Mock Exam 7

Section A

1. *Reference: HKDSE Math M2 2012 Q2*

(a) $(1 + kx)^n$

$$= 1 + n(kx) + \frac{n(n-1)(kx)^2}{2} + \cdots$$

$$= 1 + nkx + \frac{n(n-1)k^2}{2}x^2 + \cdots \qquad \text{1M + 1A}$$

$$\therefore \begin{cases} nk = -24 \text{ ....} (1) \\ \dfrac{n(n-1)k^2}{2} = 252 \text{ ....} (2) \end{cases} \qquad \text{1M}$$

From (1), $k = -\dfrac{24}{n}$ ....(3)

Substituting (3) into (2),

$$\frac{n(n-1)}{2} \times \left(-\frac{24}{n}\right)^2 = 252$$

$$576n^2 - 576n = 504n^2$$

$$72n(n-8) = 0$$

$$n = \underline{\underline{8}} \text{ or } 0 \text{ (rejected)} \qquad \text{1A}$$

Substituting $n = 8$ into (3),

$$k = -\frac{24}{8}$$

$$= \underline{\underline{-3}} \qquad \text{1A}$$

(b) Coefficient of $x^3 = C_3^8(-3)^3$

$$= \underline{\underline{-1512}} \qquad \text{1A}$$

(6)

2. For $n = 1$,

L.H.S. = $1 \times 1! = 1$

R.H.S. = $(1 + 1)! - 1 = 1$ = L.H.S.

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = m$, where m is a positive integer, that is,

$$\sum_{k=1}^{m} k \times k! = (m+1)! - 1. \qquad 1$$

When $n = k + 1$,

L.H.S.

$$= \sum_{k=1}^{m+1} k \times k!$$

$$= \sum_{k=1}^{m} k \times k! + (m+1) \times (m+1)!$$

$= [(m + 1)! - 1] + (m + 1) \times (m + 1)!$ (by the assumption) 1

$= (m + 1)!(1 + m + 1) - 1$

$= (m + 2)(m + 1)! - 1$

$= (m + 2)! - 1$

$= [(m + 1) + 1]! - 1$ = R.H.S. 1

$\therefore$ The proposition is also true for $n = m + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(5)

3. *Reference: HKDSE Math M2 2013 Q1*

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right] \qquad \text{1M}$$

$$= \lim_{h\to 0}\frac{\sin x - \sin(x+h)}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{x+(x+h)}{2}\sin\frac{x-(x+h)}{2}}{h\sin(x+h)\sin x} \qquad \text{1M}$$

$$= \lim_{h\to 0}\left[\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h)\sin x}\times\frac{-\sin\frac{h}{2}}{\frac{h}{2}}\right] \qquad \text{1M}$$

$$= \frac{\cos x}{\sin^2 x}\times(-1)$$

$$= -\csc x\cot x \qquad \text{1}$$

(4)

4. *Reference: HKCEE A. Math 2008 Q3*

(a) $\tan A = \tan\left(B+\frac{\pi}{4}\right)$

$$= \frac{\tan B + \tan\frac{\pi}{4}}{1-\tan B\tan\frac{\pi}{4}} \qquad \text{1M}$$

$$= \frac{1+\tan B}{1-\tan B} \qquad \text{1}$$

(b) $\because \quad \frac{5\pi}{8} - \frac{3\pi}{8} = \frac{\pi}{4}$

$\therefore$

$$\tan\frac{5\pi}{8} = \frac{1+\tan\frac{3\pi}{8}}{1-\tan\frac{3\pi}{8}} \quad \text{(by (a))}$$

$$\tan\left(\pi - \frac{3\pi}{8}\right) = \frac{1+\tan\frac{3\pi}{8}}{1-\tan\frac{3\pi}{8}}$$

$$-\tan\frac{3\pi}{8} = \frac{1+\tan\frac{3\pi}{8}}{1-\tan\frac{3\pi}{8}} \qquad \text{1M}$$

$$\tan^2\frac{3\pi}{8} - 2\tan\frac{3\pi}{8} - 1 = 0 \qquad \text{1M}$$

$$\tan\frac{3\pi}{8} = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$$= 1\pm\sqrt{2}$$

