Last Update: 2.5.2019

HKDSE
Mathematics
Extended Part
Module 2

Past Papers (by topic)
2012 – 2019

Content

Mathematical Induction

1. Prove, by mathematical induction, that for all positive integers n ,

   $1\times 2+2\times 5+3\times 8+\cdots+n(3n-1)=n^2(n+1)$. (5 marks)

   [HKDSE 2012 M2 #3]

2. Prove by mathematical induction, that for all positive integers n ,

   $1+\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{4n+1}{3n+1}$. (5 marks)

   [HKDSE 2013 M2 #3]

3. (a) Using mathematical induction, prove that

   $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx=\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$ for all positive integers n .

   (b) Using (a), evaluate $\sum_{k=1}^{567}\cos\frac{k\pi}{7}$. (8 marks)

   [HKDSE 2015 M2 #8]

4. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}(-1)^k k^2=\frac{(-1)^n n(n+1)}{2}$ for all positive integers n .

   (b) Using (a), evaluate $\sum_{k=3}^{333}(-1)^{k+1}k^2$. (6 marks)

   [HKDSE 2016 M2 #5]

5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}k(k+4)=\frac{n(n+1)(2n+13)}{6}$ for all positive integers n .

   (b) Using (a), evaluate $\sum_{k=333}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$. (7 marks)

   [HKDSE 2018 M2 #6]

6. (a) Using mathematical induction, prove that $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(7 marks)

[HKDSE 2019 M2 #5]

Binomial Theorem

1. It is given that $(1+ax)^n = 1+6x+16x^2 +$ terms involving higher powers of x, where n is a positive integer and a is a constant. Find the values of a and n.
(5 marks)
[HKDSE 2012 M2 #2]

2. Suppose the coefficients of x and x^2 in the expression of $(1+ax)^n$ are –20 and 180 respectively. Find the values of a and n. (4 marks)
[HKDSE 2013 M2 #2]

3. In the expansion of $(1-4x)^2(1+x)^n$, the coefficient of x is 1.
   (a) Find the value of n.
   (b) Find the coefficient of x^2. (4 marks)
[HKDSE 2014 M2 #1]

4. Expand $(5+x)^4$. Hence, find the constant term in the expansion of
$(5+x)^4\left(1-\frac{2}{x}\right)^3$. (5 marks)
[HKDSE 2016 M2 #1]

5. Let $(1+ax)^8 = \sum_{k=0}^{8} \lambda_k x^k$ and $(b+x)^9 = \sum_{k=0}^{9} \mu_k x^k$, where a and b are constants. It is given that $\lambda_2 : \mu_7 = 7:4$ and $\lambda_1 + \mu_8 + 6 = 0$. Find a. (5 marks)
[HKDSE 2017 M2 #2]

6. Expand $(x+3)^5$. Hence, find the coefficient of x^3 in the expansion of
$(x+3)^5\left(x-\frac{4}{x}\right)^2$. (5 marks)
[HKDSE 2018 M2 #2]

7. Let $\mathrm{P}(x)=\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $\mathrm{P}(x)$ is 160 . Find

(a) λ ,

(b) $\mathrm{P}'(0)$.

(5 marks)

[HKDSE 2019 M2 #2]

Compound Angle Formulae

***The following table will be provided in the HKDSE M2 paper:

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$ $2\sin A\sin B = \cos(A-B) - \cos(A+B)$	

1.

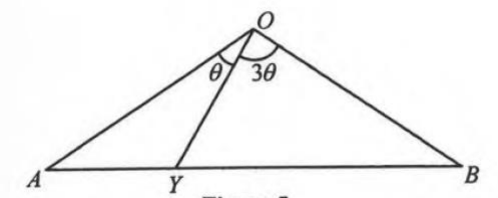


Figure 5

In Figure 5, OAB is an isosceles triangle with $OA = OB$, $AB = 1$, $AY = y$, $\angle AOY = \theta$ and $\angle BOY = 3\theta$.

(a) Show that $y = \dfrac{1}{4}\sec^2\theta$.

(b) Find the range of values of y.

[Hint: you may use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.] (6 marks)

[HKDSE 2012 M2 #10]

2. (a) Prove the identity $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$.

(b) Using (a), prove the identity $\tan y = \dfrac{\sin 8y\cos 4y\cos 2y}{(1+\cos 8y)(1+\cos 4y)(1+\cos 2y)}$.

(5 marks)

[HKDSE 2013 M2 #7]

3. (a) Prove that $\sin^2 x\cos^2 x=\dfrac{1-\cos 4x}{8}$.

(b) Let $f(x)=\sin^4 x+\cos^4 x$.

(i) Express $f(x)$ in the form $A\cos Bx+C$, where A, B and C are constants.

(ii) Solve the equation $8f(x)=7$, where $0\le x\le\dfrac{\pi}{2}$. (7 marks)

[HKDSE 2015 M2 #7]

4. (a) Prove that $x+1$ is a factor of $4x^3+2x^2-3x-1$.

(b) Express $\cos 3\theta$ in terms of $\cos\theta$.

(c) Using the results of (a) and (b), prove that $\cos\dfrac{3\pi}{5}=\dfrac{1-\sqrt{5}}{4}$. (6 marks)

[HKDSE 2016 M2 #6]

5. (a) Prove that $\sin 3x=3\sin x-4\sin^3 x$.

(b) Let $\dfrac{\pi}{4}<x<\dfrac{\pi}{2}$.

(i) Prove that $\dfrac{\sin 3\left(x-\dfrac{\pi}{4}\right)}{\sin\left(x-\dfrac{\pi}{4}\right)}=\dfrac{\cos 3x+\sin 3x}{\cos x-\sin x}$.

(ii) Solve the equation $\dfrac{\cos 3x+\sin 3x}{\cos x-\sin x}=2$. (8 marks)

[HKDSE 2017 M2 #7]

6. (a) If $\cot A=3\cot B$, prove that $\sin(A+B)=2\sin(B-A)$.

(b) Using (a), solve the equation $\cot\left(x+\dfrac{4\pi}{9}\right)=3\cot\left(x+\dfrac{5\pi}{18}\right)$, where $0\le x\le\dfrac{\pi}{2}$. (5 marks)

[HKDSE 2018 M2 #3]

Differentiation (First principles)

1. Let $f(x)=e^{2x}$. Find $f'(0)$ from first principles. (3 marks)

[HKDSE 2012 M2 #1]

2. Find $\dfrac{d}{dx}(\sin 2x)$ from first principles. (4 marks)

[HKDSE 2013 M2 #1]

3. Consider the curve $C: y=x^3-3x$.

   (a) Find $\dfrac{dy}{dx}$ from first principles.

