Mock Exam 5

Section A

1. *Reference: HKDSE Math M2 2014 Q4*

$$x = 4y - \cos y$$

$$\frac{dx}{dx} = (4 + \sin y)\frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4 + \sin y} \qquad \text{1M}$$

$$\frac{d^2 y}{dx^2} = -\frac{\cos y}{(4 + \sin y)^2}\frac{dy}{dx}$$

$$= -\frac{\cos y}{(4 + \sin y)^3} \qquad \text{1M}$$

When $\frac{d^2 y}{dx^2} = 0$,

$$\cos y = 0$$

$$y = \underline{\underline{\frac{\pi}{2}}}$$

$$x = 4\left(\frac{\pi}{2}\right) - \cos\frac{\pi}{2}$$

$$= \underline{\underline{2\pi}} \qquad \text{1A}$$

(3)

Smart Tips

Alternatively, you may find $\frac{dy}{dx}$ by using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Watch Out

$$\frac{d^2 y}{dx^2} \neq \frac{1}{\frac{d^2 x}{dy^2}}$$

2. *Reference: HKDSE Math M2 PP Q8*

(a) $\int e^x \cos 2x \, dx = \int \cos 2x d(e^x)$

$$= e^x \cos 2x - \int e^x d(\cos 2x) \qquad \text{1M}$$

$$= e^x \cos 2x + 2\int e^x \sin 2x \, dx$$

$$= e^x \cos 2x + 2\int \sin 2x d(e^x)$$

$$= e^x \cos 2x + 2\left[e^x \sin 2x - 2\int e^x \cos 2x \, dx\right]$$

$$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx \qquad \text{1M}$$

$$\therefore \quad 5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\int e^x \cos 2x \, dx = \frac{e^x(2\sin 2x + \cos 2x)}{5} + C\text{, where } C \text{ is a constant} \qquad 1$$

(b) $\int_0^{\frac{\pi}{4}} 5e^x \cos^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{5e^x(1 + \cos 2x)}{2} dx$ 1M

$$= \frac{5}{2}\int_0^{\frac{\pi}{4}} e^x \, dx + \frac{5}{2}\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx$$

$$= \frac{5}{2}\left[e^x\right]_0^{\frac{\pi}{4}} + \frac{5}{2}\left[\frac{e^x(2\sin 2x + \cos 2x)}{5}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{5}{2}(e^{\frac{\pi}{4}} - 1) + \frac{1}{2}[e^{\frac{\pi}{4}}(2) - (1)]$$

$$= \underline{\underline{\frac{7e^{\frac{\pi}{4}}}{2} - 3}} \qquad \text{1A}$$

(5)

Smart Tips

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

3. $y = \int \sin x \cos 3x \, dx$ 1M

$= \int \frac{1}{2}[\sin(x + 3x) + \sin(x - 3x)] \, dx$

$= \frac{1}{2}\int (\sin 4x - \sin 2x) \, dx$ 1M

$= -\frac{\cos 4x}{8} + \frac{\cos 2x}{4} + C$, where C is a constant 1A

Since the y-intercept is 1, we have

$$-\frac{\cos 0}{8} + \frac{\cos 0}{4} + C = 1$$

$$-\frac{1}{8} + \frac{1}{4} + C = 1$$

$$C = \frac{7}{8}$$

$\therefore$ The equation of Γ is $y = -\frac{\cos 4x}{8} + \frac{\cos 2x}{4} + \frac{7}{8}$. 1A

(4)

4. *Reference: HKDSE Math M2 2014 Q2*

(a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\left(\frac{1}{x+h} + x + h\right) - \left(\frac{1}{x} + x\right)}{h}$ 1M

$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x} + h}{h}$

$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)} + h}{h}$

$= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)} + h}{h}$

$= \lim_{h \to 0} \left[\frac{-1}{x(x+h)} + 1\right]$

$= \underline{\underline{1 - \frac{1}{x^2}}}$ 1A

(b) When $\frac{dy}{dx} > 0$,

$1 - \frac{1}{x^2} > 0$ 1M

$\frac{x^2 - 1}{x^2} > 0$

$(x + 1)(x - 1) > 0$

$\therefore$ $x < -1$ or $x > 1$ 1A

(4)

Smart Tips

Since $x^2 > 0$, multiplying both sides by x^2 does not change the inequality sign.

