香港考試及評核局
Hong Kong Examinations and Assessment Authority

按查閱資料要求提供的報告
INFORMATION PROVIDED ON DATA ACCESS REQUEST

考生編號
Candidate No.:

	評卷紀錄 (包括網上評卷系統的評分資料) MARKING RECORDS (INCLUDING DATA FROM THE ON-SCREEN MARKING SYSTEM)	
科目名稱 Subject name:	數學延伸部分（代數與微積分） Mathematics Extended Part (Algebra and Calculus) - E	
卷別組 Paper group:	Paper 1	
評卷組 Panel:	1A	1B
分部基本得分 Section raw mark:	43 M1 (out of 50)	49 M1 (out of 50)
分部調整得分 Section adjusted mark:	43	49
卷別調整得分 Paper adjusted mark:	92	
卷別組調整得分 Paper group adjusted mark:	92	
加權卷別得分 Weighted paper mark:	92	
科目得分 Subject mark:	92 (out of 100)	
科目等級 Subject level:	5**	

M1 = 第一次評卷 1st marking

基本得分 = 閱卷員評卷後所給予的分數
調整得分 = 已加入調整分數的總基本得分，以維持評分寬緊的公平性
卷別得分 = 各題或分部調整得分按比重相加的總和
加權卷別得分 = 因應各卷的分數分布及相對比重，把各卷別得分轉換至同一尺度的得分

科目得分 = 加權卷別得分的總和

Raw mark = Mark awarded by a marker
Adjusted mark = Total raw mark with adjustment made to maintain fairness between strict and lenient marking
Paper mark = Weighted sum of the adjusted marks of all questions / sections in the paper
Weighted paper mark = Paper mark converted to the same comparable scale as other papers by taking into account the spread of marks and the relative paper weighting
Subject mark = Sum of weighted paper marks

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考生編號
Candidate No.:

數學延伸部分（代數與微積分） 卷1
Mathematics Extended Part (Algebra and Calculus) - E Paper 1

試題號數 Question no.	評卷組 Panel	項目 Item	最高分數 Max. Mark	M1
1	1A	1(a)	1	1
		1(b)	3	3
2	1A	2	4	4
3	1A	3(a)(i)	2	2
		3(a)(ii)	2	2
		3(b)	2	2
4	1A	4(a)	2	2
		4(b)	4	3
5	1A	5(a)	4	4
		5(b)	3	3
6	1A	6(a)	2	1
		6(b)	5	4
7	1A	7(a)	3	3
		7(b)(i)	2	2
		7(b)(ii)	3	3
8	1A	8(a)	3	3
		8(b)(i)	2	1
		8(b)(ii)	3	0
9	1B	9(a)	3	3
		9(b)	2	2
		9(c)	2	2
		9(d)	2	2
		9(e)	3	3
10	1B	10(a)	3	3
		10(b)	3	3
		10(c)(i)	2	2
		10(c)(ii)	5	4

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考生編號
Candidate No.:

數學延伸部分（代數與微積分） 卷1
Mathematics Extended Part (Algebra and Calculus) - E Paper 1

試題號數 Question no.	評卷組 Panel	項目 Item	最高分數 Max. Mark	M1
11	1B	11(a)(i)(1)	3	3
		11(a)(i)(2)	3	3
		11(a)(ii)	3	3
		11(b)	4	4
12	1B	12(a)	2	2
		12(b)	2	2
		12(c)(i)	2	2
		12(c)(ii)(1)	3	3
		12(c)(ii)(2)	3	3

2020-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2020

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

Candidate Number

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

****

SECTION A (50 marks)

1. (a) Expand $(1-x)^4$.

   (b) Find the constant k such that the coefficient of x^2 in the expansion of $(1+kx)^9(1-x)^4$ is -3.

(4 marks)

(a) $(1-x)^4$

$= C^4_0(1)^4(-x)^0 + C^4_1(1)^3(-x)^1 + C^4_2(1)^2(-x)^2 + C^4_3(1)^1(-x)^3 + C^4_4(1)^0(-x)^4$

$= 1 - 4x + 6x^2 - 4x^3 + x^4$.

$(1+kx)^9 = 1 + C^9_1(kx) + C^9_2(kx)^2 +$ other terms involving higher powers of x

$= 1 + 9kx + 36k^2x^2 +$ other terms involving higher powers of x.