   (b) Find the range of x where C is decreasing. (5 marks)

[HKDSE 2014 M2 #2]

4. Find $\dfrac{d}{dx}(x^5+4)$ from first principles. (4 marks)

[HKDSE 2015 M2 #1]

5. Prove that $\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x+h}}=\dfrac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}$. Hence, find $\dfrac{d}{dx}\sqrt{\dfrac{3}{x}}$ from first principles. (5 marks)

[HKDSE 2016 M2 #2]

6. Find $\dfrac{d}{d\theta}\sec 6\theta$ from first principles. (5 marks)

[HKDSE 2017 M2 #1]

7. Let $f(x)=(x^2-1)e^x$. Express $f(1+h)$ in terms of h. Hence, find $f'(1)$ from first principles. (4 marks)

[HKDSE 2018 M2 #1]

8. Let $f(x)=\dfrac{10x}{7+3x^2}$. Prove that $f(1+h)-f(1)=\dfrac{4h-3h^2}{10+6h+3h^2}$. Hence, find $f'(1)$ from first principles. (4 marks)

[HKDSE 2019 M2 #1]

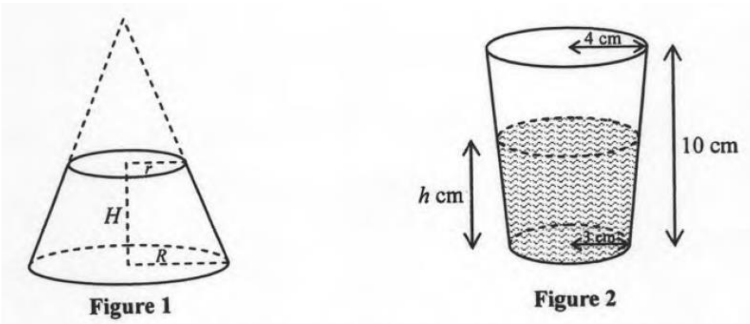
Applications of Differentiation

1. Find the minimum point(s) and asymptote(s) of the graph of $y = \dfrac{x^2 + x + 1}{x + 1}$.

(6 marks)

[HKDSE 2012 M2 #5]

2.



A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (see Figure 1). It is given that the volume of the frustum is $\dfrac{\pi}{3}H(r^2 + rR + R^2)$.

An empty glass in the form of an inverted frustum described above with height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass. Let h cm $(0 \le h \le 10)$ be the depth of the water inside the glass at time t s (see Figure 2).

(a) Show that the volume V cm^3 of water inside the glass at time t s is given by

$V = \dfrac{\pi}{300}(h^3 + 90h^2 + 2700h)$.

(b) If the volume of water in the glass is increasing at the rate 7π cm^3 s^{-1}, find the rate of increase of depth of water at the instant when $h = 5$. (6 marks)

[HKDSE 2012 M2 #6]

3. Consider a continuous function $f(x)=\dfrac{3-3x^2}{3+x^2}$. It is given that

x	$x<-1$	-1	$-1<x<0$	0	$0<x<1$	1	$x>1$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

('+' and '–' denote 'positive value' and 'negative value' respectively.)

(a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.

(b) Find the asymptote(s) of the graph of $y=f(x)$.

(c) Sketch the graph of $y=f(x)$. (6 marks)

[HKDSE 2013 M2 #5]

4.

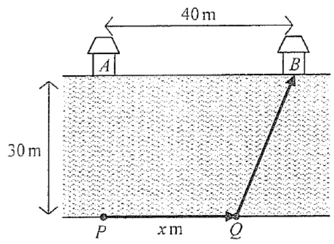


Figure 3

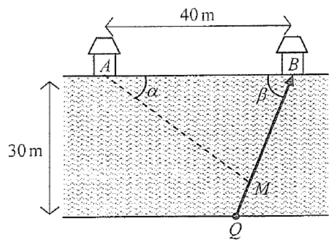


Figure 4

In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m. The width of the river is always 30 m. In the beginning, Mike stands at the starting point P in the opposite bank which is 30 m from A. Mike's wife, situated at A, is watching him running along the bank for x m at a constant speed of 7 m s^{-1} to point Q and then swimming at a constant speed of 1.4 m s^{-1} along a straight path to reach B.

(a) Let T seconds be the time that Mike travels from P to B.

(i) Express T in terms of x.

(ii) When T is minimum, show that x satisfies the equation

$2x^2-160x+3125=0$. Hence show that $QB=\dfrac{25\sqrt{6}}{2}$ m. (6 marks)

(b) In Figure 4, Mike is swimming from Q to B with QB equals to the value mentioned in (a)(ii). Let $\angle MAB=\alpha$ and $\angle ABM=\beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that $MB=\dfrac{200\tan\alpha}{\tan\alpha+2\sqrt{6}}$.

(ii) Find the rate of change of α when $\alpha=0.2$ radian. Correct your answer to 4 decimal places. (7 marks)

[HKDSE 2013 M2 #12]

5. Find the equation of the tangent to the curve $x\ln y + y = 2$ at the point where the curve cuts the y-axis. (5 marks)

[HKDSE 2014 M2 #3]

6. Let $x = 2y + \sin y$. Find $\frac{d^2y}{dx^2}$ in terms of y. (3 marks)

[HKDSE 2014 M2 #4]

7.

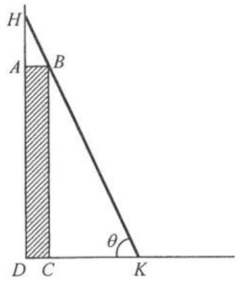


Figure 2

Thomas has a bookcase of dimensions 100 cm × 24 cm × 192 cm at the corner in his room. He wants to hang a decoration on the wall above the bookcase. Therefore, he finds a ladder to climb up. Initially, the ladder touches the wall, the edge of the top of the bookcase and the floor at the same time. Let rectangle *ABCD* be the side-view of the bookcase and *HK* be the side view of the ladder, so that $AB = 24$ cm and $BC = 192$ cm (see Figure 2). Let $\angle HKD = \theta$.

(a) Find the length of *HK* in terms of θ. (1 mark)

(b) Prove that the shortest length of the ladder is $120\sqrt{5}$ cm. (5 marks)

(c)

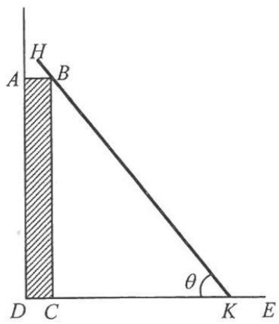


Figure 3

Suppose the length of the ladder is 270 cm. Suddenly, the ladder slides down so that the end of the ladder, K, moves towards E (see Figure 3). The ladder touches the edge of the top of the bookcase and the floor at the same time. Let x cm and y cm be the horizontal distances from H and K respectively to the wall.

