2021-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2021

# MATHEMATICS Extended Part
# Module 2 (Algebra and Calculus)
## Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

### INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.



Please stick the barcode label here.

| Candidate Number | | | | | | | | | |
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*A032E001*

## FORMULAS FOR REFERENCE

| | |
|---|---|
| $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ | $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$ |
| $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ | $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$ |
| $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ | $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$ |
| $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$ | $\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$ |
| $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ | |
| $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$ | |

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**SECTION A (50 marks)**

1. Let $f(x) = \dfrac{1}{3x^2+4}$ . Find $f'(x)$ from first principles. (4 marks)



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2. Using mathematical induction, prove that $\sum_{k=1}^{n}(3k^5+k^3)=\frac{n^3(n+1)^3}{2}$ for all positive integers $n$.

(5 marks)

Answers written in the margins will not be marked.

3. The coefficient of $x^2$ in the expansion of $(1-4x)^n$ is $240$, where $n$ is a positive integer. Find

(a) $n$,

(b) the coefficient of $x^4$ in the expansion of $(1-4x)^n\left(1+\frac{2}{x}\right)^5$.

(6 marks)

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4. (a) Prove that $\cos 2x + \cos 4x + \cos 6x = 4\cos x \cos 2x \cos 3x - 1$ .

(b) Solve the equation $\cos 4\theta + \cos 8\theta + \cos 12\theta = -1$ , where $0 \le \theta \le \frac{\pi}{2}$ .

(6 marks)

Answers written in the margins will not be marked.

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5. Define $r(x)=\dfrac{x^3-x^2-2x+3}{(x-1)^2}$ for all real numbers $x \neq 1$ .

(a) Find the asymptote(s) of the graph of $y=r(x)$ .

(b) Find $\dfrac{d}{dx}r(x)$ .

(c) Someone claims that there is only one point of inflexion of the graph of $y=r(x)$ . Do you agree? Explain your answer.

(7 marks)

Answers written in the margins will not be marked.

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6. Consider the curve $\Gamma: y=e^{2x-6}$ . Denote the normal to $\Gamma$ at the point $(3,1)$ by $L$ . Let $c$ be the $x$-intercept of $L$ . Find

(a) $c$ ;

(b) the area of the region bounded by $L$ , $\Gamma$ and the straight line $x=c$ .

(7 marks)

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7. (a) Using integration by parts, find $\int (\ln x)^2 \, dx$ .

(b) Consider the curve $C: y = \sqrt{x} \ln(x^2+1)$ , where $x \geq 0$ . Let $R$ be the region bounded by $C$, the straight line $x = 1$ and the $x$-axis. Find the volume of the solid of revolution generated by revolving $R$ about the $x$-axis.

(7 marks)

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8. Consider the system of linear equations in real variables $x$, $y$, $z$

$$(E):\begin{cases} x + (d-1)y + (d+3)z = 4-d \\ 2x + (d+2)y - z = 2d-5 \\ 3x + (d+4)y + 5z = 2 \end{cases}, \text{ where } d \in \mathbf{R} .$$

It is given that $(E)$ has infinitely many solutions.

(a) Find $d$. Hence, solve $(E)$.

(b) Someone claims that $(E)$ has a real solution $(x, y, z)$ satisfying $xy + 2xz = 3$. Is the claim correct? Explain your answer.

(8 marks)

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**SECTION B (50 marks)**

9. (a) Let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ .

(i) Find $\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\sec\theta + \tan\theta)$ .

(ii) Using the result of (a)(i), find $\int \sec\theta \, \mathrm{d}\theta$ . Hence, find $\int \sec^3\theta \, \mathrm{d}\theta$ .

(4 marks)

(b) Let $\mathrm{g}(x)$ and $\mathrm{h}(x)$ be continuous functions defined on $\mathbf{R}$ such that $\mathrm{g}(x) + \mathrm{g}(-x) = 1$ and $\mathrm{h}(x) = \mathrm{h}(-x)$ for all $x \in \mathbf{R}$ .

