HKDSE Mathematics (Extended M2)

Section (I): Foundation Knowledge

0 Surds - root of a rational number that cannot be expressed in the form of fractions

- Basic formulae
  - Definition of square root: $b = \sqrt{b}\sqrt{b}$
  - Properties of surds: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (Note: If and only if $a, b > 0$)

- Rationalization of answers

  When leaving our final answers, it is of common practice to <u>not leave surds in the denominator</u>.
  We only leave surds in the numerator. ***WE HATE SURDS IN DENOMINATOR!!!!***

  For instance, we would write the final answer as $\frac{\sqrt{5}}{5}$ instead of $\frac{1}{\sqrt{5}}$.

  Remember to rationalize your final answers or you risk losing one mark in the exams!

  Two types of rationalization strategies are discussed below:

  (1) $\frac{10}{\sqrt{6}}$

  $$= \frac{10}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

  $$= \frac{5\sqrt{6}}{3}$$

  (2) $\frac{14}{3+\sqrt{2}}$

  $$= \frac{14}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$$

  $$= \frac{14(3-\sqrt{2})}{(3)^2-(\sqrt{2})^2}$$

  $$= 6-2\sqrt{2}$$

  Purpose to multiply by $\frac{\sqrt{6}}{\sqrt{6}}$ and $\frac{3-\sqrt{2}}{3-\sqrt{2}}$:
  - ✧ Allow rationalization to be carried out in denominator while keeping the value of the fraction unchanged. ***WE HATE SURDS IN DENOMINATOR!***
  - ✧ For (2): Make use of the identity $a^2 - b^2 \equiv (a+b)(a-b)$ learnt in Junior Secondary curriculum.

  When encountering seemingly complicated terms such as $\frac{2x-3}{\sqrt{3x-1}-\sqrt{x+2}}$, don't be afraid to rationalize it. For most of the time rationalization is always the only solution.

- The technique of rationalization is important when we evaluate limits.
  Details will be discussed in Ch.4.

  Try the question below: [Quite common to see this type of questions in S4 UT1/Exam]

  (a) (i) Rationalize the denominator of $\frac{1}{\sqrt{k}+\sqrt{k+4}}$, where $k > 0$.

  (ii) Simplify $\frac{1}{\sqrt{k}+\sqrt{k+4}} + \frac{1}{\sqrt{k+4}+\sqrt{k+8}} + \frac{1}{\sqrt{k+8}+\sqrt{k+12}} \dots + \frac{1}{\sqrt{k+4(m-1)}+\sqrt{k+4m}}$ in terms of m and k, where m is a positive integer. (4 marks)

  (b) Someone claims that the value of $\frac{1}{\sqrt{4}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{12}} + \dots + \frac{1}{\sqrt{2016}+\sqrt{2020}}$ is greater than 45.
  Do you agree with the claim? Explain your answer. (2 marks)

1 Mathematical Induction

- **Propositions** are statements that are yet to be proved.
  To prove a statement, one of the important tools that we could use is Mathematical Induction.

- **Structure of Mathematical Induction (IMPORTANT! Usually 5 marks in DSE)**
  The below answering framework is obtained by summarizing the marking schemes of induction-proving Questions in HKDSE/CE AMaths Papers:

Steps of proofs using M.I.	Illustration (Prove that $1+2+3+...+n=\frac{1}{2}n(n+1)$)
[1st Mark] Prove that the statement is true for $n=1$.	Note that $1=\frac{1}{2}1(1+1)$. So, the above statement is true for $n=1$.
[2nd Mark] Assume that the statement is true for some ***positive integer*** k. It would be better if you also write the statement again.	Assume that $1+2+3+...+k=\frac{k(k+1)}{2}$ is true for some ***positive integer*** k, (Or alternatively, $1+2+3+...+k=\frac{k(k+1)}{2}$ is true for some $k\in\mathbf{Z}^{+}$.)
[3rd Mark] Using the induction assumption for the case when $n=k+1$.	$1+2+3+...+k+(k+1)$ $=\frac{k(k+1)}{2}+(k+1)$
[4th Mark] Prove that the statement is also true for $n=k+1$ if it is assumed to be true for $n=k$.	$=\frac{(k+1)(k)+2(k+1)}{2}$ $=\frac{(k+1)(k+2)}{2}$ $=\frac{(k+1)[(k+1)+1]}{2}$
[5th Mark] Concluding sentence, i.e. followthrough	Hence, the statement is also true for $n=k+1$ if it is true for $n=k$. By Mathematical Induction, the statement is true for all positive integers n.

- In the exams we are often asked to prove the results of summation of sequences, just like the one above. Sometimes, a (b) part of the question may be attached to your proof in (a). Sometimes we may use the results proved in (a) to generate new results.

- Questions that fuses M.I. and summation signs together are increasingly common in DSE exams.
  (Note: The summation sign $\sum$ will be discussed in Ch.2 on the next page.)

  This example summarizes the above two points: [2018 DSE M2 #6]

  (a) Using Mathematical Induction, prove that $\sum_{k=1}^{n}k(k+4)=\frac{n(n+1)(2n+13)}{6}$ for all positive integers n.

  (b) Using (a), evaluate $\sum_{k=333}^{555}\left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$. (7 marks)

  Can you explain how the results of (a) can be applied to (b)? The answer to (b) is 1813.

- Questions that fuses M.I. and manipulation of trigonometric identities/ matrices are common. Details will be discussed in the latter sections.

2 Binomial Theorem - General formula of expansion of a two-term polynomial to the power of n

- Factorial $n!$ and combination C_r^n (Also known as n choose r)
  (1) Factorial: $n! = n\times(n-1)\times(n-2)\times\ldots\times3\times2\times1$, where n is a positive integer.
  (2) Combination: $C_r^n = \dfrac{n!}{r!(n-r)!}$, i.e. the number of methods to choose r items from n different items.
  The value of complication is often evaluated with a calculator.

- Summation sign *Sigma* $\sum$ and Product sign *Pi* $\prod$
  (1) Summation sign: If m and n are integers and $m \le n$, then $\sum_{r=m}^{n} a_r = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$.
  For example, $\sum_{k=3}^{5}(k+4) = (3+4)+(4+4)+(5+4) = 24$.
  (2) Product sign (Outsyl): If m and n are integers and $m \le n$, then $\prod_{r=m}^{n} a_r = a_m \times a_{m+1} \times a_{m+2} \times \ldots \times a_n$.
  For example, $\prod_{k=3}^{5}(k+4) = (3+4)(4+4)(5+4) = 504$.
  Note: *$n!$ Can also be written as* $n! = \prod_{k=1}^{n} k$.

Note that the variables r and k above can be replaced by other variables p, q etc. They are called **dummy variables**

- **Binomial Theorem (IMPORTANT!)**

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r = C_0^n a^n + C_1^n a^{n-1} b^1 + \ldots + C_r^n a^{n-r} b^r + \ldots + C_n^n b^n$$ (n and r are integers)

In DSE exams, we often use this formula to find the coefficient of a certain term.
Common question types include determining **power of expansion/ coefficients of terms** in the expansion;

Example [2014 DSE #1]

In the expansion of $(1-4x)^2(1+x)^n$, the coefficient of x is 1.

(a) Find the value of positive integer n.

(b) Find the coefficient of x^2.

How do you show that the answer to (b) is -20?

◆ Determining coefficient of the binomial and power at the same time. (Harder)
Usually the techniques of employing simultaneous equations and dividing them are useful.
[e.g. 2013 DSE #2]
Suppose the coefficients of x and x^2 in the expansion of $(1+ax)^n$ are -20 and 180 respectively. Find the values of a and n. (4 marks)

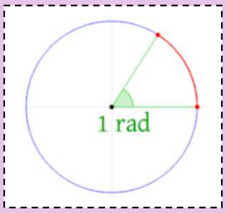
For this type of questions, we would

- first express the coefficients in terms of a and n;
  Coefficient of x = $(C_1^n)(a^1) = an = -20$; ------[1]
  coefficient of x^2 = $(C_2^n)(a^2) = \dfrac{a^2 n(n-1)}{2} = 180$. ------[2]
- then divide the equation with higher powers by that of lower powers.
  [2] ÷ [1], we have $\dfrac{a(n-1)}{2} = -9$, i.e. $an - a = -18$.

Then the values of the unknowns can be worked out easily.

3 More about Trigonometric functions

- Definition of radian:
One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle, i.e.



2π rad = 360 degrees (Or 1 rad = $\frac{180}{\pi}$ degrees)

Why radian? We could compute sector areas and arc lengths easily.
➲ Arc length $l = r\theta$
➲ Sector area $A = \frac{1}{2}r^2\theta = \frac{1}{2}rl$
Besides, it has many uses in other fields such as physics, and is the basis of many important formulae (some of which will be discussed in the following chapters).

- New trigonometric functions introduced:

JS Trigo Functions	$\sin\theta$	$\cos\theta$	$\tan\theta$
[NEW] Their reciprocals	$\frac{1}{\sin\theta} = \csc\theta$	$\frac{1}{\cos\theta} = \sec\theta$	$\frac{1}{\tan\theta} = \cot\theta$

Based on the original Trigometric identites, we could obtain new results:

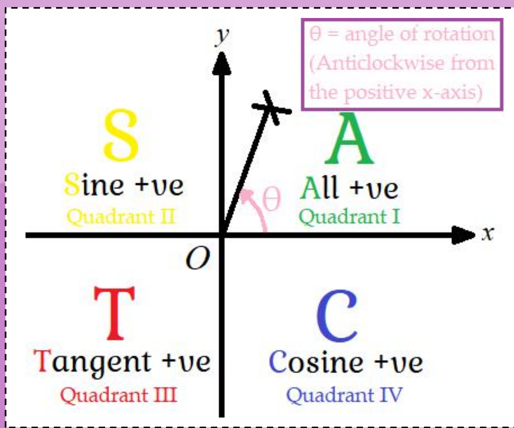
Relation	Original identities	New identities (with $\csc\theta$, $\sec\theta$ and $\cot\theta$)
Quotient relation	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
Square relation	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$ (dividing both sides by $\cos^2\theta$) $1 + \cot^2\theta = \csc^2\theta$ (dividing both sides by $\sin^2\theta$)

- Conversion formulae for trigonometric functions:
Note: This is a linkage to S4 Math Core Ch.12 (More about trigonometry), which is normally taught in May.
A slogan from LWY and YMC is quoted below:
奇變偶不變，正負看象限

where 奇偶 means the value of the integer n in the expression $n\times\frac{\pi}{2}+\theta$, where $\theta < 90°$ (Or $\frac{\pi}{2}$).

變 means $\sin\theta$ changes to $\cos\theta$; $\cos\theta$ changes to $\sin\theta$ and $\tan\theta$ changes to $\frac{1}{\tan\theta}$.

If n is odd, 變 is needed. Otherwise, 變 is not needed.
象限 means to consider the quadrant of the expression
Refer to the CAST table on the right.



[eg.] For $\sin(\frac{3\pi}{2}+\theta)$,

➢ $n = 3$ so 變 is needed.

➢ Also, $(\frac{3\pi}{2}+\theta)$ lies in quadrant IV, where $\sin\theta$ is negative.

➲ Therefore, $\sin(\frac{3\pi}{2}+\theta) = -\cos\theta$.

Note: For $\sec\theta$ and $\csc\theta$, just treat them as $\frac{1}{\cos\theta}$ and $\frac{1}{\sin\theta}$ and apply 象限 of $\cos\theta$ and $\sin\theta$ respectively.

[Concept checker] Can you explain why $\tan(\frac{\pi}{2}+\theta) = -\cot\theta$?

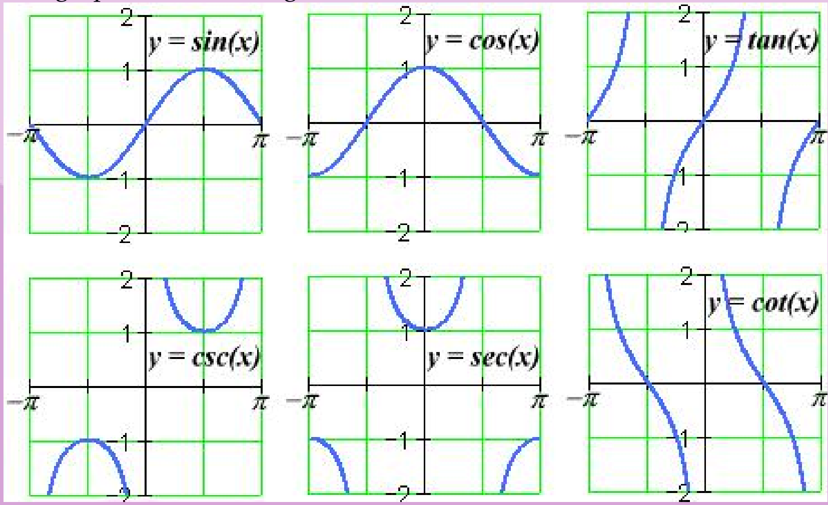
- Using the above conversions, we are often asked to find the values of θ that satisfies a certain equation.

For instance, "**Using (a), solving the equation** $\cos\left(x+\frac{4\pi}{9}\right) = 3\cot\left(x+\frac{5\pi}{18}\right)$**, where** $0 \le x \le \frac{\pi}{2}$" **~2018 DSE #3**

While finding their values, remember to look at the range of the given expression!

[e.g.] If $\sin\theta = \frac{1}{2}$ and $\frac{\pi}{2} < \theta < \pi$, instead of $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$.

- Sometimes we are also asked to determine the maximum/minimum values of certain trigonometric functions. The graphs of the six trigometric functions are shown below:



> Note that:
> - $-1 \le \sin\theta, \cos\theta \le 1$;
> - $\csc\theta, \sec\theta \ge 1$ or $\csc\theta, \sec\theta \le -1$.
> - Both $\tan\theta$ and $\cot\theta$ do not have maxima/minima.

- Sometimes we may apply the identities such as $a^2 - b^2 \equiv (a+b)(a-b)$ or $(a+b)^2 = (a-b)^2 + 4ab$; and the new trigo identities mentioned to find the values of trigonometric functions.Consider the question below:

  Suppose $\sin\theta + \cos\theta = \frac{1}{5}$, where $\frac{3\pi}{2} < \theta < 2\pi$. Find the value of $\sin\theta - \cos\theta$.

  How would you show that $\sin\theta - \cos\theta = -\frac{7}{5}$?

- **<u>Some new formulae about compound angles are also introduced (VERY IMPORTANT!)</u>**
  The following table of formulae will be provided in the HKDSE exams or your school's tests:

FORMULAS FOR REFERENCE	
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ $2\sin A \sin B = \cos(A-B) - \cos(A+B)$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$ $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
(Compound angle formula)	(Sum-to-product formula)

In addition, putting $A = B$, we obtain the double angle formulae:

(Sine) $\sin 2A = 2\sin A \cos A$

(Cos) $\cos 2A = \cos^2 A - \sin^2 A$

(Tan) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

> The formulae $\cos 2A = \cos^2 A - \sin^2 A$ is particularly useful as it can be transformed to $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$, allowing us to express $\cos^2 A$ and $\sin^2 A$ in double angles:
> - $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
> - $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

You need not memorize the formulae, but you need to get familiarised with them as they are often the most useful tools you could rely on when solving problems. The next page shows some of their applications:

These two pages summarizes most of the methods of applying trigonometric identities/sum-to-product formulae in your exams. Try to do the questions in the spaces provided to make sure you understand them!

- ❖ Carrying out trigonometric proofs, e.g. 火燒連環船

  e.g. [HKDSE SP #5]

  By considering $\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$, find the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$.

  How would you apply the formulae to obtain the result of $\frac{1}{8}$? Think about why it is called 火燒連環船.

- ❖ Carrying out Mathematical Induction that involves the manipulation of trigonometric identities

  (One of the harder M.I. types - Usually sum-to-product formulae are applied.)

  [e.g.] [HKDSE 2015 M2 - #8]

  (a) Prove that $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx = \sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$ for all positive integers n.

  (b) Hence evaluate $\sum_{1}^{567}\cos\frac{k\pi}{7}$. (8 marks)

- First principles (Differentiation) - One of the most important techniques in M2. Details will be discussed in Ch.5.
- **<u>[VERY IMPORTANT] The half-angle substitution method</u>**.

  If we let $t = \tan\frac{\theta}{2}$ (Where $-\pi < \theta < \pi$), $\frac{1}{1+t^2} = \cos^2\frac{\theta}{2}$. We can thus obtain $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$. Could you try to deduce these results yourself? Questions involving this substitution has appeared for numerous times in DSE/CE. Examples are listed below:

[DSE PP #4]

(a) Let $x = \tan\theta$. Show that $\frac{2x}{1+x^2} = \sin 2\theta$.

(b) Hence find the greatest value of $\frac{(1+x)^2}{1+x^2}$, where x is real. (5 marks)

[1993 AM Paper 2 #8]

Given $-\pi < x < \pi$ and $t = \tan\frac{x}{2}$.

(a) By expressing $\sin x$ and $\cos x$ in terms of t, show that $a\cos x + b\sin x = c$ can be transformed to $(a+c)t^2 - 2bt + (c-a) = 0$. Hence show that $a^2 + b^2 \geq c^2$ if the equation has solutions.

(b) Suppose x_1 and x_2 are the two values of x satisfying $5\cos x + 6\sin x = 7$.
Without evaluating x_1 and x_2, find

(i) $\tan\frac{x_1 + x_2}{2}$,

(ii) $\tan x_1 \tan x_2$.

Note that this method is particularly useful for integrals (discussed in Ch.7)

4 Limits and the number e - Fundamental pre-calculus topics

Definition: When x approaches a real number a, i.e. $x \to a$ and the value of a function $f(x)$ approaches a definite value L, we write $\lim_{x\to a} f(x) = L$.