Coefficient of $x^2 = (36k^2)(1) - 4(9k) + 6$

$= 36k^2 - 36k + 6 = -3$.

So, we have $36k^2 - 36k + 9 = 0$.

$k = \frac{1}{2}$ (repeated)

2. Define $f(x)=\dfrac{x}{\sqrt{2+x}}$ for all $x>-2$. Find $f'(2)$ from first principles. (4 marks)

$$f(2+h)-f(2)$$

$$=\frac{2+h}{\sqrt{2+2+h}}-\frac{2}{\sqrt{2+2}}$$

$$=\frac{2+h}{\sqrt{4+h}}-1.$$

$$f'(2)=\lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{2+h}{\sqrt{4+h}}-1\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{2+h-\sqrt{4+h}}{\sqrt{4+h}}\cdot\frac{2+h+\sqrt{4+h}}{2+h+\sqrt{4+h}}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(4+4h+h^2)-(4+h)}{(2+h)\sqrt{4+h}+(4+h)}$$

$$=\lim_{h\to 0}\frac{h^2+3h}{h}\cdot\lim_{h\to 0}\frac{1}{(2+h)\sqrt{4+h}+(4+h)}$$

$$=\lim_{h\to 0}(h+3)\cdot\lim_{h\to 0}\frac{1}{(2+h)\sqrt{4+h}+4+h}$$

$$=3\cdot\frac{1}{8}$$

$$=\frac{3}{8}.$$

Answers written in the margins will not be marked.

3. (a) Let x be an angle which is not a multiple of $30°$. Prove that

(i) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$,

(ii) $\tan x \tan(60° - x)\tan(60° + x) = \tan 3x$.

(b) Using (a)(ii), prove that $\tan 55° \tan 65° \tan 75° = \tan 85°$.

(6 marks)

(i) $\tan 3x$

$= \tan(x + 2x)$

$= \dfrac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$

$= \dfrac{\tan x + \dfrac{2\tan x}{1-\tan^2 x}}{1 - \dfrac{\tan x \cdot 2\tan x}{1 - \tan^2 x}}$

$= \dfrac{\tan x(1-\tan^2 x) + 2\tan x}{1 - \tan^2 x - 2\tan^2 x} = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(ii). $\tan x \tan(60° - x) \tan(60° + x)$

$= \tan x \cdot \dfrac{\tan 60° - \tan x}{1 + \tan 60° \tan x} \cdot \dfrac{\tan 60° + \tan x}{1 - \tan 60° \tan x}$

$= \tan x \cdot \dfrac{(\sqrt{3})^2 - (\tan x)^2}{(1)^2 - (\sqrt{3}\tan x)^2} = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

$= \tan 3x.$

(b). Put $x = 5°$. Then $\tan 5° \tan 55° \tan 65° = \tan 15°$,

i.e. $\dfrac{\tan 55° \tan 65°}{\tan 15°} = \dfrac{1}{\tan 5°}$.

It follows that $\tan 55° \tan 65° \tan 75° = \tan 85°$.

Answers written in the margins will not be marked.

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Answers written in the margins will not be marked.

4. (a) Find $\int \sin^2\theta \, d\theta$.

(b) Define $f(x) = 4x(1-x^2)^{\frac{1}{4}}$ for all $x \in [0,1]$. Denote the graph of $y = f(x)$ by G . Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

(a) $\int \sin^2\theta \, d\theta$

$= \frac{1}{2}\int (1-\cos 2\theta) \, d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

(b) Volume of solid of revolution

$= \pi \int_0^1 \left(4x(1-x^2)^{\frac{1}{4}}\right)^2 dx$

$= \pi \int_0^1 16x^2\sqrt{1-x^2} \, dx$

$= \pi \int_0^{\frac{\pi}{2}} 16\sin^2\theta \cos\theta \, (\cos\theta \, d\theta)$ (By substituting $x = \sin\theta$, where $0 \le \theta \le \frac{\pi}{2}$)

$= \pi \int_0^{\frac{\pi}{2}} 4\sin^2 2\theta \, d\theta$ ($2\sin\theta\cos\theta = \sin 2\theta$.

$= \pi \int_0^{\pi} 2\sin^2 u \, du$ (Let $u = 2\theta$)

$= \pi \left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi}$ [From (a)]

$= \frac{1}{2}\pi^2$

Answers written in the margins will not be marked.

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5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$.

