Differentiation Exercise

1. Use the definition of derivative to compute $f'(x)$ for each of the following.

   (a) $f(x) = \sqrt{4x-1}$

   (b) $f(x) = \sqrt{2+x^2}$

   (c) $f(x) = \dfrac{1}{2x+1}$

   (d) $f(x) = \cos x$

   (e) $f(x) = \tan x$

   (f) $f(x) = \ln x$

2. Find $\dfrac{dy}{dx}$ for each of the following.

   (a) $y = -8x^5 + \sqrt{3}x^3 + 2\pi x^2 - 12$

   (b) $y = \left(x^{100} + 2x^{50} - 3\right)\left(7x^8 + 20x + 5\right)$

   (c) $y = \dfrac{x^5 - x + 2}{x^3 + 7}$

   (d) $y = \left(x^3 - 2x^2 + 7x - 3\right)^4$

   (e) $y = \dfrac{1}{(3x^2+5)^4}$

   (f) $y = \sqrt{2x+7}$

   (g) $y = \left(\dfrac{x+2}{x-3}\right)^3$

   (h) $y = x^{\frac{1}{2}} \cos\left(2x^3 + x - 10\right)$

   (j) $y = \ln\left((\ln x)^5\right)$

   (k) $y = \sin^3(5x+4)$

   (l) $y = \tan^3(\ln x)$

   (m) $y = \dfrac{x}{\cos^2(2x)}$

   (n) $y = \sin^2(4x) - 4\cos\left(x^2 - 1\right) + \sin(x \ln x)$

   (o) $y = e^{3x^2+5x-2}$

   (p) $y = \ln\left(\dfrac{e^x + 2x + 1}{e^x - 3x - 1}\right)$

   (q) $y = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

   (r) $y = \log_x(2x+3)$

   (s) $y = \sin^{-1}\left(\sqrt{1-x^2}\right)$

3. Find $\dfrac{dy}{dx}$ for each of the following.

(a) $x^2 + y^2 - 4x + 6y + 12 = 0$

(b) $xy + y^2 = 1$

(c) $x + \sin y = xy$

(d) $\tan(xy) = y$

(e) $y = \tan^2(x + y)$

(f) $xy^3 - 2x^2 = xy + 5$

(g) $\sin\left(x^2 + y\right) = 3xy^2 + y^2$

(h) $x\sqrt{x + y} = 8 - xy$

(i) $x^2(x - y)^2 = x^2 - y^2$

(j) $y^2 = \dfrac{x - 1}{x + 1}$

(k) $x^4 = x^2y^2 + 2\ln y$

(l) $x^2 + y^2 = 2x\cos\left(y^3\right)$

4. Find $\dfrac{dy}{dx}$ for each of the following.

(a) $y = \sqrt[5]{\dfrac{x - 1}{x + 1}}$

(b) $y = \dfrac{x^2\sqrt[3]{7x - 14}}{(1 + x^2)^4}$

(c) $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2 + 1}}{(3x + 2)^5}$

(d) $y = \dfrac{x^3}{1 - x}\sqrt[3]{\dfrac{3 - x}{(3 + x)^2}}$

(e) $y = \sqrt[3]{\dfrac{(x + 2)(3x - 1)^4}{(2 - x)^5}}$

(f) $y = 2^{\sin x}$

(g) $y = 3^{\tan\frac{1}{x}}$

(h) $y = x^{\ln x}$

(i) $y = (\sin x)^x$

(j) $y = (1 + x)^{\frac{1}{x}}$

(n) $y = \sin\left(x^{\cos x}\right)$

5. (a) Given $y = \dfrac{u^2 - 1}{u^2 + 1}$ and $u = \sqrt[3]{x^2 + 2}$, find $\dfrac{dy}{dx}$ in terms of x.

(b) Given $y = \dfrac{1}{\sqrt{3u^2 + 4}}$ and $u = e^{-x}$, find $\dfrac{dy}{dx}$ in terms of x.