5. (a) Since $x = 10^y$, we have $y = \dfrac{\ln x}{\ln 10}$.

$\dfrac{dy}{dx} = \dfrac{1}{x \ln 10}$ 1M

Let (m, n) be the coordinates of P.

$\left.\dfrac{dy}{dx}\right|_{(m,\, n)} = \text{Slope of } L$

$\dfrac{1}{m \ln 10} = \dfrac{1}{\ln 10}$ 1M

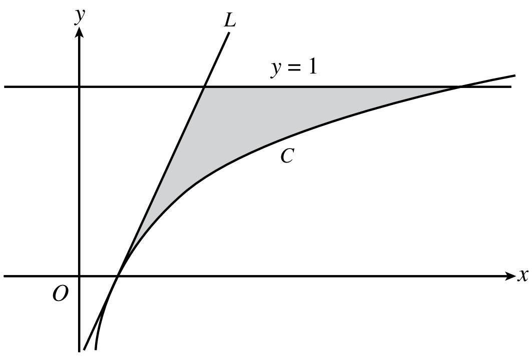
$m = 1$

Substituting $x = 1$ and $y = n$ into $y = \dfrac{\ln x}{\ln 10}$,

$y = \dfrac{\ln 1}{\ln 10} = 0$

$\therefore$ The coordinates of P are $(1, 0)$. 1A

(b)



Note that the equation of L can be rewritten as $x = y \ln 10 + 1$,
and the equation of C can be rewritten as $x = e^{y \ln 10}$.

Area

$= \int_0^1 [e^{y \ln 10} - (y \ln 10 + 1)]\, dy$ 1M

$= \left[\dfrac{e^{y \ln 10}}{\ln 10} - \dfrac{y^2 \ln 10}{2} - y \right]_0^1$ 1M

$= \underline{\underline{\dfrac{9}{\ln 10} - \dfrac{\ln 10}{2} - 1}}$ 1A

(6)

6. *Reference: HKDSE Math M2 2013 Q5*

(a) From the table, $f'(x)$ does not change sign.

$\therefore$ There are no maximum points or minimum points. 1A

From the table, $f''(x)$ changes sign at $x = -\sqrt{3}$, 0 and $\sqrt{3}$.

$f\left(-\sqrt{3}\right) = -3\sqrt{3}$, $f(0) = 0$ and $f\left(\sqrt{3}\right) = 3\sqrt{3}$

$\therefore$ The points of inflexion are $(-\sqrt{3},\ -3\sqrt{3})$, $(0, 0)$ and $(\sqrt{3},\ 3\sqrt{3})$. 1A

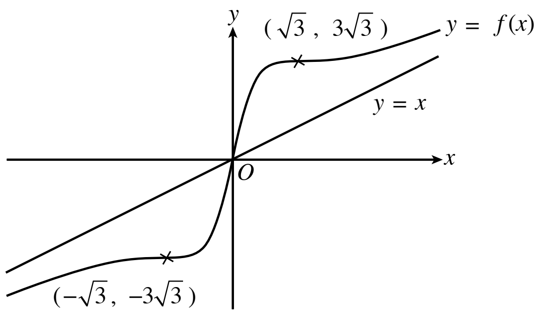
(b) Since $x^2 + 1 \neq 0$, there is no vertical asymptote.

$$f(x) = \frac{x^3 + 9x}{x^2 + 1} = \frac{x(x^2 + 1) - x + 9x}{x^2 + 1} = x + \frac{8x}{x^2 + 1}$$ 1M

When $x \to \pm\infty$, $\dfrac{8x}{x^2 + 1} \to 0$.

