# HKDSE MATH M2 2018

1. **HKDSE Math M2 2018 Q1**
   Let $f(x) = (x^2 - 1)e^x$. Express $f(1+h)$ in terms of $h$. Hence, find $f'(1)$ from first principles.
   (4 marks)

2. **HKDSE Math M2 2018 Q2**
   Expand $(x+3)^5$. Hence, find the coefficient of $x^3$ in the expansion of $(x+3)^5 \left(x - \dfrac{4}{x}\right)^2$.
   (5 marks)

3. **HKDSE Math M2 2018 Q3**

   (a) If $\cot A = 3\cot B$, prove that $\sin(A+B) = 2\sin(B-A)$.

   (b) Using (a), solve the equation $\cot(x + \frac{4\pi}{9}) = 3\cot(x + \frac{5\pi}{18})$, where $0 \le x \le \frac{\pi}{2}$.

   (5 marks)

4. **HKDSE Math M2 2018 Q4**

   (a) Using integration by parts, find $\displaystyle\int u(5^u)\,du$.

   (b) Define $f(x) = x(5^{2x})$ for all real numbers $x$. Find the area of the region bounded by the graph of $y = f(x)$, the straight line $x = 1$ and the $x$-axis.

   (6 marks)

5. **HKDSE Math M2 2018 Q5**

   (a) Using integration by substitution, find $\displaystyle\int x^3\sqrt{1+x^2}\,dx$.

   (b) At any point $(x, y)$ on the curve $\Gamma$, the slope of the tangent to $\Gamma$ is $15x^3\sqrt{1+x^2}$. The $y$-intercept of $\Gamma$ is 2. Find the equation of $\Gamma$.

   (7 marks)

6. **HKDSE Math M2 2018 Q6**

   (a) Using mathematical induction, prove that $\displaystyle\sum_{k=1}^{n} k(k+4) = \frac{n(n+1)(2n+13)}{6}$ for all positive integers $n$.

   (b) Using (a), evaluate $\displaystyle\sum_{k=333}^{555} \left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$.

   (7 marks)

7. **HKDSE Math M2 2018 Q7**
   Let $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be a non-zero real matrix such that $MX = XM$.

(a) Express $b$ and $c$ in terms of $a$.

(b) Prove that $X$ is a non-singular matrix.

(c) Denote the transpose of $X$ by $X^T$. Express $(X^T)^{-1}$ in terms of $a$.

(8 marks)

8. **HKDSE Math M2 2018 Q8**

Define $f(x) = \dfrac{A}{x^2 - 4x + 7}$ for all real numbers $x$, where $A$ is a constant. It is given that the extreme value of $f(x)$ is 4.

(a) Find $f'(x)$.

(b) Someone claims that there are at least two asymptotes of the graph of $y = f(x)$. Do you agree? Explain your answer.

(c) Find the point(s) of inflexion of the graph of $y = f(x)$.

(8 marks)

9. **HKDSE Math M2 2018 Q9**

Consider the curve $C : y = \ln \sqrt{x}$, where $x > 1$. Let $P$ be a moving point lying on $C$. The normal to $C$ at $P$ cuts the $x$-axis at the point $Q$ while the vertical line passing through $P$ cuts the $x$-axis at the point $R$.

(a) Denote the $x$-coordinate of $P$ by $r$. Prove that the $x$-coordinate of $Q$ is $\dfrac{4r^2 + \ln r}{4r}$.
(3 marks)

(b) Find the greatest area of $\triangle PQR$.
(5 marks)

(c) Let $O$ be the origin. It is given that $OP$ increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of $\triangle PQR$ increases at a rate lower than 2 square units per minute when the $x$-coordinate of $P$ is $e$. Is the claim correct? Explain your answer.
(4 marks)

10. **HKDSE Math M2 2018 Q10**

(a) (i) Prove that $\displaystyle\int \sin^4 x \, dx = \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$.

(ii) Evaluate $\displaystyle\int_0^{\pi} \sin^4 x \, dx$.

(5 marks)

(b) (i) Let $f(x)$ be a continuous function such that $f(\beta - x) = f(x)$ for all real numbers $x$, where $\beta$ is a constant. Prove that $\displaystyle\int_0^{\beta} x f(x) \, dx = \frac{\beta}{2}\int_0^{\beta} f(x) \, dx$.

(ii) Evaluate $\displaystyle\int_0^{\pi} x \sin^4 x \, dx$.

(5 marks)

(c) Consider the curve $G : y = \sqrt{x}\sin^2 x$, where $\pi \leq x \leq 2\pi$. Let $R$ be the region bounded by $G$ and the $x$-axis. Find the volume of the solid of revolution generated by revolving $R$ about the $x$-axis.
(3 marks)

11. **HKDSE Math M2 2018 Q11**

(a) Consider the system of linear equations in real variables $x$, $y$, $z$

$$(E) : \begin{cases} x & + & ay & + & 4(a+1)z & = & 18 \\ 2x & + & (a-1)y & + & 2(a-1)z & = & 20 \\ x & - & y & - & 12z & = & b \end{cases} \text{, where } a, b \in \mathbb{R}.$$

(i) Assume that $(E)$ has a unique solution.

(1) Find the range of values of $a$.

(2) Solve $(E)$.

(ii) Assume that $a = 3$ and $(E)$ is consistent.

(1) Find $b$.

(2) Solve $(E)$.

(9 marks)

(b) Consider the system of linear equations in real variables $x$, $y$, $z$

$$(F) : \begin{cases} x & + & 3y & + & 16z & = & 18 \\ x & + & y & + & 2z & = & 20 \\ x & - & y & - & 12z & = & s \\ 2x & - & 5y & - & 45z & = & t \end{cases} \text{, where } s, t \in \mathbb{R}.$$

Assume that $(F)$ is consistent. Find $s$ and $t$.
(3 marks)

12. **HKDSE Math M2 2018 Q2**
The position vectors of the points $A, B, C$ and $D$ are $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $-\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $7\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$ respectively. Denote the plane which contains $A, B$ and $C$ by $\Pi$. Let E be the projection of $D$ on $\Pi$.

(a) Find

(i) $\overrightarrow{AB} \times \overrightarrow{AC}$,

(ii) the volume of the tetrahedron $ABCD$,

(iii) $\overrightarrow{DE}$.

(5 marks)

(b) Let $F$ be a point lying on $BC$ such that $DF$ is perpendicular to $BC$.

(i) Find $\overrightarrow{DF}$.

(ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$ ? Explain your answer.

(5 marks)

(c) Find the angle between $\triangle BCD$ and $\Pi$.
(3 marks)