac{25}{9}\pi\int_{-3}^{-3+h}(9 - y^2)\,dy$$
$$= \frac{25}{9}\pi\left[9y - \frac{y^3}{3}\right]_{-3}^{-3+h} \quad \text{1A}$$
$$= \frac{25}{9}\pi\left[9(-3+h) - \frac{(-3+h)^3}{3} + 18\right]$$
$$= \underline{\underline{\frac{25}{27}\pi(9h^2 - h^3)}} \quad \text{1A}$$

Smart Tips

Note that when $x = 0$, $y = \pm 3$. Therefore the lower limit is -3. Furthermore, since h is the depth, the upper limit is $-3 + h$.

(ii)
$$\frac{dV}{dt} = \frac{25}{27}\pi(18h - 3h^2)\frac{dh}{dt} \quad \text{1M}$$
$$= \frac{25}{9}\pi(6h - h^2)\frac{dh}{dt}$$
$$30 - 5 = \frac{25}{9}\pi(6h - h^2)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{9}{\pi(6h - h^2)} \quad \text{1A}$$

When $\frac{dh}{dt}$ is minimum, $6h - h^2$ is maximum.

Consider
$$6h - h^2 = -h^2 + 6h$$
$$= -(h^2 - 6h)$$
$$= -\left[h^2 - 6h + \left(\frac{6}{2}\right)^2\right] + \left(\frac{6}{2}\right)^2$$
$$= -(h-3)^2 + 9 \quad \text{1M}$$

$\therefore$ When $h = 3$, $6h - h^2$ is the maximum.

$\therefore$ When the rate of change of the depth of water is minimum, the depth of water is 3 units. 1A

Smart Tips

The rate of change of volume of water = The rate of change of volume of water poured into the container − The rate of change of water evaporated.

(iii) $\dfrac{dV}{dt} = -\dfrac{\pi}{800}(t + 100)$

$V = -\dfrac{\pi}{800}\int (t + 100)\, dt$ 1M

$= -\dfrac{\pi t^2}{1600} - \dfrac{\pi t}{8} + C$, where C is a constant

By (b)(ii), when $t = 0$, $h = 3$.

$V = \dfrac{25}{27}\pi[9(3)^2 - 3^3] = 50\pi$

$\therefore$ $C = 50\pi$ 1A

$V = -\dfrac{\pi t^2}{1600} - \dfrac{\pi t}{8} + 50\pi$

When $V = 0$,

$-\dfrac{\pi t^2}{1600} - \dfrac{\pi t}{8} + 50\pi = 0$ 1M

$t^2 + 200t - 80\ 000 = 0$

$(t - 200)(t + 400) = 0$

$t = 200$ or -400 (rejected)

$\therefore$ The time required is 200 minutes. 1A

(11)

11. (a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 3 & 2 & 1 & c \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & -1 & 1 & c-3a \end{array}\right) \quad (R_2 - R_1 \to R_2;\ R_3 - 3R_1 \to R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 0 & c-b-2a \end{array}\right) \quad (R_3 - R_2 \to R_3)$$ 1M

$\therefore$ If $c = 2a + b$, the augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & 0 & 0 \end{array}\right)$. 1M

$\therefore$ (E) is consistent. 1

Let $z = t$, where t is any real number.

Then $x = b - t$, $y = a - b + t$. 1A

(4)

(b) Since (F) is consistent, the system of linear equations formed by the first 3 equations of (F) is also consistent, which is equivalent to (E) for $a = 2$, $b = 2$ and $c = \alpha$.

By (a), $\alpha = 2(2) + 2 = \underline{\underline{6}}$, 1A

and $x = 2 - t$, $y = t$, $z = t$, where t is any real number.

Substituting $x = 2 - t$, $y = t$ and $z = t$ into $2x + 3y - z = \beta$,

$2(2 - t) + 3t - t = \beta$ 1M

$\beta = \underline{\underline{4}}$ 1A

(3)

(c) Since (G) is consistent, by (a), we have

$$2p + q = 7$$

$$q = 7 - 2p \text{ ...... } (*)$$

Let $f = x^3 + y^2 + z$.

$$f = (q - t)^3 + (p - q + t)^2 + t$$

$$= (7 - 2p - t)^3 + (3p + t - 7)^2 + t \quad \text{(from (*))}$$ 1M

$$\frac{df}{dt} = -3(7 - 2p - t)^2 + 2(3p + t - 7) + 1$$ 1M

$$\frac{d^2 f}{dt^2} = 6(7 - 2p - t) + 2$$

$$= -12p - 6t + 44$$

Since f attains its local minimum at $z = 2$ (i.e., $t = 2$), $\left.\frac{df}{dt}\right|_{t=2} = 0$.

$$-3(7 - 2p - 2)^2 + 2(3p + 2 - 7) + 1 = 0$$

$$-3(5 - 2p)^2 + 2(3p - 5) + 1 = 0$$

$$-12p^2 + 66p - 84 = 0$$

$$-6(p - 2)(2p - 7) = 0$$

$$p = 2 \text{ or } \frac{7}{2}$$ 1M

When $p = 2$, $\left.\frac{d^2 f}{dt^2}\right|_{t=2} = -12(2) - 6(2) + 44 = 8 > 0$, which gives a local minimum.

1M

When $p = \frac{7}{2}$, $\left.\frac{d^2 f}{dt^2}\right|_{t=2} = -12\left(\frac{7}{2}\right) - 6(2) + 44 = -10 < 0$, which gives a local maximum.