$\because \quad \frac{\pi}{4} < \frac{3\pi}{8} < \frac{\pi}{2}$

$\therefore \quad \tan\frac{3\pi}{8} > 1$

$\therefore \quad \tan\frac{3\pi}{8} = \underline{\underline{1+\sqrt{2}}}$ 1A

(5)

5. *Reference: HKDSE Math M2 2012 Q9*

(a) $\int(\ln x)^2 dx = x(\ln x)^2 - \int x d[(\ln x)^2]$

$= x(\ln x)^2 - \int x \times 2\ln x \times \frac{1}{x} dx$ 1M

$= x(\ln x)^2 - 2\int \ln x\, dx$

$= x(\ln x)^2 - 2\left[x\ln x - \int x d(\ln x)\right]$

$= x(\ln x)^2 - 2x\ln x + 2\int x \times \frac{1}{x} dx$ 1M

$= x(\ln x)^2 - 2x\ln x + 2\int dx$

$= \underline{\underline{x(\ln x)^2 - 2x\ln x + 2x + C}}$, where C is a constant 1A

(b) Volume $= \pi(2)^2(e^2) - \int_1^{e^2} \pi(\ln y)^2\, dy$ 1M + 1A

$= 4\pi e^2 - \pi\left[y(\ln y)^2 - 2y\ln y + 2y\right]_1^{e^2}$ (by (a))

$= \underline{\underline{2\pi(e^2 + 1)}}$ 1A

(6)

6. *Reference: HKDSE Math M2 2013 Q4*

(a) Slope of the line $= -\frac{2}{-1}$

$= 2$

Let (a, b) be the coordinates of P.

$\left.\frac{dy}{dx}\right|_{(a,b)} = \frac{1}{a} + 1$

$2 = \frac{1}{a} + 1$ 1M

$\frac{1}{a} = 1$

$a = 1$

Substituting $(1, b)$ into $2x - y - 3 = 0$,

$2(1) - b - 3 = 0$

$b = -1$

$\therefore$ The coordinates of P are $(1, -1)$. 1A

(b) $\frac{dy}{dx} = \frac{1}{x} + 1$

$y = \int\left(\frac{1}{x} + 1\right)dx$ 1M

$= \ln|x| + x + C$

Substituting $(1, -1)$ into $y = \ln|x| + x + C$,

$-1 = \ln 1 + 1 + C$

$C = -2$

$\therefore$ The equation of the curve is $y = \ln|x| + x - 2$. 1A

(c) The equation of the normal is

$y - (-1) = -\frac{1}{2}(x - 1)$ 1M

$x + 2y + 1 = 0$ 1A

(6)

7. (a) $f'(x) = \dfrac{(x)\left(\dfrac{1}{x}\right) - (\ln x)(1)}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$ 1M

When $f'(x) = 0$,

$\dfrac{1 - \ln x}{x^2} = 0$ 1M

$\ln x = 1$

$x = e$

x	$0 < x < e$	$x = e$	$x > e$
$f'(x)$	+	0	−

$\therefore$ When $x = e$, $f(x)$ is maximum. 1M

$\therefore$ The maximum value of $f(x) = f(e)$

$= \dfrac{\ln e}{e}$

$= \dfrac{1}{e}$ 1A

(b) By (a), $f(e) \geq f(x)$ for $x > 0$.