> If $f(x)$ does not have a definite value when x approaches a, then $\lim_{x\to a} f(x)$ does not exist.
>
> (e.g.) The value of $\frac{1}{x-2}$ increases indefinitely as it approaches 2 from values greater than 2.
>
> Therefore $\lim_{x\to 2}\frac{1}{x-2}$ does not exist.

- About basic properties of limits:

  Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ and k be a constant.

  (1) Constant functions: $\lim_{x\to a} k = k$, $\lim_{x\to a} kf(x) = k$ and $\lim_{x\to a} f(x) = kL$

  (2) Addition: $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

  (3) Multiplication and division: $\lim_{x\to a}[f(x) \bullet g(x)] = \lim_{x\to a} f(x) \bullet \lim_{x\to a} g(x) = L \bullet M$

  $$\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad \text{(Given that } M \neq 0\text{)}$$

  (4) Putting limits 'inside': $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

- <u>**Evaluating limits**</u> (VERY important! This is the prerequisite of differentiation.)

  ➢ If a function $f(x)$ is continuous (i.e. its value are defined for every value of x; the graph of $y = f(x)$ does not break) at $x = a$, then $\lim_{x\to a}(x) = f(a)$.

  For instance, $\lim_{x\to 3}(3x^2 - 2x + 4) = 3(3)^2 - 2(3) + 4 = 22$.

  ➢ But what about finding the limit of a discontinuous function like $\lim_{x\to -3}\frac{x^2+4x+3}{x+3}$? Its value is undefined at $x = -3$. In this case, our target is to simplify the function such that '<u>**the denominator of the function will no longer become zero**</u>'. This is done by cancelling factors for most of the cases.

  ✧ Strategy 1: Factorization and simplify the function

  $$\lim_{x\to 1}\frac{9^x - 3^x - 6}{9^x + 3^x - 12}$$

  [TWGSS 17-18 S4UT2 #3]

  $$= \lim_{x\to 1}\frac{(3^x-3)(3^x+2)}{(3^x-3)(3^x+4)}$$

  $$= \lim_{x\to 1}\frac{3^x+2}{3^x+4} = \frac{3(1)+2}{3(1)+4} = \frac{5}{7}$$

  > [Explanation]
  >
  > Why $\frac{x+3}{x+3}$ can be cancelled?
  >
  > In this case, $x \to -3$ but $x \neq -3$.
  >
  > So, $x + 3 \neq 0$ and we can cross out the function.

  ✧ Strategy 2: Carrying out rationalization (Mentioned in Ch.0):

  $$\lim_{x\to 3}\frac{\sqrt{x+1}-\sqrt{7-x}}{\sqrt{2x+3}-\sqrt{x+6}}$$

  $$= \lim_{x\to 3}\frac{\sqrt{x+1}-\sqrt{7-x}}{\sqrt{2x+3}-\sqrt{x+6}} \times \lim_{x\to 3}\frac{\sqrt{2x+3}+\sqrt{x+6}}{\sqrt{2x+3}+\sqrt{x+6}}$$

  $$= \lim_{x\to 3}\frac{(\sqrt{x+1}-\sqrt{7-x})(\sqrt{2x+3}+\sqrt{x+6})}{(2x+3)-(x+6)}$$

  $$= \lim_{x\to 3}\left(\frac{(\sqrt{2x+3}+\sqrt{x+6})(\sqrt{x+1}-\sqrt{7-x})}{x-3} \times \frac{\sqrt{x+1}+\sqrt{7-x}}{\sqrt{x+1}+\sqrt{7-x}}\right)$$

  $$= \lim_{x\to 3}\frac{(\sqrt{2x+3}+\sqrt{x+6})[(x+1)-(7-x)]}{(x-3)(\sqrt{x+1}+\sqrt{7-x})}$$

  $$= \lim_{x\to 3}\frac{(\sqrt{2x+3}+\sqrt{x+6})}{(\sqrt{x+1}+\sqrt{7-x})} \times \lim_{x\to 3}\frac{2x-6}{x-3}$$

  $$= \frac{\sqrt{2(3)+3}+\sqrt{(3)+6}}{\sqrt{(3)+1}+\sqrt{7-(3)}} \times 2 = 3$$

  > Note:
  >
  > When evaluating limits with surds, <u>**rationalize until you could cancel the factor**</u> appearing in both numerator and denominator). Usually you only need to rationalize once but sometimes you'll need to do it twice (Like the example on the left). You'll know whether you need to do so or not after rationalizing once.

[Cont. Ch.4]

✧ Strategy 3: Applying limits of trigonometric functions

Theorem: $\lim_{x\to 0}\frac{\sin x}{x}=1$, where x is in radians [Its converse also holds.]

One may also apply the above together with trigonometric formulae to find limits of trigo functions.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$
$$=\lim_{x\to 0}\frac{1-(1-2\sin^2 x)}{x^2}$$
$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$
$$=2\times\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2\times 1^2=2$$

✧ Strategy 4: Dividing both sides by the highest power of the unknown **[For limits to infinity]**

Theorem: $\lim_{x\to\infty}\frac{1}{f(x)}=0$, given that $\lim_{x\to\infty}f(x)=\infty$.

$$\lim_{x\to\infty}\frac{\sqrt{2x^4+5x^2+1}}{3x^2+2}$$

➲ The highest power of denominator = 2

$$=\lim_{x\to\infty}\frac{\frac{\sqrt{2x^4+5x^2+1}}{x^2}}{\frac{3x^2+2}{x^2}}$$
$$=\frac{\lim_{x\to\infty}\sqrt{2+\frac{5}{x^2}+\frac{1}{x^4}}}{\lim_{x\to\infty}\left(3+\frac{2}{x^2}\right)}=\frac{\sqrt{2+0+0}}{3+0}=\frac{\sqrt{2}}{3}$$

✧ Strategy 5: Creating terms (i.e. multiplying both denominator and numerator by a number)

[TWGSS 17-18 S4EXM2 #1]

$$\lim_{x\to 0}\frac{\sin x}{\sin 5x}$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{\sin 5x}\times\frac{x}{x}\right)$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{5}\cdot\lim_{x\to 0}\frac{5x}{\sin 5x}$$
$$=1\cdot\frac{1}{5}\cdot 1=\frac{1}{5}$$

Questions related to evaluation of limits are uncommon in DSE. However, such techniques are very important when we work out derivatives of a certain function by first principles in the next chapter. Details will be discussed in Ch.5.

[Cont. Ch.4]

- Number e and exponential functions

  - As the number n increases to infinity, the number $\left(1+\frac{1}{n}\right)^n$ eventually approaches a limit and we define that limit as the number e (Euler's number; as it is discovered by Euler). By definition,

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{t\to 0}(1+t)^{\frac{1}{t}}\approx 2.71828\ldots$$

[$e=\lim_{t\to 0}(1+t)^{\frac{1}{t}}$ is obtained by replacing n by

e can also be expressed as an infinite series, i.e. $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}$.

We also have $e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. These two expansions will be discussed in detail in M1.

  - The natural logarithm $\ln x$, i.e. logarithm with base e, is introduced:

If $e^y=x$, then $y=\log_e x=\ln x$.

The properties of natural log and e are significantly important in the latter chapters.

Logarithm is introduced in S4 Maths (Core). The logarithm of a given number x is the exponent to which another fixed number, the base b, must be raised, to produce that number x. This means if $b^k=x$, then $\log_b x=k$ (b = base)

  - Limit related to e^x

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

The limit relating e^x is less likely to appear in DSE. However, questions that involves natural log/includes e^x is quite common in DSE and you should familiarise yourself with e. More properties about e will be discussed in the following chapters.

Try the following questions to make sure you fully understand!

Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{e^{x+1}-e}$

Evaluate $\lim_{x\to 0}\frac{e^{3x}-1}{x}$.

Prove, by mathematical induction, that $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$.

Hence find the limit $\lim_{n\to\infty}\sum_{1}^{k=n}\frac{k^2}{n^3}$.

Answers: 3, $\frac{1}{2e}$, $\frac{1}{3}$

Section (II): Calculus

5 Differentiation - Computing slope of a curve at a particular point

5.1 First principles

> Here, Δx and Δy are used to denote an infinitesimally small ($\to 0$) change in x and y respectively. They are pronounced as 'delta x' and 'delta y' respectively.

- For a function $y = f(x)$, its slope m at a particular point (x, y) is given by $m = \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$.

  We call m the derivative of $y = f(x)$, and denote it by $\dfrac{dy}{dx}$, y' or $f'(x)$.

  The derivative of $y = f(x)$ at $x = x_0$ is written as $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ or $f'(x_0)$.

- To find derivatives, the most fundamental way is to employ first principles. **<u>This is where our techniques learnt in Ch.4 can be used. Also note that there is always one question (FIXED) about first principles in DSE Examinations.</u>**

- In general, a question of finding derivatives with first principles is usually worth 4 marks:

Steps of first principles	Illustration (Finding $f'(1)$ for $f(x) = (x^2-1)e^x$)	Remarks
[1st Mark] Expressing $f(x_0+\Delta x) - f(x_0)$ in terms of Δx.	$f(1+\Delta x) - f(1)$ $= [(1+\Delta x)^2 - 1]e^{1+\Delta x} - (1^2-1)e^1$ $= [2\Delta x + (\Delta x)^2]e^{1+\Delta x}$	From 2019 DSE onwards, the questions will ask you to express $f(x_0+\Delta x) - f(x_0)$ in terms of Δx before finding derivative.
[2nd Mark] Copy the statement of first principles, i.e. $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$	$f'(1)$ $= \lim\limits_{\Delta x \to 0} \dfrac{f(1+\Delta x) - f(1)}{\Delta x}$ $= \lim\limits_{\Delta x \to 0} \dfrac{[2\Delta x + (\Delta x)^2]e^{1+\Delta x}}{\Delta x}$	The formula $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ should be copied in exact wording (x_0 is a given number)
[3rd Mark] Applying any of the limit-finding techniques you learnt in Ch.4: **1M WILL BE WITHHELD IF THIS STEP IS OMITTED.**	$= \lim\limits_{\Delta x \to 0} (2+\Delta x)e^{1+\Delta x}$	The techniques include: • $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ • $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ or $\lim\limits_{x \to 0} (1+\dfrac{n}{x})^x = e^n$ • Removing factors in the denominator that makes it zero (Used here)
[4th Mark] Answer	$= 2e$	This question is actually 2018 DSE #1.

Note: Sometimes there will be a 5th mark that requires the use of rationalization/sum-to-product formulae when you are simplifying the obtained expression. Those techniques are discussed in the previous chapters. Try the two questions on the next page:

- [A root function 2016 M2 #2]

  Prove that $\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}}=\frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}$. Hence find $\frac{d}{dx}\sqrt{\frac{3}{x}}$ from first principle. (5 marks)

- [Trigo functions 2017 M2 #1]

  Find $\frac{d}{d\theta}\sec 6\theta$ from first principles. (5 marks)

5.2 Techniques and rules of differentiation

Some techniques of differentiation will be discussed below:

(1) Derivative rules

Obtaining derivatives from first principles is often tedious. Thus the following formulae are introduced:

Suppose k is a constant, $f(x), g(x)$ are differentiable functions of x and n is real. Then	
$\frac{d}{dx}(k)=0$	$\frac{d}{dx}(x^n)=nx^{n-1}$
$\frac{d}{dx}[kf(x)]=k\frac{d}{dx}[f(x)]$	$\frac{d}{dx}[f(x)\pm g(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]$

Product rule:

$\frac{d}{dx}[f(x)g(x)]=f(x)\frac{d}{dx}[g(x)]+g(x)\frac{d}{dx}[f(x)]$,

i.e. $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$ If u and v are differentiable with respect to x.

The derivatives of trigonometric functions: [All trig functions starting with 'c' has negative derivative]

$\frac{d}{dx}(\sin x)=\cos x$	$\frac{d}{dx}(\cos x)=-\sin x$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\frac{d}{dx}(\cot x)=-\csc^2 x$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\frac{d}{dx}(\csc x)=-\csc x\cot x$

(2) Chain rule

Always remember to differentiate **with a respect**, i.e. to take that variable as a reference. In a DSE compulsory Chinese reading passage <論仁論孝論君子>, we learnt that “不敬，何以别乎”?

In mathematics, its English translation is **‘Without respect, how to differentiate?’**

This sentence may help you remember to differentiate with a respect - this is the foundation of Chain rule.

> Let $y=f(u)$ and $u=g(x)$, where y and u are differentiable functions of x, then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

e.g. For a function $y=(5x^2-5x+1)^6$, we may let $u=5x^2-5x+1$. Then $y=u^6$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$ ➲ Applying Chain rule

$$= \frac{d}{du}(u^6)\cdot\frac{d}{dx}(5x^2-5x+1)$$ ➲ Separate the function into two parts

$$= 6u^5\cdot(10x-5)$$

$$= 30(5x^2-5x+1)^5(2x-1)$$ ➲ Remember to subs u back to x!

Or in a simplified manner,

$$\frac{dy}{dx} = \frac{d}{dx}(5x^2-5x+1)^6$$

$$= (5x^2-5x+1)^5\times 6\times\frac{d}{dx}(5x^2-5x+1)$$

$$= 30(5x^2-5x+1)^5(2x-1)$$

> Remember to:
> - ◆ Subtract 1 from the index;
> - ◆ Multiply the index to the front;
> - ◆ Differentiate ‘inside’ with respect to x.

(3) Implicit differentiation

e.g. for the function $x^3+x^2y^2-y^3=1$,	Steps of Logarithmic differentiation
$3x^2+2xy^2+2x^2y\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$	Differentiate both sides with respect to a variable (e.g. x)
$\frac{dy}{dx}=\frac{3x^2+2xy^2}{3y^2-2x^2y}$	Express $\frac{dy}{dx}$ in terms of x and y by grouping terms. (Steps are skipped here)
$\left.\frac{dy}{dx}\right\vert_{(1,1)}=\frac{3(1)^2+2(1)(1)^2}{3(1)^2-2(1)^2(1)}=5$	If $\frac{dy}{dx}$ at a given point (e.g. (1,1)) is required, just substitute the values in.

(4) Differentiation of exponential functions and logarithmic differentiation

$\frac{d}{dx}(e^x)=e^x$	$\frac{d}{dx}(\ln x)=\frac{1}{x}$

Could you try to use the limit-finding techniques in Chapter 4 to try to prove these two? The techniques applied are quite useful in DSE. Also try this question below:

[DSE 2012 M2 #1]

Let $f(x)=e^{2x}$. Find $f'(0)$ from first principles. (3 marks)

The steps of logarithmic differentiation is shown below:

e.g. Finding $\frac{dy}{dx}$ for $y=\frac{\sqrt[3]{4x+1}}{(6x+7)^4}$	Steps of Logarithmic differentiation
$\ln y=\ln\left(\frac{\sqrt[3]{4x+1}}{(6x+7)^4}\right)$	Taking log on both sides
$\ln y=\frac{1}{3}\ln(4x+1)-4\ln(6x+7)$	Applying $\ln(p^n)=n\ln p$ and $\ln(\frac{q}{r})=\ln q-\ln r$ to simplify the power and divisions
$\frac{1}{y}\frac{dy}{dx}=\frac{4}{3(4x+1)}-\frac{24}{6x+7}$	Differentiating both sides with $\frac{d}{dx}(\ln x)=\frac{1}{x}$
$\frac{dy}{dx}=\frac{\sqrt[3]{4x+1}}{(6x+7)^4}\left[\frac{4}{3(4x+1)}-\frac{24}{6x+7}\right]$	Obtaining the answer by putting y to RHS and converting it in terms of x

5.3 Second derivative

As its name suggests, the second derivative of a function means the expression obtained after differentiating it two times. The notation of $\frac{d^2y}{dx^2}$ or $y"$ is usually used. It is quite useful in determining some special points on a graph. Details will be discussed in Ch.6. Try this question below:

Note that in general, $\frac{d^2y}{dx^2}\neq(\frac{dy}{dx})^2$!

Let $y=x^n\sin x$, where n is a positive integer.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) If $x\frac{d^2y}{dx^2}-2\frac{dy}{dx}+xy=28x^{n-1}\sin x+2(n-1)x^n\cos x$, find n.

DSE 2012 M1 #4

Let $y=\sqrt[3]{\frac{3x-1}{x-2}}$ where $x>2$.

(a) Use logarithmic differentiation to express $\frac{1}{y}\cdot\frac{dy}{dx}$ in terms of x.

(b) Using (a), find $\frac{d^2y}{dx^2}$ when $x=3$.

(6 marks)

6 Application of Differentiation

THIS CHAPTER IS VERY IMPORTANT AND ACCOUNTS FOR OVER 20 MARKS IN DSE EXAMS.

- Differentiation is applied to find the equations of tangents and normals to a graph. As $f'(x)$ simply tells us the slope of the tangent at a certain point on a graph, we may apply point-slope form, i.e. $y = y_1 = m(x - x_1)$ to find the desired equation(s).

> Note that normals are perpendicular to tangents.

When finding the tangents and normals…

- [Target 1] Work out the contact point of the tangents/normals
- [Target 2] Work out the slope of tangents/normals by considering $f'(x)$.

⚠ Remember the slope of normal is $-\dfrac{1}{f'(x)}$ as normals are perpendicular to tangents.