$\therefore$ $y = x$ is an oblique asymptote. 1A

(c)



1A for the shape
1A for all correct
(6)

7. (a) The augmented matrix is

$$\left(\begin{array}{ccc|c} m & 1 & 1 & 1 \\ 1 & m & 1 & 2 \\ 1 & 1 & m & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & m & 4 \\ 1 & m & 1 & 2 \\ m & 1 & 1 & 1 \end{array}\right) \qquad (R_1 \leftrightarrow R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & m & 4 \\ 0 & m-1 & 1-m & -2 \\ 0 & 1-m & 1-m^2 & 1-4m \end{array}\right) \qquad (R_2 - R_1 \to R_2;\ R_3 - mR_1 \to R_3)$$ 1M

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & m & 4 \\ 0 & m-1 & 1-m & -2 \\ 0 & 0 & 2-m-m^2 & -1-4m \end{array}\right) \qquad (R_3 + R_2 \to R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & m & 4 \\ 0 & m-1 & 1-m & -2 \\ 0 & 0 & (m+2)(1-m) & -1-4m \end{array}\right)$$ 1M

When (E) has a unique solution, $(m + 2)(1 - m) \neq 0$ and $m - 1 \neq 0$.

$\therefore$ $m < -2$ or $-2 < m < 1$ or $m > 1$ 1A

(b) Suppose (E) has infinitely many solutions. Then we have

$$\begin{cases}(m+2)(1-m)=0\\ -1-4m=0\end{cases}$$

$$\begin{cases}m=-2 \text{ or } 1\\ m=-\dfrac{1}{4}\end{cases}\text{, which is impossible} \qquad \text{1M}$$

$\therefore$ (E) cannot have infinitely many solutions.

$\therefore$ The claim is agreed. 1A

(5)

8. *Reference: HKDSE Math M2 PP Q5*

(a)

$$\begin{aligned}\cos(x+k)-\cos(x-k)&=\sin x\\ -2\sin\frac{(x+k)+(x-k)}{2}\sin\frac{(x+k)-(x-k)}{2}&=\sin x\\ -2\sin x\sin k&=\sin x \qquad \text{1M}\\ \therefore \quad \sin k&=-\frac{1}{2}\\ k&=\underline{\underline{-\frac{\pi}{6}}} \qquad \text{1A}\end{aligned}$$

(b)

$$\begin{vmatrix}\cos\frac{-\pi}{6} & 0 & \cos\frac{\pi}{6}\\ \cos\frac{\pi}{6} & 2\sqrt{3} & \cos\frac{\pi}{2}\\ \cos\frac{\pi}{12} & 2\sqrt{2} & \cos\frac{5\pi}{12}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\left(0-\frac{\pi}{6}\right) & 0 & \cos\left(0+\frac{\pi}{6}\right)\\ \cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) & 2\sqrt{3} & \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\\ \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) & 2\sqrt{2} & \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\end{vmatrix}$$

$$=\begin{vmatrix}\cos\left(0-\frac{\pi}{6}\right)-\cos\left(0+\frac{\pi}{6}\right) & 0 & \cos\left(0+\frac{\pi}{6}\right)\\ \cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) & 2\sqrt{3} & \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\\ \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) & 2\sqrt{2} & \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\end{vmatrix} \quad (C_1-C_3\to C_1) \qquad \text{1M}$$

$$=\begin{vmatrix}\sin 0 & 0 & \cos\left(0+\frac{\pi}{6}\right)\\ \sin\frac{\pi}{3} & 2\sqrt{3} & \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\\ \sin\frac{\pi}{4} & 2\sqrt{2} & \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\end{vmatrix} \quad \text{(by (a))} \qquad \text{1M}$$

$$= \begin{vmatrix} 0 & 0 & \cos\left(0+\frac{\pi}{6}\right) \\ \frac{\sqrt{3}}{2} & 2\sqrt{3} & \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) \\ \frac{\sqrt{2}}{2} & 2\sqrt{2} & \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) \end{vmatrix}$$

$$= \cos\frac{\pi}{6}\times\frac{\sqrt{3}}{2}\times 2\sqrt{2} - \cos\frac{\pi}{6}\times 2\sqrt{3}\times\frac{\sqrt{2}}{2}$$

