Chapter 5

Applications of Differentiation

1. **HKDSE Math M2 Sample Paper Q2**
   A snowball in a shape of sphere is melting with its volume decreasing at a constant rate of 4 cm^3s^{-1}.
   When its radius is 5 cm, find the rate of change of its radius.
   (4 marks)

2. **HKDSE Math M2 Sample Paper Q6**
   Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$.
   Find the equation of the tangent to C at the point $(1, 1)$.
   (5 marks)

3. **HKDSE Math M2 Sample Paper Q12**
   Let $f(x) = \dfrac{4}{x-1} - \dfrac{4}{x+1} - 1$, where $x \neq \pm 1$.

   (a) (i) Find the x- and y-intercept(s) of the graph of $y = f(x)$.
   (ii) Find $f'(x)$ and prove that
   $$f''(x) = \frac{16(3x^2+1)}{(x-1)^3(x+1)^3}$$
   for $x \neq \pm 1$.
   (iii) For the graph of $y = f(x)$, find all the extreme points and show that there are no points of inflexion.
   (6 marks)

   (b) Find all the asymptote(s) of the graph of $y = f(x)$.
   (2 marks)

   (c) Sketch the graph of $y = f(x)$.
   (3 marks)

   (d) Let S be the area bounded by the graph of $y = f(x)$, the straight lines $x = 3$, $x = a$ $(a > 3)$ and $y = -1$.
   Find S in terms of a. Deduce that $S < 4\ln 2$.
   (3 marks)

4. **HKDSE Math M2 Practice Paper Q7**
   Let $f(x) = e^x(\sin x + \cos x)$.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Find the value of x such that $f''(x) - f'(x) + f(x) = 0$ for $0 \leq x \leq \pi$.

   (5 marks)

5. **HKDSE Math M2 Practice Paper Q9**
   Find the equations of the two tangents to the curve $x^2 - xy - 2y^2 - 1 = 0$ which are parallel to the straight line $y = 2x + 1$.
   (6 marks)

6. **HKDSE Math M2 2012 Q5**
   Find the minimum point(s) and asymptote(s) of the graph of $y = \dfrac{x^2+x+1}{x+1}$.
   (6 marks)

7. **HKDSE Math M2 2012 Q6**

A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (See Figure 1). It is given that the volume of the frustum is $\frac{\pi}{3}H(r^2 + rR + R^2)$. An empty glass is in the form of an inverted frustum described above with height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass. Let h cm $(0 \le h \le 10)$ be the depth of the water inside the glass at time t s (see Figure 2).

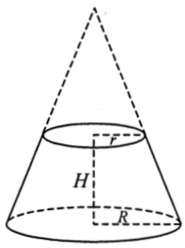


Figure 1

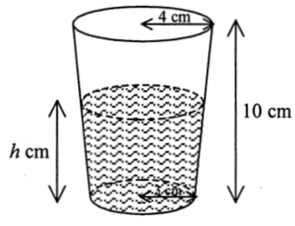


Figure 2

(a) Show that the volume V cm^3 of water inside the glass at time t s is given by

$$V = \frac{\pi}{300}(h^3 + 90h^2 + 2700h).$$

(b) If the volume of water in the glass is increasing at the rate 7π cm^3 s^{-1}, find the rate of increase of depth of water at the instant when $h = 5$.

(6 marks)

8. **HKDSE Math M2 2012 Q14**
Consider the curve $\Gamma : y = kx^p$, where $k > 0$, $p > 0$. In Figure 7, the tangent to Γ at $A(a, ka^p)$ cuts the x-axis at $B(-a, 0)$, where $a > 0$.

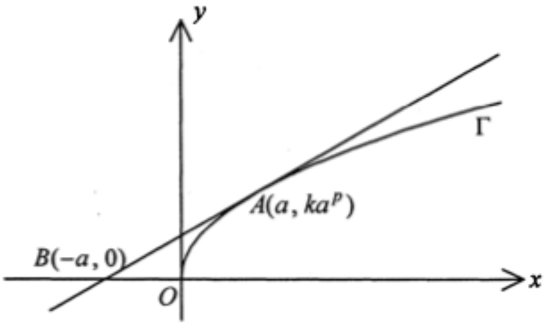


Figure 7

(a) Show that $p = \dfrac{1}{2}$.
(3 marks)

(b) Suppose that $a = 1$. As shown in Figure 8, the circle C, with radius 2 and centre on the y-axis, touches Γ at point A.