Sometimes simultaneous equations may also be applied to find the tangents ***if they pass through a certain point***. See the example below:

Question: Find the equation of tangents to $2x^2 + 3y^2 = 5$ which passes through $(0, -\frac{5}{3})$.	Illustration of techniques (You could try to remember this flow)
Let the point(s) of contact be (a, b). As (a, b) lies on $2x^2 + 3y^2 = 5$, we have $2a^2 + 3b^2 = 5$ ------------(1)	➢ Letting the point(s) of contact and substitute it into the curve [Step 1]
$2x^2 + 3y^2 = 5$, $4x + 6y\dfrac{dy}{dx} = 0$ $\dfrac{dy}{dx} = -\dfrac{2x}{3y}$; $\left.\dfrac{dy}{dx}\right\|_{(a,b)} = -\dfrac{2a}{3b}$	➢ Find the slope of tangents in terms of the coordinates of the contact point [Step 2]
So, we have $b + \dfrac{5}{3} = -(\dfrac{2a}{3b})(a - 0)$. $3b^2 + 5b + 2a^2 = 0$ ------------(2)	➢ Writing the equation of tangent in terms of the contact point; thus **forming an equation that can eliminate the terms of that in step 1**. [Step 3]
Solving (1) and (2), $b = -1$ and $a = \pm 1$ For $a = 1$ and $b = -1$, $\dfrac{dy}{dx} = \dfrac{2}{3}$. Required equation: $y + \dfrac{5}{3} = (\dfrac{2}{3})(x - 0)$ $2x - 3y - 5 = 0$ For $a = -1$ and $b = -1$, $\dfrac{dy}{dx} = -\dfrac{2}{3}$. Required equation: $y + \dfrac{5}{3} = (-\dfrac{2}{3})(x - 0)$ $2x + 3y + 5 = 0$	➢ Solving the equations to find a and b; hence finding the require slopes and equations for all possible values obtained [Step 4]

- We may also employ differentiation to find rates of changes. For most of the times we will have to express a certain expression in terms of one variable only, and differentiate both sides with time (i.e. $\dfrac{d}{dt}$) to find rates of changes.

> You're encouraged to do more drills on this topic as it is frequently examined.

The following few questions may serve as your test of understanding of rate of changes.

[2019 DSE M2 #3]
A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm^3 of liquid X. It is found that during the experiment, $\frac{dV}{dt}=-2t$, where V cm^3 is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment. The volume of remaining liquid X is a vessel after t hours is given by $V=580-t^2$.

It is given that $V=h^2+24h$, where h cm is the depth of liquid X in the vessel. Find $\frac{dh}{dt}$ when $t=18$. (3 marks)

[2016 DSE M2 #3]
Consider the curve $y=2e^x$ where $x>0$. It is given that P is a point lying on C. The horizontal line passing through P cuts the y-axis at the point Q. Let O be the origin. Denote the x-coordinate of P by u.

(a) Express the area of ΔOPQ in terms of u.

(b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of ΔOPQ when $u=4$. (5 marks)

[2017 DSE M2 #6]
A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

(a) Let A cm^2 and h cm be the wet curved surface area and the depth of water in the container respectively. Prove that $A=\frac{15}{16}\pi h^2$.

(b) The depth of water in the container increases at a constant rate of $\frac{3}{\pi}$ cm/s. Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm^2.

(7 marks)

- Guide to graph-sketching:
  There are always a Section B question related to graph-sketching in DSE exam. Below is a summary of all the essential features of DSE curve-sketching with the example of $y=\dfrac{x^2+x+1}{x+1}$ being used.

(1) x- and y- intercepts:

Just like how we solve for $x=0$ or $y=0$ in core.

For instance, $y=\dfrac{x^2+x+1}{x+1}$ has no x-intercepts since $x^2+x+1=(x+\frac{1}{2})^2+\frac{3}{4}>0$.

The y-intercept of $y=\dfrac{x^2+x+1}{x+1}$ is 1 since $\dfrac{(0)^2+(0)^1+1}{(0)+1}=1$.

(2) Maximum and Minimum (pl. maxima and minima), or turning point(s)

They are the points where the function 'turns', i.e. changes from increasing to decreasing/ from decreasing to increasing.
In Chinese terms, "止跌回升" or "由升轉跌".

Those points can be tested by the first derivative test. (Usually 4 marks)
Procedures are listed below:

> A function $y=f(x)$ is
> - increasing if $f'(x)\ge 0$,
>   **<u>strictly</u>** increasing if $f'(x)>0$;
> - decreasing if $f'(x)\le 0$,
>   **<u>strictly</u>** decreasing if $f'(x)<0$.
>
> Don't mix up increasing/decreasing with **<u>STRICTLY</u>** increasing/decreasing!

<table>
<tr><th>Example</th><th>Steps</th></tr>
<tr><td>$y=\dfrac{x^2+x+1}{x+1}$
$y=x+\dfrac{1}{x+1}$
$\dfrac{dy}{dx}=1-\dfrac{1}{(x+1)^2}$</td><td>[1st Mark]
Find $\dfrac{dy}{dx}$ in terms of x.</td></tr>
<tr><td>When $\dfrac{dy}{dx}=0$, $1-\dfrac{1}{(x+1)^2}=0 \Leftrightarrow x=1$ or $x=-1$</td><td>[2nd Mark]
Solve $\dfrac{dy}{dx}=0$</td></tr>
<tr><td>
<table>
<tr><td>x</td><td>$x<-2$</td><td>-2</td><td>$-2<x<-1$</td><td>-1</td><td>$-1<x<0$</td><td>0</td><td>$x>0$</td></tr>
<tr><td>y</td><td>--</td><td>-3</td><td>--</td><td>undefined</td><td>--</td><td>1</td><td>------</td></tr>
<tr><td>$\dfrac{dy}{dx}$</td><td>+</td><td>0</td><td>−</td><td>undefined</td><td>−</td><td>0</td><td>+</td></tr>
</table>
Do not miss out that $\left.\dfrac{dy}{dx}\right|_{x=-1}$ is undefined! Other you risk losing the mark for testing.</td><td>[3rd Mark]
Test if the sign of $f'(x)$ changes for the value(s) of x found.

$+\to 0\to -$: Maximum

$-\to 0\to +$: Minimum</td></tr>
<tr><td>The minimum is (0, 1) and the maximum is (-2, -3).</td><td>[4th Mark]
Answer</td></tr>
</table>

Note: Maximum/minimum of a function can also be tested with the second derivative test.
No test charts are required to be drawn.

- If $f(x)=0$ and $f''(x)<0$, then the point is a maximum
- If $f(x)=0$ and $f''(x)>0$, then the point is a minimum

> Simplifying here makes differentiation easier as $\dfrac{d}{dx}(\dfrac{x^2+x+1}{x+1})$ would be quite tedious.

[e.g.] For $y=\dfrac{x^2+x+1}{x+1}$, $\dfrac{d^2y}{dx^2}=\dfrac{2}{(x+1)^3}$.

When $x=-2$, $y=0$ and $\dfrac{d^2y}{dx^2}=-2<0$. So $(-2,0)$ is a maximum.

When $x=0$, $y=1$ and $\dfrac{d^2y}{dx^2}=2>0$. So $(0,1)$ is a minimum.

Guide to graph-sketching (Cont'd)

(3) Concavity and inflection/ inflexion points.

Points of inflection are points where a function's concavity changes (The sign of $f''(x)$ changes). To find inflection points, we could carry out the second derivative test.

A function $y = f(x)$ is said to be
- ✧ Concave upwards if its slope $f'(x)$ is increasing, i.e. $f''(x) > 0$;
- ✧ Concave downwards if its slope $f'(x)$ is decreasing, i.e. $f''(x) < 0$;

(e.g.) For $y = \dfrac{x^2+x+1}{x+1}$, $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+1)^3}$.

There are no values of x that satisfies $\dfrac{d^2y}{dx^2} = 0$.

Hence there are no inflection points for $y = \dfrac{x^2+x+1}{x+1}$.

Also note that it is completely normal for a function to not have any extrema/ inflection points.

(4) Asymptotes (漸近綫, this translation may help you understand the meaning of such lines)

(i) **Vertical** asymptotes

If $\lim\limits_{x\to a} f(x) = \infty$ (does not exist), then $x = a$ is a vertical asymptote of $y = f(x)$.

(ii) **Horizontal OR oblique** asymptotes (usually only one of these two exists for each graph)

If $\lim\limits_{x\to\infty}[f(x) - g(x)] = 0$, then

- ◆ $y = g(x)$ is a horizontal (橫) asymptote of $y = f(x)$ (If $g(x)$ is a constant function);
- ◆ $y = g(x)$ is an oblique (斜) asymptote of $y = f(x)$ (If $g(x)$ is a linear function).

For example, for the function $y = \dfrac{x^2+x+1}{x+1} = x + \dfrac{1}{x+1}$,

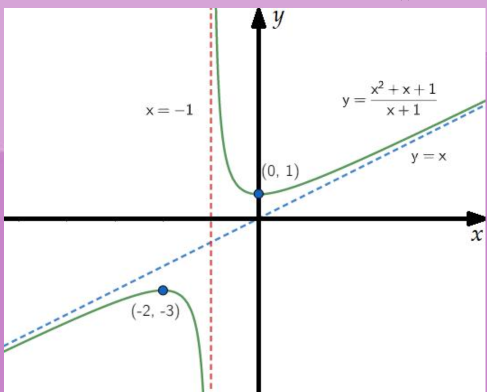
(i) $\lim\limits_{x\to -1} y = \infty$. ➲ So $x = -1$ is a vertical asymptote.

(ii) Also, $\lim\limits_{x\to\infty}(y - x) = 0$. ➲ So $y = x$ is an oblique asymptote.

(5) 萬事俱備，只欠東風 sketch graph

The graph you sketch should include all the four elements listed above.

For instance, for the graph of $y = \dfrac{x^2+x+1}{x+1}$,

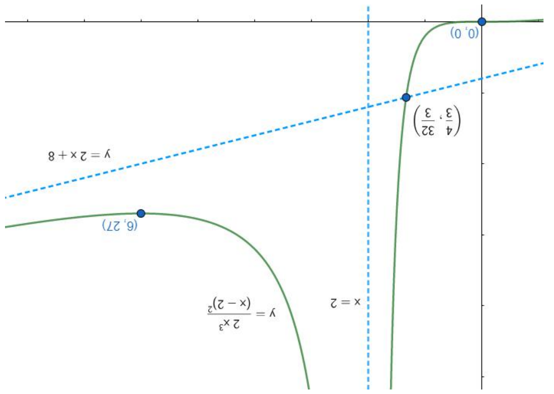


Note:
Usually there are two marks in a curve-sketching question:
- ➢ [1st Mark] Labelling your graph. (Both the graph and the asymptotes!)
- ➢ [2nd Mark] Sketching the shape carefully (e.g. Extrema and increasing/decreasing slope for inflection points)

[Checkpoint]
(18-19 TWGSS M2 F4 Exam #11)

For the graph of $y = \dfrac{2x^3}{(x-2)^2}$, find

(a) its x- and y- intercepts,
(b) its first and second derivatives,
(c) its turning point(s) and point(s) of inflection,
(d) its asymptotes, if any;
(e) the points where each asymptote intersects with the curve.
(f) Sketch the graph of the curve.



When you're done, turn the notes upside down to see the answers to the question on the left.

5 Indefinite integration - Reverse of differentiation

- Definition of integration:

  If $\frac{d}{dx}[F(x)] = g(x)$, then $\int g(x)dx = F(x) + C$

  [e.g.] 2013 M2 #11(a)

  Let $0 < \theta < \frac{\pi}{2}$. By finding $\frac{d}{d\theta}(\sec\theta + \tan\theta)$ or otherwise, show that $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$.

Note:

$\int$ Here represents the integral sign.

C is an arbitary constant.

It is added because the derivative of $F(x) + C$ is always $g(x)$ regardless of the value of C.

In other words, C can be any constant.

➲ Could you work that out yourself?

➢ As **<u>integration is the reverse process of differentiation</u>**, we may refer to the differentiation formulae discussed in Ch.5:

Suppose k is a constant, $f(x), g(x)$ are differentiable functions of x and n is real. Then

$\frac{d}{dx}(kx) = k$	$\int kdx = kx + C$
$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$	$\int kf(x)dx = k\int f(x)dx$
$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx}(x^n) = nx^{n-1}$ ($n \neq 0$)	$\int x^{n-1}dx = \int \frac{x^n}{n} + C$ ($n \neq 0$) i.e. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ ($n \neq -1$)
$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$ $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \cos xdx = \sin x + C$ $\int \sin xdx = -\cos x + C$ $\int \sec^2 xdx = \tan x + C$ $\int \csc^2 xdx = -\cot x + C$ $\int \sec x \tan xdx = \sec x + C$ $\int \csc x \cot xdx = -\csc x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\|x\| + C$
Note that the absolute value function $\|x\|$ is added here since the number inside a log function must be greater than 0.	The absolute value of a number must be non-negative, i.e. $\|x\| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$.

Some common methods of integration will be discussed here:

- Integration by substitution

Example (Finding $\int x\sqrt{x^2+2}dx$)	Reminders
Let $u=x^2+2$. Then $du=2xdx$.	➲ Write down your substitution and its derivative w.r.t. x.
$\int x\sqrt{x^2+2}dx$ $=\int \frac{\sqrt{x^2+2}}{2}\cdot 2xdx$	➲ Work out an expression that corresponds to du (In this case, $du=2xdx$ so you could take out the x in the integrand)
$=\int \frac{\sqrt{u}}{2}du$	➲ Apply the substitution. (Remember: both u and du should be converted to x and dx respectively!)
$=\frac{1}{3}u^{\frac{3}{2}}+C = \frac{1}{3}(x^2+2)^{\frac{3}{2}}+C$	➲ Leave your answers in terms of x instead of u.

Or in a simplified manner,

$$\begin{aligned}&\int x\sqrt{x^2+2}dx\\ &= \int \sqrt{x^2+2}\cdot\frac{1}{2}d(x^2+2)\\ &= \frac{1}{3}(x^2+2)^{\frac{3}{2}}+C\end{aligned}$$

If you are still unfamiliar with integrals, you're suggested to follow the steps above first. However, it is clear that presenting in the way on the left is more concise and time-saving.

[Checkpoint]

Could you show that $\int \frac{2x^3}{\sqrt{x^2+4}}dx=\frac{2}{3}(x^2+4)^{\frac{3}{2}}-8\sqrt{x^2+4}+C$?

- Partial fractions (Less common)

Sometimes we could break down an integral with a denominator of high degree, e.g. [1984 AM #7]

$$\begin{aligned}&\int \frac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}dx\\ &= \int\left[\frac{1}{x^3}+\frac{3}{(2-3x)^2}\right]dx\\ &= -\frac{1}{2x^2}+\frac{1}{(2-3x)}+C\end{aligned}$$

Partial fractions are more common in AL Pure Maths, but it has not yet appeared in DSE. Usually in part (a) of the question, you are guided to express (resolve) a complicated rational function in terms of simple partial fractions, and then you will integrate the resolved function in (b).

HKDSE Mathematics (Extended M2)

- Integration by parts

  $\int udv = uv - \int vdu$

Note that the formula for integration by parts is directly derived from product rule.

Sometimes we may have to apply integration by-parts for more than once, or even have to rearrange terms.

Example (Finding $\int e^x \cos xdx$)	Reminders
$\int e^x \cos dx$	
$= \int \cos xd(e^x)$	➲ 'u' and 'v' should be chosen carefully. Otherwise the integral may not be found.
$= e^x \cos x - \int e^x d(\cos x)$	➲ Apply integration by parts
$= e^x \cos x + \int \sin xd(e^x)$	
$= e^x \cos x + e^x \sin x - \int e^x d(\sin x)$	➲ Apply integration by parts again
$= e^x(\cos x + \sin x) - \int e^x \cos xdx$	
$\therefore 2\int e^x \cos xdx = e^x(\cos x + \sin x) + C_1$	➲ **<u>Rearranging terms (AN IMPORTANT TECHNIQUE!)</u>**
$\int e^x \cos xdx = \frac{1}{2}e^x(\cos x + \sin x) + C$	➲ Don't forget to include constants in your answer!

- Integration with trigonometric functions

  Usually, we would apply double angle/product-to-sum formulae introduced in Ch.3.

  ➢ OUR ULTIMATE GOAL: Express the integrand in **<u>only one type of trigo functions</u>**, e.g. $\int \tan^7 xd(\tan x)$, so that they can be easily integrated.

  ➢ OUR STRATEGY: **'consume' dx** with trigonometric functions and applying **trigonometric identities.** For example, replacing $\sec^2 xdx$ with $d(\tan x)$ and applying those identities learnt in Ch.3.

