Mock Exam 3

Section A

1. *Reference: HKDSE Math M2 2014 Q1*

(a) $(1 + 2x)^2(1 - x)^n$

$= (1 + 4x + 4x^2)\left[1 - nx + \frac{n(n-1)}{2}x^2 + \cdots\right]$ 1M

Coefficient of $x = 4 - n$

$\therefore \quad 4 - n = -7$

$n = \underline{\underline{11}}$ 1A

(b) Coefficient of $x^2 = 4 - 4n + \frac{n(n-1)}{2}$ 1M

$= 4 - 4(11) + \frac{11(11-1)}{2}$

$= \underline{\underline{15}}$ 1A

(4)

2. *Reference: HKDSE Math M2 PP Q6*

$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)$ 1M

$= \lim_{h\to 0}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$ 1M

$= \lim_{h\to 0}\left[\frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}\right]$

$= \lim_{h\to 0}\frac{1}{\sqrt{x+h} + \sqrt{x}}$ 1A

$= \underline{\underline{\frac{1}{2\sqrt{x}}}}$ 1A

(4)

3. *Reference: HKDSE Math M2 PP Q4*

(a) $\frac{x^2 - 1}{x^2 + 1} = \frac{\cot^2\theta - 1}{\cot^2\theta + 1}$

$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$ 1M

$= \cos^2\theta - \sin^2\theta$

$= \cos 2\theta$ 1

(b) $\frac{2(x+1)(x-1)}{x^2 + 1} = \frac{2(x^2 - 1)}{x^2 + 1}$ 1M

Since x is real, we let $x = \cot\theta$ for some θ.

$\therefore \quad \frac{2(x+1)(x-1)}{x^2 + 1} = 2\cos 2\theta$ (by (a)) 1M

$-1 \le \cos 2\theta \le 1$

$-2 \le 2\cos 2\theta \le 2$

$\therefore$ The least value of $\frac{2(x+1)(x-1)}{x^2 + 1}$ is -2. 1A

(5)

4. *Reference: HKDSE Math M2 2015 Q2*

(a) $f'(x) = xk(e^{kx}) + (1)e^{kx}$

$= \underline{\underline{kxe^{kx} + e^{kx}}}$ 1A

$f''(x) = k(kxe^{kx} + e^{kx}) + ke^{kx}$ 1M

$= \underline{\underline{k^2xe^{kx} + 2ke^{kx}}}$ 1A

(b) $f''(x) - 2kf'(x) + 4f(x)$

$= k^2xe^{kx} + 2ke^{kx} - 2k(kxe^{kx} + e^{kx}) + 4xe^{kx}$ (by (a))

$= (4 - k^2)xe^{kx}$ 1M

Since $f''(x) - 2kf'(x) + 4f(x) = 0$ for all real x, we have

$4 - k^2 = 0$

$k = \underline{\underline{\pm 2}}$ 1A

(5)

5. *Reference: HKDSE Math M2 2015 Q4*

(a) Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\ d\theta$. 1M

$\int \frac{dx}{x^2 + 9} = \int \frac{3\sec^2\theta\ d\theta}{9\sec^2\theta}$ 1M

$= \frac{1}{3}\int d\theta$

$= \frac{1}{3}\theta + C$

$= \underline{\underline{\frac{1}{3}\tan^{-1}\frac{x}{3} + C}}$, where C is a constant 1A

(b) $y = \int \frac{x^2}{x^2 + 9}\,dx$

$= \int \frac{x^2 + 9 - 9}{x^2 + 9}\,dx$ 1M

$= \int \left(1 - \frac{9}{x^2 + 9}\right)dx$

$= x - 9\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right) + C$ (by (*a*))

$= x - 3\tan^{-1}\frac{x}{3} + C$ 1M

Since Γ passes through the point (0, 6), we have $C = 6$. 1M

$\therefore$ The equation of Γ is $y = x - 3\tan^{-1}\frac{x}{3} + 6$. 1A

(7)

6. *Reference: HKCEE A. Math 2006 Q13*

(a) From the graph, we have

x	$a < x < 0$	$x = 0$	$0 < x < b$	$x = b$	$b < x < c$
$f'(x)$	–	0	+	0	–

$\therefore$ The x-coordinate of the maximum point is b. 1A

The x-coordinate of the minimum point is 0. 1A

(b) (i) $\int_0^b f'(x)\,dx = \text{Area of } R_2$

$f(b) - f(0) = 6$ 1M

$f(b) - 1 = 6$

$f(b) = \underline{\underline{7}}$ 1A

(ii) $\text{Area of } R_1 = \text{Area of } R_3$

$-\int_a^0 f'(x)\,dx = -\int_b^c f'(x)\,dx$

$f(a) - f(0) = f(b) - f(c)$ 1M

$f(a) - 1 = 7 - f(a)$ 1M

$f(a) = \underline{\underline{4}}$ 1A

(7)