(i) When $CK = 160$ cm, the rate of change of θ is -0.1 rad s^{-1}. Find the rate of change of x at this moment, correct to 4 significant figures.

(ii) Thomas claims that K is moving towards E at a speed faster than the horizontal speed H is leaving the wall. Do you agree? Explain your answer. (6 marks)

[HKDSE 2014 M2 #10]

8. Let $y = x\sin x + \cos x$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Let k be a constant such that $x\frac{d^2y}{dx^2} + k\frac{dy}{dx} + xy = 0$ for all real values of x .

Find the value of k . (5 marks)

[HKDSE 2015 M2 #2]

9. Define $f(x) = \frac{x^2+12}{x-2}$ for all $x \neq 2$.

(a) Find $f'(x)$. (2 marks)

(b) Prove that the maximum value and the minimum value of $f(x)$ are –4 and 12 respectively. (4 marks)

(c) Find the asymptote(s) of the graph of $y = f(x)$. (3 marks)

(d) Find the area of the region bounded by $y = f(x)$ and the horizontal line $y = 14$. (4 marks)

[HKDSE 2015 M2 #9]

10. Define $f(x) = \frac{2x^2+x+1}{x-1}$ for all $x \neq 1$. Denote the graph of $y = f(x)$ by G .

Find

(a) the asymptote(s) of G ,

(b) the slope of the normal to G at the point $(2,11)$. (7 marks)

[HKDSE 2016 M2 #4]

11. Let a and b be constants. Define $f(x) = x^3 + ax^2 + bx + 5$ for all real numbers x .

Denote the curve $y = f(x)$ by C . It is given that $P(-1,10)$ is a turning point of C .

(a) Find a and b . (3 marks)

(b) Is P a maximum point of C ? Explain your answer. (2 marks)

(c) Find the minimum value(s) of $f(x)$. (2 marks)

(d) Find the point(s) of inflexion of C . (2 marks)

(e) Let L be the tangent to C at P . Find the area of the region bounded by C and L . (4 marks)

[HKDSE 2016 M2 #9]

12. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.
   (a) Let A cm^2 be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A=\dfrac{15}{16}\pi h^2$.
   (b) The depth of water in the container increases at a constant rate $\dfrac{3}{\pi}$ cm/s. Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm^3. (7 marks)

[HKDSE 2017 M2 #6]

13. Let $f(x)=\dfrac{x^2-5x}{x+4}$ for all $x\neq -4$. Denote the graph of $y=f(x)$ by G.
   (a) Find the asymptote(s) of G. (3 marks)
   (b) Find $f'(x)$. (2 marks)
   (c) Find the maximum point(s) and the minimum point(s) of G. (4 marks)
   (d) Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis. (4 marks)

[HKDSE 2017 M2 #9]

14. Define $f(x)=\dfrac{A}{x^2-4x+7}$ for all real numbers x, where A is a constant. It is given that the extreme value of $f(x)$ is 4.
   (a) Find $f'(x)$.
   (b) Someone claims that there are at least two asymptotes of the graph of $y=f(x)$. Do you agree? Explain your answer.
   (c) Find the point(s) of inflexion of the graph of $y=f(x)$. (8 marks)

[HKDSE 2018 M2 #8]

15. Consider the curve $C: y=\ln\sqrt{x}$, where $x>1$. Let P be a moving point lying on C. The normal to C at P cuts the x-axis at the point Q while the vertical line passing through P cuts the x-axis at the point R.

    (a) Denote the x-coordinate of P by r. Prove that the x-coordinate of Q is $\dfrac{4r^2+\ln r}{4r}$. (3 marks)

    (b) Find the greatest area of ΔPQR. (5 marks)

    (c) Let O be the origin. It is given that OP increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of ΔPQR increases at a rate lower than 2 square units per minute when the x-coordinate of P is e. Is the claim correct? Explain your answer. (4 marks)

[HKDSE 2018 M2 #9]

16. A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm^3 of liquid X. The researcher finds that during the experiment, $\dfrac{\mathrm{d}V}{\mathrm{d}t}=-2t$, where V cm^3 is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

    (a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

    (b) It is given that $V=h^2+24h$, where h cm is the depth of liquid X in the vessel. Find the value of $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ when $t=18$.

(6 marks)

[HKDSE 2019 M2 #3]

Technique of Integration

1. (a) Find $\int \frac{x+1}{x} \mathrm{d}x$.

   (b) Using the substitution $u = x^2 - 1$, find $\int \frac{x^3}{x^2-1} \mathrm{d}x$. (5 marks)

[HKDSE 2012 M2 #4]

2. (a) Find $\int \frac{\mathrm{d}x}{\sqrt{9-x}}$, where $x < 9$.

   (b) Using integration by substitution, find $\int \frac{\mathrm{d}x}{\sqrt{9-x^2}}$, where $-3 < x < 3$.

(6 marks)

[HKDSE 2014 M2 #5]

3. (a) Find $\int \frac{1}{e^{2u}} \mathrm{d}u$.

   (b) Using integration by substitution, evaluate $\int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}} \mathrm{d}x$. (7 marks)

[HKDSE 2015 M2 #3]

4. (a) Using integration by parts, find $\int e^x \sin \pi x \ \mathrm{d}x$.

   (b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x \ \mathrm{d}x$.

(7 marks)

[HKDSE 2019 M2 #7]

Applications of Integration

1. (a) Using integration by parts, find $\int x\sin x\,dx$.

   (b)

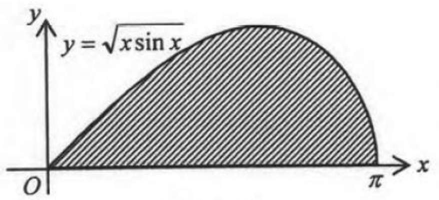


Figure 4

Figure 4 shows the shaded region bounded by the curve $y=\sqrt{x\sin x}$ for $0\le x\le\pi$ and the x-axis. Find the volume of the solid generated by revolving the region about the x-axis. (4 marks)

[HKDSE 2012 M2 #9]

2. (a) (i) Suppose $\tan u = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$, where $\dfrac{-\pi}{2} < u < \dfrac{\pi}{2}$.

Show that $u = \dfrac{-\pi}{5}$.

(ii) Suppose $\tan v = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$. Find v, where $\dfrac{-\pi}{2} < v < \dfrac{\pi}{2}$. (4 marks)

(b) (i) Express $x^2 + 2x\cos\dfrac{2\pi}{5} + 1$ in the form $(x+a)^2 + b^2$, where a and b are constants.

