Binomial Theorem Note

1 Binomial Theorem

Concept.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Where,

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r} = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2 Exercise

1. Warm up:

   (a) Expand $(1+x)^7$ in ascending powers of x up to x^4.

   (b) Expand $(2x^2+1)^5$ in ascending powers of x up to x^7.

   (c) Expand $(9-x)^6$ in descending powers of x up to x^4.

   (d) Find the coefficient of x^3y^5 in the expansion $(2x-y)^8$.

   Ans:

   (a) $1+7x+21x^2+35x^3+35x^4+\cdots$

   (b) $1+10x^2+40x^4+80x^6+\cdots$

   (c) $x^6-54x^5+1215x^4+\cdots$

   (d) -448

2. **HKCEE A. Maths 2004 Q3**

   (a) Expand $(1+2x)^6$ in ascending powers of x up to the terms x^3.

   (b) Find the constant term in the expansion of $\left(1-\frac{1}{x}+\frac{1}{x^2}\right)(1+2x)^6$.

   Ans:

   (a) $a+12x+60x^2+160x^3+\cdots$

   (b) 49

3. **HKCEE A.Maths 2009 Q11**

   In the binomial expansion of $\left(x^2+\frac{1}{x}\right)^{20}$, find

(a) the coefficient of x^{16}.

(b) the constant term.

Ans:

(a) 125970

(b) 0

4. **HKDSE M2 2012 Q2**
It is given that

$$(1 + ax)^n = 1 + 6x + 16x^2 + \text{terms involving higher powers of } x$$

where n is a positive integer and a is a constant. Find the values of a and n.
Ans: $a = 2/3, n = 9$

5. **HKDSE M2 2013 Q2**
Suppose the coefficient of x and x^2 in the expansion of $(1 + ax)^n$ are -20 and 180 respectively. Find the values of a and n.
Ans: $a = -2, n = 10$

6. **HKDSE M2 2014 Q1**
In the expansion of $(1 - 4x)^2(1 + x)^n$, the coefficient of x is 1.

(a) Find the value of n.

(b) Find the coefficient of x^2.

Ans:

(a) $n = 9$

(b) Coefficient of $x^2 = -20$

7. Given

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17} = a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17}$$

where $y = x + 1$ and $a_k's$ are constants. Find a_2.
Ans: $a_2 = \sum_{k=2}^{17} k(k-1) = 816$

8. Let $(1 + x)^{100} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{99} x^{99} + a_{100} x^{100}$.90

(a) Expand $(1 - x)^{100}$ in terms of $a_0, a_1 \cdots, a_{100}$.

(b) Using (a), or otherwise, show that

i. $a_0 - a_1 + a_2 - a_3 + \cdots - a_{99} + a_{100} = 0$

ii. $a_0 + a_2 + a_4 + \cdots + a_{98} + a_{100} = 2^{99}$

(c) Using (a), or otherwise, show that

$$a_1 - 2a_2x + 3a_3x^2 - \cdots + 99a_{99}x^{98} - 100a_{100}x^{99} = 100(1-x)^{99}$$

Ans:

(a) $a_0 - a_1x + a_2x^2 + \cdots - a_{99}x^{99} + a_{100}x^{100}$