7. *Reference: HKDSE Math M2 PP Q2*

The homogeneous system of linear equations has non-trivial solutions if and only if

$$\begin{vmatrix} 1 & 3 & k \\ 1 & -k & -4 \\ 3 & 5 & -9 \end{vmatrix} = 0$$ 1M + 1A

$\therefore$ $(1)(-k)(-9) + (3)(-4)(3) + (k)(1)(5)$

$- (3)(-k)(k) - (5)(-4)(1) - (-9)(1)(3) = 0$

$9k - 36 + 5k + 3k^2 + 20 + 27 = 0$ 1M

$3k^2 + 14k + 11 = 0$

$(k + 1)(3k + 11) = 0$

$\therefore$ $k = \underline{\underline{-1}}$ or $k = \underline{\underline{-\frac{11}{3}}}$ 1A

(4)

8. *Reference: HKDSE Math M2 2015 Q6*

(a) Note that $|A^T| = |A|$ and $|-A| = (-1)^3|A| = -|A|$. 1M

From $A^T = -A$, we have $|A^T| = |-A|$. Therefore, $|A| = -|A|$.

So, we have $2|A| = 0$.

$\therefore$ $|A| = 0$ 1

Smart Tips

For $n \times n$ matrix A and constant k, we have $|kA| = k^n|A|$.

(b) (i) $I - M$

$$= \begin{pmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{pmatrix}$$

$\therefore$ $(I - M)^T = -(I - M)$ 1M

By (a), we have $|I - M| = 0$. 1

(ii) Note that $I - M^{-1} = -M^{-1}(I - M)$. 1M

$|I - M^{-1}|$

$= |-M^{-1}||I - M|$

$= |-M^{-1}|(0)$ (by (b)(i))

$= 0$

$\therefore$ $I - M^{-1}$ is a singular matrix.

$\therefore$ The claim is agreed. 1A

(6)

9. (a) Since R divides PQ in the ratio 3 : 1,

$$\overrightarrow{OR} = \frac{\overrightarrow{OP} + 3\overrightarrow{OQ}}{3+1} \quad \text{1M}$$

$$= \frac{4\mathbf{i} + \mathbf{j} + 3(12\mathbf{i} - 3\mathbf{j})}{4}$$

$$= 10\mathbf{i} - 2\mathbf{j} \quad \text{1A}$$

$$|\overrightarrow{OR}| = \sqrt{10^2 + (-2)^2} = \sqrt{104} \quad \text{1A}$$

$\therefore$ The required unit vector $= \dfrac{10\mathbf{i} - 2\mathbf{j}}{\sqrt{104}}$ 1M

$$= \underline{\underline{\frac{5}{\sqrt{26}}\mathbf{i} - \frac{1}{\sqrt{26}}\mathbf{j}}} \quad \text{1A}$$

(b) $$(4\mathbf{i} + \mathbf{j}) \cdot \left(\frac{5}{\sqrt{26}}\mathbf{i} - \frac{1}{\sqrt{26}}\mathbf{j}\right) = \sqrt{4^2 + 1^2}(1)\cos\angle ROP \quad \text{1M}$$

$$\frac{20}{\sqrt{26}} - \frac{1}{\sqrt{26}} = \sqrt{17}\cos\angle ROP$$

$$\cos\angle ROP = \frac{19}{\sqrt{26} \times \sqrt{17}} \quad \text{1M}$$

$$\angle ROP = \underline{\underline{25^\circ}} \quad (\textit{cor. to the nearest degree}) \quad \text{1A}$$

(8)

Section B

10. *Reference: HKDSE Math M2 SP Q11*

(a) (i) $$AB = \begin{pmatrix} 0 & 0 & 1 \\ 4 & 4 & 4 \\ 5 & 4 & 5 \end{pmatrix}\begin{pmatrix} 4 & 4 & -4 \\ 0 & -5 & 4 \\ -4 & 0 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}}} \quad \text{1A}$$

$$BA = \begin{pmatrix} 4 & 4 & -4 \\ 0 & -5 & 4 \\ -4 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 4 & 4 & 4 \\ 5 & 4 & 5 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}}} \quad \text{1A}$$

