HKDSE MATH M2 2022

1. **HKDSE Math M2 2022 Q1**
   Let $g(x) = \frac{1}{\sqrt{5x+4}}$, where $x > 0$. Prove that $g(1+h) - g(1) = \frac{-5h}{3\sqrt{5h+9}(3+\sqrt{5h+9})}$. Hence, find $g'(1)$ from first principles.
   (4 marks)

2. **HKDSE Math M2 2022 Q2**
   Let $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

   (a) Prove that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\csc\theta$.

   (b) Solve the equation $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 5$.

   (5 marks)

3. **HKDSE Math M2 2022 Q3**

   (a) Using mathematical induction, prove that $\sum_{k=1}^{2n}(-1)^k k^2 = n(2n+1)$ for all positive integers n.

   (b) Using (a), evaluate $\sum_{k=11}^{100}(-1)^k k^2$.

   (7 marks)

4. **HKDSE Math M2 2022 Q4**
   Let $y = (7x - 2x^2)e^{-x}$.

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   (b) Someone claims that there are two points of inflexion of the graph of $y = (7x - 2x^2)e^{-x}$. Do you agree? Explain your answer.

   (6 marks)

5. **HKDSE Math M2 2022 Q5**
   Let n be an integer greater than 1. Define $(a+x)^n = \sum_{k=0}^{n}\mu_k x^k$, where a is a constant. It is given that $\mu_2 = -10$.

   (a) Explain why a is a negative number and n is an odd number.

   (b) Let $(bx-1)^n = \sum_{k=0}^{n}\lambda_k x^k$, where b is a constant. If $\lambda_0 = \mu_0$ and $\lambda_1 = 2\mu_1$, find a, b and n.

   (6 marks)

6. **HKDSE Math M2 2022 Q6**

   (a) Using integration by substitution, prove that $\displaystyle\int \frac{1}{x^2+2x+5}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$ + constant.

   (b) At any point (x, y) on the curve G, the slope of the tangent to G is $\dfrac{2x+1}{x^2+2x+5}$. Given that G passes through the point $(-3, \ln 2)$, does G pass through the point $\left(-1, \dfrac{-\pi}{8}\right)$? Explain your answer.

   (7 marks)

7. **HKDSE Math M2 2022 Q7**

   Consider the curve $\Gamma : y = \ln(x+2)$, where $x > 0$. Let P be a moving point on Γ with h as its x-coordinate. Denote the tangent to Γ at P by L and the area of the region bounded by Γ, L and the y-axis by A square units.

   (a) Prove that $A = \dfrac{h^2+4h}{2h+4} - 2\ln(h+2) + 2\ln 2$.

   (b) If $h = 3^{-t}$, where t is the time measured in seconds, find the rate of change of A when $t = 1$.

   (8 marks)

8. **HKDSE Math M2 2022 Q8**

   Consider the system of linear equations in real variables x, y and z

$$(E) : \begin{cases} ax + 2y - z = 4k \\ -x + ay + 2z = 4 \\ 2x - y + az = k^2 \end{cases}, \text{where } a, k \in \mathbb{R}$$

   (a) Assume that (E) has a unique solution. Express y in terms of a and k.

   (b) Assume that (E) has infinitely many solutions. Solve (E).

   (7 marks)

9. **HKDSE Math M2 2022 Q9**

   Let $f(x) = \dfrac{x^2+3x}{x-1}$, where $x \neq 1$. Denote the graph of $y = f(x)$ by H.

   (a) Find the asymptote(s) of H.
   (3 marks)

   (b) Find the maximum point(s) and minimum point(s) of H.
   (4 marks)

   (c) Sketch H.
   (3 marks)

   (d) Let R be the region bounded by H and the straight line $y = 10$. Find the volume of the solid of revolution generated by revolving R about the straight line $y = 10$.
   (3 marks)

10. **HKDSE Math M2 2022 Q10**

Let $g(x) = \cos^2 x \cos 2x$.

(a) Prove that $\displaystyle\int g(x)\,dx = \frac{\sin 2x \cos^2 x}{2} + \frac{1}{2}\int \sin^2 2x\,dx$.
(2 marks)

(b) Evaluate $\displaystyle\int_0^{\pi} g(x)\,dx$.
(2 marks)

(c) Using integration by substitution, evaluate $\displaystyle\int_0^{\pi} xg(x)\,dx$.
(4 marks)

(d) Evaluate $\displaystyle\int_{-\pi}^{2\pi} xg(x)\,dx$.
(4 marks)

11. **HKDSE Math M2 2022 Q11**

(a) Let n be a positive integer. Denote the 2×2 identify matrix by I.

(i) Let A be a 2×2 matrix. Simplify $(I - A)(I + A + A^2 + \cdots + A^n)$.

(ii) Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where θ is not a multiple of 2π.

It is given that $A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$.

(1) Prove that $(I - A)^{-1} = \dfrac{1}{2\sin\frac{\theta}{2}} \begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$

(2) Using the result of (a)(i) and (a)(ii)(1),

prove that $I + A + A^2 + \cdots + A^n = \dfrac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \begin{pmatrix} \cos\frac{n\theta}{2} & -\sin\frac{n\theta}{2} \\ \sin\frac{n\theta}{2} & \cos\frac{n\theta}{2} \end{pmatrix}$.

(7 marks)

(b) Using (a)(ii), evaluate

(i) $\cos\dfrac{5\pi}{18} + \cos\dfrac{5\pi}{9} + \cos\dfrac{5\pi}{6} + \cdots + \cos 25\pi$;

(ii) $\cos^2\dfrac{\pi}{7} + \cos^2\dfrac{2\pi}{7} + \cos^2\dfrac{3\pi}{7} + \cdots + \cos^2 7\pi$.

(6 marks)

12. **HKDSE Math M2 2022 Q12**

Consider $\triangle ABC$. Denote the origin by O.

(a) Let D be a point lying on BC such that AD is the angle bisector of $\angle BAC$. Define $BC = a$, $AC = b$ and $AB = c$.

(i) Using the fact that $BD : DC = c : b$, prove that $\overrightarrow{AD} = -\overrightarrow{OA} + \dfrac{b}{b+c}\overrightarrow{OB} + \dfrac{c}{b+c}\overrightarrow{OC}$.

(ii) Let E be a point lying on AC such that BE is the angle bisector of $\angle ABC$.
Define $\overrightarrow{OJ} = \dfrac{a}{a+b+c}\overrightarrow{OA} + \dfrac{b}{a+b+c}\overrightarrow{OB} + \dfrac{c}{a+b+c}\overrightarrow{OC}$.
Prove that J lies on AD. Hence, deduce that AD and BE intersect at J.

(7 marks)

(b) Suppose that $\overrightarrow{OA} = 35\mathbf{i}+9\mathbf{j}+\mathbf{k}$, $\overrightarrow{OB} = 40\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\overrightarrow{OC} = -3\mathbf{j}+\mathbf{k}$. Let I be the incentre of $\triangle ABC$.

(i) Find $\overrightarrow{OI}$.

(ii) By considering $\overrightarrow{AI} \times \overrightarrow{AB}$, find the radius of the inscribed circle of $\triangle ABC$.

(5 marks)