$\therefore \dfrac{\ln x}{x} \leq \dfrac{1}{e}$

$e \ln x \leq x$ 1M

$\ln x^e \leq x$

$e^x \geq x^e$ 1

(6)

8. *Reference: HKDSE Math M2 2012 Q7*

(a) Area $= \dfrac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$= \dfrac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & -2 \\ 3 & 1 & 0 \end{vmatrix}\right\|$ 1M

$= \dfrac{1}{2}\left|2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\right|$

$= \dfrac{1}{2}\sqrt{2^2 + (-6)^2 + 2^2}$

$= \sqrt{11}$ 1A

(b) Volume of the tetrahedron $OABC = \dfrac{1}{6}\left|(\overrightarrow{OA} \times \overrightarrow{OB}) \cdot \overrightarrow{OC}\right|$

$= \dfrac{1}{6}\left|(2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 3\mathbf{k})\right|$ 1M

$= \dfrac{1}{6}\left|(2)(4) + (-6)(-1) + (2)(3)\right|$

$= \dfrac{10}{3}$

Analysis

The required distance is the height of the tetrahedron *OABC* with respect to the base *OAB*.

Let h be the distance between point C and the plane OAB.

$$\frac{1}{3} \times \sqrt{11} \times h = \frac{10}{3} \qquad \text{1M}$$

$$h = \frac{10\sqrt{11}}{11}$$

$\therefore$ The distance between point C and the plane OAB is $\dfrac{10\sqrt{11}}{11}$. 1A

(5)

9. (a) $|A| = 21 + 60 - 4k + 5 - 9k - 112$

$= -13k - 26$ 1M

$$A^{-1} = \frac{1}{-13k-26}\begin{pmatrix} \begin{vmatrix} 1 & -4 \\ -4 & 3 \end{vmatrix} & -\begin{vmatrix} k & -4 \\ -5 & 3 \end{vmatrix} & \begin{vmatrix} k & 1 \\ -5 & -4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ -4 & 3 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ -5 & 3 \end{vmatrix} & -\begin{vmatrix} 7 & 3 \\ -5 & -4 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} 7 & 1 \\ k & -4 \end{vmatrix} & \begin{vmatrix} 7 & 3 \\ k & 1 \end{vmatrix} \end{pmatrix}^T \qquad \text{1M}$$

$$= \frac{1}{-13k-26}\begin{pmatrix} -13 & 20-3k & 5-4k \\ -13 & 26 & 13 \\ -13 & 28+k & 7-3k \end{pmatrix}^T$$

$$= \frac{1}{13k+26}\begin{pmatrix} 13 & 13 & 13 \\ 3k-20 & -26 & -28-k \\ 4k-5 & -13 & 3k-7 \end{pmatrix} \qquad \text{1}$$

(b) $$A^{-1} = \frac{1}{13k+26}\begin{pmatrix} 13 & 13 & 13 \\ 3k-20 & -26 & -28-k \\ 4k-5 & -13 & 3k-7 \end{pmatrix}$$

$$= \frac{1}{k+2}\begin{pmatrix} 1 & 1 & 1 \\ \dfrac{3k-20}{13} & -2 & \dfrac{-28-k}{13} \\ \dfrac{4k-5}{13} & -1 & \dfrac{3k-7}{13} \end{pmatrix}$$

Take $k = 11$, we have $A^{-1} = \dfrac{1}{13}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & 2 \end{pmatrix}$. 1M

The given system of linear equations can be rewritten as

$$13A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}A\begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 7 & 3 & 1 \\ 11 & 1 & -4 \\ -5 & -4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} \qquad \text{1M + 1A}$$

$$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

$\therefore$ The solution is $x = 2$, $y = 4$, $z = -1$. 1A

(7)

Section B

10. (a) $DO = DC$ (*diag. of // gram*)

$CD : DE = 1 : 2$

$\therefore \quad \overrightarrow{OE} = -\frac{1}{2}\overrightarrow{OC}$

$= -\frac{1}{2}(6\mathbf{i} + 8\mathbf{j})$

$= \underline{\underline{-3\mathbf{i} - 4\mathbf{j}}}$ 1A

(1)

(b) $\overrightarrow{AC} = \overrightarrow{OB}$

$= m\mathbf{i} + n\mathbf{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (6\mathbf{i} + 8\mathbf{j}) - (m\mathbf{i} + n\mathbf{j})$