(ii) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2 + 2x\cos\frac{2\pi}{5} + 1}\,dx$. (6 marks)

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2 + 2x\cos\frac{7\pi}{5} + 1}\,dx$. (3 marks)

[HKDSE 2012 M2 #13]

3. Consider the curve $\Gamma: y = kx^p$, where $k > 0$, $p > 0$. In Figure 7, the tangent to Γ at $A(a, ka^p)$ cuts the x-axis at $B(-a, 0)$, where $a > 0$.

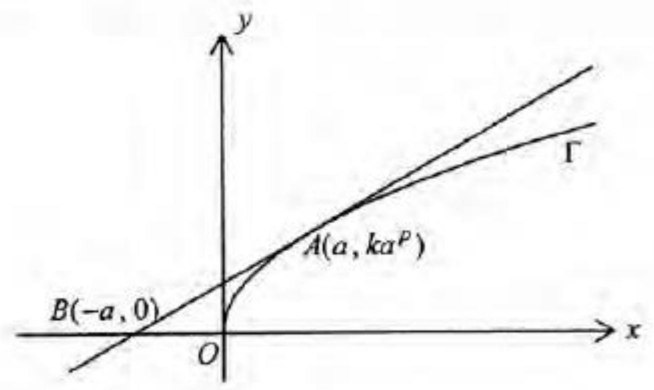


Figure 7

(a) Show that $p = \frac{1}{2}$. (3 marks)

(b) Suppose that $a = 1$. As shown in Figure 8, the circle C, with radius 2 and centre on the y-axis, touches Γ at point A.

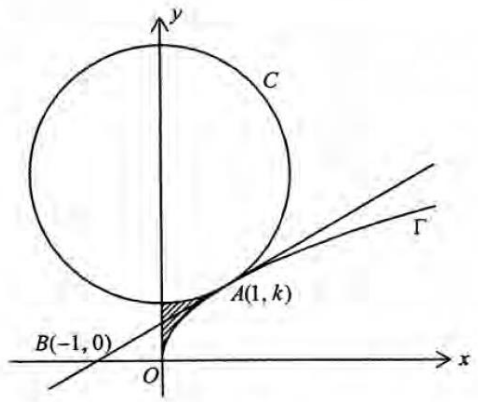


Figure 8

(i) Show that $k = \frac{2\sqrt{3}}{3}$.

(ii) Find the area of the shaded region bounded by Γ, C and the y-axis.

(9 marks)

[HKDSE 2012 M2 #14]

4. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = e^x - 1$. It is given that the curve passes through the point $(1, e)$.

(a) Find the equation of the curve.

(b) Find the equation of the tangent to the curve at the point where the curve cuts the y-axis. (5 marks)

[HKDSE 2013 M2 #4]

5.

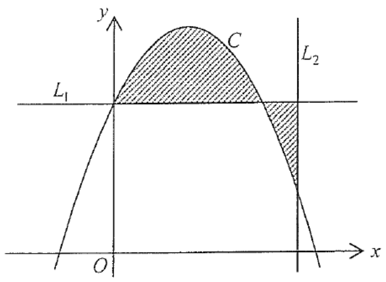


Figure 1

Figure 1 shows the shaded region with boundaries $C: y = \frac{-x^2}{2} + 2x + 4$, $L_1: y = 4$ and $L_2: x = 5$. It is given that C intersects L_1 at $(0, 4)$ and $(4, 4)$.

(a) Find the area of the shaded region.

(b) Find the volume of the solid of revolution when the shaded region is revolved about L_1. (6 marks)

[HKDSE 2013 M2 #6]

6. (a) Let $0 < \theta < \frac{\pi}{2}$. By finding $\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\sec\theta + \tan\theta)$, or otherwise, show that

$\int \sec\theta \mathrm{d}\theta = \ln(\sec\theta + \tan\theta) + C$, where C is any constant. (2 marks)

(b) (i) Using (a) and a suitable substitution, show that

$$\int \frac{\mathrm{d}u}{\sqrt{u^2-1}} = \ln(u + \sqrt{u^2-1}) + C \quad \text{for} \quad u > 1.$$

(ii) Using (b)(i), show that $\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\mathrm{d}x = \ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$.

(5 marks)

(c) Let $t = \tan\phi$. Show that $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{1+t^2}$. Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1+2\cos^2\phi}}\mathrm{d}\phi$.

(5 marks)

[HKDSE 2013 M2 #11]

7. (a) Find $\int xe^{-x}\mathrm{d}x$.

(b)

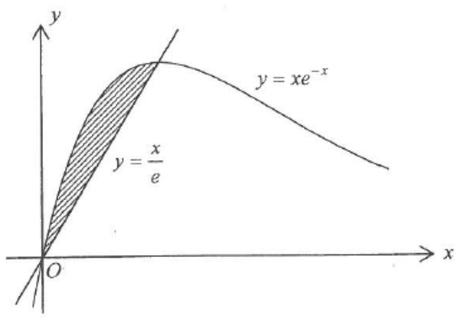


Figure 1

Figure 1 shows the shaded region bounded by the curve $y = xe^{-x}$ and the straight line $y = \frac{x}{e}$. Find the area of the shaded region. (6 marks)

[HKDSE 2014 M2 #6]

8. (a) Prove that $1-\cos 4\theta - 2\cos 2\theta \sin^2 2\theta = 16\cos^2\theta \sin^4\theta$. (2 marks)

(b) Show that $\int_0^{n\pi} \cos^2 x \sin^4 x \, dx = \frac{n\pi}{16}$, where n is a positive integer.

(4 marks)

(c) Let $f(x)$ be a continuous function such that $f(k-x) = f(x)$, where k is a constant. Show that $\int_0^k x f(x)\, dx = \frac{k}{2}\int_0^k f(x)\, dx$. (4 marks)

(d)

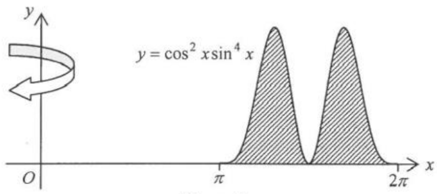


Figure 5

Figure 5 shows the shaded region bounded by the curve $y = \cos^2 x \sin^4 x$ and the x-axis, where $\pi \le x \le 2\pi$. Find the volume of the solid of revolution when the shaded region is revolved about the y-axis. (4 marks)

[HKDSE 2014 M2 #13]

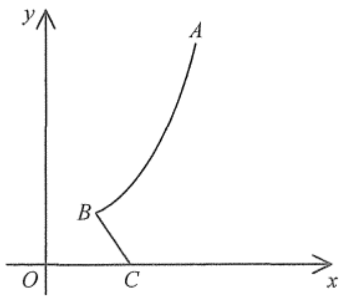
9. (a) Using integration by parts, find $\int x^2 \ln x \, dx$.