Below shows a few common types of integrands and methods of dealing with them:

Integrals in the form of $\int \sin^m x\cos^n xdx$	Remarks
m odd: Apply $u = \cos x$ to 'comsume' $\sin xdx$ e.g. $\int \sin x\cos^6 xdx = \int -\cos^6 xd(\cos x)$	Identities used: ➢ $\sin^2 x + \cos^2 x = 1$ (to interconvert even powers of sin and cos) ➢ $2\sin x\cos x = \sin 2x$, $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$ (to reduce degrees of sin/cos functions)
n odd: Apply $u = \sin x$ to 'comsume' $\cos xdx$ e.g. $\int \sin^8 x\cos^3 xdx = \int \sin^8 x(1-\sin^2 x)d(\sin x)$	
m, n even: Use identities to reduce degrees e.g. $\int \sin^4 x\cos^2 xdx = \int 4\sin^2 2x \cdot \frac{1}{2}(1-\cos 2x)dx$	

Integrals in the form of $\int \tan^m x\sec^n xdx$	Remarks
m odd: Apply $u = \sec x$ to 'comsume' $\sec x\tan xdx$ e.g. $\int \tan^3 x\sec^4 xdx = \int (\sec^2 x - 1)\sec^3 xd(\sec x)$	Identities used: $\tan^2 x + 1 = \sec^2 x$ (to interconvert even powers of tan and sec)
n even: Apply $u = \tan x$ to 'comsume' $\sec^2 dx$ e.g. $\int \tan^3 x\sec^4 xdx = \int \tan^3 x(\tan^2 x + 1)d(\tan x)$	
m even, n odd: Use by parts and rearrange terms [Refer to the example of $\int e^x \cos xdx$ on P.17]	

[Cont'd] Integrals in the form of $\int \cot^m x \csc^n x dx$	Remarks
m odd: Apply $u=\csc x$ to 'comsume' $\cot x \csc x dx$ e.g. $\int \cot x \csc^4 x dx = -\int \csc^3 x d(\csc x)$	Identities used: $\tan^2 x + 1 = \sec^2 x$ (to interconvert even powers of tan and sec)
n even: Apply $u=\cot x$ to 'comsume' $\csc^2 dx$ e.g. $\int \cot^5 x \csc^2 x dx = -\int \cot^5 x d(\cot x)$	
m even, n odd: Use by parts and rearrange terms [Refer to the example of $\int e^x \cos x dx$ on P.17]	And be careful of the negative signs. $d(\cot x) = -\csc^2 x dx$ and $d(\csc x) = -\cot x \csc x dx$

- Integration using trigonometric substitution
  Now that integration with trigonometric functions is discussed, we may apply trigonometric substitution to simplify integrands.
  - ➢ Be careful of the range of the trigonometric functions you are applying!

The purpose of using trigonometric substitution is to 'erase' the radical sign in the integrand by using trigonometric identities, e.g. $\sqrt{4-x^2}$ can be transformed to $2\cos\theta$ with the substitution $x = 2\sin\theta$.

Integrand	Trigonometric substitution and its range	Substitution result
$\sqrt{a^2-x^2}$	Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$dx = a\cos\theta d\theta$ and $\sqrt{a^2-x^2} = a\cos\theta$
$\sqrt{a^2+x^2}$	Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$dx = a\sec^2\theta d\theta$ and $\sqrt{a^2+x^2} = a\sec\theta$
$\sqrt{x^2-a^2}$	Let $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ and $\pi \le \theta < \frac{3\pi}{2}$	$dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$

Note: $\cos\theta / \csc\theta / \cot\theta$ substitutions are possible but they are not recommended here.
($\because$ Their derivative gives negative signs and increases our chance of having careless mistakes.)

Sometimes, the method of completing square is also applied alongside with trigonometric substitution.

[e.g.] For $\int \frac{dx}{x^2-8x+25} = \int \frac{dx}{(x-4)^2+3^2}$, we may let $x-4 = 3\tan\theta$. Then $dx = 3\sec^2\theta d\theta$.

Then $\int \frac{dx}{x^2-8x+25} = \int \frac{3\sec^2\theta d\theta}{9\sec^2\theta} = \frac{\theta}{3} = \frac{\tan^{-1}(x-4)}{3} + C$.

(Checkpoint)
Find
(i) $\int \sin^4 x \cos^3 x dx$,
(ii) $\int \sec^3 x dx$, and
(iii) $\int \cot^2 x \csc^6 x dx$.

Answers:
$\int \sin^4 x \cos^3 x dx = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$
$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$
$\int \cot^2 x \csc^6 x dx = -\frac{1}{7}\cot^7 x - \frac{2}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C$

- Applications of indefinite integration
  - As a reverse process of differentiation

    With integration, we may **find equation of curves from their given slopes.**

    NOTE: this type of questions are common in DSE! See the example below:

    [DSE 2018 M2 #5]

    (a) Using integration by substitution, find $\int x^3\sqrt{1+x^2}\,dx$.

    (b) At any point (x, y) on the curve Γ, the slope of the tangent to Γ is $15x^3\sqrt{1+x^2}$. The y-intercept of Γ is 2. Find the equation of Γ. (7 marks)

    Could you obtain the answer $3\left(\sqrt{1+x^2}\right)^5-5\left(\sqrt{1+x^2}\right)^3+4$ to (b)?

  - As an integral of rate of change
    - Finding displacement s from velocity v (speed) and finding velocity from acceleration a, i.e. $s=\int v\,dt$; $v=\int a\,dt$

      If you are studying physics, try to relate this part to Book 2 (Force and Motion)'s Chapter about uniformly accelerated motion. Indeed the equations of uniformly accelerated motion are derived from integration.

    - From change of volume/other quantities, we could derive the original quantity of volume/other quantities at $t=0$ by integrating the rate of change and examining the constant of integration. See the example below:

      [DSE 2019 M2 #3]

      A researcher performs an 24-hour experiment to study the rate of change of the volume of liquid X in a vessel. At the start of the experiment, the vessel contains 580 cm^3 of liquid X . It is found that during the experiment, $\frac{dV}{dt}=-2t$, where V cm^3 is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

      (a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer. (3 marks)

      How could you justify his claim?

6 Definite integration and its application

- Fundamental Theorem of Calculus

  Suppose $\frac{d}{dx}[f(x)] = f'(x)$. Then

  $\int_a^b f'(x)dx = f(b) - f(a)$.

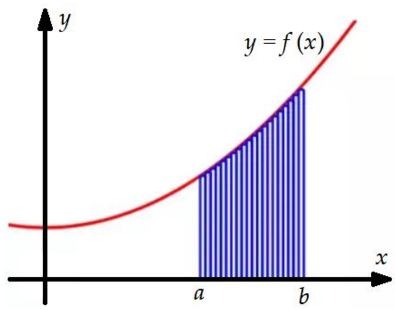
More about the proof on the left:
We divide the area between a and b into n small rectangular parts.
Then area of each rectangle

$= f(x_k)\frac{b-a}{n}$

Area under curve = sum of area of rectangles when n tends to infinity,

i.e. $\int_a^b f'(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$

$= f(b) - f(a)$



- Properties of definite integrals:

  (1) Reversing limits:

  $\int_b^a f(x)dx = -\int_a^b f(x)dx$

  (2) Dividing into sections of integrals:

  $\int_a^b f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx$

  (3) Renaming functions:

  $\int_a^b f(x)dx = \int_a^b f(u)du$

The three properties discussed here are extremely useful when tackling integration problems. Details will be discussed on the next page.

- Recapping Ch.7's indefinite integration

  ➢ Integration by substitution:

  We carry integration in the same manner as we did in definite integration.
  <u>However, the upper and lower limits should also be changed</u> if the variable is changed.

Example (Evaluating $\int_1^3 x\sqrt{2x^2+7}dx$)	Reminders
Let $u = 2x^2+7$. Then $du = 4xdx$. When $x=1, u=9$. When $x=3, u=25$.	➲ Writing down the substitution, $\frac{du}{dx}$ **<u>and also the change in upper/lower limits</u>**.
Then $\int_1^3 x\sqrt{2x^2+7}dx = \frac{1}{4}\int_9^{25}\sqrt{u}du$	➲ Changing the limits and variable
$= \left[\frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25}$	➲ $[f(x)]_q^p$ means $f(p) - f(q)$.
$= \frac{1}{6}\cdot 25^{\frac{3}{2}} - \frac{1}{6}\cdot 9^{\frac{3}{2}} = \frac{49}{3}$	

➢ Definite integration by parts: Nothing special except the newly added upper/lower limits.

$\int_a^b udv = [uv]_a^b - \int_a^b vdu$

[Practice] DSE 2015 M2 #3

(a) Find $\int \frac{1}{e^{2u}}du$.

(b) Using integration by substitution,
evaluate $\int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}}dx$. (7 marks)

The answer to (b) is $\frac{1}{e^2} - \frac{1}{e^6}$.

- [奪分神技] Definite integration techniques for more complicated functions
  In DSE M2 Section B, there is always a long question about definite integration that carries 12-13 marks.
  It is important that you grasp the ideas and techniques here and are able to employ them in DSE.

In fact, all the integral questions in public exams can be sorted into **THREE types**. The purpose of those charts and techniques below is to make you familiar with the functions that you are going to deal with in DSE.

> Each of the below proofs/techniques has appeared for more than 5 times in public exams, i.e. 翻炒再翻炒.
> Try to master these techniques and grab those easy marks in DSE!

(1) 'Reflection of functions':

"Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, where a is a positive constant and $f(x)$ a continuous function."

The above 2-mark proof question has appeared for many times in public exams.
The table below serves to guide you through this proof with a standard and simple method..

Proving $\int_0^a f(x)dx = \int_0^a f(a-x)dx$	Remarks
Let $u=a-x$. Then $du=-dx$. When $u=a, x=0$. When $u=0, x=a$.	➲ Mention substitution, relationship between du and dx [And optional: the new integrand limits]
$\int_0^a f(x)dx = -\int_a^0 f(a-u)du$	
$= \int_0^a f(a-u)du$ $= \int_0^a f(a-x)dx$	➲ Reversing the limit of the integrand, and then rename the integrand.

In particular, if $f(x)=f(a-x)$, then $\int_0^a xf(x)dx = \frac{a}{2}\int_0^a f(x)dx$. Could you employ a similar method to prove this?

(2) Half-angle substitution. [Mentioned in Ch.3, see P.6 of this notes.]

Let $t=\tan\frac{\theta}{2}$. Then $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$.

We also have $dt = \frac{1}{2}(t^2+1)d\theta$.

> Purpose of this substitution:
> To convert integrals of complicated trigonometric functions into computable rational function integrals.

It may be hard for us to find integrals like $\int_{-\frac{\pi}{2}}^0 \frac{dx}{3+2\sin x+\cos x}$ directly. Half-angle substitution is used in this case.

Finding $\int_{-\frac{\pi}{2}}^0 \frac{dx}{3+2\sin x+\cos x}$ (1994 AM #10)	Remarks
For the integral $\int_{-\frac{\pi}{2}}^0 \frac{dx}{3+2\sin x+\cos x}$, let $t=\tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{2dt}{t^2+1} = dx$.	➲ Mention substitution and $\frac{du}{dx}$. Usually the questions will ask you to prove the double-angle identities in previous parts.
$\int_{-\frac{\pi}{2}}^0 \frac{dx}{3+2\sin x+\cos x} = \int_{-1}^0 \frac{1}{3+2(\frac{2t}{1+t^2})+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{t^2+1}$ $= 2\int_{-1}^0 \frac{t^2+1}{2t^2+4t+4}\cdot\frac{dt}{t^2+1}$	➲ The $\frac{1}{t^2+1}$ generated by this substitution should be eliminated in the process. You could check if you are proceeding in the right way with this.
$= 4\int_{-1}^0 \frac{dt}{1+(1+t)^2}$	➲ Apply completing square where necesasry. See P.19.
$= 4\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2 u}\cdot\sec^2 u\,du$ (By letting $1+t=\tan u$)	➲ Sometimes you may need to use another substitution again.
$= 4[u]_0^{\frac{\pi}{4}} = \pi$	

(3) 'Finding their another half 搵另一半脫毒'
[For logarithmic and exponential integrals ONLY!]

The complexity of functions like $\int_0^{2\pi} \frac{dx}{e^{\sin x}+1}$ is "毒".

We could employ substitutions and generate their '另一半' to eliminate them and thus "脫毒".

The substitution you use should
- Reverse the limits of your integral
  (➲ So the subs. $u = a - x$ is often applied)
- Generate a suitable "另一半" that perfectly combines with the original integrand to form simple integrals/constants (See below).

Finding $\int_0^{2\pi} \frac{dx}{e^{\sin x}+1}$ (1996 PM #3)	Remarks
For $\int_0^{2\pi} \frac{dx}{e^{\sin x}+1}$, let $x = 2\pi - u$. Then $dx = -du$.	
$\int_0^{2\pi} \frac{dx}{e^{\sin x}+1} = -\int_{2\pi}^{0} \frac{du}{e^{\sin(-u)}+1}$	
$= \int_0^{2\pi} \frac{du}{e^{-\sin u}+1}$	➲ Reversing limits
$= \int_0^{2\pi} \frac{(e^{\sin u})du}{1+e^{\sin u}}$	➲ [IMPORTANT!] 上下乘一啲野既 technique
$= \int_0^{2\pi} \frac{e^{\sin x}dx}{e^{\sin x}+1}$	➲ "另一半" generated (合埋係 constant)
$\therefore 2\int_0^{2\pi} \frac{dx}{e^{\sin x}+1} = \int_0^{2\pi} \frac{dx}{e^{\sin x}+1} + \int_0^{2\pi} \frac{e^{\sin x}dx}{e^{\sin x}+1}$ $= \int_0^{2\pi} \frac{(1+e^{\sin x})}{1+e^{\sin x}}dx = [x]_0^{2\pi} = 2\pi$ Thus, we have $\int_0^{2\pi} \frac{dx}{e^{\sin x}+1} = \pi$.	➲ 搵另一半加埋脫毒!!! :)

The 'finding another half' method mentioned in the previous page also applies for natural log functions.

Finding $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$	Remarks
For $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$, let $x = \frac{\pi}{4} - u$. Then $dx = -du$.	
$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = -\int_{\frac{\pi}{4}}^{0} \ln(1+\tan(\frac{\pi}{4}-u))du$	
$= \int_0^{\frac{\pi}{4}} \ln(1+\frac{1-\tan u}{1+\tan u})du$	➲ Reversing limits
$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$	➲ "另一半" generated ($\ln x + \ln(\frac{k}{x})$ 合埋係 constant)
$\therefore 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$ $= \int_0^{\frac{\pi}{4}} (\ln 2)dx = [x\ln 2]_0^{\frac{\pi}{4}} = \frac{\pi \ln 2}{4}$ So, we have $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \frac{\pi \ln 2}{8}$.	➲ 搵另一半加埋脫毒!!! :) [Actually, you've already obtained 9 marks in 2016 DSE M2 with the above techniques!]

REMARK: If the above techniques still don't help, try considering integration by-parts.
Maybe you would eventually find the way to complete the question!

[Checkpoint]
What substitution would you use for the following? And through what means will you complete them?

Question 1 [AL 1990 PM II #3]
Let $f(x)$ and $g(x)$ be functions satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=K$ for some constant K. Show that $\int_0^a f(x)g(x)dx=\frac{K}{2}\int_0^a f(x)dx$. Hence evaluate $\int_0^\pi x\sin x\cos^4 xdx$.

Question 2 [CE 1992 AM II #8]
(b) Evaluate $\int_0^1 \frac{dt}{t^2+3}$ by using a suitable substitution.

(c) Let $t=\tan\frac{x}{2}$. Express $\cos x$ in terms of t. Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Question 3 [DSE 2019 M2 #10 (c)]

Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x)=-f(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} f(x)\ln(1+e^x)dx = \int_0^a xf(x)dx$ for any $a \in \mathbf{R}$. (4 marks)

Question 4 [DSE 2020 M2 #10 (a)(b)]

(a) Using integration by substitution, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)dx$. (3 marks)

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)dx$. (3 marks)

- Miscellaneous properties of functions (Less important)
  (i) For even functions, i.e. $f(-x)=f(x)$,
  $$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx .$$
  (ii) For odd functions, i.e. $f(-x)=-f(x)$,
  $$\int_{-a}^{a} f(x)dx = 0 .$$
  (iii) For periodic functions with period t, i.e. $f(t+x)=f(x)$,
  $$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx .$$

Those properties might be useful at times. (especially for the 5*/5** questions!)

- Application of definite integration

(I) Area between two curves (Remember: 大減細!)

➢ 用 x-axis in:

Area $= \int_a^b [f(x) - g(x)]\,dx$,

[where $y = f(x)$ is above $y = g(x)$]

➢ 用 y-axis in:

Area $= \int_c^d [f(y) - g(y)]\,dy$,

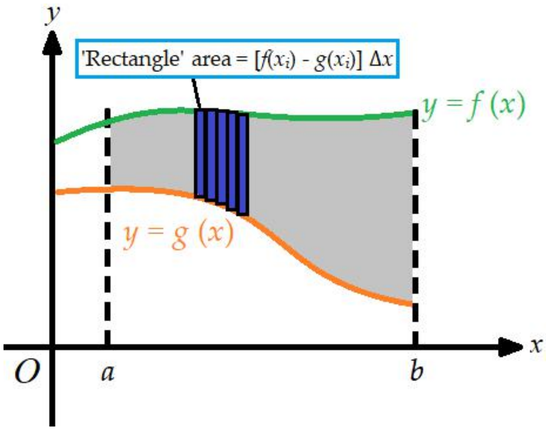
[Where $x = f(y)$ is on the right of $x = g(y)$]

[Note: integrating the area along each axis gives the same result. However, we should be wise when choosing the axis to integrate along.

e.g. To obtain the area bounded by $y^2 = 9x$ and the line $y = 3x - 6$, we would rather choose to integrate along the y-axis. (why?)

Could you answer the above question yourself?

Extra information:



Area of the shaded region above

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} - \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\frac{b-a}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i) - g(x_i)]\Delta x$$

$$= \int_a^b [f(x) - g(x)]dx$$

(2) Volume of solid of revolution

There are two methods to obtain the volume of solid of revolution.