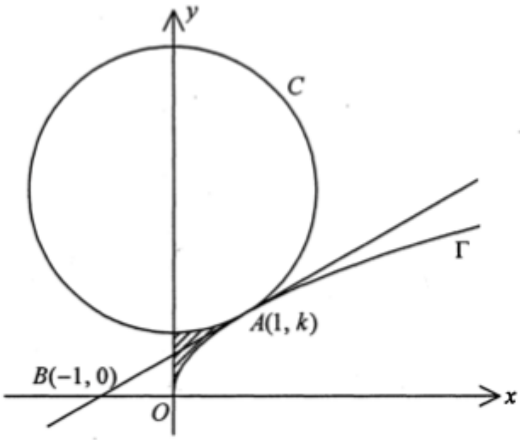


Figure 8

(i) Show that $k = \dfrac{2\sqrt{3}}{3}$.
(ii) Find the area of the shaded region bounded by Γ, C and the y-axis.
(9 marks)

9. **HKDSE Math M2 2013 Q5**
Consider a continuous function $f(x) = \dfrac{3 - 3x^2}{3 + x^2}$. It is given that

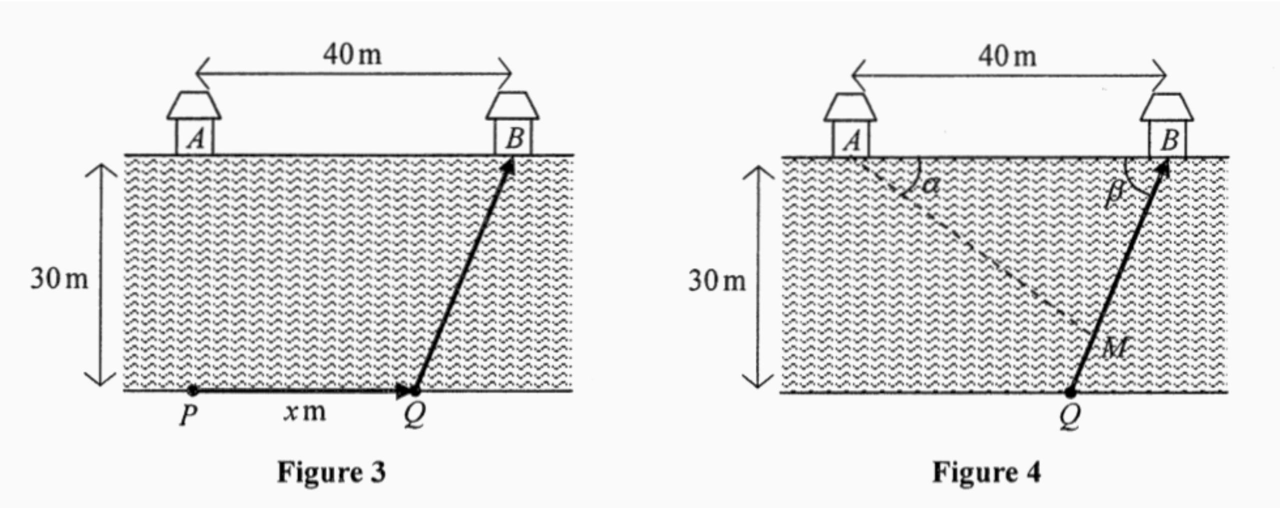
x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

('+' and '−' denote 'positive value' and 'negative value' respectively.)

(a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.
(b) Find the asymptote(s) of the graph of $y = f(x)$.
(c) Sketch the graph of $y = f(x)$.

(6 marks)

10. **HKDSE Math M2 2013 Q12**
In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m. The width of the river is always 30 m. In the beginning, Mike stands at the starting point P in the opposite bank which is 30 m from A. Mike's wife, situated at A, is watching him running along the bank for x m at a constant speed of 7 m s^{-1} to point Q and then swimming at a constant speed of 1.4 m s^{-1} along a straight path to reach B.



(a) Let T seconds be the time that Mike travels from P to B.

(i) Express T in terms of x.

(ii) When T is minimum, show that x satisfies the equation $2x^2 - 160x + 3125 = 0$.
Hence show that

$$QB = \frac{25\sqrt{6}}{2} \text{ m.}$$

(6 marks)

(b) In Figure 4, Mike is swimming from Q to B with QB equal to the value mentioned in (a)(ii).
Let $\angle MAB = \alpha$ and $\angle ABM = \beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that

$$MB = \frac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}}.$$

(ii) Find the rate of change of α when $\alpha = 0.2$ radian. Correct your answer to 4 decimal places.