(c) Given $y = 2x^2 + 1$ and $u = 2x - 1$, find $\dfrac{dy}{du}$ in terms of u.

6. (a) If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$ and $g'(5) = 6$, find $F'(5)$.

   (b) Find f' in terms of g and/or g' for each of the following.

      (i) $f(x) = (g(x))^2$

      (ii) $f(x) = \sin(g(x))$

      (iii) $f(x) = g(\sin x)$

   (c) Let f and g be two differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + (f(x))^2$. Show that $g'(x) = \dfrac{1}{1+x^2}$.

7. (a) Find the equation of the tangent line to the graph of $y = x^2 - \cos x - 1$ at $x = \pi$.

   (b) Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ at which the tangent lines are horizontal.

   (c) Find the equations of both lines that are tangent to the curve $y = 1 + x^3$ and are parallel to the line $12x - y = 1$.

   (d) Find the equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$.

8. (a) For what values of a and b is the line $2x + y = b$ tangent to the curve $y = ax^2$ when $x = 2$?

   (b) For what value(s) of c is the curve $y = \dfrac{c}{x+1}$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

9. The equation of a curve C is $y = x^3 - 2x^2 - 4$.

   (a) Find the point(s) on C at which the tangent line(s) to C

      (i) is/are parallel to the line $4x - y = 3$.

      (ii) cut(s) the x-axis at the point $(-1, 0)$.

   (b) Write down the equations of the corresponding tangent lines in (a)(i) and (ii).

10. Let $f(x) = 2x^3 + 5x^2 - 12$.

   (a) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

   (b) Find the value(s) of m for which the line $y = mx$ is tangent to the graph of $y = f(x)$.

11. (a) A curve is defined by the equation $2x^4 - 2x^2y^2 - y^3 + 1 = 0$ and $(1, 1)$ is a point on the curve. Find the slope of the tangent line to the curve at this point.

   (b) Let C be the curve defined by the equation $x = 2y^2 - y^3$. Find the equation(s) of the tangent line(s) to C such that the slope of each tangent line is one.

12. (a) If $f(x) = x^3 + x - 9$, find $(f^{-1})'(1)$.

   (b) If $g(x) = 2x + \cos x$, find $(g^{-1})'(1)$.

(c) If $h(x) = \dfrac{4x^3}{x^2+1}$, find $(h^{-1})'(2)$.

13. Let f be a differentiable function such that $f(0) = 3$, $f'(0) = 5$, $f(1) = 7$, $f'(1) = 9$, $f\left(\dfrac{\pi}{2}\right) = 11$ and $f'\left(\dfrac{\pi}{2}\right) = 15$.

    (a) Compute $g'(7)$, where $g(x) = \dfrac{1}{f^{-1}(x)}$ and f^{-1} is the inverse function of f.

    (b) Find the derivative of h at $x = 0$, where $h(x) = f\left(\cos^{-1}\dfrac{x}{3}\right)$.

14. Find $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$ for each of the following functions.

    (a) $y = \ln x$

    (b) $y = \dfrac{1}{ax+b}$, where a and b are constants with $a \neq 0$.

    (c) $y = e^x(x-1)$

    (d) $y = \sin^2 x$

15. Find $\dfrac{d^2y}{dx^2}$ in terms of x and y for each of the following. Hence evaluate $\dfrac{d^2y}{dx^2}$ at the point $(0,1)$.

    (a) $x^2 + 4y^2 = 4$

    (b) $x^3 - 3xy + y^3 = 1$

16. (a) If $y = \dfrac{x+a}{x+b}$, where a and b are constants, show that $2\left(\dfrac{dy}{dx}\right)^2 + (1-y)\dfrac{d^2y}{dx^2} = 0$.

    (b) If $y = \dfrac{1}{\sqrt{1+x^2}}$, show that $\left(1+x^2\right)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 0$.