$= \underline{\underline{0}}$ 1A

Alternative Solution:

$$\begin{vmatrix} \cos\frac{-\pi}{6} & 0 & \cos\frac{\pi}{6} \\ \cos\frac{\pi}{6} & 2\sqrt{3} & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{12} & 2\sqrt{2} & \cos\frac{5\pi}{12} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 2\sqrt{3} & 0 \\ \cos\frac{\pi}{12} & 2\sqrt{2} & \cos\frac{5\pi}{12} \end{vmatrix}$$

$$= \frac{\sqrt{3}}{2}\times 2\sqrt{3}\times\cos\frac{5\pi}{12} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}\times 2\sqrt{2} - \frac{\sqrt{3}}{2}\times 2\sqrt{3}\times\cos\frac{\pi}{12}$$

$$= 3\left(\cos\frac{5\pi}{12} - \cos\frac{\pi}{12}\right) + \frac{3\sqrt{2}}{2}$$

$$= -3\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right] + \frac{3\sqrt{2}}{2}$$ 1M

$$= -3\sin\frac{\pi}{4} + \frac{3\sqrt{2}}{2} \quad \text{(by (a))}$$ 1M

$$= -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}$$

$= \underline{\underline{0}}$ 1A

(5)

9. (a) $\overrightarrow{BA}\times\overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 1 \\ 2x & x & 14 \end{vmatrix}$

$= (14 - x)\mathbf{i} + (2x - 70)\mathbf{j} + 3x\mathbf{k}$ 1A

The area of $ABCD = \left|\overrightarrow{BA}\times\overrightarrow{BC}\right|$

$= \sqrt{(14-x)^2 + (2x-70)^2 + (3x)^2}$

$= \underline{\underline{\sqrt{14x^2 - 308x + 5096}}}$ 1A

(b) The area of $ABCD = \sqrt{14x^2 - 308x + 5096}$

$= \sqrt{14(x-11)^2 + 3402}$ 1M

$\therefore$ When $x = 11$, the area of $ABCD$ attains its minimum value.

When $x = 11$, we have

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{BC} &= (5\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (22\mathbf{i} + 11\mathbf{j} + 14\mathbf{k}) \\ &= 5(22) + 11 + 14 \\ &= 135 \\ &\neq 0\end{aligned}$$ 1M

$\therefore$ $ABCD$ is not a rectangle. 1A

(5)

10. (a) For $n = 1$,

$$\text{L.H.S.} = X^1 = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\text{R.H.S.} = \begin{pmatrix} 1 & 0 \\ \dfrac{3^1 - 1}{2} & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = \text{L.H.S.}$$

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = k$, where k is a positive integer, i.e.,

$$X^k = \begin{pmatrix} 1 & 0 \\ \dfrac{3^k - 1}{2} & 3^k \end{pmatrix},$$

when $n = k + 1$,

$$\begin{aligned}\text{L.H.S.} &= X^{k+1} \\ &= X^k X \\ &= \begin{pmatrix} 1 & 0 \\ \dfrac{3^k - 1}{2} & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \quad \text{(by the assumption)} \\ &= \begin{pmatrix} 1 & 0 \\ \dfrac{3^k - 1}{2} + 3^k & 3^{k+1} \end{pmatrix} \quad 1 \\ &= \begin{pmatrix} 1 & 0 \\ \dfrac{3^{k+1} - 1}{2} & 3^{k+1} \end{pmatrix} \quad 1 \\ &= \text{R.H.S.}\end{aligned}$$

$\therefore$ The proposition is also true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(b) Note that $X^3 = \begin{pmatrix} 1 & 0 \\ \dfrac{3^3-1}{2} & 3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 13 & 27 \end{pmatrix} = Y$ 1M

$\therefore \quad XY^3 = X(X^3)^3 = X^{10}$

$$\begin{aligned} \therefore \quad |XY^3| &= |X^{10}| \\ &= |X|^{10} \\ &= [(1)(3)-(0)(1)]^{10} \\ &= 3^{10} \\ &= \underline{\underline{59\ 049}} \end{aligned}$$

