Level 1 Module 2
exemplar with comments

2018-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2018

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

1. Let $f(x) = (x^2 - 1)e^x$. Express $f(1+h)$ in terms of h. Hence, find $f'(1)$ from first principles. (4 marks)

$f(x) = (x^2-1)e^x$

$f(1+h) = ((1+h)^2 - 1)e^{1+h}$

$= (1^2 + 2h + h^2 - 1)e^{1+h}$

$= (h^2 + 2h)e^{1+h}$

~~f(1+h)~~

$f'(x) = \lim_{h \to 0} ((x+h)^2 - 1)e^{x+h}$

$= \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 - 1)(e^x \cdot e^h)}{h}$

$= 2xe^x + (x^2-1)e^x$

$f'(1) = 2e$

2. Expand $(x+3)^5$. Hence, find the coefficient of $\underline{\underline{x^3}}$ in the expansion of $(x+3)^5\left(x-\frac{4}{x}\right)^2$. (5 marks)

$(x+3)^5$

$= C_0^5 x^5 3^0 + C_1^5 x^4 3^1 + C_2^5 x^3 3^2 + C_3^5 x^2 3^3 + C_4^5 x^1 3^4$
$+ C_5^5 x^0 3^5$

$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243.$

$(x-\frac{4}{x})^2 = x^2 - 2(\frac{4}{x})(x) + (\frac{4}{x})^2$
$= x^2 - 8 + \frac{16}{x^2}$

Coefficient of x^3 : $16 + 90(-8) + 405$
$= -299$

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3. (a) If $\cot A = 3\cot B$, prove that $\sin(A+B) = 2\sin(B-A)$.

(b) Using (a), solve the equation $\cot\left(x+\frac{4\pi}{9}\right) = 3\cot\left(x+\frac{5\pi}{18}\right)$, where $0 \le x \le \frac{\pi}{2}$.

(5 marks)

(a) $\cot A = 3\cot B$

$$\frac{1}{\tan A} = \frac{3}{\tan B}$$

$$\frac{\tan B}{\tan A} = 3$$

$$\frac{\sin B}{\cos B} \times \frac{\cos A}{\sin A} = 3$$

$$\frac{2}{2}\frac{\sin B\cos A}{\cos B\sin A} = 3$$

$$\frac{\sin(B+A)+\sin(B-A)}{\sin(A+B)+\sin(A-B)} = 3$$

$$\sin(A+B)+\sin(B-A) = 3\sin(A+B)+3\sin(A-B)$$

$$-\cancel{4}^{2}\sin(A-B) = \cancel{2}\sin(A+B)$$

$$2\sin(B-A) = \sin(A+B)$$

(b) $\cot\left(x+\frac{4\pi}{9}\right) = 3\cot\left(x+\frac{5\pi}{18}\right)$

By (a).

$$2\sin\left(\cancel{x}+\frac{5\pi}{18}-\cancel{x}-\frac{4\pi}{9}\right) = \sin\left(x+\frac{5\pi}{18}+x+\frac{4\pi}{9}\right)$$

$$2\sin\left(\frac{5\pi-8\pi}{18}\right) = \sin\left(2x+\frac{5\pi+8\pi}{18}\right)$$

$\sin^{-1}(-1)$

$$-1 = \sin\left(2x+\frac{13\pi}{18}\right)$$

$=\frac{-\pi}{2}$ or $\frac{3\pi}{4}$ (rej.)

$$\frac{3\pi}{4} = 2x+\frac{13\pi}{18})$$

$$2x = \frac{1}{36}$$

$$x = \frac{1}{72}$$

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4. (a) Using integration by parts, find $\int u(5^u)\,du$.

(b) Define $f(x) = x(5^{2x})$ for all real numbers x . Find the area of the region bounded by the graph of $y = f(x)$, the straight line $x = 1$ and the x-axis.

(6 marks)

(a) $\int u(5^u)\,du$.

$u = u$. $du = du$ $dv = 5^u du$ $v = \frac{5^{u+1}}{u+1}$

$= \frac{u5^{u+1}}{u+1} - \int \frac{5^{u+1}}{u+1} d(u+1)$

$= \frac{u5^{u+1}}{u+1} - 5^{u+1} \ln|u+1| + c$

(b) $\int_0^1 x5^{2x} dx$

$= \left[\frac{x5^{2x+1}}{2x+1}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{5^{2x+1}}{2x+1} d(2x+1)$

$= \left[\frac{x5^{2x+1}}{2x+1}\right]_0^1 - \frac{1}{2} \cdot \left[5^{2x+1} \ln|2x+1|\right]_0^1$

$= \frac{125}{3} - \frac{1}{2}(5^3 \ln|3| - 5 \ln|1|)$

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5. (a) Using integration by substitution, find $\int x^3\sqrt{1+x^2}\,dx$.

(b) At any point (x, y) on the curve Γ , the slope of the tangent to Γ is $15x^3\sqrt{1+x^2}$. The y-intercept of Γ is 2 . Find the equation of Γ .