(7 marks)

(a) Note that $\frac{1}{1(1+1)(1+2)} = \frac{1}{6} = \frac{1(1+3)}{4(1+1)(1+2)}$.

So, the statement is true for $n=1$.

Assume $\sum_{k=1}^{p} \frac{1}{k(k+1)(k+2)} = \frac{p(p+3)}{4(p+1)(p+2)}$

for some $p \in \mathbb{Z}^+$.

Then for $n=p+1$,

$$\sum_{k=1}^{p+1} \frac{1}{k(k+1)(k+2)}$$

$$= \frac{p(p+3)}{4(p+1)(p+2)} + \frac{1}{(p+1)(p+2)(p+3)}$$ By using induction assumption

$$= \frac{p(p+3)^2 + 4}{4(p+1)(p+2)(p+3)}$$

$$= \frac{p^3+6p^2+9p+4}{4(p+1)(p+2)(p+3)}$$

$$= \frac{(p+1)(p^2+5p+4)}{4(p+1)(p+2)(p+3)}$$

$$= \frac{(p+1)(p+4)}{4(p+2)(p+3)} = \frac{(p+1)[(p+1)+3]}{4[(p+1)+1][(p+1)+2]}$$

Thus assuming the statement is true for $n=p$, it is also true for $n=p+1$.

By mathematical induction, $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all $n \in \mathbb{Z}^+$.

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(b) $\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$

$= 50\left[\sum_{k=1}^{123} \frac{1}{k(k+1)(k+2)} - \sum_{k=1}^{3} \frac{1}{k(k+1)(k+2)}\right]$.

$= 50\left[\frac{123(123+3)}{4(123+1)(123+2)} - \frac{3(3+3)}{4(3+1)(3+2)}\right]$.

$= 50\left[\frac{7749}{31000} - \frac{9}{40}\right]$.

$= \frac{387}{310}$.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

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6. Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the origin by O . Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

(a) Define $S = u^2 + v^2$. Does S increase at a constant rate? Explain your answer.

(b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q . Find the rate of change of the area of ΔOPQ when $u = 2$.

(7 marks)

(a) let the area of a circle with diameter OP be A.

$A = \pi (OP)^2$

$= \pi (u^2 + v^2)$.

So, we have $\frac{dA}{dt} = \frac{\pi d(u^2+v^2)}{dt} = 5\pi$.

$= \frac{\pi dS}{dt}$.

Thus, S increases at a constant rate, i.e. 5 square units per second.

(b) • Put $x = u$ into $C_1 : y = 2^{x-1}$.

Then $y = 2^{u-1}$, i.e. coordinates of $P: (u, 2^{u-1})$.

• Put $x = u$ into $C_2 : y = 2^x$.

Then $y = 2^u$, i.e. coordinates of Q: $(u, 2^u)$.

• Area of ΔOPQ

$= \frac{1}{2}$(horizontal distance between O and P)(PQ).

$= \frac{1}{2}(u)[2^u - 2^{u-1}]$

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i.e. area of $\Delta OPQ = \frac{1}{4} u 2^u$ (sq. units)

$$\frac{d(\text{area of } \Delta OPQ)}{dt} = \left(\frac{1}{4} \cdot 2^u + \frac{1}{4\ln 2} \cdot u 2^u\right) \frac{du}{dt}$$

$$= \left(\frac{(\ln 2 + u) 2^u}{4 \ln 2}\right)$$

Note that $\frac{ds}{dt} = \frac{d(u^2+v^2)}{dt}$

$$5 = \frac{d(u^2 + 2^{u-1})}{dt}$$

$$5 = \left(2u + \frac{2^{u-1}}{\ln 2}\right) \frac{du}{dt}$$

$$\left.\frac{du}{dt}\right|_{u=2} = \frac{5}{2(2) + \frac{2^{2-1}}{\ln 2}}$$

$$= \frac{5}{4 + \frac{2}{\ln 2}} = \frac{5 \ln 2}{4 \ln 2 + 2}$$

$$\left.\frac{d(\text{area of } \Delta OPQ)}{dt}\right|_{u=2} = \left(\frac{(\ln 2 + 2) 2^2}{4 \ln 2}\right) \cdot \left(\frac{5 \ln 2}{4 \ln 2 + 2}\right)$$

$$= \left(\frac{\ln 2 + 2}{\ln 2}\right) \cdot \left(\frac{5 \ln 2}{4 \ln 2 + 2}\right)$$

$$= \frac{5(\ln 2)^2 + 10 \ln 2}{4(\ln 2)^2 + 2 \ln 2} \text{ (sq. units s}^{-1}\text{)}$$

Answers written in the margins will not be marked.