(7 marks)

11. **HKDSE Math M2 2014 Q2**
Consider the curve $C : y = x^3 - 3x$.

(a) Find $\frac{dy}{dx}$ from first principles.

(b) Find the range of x where C is decreasing.

(5 marks)

12. **HKDSE Math M2 2014 Q3**
Find the equation of tangent to the curve $x \ln y + y = 2$ at the point where the curve cuts the y-axis.
(5 marks)

13. **HKDSE Math M2 2014 Q4**
Let $x = 2y + \sin y$. Find $\frac{d^2y}{dx^2}$ in terms of y.
(3 marks)

14. **HKDSE Math M2 2014 Q10**

Thomas has a bookcase of dimensions 100 cm $\times$ 24 cm $\times$ 192 cm at the corner in his room. He wants to hang a decoration on the wall above the bookcase. Therefore, he finds a ladder to climb up. Initially, the ladder touches the wall, the edge of the top of the bookcase and the floor at the same time. Let rectangle $ABCD$ be the side-view of the bookcase and HK be the side-view of the ladder, so that $AB = 24$ cm and $BC = 192$ cm (see Figure 2). Let $\angle HKD = \theta$.

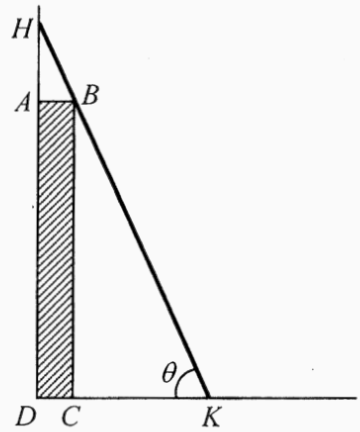


Figure 2

(a) Find the length of HK in terms of θ.
(1 marks)

(b) Prove that the shortest length of the ladder is $120\sqrt{5}$ cm.
(5 marks)

(c) Suppose the length of the ladder is 270 cm. Suddenly, the ladder slides down so that the end of the ladder, K, moves towards E (see Figure 3). The ladder touches the edge of the top of the bookcase and the floor at the same time. Let x cm and y cm be the horizontal distances from H and K respectively to the wall.

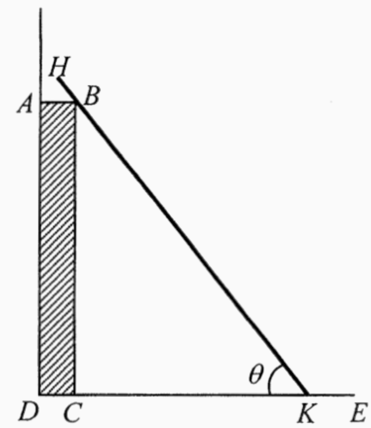


Figure 3

(i) When $CK = 160$ cm, the rate of change of θ is -0.1 rad s^{-1}. Find the rate of change of x at this moment,
correct to 4 significant figures.

(ii) Thomas claims that K is moving towards E at a speed faster than the horizontal speed H is leaving the wall. Do you agree? Explain your answer.

(6 marks)

15. **HKDSE Math M2 2015 Q2**
Let $y = x\sin x + \cos x$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Let k be a constant such that $x\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + xy = 0$ for all real values of x. Find the value of k.

(5 marks)

16. **HKDSE Math M2 2015 Q9**
Define $f(x) = \dfrac{x^2+12}{x-2}$ for all $x \neq 2$.

(a) Find $f'(x)$.
(2 marks)

(b) Prove that the maximum value and the minimum value of $f(x)$ are -4 and 12 respectively.
(4 marks)

(c) Find the asymptote(s) of the graph of $y = f(x)$.
(3 marks)

(d) Find the area of the region bounded by the graph of $y = f(x)$ and the horizontal line $y = 14$.
(4 marks)

17. **HKDSE Math M2 2016 Q3**
Consider the curve $C : y = 2e^x$, where $x > 0$. It is given that P is a point lying on C. The horizontal line which passes through P cuts the y-axis at the point Q. Let O be the origin. Denote the x-coordinate of P by u.

(a) Express the area of $\triangle OPQ$ in terms of u.

(b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of $\triangle OPQ$ when $u = 4$.