$= (6 - m)\mathbf{i} + (8 - n)\mathbf{j}$ 1A

$\because \quad AC \perp BC$ ($\angle$ *in semicircle*)

$\therefore \quad \overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ 1M

$m(6 - m) + n(8 - n) = 0$

$m^2 + n^2 = 6m + 8n$ ...... (1) 1M

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \overrightarrow{OB} - \overrightarrow{BC}$

$= (m\mathbf{i} + n\mathbf{j}) - [(6 - m)\mathbf{i} + (8 - n)\mathbf{j}]$

$= (2m - 6)\mathbf{i} + (2n - 8)\mathbf{j}$

$\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB}$

$= (-3\mathbf{i} - 4\mathbf{j}) - (m\mathbf{i} + n\mathbf{j})$

$= (-3 - m)\mathbf{i} + (-4 - n)\mathbf{j}$

$\because \quad AB \perp BE$ (*tangent* $\perp$ *radius*)

$\therefore \quad \overrightarrow{AB} \cdot \overrightarrow{BE} = 0$

$(2m - 6)(-3 - m) + (2n - 8)(-4 - n) = 0$

$m^2 + n^2 = 25$ ...... (2) 1M

Substituting (1) into (2),

$6m + 8n = 25$

$n = \frac{25 - 6m}{8}$ ...... (3)

Substituting (3) into (2),

$$m^2 + \left(\frac{25 - 6m}{8}\right)^2 = 25$$

$64m^2 + 625 - 300m + 36m^2 = 1600$

$4m^2 - 12m - 39 = 0$ 1M

$$m = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(-39)}}{2(4)}$$

$= \underline{\underline{\frac{3 + 4\sqrt{3}}{2}}}$ or $\frac{3 - 4\sqrt{3}}{2}$ (rejected) 1A

Substituting $m = \dfrac{3+4\sqrt{3}}{2}$ into (3),

$$n = \frac{25 - 6\left(\dfrac{3+4\sqrt{3}}{2}\right)}{8}$$
$$= \underline{\underline{\frac{4-3\sqrt{3}}{2}}}$$ 1A

(7)

Analysis

Note that $\angle ABE = \dfrac{\pi}{2}$.
If O is the incentre of ΔABE, then OB bisects $\angle ABE$, i.e., $\angle OBE = \dfrac{\pi}{4}$.

(c) $\overrightarrow{OB} = \left(\dfrac{3+4\sqrt{3}}{2}\right)\mathbf{i} + \left(\dfrac{4-3\sqrt{3}}{2}\right)\mathbf{j}$

$$\left|\overrightarrow{OB}\right| = \sqrt{\left(\frac{3+4\sqrt{3}}{2}\right)^2 + \left(\frac{4-3\sqrt{3}}{2}\right)^2}$$
$$= 5$$

$$\overrightarrow{EB} = \left(\frac{3+4\sqrt{3}}{2} + 3\right)\mathbf{i} + \left(\frac{4-3\sqrt{3}}{2} + 4\right)\mathbf{j}$$
$$= \left(\frac{9+4\sqrt{3}}{2}\right)\mathbf{i} + \left(\frac{12-3\sqrt{3}}{2}\right)\mathbf{j}$$

$$\left|\overrightarrow{EB}\right| = \sqrt{\left(\frac{9+4\sqrt{3}}{2}\right)^2 + \left(\frac{12-3\sqrt{3}}{2}\right)^2}$$
$$= 5\sqrt{3}$$

$$\overrightarrow{OB} \cdot \overrightarrow{EB} = \left(\frac{3+4\sqrt{3}}{2}\right)\left(\frac{9+4\sqrt{3}}{2}\right) + \left(\frac{4-3\sqrt{3}}{2}\right)\left(\frac{12-3\sqrt{3}}{2}\right)$$
$$= \frac{75}{2}$$

$$\cos\angle OBE = \frac{\overrightarrow{OB} \cdot \overrightarrow{EB}}{\left|\overrightarrow{OB}\right|\left|\overrightarrow{OE}\right|}$$
$$= \frac{\frac{75}{2}}{5 \times 5\sqrt{3}}$$ 1M + 1A
$$= \frac{3}{2\sqrt{3}}$$
$$= \frac{\sqrt{3}}{2}$$