7. Let $f(x)$ be a continuous function defined on $\mathbf{R}$. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $(1, 2)$ and $f'(x) = -2x + 8$ for all $x \in \mathbf{R}$.

(a) Find the equation of Γ.

(b) Let L be a tangent to Γ such that L passes through the point $(5, 14)$ and the slope of L is negative. Denote the point of contact of Γ and L by P. Find

(i) the coordinates of P,

(ii) the equation of the normal to Γ at P.

(8 marks)

(a) $f'(x) = -2x+8.$

$f(x) = \int f'(x)\,dx$

$= -x^2+8x+C.$

Γ passes through $(1,2)$

So, $2 = -(1)+8(1)+C.$

$C = -5.$

$\therefore$ equation of Γ : $y = -x^2+8x-5.$

(b). Let the coordinates of P be (a,b).

(i) Then

① Slope of $L = f'(x)\big|_{x=a} < 0,$

i.e. $\dfrac{14-b}{5-a} = -2a+8 < 0.$

$14-b = 2a^2-18a+40$ — ☆

② P lies on Γ.

So, $b = -a^2+8a-5.$ — (D)

sub (D) into ☆,

$14-(-a^2+8a-5) = 2a^2-18a+40.$

$a^2-10a+21=0 \Leftrightarrow a=3$ or $a=7.$

Answers written in the margins will not be marked.

Note that when $a=3$, slope of $L = -2(3)+8 = 2 > 0$.

and when $a=7$, slope of $L = -2(7)+8 = -6 < 0$.

Thus the case $a=3$ should be rejected.

Put $a=7$, $b = -(7)^2 + 8(7) - 5 = 2$

$\therefore$ Coordinates of P: $(7, 2)$.

(b)(ii). Slope of the normal to Γ at P

$= -\dfrac{1}{-6} = \dfrac{1}{6}$.

Required Equation:

$\dfrac{y-2}{x-7} = \dfrac{1}{6}$.

$x - 6y + 5 = 0$.

Answers written in the margins will not be marked.

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8. Define $P=\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Let $M=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}$ such that $|M|=1$ and $PM=MQ$, where a, b and c are real numbers.

~~(a)~~ (a) Find a, b and c.

(b) Define $R=\begin{pmatrix}6 & 2\\ -15 & -5\end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b)(i), prove that $(\alpha P+\beta R)^{99}=\alpha^{99}P+\beta^{99}R$ for any real numbers α and β.

(8 marks)

(a) $|M|=1$

so, $c-ab=1$ — ①.

$PM=MQ$.

$$\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix}\begin{pmatrix}1 & a\\ b & c\end{pmatrix}=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

$$\begin{pmatrix}-5-2b & -5a-2c\\ 15+6b & 15a+6c\end{pmatrix}=\begin{pmatrix}1 & 0\\ b & 0\end{pmatrix}$$

By comparing the entries,

$15+6b=b$ — ②

$-5a-2c=0$ — ③.

From ②, $5b=-15 \Leftrightarrow b=-3$.

From ③, $a=-\frac{2}{5}c$. — ④.

Putting ④ into ①,

$c+\frac{2}{5}c(-3)=1$.

$-\frac{1}{5}c=1$.

$c=-5$. so $a=2$.

Therefore,
$\begin{cases}a=2\\ b=-3\\ c=-5.\end{cases}$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

(b) $M = \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$.

$M^{-1} = \dfrac{1}{\begin{vmatrix} 1 & 2 \\ -3 & -5 \end{vmatrix}} \begin{pmatrix} -5 & 3 \\ -2 & 1 \end{pmatrix}^T$

$= \begin{pmatrix} -5 & -2 \\ 3 & 1 \end{pmatrix}$.

$M^{-1}RM$

$= \begin{pmatrix} -5 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$

$= \begin{pmatrix} -60 & -20 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$

$= \begin{pmatrix} 0 & 20 \\ 0 & 1 \end{pmatrix}$. Note that $\begin{pmatrix} 0 & 20 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 20 \\ 0 & 1 \end{pmatrix}$ for all positive integers n,

(b)(ii). i.e. $(M^{-1}RM)^n = M^{-1}R^nM$

$= M^{-1}RM$.