$\therefore \quad p = \underline{\underline{2}}$ 1A

$q = 7 - 2(2)$

$= \underline{\underline{3}}$

(5)

12. *Reference: HKDSE Math M2 2016 Q12*

(a) Since OC is the angle bisector of $\angle AOB$, O, A, B and C lie on the same plane.

$$\begin{vmatrix} 3 & 4 & 0 \\ \frac{32}{5} & -\frac{24}{5} & 6 \\ m & n & 15 \end{vmatrix} = 0$$ 1M

$$24m - 18n - 600 = 0$$

$$4m - 3n - 100 = 0 \text{ ...... (1)}$$

Smart Tips

When O, A, B and C lie on the same plane, the volume of the parallelepiped formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is 0, i.e., $\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC}) = 0$.

Also, $\angle AOC = \angle BOC$.

$$\cos\angle AOC = \cos\angle BOC$$

$$\frac{\overrightarrow{OA}\cdot\overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|} = \frac{\overrightarrow{OB}\cdot\overrightarrow{OC}}{|\overrightarrow{OB}||\overrightarrow{OC}|}$$

$$\frac{(3\mathbf{i}+4\mathbf{j})\cdot(m\mathbf{i}+n\mathbf{j}+15\mathbf{k})}{5\sqrt{m^2+n^2+15^2}} = \frac{\left(\frac{32}{5}\mathbf{i}-\frac{24}{5}\mathbf{j}+6\mathbf{k}\right)\cdot(m\mathbf{i}+n\mathbf{j}+15\mathbf{k})}{10\sqrt{m^2+n^2+15^2}} \qquad \text{1M}$$

$$\frac{3m+4n}{5} = \frac{\frac{32m}{5}-\frac{24n}{5}+90}{10}$$

$$30m+40n = 32m-24n+450$$

$$m = 32n-225 \text{ ...... (2)}$$

Substituting (2) into (1),

$$4(32n-225)-3n-100=0$$

$$125n = 1000$$

$$n = \underline{\underline{8}} \qquad \text{1A}$$

Substituting $n = 8$ into (2),

$$m = 32(8)-225$$

$$= \underline{\underline{31}} \qquad \text{1A}$$

(4)

(b) (i) $\overrightarrow{OA}\times\overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ \frac{32}{5} & -\frac{24}{5} & 6 \end{vmatrix}$

$= 24\mathbf{i}-18\mathbf{j}-40\mathbf{k}$

Note that $\overrightarrow{DE} \,//\, \overrightarrow{OA}\times\overrightarrow{OB}$.

$\therefore$ $\overrightarrow{DE} = 24t\mathbf{i}-18t\mathbf{j}-40t\mathbf{k}$, where t is a non-zero real number.

$\overrightarrow{OE} = \overrightarrow{OD}+\overrightarrow{DE}$

$$= 24t\mathbf{i}+\left(\frac{15}{8}-18t\right)\mathbf{j}+\left(\frac{17}{3}-40t\right)\mathbf{k} \qquad \text{1M}$$

Since $\overrightarrow{DE}\perp\overrightarrow{OE}$,

$$\overrightarrow{DE}\cdot\overrightarrow{OE} = 0$$

$$(24t)^2-18t\left(\frac{15}{8}-18t\right)-40t\left(\frac{17}{3}-40t\right)=0 \qquad \text{1M}$$

$$2500t^2-\frac{3125t}{12}=0$$

$$t = \frac{5}{48} \text{ or } 0 \text{ (rejected)}$$

$$\therefore \quad \overrightarrow{DE} = 24\left(\frac{5}{48}\right)\mathbf{i}-18\left(\frac{5}{48}\right)\mathbf{j}-40\left(\frac{5}{48}\right)\mathbf{k}$$

$$= \underline{\underline{\frac{5}{2}\mathbf{i}-\frac{15}{8}\mathbf{j}-\frac{25}{6}\mathbf{k}}} \qquad \text{1A}$$

(ii) $\overrightarrow{OE} = 24\left(\frac{5}{48}\right)\mathbf{i} + \left[\frac{15}{8} - 18\left(\frac{5}{48}\right)\right]\mathbf{j} + \left[\frac{17}{3} - 40\left(\frac{5}{48}\right)\right]\mathbf{k}$

$= \frac{5}{2}\mathbf{i} + \frac{3}{2}\mathbf{k}$

From (a), $\overrightarrow{OC} = 31\mathbf{i} + 8\mathbf{j} + 15\mathbf{k}$.

Note that when E is the incentre of ΔOAB, E lies on the angle bisector of $\angle AOB$, i.e., E lies on OC.

Since the $\mathbf{j}$ component of $\overrightarrow{OE}$ is 0 while that of $\overrightarrow{OC}$ is not, it is impossible to find a real number λ such that $\overrightarrow{OE} = \lambda\overrightarrow{OC}$. 1M

$\therefore$ E does not lie on OC.

$\therefore$ The claim is disagreed. 1A

Analysis
When E is the incentre, E is the point of intersection of the three angle bisectors of the triangle.

(iii) $\left|\overrightarrow{OE}\right| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

$= \frac{\sqrt{34}}{2}$

$\overrightarrow{EA} = \overrightarrow{OA} - \overrightarrow{OE}$

$= \frac{1}{2}\mathbf{i} + 4\mathbf{j} - \frac{3}{2}\mathbf{k}$

$\left|\overrightarrow{EA}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + 4^2 + \left(-\frac{3}{2}\right)^2}$

$= \frac{\sqrt{74}}{2}$

Since $\left|\overrightarrow{EA}\right| \neq \left|\overrightarrow{OE}\right|$, E is not the circumcentre of ΔOAB. 1M

$\therefore$ The claim is disagreed. 1A

(7)

Analysis
When E is the circumcentre of ΔOAB, a circle with E as the centre and passing through O, A and B can be drawn, i.e., $EO = EA = EB$.