(ii) From (a)(i), $AB = BA = -4I$, where I is the 3×3 identity matrix.

$$\therefore \quad A\left(-\frac{1}{4}B\right) = \left(-\frac{1}{4}B\right)A = I$$

$$\therefore \quad A^{-1} = -\frac{1}{4}B \quad \text{1M}$$

$$= \underline{\underline{\begin{pmatrix} -1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 1 & 0 & 0 \end{pmatrix}}} \quad \text{1A}$$

(4)

(b) (i) $$ACA^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 4 & 4 & 4 \\ 5 & 4 & 5 \end{pmatrix}\begin{pmatrix} -3 & -4 & -4 \\ 5 & 6 & 5 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 8 & 8 & 8 \\ 5 & 4 & 5 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}}$$ 1A

(ii) $$\det (ACA^{-1}) = (1)(2)(1)$$

$$\det A \times \det C \times \det A^{-1} = 2$$ 1M

$$\det C \times \det (AA^{-1}) = 2$$

$$\det C = 2$$

$$\neq 0$$

$\therefore$ C is invertible. 1

(iii) By (b)(i), $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$D^{-1} = \frac{1}{2}\begin{pmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ 1M

$\because$ D^{-1} is a diagonal matrix.

$$\therefore \quad (D^{-1})^{2016} = \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^{2016}} & 0 \\ 0 & 0 & 1 \end{pmatrix}}}$$ 1A

Note that $D^{-1} = (ACA^{-1})^{-1} = (A^{-1})^{-1}(AC)^{-1} = AC^{-1}A^{-1}$.

$\therefore \quad (D^{-1})^{2016} = A(C^{-1})^{2016}\,A^{-1}$

$$(C^{-1})^{2016} = A^{-1}(D^{-1})^{2016}A$$

$$= \begin{pmatrix} -1 & -1 & 1 \\ 0 & \frac{5}{4} & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^{2016}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 4 & 4 & 4 \\ 5 & 4 & 5 \end{pmatrix} \qquad \text{1M}$$

$$= \begin{pmatrix} -1 & -\frac{1}{2^{2016}} & 1 \\ 0 & \frac{5}{4(2^{2016})} & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 4 & 4 & 4 \\ 5 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5-\frac{4}{2^{2016}} & 4-\frac{4}{2^{2016}} & 4-\frac{4}{2^{2016}} \\ \frac{5}{2^{2016}}-5 & \frac{5}{2^{2016}}-4 & \frac{5}{2^{2016}}-5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 5-\frac{1}{2^{2014}} & 4-\frac{1}{2^{2014}} & 4-\frac{1}{2^{2014}} \\ \frac{5}{2^{2016}}-5 & \frac{5}{2^{2016}}-4 & \frac{5}{2^{2016}}-5 \\ 0 & 0 & 1 \end{pmatrix}}} \qquad \text{1A}$$

(7)

11. *Reference: HKDSE Math M2 PP Q12*

(a) (i) $\overrightarrow{OP} = (1-r)\overrightarrow{OA} + r\overrightarrow{OB}$

$= (1-r)(-\mathbf{i}-\mathbf{k}) + r(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$

$= (4r-1)\mathbf{i} + 2r\mathbf{j} + (2r-1)\mathbf{k}$ 1A

$\overrightarrow{OQ} = (1-s)\overrightarrow{OC} + s\overrightarrow{OD}$

$= (1-s)(-\mathbf{i}+3\mathbf{j}-\mathbf{k}) + s(\mathbf{i}+\mathbf{j}+3\mathbf{k})$

$= (2s-1)\mathbf{i} + (3-2s)\mathbf{j} + (4s-1)\mathbf{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \underline{\underline{(2s-4r)\mathbf{i} + (3-2s-2r)\mathbf{j} + (4s-2r)\mathbf{k}}}$

1A

(ii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\because \quad \overrightarrow{PQ}\cdot\overrightarrow{AB} = 0$ 1M

$\therefore \quad [(2s-4r)\mathbf{i} + (3-2s-2r)\mathbf{j} + (4s-2r)\mathbf{k}]\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 0$

$(8s-16r) + (6-4s-4r) + (8s-4r) = 0$

$6 + 12s - 24r = 0$

$2s = 4r - 1$ ...... (1)