(5 marks)

18. **HKDSE Math M2 2016 Q4**
Define $f(x) = \dfrac{2x^2+x+1}{x-1}$ for all $x \neq 1$. Denote the graph of $y = f(x)$ by G. Find

(a) the asymptote(s) of G,

(b) The slope of the normal to G at the point $(2, 11)$.

(7 marks)

19. **HKDSE Math M2 2016 Q9**
Let a and b be constants. Define $f(x) = x^3 + ax^2 + bx + 5$ for all real numbers x. Denote the curve $y = f(x)$ by C. It is given that $P(-1, 10)$ is a turning point of C.

(a) Find a and b.
(3 marks)

(b) Is P a maximum point of C? Explain your answer.
(2 marks)

(c) Find the minimum value(s) of $f(x)$.
(2 marks)

(d) Find the point(s) of inflexion of C.
(2 marks)

(e) Let L be the tangent to C at P. Find the area of the region bounded by C and L.
(4 marks)

20. **HKDSE Math M2 2017 Q6**
A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

(a) Let A cm^2 be the wet curve surface area of the container and h cm be the depth of water in the container. Prove that $A = \dfrac{15}{16}\pi h^2$.

(b) The depth of water in the container increases at a constant rate of $\dfrac{3}{\pi}$ cm/s. Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm^3.

(7 marks)

21. **HKDSE Math M2 2017 Q8**
Let $f(x)$ be a continuous function defined on $\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $P(e^3, 7)$ and $f'(x) = \dfrac{1}{x}\ln x^2$ for all $x > 0$. Find

(a) the equation of the tangent to Γ at P,

(b) the equation of Γ,

(c) the point(s) of inflexion of Γ.

(8 marks)

22. **HKDSE Math M2 2017 Q9**
Define $f(x) = \dfrac{x^2 - 5x}{x + 4}$ for all $x \neq -4$. Denote the graph of $y = f(x)$ by G.

(a) Find the asymptote(s) of G.
(3 marks)

(b) Find $f'(x)$.
(2 marks)

(c) Find the maximum point(s) and the minimum point(s) of G.
(4 marks)

(d) Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.
(4 marks)

23. **HKDSE Math M2 2018 Q8**
Define $f(x) = \dfrac{A}{x^2 - 4x + 7}$ for all real numbers x, where A is a constant.
It is given that the extreme value of $f(x)$ is 4.

(a) Find $f'(x)$.

(b) Someone claims that there are at least two asymptotes of the graph of $y = f(x)$. Do you agree? Explain your answer.

(c) Find the point(s) of inflexion of the graph of $y = f(x)$.

(8 marks)

24. **HKDSE Math M2 2018 Q9**
Consider the curve $C : y = \ln\sqrt{x}$, where $x > 1$. Let P be a moving point lying on C. The normal to C at P cuts the x-axis at the point Q while the vertical line passing through P cuts the x-axis at the point R.

(a) Denote the x-coordinate of P by r. Prove that the x-coordinate of Q is $\dfrac{4r^2 + \ln r}{4r}$.
(3 marks)

(b) Find the greatest area of $\triangle PQR$.
(5 marks)

(c) Let O be the origin. It is given that OP increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of $\triangle PQR$ increases at a rate lower than 2 square units per minute when the x-coordinate of P is e. Is the claim correct? Explain your answer.
(4 marks)

25. **HKDSE Math M2 2019 Q3**
A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580cm^3 of liquid X. The researcher finds that during the experiment, $\frac{dV}{dt} = -2t$, where $V\text{cm}^3$ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

(a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

(b) It is given that $V = h^2 + 24h$, where h cm is the depth of liquid X in the vessel. Find the value of $\frac{dh}{dt}$ when $t = 18$.

(6 marks)

26. **HKDSE Math M2 2019 Q4**
Define $g(x) = \frac{\ln x}{\sqrt{x}}$ for all $x \in (0, 99)$. Denote the graph of $y = g(x)$ by G.

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G, the x-axis and the vertical line passing through the maximum point of G. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

27. **HKDSE Math M2 2019 Q8**
Let $h(x)$ be a continuous function defined on $\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \frac{2x^2 - 7x + 8}{x}$ for all $x > 0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = h(x)$ by H. It is given that H passes through the point $(1, 3)$. Find
  (i) the equation of H,
  (ii) the point(s) of inflexion of H.