1M

1A

(7)

Section B

11. *Reference: HKDSE Math M2 2013 Q11*

(a) (i) Let $u = a \tan y$. Then $du = a \sec^2 ydy$. 1M

$$\begin{aligned} \int \frac{du}{u^2+a^2} &= \int \frac{a\sec^2 ydy}{a^2\tan^2 y + a^2} \\ &= \frac{1}{a}\int dy \\ &= \frac{1}{a}y + C \\ &= \frac{1}{a}\tan^{-1}\frac{u}{a} + C \text{, where } C \text{ is a constant} \end{aligned}$$

1M

1

(ii) $$\begin{aligned} \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{x^2-\sqrt{3}x+1} &= \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\left(x-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \left[\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right]_0^{\frac{1}{\sqrt{3}}} \quad \text{(by (a)(i))} \\ &= \left[2\tan^{-1}(2x-\sqrt{3})\right]_0^{\frac{1}{\sqrt{3}}} \\ &= -\frac{\pi}{3} - \left(-\frac{2\pi}{3}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

1M

1M

1

(6)

(b) Let $x = \tan\alpha$. Then $dx = \sec^2\alpha\, d\alpha$. 1M

$d\alpha = \dfrac{dx}{1+x^2}$ 1A

When $\alpha = 0$, $x = 0$; when $\alpha = \dfrac{\pi}{6}$, $x = \dfrac{1}{\sqrt{3}}$.

$$\int_0^{\frac{\pi}{6}} \frac{(\tan^2\alpha + \sqrt{3}\tan\alpha + 1)(\tan^2\alpha + 1)}{\tan^4\alpha - \tan^2\alpha + 1}\, d\alpha$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{(x^2 + \sqrt{3}x + 1)(x^2 + 1)}{x^4 - x^2 + 1} \frac{dx}{1+x^2} \qquad \text{1M + 1M}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{x^2 + \sqrt{3}x + 1}{x^4 - x^2 + 1}\, dx$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{x^2 + \sqrt{3}x + 1}{x^4 + 2x^2 + 1 - 3x^2}\, dx$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{x^2 + \sqrt{3}x + 1}{(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)}\, dx \qquad \text{1M}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{x^2 - \sqrt{3}x + 1}\, dx$$

$$= \underline{\underline{\frac{\pi}{3}}} \qquad \text{1A}$$

(6)

12. *Reference: HKDSE Math M2 2014 Q12*

(a) (i) $\det A = 1 - p$

$$A^{-1} = \frac{1}{1-p}\begin{pmatrix} 1 & -p \\ -1 & 1 \end{pmatrix}^T \qquad \text{1M}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1 & -1 \\ -p & 1 \end{pmatrix} \qquad \text{1A}$$

$$A^{-1}MA = \frac{1}{1-p}\begin{pmatrix} 1 & -1 \\ -p & 1 \end{pmatrix}\begin{pmatrix} k+1 & -1 \\ k & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ p & 1 \end{pmatrix}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1 & -1 \\ -kp - p + k & p \end{pmatrix}\begin{pmatrix} 1 & 1 \\ p & 1 \end{pmatrix} \qquad \text{1M + 1A}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1-p & 0 \\ -kp - p + k + p^2 & -kp + k \end{pmatrix}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1-p & 0 \\ (1-p)(k-p) & k(1-p) \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ k-p & k \end{pmatrix}}} \qquad \text{1A}$$

(ii) Since $p = k$, we have $A^{-1}MA = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

$$(A^{-1}MA)^n = \underbrace{(A^{-1}MA) \times (A^{-1}MA) \times \cdots \times (A^{-1}MA)}_{n \text{ times}}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^n = A^{-1}M^nA \qquad \text{1M}$$

$$\therefore \quad M^n = A\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^n A^{-1}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1 & 1 \\ p & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -p & 1 \end{pmatrix} \qquad \text{1M}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1 & p^n \\ p & p^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -p & 1 \end{pmatrix}$$

$$= \frac{1}{1-p}\begin{pmatrix} 1-p^{n+1} & p^n-1 \\ p-p^{n+1} & p^n-p \end{pmatrix} \qquad \text{1}$$

