Past paper

1. **HKDSE Math M2 Sample Paper Q9**
   Let $\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$. Figure 2 shows the parallelepiped $OADBECFG$ formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

   (a) Find the area of the parallelogram $OADB$.

   (b) Find the volume of the parallelepiped $OADBECFG$.

   (c) If C' is a point different from C such that the volume of the parallelepiped formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC'}$ is the same as that of $OADBECFG$, find a possible vector of $\overrightarrow{OC'}$.

   (6 marks)

2. **HKDSE Math M2 Sample Paper Q14**
In Figure 3, $\triangle ABC$ is an acute-angled triangle, where O and H are the circumcentre and orthocentre respectively. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OH} = \mathbf{h}$.

(a) Show that

$$(\mathbf{h} - \mathbf{a}) // (\mathbf{b} + \mathbf{c}).$$

(3 marks)

(b) Let $\mathbf{h} - \mathbf{a} = t(\mathbf{b} + \mathbf{c})$, where t is a non-zero constant.
Show that

(i) $t(\mathbf{b} + \mathbf{c}) + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a})$ for some scalar s,

(ii) $(t - 1)(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 0$.

(5 marks)

(c) Express $\mathbf{h}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
(2 marks)

3. **HKDSE Math M2 Practice Paper Q12**
Let $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (see Figure 2). Let M and N be points on the straight lines AB and OC respectively such that $AM : MB = a : (1 - a)$ and $ON : NC = b : (1 - b)$, where $0 < a < 1$ and $0 < b < 1$. Suppose that MN is perpendicular to both AB and OC.

(a) (i) Show that

$$\overrightarrow{MN} = (a + b - 1)\mathbf{i} + (b - a)\mathbf{j} + b\mathbf{k}.$$

(ii) Find the values of a and b.

(iii) Find the shortest distance between straight lines AB and OC.

(8 marks)

(b) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Let G be the projection of O on the plane ABC, find the coordinates of the intersecting point of the two straight lines OG and MN.

(5 marks)

4. **HKDSE Math M2 2012 Q7**

Figure 3 shows a parallelepiped $OADBECFG$. Let $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OC} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

(a) Find the area of the parallelogram $OADB$.

(b) Find the distance between point C and the plane $OADB$.

(5 marks)

5. **HKDSE Math M2 2012 Q12**

Figure 6 shows an acute angled scalene triangle ABC, where D is the mid-point of AB, G is the centroid and O is the circumcentre. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Express $\overrightarrow{AG}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
(3 marks)

(b) It is given that E is a point on AB such that CE is an altitude. Extend OG to meet CE at F.

(i) Prove that $\triangle DOG \sim \triangle CFG$.
Hence find $FG : GO$.

(ii) Show that $\overrightarrow{AF} = \mathbf{b} + \mathbf{c}$.
Hence prove that F is the orthocentre of $\triangle ABC$.

(9 marks)

6. **HKDSE Math M2 2013 Q10**

Let $\overrightarrow{OA} = 2\mathbf{i}$ and $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN : NA = k : 1$. BM intersects ON at P (see Figure 2).

(a) Express $\overrightarrow{ON}$ in terms of k.

(b) If A, N, P and M are concyclic, find the value of k.

(5 marks)

7. **HKDSE Math M2 2013 Q14**

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = 90°$. P is a variable point such that $\overrightarrow{AP} \cdot \overrightarrow{BP} + \overrightarrow{BP} \cdot \overrightarrow{CP} + \overrightarrow{CP} \cdot \overrightarrow{AP} = 0$. Let D be the fixed point such that $\overrightarrow{OD} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OD} = \mathbf{d}$.

(a) (i) Show that $\overrightarrow{AP} \cdot \overrightarrow{BP} = \mathbf{p} \cdot \mathbf{p} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p}$.

(ii) Using (a)(i), show that $\mathbf{p} \cdot \mathbf{p} = 2\mathbf{p} \cdot \mathbf{d}$.

(iii) Show that $|\mathbf{p} - \mathbf{d}| = |\mathbf{d}|$.

Hence show that P lies on the sphere centred at D with fixed radius.

(8 marks)

(b) (i) Alice claims that O lies on the sphere mentioned in (a)(iii). Do you agree? Explain your answer.

(ii) Suppose P_1, P_2 and P_3 are three distinct points on the sphere in (a)(iii) such that $\overrightarrow{DP_1} \times \overrightarrow{DP_2} = \overrightarrow{DP_2} \times \overrightarrow{DP_3}$. Alice claims that the radius of the circle passing through P_1, P_2 and P_3 is OD.