$\therefore$ $\angle OBE = \dfrac{\pi}{6} \neq \dfrac{\pi}{4}$ 1M

$\therefore$ O is not the incentre of ΔABE. 1A

(4)

11. (a) $\begin{vmatrix} p & q & 1 \\ p^2 & q^2 & 1 \\ p^3 & q^3 & 1 \end{vmatrix} = pq\begin{vmatrix} 1 & 1 & 1 \\ p & q & 1 \\ p^2 & q^2 & 1 \end{vmatrix}$

$= pq\begin{vmatrix} 1 & 1 & 1 \\ p-1 & q-1 & 0 \\ p^2-1 & q^2-1 & 0 \end{vmatrix}$ $(R_2 - R_1 \to R_2;$ $R_3 - R_1 \to R_3)$ 1M

$= pq\begin{vmatrix} p-1 & q-1 \\ p^2-1 & q^2-1 \end{vmatrix}$ 1M

$= pq\begin{vmatrix} p-1 & q-1 \\ (p-1)(p+1) & (q-1)(q+1) \end{vmatrix}$

$= pq(p-1)(q-1)\begin{vmatrix} 1 & 1 \\ p+1 & q+1 \end{vmatrix}$

$= pq(1-p)(1-q)(q-p)$ 1

(3)

(b) (i) If (E) does not have unique solution, then

$$\begin{vmatrix} h & 3 & 1 \\ h^2 & 9 & 1 \\ h^3 & 27 & 1 \end{vmatrix} = 0$$ 1M

$(h)(3)(1-h)(1-3)(3-h) = 0$ (by (a))

$h = \underline{\underline{0, 1 \text{ or } 3}}$ 1A

(ii) The augmented matrix is

$\left(\begin{array}{ccc|c} 3 & 3 & 1 & a \\ 9 & 9 & 1 & b \\ 27 & 27 & 1 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 3 & 1 & a \\ 0 & 0 & -2 & b-3a \\ 0 & 0 & -8 & c-9a \end{array}\right)$ $(R_2 - 3R_1 \to R_2;$ $R_3 - 9R_1 \to R_3)$ 1M

$\sim \left(\begin{array}{ccc|c} 3 & 3 & 1 & a \\ 0 & 0 & -2 & b-3a \\ 0 & 0 & 0 & c-4b+3a \end{array}\right)$ $(R_3 - 4R_2 \to R_3)$ 1A

$\therefore$ Since (E) is consistent, $c - 4b + 3a = 0$. 1

(5)

(c) Take $h = 3$, $a = 2$, $b = 1$, $c = -2$. Then we have

$$(E): \begin{cases} 3x + 3y + z = 2 \\ 9x + 9y + z = 1 \\ 27x + 27y + z = -2 \end{cases}$$

By (b)(ii), the augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & 3 & 1 & 2 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $y = t$, where t is any real number.

Then $x = -t - \dfrac{1}{6}$, $y = t$ and $z = \dfrac{5}{2}$. 1A

Substituting $x=-t-\frac{1}{6}$, $y=t$ and $z=\frac{5}{2}$ into $3x+6y-2z=0$,

$$3\left(-t-\frac{1}{6}\right)+6t-2\left(\frac{5}{2}\right)=0 \qquad \text{1M}$$

$$3t=\frac{11}{2}$$

$$t=\frac{11}{6}$$

$$\therefore \quad x=-\frac{11}{6}-\frac{1}{6}=-2,\ y=\frac{11}{6},\ z=\frac{5}{2} \qquad \text{1A}$$

(3)