So, we have $M(M^{-1}R^nM)M^{-1} = M(M^{-1}RM)M^{-1}$

i.e. $R^n = R$. by using (i)

$M^{-1}PM = M^{-1}(MQ) = Q$.

So. $P^n = M(M^{-1}PM)^nM^{-1} = MQ^nM^{-1} = MQM^{-1}$ as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

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SECTION B (50 marks)

9. Let $f(x) = \dfrac{(x+4)^3}{(x-4)^2}$ for all real numbers $x \neq 4$. Denote the graph of $y = f(x)$ by H.

(a) Find the asymptote(s) of H. (3 marks)

(b) Find $f''(x)$. (2 marks)

(c) Someone claims that there are two turning points of H. Do you agree? Explain your answer. (2 marks)

(d) Find the point(s) of inflexion of H. (2 marks)

(e) Find the area of the region bounded by H, the x-axis and the y-axis. (3 marks)

(a) Note that $\lim\limits_{x \to 4} \dfrac{(x+4)^3}{(x-4)^2} = \infty$

So, the vertical asymptote of H is $x=4$.

$$f(x) = \frac{(x+4)^3}{(x-4)^2} = \frac{x^3+12x^2+48x+64}{x^2-8x+16}$$

$$= \frac{(x+20)(x^2-8x+16)+192x-256}{x^2-8x+16}$$

$$= x+20+\frac{192x-256}{(x-4)^2}.$$

$$\lim_{x \to \infty}\left[f(x)-(x+20)\right] = \lim_{x \to \infty} \frac{192x-256}{(x-4)^2} = 0.$$

So, $y = x+20$ is an oblique asymptote of H.

(b) $f(x) = x+20+64\dfrac{(3x-4)}{(x-4)^2}$.

Note that $3x-4 = 3(x-4)+8$.

$$f(x) = x+20+\frac{192}{x-4}+\frac{512}{(x-4)^2}.$$

$$f'(x) = 1-\frac{192}{(x-4)^2}-\frac{1024}{(x-4)^3}$$

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

$$f''(x) = \frac{384}{(x-4)^3} + \frac{3072}{(x-4)^4}$$

$$= \frac{384x + 1536}{(x-4)^4}$$

(c) For $f'(x) = 0$,

$$1 - \frac{192}{(x-4)^2} - \frac{1024}{(x-4)^3} = 0$$

$$\frac{192x - 768 + 1024}{(x-4)^3} = 1$$

$$(x-4)^3 = 192x + 256$$

$$x^3 - 12x^2 + 48x - 64 = 192x + 256$$

$$x^3 - 12x^2 - 144x - 320 = 0$$

$$(x-20)(x+4)^2 = 0$$

$$x = 20 \text{ or } x = -4$$

x	$x<-4$	$x=-4$	$-4<x<4$	$x=4$	$4<x<20$	$x=20$	$x>20$
$f(x)$	↗	0	↗	undefined	↘	54	↗
$f'(x)$	+	0	+	undefined	−	0	+

Thus, there is only one turning point of H.
The claim is disagreed.

Answers written in the margins will not be marked.

(d). $f''(x) = \dfrac{384x + 1536}{(x-4)^4}$

When $f''(x) = 0$, $384x + 1536 = 0 \Leftrightarrow x = -4$.

x	$x<-4$	$x=-4$	$-4<x<4$	$x=4$	$x>4$
$f(x)$	↗	0	↗	undefined	↗
$f''(x)$	$-$	0	$+$	undefined	$+$

Thus, the point of inflexion of H is $(-4, 0)$.

(e) When $x = -4$, $f(x) = \dfrac{(-4+4)^3}{(-4-4)^2} = 0$.

Area of the shaded region

$$= \int_{-4}^{0} \frac{(x+4)^3}{(x-4)^2}\,dx$$

$$= \int_{-4}^{0} \left(x + 20 + \frac{192}{x-4} + \frac{512}{(x-4)^2}\right) dx \quad \text{[From (b)]}$$

$$= \left[\frac{1}{2}x^2 + 20x + 192\ln|x-4| - \frac{512}{x-4}\right]_{-4}^{0}$$

$$= (192\ln 4 + 128) - (8 - 80 + 192\ln 8 + 64)$$

$$= 136 - 192\ln 2$$ //

10. (a) Using integration by substitution, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$.