(b) At any point (x, y) on the curve Γ, the slope of the tangent to Γ is $9x^2 \ln x$. It is given that Γ passes through the point $(1, 4)$. Find the equation of Γ.

(7 marks)

[HKDSE 2015 M2 #4]

10. (a) In the figure, the curve Γ consists of the curve AB, the line segments BC and CO, where O is the origin, B lies in the first quadrant and C lies on the x-axis. The equations of AB and BC are $x^2-4y+8=0$ and $3x+y-9=0$ respectively.



(i) Find the coordinates of B.

(ii) Let h be the y-coordinate of A, where $h>3$. A cup is formed by revolving Γ about the y-axis. Prove that the capacity of the cup is $\pi(2h^2-8h+25)$. (7 marks)

(b) A cup described in (a)(ii) is placed on a horizontal table. The radii of the base and the tip of the cup are 3 cm and 6 cm respectively.

(i) Find the capacity of the cup.

(ii) Water is poured into the cup at a constant rate of 24π cm^3/s. Find the rate of change of the depth of water when the volume of water in the cup is 35π cm^3. (6 marks)

[HKDSE 2015 M2 #12]

11. Consider the curve $C: y=2e^x$, where $x>0$. It is given that P is a point lying on C. The horizontal line which passes through P cuts the y-axis at the point Q. Let O be the origin. Denote the x-coordinate of P by u.

(a) Express the area of ΔOPQ in terms of u.

(b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of ΔOPQ when $u=4$.

(5 marks)

[HKDSE 2016 M2 #3]

12. (a) Using integration by substitution, find $\int (1+\sqrt{t+1})^2 \mathrm{d}t$.

(b) Consider the curve $\Gamma: y = 4x^2 - 4x$, where $1 \le x \le 4$. Let R be the region bounded by Γ, the straight line $y = 48$ and the two axes. Find the volume of the solid of revolution generated by revolving R about the y-axis.

(8 marks)

[HKDSE 2016 M2 #7]

13. (a) Let $\mathrm{f}(x)$ be a continuous function defined on the interval $[0, a]$, where a is a positive constant. Prove that $\int_0^a \mathrm{f}(x)\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\mathrm{d}x$. (3 marks)

(b) Prove that $\int_0^{\frac{\pi}{4}} (1+\tan x)\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)\mathrm{d}x$. (3 marks)

(c) Using (b), prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x = \frac{\pi \ln 2}{8}$. (3 marks)

(d) Using integration by parts, evaluate $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\mathrm{d}x$. (3 marks)

[HKDSE 2016 M2 #10]

14. (a) Using integration by parts, find $\int x^2 e^{-x}\mathrm{d}x$.

(b) Find the area of the region bounded by the graph $y = x^2 e^{-x}$, the x-axis and the straight line $x = 6$. (6 marks)

[HKDSE 2017 M2 #4]

15. Let $\mathrm{f}(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Denote the curve $y = \mathrm{f}(x)$ by Γ. It is given that Γ passes through the point $P(e^3, 7)$ and $\mathrm{f}'(x) = \frac{1}{x}\ln x^2$ for all $x > 0$. Find

(a) the equation of the tangent to Γ at P,

(b) the equation of Γ,

(c) the point(s) of inflexion of Γ. (8 marks)

[HKDSE 2017 M2 #8]

16. (a) Using $\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$, evaluate $\int_0^1 \frac{1}{x^2+2x+3}\,\mathrm{d}x$.

(3 marks)

(b) (i) Let $0\le\theta\le\frac{\pi}{4}$. Prove that $\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$.

(ii) Using the substitution $t=\tan\theta$, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$.

(5 marks)

(c) Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta=\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$. (2 marks)

(d) Evaluate $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$. (3 marks)

[HKDSE 2017 M2 #11]

17. (a) Using integration by parts, find $\int u(5^u)\,\mathrm{d}u$.

(b) Define $\mathrm{f}(x)=x(5^{2x})$ for all real numbers x. Find the area of the region bounded by the graph of $y=\mathrm{f}(x)$, the straight line $x=1$ and the x-axis.

(6 marks)

[HKDSE 2018 M2 #4]

18. (a) Using integration by substitution, find $\int x^3\sqrt{1+x^2}\,\mathrm{d}x$.

(b) At any point (x, y) on the curve Γ, the slope of the tangent to Γ is $15x^3\sqrt{1+x^2}$. The y-intercept of Γ is 2. Find the equation of Γ. (7 marks)

[HKDSE 2018 M2 #5]

19. (a) (i) Prove that $\int \sin^4 x \, dx = \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$.

(ii) Evaluate $\int_0^{\pi} \sin^4 x \, dx$. (5 marks)

(b) (i) Let $f(x)$ be a continuous function such that $f(\beta - x) = f(x)$ for all real numbers x, where β is a constant. Prove that

$$\int_0^{\beta} x f(x) dx = \frac{\beta}{2}\int_0^{\beta} f(x) dx.$$

(ii) Evaluate $\int_0^{\pi} x \sin^4 x dx$. (5 marks)

(c) Consider the curve $G : y = \sqrt{x}\sin^2 x$, where $\pi \le x \le 2\pi$. Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis. (3 marks)

[HKDSE 2018 M2 #10]

20. Define $g(x) = \frac{\ln x}{\sqrt{x}}$ for all $x \in (0, 99)$. Denote the graph of $y = g(x)$ by G.

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G, the x-axis and the vertical line passing through the maximum point of G. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

[HKDSE 2019 M2 #4]

21. Let $h(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \frac{2x^2 - 7x + 8}{x}$ for all $x > 0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = h(x)$ by H. It is given that H passes through the point $(1, 3)$. Find

(i) the equation of H,

(ii) the point(s) of inflexion of H.

(8 marks)

[HKDSE 2019 M2 #8]

22. Consider the curve $\Gamma : y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$. Denote the tangent to Γ at $x = 3$ by L .

(a) Find the equation of L . (3 marks)

(b) Let C be the curve $y = \sqrt{4 - x^2}$, where $0 < x < 2$. It is given that L is a tangent to C . Find

(i) the point(s) of contact of L and C ;

(ii) the point(s) of intersection of C and Γ ;

(iii) the area of the region bounded by L , C and Γ .