12. *Reference: HKDSE Math M2 2013 Q12*

(a) (i) $y=\sqrt{x^2+6^2}+\sqrt{(11-x)^2+4^2}$

$$=\underline{\underline{\sqrt{x^2+36}+\sqrt{x^2-22x+137}}} \qquad \text{1A}$$

(ii) $\frac{dy}{dx}=\frac{2x}{2\sqrt{x^2+36}}+\frac{2x-22}{2\sqrt{x^2-22x+137}}$

$$=\frac{x}{\sqrt{x^2+36}}+\frac{x-11}{\sqrt{x^2-22x+137}} \qquad \text{1M}$$

When $\frac{dy}{dx}=0$,

$$\frac{x}{\sqrt{x^2+36}}+\frac{x-11}{\sqrt{x^2-22x+137}}=0$$

$$\frac{x}{\sqrt{x^2+36}}=-\frac{x-11}{\sqrt{x^2-22x+137}} \qquad \text{1M}$$

$$\frac{x^2}{x^2+36}=\frac{x^2-22x+121}{x^2-22x+137}$$

$$x^4-22x^3+137x^2=x^4+36x^2-22x^3-792x+121x^2+4356$$

$$-20x^2+792x-4356=0$$

$$5x^2-198x+1089=0$$

$$(5x-33)(x-33)=0$$

$$x=\frac{33}{5} \text{ or } 33 \text{ (rejected)}$$

x	$0<x<\frac{33}{5}$	$x=\frac{33}{5}$	$x>\frac{33}{5}$
$\frac{dy}{dx}$	$-$	0	$+$

1M

$\therefore$ When $x=\frac{33}{5}$, then length of the road attains its minimum. 1A

$$\text{Minimum length}=\left[\sqrt{\left(\frac{33}{5}\right)^2+36}+\sqrt{\left(\frac{33}{5}\right)^2-22\left(\frac{33}{5}\right)+137}\right]\text{km}$$

$$=\left(\sqrt{\frac{1989}{25}}+\sqrt{\frac{884}{25}}\right)\text{km}$$

$$=\left(\frac{3\sqrt{221}}{5}+\frac{2\sqrt{221}}{5}\right)\text{km}$$

$$=\underline{\underline{\sqrt{221}\text{ km}}} \qquad \text{1A}$$

(6)

(b) (i) In ΔBCX,

$$\sin\alpha = \frac{BC}{BX} = \frac{4}{\frac{2\sqrt{221}}{5}} = \frac{10}{\sqrt{221}}$$ 1A

$$\cos\alpha = \frac{CX}{BX} = \frac{11-\frac{33}{5}}{\frac{2\sqrt{221}}{5}} = \frac{11}{\sqrt{221}}$$

$$\frac{XY}{\sin\beta} = \frac{CX}{\sin(\pi-\alpha-\beta)}$$ 1M

$$XY = \frac{22\sin\beta}{5\sin(\alpha+\beta)} \text{ km}$$

$$= \frac{22\sin\beta}{5\sin\alpha\cos\beta + 5\cos\alpha\sin\beta} \text{ km}$$ 1M

$$= \frac{22}{5\sin\alpha\cot\beta + 5\cos\alpha} \text{ km}$$

$$= \frac{22}{5\left(\frac{10}{\sqrt{221}}\right)\cot\beta + 5\left(\frac{11}{\sqrt{221}}\right)} \text{ km}$$

$$= \frac{22\sqrt{221}}{50\cot\beta + 55} \text{ km}$$ 1

(ii) Let L km $= XY$.

$$\frac{dL}{dt} = \frac{-22\sqrt{221}(-50\csc^2\beta)}{(50\cot\beta+55)^2} \times \frac{d\beta}{dt}$$ 1M

Note that $\frac{dL}{dt} = 50$.