(8)

(b) Take $p = k = 2$. Then, we have

$$\begin{pmatrix} x_n \\ 2x_{n-1} \end{pmatrix} = M\begin{pmatrix} x_{n-1} \\ 2x_{n-2} \end{pmatrix}$$

$$= M^2\begin{pmatrix} x_{n-2} \\ 2x_{n-3} \end{pmatrix} \qquad \text{1M}$$

$$\vdots$$

$$= M^{n-2}\begin{pmatrix} x_2 \\ 2x_1 \end{pmatrix} \qquad \text{1A}$$

$$= \frac{1}{1-2}\begin{pmatrix} 1-2^{n-1} & 2^{n-2}-1 \\ 2-2^{n-1} & 2^{n-2}-2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \text{1M}$$

$$\therefore \quad x_n = \underline{\underline{3(2^{n-1})-3}} \qquad \text{1A}$$

(4)

13. (a) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{3}$

$= 7 \times 12 \times \frac{1}{2}$

$= 42$ 1A

Since $\overrightarrow{CB} \parallel \overrightarrow{OA} \parallel \overrightarrow{DO}$, we have $\overrightarrow{CB} = p\mathbf{a}$ and $\overrightarrow{DO} = q\mathbf{a}$, where p and q are scalars.

$$\begin{aligned}\overrightarrow{DC} &= \overrightarrow{DO} + \overrightarrow{OB} + \overrightarrow{BC} \\ &= q\mathbf{a} + \mathbf{b} - p\mathbf{a} \\ &= \mathbf{b} + k\mathbf{a}, \text{ where } k = q - p\end{aligned}$$

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{DC}\right|$$

$$|\mathbf{b} - \mathbf{a}| = |\mathbf{b} + k\mathbf{a}| \qquad \text{1M}$$

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{b} + k\mathbf{a}|^2$$

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = (\mathbf{b} + k\mathbf{a}) \cdot (\mathbf{b} + k\mathbf{a})$$

$$|\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2 = |\mathbf{b}|^2 + 2k\mathbf{a} \cdot \mathbf{b} + k^2|\mathbf{a}|^2 \qquad \text{1M}$$

$$(1 - k^2)|\mathbf{a}|^2 = 2(k + 1)\mathbf{a} \cdot \mathbf{b}$$

$$(1 - k^2)(7^2) = 2(k + 1)(42)$$

$$7k^2 + 12k + 5 = 0$$

$$(7k + 5)(k + 1) = 0 \qquad \text{1M}$$

$$k = -\frac{5}{7} \text{ or } -1 \text{ (rejected)}$$

$$\therefore \quad \overrightarrow{DC} = \underline{\underline{\mathbf{b} - \frac{5}{7}\mathbf{a}}} \qquad \text{1A}$$

(5)

Smart Tips

When $k = -1$, $\overrightarrow{DC} = \overrightarrow{AB}$. Therefore, $k = -1$ should be rejected.

(b) (i) $\because$ $PQ \parallel DB$

$\therefore$ $\Delta CPQ \sim \Delta CDB$ (*AAA*)

$$\therefore \quad \frac{CP}{CD} = \frac{CQ}{CB} \quad (\textit{corr. sides}, \sim \Delta s)$$

$$\frac{1}{1 + r} = \frac{r}{1 + r} \qquad \text{1M}$$

$$r = \underline{\underline{1}} \qquad \text{1A}$$

(ii)
$$\begin{aligned}\overrightarrow{DB} &= \overrightarrow{DC} + \overrightarrow{CB} \\ &= \left(\mathbf{b} - \frac{5}{7}\mathbf{a}\right) + p\mathbf{a} \\ &= \mathbf{b} + \left(p - \frac{5}{7}\right)\mathbf{a}\end{aligned}$$

