HKDSE MATH M2 2021

1. **HKDSE Math M2 2021 Q1**
   Let $f(x) = \dfrac{1}{3x^2+4}$. Find $f'(x)$ from first principles.
   (4 marks)

2. **HKDSE Math M2 2021 Q2**
   Using mathematical induction, prove that $\displaystyle\sum_{k=1}^{n}(3k^5+k^3) = \frac{n^3(n+1)^3}{2}$ for all positive integers n.
   (5 marks)

3. **HKDSE Math M2 2021 Q3**
   The coefficient of x^2 in the expansion of $(1-4x)^n$ is 240, where n is a positive integer. Find

   (a) n,

   (b) the coefficient of x^4 in the expansion of $(1-4x)^n\left(1+\dfrac{2}{x}\right)^5$.

   (6 marks)

4. **HKDSE Math M2 2021 Q4**

   (a) Prove that $\cos 2x + \cos 4x + \cos 6x = 4\cos x \cos 2x \cos 3x - 1$.

   (b) Solve the equation $\cos 4\theta + \cos 8\theta + \cos 12\theta = -1$, where $0 \le \theta \le \dfrac{\pi}{2}$.

   (6 marks)

5. **HKDSE Math M2 2021 Q5**
   Define $r(x) = \dfrac{x^3 - x^2 - 2x + 3}{(x-1)^2}$ for all real numbers $x \neq 1$.

   (a) Find the asymptote(s) of the graph of $y = r(x)$.

   (b) Find $\dfrac{d}{dx}r(x)$.

   (c) Someone claims that there is only one point of inflexion of the graph of $y = r(x)$. Do you agree? Explain your answer.

   (7 marks)

6. **HKDSE Math M2 2021 Q6**
   Consider the curve $\Gamma : y = e^{2x-6}$. Denote the normal to Γ at the point $(3, 1)$ by L. Let c be the x-intercept of L. Find

   (a) c;

   (b) the area of the region bounded by L, Γ and the straight line $x = c$.

   (7 marks)

7. **HKDSE Math M2 2021 Q7**

   (a) Using integration by parts, find $\int (\ln x)^2\, dx$.

   (b) Consider the curve $C : y = \sqrt{x}\ln(x^2+1)$, where $x \geq 0$. Let R be the region bounded by C, the straight line $x = 1$ and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

   (7 marks)

8. **HKDSE Math M2 2021 Q8**

   Consider the system of linear equations in real variables x, y, z

$$(E) : \begin{cases} x + (d-1)y + (d+3)z = 4-d \\ 2x + (d+2)y - z = 2d-5 \\ 3x + (d+4)y + 5z = 2 \end{cases}, \text{ where } d \in \mathbb{R}.$$

   It is given that (E) has infinitely many solutions.

   (a) Find d. Hence, solve (E).

   (b) Someone claims that (E) has a real solution (x, y, z) satisfying $xy + 2xz = 3$. Is the claim correct? Explain your answer.

   (8 marks)

9. **HKDSE Math M2 2021 Q9**

   (a) Let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

      (i) Find $\frac{d}{d\theta}\ln(\sec\theta + \tan\theta)$.

      (ii) Using the result of (a)(i), find $\int \sec\theta\, d\theta$. Hence, find $\int \sec^3\theta\, d\theta$.

      (4 marks)

   (b) Let $g(x)$ and $h(x)$ be continuous functions defined on $\mathbb{R}$ such that $g(x) + g(-x) = 1$ and $h(x) = h(-x)$ for all $x \in \mathbb{R}$.
   Using integration by substitution, prove that $\int_{-a}^{a} g(x)h(x)\, dx = \int_0^a h(x)\, dx$ for any $a \in \mathbb{R}$.
   (3 marks)

   (a) Evaluate $\int_{-1}^{1} \frac{3^x x^2}{(3^x + 3^{-x})\sqrt{x^2+1}}\, dx$.
   (5 marks)

10. **HKDSE Math M2 2021 Q10**

    Denote the graph of $y = \sqrt{x^2+36}$ and the graph of $y = -\sqrt{(20-x)^2+16}$ by F and G respectively, where $0 < x < 20$. Let P be a moving point on F. The vertical line passing through P cuts G at the point Q. Denote the x-coordinate of P by u. It is given that the length of PQ attains its minimum value when $u = a$.

(a) Find a.

(4 marks)

(b) The horizontal line passing through P cuts the y-axis at the point R while the horizontal line passing through Q cuts the y-axis at the point S.

(i) Someone claims that the area of the rectangle $PQSR$ attains its minimum value when $u = a$. Do you agree? Explain your answer.

(ii) The length of OP increases at a constant rate of 28 units per minute. Find the rate of change of the perimeter of the rectangle $PQSR$ when $u = a$.

(9 marks)

11. **HKDSE Math M2 2021 Q11**

Define $P = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, where $\frac{\pi}{2} < \theta < \pi$.

(a) Let $A = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$.

Prove that $PAP^{-1} = \begin{pmatrix} -\alpha\cos 2\theta + \beta\sin 2\theta & -\beta\cos 2\theta - \alpha\sin 2\theta \\ -\beta\cos 2\theta - \alpha\sin 2\theta & \alpha\cos 2\theta - \beta\sin 2\theta \end{pmatrix}$.

(3 marks)

(b) Let $B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.

(i) Find θ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where λ, $\mu \in \mathbb{R}$.

(ii) Using the result of (b)(i), prove that $B^n = 2^{n-2}\begin{pmatrix} (-1)^n + 3 & \sqrt{3}(-1)^{n+1} + \sqrt{3} \\ \sqrt{3}(-1)^{n+1} + \sqrt{3} & 3(-1)^n + 1 \end{pmatrix}$ for any positive integer n.

(iii) Evaluate $(B^{-1})^{555}$.

(9 marks)

12. **HKDSE Math M2 2021 Q12**

The position vectors of the points A, B, C and D are $t\mathbf{i}+14\mathbf{j}+s\mathbf{k}$, $12\mathbf{i}-s\mathbf{j}-2\mathbf{k}$, $(s+2)\mathbf{i}-16\mathbf{j}+10\mathbf{k}$ and $-t\mathbf{i}+(s+2)\mathbf{j}+14\mathbf{k}$ respectively, where s, $t \in \mathbb{R}$. Suppose that $\overrightarrow{AB}$ is parallel to $5\mathbf{i}-4\mathbf{j}-2\mathbf{k}$. Denote the plane which contains A, B and C by Π.

(a) Find

(i) s and t,

(ii) the area of $\triangle ABC$,

(iii) the volume of the tetrahedron $ABCD$,

(iv) the shortest distance from D to Π.

(9 marks)

(b) Let E be the projection of D on Π. Is E the circumcentre of $\triangle ABC$? Explain your answer.

(4 marks)