Do you agree? Explain your answer.

(4 marks)

8. **HKDSE Math M2 2014 Q8**

Let $\overrightarrow{OP} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OR} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OQ}$.
Hence find the volume of tetrahedron $OPQR$.

(b) Find the acute angle between the plane OPQ and the line OR, correct to the nearest $0.1°$.

(8 marks)

9. **HKDSE Math M2 2014 Q11**

In Figure 4, C and D are points on OB and OA respectively such that $AD : DO = OC : CB = t : 1 - t$, where $0 < t < 1$. BD and AC intersect at E such that $AE : EC = m : 1$ and $BE : ED = n : 1$, where m and n are positive. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) By considering $\triangle OAC$, express $\overrightarrow{OE}$ in terms of m, t, $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering $\triangle OBD$, express $\overrightarrow{OE}$ in terms of n, t, $\mathbf{a}$ and $\mathbf{b}$.

(iii) Show that $m = \dfrac{t}{(1-t)^2}$ and $n = \dfrac{1-t}{t^2}$.

(iv) Chris claims that

"if $m = n$, then E is the centroid of $\triangle OAB$."

Do you agree? Explain your answer.

(9 marks)

(b) It is given that $OA = 1$ and $OB = 2$. Francis claims that

"if AC is perpendicular to OB, then BD is always perpendicular to OA."

Do you agree? Explain your answer.

(4 marks)

10. **HKDSE Math M2 2015 Q10**

OAB is a triangle. P is the mid-point of OA. Q is a point lying on AB such that $AQ : QB = 1 : 2$ while R is a point lying on OB such that $OR : RB = 3 : 1$. PR and OQ intersect at C.

(a) (i) Let t be a constant such that $PC : CR = t : (1 - t)$.
By expressing $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, find the value of t.

(ii) Find $CQ : OQ$.

(7 marks)

(b) Suppose that $\overrightarrow{OA} = 20\mathbf{i} - 6\mathbf{j} - 12\mathbf{k}$, $\overrightarrow{OB} = 16\mathbf{i} - 16\mathbf{j}$ and $\overrightarrow{OD} = \mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, where O is the origin. Find

(i) the area of $\triangle OAB$,

(ii) the volume of tetrahedron $ABCD$.

(5 marks)

11. **HKDSE Math M2 2016 Q12**

Let $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OP} = \mathbf{i} + t\mathbf{j}$, where t is a constant and O is the origin. It is given that P is equidistant from A and B.

(a) Find t.

(3 marks)

(b) Let $\overrightarrow{OC} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$. Denote the plane which contains A, B and C by Π.

(i) Find a unit vector which is perpendicular to Π.

(ii) Find the angle between CD and Π.

(iii) It is given that E is a point lying on Π such that $\overrightarrow{DE}$ is perpendicular to Π. Let F be a point such that $\overrightarrow{PF} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$. Describe the geometric relationship between D, E and F. Explain your answer.

(10 marks)

12. **HKDSE Math M2 2017 Q3**

P is a point lying on AB such that $AP : PB = 3 : 2$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in therms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}| = 45$, $|\mathbf{b}| = 20$ and $\cos\angle AOB = \dfrac{1}{4}$. Find

(i) $\mathbf{a} \cdot \mathbf{b}$,

(ii) $|\overrightarrow{OP}|$.

(5 marks)

13. **HKDSE Math M2 2017 Q10**

ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE : EC = 1 : r$. AB produced and DE produced meet at the point F. It is given that $DE : EF = 1 : 10$.
Let $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{OC} = 8\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, where O is the origin.

(a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r.

(4 marks)

(b) (i) Find $\overrightarrow{AD} \cdot \overrightarrow{DE}$.

(ii) Are B, D, C and F concyclic? Explain your answer.

(5 marks)

(c) Let $\overrightarrow{OP} = 3\mathbf{i} + 10\mathbf{j} - 4\mathbf{k}$. Denote the circumcentre of $\triangle BCF$ by Q.
Find the volume of the tetrahedron $ABPQ$.

(3 marks)

14. **HKDSE Math M2 2018 Q12**

The position vectors of the points A, B, C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A, B and C by Π. Let E be the projection of D on Π.

(a) Find

(i) $\overrightarrow{AB} \times \overrightarrow{AC}$,

(ii) the volume of the tetrahedron $ABCD$,

(iii) $\overrightarrow{DE}$.

(5 marks)

(b) Let F be a point lying on BC such that DF is perpendicular to BC.

(i) Find $\overrightarrow{DF}$.

(ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$? Explain your answer.