Using integration by substitution, prove that $\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\,\mathrm{d}x = \int_{0}^{a} \mathrm{h}(x)\,\mathrm{d}x$ for any $a \in \mathbf{R}$ .

(3 marks)

(c) Evaluate $\int_{-1}^{1} \frac{3^x x^2}{(3^x + 3^{-x})\sqrt{x^2+1}}\,\mathrm{d}x$ . (5 marks)

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10. Denote the graph of $y=\sqrt{x^2+36}$ and the graph of $y=-\sqrt{(20-x)^2+16}$ by $F$ and $G$ respectively, where $0<x<20$ . Let $P$ be a moving point on $F$ . The vertical line passing through $P$ cuts $G$ at the point $Q$ . Denote the $x$-coordinate of $P$ by $u$ . It is given that the length of $PQ$ attains its minimum value when $u=a$ .

(a) Find $a$ . (4 marks)

(b) The horizontal line passing through $P$ cuts the $y$-axis at the point $R$ while the horizontal line passing through $Q$ cuts the $y$-axis at the point $S$ .

(i) Someone claims that the area of the rectangle $PQSR$ attains its minimum value when $u=a$ . Do you agree? Explain your answer.

(ii) The length of $OP$ increases at a constant rate of $28$ units per minute. Find the rate of change of the perimeter of the rectangle $PQSR$ when $u=a$ .

(9 marks)

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11. Define $P=\begin{pmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{pmatrix}$, where $\dfrac{\pi}{2}<\theta<\pi$ .

(a) Let $A=\begin{pmatrix}\alpha & \beta\\ \beta & -\alpha\end{pmatrix}$, where $\alpha,\beta\in\mathbf{R}$ .

Prove that $PAP^{-1}=\begin{pmatrix}-\alpha\cos 2\theta+\beta\sin 2\theta & -\beta\cos 2\theta-\alpha\sin 2\theta\\ -\beta\cos 2\theta-\alpha\sin 2\theta & \alpha\cos 2\theta-\beta\sin 2\theta\end{pmatrix}$ . (3 marks)

(b) Let $B=\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$ .

(i) Find $\theta$ such that $PBP^{-1}=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, where $\lambda,\mu\in\mathbf{R}$ .

(ii) Using the result of (b)(i), prove that $B^n=2^{n-2}\begin{pmatrix}(-1)^n+3 & \sqrt{3}(-1)^{n+1}+\sqrt{3}\\ \sqrt{3}(-1)^{n+1}+\sqrt{3} & 3(-1)^n+1\end{pmatrix}$ for any positive integer $n$ .

(iii) Evaluate $(B^{-1})^{555}$ .

(9 marks)

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12. The position vectors of the points $A$, $B$, $C$ and $D$ are $t\mathbf{i}+14\mathbf{j}+s\mathbf{k}$, $12\mathbf{i}-s\mathbf{j}-2\mathbf{k}$, $(s+2)\mathbf{i}-16\mathbf{j}+10\mathbf{k}$ and $-t\mathbf{i}+(s+2)\mathbf{j}+14\mathbf{k}$ respectively, where $s, t \in \mathbf{R}$. Suppose that $\overrightarrow{AB}$ is parallel to $5\mathbf{i}-4\mathbf{j}-2\mathbf{k}$. Denote the plane which contains $A$, $B$ and $C$ by $\Pi$.

(a) Find

(i) $s$ and $t$,

(ii) the area of $\Delta ABC$,

(iii) the volume of the tetrahedron $ABCD$,

(iv) the shortest distance from $D$ to $\Pi$.

(9 marks)

(b) Let $E$ be the projection of $D$ on $\Pi$. Is $E$ the circumcentre of $\Delta ABC$? Explain your answer. (4 marks)