➢ Disc method

Volume of a solid of revolution about x-axis:

$$V = \pi\int_a^b y^2 \; dx$$

Volume of a solid of revolution about y-axis:

$$V = \pi\int_c^d x^2 \; dy$$

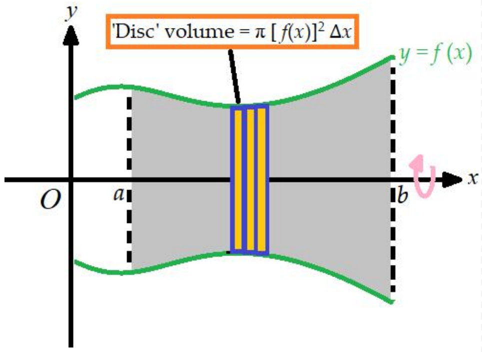
Volume of a hollow solid of revolution:

$V = \pi\int_a^b ({y_1}^2 - {y_2}^2)dx$ (revolving abt x-axis) OR

$V = \pi\int_c^d ({x_1}^2 - {x_2}^2)dy$ (revolving abt y-axis),

where x_1, y_1 and x_2, y_2 represents the 'outer' and 'inner' function respectively.

Extra information:



Volume of solid of revolution about x-axis

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\pi[f(x_i)]^2\frac{b-a}{n}$$

$$= \int_a^b \pi[f(x)]^2\,dx = \pi\int_a^b y^2 dx$$

■ Volume of solid of revolution about a line parallel to a coordinate axis :

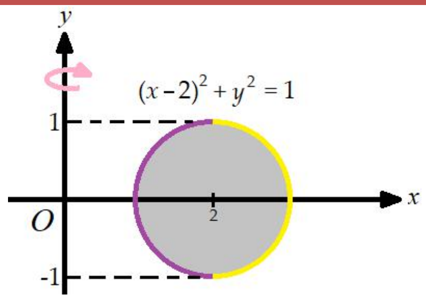
$V = \pi\int_a^b (y-k)^2 \; dx$ (If revolved about a horizontal line $y = k$, i.e. parallel to x-axis)

$V = \pi\int_c^d (x-m)^2 \; dy$ (If revolved about a vertical line $x = m$, i.e. parallel to x-axis)

Note that the independent variable (the use of dx or dy) is determined by the axis of revolution.

➲ Sometimes we may need to rewrite the equations, e.g. expressing y in terms of x.

The points mentioned in the previous page can be illustrated by the following question:

Volume of the solid of revolution if the shaded region bounded by the circle $(x-2)^2+y^2=1$ is revolved about the y-axis.	Remarks
Rewrite the equation as $x-2=\pm\sqrt{1-y^2}$, i.e. $x=2+\sqrt{1-y^2}$ (yellow in the figure) and $x=2-\sqrt{1-y^2}$ (purple part in the figure) The required volume could be treated as the volume generated by revolving $x=2+\sqrt{1-y^2}$ about the y-axis (outer) minus that of $x=2-\sqrt{1-y^2}$ revolved about the y-axis (inner).	 ➲ Expressing y in terms of x. Be careful of the $\pm$ sign! ($\because$ square rt taken)
Required volume $= \pi\int_{-1}^{1}\left[(2+\sqrt{1-y^2})^2-(2-\sqrt{1-y^2})^2\right]dy$	➲ Volume = $\pi\int_c^d\left(x_1{}^2-x_2{}^2\right)dy$
$= \pi\int_{-1}^{1}(4)(2)(\sqrt{1-y^2})dy$	➲ Sometimes sensible uses of identities could save your time of simplifying. Here $(a+b)^2-(a-b)^2=4ab$ is used.
$= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\,d(\sin\theta)$ (letting $y=\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$) $= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta d\theta$	➲ You may also have to apply the substitution techniques learnt in Ch.7. Here trigonometric substitution is applied to simplify $\sqrt{1-y^2}\,dy$.
$= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta$ $= 8\pi\left[\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ $= 8\pi(\frac{\pi}{4}+\frac{\pi}{4})=4\pi^2$	➲ Other identities, e.g. $\cos^2\theta=\frac{1+\cos 2\theta}{2}$ may also be useful. (See P.18)

➢ The another method is shell method.
(Outsyl from 2016 onwards)

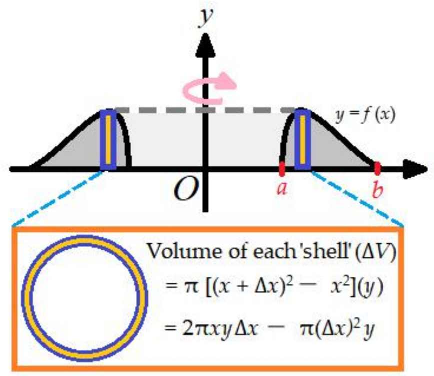
Volume of revolution along x-axis = $2\pi\int_a^b xy\ dy$

Volume of revolution along y-axis = $2\pi\int_a^b xy\ dx$

Notice that for the shell method, **the axis of rotation and the variable of integration are on a different axis**. Even though we are rotating about the y-axis, we are actually integrating the x-axis.

- Shell method is preferred over disk method when:
  - ✓ a function $f(x)$ is rotated about the y-axis; or $f(y)$ rotated about the x-axis, i.e. different function variable and axis of rotation;
  - ✓ $f(x)^2$ is hard to integrate but $xf(x)$ is easy to integrate (especially by parts).

Extra information:



Volume of solid obtained by revolving the shaded (deeper grey) region about y-axis

$$= \lim_{n\to\infty}\sum_{i=1}^{n}[2\pi x_i y_i\Delta x-\pi(\Delta x)^2 y_i]$$

$$= 2\pi\int_a^b xy\,dx-\pi\int_a^b \Delta x y_i dx$$

$$= 2\pi\int_a^b xydx \qquad (\because \Delta x=\frac{b-a}{n}\to 0)$$

[Checkpoint]

Question 1 [2013 M2 #6]

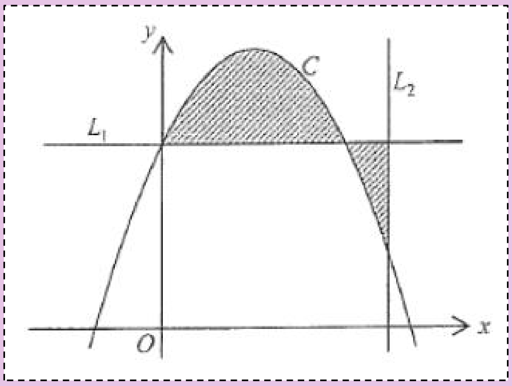
The figure on the right shows the shaded region with boundaries

$C: y = -\frac{x^2}{2} + 2x + 4$, $L_1 : y = 4$ and $L_2 : x = 5$.

C intersects L_1 at $(0,4)$ and $(4,4)$.

(a) Find the area of the shaded region.

(b) Find the volume of solid of revolution when the shaded region is revolved about L_1. (6 marks)



Question 2 [2015 M2 #12 (a)]

In the figure on the right, the curve Γ consists of the curve

- $AB: x^2 - 4y + 8 = 0$, and
- $BC = 3x + y - 9 = 0$,

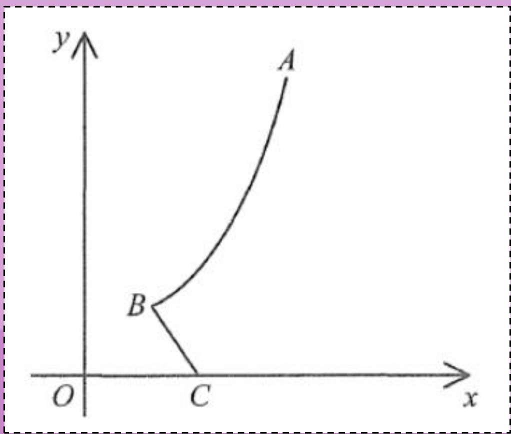
where O is the origin, B lies in the first quadrant and C lies on the x-axis.

(i) Find the coordinates of B.

(ii) Let h be the y-coordinate of A (where $h > 3$). A cup is formed by revolving Γ about the y-axis.

Prove that the capacity of the cup is $\pi(2h^2 - 8h + 25)$.

(7 marks)



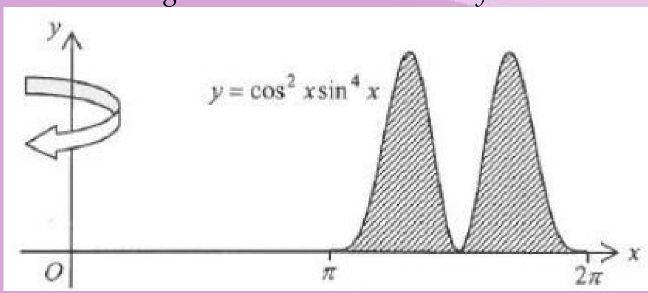
Question 3 [2014 M2 #13]

(a) Prove that $1-\cos 4\theta - 2\cos 2\theta \sin^2 2\theta = 16\cos^2\theta \sin^4\theta$. (2 marks)

(b) Show that $\int_0^{n\pi} \cos^2 \sin^4 x dx = \frac{n\pi}{16}$, where $n \in \mathbf{Z}^+$. (4 marks)

(c) Let $f(x)$ be a continuous function such that $f(k-x) = f(x)$, where k is a constant. Show that $\int_0^k xf(x)dx = \frac{k}{2}\int_0^k f(x)dx$. (4 marks)

(d) The below figure shows the shaded region bounded by the curve $y = \cos^2 x \sin^4 x$ and the x-axis, where $\pi \le x \le 2\pi$. Find the volume of the solid of revolution when the shaded region is revolved about the y-axis. (4 marks)



Answers:

Q1 (a) $\frac{13}{2}$, (b) $\frac{125\pi}{12}$

Q2 (i) (2, 3) (ii) obtained by dividing the cup into two parts along $y = 3$

Q3 The shell method is employed to get the answer to (d) $\frac{3\pi^3}{16}$

Section (III): Algebra

9 Matrices and Determinants

9.1 Introduction to matrices

- Matrices are rectangular arrays of real numbers arranged in the form $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
- The order (dimensions) of a matrix with m rows (橫) and n columns (直) is $m \times n$.
- Different types of matrices exist. However our syllabus mainly focuses on **square matrices**, i.e. matrices with the same number of rows and columns.

Matrices are usually represented by capital letters, e.g. M.

Different types of matrices ($m, n \in \mathbf{N}^+$)

Names	Orders	Orders
Row matrix (vector)	$\begin{pmatrix} 1 & 3 & -4 \end{pmatrix}$	$1 \times n$ (only 1 row)
Column matrix (vector)	$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$	$m \times 1$ (only 1 column)
Square matrix	$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 4 & 0 \\ -1 & 2 & 0 \end{pmatrix}$	$n \times n$
Identity matrix I (all diag elem=1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$n \times n$
Zero matrix $\mathbf{0}$ (all elements=0)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$n \times n$

9.2 Operation of matrices

- Addition and subtraction: Same as arithmetic operations (Must be of the **SAME ORDER**)

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] \pm \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \pm \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] = \begin{pmatrix} a \pm e \pm i & b \pm f \pm j \\ c \pm g \pm k & d \pm h \pm l \end{pmatrix}$$

- ***Scalar*** multiplication (***Scalar*** means a number): Also same as arithmetic operation, e.g.

$$4\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 \times 4 & 1 \times 4 \\ -3 \times 4 & 2 \times 4 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ -12 & 8 \end{pmatrix}$$

- ***Matrix multiplication*** (PAY ATTENTION! A completely different operation from the arithmetic one.)

Each element in the product matrix AB is the sum of each element in the corresponding row of A (橫) times each element in the corresponding column of B (直), i.e. 橫乘直. See the examples below:

➲ Example 1: Square matrices

For two matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

$$AB = \underset{\text{橫}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underset{\text{直}}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \underset{\text{橫乘直}}{\begin{pmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{pmatrix}} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$\neq$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e \times a + f \times c & e \times b + f \times d \\ g \times a + h \times c & g \times b + h \times d \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}$$

So, we have $AB \neq BA$.

In general, matrix multiplication are **NOT COMMUTATIVE** as $AB \neq BA$.
$AB = \mathbf{0}$ does not imply $A = \mathbf{0} / B = \mathbf{0}$.

In cases that $AB = BA$, A and B are called ***commuting*** matrices.

➲ Example 2: Non-square matrices

For two matrices $C = \begin{pmatrix} 1 & 5 & -2 \\ -2 & -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -3 & 4 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$,

$$CD = \begin{pmatrix} 1 \times (-3) + 5 \times 2 + (-2) \times 1 & 1 \times 4 + 5 \times 0 + (-2) \times (-1) \\ (-2) \times (-3) + (-1) \times 2 + 1 \times 1 & -2 \times 4 + (-1) \times 0 + 1 \times (-1) \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 5 & -9 \end{pmatrix}$$

$$DC = \begin{pmatrix} (-3)(1) + (4)(-2) & (-3)(5) + (4)(-1) & (-3)(-2) + (4)(1) \\ (2)(1) + (0)(-2) & (2)(5) + (0)(-1) & (2)(-2) + (0)(1) \\ (1)(1) + (-1)(-2) & (1)(5) + (-1)(-1) & (1)(-2) + (-1)(1) \end{pmatrix} = \begin{pmatrix} -11 & -19 & 10 \\ 2 & 10 & -4 \\ 3 & 6 & -3 \end{pmatrix}$$

So, we have $\boxed{CD \neq DC}$.

Can you explain why CD is a 2×2 matrix but DC is a 3×3 matrix?

It should also be noted that the number of columns of A must be equal to the number of rows of B in matrix multiplication.

➲ Otherwise the product is undefined.

- Transpose M^T: A operator that ***'flips' a matrix over its diagonal***. An example is shown below:

$$\begin{pmatrix}1 & 3\\ 2 & -1\\ 4 & 0\end{pmatrix}^T = \begin{pmatrix}1 & 2 & 4\\ 3 & -1 & 0\end{pmatrix}$$

Flip!

> Transposes is useful in the latter sections of this chapter (inverse). Just 'Flip'!

It should also be noted that sometimes operation of matrices will appear together with Mathematical Induction that you have learnt in Ch.1. The methods of solving these type of questions are similar to those in normal MIs. Try the questions below (Q2(b) for Matrix MI):

Checkpoint 9.2 (Familiarising yourself with matrices)

Q1 [2018 M2 #7(a)]

Define $M = \begin{pmatrix}7 & 3\\ -1 & 5\end{pmatrix}$. Let $X = \begin{pmatrix}a & 6a\\ b & c\end{pmatrix}$ be a non-zero real matrix such that M, X are *commuting*.

Express b and c in terms of a. (3 marks)

Q2 [2019 M2 #11 (a)(b)]

Let $M = \begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$. Denote the 2×2 identity matrix by I.

> Note that M^2 can be written as MM, and that M^n can be written as MM^{n-1}, etc.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$. (3 marks)

(b) Prove that $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ for all positive integers n. (4 marks)

Answers: Q1 $b=-2a$ and $c=-3a$; Q2 (a) $a=-4$ and $b=5$

9.3 Determinants (for square matrices) - ***VERY IMPORTANT!***

The determinant of $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is denoted by $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$, $|A|$, Δ_A or $\det(A)$.

Calculation of determinants

- For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

- For a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, its determinant is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Note 1 ▶ $$= +a\boxed{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}\boxed{-b\begin{vmatrix} d & f \\ g & i \end{vmatrix}} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Note 2 ▶ $$= +a\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} - b\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix} + c\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix}$$

Note 1:

$\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ is called the minor of entry a , and

$-\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ is called the cofactor of entry b .

Note that minors do not include the '+' pr '-' signs but cofactors do.

Note 2:

The minor of an entry is a determinant of order $(n-1)$ which is obtained by ***deleting the corresponding row and column of the original determinant***.

Note 3:

Determining signs of cofactor:

You may memorize the 'chessboard method'.

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ Like the black-and-white pattern of chessboard, '+' and '-' signs is added to cofactors **alternately**.

[Cont'd] Properties of determinants (Here, only det. of order 3 are illustrated; the rules hold for all orders)

(1) $\det A = \det A^T$; $\det(AB) = \det A \det B$ and $\det(A^n) = (\det A)^n$, where A and B are matrices.

(2) ***Interchanging rows of determinants:*** ➲ **Add a negative sign**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$$

(3) ***Factorizing determinants***

$$\begin{vmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

To factorize a determinant, the factor must be ***present on the same row or column.***

Also note that for a $n \times n$ matrix, $\det(kA) = k^n \det(A)$. Can you explain why?

➲ This implies that if the entries of a row/column of a matrix are all 0, its determinant = 0. ***Why?***

(4) ***Subtracting or adding multiples of rows/ columns from another*** ➲ **Remains unchanged.**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b-\mu a & c \\ d & e-\mu d & f \\ g & h-\mu g & i \end{vmatrix}$$

Together with the determinant calculation discussed in the previous page, we should employ those techniques listed above in order to calculate a determinant effectively. See the example on the next page:

[PP M2 #5]

(a) It is given that $\cos(x+1)+\cos(x-1)=k\cos x$ for any real x. Find the value of k.
[***Can you obtain the result*** $k=2\cos 1$ ***with what you've learnt in Trigonometry?***]

(b) Without using a calculator, find the value of $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$. (6 marks)

Steps	Technique used
$\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix} = \begin{vmatrix} \cos 1+\cos 3 & \cos 2 & \cos 3 \\ \cos 4+\cos 6 & \cos 5 & \cos 6 \\ \cos 7+\cos 9 & \cos 8 & \cos 9 \end{vmatrix}$	[1M] Using (4): Adding C3 to C1 ***Why this choice? Because we should make one of the entries to be in the form*** $\cos(x+1)+\cos(x-1)$ ***to use (a).***
$= \begin{vmatrix} 2\cos 1\cos 2 & \cos 2 & \cos 3 \\ 2\cos 1\cos 5 & \cos 5 & \cos 6 \\ 2\cos 1\cos 8 & \cos 8 & \cos 9 \end{vmatrix}$	[1M] Using (a)
$= 2\cos 1\begin{vmatrix} \cos 2 & \cos 2 & \cos 3 \\ \cos 5 & \cos 5 & \cos 6 \\ \cos 8 & \cos 8 & \cos 9 \end{vmatrix}$	[1M] Factorizing by (3)
$= 2\cos 1\begin{vmatrix} 0 & \cos 2 & \cos 3 \\ 0 & \cos 5 & \cos 6 \\ 0 & \cos 8 & \cos 9 \end{vmatrix} = 0$	[1A] By (3) [or (4)]

[Checkpoint 9.3] Can you try to employ what you have learnt to prove the following?