(9 marks)

[HKDSE 2019 M2 #9]

23. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2 + \cos 2x} = \frac{\sec^2 x}{2 + \sec^2 x}$. (1 mark)

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2 + \cos 2x}\,\mathrm{d}x$. (3 marks)

(c) Let $\mathrm{f}(x)$ be a continuous function defined on $\mathbf{R}$ such that $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all $x \in \mathbf{R}$. Prove that

$\int_{-a}^{a} \mathrm{f}(x)\ln(1 + e^x)\,\mathrm{d}x = \int_0^a x\mathrm{f}(x)\mathrm{d}x$ for any $a \in \mathbf{R}$. (4 marks)

(d) Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2 + \cos 2x)^2}\ln(1 + e^x)\,\mathrm{d}x$. (5 marks)

[HKDSE 2019 M2 #10]

Matrices

1. (a) Solve the equation $\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0$ ---------(*). (2 marks)

   (b) Let x_1, x_2 $(x_1 < x_2)$ be the roots of (*). Let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$. It is given that $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix}$ and $|P| = 1$, where a , b and c are constants.

   (i) Find P .

   (ii) Evaluate $P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P$.

   (iii) Using (b)(ii), evaluate $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$. (11 marks)

[HKDSE 2012 M2 #11]

2. Let M be the matrix $\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}$, where $k \neq 0$.

   (a) Find M^{-1} .

   (b) If $M\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find the value of k . (5 marks)

[HKDSE 2013 M2 #8]

3. For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\text{tr}(M) = a + d$. Let A and B be 2×2 matrices such that $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\text{tr}(MN) = \text{tr}(NM)$.

(ii) Show that $\text{tr}(A) = 4$.

(iii) Find the value of $|A|$. (6 marks)

(b) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

(i) Prove that $\begin{vmatrix} p - \lambda_1 & 1 \\ r & s - \lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p - \lambda_2 & 1 \\ r & s - \lambda_2 \end{vmatrix} = 0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation

$$\lambda^2 - \text{tr}(C) \cdot \lambda + |C| = 0 .$$

(5 marks)

(c) Find two values of λ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$. (2 marks)

[HKDSE 2013 M2 #13]

4. Let $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}$.

(a) Prove, by mathematical induction, that for all positive integers n,
$A^{n+1}=2^n A$.

(b) Using the result of (a), Willy proceeds in the following way:

$$A^2=2A$$
$$A^2A^{-1}=2AA^{-1}$$
$$A=2I$$

Explain why Willy arrives at a wrong conclusion. (7 marks)

[HKDSE 2014 M2 #7]

5. Let $M=\begin{pmatrix}k-1 & k\\ 1 & 0\end{pmatrix}$ and $A=\begin{pmatrix}1 & p\\ -1 & 1\end{pmatrix}$, where k and p are real numbers and $p\neq -1$.

(a) (i) Find A^{-1} in terms of p.

(ii) Show that $A^{-1}MA=\begin{pmatrix}-1 & k=p\\ 0 & k\end{pmatrix}$.

(iii) Suppose $p=k$. Using (ii), find M^n in terms of k and n, where n is a positive integer. (8 marks)

(b) A sequence is defined by $x_1=0$, $x_2=1$ and $x_n=x_{n-1}+2x_{n-2}$ for $n=3,4,5\ldots$. It is known that this sequence can be expressed in the matrix form $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{n-1}\\ x_{n-2}\end{pmatrix}$. Using the result of (a)(iii), express x_n in terms of n. (3 marks)

[HKDSE 2014 M2 #12]

6. (a) Let M be a 3×3 real matrix such that $M^T=-M$, where M^T is the transpose of M. Prove that $|M|=0$.

(b) Let $A=\begin{pmatrix}-1 & a & b\\ -a & -1 & -8\\ -b & 8 & -1\end{pmatrix}$, where a and b are real numbers. Denote the 3×3 identity matrix by I.

(i) Using (a), or otherwise, prove that $|A+I|=0$.

(ii) Someone claims that A^3+I is a singular matrix. Do you agree? Explain your answer. (6 marks)

[HKDSE 2015 M2 #6]

7. (a) Let λ and μ be real numbers such that $\mu-\lambda\neq 2$. Denote the 2×2 identity matrix by I. Define $A=\dfrac{1}{\lambda-\mu+2}(I-\mu I-M)$, where $M=\begin{pmatrix}\lambda & 1\\ \lambda-\mu+1 & \mu\end{pmatrix}$.

(i) Evaluate AB, BA and $A+B$.

(ii) Prove that $A^2=A$ and $B^2=B$.

(iii) Prove that $M^n=(\lambda+1)^nA+(\mu-1)^nB$ for all positive integers n.

(8 marks)

(b) Using (a), or otherwise, evaluate $\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}$. (4 marks)

[HKDSE 2015 M2 #11]

8. Let n be a positive integer.

(a) Define $A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. Evaluate

(i) A^2,

(ii) A^n,

(iii) $(A^{-1})^n$.

(b) Evaluate

(i) $\sum_{k=0}^{n-1} 2^k$,

(ii) $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}^n$. (8 marks)

[HKDSE 2016 M2 #8]

9. Let $A=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Using mathematical induction, prove that $A^n = 3^n I + 3^{n-1} n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ for all positive integers n. (4 marks)

(b) Let $B=\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}$.

(i) Define $P=\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n = 3^n I + 3^{n-1} n\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}$ for any positive integer n.

(iii) Does there exist a positive integer m such that $\left|A^m - B^m\right| = 4m^2$?

Explain your answer. (8 marks)

[HKDSE 2017 M2 #12]

10. Define $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be non-zero real matrix such that $MX = XM$.

(a) Express b and c in terms of a.

(b) Prove that X is a non-singular matrix.

(c) Denote the transpose of X by X^T. Express $(X^T)^{-1}$ in terms of a.

(8 marks)

[HKDSE 2018 M2 #7]

11. Let $M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$.

(3 marks)

(b) Prove that $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ for all positive integers n.

(4 marks)

(c) Does there exist a pair of 2×2 matrices A and B such that $(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$ for all positive integers n? Explain your answer.

(5 marks)

[HKDSE 2019 M2 #11]

Systems of Linear Equations

1. (a) Solve the following system of linear equations: $\begin{cases} x+y+z=0 \\ 2x-y+5z=6 \end{cases}$.

   (b) Using (a), or otherwise, solve the following system of linear equations:

   $\begin{cases} x+y+z=0 \\ 2x-y+5z=6 \\ x-y+\lambda z=4 \end{cases}$, where λ is a constant. (5 marks)

   [HKDSE 2012 M2 #8]

2. Consider the following system of linear equations in x , y and z

   $(E):\begin{cases} x-ay+z=2 \\ 2x+(1-2a)y+(2-b)z=a+4 \\ 3x+(1-3a)y+(3-ab)z=4 \end{cases}$, where a and b are real numbers.