(3 marks)

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\,dx$. (3 marks)

(c) (i) Using $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot\frac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \dfrac{x\csc^2 x}{\cot x - 1}\,dx = \frac{\pi}{8}\ln(2+\sqrt{3})$.

(7 marks)

(a) Let $u = \frac{\pi}{4} - x$. Then $du = -dx$.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right) dx$$

$$= -\int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln(\sin(u))\, du.$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin u)\, du. = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\, dx.$$

(b). $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\, dx$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\frac{\cos x - \sin x}{\sin x}\right) dx. = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x - \sin x)\, dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x).\, dx.$$

Note that $\sin\left(\frac{\pi}{4} - x\right) = \sin\frac{\pi}{4}\cos x - \sin x \cos\frac{\pi}{4}$

$= \frac{\sqrt{2}}{2}(\cos x - \sin x).$

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So, we have

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\,dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cos x - \sin x)\,dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{4} - x\right)\right)\right)dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx.$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\sqrt{2}\,dx + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$

(By (a)).

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln 2\,dx.$$

$$= \frac{1}{2}\Big[x\ln 2\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{24}\ln 2$$

(c)(i). $\cot\frac{\pi}{12} = \cot\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \frac{\cot\frac{\pi}{4}\cot\frac{\pi}{6} + 1}{\cot\frac{\pi}{6} - \cot\frac{\pi}{4}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}.$$

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(ii) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\csc^2 x}{\cot x-1}\,dx$

Note that $\dfrac{d}{dx}\ln(\cot x-1) = -\dfrac{\csc^2 x}{\cot x-1}$.

So, $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\csc^2 x}{\cot x-1} = -\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} x\,d\ln(\cot x-1)$

$\displaystyle= \Big[-x\ln(\cot x-1)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}} + \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x-1)\,dx$

$= -\frac{\pi}{6}\ln\left(\cot\frac{\pi}{6}-1\right) + \frac{\pi}{12}\ln\left(\cot\frac{\pi}{12}-1\right)$

$+ \frac{1}{2}\ln 2$ (From (b))

$= -\frac{\pi}{6}\ln(\sqrt{3}-1) + \frac{\pi}{12}\ln(\sqrt{3}+1) + \frac{\pi}{24}\ln 2$

$= \left(\frac{\pi}{6}\ln(\sqrt{3}+1) - \frac{\pi}{6}\ln 2\right) + \frac{\pi}{12}\ln(\sqrt{3}+1) + \frac{\pi}{24}\ln 2$

$= \frac{\pi}{4}\ln(\sqrt{3}+1) - \frac{\pi}{8}\ln 2$

$= \frac{\pi}{8}\ln\frac{(\sqrt{3}+1)^2}{2} = \frac{\pi}{8}\ln\frac{(4+2\sqrt{3})}{2}$

$= \frac{\pi}{8}\ln(2+\sqrt{3})$

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11. (a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = k \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h, k \in \mathbf{R}.$$

(i) Assume that (E) has a unique solution.

(1) Prove that $h \neq -3$.

(2) Solve (E).

(ii) Assume that $h = -3$ and (E) is consistent.

(1) Prove that $k = -2$.

(2) Solve (E).

(9 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = -2 \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h \in \mathbf{R}.$$

Someone claims that there are at least two values of h such that (F) has a real solution (x, y, z) satisfying $3x^2 + 4y^2 - 7z^2 = 1$. Do you agree? Explain your answer. (4 marks)

(i) $\Delta = \begin{vmatrix} 1 & -1 & -2 \\ 1 & -2 & h \\ 4 & h & -7 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 \\ 0 & -1 & h+2 \\ 4 & h & -7 \end{vmatrix}$

$= \begin{vmatrix} -1 & h+2 \\ h & -7 \end{vmatrix} + 4\begin{vmatrix} -1 & -2 \\ -1 & h+2 \end{vmatrix}$

$= 7 - (h^2+2h) - 4(h+2) - 8$.

$= -h^2 - 6h - 9 = -(h+3)^2 \neq 0$.

So, we have $h \neq -3$.