(7 marks)

(a) $\int x^3 \sqrt{1+x^2}\, dx$

$= \int x^3 (1+x)\, dx$

let $u = 1+x$ $du = dx$

$x = u-1$

$= \int (u-1)^3\, u\, du$

$= u \cdot \frac{(u-1)^4}{4} - \int \frac{(u-1)^4}{4}\, du$

$= \frac{u(u-1)^4}{4} - \frac{(u-1)^5}{20}$

$= \frac{(x+1)(x+1-1)^4}{4} - \frac{(x+1-1)^5}{20}$

$= \frac{(x+1)(x)}{4} - \frac{x^5}{20}$

$= \frac{5x^2+5x-x^5}{20}$.

(b)

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6. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} k(k+4) = \frac{n(n+1)(2n+13)}{6}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=333}^{555} \left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$.

(7 marks)

(a) let $p(n)$ be $\sum_{k=1}^{n} k(k+4) = \frac{n(n+1)(2n+13)}{6}$

when $n=1$, LHS $= 5 =$ RHS

$\therefore$ $p(1)$ is true

Assume $p(m)$ is also true for some positive integers. when $n=m+1$,

LHS $= \frac{m(m+1)(2m+13)}{6} + (m+1)(m+5)$

$= \frac{m(m+1)(2m+13) + 6(m+1)(m+5)}{6}$

$= \frac{(m+1)[m(2m+13) + 6(m+5)]}{6}$

$= \frac{(m+1)(2m^2+13m+6m+30)}{6}$

$= \frac{(m+1)(m+2)(2m+15)}{6}$ = RHS

$\therefore$ $p(m+1)$ is also true.

$\therefore$ By Mathematical induction, the results follows.

Answers written in the margins will not be marked.

(b) $\displaystyle\sum_{k=333}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$

$\displaystyle= \sum_{k=1}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right) - \sum_{k=1}^{332}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$

$= 57755890 - 12474402$

$= 45.2814.88$

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7. Define $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be a non-zero real matrix such that $MX = XM$.

(a) Express b and c in terms of a .

(b) Prove that X is a non-singular matrix.

(c) Denote the transpose of X by X^T . Express $(X^T)^{-1}$ in terms of a .

(8 marks)

(a) MX

$= \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$

(b) X is a non-zero matrix

$\therefore \det(X) \neq 0$

$\therefore X$ is a non-singular matrix

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8. Define $f(x)=\dfrac{A}{x^2-4x+7}$ for all real numbers x, where A is a constant. It is given that the extreme value of $f(x)$ is 4.

(a) Find $f'(x)$.

(b) Someone claims that there are at least two asymptotes of the graph of $y=f(x)$. Do you agree? Explain your answer.

(c) Find the point(s) of inflexion of the graph of $y=f(x)$.

(8 marks)

(a) $f(x)=\dfrac{A}{x^2-4x+7}$ $\quad A=7$

extreme value $=4$.

$f'(x)=\dfrac{-(7)(2x-4)}{(x^2-4x+7)^2}$

$=\dfrac{-14x+28}{(x^2-4x+7)^2}$

(b) $x^2-4x+7=0$ (No solution) $\rightarrow$ No vertical asymptote

i.e. It is not possible for at least two asymptotes on the graph. So I don't agree.

(c) $f'(x)=0$.

$-14x+28=0$

$x=2$.

$f''(x)=0$.

$\underbrace{(x^2-4x+7)}_{\text{no solution}}\underbrace{(42x^2-168x+126)}_{x=3 \text{ or } x=1.}=0$

$f(3)=\dfrac{7}{4}$ $\quad$ Inflexion point is $\dfrac{7}{4}$.

$f(1)=\dfrac{7}{4}$

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SECTION B (50 marks)

9. Consider the curve $C: y = \ln\sqrt{x}$, where $x > 1$. Let P be a moving point lying on C . The normal to C at P cuts the x-axis at the point Q while the vertical line passing through P cuts the x-axis at the point R .

~~$(r, 0)$~~

(a) Denote the x-coordinate of P by r . Prove that the x-coordinate of Q is $\dfrac{4r^2 + \ln r}{4r}$. (3 marks)

(b) Find the greatest area of ΔPQR . (5 marks)

(c) Let O be the origin. It is given that OP increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of ΔPQR increases at a rate lower than 2 square units per minute when the x-coordinate of P is e . Is the claim correct? Explain your answer. (4 marks)

(a)

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10. (a) (i) Prove that $\int \sin^4 x \, dx = \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$.

(ii) Evaluate $\int_0^\pi \sin^4 x \, dx$.

(5 marks)

(b) (i) Let $f(x)$ be a continuous function such that $f(\beta - x) = f(x)$ for all real numbers x, where β is a constant. Prove that $\int_0^\beta x f(x) \, dx = \frac{\beta}{2}\int_0^\beta f(x) \, dx$.