$\because$ $AB \perp DB$ ($\angle$ *in semicircle*)

$\therefore$
$$\overrightarrow{AB} \cdot \overrightarrow{DB} = 0$$

$$(\mathbf{b} - \mathbf{a}) \cdot \left[\mathbf{b} + \left(p - \frac{5}{7}\right)\mathbf{a}\right] = 0 \qquad \text{1M}$$

$$|\mathbf{b}|^2 - \mathbf{a} \cdot \mathbf{b} + \left(p - \frac{5}{7}\right)\mathbf{a} \cdot \mathbf{b} - \left(p - \frac{5}{7}\right)|\mathbf{a}|^2 = 0$$

$$12^2 - 42 + \left(p - \frac{5}{7}\right)(42) - \left(p - \frac{5}{7}\right)(7^2) = 0 \qquad \text{1M}$$

$$7\left(p - \frac{5}{7}\right) = 102$$

$$p = \frac{107}{7}$$

$$\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BQ} + \overrightarrow{QP}$$
$$= (\mathbf{b} - \mathbf{a}) - \frac{1}{2}\left(\frac{107}{7}\mathbf{a}\right) - \frac{1}{2}\left(\mathbf{b} + \frac{102}{7}\mathbf{a}\right)$$
$$= \frac{1}{2}\mathbf{b} - \frac{223}{14}\mathbf{a}$$ 1A

Let θ be the angle between $\overrightarrow{AP}$ and $\overrightarrow{OB}$.

$$\left|\overrightarrow{AP}\right|^2 = \left(\frac{1}{2}\mathbf{b} - \frac{223}{14}\mathbf{a}\right) \cdot \left(\frac{1}{2}\mathbf{b} - \frac{223}{14}\mathbf{a}\right)$$
$$= \frac{1}{4}|\mathbf{b}|^2 - \frac{223}{14}\mathbf{a} \cdot \mathbf{b} + \frac{49\ 729}{196}|\mathbf{a}|^2$$
$$= \frac{1}{4}(12^2) - \frac{223}{14}(42) + \frac{49\ 729}{196}(7^2)$$
$$= \frac{47\ 197}{4}$$

$$\left|\overrightarrow{AP}\right| = \frac{\sqrt{47\ 197}}{2}$$ 1A

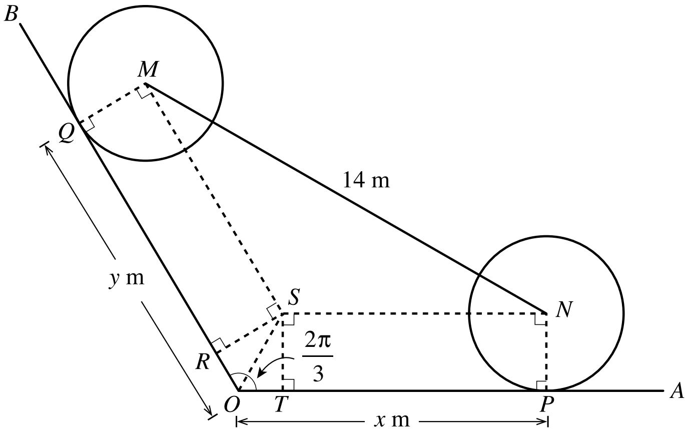
$$\overrightarrow{AP} \cdot \overrightarrow{OB} = \left|\overrightarrow{AP}\right|\left|\overrightarrow{OB}\right|\cos\theta$$
$$\left(\frac{1}{2}\mathbf{b} - \frac{223}{14}\mathbf{a}\right) \cdot \mathbf{b} = \frac{\sqrt{47\ 197}}{2} \times 12 \times \cos\theta$$ 1M
$$\frac{1}{2}|\mathbf{b}|^2 - \frac{223}{14}\mathbf{a} \cdot \mathbf{b} = \frac{\sqrt{47\ 197}}{2} \times 12 \times \cos\theta$$
$$\frac{1}{2}(12^2) - \frac{223}{14}(42) = \frac{\sqrt{47\ 197}}{2} \times 12 \times \cos\theta$$
$$\cos\theta = -\frac{199}{2\sqrt{47\ 197}}$$
$$\theta \approx 117.258\ 153\ 8^\circ$$

The required acute angle $\approx 180^\circ - 117.258\ 153\ 8^\circ$

$= \underline{\underline{63^\circ}}$ (*cor. to the nearest degree*) 1A

(8)

14 (a)



Construct rectangles *QMSR* and *PNST* as shown in the figure.