   It is given that (E) has infinitely many solutions.

   (a) Find the values of a and b .

   (b) Solve (E). (5 marks)

   [HKDSE 2013 M2 #9]

3. (a) Solve the system of linear equations $\begin{cases} x+y+z=100 \\ x+6y+10z=200 \end{cases}$.

   (b) In a store, the prices of each of small, medium and large marbles are \$0.5, \$3 and \$5 respectively. Audrey plans to spend all \$100 for exactly 100 marbles, which include m small marbles, n medium marbles and k large marbles. Audrey claims that there is only one set of combination of m , n and k . Do you agree? Explain your answer. (6 marks)

   [HKDSE 2014 M2 #9]

4. Solve the following systems of linear equations in real variables x , y , z :

(a) $\begin{cases} x+y+z=2 \\ 2x+3y-3z=4 \end{cases}$;

(b) $\begin{cases} x+y+z=2 \\ 2x+3y-3z=4 \\ 3x+2y+kz=6 \end{cases}$, where k is a real constant. (6 marks)

[HKDSE 2015 M2 #5]

5. (a) Consider the following system of linear equations in real variables x , y , z

$$(E):\begin{cases} x+y-z=3 \\ 4x+6y+az=b \\ 5x+(1-a)y+(3a-1)z=b-1 \end{cases}$$, where a and b are real numbers.

(i) Assume that (E) has a unique solution.

(1) Prove that $a\neq -2$ and $a\neq -12$.

(2) Solve (E)

(ii) Assume that $a=-2$ and (E) is consistent.

(1) Find b .

(2) Solve (E). (9 marks)

(b) Is there a real solution of the system of linear equations $\begin{cases} x+y-z=3 \\ 2x+3y-z=7 \\ 5x+3y-7z=13 \end{cases}$

satisfying $x^2+y^2-6z^2>14$? Explain your answer. (3 marks)

[HKDSE 2016 M2 #11]

6. Consider the following system of linear equations in real variables x , y , z

$$(E):\begin{cases} x+2y-z=11 \\ 3x+8y-11z=49 \\ 2x+3y+hz=k \end{cases}, \text{ where } h,k\in\mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h .

(ii) Express z in terms of h and k .

(b) Assume (E) has infinitely many solutions. Solve (E). (6 marks)

[HKDSE 2017 M2 #5]

7. (a) Consider the system of linear equations in real variables x , y , z

$$(E):\begin{cases} x+ay+4(a+1)z=18 \\ 2x+(a-1)y+2(a-1)z=20 \\ x-y-12z=b \end{cases}, \text{ where } a,b\in\mathbf{R}.$$

(i) Assume that (E) has a unique solution.

(1) Find the range of values of a .

(2) Solve (E).

(ii) Assume that $a=3$ and (E) is consistent.

(1) Find b .

(2) Solve (E). (9 marks)

(b) Consider the system of linear equations in real variables x , y , z

$$(F):\begin{cases} x+3y+16z=18 \\ x+y+2z=10 \\ x-y-12z=s \\ 2x-5y-45z=t \end{cases}, \text{ where } s,t\in\mathbf{R}.$$

Assume that (F) is consistent. Find s and t . (3 marks)

[HKDSE 2018 M2 #11]

8. Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x-2y-2z=\beta \\ 5x+\alpha y+\alpha z=5\beta \\ 7x+(\alpha-3)y+(2\alpha+1)z=8\beta \end{cases} \quad \text{, where } \alpha,\beta\in\mathbf{R} .$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α .

(ii) Express y in terms of α and β .

(b) Assume that $\alpha=-4$. If (E) is inconsistent, find the range of values of β .

(7 marks)

[HKDSE 2019 M2 #6]

Vectors

1.

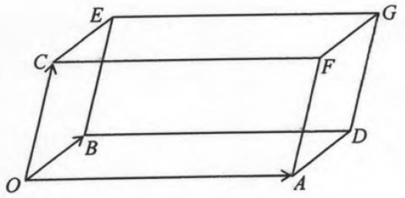


Figure 3

Figure 3 shows a parallelpiped $OADBECFG$. Let $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OC} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

(a) Find the area of the parallelogram $OADB$.

(b) Find the distance between point C and the plane $OADB$. (5 marks)

[HKDSE 2012 M2 #7]

2.

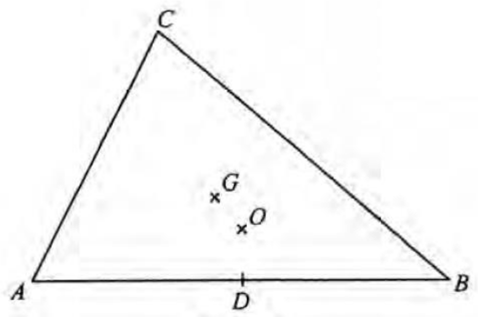


Figure 6

Figure 6 shows an acute angled scalene triangle ABC , where D is the mid-point of AB , G is the centroid and O is the circumcenter. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Express AG in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (3 marks)

(b) It is given that E is a point on AB such that CE is an altitude. Extend OG to meet CE at F .

(i) Prove that $\Delta DOG \sim \Delta CFG$. Hence find $FG : GO$.

(ii) Show that $\overrightarrow{AF} = \mathbf{b} + \mathbf{c}$. Hence prove that F is the orthocentre of ΔABC . (9 marks)

[HKDSE 2012 M2 #12]

3.

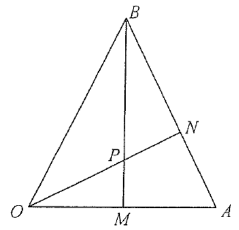


Figure 2

Let $\overrightarrow{OA}=2\mathbf{i}$ and $\overrightarrow{OB}=\mathbf{i}+2\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN:NA=k:1$. BM intersects ON at P (see Figure 2).

(a) Express $\overrightarrow{ON}$ in terms of k.

(b) If A, N, P and M are concyclic, find the value of k. (5 marks)

[HKDSE 2013 M2 #10]

4.

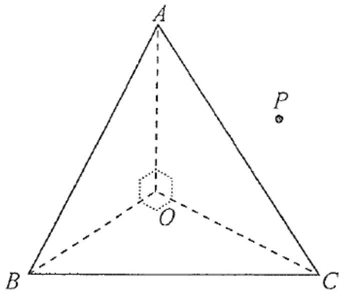


Figure 5

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = 90°$. P is a variable point such that $\overrightarrow{AP} \cdot \overrightarrow{BP} + \overrightarrow{BP} \cdot \overrightarrow{CP} + \overrightarrow{CP} \cdot \overrightarrow{AP} = 0$. Let D be the fixed point such that $\overrightarrow{OD} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OD} = \mathbf{d}$.