(ii) Evaluate $\int_0^\pi x \sin^4 x \, dx$.

(5 marks)

(c) Consider the curve $G: y = \sqrt{x}\sin^2 x$, where $\pi \le x \le 2\pi$. Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(3 marks)

(a)(i) $\int \sin^4 x \, dx$

let $u = \sin^2 x$

$du = \cancel{\cos x \, dx} \; 2\sin x \cos x \, dx$

$= \int (\sin^2 x)^2 \, dx$

$= \int \frac{u^2 \, du}{2 \sin x \cos x}$

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11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + ay + 4(a+1)z = 18 \\ 2x + (a-1)y + 2(a-1)z = 20 \\ x - y - 12z = b \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Assume that (E) has a unique solution. $\Delta \neq 0$.

(1) Find the range of values of a .

(2) Solve (E) .

(ii) Assume that $a = 3$ and (E) is consistent.

(1) Find b .

(2) Solve (E) .

(9 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F): \begin{cases} x + 3y + 16z = 18 \\ x + y + 2z = 10 \\ x - y - 12z = s \\ 2x - 5y - 45z = t \end{cases}, \text{ where } s, t \in \mathbf{R}.$$

Assume that (F) is consistent. Find s and t .

(3 marks)

(a)(i)(1) (E): $\left|\begin{matrix} 1 & a & 4(a+1) \\ 2 & a-1 & 2(a-1) \\ 1 & -1 & -12 \end{matrix}\right| \begin{matrix} 1 & a \\ 2 & a-1 \\ 1 & -1 \end{matrix}$

$\det(E) = (a-1)(-12) + a(2a-2) + (4a+4)(2)(-1)$
$- ((a-1)(4a+4) + (-1)(2a-2) + (-12)(2a)) \neq 0$

$0 \neq -12a + 12 + 2a^2 \cancel{-2a} - 8a - 8$
$- (4a^2 \cancel{+4a} \cancel{-4a} - 4 \cancel{-2a} + 2 - 24a)$

$0 \neq -2a^2 + 4a + 6$

$a \neq -1$ ~~or~~ and $a \neq 3$

range of a : any real numbers except -1 and 3 .

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(2) $x + ay + 4(a+1)z = 18$ — ①

$2x + (a-1)y + 2(a-1)z = 20$ — ②

$x - y - 12z = b$ — ③

③ − ① : $-(1+a)y - (4a+16)z = b - 18$ — ④

② − ①×2 : $-(a+1)y - (6a+10)z = -16$. — ⑤

⑤ − ④ : $10az = -b + 2$

$$z = \frac{-b+2}{10a}$$

Let $z = t$

$$-(1+a)y = b - 18 + (4a+10)z$$

$$y = \frac{b-18+(4a+10)t}{-(1+a)}$$

$$x = b + \frac{b-18+(4a+10)t}{-(1+a)} + 12t$$

Solution set : $\left\{\left(b + \frac{b-18+(4a+10)t}{-(1+a)} + 12t, \frac{b-18+(4a+10)t}{-(1+a)}, t\right) : t \in R\right\}$

(ii) ①
$$\begin{cases} x + 3y + 16z = 18 \\ 2x + 2y + 4z = 20 \\ x - y - 12z = b. \end{cases}$$

$$z = \frac{-b+2}{10(3)} \rightarrow -(30z - 2) = b.$$

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12. The position vectors of the points A, B, C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A, B and C by Π. Let E be the projection of D on Π.

(a) Find

(i) $\overrightarrow{AB}\times\overrightarrow{AC}$,

(ii) the volume of the tetrahedron $ABCD$,

(iii) $\overrightarrow{DE}$.

(5 marks)

(b) Let F be a point lying on BC such that DF is perpendicular to BC.

(i) Find $\overrightarrow{DF}$.

(ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$? Explain your answer.

(5 marks)

(c) Find the angle between ΔBCD and Π. (3 marks)

(a) (i) $\overrightarrow{AB} \times \overrightarrow{AC} = -8i - 40j - 12k$

$\overrightarrow{AB} = (4i - 3j + k) + (-i + 3j - 3k)$

$= (3i - 2k)$

$\overrightarrow{AC} = (4i - 3j + k) + (7i - j + 5k)$

$= 11i - 4j + 6k$.

(ii) volume $= \frac{1}{6}|(-8i - 40j - 12k) \cdot \overrightarrow{AD})|$

$\overrightarrow{AD} = (4i - 3j + k) + (3i - 2j - 5k)$

$= 7i - 5j - 4k$

$= \frac{1}{6}|(-56i + 200j + 48k)|$

$= 35.5$

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END OF PAPER

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Comments

The candidate communicates and expresses simple ideas using mathematical language and notations.

He/She is able to apply binomial theorem to find the coefficients in Question 2 and is also able to complete the proof by mathematical induction in Question 6(a).

It can be concluded that the candidate demonstrates elementary knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by performing straightforward procedures according to direct instructions.