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\because \quad \overrightarrow{PQ}\cdot\overrightarrow{CD} = 0$ 1M

$\therefore \quad [(2s-4r)\mathbf{i} + (3-2s-2r)\mathbf{j} + (4s-2r)\mathbf{k}]\cdot(2\mathbf{i}-2\mathbf{j}+4\mathbf{k}) = 0$

$(4s-8r) + (-6+4s+4r) + (16s-8r) = 0$

$24s - 12r - 6 = 0$

$4s - 2r - 1 = 0$ ...... (2)

Substituting (1) into (2),

$$2(4r-1)-2r-1=0$$
$$6r-3=0$$
$$r=\frac{1}{2} \qquad \text{1A}$$

$$\therefore \quad s=\frac{1}{2}\left(4\cdot\frac{1}{2}-1\right)=\frac{1}{2} \qquad \text{1A}$$

$$\overrightarrow{PQ}=(2s-4r)\mathbf{i}+(3-2s-2r)\mathbf{j}+(4s-2r)\mathbf{k}$$
$$=-\mathbf{i}+\mathbf{j}+\mathbf{k}$$

The shortest distance between the straight lines AB and CD

$=\left|\overrightarrow{PQ}\right|$

$=\sqrt{(-1)^2+1^2+1^2}$ 1M

$=\underline{\underline{\sqrt{3}}}$ 1A

(8)

(b) (i) $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$=3\mathbf{j}$

$\overrightarrow{AB}\times\overrightarrow{AC}=(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})\times(3\mathbf{j})$

$=\underline{\underline{-6\mathbf{i}+12\mathbf{k}}}$ 1A

(ii) Since $\overrightarrow{RD}\parallel(\overrightarrow{AB}\times\overrightarrow{AC})$, let $\overrightarrow{RD}=t(\overrightarrow{AB}\times\overrightarrow{AC})$, where t is a constant. 1M

Analysis

Since $\overrightarrow{AR}$ lies on the plane ABC, $\overrightarrow{AR}\perp(\overrightarrow{AB}\times\overrightarrow{AC})$. We have $\overrightarrow{AR}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$.

$\therefore \quad \overrightarrow{RD}=-6t\mathbf{i}+12t\mathbf{k}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$

$=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$

$\overrightarrow{AR}=\overrightarrow{AD}+\overrightarrow{DR}$

$=(2\mathbf{i}+\mathbf{j}+4\mathbf{k})-(-6t\mathbf{i}+12t\mathbf{k})$

$=(2+6t)\mathbf{i}+\mathbf{j}+(4-12t)\mathbf{k}$ 1A

$\because \quad \overrightarrow{AR}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$ 1M

$$\therefore \quad [(2+6t)\mathbf{i}+\mathbf{j}+(4-12t)\mathbf{k}]\cdot(-6\mathbf{i}+12\mathbf{k})=0$$
$$(-12-36t)+(48-144t)=0$$
$$t=\frac{1}{5}$$

$\overrightarrow{OR}=\overrightarrow{OD}+\overrightarrow{DR}$

$=(\mathbf{i}+\mathbf{j}+3\mathbf{k})-\left(-\frac{6}{5}\mathbf{i}+\frac{12}{5}\mathbf{k}\right)$

$=\frac{11}{5}\mathbf{i}+\mathbf{j}+\frac{3}{5}\mathbf{k}$

Hence the coordinates of R are $\left(\frac{11}{5},1,\frac{3}{5}\right)$. 1A

(5)

12. (a) When $x = 0$, $y = 0$.

When $y = 0$, $x = 0$.

$\therefore$ The x-intercept is 0 and the y-intercept is 0. 1A

(1)

(b) $f'(x) = \dfrac{(x^3+2)(4x^3) - x^4(3x^2)}{(x^3+2)^2}$ 1M

$= \underline{\underline{\dfrac{x^6+8x^3}{(x^3+2)^2}}}$ 1A

$f''(x) = \dfrac{(x^3+2)^2(6x^5+24x^2) - (x^6+8x^3)\times 2(x^3+2)(3x^2)}{(x^3+2)^4}$

$= \dfrac{6x^2(x^3+2)[(x^3+2)(x^3+4) - (x^6+8x^3)]}{(x^3+2)^4}$ 1M

$= \dfrac{6x^2(x^3+2)(x^6+6x^3+8-x^6-8x^3)}{(x^3+2)^4}$

$= \dfrac{12x^2(4-x^3)}{(x^3+2)^3}$ 1

(4)