(5 marks)

(c) Find the angle between $\triangle BCD$ and Π.

(3 marks)

15. **HKDSE Math M2 2019 Q12**

Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $|\overrightarrow{AC}| = |\overrightarrow{BC}|$.

(a) Find t.
(3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.
(2 marks)

(c) Find the volume of the pyramid $OABC$.
(2 marks)

(d) Denote the plane which contains A, B and C by Π. It is given that P, Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OQ} = r\mathbf{k}$. Let D be the projection of O on Π.

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(ii) Let E be a point such that $\overrightarrow{OE} = \dfrac{1}{p}\mathbf{i} + \dfrac{1}{q}\mathbf{j} + \dfrac{1}{r}\mathbf{k}$. Describe the geometric relationship between D, E and O. Explain your answer.

(6 marks)

16. **HKDSE Math M2 2020 Q12**

Let $\overrightarrow{OP} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$, where O is the origin.
R is a point lying on PQ such that $PR : RQ = 1 : 3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$.
(2 marks)

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$.
(2 marks)

(c) Let N be a point such that $\overrightarrow{ON} = \lambda(\overrightarrow{OP} \times \overrightarrow{OR})$, where λ is a real number.

(i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

(ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i} + \mu\mathbf{j} - 10\mathbf{k}$.

(1) Find λ and μ.

(2) Denote the angle between $\triangle OPQ$ and $\triangle NPQ$ by θ. Find $\tan\theta$.

(8 marks)

17. **HKDSE Math M2 2021 Q12**

The position vectors of the points A, B, C and D are $t\mathbf{i}+14\mathbf{j}+s\mathbf{k}$, $12\mathbf{i}-s\mathbf{j}-2\mathbf{k}$, $(s+2)\mathbf{i}-16\mathbf{j}+10\mathbf{k}$ and $-t\mathbf{i}+(s+2)\mathbf{j}+14\mathbf{k}$ respectively, where s, $t \in \mathbb{R}$. Suppose that $\overrightarrow{AB}$ is parallel to $5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. Denote the plane which contains A, B and C by Π.

(a) Find

(i) s and t,

(ii) the area of $\triangle ABC$,

(iii) the volume of the tetrahedron $ABCD$,

(iv) the shortest distance from D to Π.

(9 marks)

(b) Let E be the projection of D on Π. Is E the circumcentre of $\triangle ABC$? Explain your answer.
(4 marks)

18. **HKDSE Math M2 2022 Q12**

Consider $\triangle ABC$. Denote the origin by O.

(a) Let D be a point lying on BC such that AD is the angle bisector of $\angle BAC$.
Define $BC = a$, $AC = b$ and $AB = c$.

(i) Using the fact that $BD : DC = c : b$, prove that

$$\overrightarrow{AD} = -\overrightarrow{OA} + \frac{b}{b+c}\overrightarrow{OB} + \frac{c}{b+c}\overrightarrow{OC}.$$

(ii) Let E be a point lying on AC such that BE is the angle bisector of $\angle ABC$.
Define

$$\overrightarrow{OJ} = \frac{a}{a+b+c}\overrightarrow{OA} + \frac{b}{a+b+c}\overrightarrow{OB} + \frac{c}{a+b+c}\overrightarrow{OC}.$$

Prove that J lies on AD. Hence, deduce that AD and BE intersect at J.

(7 marks)

(b) Suppose that $\overrightarrow{OA} = 35\mathbf{i}+9\mathbf{j}+\mathbf{k}$, $\overrightarrow{OB} = 40\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\overrightarrow{OC} = -3\mathbf{j}+\mathbf{k}$. Let I be the incentre of $\triangle ABC$.

(i) Find $\overrightarrow{OI}$.

(ii) By considering $\overrightarrow{AI} \times \overrightarrow{AB}$, find the radius of the inscribed circle of $\triangle ABC$.

(5 marks)

19. **HKDSE Math M2 2023 Q10**

Let O be the origin. The position vectors of P and Q are $-2\mathbf{i} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. Denote the circle passing through O, P and Q by C. Let R be a point lying on PQ such that OR is perpendicular to OQ.

(a) By considering the ratio of PR to RQ, find $\overrightarrow{OR}$.
(3 marks)

(b) OR produced meets C at another point S. Find $\overrightarrow{OS}$.
(3 marks)

(c) Let Π be the plane which contains C.

(i) Find a non-zero vector which is perpendicular to Π.

(ii) Let G be the center of C. Denote the projection of point $A(-6, -22, 2)$ on Π by B. Describe the geometric relationship between O, B and G. Explain your answer.

(6 marks)