[AL 1991 PM I #1]

Show that $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a-c)(a-b)(b-c)(a+b+c)$. (4 marks)

9.4 Inverse matrices

For any square matrix A of order n, if there exists a square matrix B such that $AB = BA = I_n$, then B is called an ***inverse matrix*** of A. We write $B = A^{-1}$.

If the inverse of A can be found (i.e. exists), A is **invertible/non-singular**. Otherwise we say that A is **non-invertible/singular**.

> Not all matrices are invertible. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the zero matrix $\mathbf{0}$ is non-invertible. For invertible matrices P, $\det P \neq 0$. (Discussed below)

To find the inverse of a matrix P, i.e. P^{-1}, we may:

- Show that another matrix Q satisfy $PQ = QP = I_n$.

  [e.g.1] For $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$,

  $$AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 .$$

  $$BA = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 .$$

  ➲ So, $AB = BA = I_2$ and $B = A^{-1}$.

> Both $AB = I$ and $BA = I$ should be shown if questions like this are featured in your exam.

- Express the identity matrix in terms of the given matrix. (Usually guided by the question).

  [e.g.2] Let $M = \begin{pmatrix} -8 & -4 & 4 \\ -4 & 7 & 1 \\ 4 & 1 & 7 \end{pmatrix}$. Find $M^2 + 2M$ and hence find M^{-1}.

  $$M^2 + 2M = \begin{pmatrix} -8 & -4 & 4 \\ -4 & 7 & 1 \\ 4 & 1 & 7 \end{pmatrix}\begin{pmatrix} -8 & -4 & 4 \\ -4 & 7 & 1 \\ 4 & 1 & 7 \end{pmatrix} + 2\begin{pmatrix} -8 & -4 & 4 \\ -4 & 7 & 1 \\ 4 & 1 & 7 \end{pmatrix}$$

  $$= \begin{pmatrix} 96 & 8 & -8 \\ 8 & 66 & -2 \\ -8 & -2 & 66 \end{pmatrix} + \begin{pmatrix} -16 & -8 & 8 \\ -8 & 14 & 2 \\ 8 & 2 & 14 \end{pmatrix} = \begin{pmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 80 \end{pmatrix} = 80I$$

  So, we have $\frac{1}{80}M^2 + \frac{2}{80}M = I$, i.e. $\frac{1}{80}M(M) + \frac{2}{80}M(I) = I$ or $M(\frac{1}{80}M + \frac{1}{40}I) = I$.

  Similarly, $(\frac{1}{80}M + \frac{1}{40}I)(M) = \frac{1}{80}M^2 + \frac{1}{40}M = I$. So we have $M^{-1} = \frac{1}{80}M + \frac{1}{40}I$,

  i.e. $M^{-1} = \frac{1}{80}\begin{pmatrix} -8 & -4 & 4 \\ -4 & 7 & 1 \\ 4 & 1 & 7 \end{pmatrix} + \frac{1}{40}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{80}\begin{pmatrix} -6 & -4 & 4 \\ -4 & 9 & 1 \\ 4 & 1 & 9 \end{pmatrix}$.

- ***Apply the general formula for inverses (Preferred) [KEYPOINT!!]***

  For any ***invertible*** square matrix A,

  $A^{-1} = \frac{1}{\det A}\operatorname{adj} A$ ($\operatorname{adj} A$ is the ***adjoint*** matrix of A)

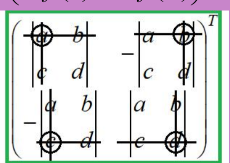
> Note that $\det A \neq 0$ for invertible matrices. Otherwise the inverse matrix cannot be found.

But what is adjoint matrix? Below shows the steps in finding the adjoint matrix.

$$\operatorname{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} cof(a) & cof(b) \\ cof(c) & cof(d) \end{pmatrix}^T$$

➣ First cofactor, then transpose

=



$$= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

➲ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

[Cont. General formula]

Similarly, the same rule applies for 3×3 matrices (and matrices of other orders as well): [1996 PM I #1]

$$\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}^{-1}=\frac{1}{\begin{vmatrix}0&0&-2\\1&2&1\\1&0&3\end{vmatrix}}\begin{pmatrix}cof(a_{11})&cof(a_{12})&cof(a_{13})\\cof(a_{21})&cof(a_{22})&cof(a_{23})\\cof(a_{31})&cof(a_{32})&cof(a_{33})\end{pmatrix}^T$$

Usually, finding the inverse of a 3×3 matrix worths 3 marks:
[1M] Determinant
[1M] Cofactor + Transpose
[1A] Answer

$$=\frac{1}{2\begin{vmatrix}0&-2\\1&3\end{vmatrix}}\begin{pmatrix}\begin{vmatrix}2&1\\0&3\end{vmatrix}&-\begin{vmatrix}1&1\\1&3\end{vmatrix}&\begin{vmatrix}1&2\\1&0\end{vmatrix}\\-\begin{vmatrix}0&-2\\0&3\end{vmatrix}&\begin{vmatrix}0&-2\\1&3\end{vmatrix}&-\begin{vmatrix}0&0\\1&0\end{vmatrix}\\\begin{vmatrix}0&-2\\2&1\end{vmatrix}&-\begin{vmatrix}0&-2\\1&1\end{vmatrix}&\begin{vmatrix}0&0\\1&2\end{vmatrix}\end{pmatrix}^T$$

$$=\frac{1}{4}\begin{pmatrix}6&-2&-2\\0&2&0\\4&-2&0\end{pmatrix}^T=\begin{pmatrix}\frac{3}{2}&0&1\\-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&0&0\end{pmatrix}$$

[Checkpoint 9.4] Try the three questions below:

[Q1] Let $A=\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}$.

(a) Evaluate $A^3-5A^2+8A-4I$.

(b) Hence find A^{-1}. (6 marks)

[The answer to (b) is just shown above ;)]

[Q2] Let M be the matrix $\begin{pmatrix}1&k&0\\0&1&1\\k&0&0\end{pmatrix}$, where $k\neq0$.

(a) Find M^{-1}.

(b) If $M\begin{pmatrix}x\\1\\z\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}$, find k. (5 marks)

[Q3] Let $A=\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$.

(a) Prove, by M.I., that $A^{n+1}=2^nA$ for all positive integers n.

(b) Using the results of (a), Willy proceeds in the following way:

$$A^2=2A$$
$$A^2A^{-1}=2AA^{-1}$$
$$A=2I$$

Explain why Willy arrives at a wrong conclusion. (7 marks)

Answers: [Q2 - 2013 M2 #8] $\frac{1}{k^2}\begin{pmatrix}0&0&k\\k&0&-1\\-k&k^2&1\end{pmatrix}$; $k=1$ (repeated) [Q3 - 2014 M2 #7] (b) $\det A=0$

9.5 Three types commonly encountered in matrix [Especially important for high-achievers]
Although instructions will usually be given to guide you through the questions in DSE, you should know what is ***exactly*** going on in these questions rather than just being an exam machine…

(1) Diagonalization of matrices

The matrix A is expressed in the form of DPD^{-1}, where P is a diagonal matrix. (Alternatively, we may let $D^{-1}AD=P$ but they're essentially the same)

➲ Allows the easy computation of A^n based on the ***properties of diagonal matrices***.

Diagonal matrices are $n\times n$ square matrices in the form $\begin{pmatrix}a&0\\0&b\end{pmatrix}$, $\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}$ etc.

The fact that $\begin{pmatrix}a&0\\0&b\end{pmatrix}^n=\begin{pmatrix}a^n&0\\0&b^n\end{pmatrix}$ for all $n\in\mathbf{N}^+$ makes it particularly useful to raise the power of a given matrix. Note that the above fact is assumed to be known for your DSE exam.

One example shown below: [1994 PM I #1]

Let $A=\begin{pmatrix}3 & 8\\ 1 & 5\end{pmatrix}$ and $P=\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}$ be invertible matrices.

(a) Find $P^{-1}AP$.

(b) Evaluate A^k for any positive integer k. (6 marks)

(a) $$P^{-1}AP = \left[\frac{1}{(2)(1)-(-4)(1)}\begin{pmatrix}1 & -1\\ 4 & 2\end{pmatrix}^T\right]\begin{pmatrix}3 & 8\\ 1 & 5\end{pmatrix}\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}3 & 8\\ 1 & 5\end{pmatrix}\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix}7 & 28\\ -1 & 2\end{pmatrix}\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}42 & 0\\ 0 & 6\end{pmatrix}=\begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}$$

(b) Note that $P^{-1}A^kP = \underbrace{(P^{-1}AP)(P^{-1}AP)...(P^{-1}AP)}_{k\text{ times}}$.

This method (highlighted) is often applied in DSE. Make sure to remember that!

$$A^k = P(P^{-1}AP)^kP^{-1} = \begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}\begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}^n\frac{1}{6}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix}2(7^n)-4(0) & 2(0)-4(1)\\ 1(7^n)+1(0) & 1(0)+1(1)\end{pmatrix}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix}(2\cdot 7^n)(1)+(-4)(-1) & 2\cdot 7^n(4)-4(2)\\ 7^n(1)+1(-1) & 7^n(4)+1(2)\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2\cdot 7^n+4 & 8\cdot 7^n-8\\ 7^n-1 & 4\cdot 7^n+2\end{pmatrix}$$

(2) Eigenvalues and eigenvectors (The process and theorem lies out of the scope)

- The roots of the equation $\det(A-kI)=0$ (k_1, k_2, …, k_n) are the entries of the diagonal matrix obtained by diagonalizing A,

i.e. $\begin{pmatrix}k_1 & 0 & \cdots & 0\\ 0 & k_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & k_n\end{pmatrix}=P^{-1}AP$. ($\Rightarrow$ $A^n = P\begin{pmatrix}k_1 & 0 & \cdots & 0\\ 0 & k_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & k_n\end{pmatrix}^n P^{-1}$)

- P can be obtained by performing some matrix operations with the values of k obtained. [Not discussed in detail in the syl; just try out Q2 of CP9.5 and you'll understand the flow.]

Also note that you are only required to evaluate 2×2 matrices with this method in DSE/AL PM.

(3) Expressing the general term of a sequence in matrix form (a.k.a. recurrence relationship)

e.g. For a sequence $\{x_n\}$ defined by $x_n = x_{n-1}+2x_{n-2}$ where $n\in\mathbf{N}^+$, [2014 M2 #12]

- we can write it in the **matrix form** $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{n-1}\\ x_{n-2}\end{pmatrix}$.
- By expressing the k th term in the matrix in terms of the previous terms, e.g. $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{n-2}\\ x_{n-3}\end{pmatrix}$, we would eventually get $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}^{n-2}\begin{pmatrix}x_2\\ x_1\end{pmatrix}$.
- Given the initial terms (e.g. $x_1=0$ and $x_2=1$) and after evaluating the matrix $\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}$ to the power n, we can find the general term x_n in terms of n (usually n = power).

Checkpoint 9.5 - Each of the question corresponds to each of the strategies mentioned!

[Q1] DSE 2017 M2 #12

Let $A=\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Using M.I., prove that $A^n=3^nI+3^{n-1}n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ for all positive integers n. (4 marks)

(b) Let $B=\begin{pmatrix}5 & 1\\ -4 & 1\end{pmatrix}$.

(i) Define $P=\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n=3^nI+3^{n-1}n\begin{pmatrix}2 & 1\\ -4 & -2\end{pmatrix}$ for any positive integer n.

(iii) Does there exist a positive integer m such that $\left|A^m-B^m\right|=4m^2$? Explain. (8 marks)

[Q2] DSE 2012 M2 #11

(a) Solve the equation $\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0$ --------(*) (2 marks)

(b) Let x_1, x_2 (where $x_1 < x_2$) be the roots of (*). Also let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$. It is given that $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix}$ and $|P| = 1$, where a, b and c are constants.

(i) Find P.

(ii) Evaluate $P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P$. Hence evaluate $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$. (11 marks)

[Q3] Deducing the general term of the fibonacci sequence:

Let $A=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ and $P=\begin{pmatrix}\alpha & \beta\\ 1 & 1\end{pmatrix}$. Suppose α, β be the roots of $\det[A-xI_2]=0$, where $\alpha>\beta$.

(a) Show that $P^{-1}AP=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$. Hence show that $A^n=\dfrac{1}{\alpha-\beta}\begin{pmatrix}\alpha^{n+1}-\beta^{n+1} & \alpha^n-\beta^n\\ \alpha^n-\beta^n & \alpha^{n-1}-\beta^{n-1}\end{pmatrix}$.

(b) The fibonacci sequence $\{F_n\}$ is defined by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n\ge 3\in\mathbf{N}^+$.

By writing $\{F_n\}$ in matrix form, show that $F_n=\dfrac{1}{\alpha-\beta}(\alpha^n-\beta^n)=\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$.

Q1 - 17M2 #12(b) (i) $P^{-1}BP=A$; (iii) No, because $\det(A^m-B^m)=-4m^2(9^{m-1})$ and $-1<0<9^{m-1}$

Q2 - 12M2 #11(a) -1 or 5 (b) $P=\begin{pmatrix}2/3 & 1\\ -1/3 & 1\end{pmatrix}$; $\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}$, $\begin{pmatrix}(5^{12}+2)/3 & (2\cdot 5^{12}-2)/3\\ (5^{12}-1)/3 & (2\cdot 5^{12}+1)/3\end{pmatrix}$

10 System of linear equations

Now that we have learnt some basic operations and properties of matrices, we may employ it in ***system of linear equations.***

In Math Core syllabus, we are taught to solve linear equations ***in two unknowns*** with elimination/substitution.
For example for the equation $\begin{cases} 2x+3y=7 & ---(*) \\ x-y=1 & ---(**) \end{cases}$, we will perform $(*)+3\times(**)$ and obtain $x=2$, $y=1$.

➲ But wait! We can also solve linear equations by matrices!
The equations $\begin{cases} 2x+3y=7 & ---(*) \\ x-y=1 & ---(**) \end{cases}$ can be transformed to $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ 1 \end{pmatrix}$.
We can thus employ matrix-related methods, which are to be discussed, to find the solutions.

This is particularly useful when the number of unknowns ***is more than two***. THREE IN M2 SYLLABUS.
Before starting our discussion on different methods of solving linear equations with matrices, some terminologies should be introduced:

10.1 Theory of Systems of Linear Equations

■ A system of linear equations in a certain number of unknowns can be written as $A\mathbf{x}=\mathbf{b}$, where A is called the coefficient matrix; $\mathbf{x}$ the unknown matrix/vector, and $\mathbf{b}$ the constant vector.

> For the case in the introduction, $A=\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$, $\mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b}=\begin{pmatrix} 7 \\ 1 \end{pmatrix}$.

■ Regarding the nature of the solutions of the system:
- ◆ A system is said to be ***consistent*** if it has solution(s); and
- ◆ ***inconsistent*** if not.

If there is only one set of solution, we call the solution set ***unique***.

> It should be noted that a system of linear equations should either:
> - ➢ has **NO** solutions;
> - ➢ has **ONE** (unique) solution; or
> - ➢ has **INFINITELY MANY** solutions.

e.g. $\begin{cases} x-y=0 \\ x+y=2 \end{cases}$ has a unique solution; whereas $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$ has infinitely many solutions;

and $\begin{cases} x+y=3 \\ 2x+2y=5 \end{cases}$ has none.

■ Two types of linear equations are discussed in our syllabus:
- ◆ A **homogeneous** system: A system in the form $A\mathbf{x}=\mathbf{0}$, e.g. $\begin{cases} 2x+3y=0 \\ 6x-5y=0 \end{cases}$
  - ➢ $\mathbf{x}=\mathbf{0}$ (e.g. $x=y=z=0$), i.e. the **trivial solution**, is immediately seen. Thus the system is consistent and we wish to find other solutions other than the trivial one.
- ◆ A **non-homogeneous** system: A system in the form $A\mathbf{x}=\mathbf{b}$ ($\mathbf{b}\neq\mathbf{0}$), e.g. $\begin{cases} 2x+3y=7 \\ x-y=1 \end{cases}$
  - ➢ We wish to find the conditions for its consistency (and for the solutions' uniqueness).

A total of three methods are discussed in the syllabus, namely
- ✓ Inverse matrix
- ✓ Cramer's Rule; and
- ✓ Gaussian Elimination.

They are to be discussed in the following sections.

> Note that we won't use inverse matrix and Cramer's Rule for homogeneous systems.

In the following sections, we shall use the non-homogeneous system $\begin{cases} 5x-y+z=3 \\ 2x+5y+3z=7 \\ x+y-z=9 \end{cases}$ as an example.

10.2 Solving linear equations with inverse matrices

For $A\mathbf{x}=\mathbf{b}$, provided that A is invertible, then $\mathbf{x}=A^{-1}\mathbf{b}$ and this solution to the system is unique.