(c) (i) When $f'(x) = 0$,

$x^6 + 8x^3 = 0$

$x^3(x^3+8) = 0$

$x^3 = 0$ or $x^3 = -8$

$x = 0$ or -2 1M

x	$x<-2$	$x=-2$	$-2 < x < -2^{\frac{1}{3}}$	$-2^{\frac{1}{3}} < x < 0$	$x=0$	$x>0$
$f'(x)$	+	0	–	–	0	+

$f(0) = 0$

$f(-2) = -\dfrac{8}{3}$

$\therefore$ The maximum point is $\left(-2, -\dfrac{8}{3}\right)$, the minimum point is $(0, 0)$. 1A

When $f''(x) = 0$,

$12x^2(4-x^3) = 0$

$x = 0$ or $2^{\frac{2}{3}}$

x	$-2^{\frac{1}{3}} < x < 0$	$x=0$	$0 < x < 2^{\frac{2}{3}}$	$x = 2^{\frac{2}{3}}$	$x > 2^{\frac{2}{3}}$
$f''(x)$	+	0	+	0	–

$f(2^{\frac{2}{3}}) = \dfrac{2^{\frac{5}{3}}}{3}$

$\therefore$ The point of inflexion is $\left(2^{\frac{2}{3}}, \dfrac{2^{\frac{5}{3}}}{3}\right)$. 1A

(ii) $\because$ The denominator $x^3 + 2$ is 0 when $x = -2^{\frac{1}{3}}$.

$\therefore$ The vertical asymptote is $x = -2^{\frac{1}{3}}$. 1A

WatchOut

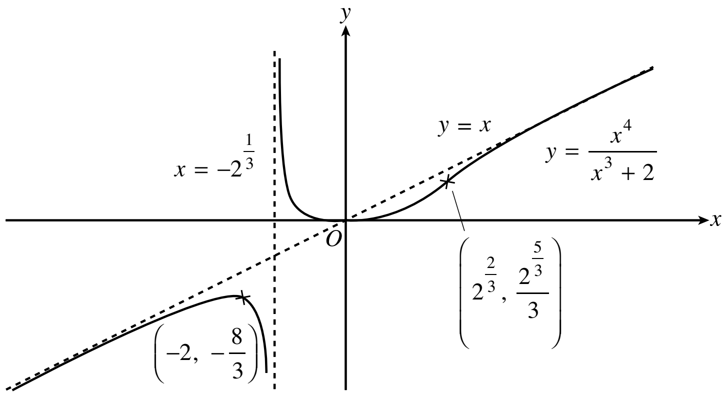
Note that $f''(0) = 0$, the second derivative test cannot be applied.

$$f(x) = \frac{x^4}{x^3+2} = \frac{x^4+2x-2x}{x^3+2} = x - \frac{2x}{x^3+2}$$

Note that $\lim_{x\to\infty} \frac{2x}{x^3+2} = 0$

$\therefore$ The oblique asymptote is $y = x$. 1A

(iii)



1A for the shape of the graph
1A for the asymptotes
1A for all correct
(8)

13. *Reference: HKDSE Math M2 PP Q14*

(a) The volume of the solid of revolution $= \pi\int_4^{4+h} y\,dy$ 1M

$$= \pi\left[\frac{y^2}{2}\right]_4^{4+h}$$

$$= \frac{\pi}{2}[(4+h)^2 - 4^2]$$

$$= \frac{\pi}{2}(h^2 + 8h + 16 - 16)$$

$$= \frac{\pi}{2}(h^2 + 8h)$$ 1

(2)

(b) (i) By (a), $V = V_1 + \frac{\pi}{2}(h^2 + 8h)$ for $0 \le h \le 8$,

where V_1 cm^3 is the capacity of the frustum and $h = H - 4$ for $4 \le H \le 12$.