(a) (i) Show that $\overrightarrow{AP} \cdot \overrightarrow{BP} = \mathbf{p} \cdot \mathbf{p} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p}$.

(ii) Using (a)(i), show that $\mathbf{p} \cdot \mathbf{p} = 2\mathbf{p} \cdot \mathbf{d}$.

(iii) Show that $|\mathbf{p} - \mathbf{d}| = |\mathbf{d}|$. Hence show that P lies on the sphere centred at D with fixed radius. (8 marks)

(b) (i) Alice claims that O lies on the sphere mentioned in (a)(iii). Do you agree? Explain your answer.

(ii) Suppose P_1, P_2 and P_3 are three distinct points on the sphere in (a)(iii) such that $\overrightarrow{DP_1} \times \overrightarrow{DP_2} = \overrightarrow{DP_2} \times \overrightarrow{DP_3}$. Alice claims that the radius of the circle passing through P_1, P_2 and P_3 is OD. Do you agree? Explain your answer. (4 marks)

[HKDSE 2013 M2 #14]

5. Let $\overrightarrow{OP} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OR} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OQ}$. Hence find the volume of tetrahedron $OPQR$.

(b) Find the acute angle between the plane OPQ and the line OR, correct to the nearest 0.1°. (8 marks)

[HKDSE 2014 M2 #8]

6.

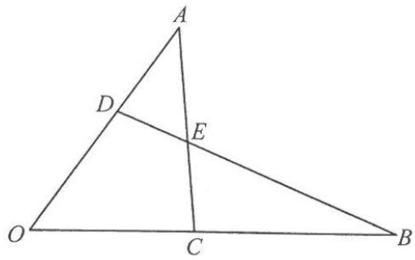


Figure 4

In Figure 4, C and D are points on OB and OA respectively such that $AD : DO = OC : CB = t : 1-t$, where $0 < t < 1$. BD and AC intersect at E such that $AE : EC = m : 1$ and $BE : ED = n : 1$, where m and n are positive. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) By considering ΔOAC, express $\overrightarrow{OE}$ in terms of m, t, $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering ΔOBD, express $\overrightarrow{OE}$ in terms of n, t, $\mathbf{a}$ and $\mathbf{b}$.

(iii) Show that $m = \dfrac{t}{(1-t)^2}$ and $n = \dfrac{1-t}{t^2}$.

(iv) Chris claims that "if $m = n$, then E is the centroid of ΔOAB." Do you agree? Explain your answer. (9 marks)

(b) It is given that $OA = 1$ and $OB = 2$. Francis claims that "if AC is perpendicular to OB, then BD is always perpendicular to OA." Do you agree? Explain your answer. (4 marks)

[HKDSE 2014 M2 #11]

7. OAB is a triangle. P is the mid-point of OA . Q is a point lying on AB such that $AQ:QB=1:2$ while R is a point lying on OB such that $OR:RB=3:1$. PR and OQ intersect at C .

   (a) (i) Let t be a constant such that $PC:CR=t:(1-t)$. By expressing $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, find the value of t .

   (ii) Find $CQ:OQ$. (7 marks)

   (b) Suppose that $\overrightarrow{OA}=20\mathbf{i}-6\mathbf{j}-12\mathbf{k}$, $\overrightarrow{OB}=16\mathbf{i}-16\mathbf{j}$ and $\overrightarrow{OD}=\mathbf{i}+3\mathbf{j}-6\mathbf{k}$, where O is the origin. Find

   (i) the area of ΔOAB ,

   (ii) the volume of the tetrahedron $ABCD$. (5 marks)

[HKDSE 2015 M2 #10]

8. Let $\overrightarrow{OA}=2\mathbf{j}+2\mathbf{k}$, $\overrightarrow{OB}=4\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\overrightarrow{OP}=\mathbf{i}+t\mathbf{j}$, where t is a constant and O is the origin. It is given that P is equidistant from A and B .

   (a) Find t . (3 marks)

   (b) Let $\overrightarrow{OC}=2\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OD}=3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$. Denote the plane which contains A , B and C by Π .

   (i) Find a unit vector which is perpendicular to Π .

   (ii) Find the angle between CD and Π.

   (iii) It is given that E is a point lying on Π such that DE is perpendicular to Π . Let F be a point such that $\overrightarrow{PF}=\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$. Describe the geometric relationship between D , E and F . Explain your answer.

   (10 marks)

[HKDSE 2016 M2 #12]

9. P is a point lying on AB such that $AP:PB=3:2$. Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, where O is the origin.
   (a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (b) It is given that $|\mathbf{a}|=45$, $|\mathbf{b}|=20$ and $\cos\angle AOB=\frac{1}{4}$. Find
      (i) $\mathbf{a}\cdot\mathbf{b}$,
      (ii) $\left|\overrightarrow{OP}\right|$. (5 marks)

[HKDSE 2017 M2 #3]

10. ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE:EC=1:r$. AB produced and DE produced meet at the point F. It is given that $BE:EF=1:10$. Let $\overrightarrow{OA}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\overrightarrow{OB}=4\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\overrightarrow{OC}=8\mathbf{i}-3\mathbf{j}-2\mathbf{k}$, where O is the origin.
    (a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r. (4 marks)
    (b) (i) Find $\overrightarrow{AD}\cdot\overrightarrow{DE}$.
       (ii) Are B, D, C and F concyclic? Explain your answer. (5 marks)
    (c) Let $\overrightarrow{OP}=3\mathbf{i}+10\mathbf{j}-4\mathbf{k}$. Denote the circumcenter of ΔBCF by Q. Find the volume of the tetrahedron $ABPQ$. (3 marks)

[HKDSE 2017 M2 #10]

11. The position vectors of the points A, B, C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A, B and C by Π. Let E be the projection of D on Π.
    (a) Find
       (i) $\overrightarrow{AB}\times\overrightarrow{AC}$,
       (ii) the volume of the tetrahedron $ABCD$,
       (iii) $\overrightarrow{DE}$. (5 marks)
    (b) Let F be a point lying on BC such that DF is perpendicular to BC.
       (i) Find $\overrightarrow{DF}$.
       (ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$? Explain your answer. (5 marks)
    (c) Find the angle between ΔBCD and Π. (3 marks)

[HKDSE 2018 M2 #12]

12. Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$.

(a) Find t . (3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (2 marks)

(c) Find the volume of the pyramid $OABC$. (2 marks)

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OR} = r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer.

(6 marks)

[HKDSE 2019 M2 #12]