We may thus apply the techniques we learnt to find the inverse matrices. The solving steps are as follows:

<1> Rewritting the equation in matrix form

$$\begin{cases} 5x-y+z=3 \\ 2x+5y+3z=7 \\ x+y-z=9 \end{cases} \text{ is equivalent to } \begin{pmatrix} 5 & -1 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}.$$

<2> Finding inverse

Inverse matrix can be used only when $\det A \neq 0$!

$$\begin{vmatrix} 5 & -1 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & -1 \end{vmatrix}=\begin{vmatrix} 5 & -1 & 1 \\ 2 & 5 & 3 \\ 6 & 0 & 0 \end{vmatrix}=6\begin{vmatrix} -1 & 1 \\ 5 & 3 \end{vmatrix}=-48$$

$$\begin{pmatrix} 5 & -1 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & -1 \end{pmatrix}^{-1}=\frac{1}{-48}\begin{pmatrix} \begin{vmatrix} 5 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & -1 \\ 2 & 5 \end{vmatrix} \end{pmatrix}^T=\frac{1}{-48}\begin{pmatrix} -8 & 5 & -3 \\ 0 & -6 & -6 \\ -8 & -13 & 27 \end{pmatrix}^T=-\frac{1}{48}\begin{pmatrix} -8 & 0 & -8 \\ 5 & -6 & -13 \\ -3 & -6 & 27 \end{pmatrix}$$

<3> Solving for (x, y, z) by multiplying the inverse to both sides

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 & -1 & 1 \\ 2 & 5 & 3 \\ 1 & 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}=-\frac{1}{48}\begin{pmatrix} -8 & 0 & -8 \\ 5 & -6 & -13 \\ -3 & -6 & 27 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

➲ So, we have $x=2$, $y=3$ and $z=-4$.

It is noteworthy that inverse matrices are seldom applied in solving systems of linear equations due to complexity of the steps of finding inverses.

10.3 Cramer's Rule (Condition: For unique solutions; $\det A \neq 0$)

For a system of linear equations $\begin{cases} a_1x+a_2y+a_3z=p \\ b_1x+b_2y+b_3z=q \\ c_1x+c_2y+c_3z=r \end{cases}$, $\text{Det}A=\Delta=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Write $\Delta_x=\begin{vmatrix} p & a_2 & a_3 \\ q & b_2 & b_3 \\ r & c_2 & c_3 \end{vmatrix}$, $\Delta_y=\begin{vmatrix} a_1 & p & a_3 \\ b_1 & q & b_3 \\ c_1 & r & c_3 \end{vmatrix}$ and $\Delta_z=\begin{vmatrix} a_1 & a_2 & p \\ b_1 & b_2 & q \\ c_1 & c_2 & r \end{vmatrix}$.

Δ_x, Δ_y and Δ_z are obtained by replacing the 1st, 2nd and 3rd ***column*** of Δ with the constant vector $\mathbf{b}$.

If $\Delta \neq 0$, then $x=\dfrac{\Delta_x}{\Delta}, y=\dfrac{\Delta_y}{\Delta}, z=\dfrac{\Delta_z}{\Delta}$ is a unique solution to the system.

[Example] For $\begin{cases} 5x-y+z=3 \\ 2x+5y+3z=7 \\ x+y-z=9 \end{cases}$, $\Delta=-48\neq 0$. So the system has a unique solution.

$$\Delta_x=\begin{vmatrix} 3 & -1 & 1 \\ 7 & 5 & 3 \\ 9 & 1 & -1 \end{vmatrix}=-96,\ \Delta_y=\begin{vmatrix} 5 & 3 & 1 \\ 2 & 7 & 3 \\ 1 & 9 & -1 \end{vmatrix}=-144 \text{ and } \Delta_z=\begin{vmatrix} 5 & -1 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 9 \end{vmatrix}=192.$$

➲ $x=\dfrac{\Delta_x}{\Delta}=2, y=\dfrac{\Delta_y}{\Delta}=3, z=\dfrac{\Delta_z}{\Delta}=-4$ is a unique solution.

It should be pointed out that variables other than x, y, z is always introduced in DSE. See the following:

[DSE 2016 M2 #11]

(a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x+y-z=3 \\ 4x+6y+az=b \\ 5x+(1-a)y+(3a-1)z=b-1 \end{cases},$$

where a and b are real numbers.

(i) Assume that (E) has a unique solution.

(1) Prove that $a \neq -2$ and $a \neq -12$.

(2) Solve (E). (6 marks)

[Can you obtain $x=\dfrac{3a^2-ab+50a+6b-24}{(a+2)(a+12)}$, $y=\dfrac{2(ab-10a+8)}{(a+2)(a+12)}$ and $z=\dfrac{ab-12a+6b-80}{(a+2)(a+12)}$?]

➲ In such cases, just treat a and b as constants and express your answer in terms of a and b. As steps involved in Cramer's rule are usually quite tedious (lots of determinants!), you should ***CHECK YOUR STEPS AND ANSWERS CAREFULLY***.

It should be concluded that the determinant test of the coefficient matrix indicates whether the system has a unique solution.

- If $\Delta \neq 0$, the system has a unique solution; inverse matrix (10.2) or Cramer's Rule (10.3) can be used;
- If $\Delta = 0$, we should use Gaussian Elimination (10.4) discussed in the next section.

Checkpoint 10.3

[DSE 2019 M2 #6]

Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x-2y-2z=\beta \\ 5x+\alpha y+\alpha z=5\beta \\ 7x+(\alpha-3)y+(2\alpha+1)z=8\beta \end{cases}, \text{ where } \alpha, \beta \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α.

(ii) Express y in terms of α and β. (5 marks)

Answers:

(i) $\alpha \neq -4$ and $\alpha \neq -10$

(ii) $y=-\dfrac{\beta}{\alpha+4}$

But… What if $\Delta = 0$? Or when the number of unknowns aren't equal to the number of equations?

10.4 Gaussian Elimination and homogeneous equations

This method can be used for cases when $\Delta = 0$, or when the system is homogeneous.
In Gaussian Elimination, we write the equation in the form of augmented matrix.

e.g. The system $\begin{cases} 5x - y + z = 3 \\ 2x + 5y + 3z = 7 \\ x + y - z = 9 \end{cases}$ can be written as $\left(\begin{array}{ccc|c} 5 & -1 & 1 & 3 \\ 2 & 5 & 3 & 7 \\ 1 & 1 & -1 & 9 \end{array}\right)$,

We may perform elementary ***ROW*** operations on augmented matrices, i.e.

- Interchanging two rows of the augmented matrix;
- Multiplication of rows by a non-zero number;
- Adding the multiple of a row to another.

The operations are similar to those performed in determinants, except that for interchanging rows. Remember! ~~Column operations~~
Only row operations are possible here.

We wish to transform the augmented matrix in the row echelon form, i.e. $\left(\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right)$.

This allows back substitution to be performed to find the solutions.

For $\begin{cases} 5x - y + z = 3 \\ 2x + 5y + 3z = 7 \\ x + y - z = 9 \end{cases}$, $\left(\begin{array}{ccc|c} 5 & -1 & 1 & 3 \\ 2 & 5 & 3 & 7 \\ 1 & 1 & -1 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 2 & 5 & 3 & 7 \\ 5 & -1 & 1 & 3 \end{array}\right)$ ➲ Interchanging rows

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & 5 & -11 \\ 0 & -6 & 6 & -42 \end{array}\right)$ ➲ Subtracting multiples of row 1

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 3 & 5 & -11 \\ 0 & 0 & 16 & -64 \end{array}\right)$ ➲ Adding multiples of row 2

Note that the symbol $A \sim B$ means that the systems A and B have the same solution.

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & 1 & -4 \end{array}\right)$ ➲ Diving rows 2 and 3 (Coef. = 1)

Thus the system is reduced to $\begin{cases} x + y - z = 9 \\ y + \frac{5}{3}z = -\frac{11}{3} \\ z = -4 \end{cases}$, from which we can get $x = 2$, $y = 3$ and $z = -4$.

$\boxed{\text{Gaussian elimination is particularly useful (THE ONLY WAY!) to solve systems with determinant 0 .}}$

See the following example:

For the system $\begin{cases} x + y - 3z = 6 \\ 2x - 2y + z = -3 \\ x - 3y + 4z = -9 \end{cases}$, $\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & 1 \\ 1 & -3 & 4 \end{vmatrix} = 0$. Solving by Cramer's rule or inverse is impossible.

Using Gaussian elimination, $\left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 2 & -2 & 1 & -3 \\ 1 & -3 & 4 & -9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & -4 & 7 & -15 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right)$

For a system of n unknowns and n equations with infinitely many solutions, one of the rows must be able to be reduced to $0 = 0$ (i.e. rendered ***redundant***.)

$\sim \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -\frac{7}{4} & \frac{15}{4} \\ 0 & 0 & 0 & 0 \end{array}\right)$, i.e. $\begin{cases} x + y - 3z = 6 \\ y - \frac{7}{4}z = \frac{15}{4} \\ 0 = 0 \end{cases}$

In such cases, we may let $z = t \quad (t \in \mathbf{R})$.

Then $y = \frac{15}{4} + \frac{7}{4}t$ and $x = 6 - 3t + \left(\frac{7}{4}t + \frac{15}{4}\right) = \frac{5}{4}t + \frac{9}{4}$ by back substitution.

So, the ***general*** solutions of the system are $(x, y, z) = \left\{\left(\frac{5}{4}t + \frac{9}{4}, \frac{7}{4}t + \frac{15}{4}, t\right) : t \in \mathbf{R}\right\}$.

For some systems reducible to one equation, e.g. $\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \\ 3x + 3y + 3z = 3 \end{cases}$, we may let $y = s$ and $z = t$, i.e. introduce two variables.

Question to ponder:

If $\begin{cases} x + 4y + z = 1 \\ 3x + 2y + 2z = h \\ 2x - 2y - kz = 2 \end{cases}$ has infinitely many solutions, find the values of h and k.

➲ Can you show that $h = 3$ and $k = -1$ by using Gaussian Elimination?

Sometimes, questions will further ask you to carry out operations with the general solutions obtained. See the example below: [AL 2008 PM I #7]

[in (a), the solution of the system below, $(x, y, z) = \left\{\left(\frac{1+5t}{4}, y, \frac{1-7t}{4}\right) : t \in \mathbf{R}\right\}$, is obtained.

(b) Find the greatest value of $2x^2 + 15y^2 - 10z^2$, where x, y and z are real numbers satisfying

$$\begin{cases} x + 4y + 3z = 1 \\ x - 3y - z = 0 \\ 3x - 2y + z = 1 \end{cases}.$$

(4 marks)

Putting $x = \frac{1+5t}{4}$, $y = t$ and $z = \frac{1-7t}{4}$ in $2x^2 + 15y^2 - 10z^2$, we have	1M for using the result of (a)(ii)(1)
$2x^2 + 15y^2 - 10z^2$ $= \frac{-25}{2}t^2 + 10t - \frac{1}{2}$ $= \frac{-25}{2}\left(t - \frac{2}{5}\right)^2 + \frac{3}{2}$	1M + 1A or equivalent
Thus, the greatest value is $\frac{3}{2}$.	1A
	----------(4)

We may need to substitute the results obtained from the previous parts, and apply strategies we have learnt in core, such as the completing square method.

Questions like this nowadays worth 3 marks.

But what about **homogeneous equations**? [Note: Not yet appeared in DSE]

For a system of n homogeneous linear equations in n unknowns $AX = \mathbf{0}$,

◆ If $|A| \neq 0$, then all the elements in the unknown matrix (x, y, z ...) must be 0

➲ The system only has trivial solutions if $|A| \neq 0$.

◆ If $|A| = 0$, then the system has trivial solutions and infinitely many non-trivial solutions.

➲ We can find those non-trivial solutions with Gaussian Elimination mentioned previously.

Checkpoint 10.4 - Try out the following to make sure that you completely understand the solving strategies!

Q1 [DSE 2012 M2 #8]

(a) Solve the system of linear equations $\begin{cases} x+y+z=0 \\ 2x-y+5z=6 \end{cases}$.

(b) Using (a) or otherwise, solve $\begin{cases} x+y+z=0 \\ 2x-y+5z=6 \\ x-y+\lambda z=4 \end{cases}$, where λ is a constant. (5 marks)

Q2 [DSE 2014 M2 #9]

(a) Solve $\begin{cases} x+y+z=100 \\ x+6y+10z=200 \end{cases}$.

(b) In a store, the prices of each of small, medium and large marbles are \$0.5, \$3 and \$5 respectively. Audrey plans to spend all \$100 for exactly 100 marbles,which include m small marbles, n medium marbles and k large marbles. Audrey claims that there is only one set of combination of m, n, k. Do you agree? Explain your answer. (6 marks)

Q3 [DSE 2018 M2 #11]

(a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x+ay+4(a+1)z=18 \\ 2x+(a-1)y+2(a-1)z=20 \\ x-y-12z=b \end{cases}, \text{ where } a,b\in\mathbf{R}.$$

(i) Assume that (E) has a unique solution.
Find the range of values of a and solve (E) in this case.

(ii) Assume that $a=3$ and (E) is consistent.
Find b and solve (E) in this case. (9 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F):\begin{cases} x+3y+16z=18 \\ x+y+2z=10 \\ x-y-12z=s \\ 2x-5y-45z=t \end{cases}, \text{ where } s,t\in\mathbf{R}.$$

Assume that (F) is consistent. Find s and t. (3 marks)

Answers:

Q1 $(2,-2,0)$ for $\lambda\neq 3$; $(2-2t,t-2,t)$ for $\lambda=3$

Q2 $(x,y,z)=\left\{(80+\frac{4t}{5},20-\frac{9t}{5},t):t\in\mathbf{R}\right\}$; Possible $t=0,5,10$; disagreed

Q3 (i) $a\neq 3$ and $a\neq -1$; $x=\frac{a^2b+ab+10a-2b-50}{(a-3)(a+1)}$ $y=\frac{-3ab+22a-5b-38}{(a-3)(a+1)}$ $z=\frac{b-2}{2(a-3)}$

(ii) $b=2$ and $(x,y,z)=\{(5u+6,-7u+4,u):u\in\mathbf{R}\}$ (b) $s=2$, $t=-8$.

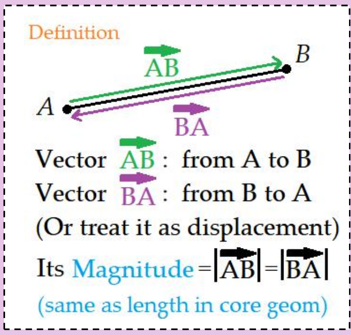
11 Introduction to vectors

11.1 What are vectors?

Vectors have both magnitudes and directions.

They are represented by arrows overbar $\overrightarrow{AB}$.

- The magnitude of a vector, represented by $\left|\overrightarrow{AB}\right|$, represents the distance from A to B.
- Vectors are **<u>straight lines</u>** (shortest distance).
  The direction of the vector $\overrightarrow{AB}$ is from A to B.



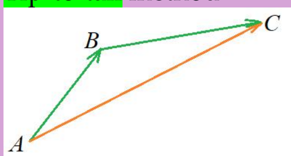
Note: Two vectors are equal if and only if they have the **<u>same magnitude and direction</u>**.
For example, $\overrightarrow{XY} = -\overrightarrow{YX}$ as they have the same magnitude but OPPOSITE directions.

In the following sections, vectors $\vec{a}$ and $\vec{b}$ would be written as $\mathbf{a}$ and $\mathbf{b}$ for simplicity.
However you are required to write them in the arrow-overbar forms in exams.

11.2 Operations of vectors

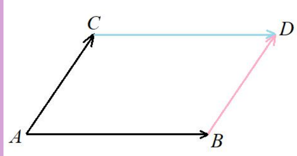
(1) Addition

- Tip-to-tail method



We only care about the initial and final positions of the vector.
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

- Parallelogram method



ACDB is a parallelogram.

Vectors can be 'translated' as long as both the direction and magnitude of the new vector remains the same.
$\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$

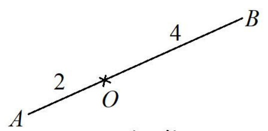
$\boxed{\overrightarrow{AB}} + \boxed{\overrightarrow{AC}} = \boxed{\overrightarrow{AB}} + \boxed{\overrightarrow{BD}}$
$= \boxed{\overrightarrow{CD}} + \boxed{\overrightarrow{AC}}$
$= \overrightarrow{AD}$

- Subtraction of vectors is treated as the negative sum of vectors.
  For instance, $\overrightarrow{AD} - \overrightarrow{BD} = \overrightarrow{AD} + (-\overrightarrow{BD}) = \overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$.

(2) For **<u>PARALLEL</u>** vectors - Scalar multiplication

Parallel vectors $\mathbf{u}$ and $\mathbf{v}$ can be expressed in the form $\mathbf{u} = k\mathbf{v}$, where $\mathbf{k}$ is a constant.

- If $k > 0$, $\mathbf{u}, \mathbf{v}$ are along the same direction.
- If $k < 0$, $\mathbf{u}, \mathbf{v}$ are of opposite directions.



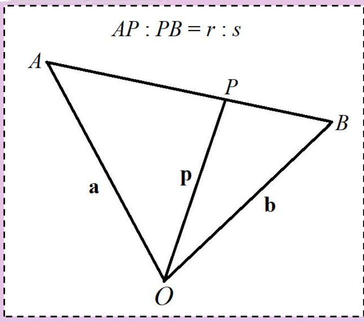
In the figure, $\left|\overrightarrow{AO}\right| = 2$ and $\left|\overrightarrow{OB}\right| = 4$.

$\overrightarrow{AO}$ and $\overrightarrow{AB}$ are along the same directions.
Thus we write $\overrightarrow{AB} = 3\,\overrightarrow{AO}$.
$\overrightarrow{OA}$ and $\overrightarrow{OB}$ are of opposite directions.
Thus we write $\overrightarrow{OB} = -2\,\overrightarrow{OA}$.