When CY is the shortest, then $CY \perp XB$, we have $\beta = \frac{\pi}{2} - \alpha$,

$$\csc^2\beta = \csc^2\left(\frac{\pi}{2}-\alpha\right)$$
$$= \sec^2\alpha$$
$$= \frac{221}{121}$$ 1A

$$\cot\beta = \cot\left(\frac{\pi}{2}-\alpha\right)$$
$$= \tan\alpha$$
$$= \frac{10}{11}$$

$$\therefore \quad \frac{-22\sqrt{221}\left(-50\times\frac{221}{121}\right)}{\left[50\left(\frac{10}{11}\right)+55\right]^2} \times \frac{d\beta}{dt} = 50$$ 1M

$$\frac{d\beta}{dt} = \frac{5525}{22\sqrt{221}}$$
$$= \frac{25\sqrt{221}}{22}$$

$\therefore$ The rate of change of β is $\frac{25\sqrt{221}}{22}$ radians per hour. 1A

(8)

13. (a) Let $u = T - x$. Then $du = -dx$.

When $x = 0$, $u = T$;

when $x = T$, $u = 0$.

$$\int_0^T xf(x)\,dx = \int_T^0 (T-u)f(T-u)(-\,du) \qquad \text{1M}$$

$$= \int_0^T (T-u)f(u)\,du \qquad \text{1A}$$

$$= \int_0^T (T-x)f(x)\,dx$$

$$= T\int_0^T f(x)\,dx - \int_0^T xf(x)\,dx \qquad \text{1M}$$

$\therefore$

$$2\int_0^T xf(x)\,dx = T\int_0^T f(x)\,dx$$

$$\int_0^T xf(x)\,dx = \frac{T}{2}\int_0^T f(x)\,dx \qquad \text{1}$$

(4)

(b) (i) Let $u = \cos x$. Then $du = -\sin x\,dx$. 1M

When $x = 0$, $u = 1$;

when $x = \pi$, $u = -1$.

$$\int_0^\pi \frac{\sin x\cos^2 x}{1+\cos^2 x}\,dx = -\int_1^{-1}\frac{u^2}{1+u^2}\,du$$

$$= \int_{-1}^1 \frac{u^2}{1+u^2}\,du \qquad \text{1M}$$

$$= \int_{-1}^1 \frac{1+u^2-1}{1+u^2}\,du$$

$$= \int_{-1}^1 du - \int_{-1}^1 \frac{du}{1+u^2}$$

$$= [u]_{-1}^1 - \int_{-1}^1 \frac{du}{1+u^2}$$

$$= 2 - \int_{-1}^1 \frac{dx}{1+x^2} \qquad \text{1}$$

(ii) Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. 1M

When $x = -1$, $\theta = -\frac{\pi}{4}$;

when $x = 1$, $\theta = \frac{\pi}{4}$.

$$\int_0^{\pi} \frac{\sin x\cos^2 x}{1+\cos^2 x}\,dx = 2 - \int_{-1}^{1}\frac{dx}{1+x^2}$$

$$= 2 - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^2\theta d\theta}{1+\tan^2\theta} \quad \text{1M}$$

$$= 2 - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^2\theta d\theta}{\sec^2\theta}$$

$$= 2 - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta$$

$$= 2 - [\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \underline{\underline{2 - \frac{\pi}{2}}} \quad \text{1A}$$

(6)

(c) Let $f(x) = \frac{\sin x\cos^2 x}{1+\cos^2 x}$.

$$f(\pi - x) = \frac{\sin(\pi - x)\cos^2(\pi - x)}{1+\cos^2(\pi - x)}$$

$$= \frac{\sin x\cos^2 x}{1+\cos^2 x}$$

$$= f(x)$$

1M

Smart Tips

Before we use the result of (a), we must point out that the function fulfills the requirement.

$$\int_0^{\pi}\frac{x\sin x\cos^2 x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x\cos^2 x}{1+\cos^2 x}\,dx \quad \text{(by (a))} \quad \text{1M}$$

$$= \frac{\pi}{2}\left(2 - \frac{\pi}{2}\right) \quad \text{(by (b))}$$

$$= \underline{\underline{\pi - \frac{\pi^2}{4}}} \quad \text{1A}$$

(3)