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(a)(i)(2)

$$\Delta_x = \begin{vmatrix} 1 & -1 & -2 \\ k & -2 & h \\ 7 & h & -7 \end{vmatrix}$$

$$= 1\begin{vmatrix} -2 & h \\ h & -7 \end{vmatrix} - k\begin{vmatrix} -1 & -2 \\ h & -7 \end{vmatrix} + 7\begin{vmatrix} -1 & -2 \\ -2 & h \end{vmatrix}.$$

$$= 1(14-h^2) - k(7+2h) + 7(-h-4).$$

$$= -2hk - 7k - 7h - 28 + 14 - h^2$$

$$= -h^2 - 2hk - 7k - 7h - 14$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & -2 \\ 1 & k & h \\ 4 & 7 & -7 \end{vmatrix}$$

$$= 1\begin{vmatrix} k & h \\ 7 & -7 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 7 & -7 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ k & h \end{vmatrix}$$

$$= (-7k-7h) - (-7+14). + 4(h+2k).$$

$$= -3h + k - 7$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & k \\ 4 & h & 7 \end{vmatrix}.$$

$$= 1\begin{vmatrix} -2 & k \\ h & 7 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ h & 7 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ -2 & k \end{vmatrix}.$$

$$= (-14-hk) - 1(-7-h) + 4(-k+2).$$

$$= -hk + h - 4k + 8 + 7 - 14 = -hk + h - 4k + 1.$$

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So, The solutions are
$$\begin{cases} x = \dfrac{\Delta x}{\Delta} = \dfrac{h^2+2hk+7k+7h+14}{(h+3)^2} \\ y = \dfrac{\Delta y}{\Delta} = \dfrac{3h-k+7}{(h+3)^2} \\ z = \dfrac{\Delta z}{\Delta} = \dfrac{hk-h+4k-1}{(h+3)^2}. \end{cases}$$

(ii)(1) For $h=-3$ the system corresponds to

$$\begin{cases} x-y-2z=1 \\ x-2y-3z=k \\ 4x-3y-7z=7. \end{cases}$$

Its augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 1 & -2 & -3 & k \\ 4 & -3 & -7 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 1 & 1-k \\ 1 & -2 & -3 & k \\ 0 & 5 & 5 & 7-4k \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & 1 & 1-k \\ 1 & -2 & -3 & k \\ 0 & 0 & 0 & 2+k \end{array}\right).$$

$$\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}.$$

As the last row corresponds to $0=0$,
$2+k=0 \Leftrightarrow k=-2.$

(2) Let $z=t$. Then $y=3-t$. and $x=4+t$.
$\therefore$ Solution set of (E): $(x,y,z) = \{(4+t, 3-t, t) : t\in\mathbb{R}\}$

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(b) ① If $h=-3$ (F) corresponds to (E) with the case $k=-2$.

By (a)(ii)(2), $\begin{cases} x=4+t \\ y=3-t \\ z=t. \end{cases}$

$3x^2+4y^2-7z^2$

$=3(t^2+8t+16)+4(3-t)^2-7t^2$

$=3t^2+24t+48+4t^2-24t+36-7t^2$

$=84\neq 1$

② If $h\neq -3$, with $k=-2$,

By (a)(i)(2)
The solutions of (F) are

$$\begin{cases} x=\dfrac{h^2-4h-14+7h+14}{(h+3)^2}=\dfrac{h}{h+3} \\ y=\dfrac{3h+9}{(h+3)^2}=\dfrac{3}{h+3} \\ z=\dfrac{-2h-h-9}{(h+3)^2}=\dfrac{-3}{h+3} \end{cases}$$ provided that $h\neq -3$

$3x^2+4y^2-7z^2$

$=\left[3h^2+4(3)^2-7(-3)^2\right]\dfrac{1}{(h+3)^2}$

Two values of h.

when $3x^2+4y^2-7z^2=1$, $(h+3)^2=3h^2-27$

$(h+3)^2=(h+3)(3h-9)$

$h=-3$ (rej.) OR $\boxed{h=6}$

Thus there is only ONE value of h, i.e. $h=6$ satisfying the constraint. $\therefore$ Disagree.

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12. Let $\overrightarrow{OP} = \mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i}-7\mathbf{j}-4\mathbf{k}$, where O is the origin. R is a point lying on PQ such that $PR:RQ=1:3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$. (2 marks)

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$. (2 marks)

(c) Let N be a point such that $\overrightarrow{ON} = \lambda(\overrightarrow{OP} \times \overrightarrow{OR})$, where λ is a real number.

(i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

(ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i}+\mu\mathbf{j}-10\mathbf{k}$.