Differentiating both sides with respect to t,

$$\frac{dV}{dt} = \frac{\pi}{2}\left(2h\frac{dh}{dt} + 8\frac{dh}{dt}\right)$$ 1A

When the depth of milk is 7 cm, i.e. $h = 3$,

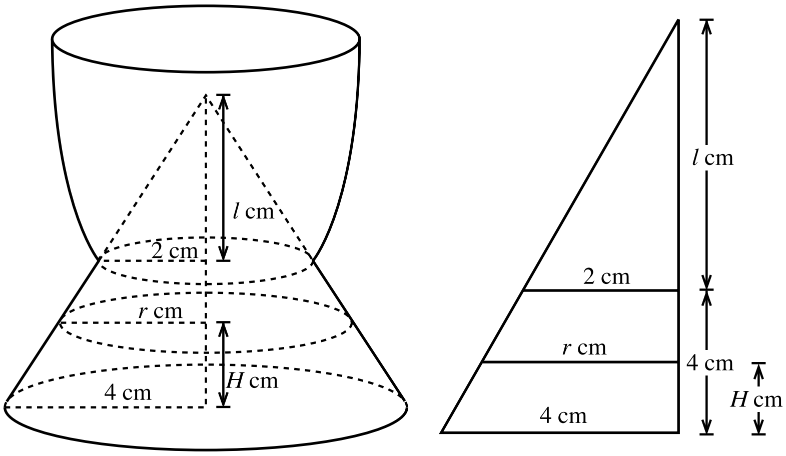
$$8 = \frac{\pi}{2}\left[2(3)\frac{dh}{dt} + 8\frac{dh}{dt}\right]$$

$$8 = \frac{\pi}{2}(14)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{7\pi}$$

$\therefore$ The rate of increase of the depth of milk is $\frac{8}{7\pi}$ cm s^{-1}. 1A

(ii) Let r cm and l cm be the lengths as shown in the figure.



By similar triangles,

$$\frac{l}{l+4} = \frac{2}{4} \quad \text{1M}$$

$$2l = l + 4$$

$$l = 4 \quad \text{1A}$$

By similar triangles,

$$\frac{r}{2} = \frac{4+4-H}{4}$$

$$r = \frac{8-H}{2} \quad \text{1A}$$

$$\begin{aligned}\therefore \quad V &= \frac{\pi}{3}(4^2)(l+4) - \frac{\pi}{3}r^2(l+4-H) \\ &= \frac{\pi}{3}(4^2)(8) - \frac{\pi}{3}\left(\frac{8-H}{2}\right)^2(8-H) && \text{1M} \\ &= \frac{\pi}{3}(128) - \frac{\pi}{3}\left(\frac{1}{4}\right)(8-H)^3 \\ &= \frac{\pi}{3}(128) - \frac{\pi}{3}(128)\left(1-\frac{H}{8}\right)^3 \\ &= \frac{128\pi}{3}\left[1-\left(1-\frac{H}{8}\right)^3\right] && \text{1}\end{aligned}$$

(iii) The volume of the upper portion

$= \frac{\pi}{2}[8^2 + 8(8)]$ cm^3

$= 64\pi$ cm^3

The volume of the milk that leaks out after 130 seconds

$= \frac{\pi}{2} \times 130$ cm^3

$= 65\pi$ cm^3 $>$ 64π cm^3

$\therefore$ The depth of the remaining milk is less than 4 cm.

$$64\pi + \frac{128\pi}{3}\left[1 - \left(1 - \frac{4}{8}\right)^3\right] - 65\pi = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$$ 1M

$$64\pi + \frac{112\pi}{3} - 65\pi = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$$

$$\frac{109\pi}{3} = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$$

$$\frac{109}{128} = 1 - \left(1 - \frac{H}{8}\right)^3$$

$$\left(1 - \frac{H}{8}\right)^3 = \frac{19}{128}$$

$$1 - \frac{H}{8} = \left(\frac{19}{128}\right)^{\frac{1}{3}}$$ 1A

$$V = \frac{128\pi}{3}\left[1 - \left(1 - \frac{H}{8}\right)^3\right]$$

Differentiating both sides with respect to t,

$$\frac{dV}{dt} = \frac{128\pi}{3}\left[0 - 3\left(1 - \frac{H}{8}\right)^2\left(-\frac{1}{8}\right)\frac{dH}{dt}\right]$$

$$= 16\pi\left(1 - \frac{H}{8}\right)^2\frac{dH}{dt}$$ 1A

After 130 seconds,

$$-\frac{\pi}{2} = 16\pi\left(\frac{19}{128}\right)^{\frac{2}{3}}\frac{dH}{dt}$$

$$\frac{dH}{dt} = -\left(\frac{1}{32}\right)\left(\frac{128}{19}\right)^{\frac{2}{3}}$$

$$= -0.111 \quad (cor.\ to\ 3\ sig.\ fig.)$$

∴ The rate of decrease of the depth of milk is 0.111 cm s^{-1}. 1A

(11)