In ΔSRO and ΔSTO,

$\angle SRO = \angle STO = 90°$ (*given*)

$RS = TS = \sqrt{3}$ m (*given*)

$OS = OS$ (*common side*)

$\therefore$ $\Delta SRO \cong \Delta STO$ (*RHS*)

$\therefore$ $OR = OT = \dfrac{\sqrt{3}}{\tan\dfrac{\pi}{3}}$ m $= 1$ m 1A

Consider ΔMNS.

$NS = (x - 1)$ m

$MS = (y - 1)$ m

By the cosine formula,

$$(x-1)^2 + (y-1)^2 - 2(x-1)(y-1)\cos\frac{2\pi}{3} = 14^2 \qquad \text{1M}$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 + xy - x - y + 1 = 196$$

$$x^2 + y^2 - 3x - 3y + xy = 193 \quad ......(1)$$

Differentiating (1) with respect to x, we have

$$2x + 2y\frac{dy}{dx} - 3 - 3\frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \qquad \text{1M}$$

$$\frac{dy}{dx} = -\frac{2x + y - 3}{x + 2y - 3} \qquad 1$$

(4)

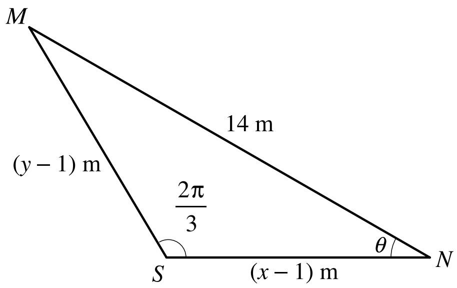
(b) When $x = 11$,

$$11^2 + y^2 - 3(11) - 3y + 11y = 193 \quad \text{1M}$$

$$y^2 + 8y - 105 = 0$$

$$(y - 7)(y + 15) = 0$$

$$y = 7 \text{ or } -15 \text{ (rejected)} \quad \text{1A}$$



By the sine formula, we have

$$\frac{y-1}{\sin\theta} = \frac{14}{\sin\frac{2\pi}{3}} \quad \text{1M}$$

$$y = \frac{28}{\sqrt{3}}\sin\theta + 1 \quad \text{......(2)}$$

When $y = 7$, $\sin\theta = \frac{3\sqrt{3}}{14}$ and

$$\cos\theta = \sqrt{1 - \left(\frac{3\sqrt{3}}{14}\right)^2} = \frac{13}{14} \quad \text{1A}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= -\frac{2x + y - 3}{x + 2y - 3} \times (-2) \quad \text{1M}$$

When $x = 11$ and $y = 7$,

$$\frac{dy}{dt} = -\frac{2(11) + 7 - 3}{11 + 2(7) - 3} \times (-2)$$

$$= \frac{26}{11} \quad \text{1A}$$

Differentiating (2) with respect to t, we have

$$\frac{dy}{dt} = \frac{28}{\sqrt{3}}\cos\theta\frac{d\theta}{dt} \quad \text{1M}$$

When $y = 7$,

$$\frac{dy}{dt} = \frac{28}{\sqrt{3}} \times \frac{13}{14} \times \frac{d\theta}{dt} = \frac{26}{\sqrt{3}} \times \frac{d\theta}{dt}$$

$$\therefore \quad \frac{26}{\sqrt{3}} \times \frac{d\theta}{dt} = \frac{26}{11} \quad \text{1M}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{3}}{11}$$

$\therefore$ The rate of increase of θ is $\frac{\sqrt{3}}{11}$ radian per second. 1A

(9)