(1) Find λ and μ .

(2) Denote the angle between ΔOPQ and ΔNPQ by θ . Find $\tan\theta$.

(8 marks)

(a). $\overrightarrow{OR}$

$= \frac{3}{4}\overrightarrow{OP} + \frac{1}{4}\overrightarrow{OQ}.$

$= \frac{3}{4}(\vec{i}+\vec{j}+4\vec{k}) + \frac{1}{4}(5\vec{i}-7\vec{j}-4\vec{k})$

$= 2\vec{i}-\vec{j}+2\vec{k}.$

$\overrightarrow{OP} \times \overrightarrow{OR}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}(2+4) - \vec{j}(2-8) + \vec{k}(-1-2)$

$= \underline{\underline{6\vec{i}+6\vec{j}-3\vec{k}}}.$

(b). Area of OPSR

$= |\overrightarrow{OP} \times \overrightarrow{OR}|$

$= \sqrt{(6)^2+(6)^2+(-3)^2} = 9.$

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(c). $\overrightarrow{ON} = \lambda \overrightarrow{OP} \times \overrightarrow{OR}$

$= 6\lambda\vec{i} + 6\lambda\vec{j} - 3\lambda\vec{k}$.

$\overrightarrow{NR} = \overrightarrow{OR} - \overrightarrow{ON}$

$= (2-6\lambda)\vec{i} + (-1-6\lambda)\vec{j} + (2+3\lambda)\vec{k}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 4\vec{i} - 8\vec{j} - 8\vec{k}$.

$\overrightarrow{NR} \cdot \overrightarrow{PQ}$

$= [(2-6\lambda)\vec{i} + (-1-6\lambda)\vec{j} + (2+3\lambda)\vec{k}] \cdot (4\vec{i} - 8\vec{j} - 8\vec{k})$

$= (8-24\lambda) + (8+48\lambda) - (16-24\lambda) = 0$.

Thus, $\overrightarrow{NR}$ is perpendicular to $\overrightarrow{PQ}$.

(ii) $\overrightarrow{NQ} = \overrightarrow{OQ} - \overrightarrow{ON}$

$= (5-6\lambda)\vec{i} + (-7-6\lambda)\vec{j} + (3\lambda-4)\vec{k}$.

As $\overrightarrow{NQ}$ is parallel to $11\vec{i} + \mu\vec{j} - 10\vec{k}$, we have

$$\frac{11}{5-6\lambda} = \frac{\mu}{-7-6\lambda} = \frac{-10}{3\lambda-4} \quad \text{---} \quad (*)$$

Solving $(*)$, $\lambda = \frac{2}{9}$ and $\mu = -25$.

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(c)(ii)
(2). $\vec{NQ} = \frac{11}{3}\vec{i} - \frac{25}{3}\vec{j} - \frac{10}{3}\vec{k}$.

A vector perpendicular to NPQ $= \vec{NQ} \times \vec{PQ}$.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{11}{3} & -\frac{25}{3} & -\frac{10}{3} \\ 4 & -8 & -8 \end{vmatrix} = \vec{i}\left(\frac{200}{3} - \frac{80}{3}\right) - \vec{j}\left(-\frac{88}{3} + \frac{40}{3}\right) + \vec{k}\left(-\frac{88}{3} + \frac{100}{3}\right)$$

$$= 40\vec{i} + 16\vec{j} + 4\vec{k}.$$

A vector $\perp$ to OPQ $= \vec{OP} \times \vec{OQ}$.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ 5 & -7 & -4 \end{vmatrix} = \vec{i}(-4+28) - \vec{j}(-4-20) + \vec{k}(-7-5)$$

$$= 24\vec{i} + 24\vec{k} - 12\vec{k}.$$

So, $\cos\theta = \left| \frac{(\vec{OP} \times \vec{OQ}) \cdot (\vec{NQ} \times \vec{PQ})}{|\vec{OP} \times \vec{OQ}| \, |\vec{NQ} \times \vec{PQ}|} \right|$ as $\theta < 90°$.

$$= \frac{1296}{(36)(\sqrt{1872})} \Leftrightarrow \tan\theta = \sqrt{\frac{1}{\cos^2\theta} - 1}$$

$$= \sqrt{\frac{(1872)(1296)}{1296^2} - 1}$$

$$= \frac{2}{3}$$

END OF PAPER

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