(8 marks)

28. **HKDSE Math M2 2019 Q9**
Consider the curve $\Gamma : y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$. Denote the tangent of Γ at $x = 3$ by L.

(a) Find the equation of L.
(3 marks)

(b) Let C be the curve $y = \sqrt{4 - x^2}$, where $0 < x < 2$. It is given that L is a tangent to C. Find
  (i) the point(s) of contact of L and C;
  (ii) the point(s) of intersection of C and Γ;
  (iii) the area of region bounded by L, C and Γ.
  (9 marks)

29. **HKDSE Math M2 2020 Q6**
Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the region by O. Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

(a) Define $S = u^2 + v^2$. Does S increase at a constant rate? Explain your answer.

(b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q. Find the rate of change of the area of $\triangle OPQ$ when $u = 2$.

(7 marks)

30. **HKDSE Math M2 2020 Q9**
Let $f(x) = \dfrac{(x+4)^3}{(x-4)^2}$ for all real numbers $x \neq 4$. Denote the graph of $y = f(x)$ by H.

(a) Find the asymptote(s) of H.
(3 marks)

(b) Find $f''(x)$.
(2 marks)

(c) Someone claims that there are two turning points of H. Do you agree? Explain your answer.
(2 marks)

(d) Find the point(s) of inflexion of H.
(2 marks)

(e) Find the area of the region bounded by H, the x-axis and the y-axis.
(3 marks)

31. **HKDSE Math M2 2021 Q5**
Define $r(x) = \dfrac{x^3 - x^2 - 2x + 3}{(x-1)^2}$ for all real numbers $x \neq 1$.

(a) Find the asymptote(s) of the graph of $y = r(x)$.

(b) Find $\dfrac{d}{dx}r(x)$.

(c) Someone claims that there is only one point of inflexion of the graph of $y = r(x)$. Do you agree? Explain your answer.

(7 marks)

32. **HKDSE Math M2 2021 Q6**
Consider the curve $\Gamma : y = e^{2x-6}$. Denote the normal to Γ at the point $(3, 1)$ by L.
Let c be the x-intercept of L. Find

(a) c;

(b) the area of the region bounded by L, Γ and the straight line $x = c$.

(7 marks)

33. **HKDSE Math M2 2021 Q10**
Denote the graph of $y = \sqrt{x^2 + 36}$ and the graph of $y = -\sqrt{(20-x)^2 + 16}$ by F and G respectively, where $0 < x < 20$. Let P be a moving point on F. The vertical line passing through P cuts G at the point Q. Denote the x-coordinate of P by u. It is given that the length of PQ attains its minimum value when $u = a$.

(a) Find a.
(4 marks)

(b) The horizontal line passing through P cuts the y-axis at the point R while the horizontal line passing through Q cuts the y-axis at the point S.

(i) Someone claims that the area of the rectangle $PQSR$ attains its minimum value when $u = a$. Do you agree? Explain your answer.

(ii) The length of OP increases at a constant rate of 28 units per minute. Find the rate of change of the perimeter of the rectangle $PQSR$ when $u = a$.

(9 marks)

34. **HKDSE Math M2 2022 Q4**
Let $y = (7x - 2x^2)e^{-x}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Someone claims that there are two points of inflexion of the graph of $y = (7x - 2x^2)e^{-x}$. Do you agree? Explain your answer.

(6 marks)

35. **HKDSE Math M2 2022 Q7**
Consider the curve $\Gamma : y = \ln(x+2)$, where $x > 0$. Let P be a moving point on Γ with h as its x-coordinate. Denote the tangent to Γ at P by L and the area of the region bounded by Γ, L and the y-axis by A square units.

(a) Prove that $A = \frac{h^2 + 4h}{2h + 4} - 2\ln(h+2) + 2\ln 2$.

(b) If $h = 3^{-t}$, where t is the time measured in seconds, find the rate of change of A when $t = 1$.

(8 marks)

36. **HKDSE Math M2 2022 Q9**
Let $f(x) = \frac{x^2 + 3x}{x - 1}$, where $x \neq 1$. Denote the graph of $y = f(x)$ by H.

(a) Find the asymptote(s) of H.
(3 marks)

(b) Find the maximum point(s) and minimum point(s) of H.
(4 marks)

(c) Sketch H.
(3 marks)

(d) Let R be the region bounded by H and the straight line $y = 10$. Find the volume of the solid of revolution generated by revolving R about the straight line $y = 10$.
(3 marks)