(3) For **<u>NON-PARALLEL</u>** vectors - compare their coefficients!

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors NOT parallel to each other and that $\lambda_1\mathbf{a} + \mu_1\mathbf{b} = \lambda_2\mathbf{a} + \mu_2\mathbf{b}$ for some constant $\lambda_1, \lambda_2, \mu_1, \mu_2$, then $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

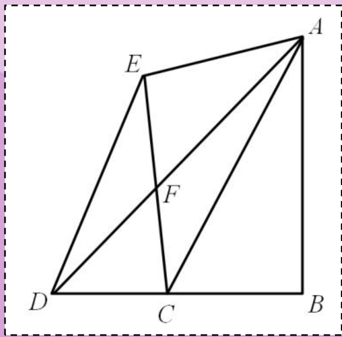
(4) Point of division of line segments
If A, P, B are collinear such that $AP : PB = r : s$,
then $\mathbf{p} = \dfrac{s\mathbf{a} + r\mathbf{b}}{s + r}$



Can you prove the results above by considering $\overrightarrow{OA} + \overrightarrow{AP}$ *?*

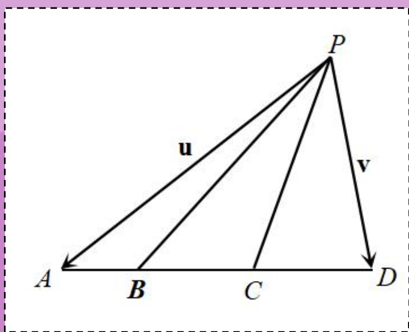
With those four rules in mind, you can now solve some geometry problems with vectors!

[Practice 1]
The figure shows the quadrilateral *ABDE*. *C* is a point on *BD*.
AD and *CE* meet at *F*. Simplify



(a) $\overrightarrow{AC} + \overrightarrow{FA}$,
(b) $\overrightarrow{EF} - \overrightarrow{AF}$,
(c) $\overrightarrow{EA} + \overrightarrow{CD} + \overrightarrow{AC}$,
(d) $\overrightarrow{DF} + \overrightarrow{AB} - \overrightarrow{DB}$.

[Practice 2]
In the figure, *B* and *C* are two points on *AD* such that
$AB : BC : CD = 2 : 3 : 5$. If $\overrightarrow{PA} = \mathbf{u}$ and $\overrightarrow{PD} = \mathbf{v}$,
express, in terms of $\mathbf{u}$ and $\mathbf{v}$,



(a) $\overrightarrow{AD}$,
(b) $\overrightarrow{BD}$,
(c) $\overrightarrow{PB}$.
(You will probably understand how Rule 4 is derived from!)

Answers [Practice 1] $\overrightarrow{FC}$; $\overrightarrow{EA}$; $\overrightarrow{ED}$; $\overrightarrow{AF}$ [Practice 2] $\mathbf{v} - \mathbf{u}$; $\frac{4}{5}(\mathbf{v} - \mathbf{u})$; $\frac{4}{5}\mathbf{u} + \frac{1}{5}\mathbf{v}$

[Practice 3]

In $\triangle ABC$, $\overrightarrow{AB} = m\mathbf{p}$, $\overrightarrow{BC} = 3\mathbf{p} + n\mathbf{q}$ and $\overrightarrow{AC} = 5\mathbf{p} + 5\mathbf{q}$, where **u** and **v** are two non-zero and non-parallel vectors. Find the values of m and n.

[Practice 4] DSE 2014 M2 #11 (a)

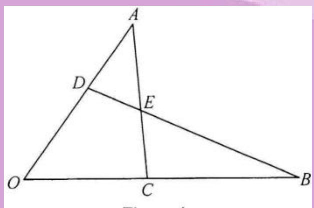


Figure 4

In Figure 4, C and D are points on OB and OA respectively such that $AD : DO = OC : CB = t : (1-t)$, where $0 < t < 1$. BD and AC intersect at E such that $AE : EC = m : 1$ and $BE : ED = n : 1$, where m and n are positive. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) By considering ΔOAC, express $\overrightarrow{OE}$ in terms of m, t, $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering ΔOBD, express $\overrightarrow{OE}$ in terms of n, t, $\mathbf{a}$ and $\mathbf{b}$.

(iii) Show that $m = \dfrac{t}{(1-t)^2}$ and $n = \dfrac{1-t}{t^2}$.

(iv) Chris claims that if $m = n$, then E is the centroid of ΔOAB. Do you agree? Explain your answer.

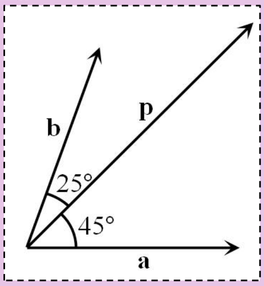
(9 marks)

Answers: [Practice 3] $m = 2$, $n = 5$

[Practice 4] $\overrightarrow{OE} = \dfrac{\mathbf{a} + mt\mathbf{b}}{1+m}$; $\overrightarrow{OE} = \dfrac{n(1-t)\mathbf{a} + \mathbf{b}}{1+n}$; yes as $t = \dfrac{1}{2}$, i.e. CA and DB medians

[Practice 5]

In the figure, **a**, **b** and **p** are three vectors in the same plane, where $|\mathbf{a}| = |\mathbf{b}| = 2$ and $|\mathbf{p}| = 3$. Express **p** as the sum of scalar multiples of **a** and **b**, where the scalars are correct to 3 decimal places.



[Practice 6] DSE 2015 M2 #10 (a)

OAB is a triangle. P is the mid-point of OA. Q is a point lying on AB such that $AQ:QB = 1:2$ while R is a point lying on OB such that $OR:RB = 3:1$. PR and OQ intersect at C.

(i) Let t be a constant such that $PC:CR = t:(1-t)$.

By expressing $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, find the value of t.

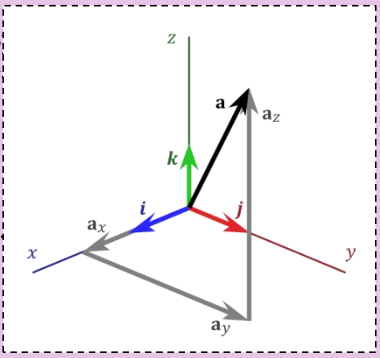
(ii) Find $CQ:OQ$. (7 marks)

Answers [Practice 5] $\mathbf{p} = 0.675\mathbf{a} + 1.129\mathbf{b}$

[Practice 6] $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB}$, $t = \frac{1}{4}$; $7:16$

11.3 Expressing vectors in coordinate systems

Vectors can be represented on 2D coordinate planes or 3D coordinate s



- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are **unit base vectors (i.e. magnitude = 1)** along the x-, y and z-direction respectively.
- In 2D planes, only $\mathbf{i}$, $\mathbf{j}$ are included; but $\mathbf{k}$ is also included in 3D systems.
- The origin is defined as $0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$ as its coordinates are $(0,0,0)$.

Between the points $M(m_1,m_2,m_3)$ and $N(n_1,n_2,n_3)$ on a coordinate system,

- $\overrightarrow{MN}=(n_1-m_1)\mathbf{i}+(n_2-m_2)\mathbf{j}+(n_3-m_3)\mathbf{k}$.
- $\left|\overrightarrow{MN}\right|=\sqrt{(n_1-m_1)^2+(n_2-m_2)^2+(n_3-m_3)^2}$ = length of MN, i.e. distance between M and N.
- The position vector of M refers to $\overrightarrow{OM}$, where O is the origin.
  For example, if the coordinates of M are $(3,2,-7)$, then $\overrightarrow{OM}=3\mathbf{i}+2\mathbf{j}-7\mathbf{k}$.
  - We could write $\overrightarrow{MN}=\overrightarrow{ON}-\overrightarrow{OM}$ to find $\overrightarrow{MN}$ if the position vectors of M, N are given.

A unit vector $\hat{\mathbf{v}}$ refers to a vector with **magnitude 1**. It can be expressed as $\hat{\mathbf{v}}=\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

Note that sometimes you are required to find the unit vectors in HKDSE (together with the techniques to be learnt in section 12.) The relevant information should be discussed later.

Below are some HKDSE questions related to vectors in coordinate systems. You may also need to apply the operation rules in 11.2 to help you out:

[Practice 7] DSE 2017 M2 #10 (a)

ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE:EC=1:r$. AB produced and DE produced meet at the point F. It is given that $DE:EF=1:10$.

Let $\overrightarrow{OA}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\overrightarrow{OB}=4\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\overrightarrow{OC}=8\mathbf{i}-3\mathbf{j}-2\mathbf{k}$, where O is the origin.

By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r. (4 marks)

Answer: $r=\dfrac{6}{5}$

[Practice 8] DSE 2016 M2 #12(a)

Let $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OP} = \mathbf{i} + t\mathbf{j}$, where t is a constant and O is the origin. It is given that P is equidistant from A and B . Find t . (3 marks)

[Practice 9] DSE 2020 M2 #12(c)(ii) Modified

Let $\overrightarrow{OQ} = 5\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{ON} = \lambda(6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$ where λ is a real number. μ is a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i} + \mu\mathbf{j} - 10\mathbf{k}$. Find λ and μ . (4 marks)

Answers

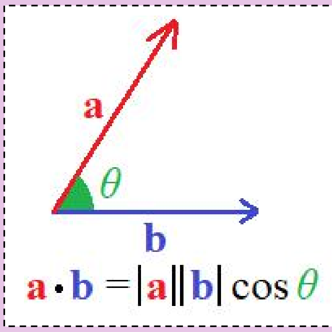
[Practice 8] $t = -1$

[Practice 9] $\lambda = \dfrac{2}{9}$, $\mu = -25$

12 Scalar and vector products of vector

12.1 Scalar products (a.k.a. dot product)

Scalars are quantities that **DOES NOT include direction**.



(1) For two vectors $\mathbf{a}$ and $\mathbf{b}$, their scalar (dot) product is given by $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between them.

The formula also yields the following:

$\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$	$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}$
$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$	$\lvert\mathbf{a}-\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2-2\lvert\mathbf{a}\cdot\mathbf{b}\rvert$

When $\mathbf{a}\cdot\mathbf{b}=0$, either $\mathbf{a}$ or $\mathbf{b}$ = $\mathbf{0}$ or $\mathbf{a}\perp\mathbf{b}$ (as $\theta=90°$) if $\mathbf{a},\mathbf{b}\neq\mathbf{0}$ [IMPORTANT!]

► The dot product serves as a test of perpendicularity and you are often required to apply it in DSE.

$\mathbf{a}$ and $\mathbf{b}$ are said to be **orthogonal** if $\mathbf{a}\perp\mathbf{b}$.

(2) For 3-D vectors in a rectangular coordinate system…

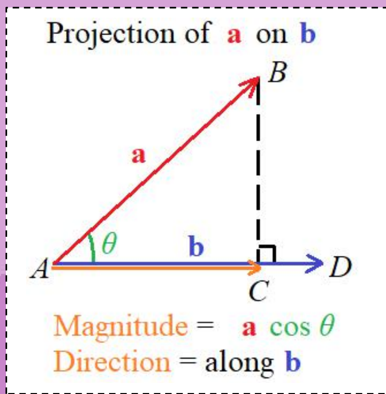
(a) $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$ ($\theta=0°$) and $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ ($\theta=90°$)

(b) For $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ with θ the angle between them,

❖ $\mathbf{a}\cdot\mathbf{b}=x_1x_2+y_1y_2+z_1z_2$

❖ $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

(3) When a vector $\mathbf{a}$ is projected onto another direction $\mathbf{b}$, its magnitude and direction will change.



- Magnitude: $|\mathbf{a}|\cos\theta=\dfrac{|\mathbf{a}||\mathbf{b}|\cos\theta}{|\mathbf{b}|}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$
- Direction: along $\mathbf{b}$, i.e. $\hat{\mathbf{b}}=\dfrac{\mathbf{b}}{|\mathbf{b}|}$ should be multiplied.

So, the projection of $\mathbf{a}$ on $\mathbf{b}$ is given by $\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\dfrac{\mathbf{b}}{|\mathbf{b}|}$.

In recent years, projection questions are typically combined with cross product (Sect 12.2). We shall discuss this in Section 12.2.

You are often required to make use of the concept of dot product to help you solve problems in DSE. Try out the following questions:

[Practice 10] DSE 2017 M2 #3

P is a point lying on AB such that $AP : PB = 3 : 2$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}| = 45$, $|\mathbf{b}| = 20$ and $\cos\angle AOB = \dfrac{1}{4}$. Find

(i) $\mathbf{a} \cdot \mathbf{b}$,

(ii) $\left|\overrightarrow{OP}\right|$. (5 marks)

[Practice 11] DSE 2017 M2 #10 (b)

ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE : EC = 5 : 6$. AB is produced and DE produced and DE produced meet at the point F. It is given that $BE : EF = 1 : 10$. Let $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OC} = 8\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, where O is the origin.

(i) Find $\overrightarrow{AD} \cdot \overrightarrow{DE}$.

(ii) Are B, D, C and F concyclic? Explain your answer. (5 marks)

Answers: [Practice 10] $\overrightarrow{OP} = \dfrac{2}{5}\mathbf{a} + \dfrac{3}{5}\mathbf{b}$; $\mathbf{a} \cdot \mathbf{b} = 225$; $\left|\overrightarrow{OP}\right| = 24$

[Practice 11] $\overrightarrow{AD} \cdot \overrightarrow{DE} = 0$, concyclic as $\angle CBF = 90° = \angle CDF$

[Practice 12] DSE 2012 M2 #12

Figure 6 shows an acute angled triangle ABC, where D is the mid-point of AB, G is the centroid and O the circumcentre. Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$.

(a) Express AG in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (3 marks)

(b) It is given that E is a point on AB such that CE is an altitude.
Extend OG to meet CE at F.

(i) Prove that $\Delta DOG \sim \Delta CFG$. Hence find $FG : GO$.

(ii) Show that $\overrightarrow{AF}=\mathbf{b}+\mathbf{c}$. Hence prove that F is the orthocentre of ΔABC. (9 marks)

Answer: $\overrightarrow{AG}=\dfrac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3}$; $FG:GO=1:2$

[Practice 13] DSE 2013 M2 #14 (a)

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = 90°$. P is a variable point such that $\overrightarrow{AP}\cdot\overrightarrow{BP}+\overrightarrow{BP}\cdot\overrightarrow{CP}+\overrightarrow{CP}\cdot\overrightarrow{AP}=0$. Let D be the fixed point such that $\overrightarrow{OD}=\dfrac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$.

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OP}=\mathbf{p}$ and $\overrightarrow{OD}=\mathbf{d}$.

(i) Show that $\overrightarrow{AP}\cdot\overrightarrow{BP}=\mathbf{p}\cdot\mathbf{p}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{p}$.

(ii) Using (i), show that $\mathbf{p}\cdot\mathbf{p}=2\mathbf{p}\cdot\mathbf{d}$.

(iii) Show that $|\mathbf{p}-\mathbf{d}|=|\mathbf{d}|$.
Hence, show that P lies on the sphere centred at D with fixed radius. (8 marks)

[Practice 14] DSE 2018 M2 #12 (b)

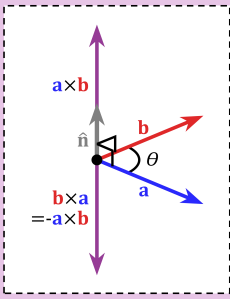
The position vectors of the points A, B, C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A, B and C by Π. Let E be a point such that $\overrightarrow{DE}=-\dfrac{32}{13}\mathbf{i}-\dfrac{8}{13}\mathbf{j}+\dfrac{28}{13}\mathbf{k}$, and F a point lying on BC such that $DF\perp BC$.

(i) Find $\overrightarrow{DF}$.

(ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$? Explain your answer. (5 marks)

Answer: [Practice 14] $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$

12.2 Vector products (a.k.a. cross product) $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$



Vectors are quantities that includes BOTH direction and magnitude.

(1) For two vectors $\mathbf{a}$ and $\mathbf{b}$, their vector (cross) product is given by $\mathbf{a}\times\mathbf{b}=(|\mathbf{a}||\mathbf{b}|\cos\theta)\hat{\mathbf{n}}$, where θ is the angle between them.

$\mathbf{n}$ is a normal vector perpendicular to BOTH $\mathbf{a}$ and $\mathbf{b}$.
Its direction can be found by applying the ***left-hand rule***.

Readers studying physics, do you find some similarities between the vector product and the Fleming's left hand rule which gives the direction of the magnetic force F ?

Indeed, the rule can be mathematically expressed as $\vec{F}=I\vec{l}\times\vec{B}$!

- ❖ Middle finger: along $\mathbf{a}$
- ❖ Index finger: along $\mathbf{b}$
- ❖ Thumb: along $\mathbf{n}$ (and $\mathbf{a}\times\mathbf{b}$)

This formula also yields the following:

$\mathbf{a}\times\mathbf{a}=\mathbf{0}$	$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$	$\lvert\vec{\mathbf{a}}\times\vec{\mathbf{b}}\rvert^2=\lvert\mathbf{a}\rvert^2\lvert\mathbf{b}\rvert^2-(\mathbf{a}\cdot\mathbf{b})^2$

When $\mathbf{a}\times\mathbf{b}=0$, either $\mathbf{a}$ or $\mathbf{b}$ = $\mathbf{0}$ or $\mathbf{a}/\!/\mathbf{b}$ (as $\theta=0°$) if $\mathbf{a},\mathbf{b}\neq\mathbf{0}$.

(2) For 3-D vectors in a rectangular coordinate system…

(a) $\mathbf{i}\cdot\mathbf{j}=\mathbf{k}$, $\mathbf{j}\cdot\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\cdot\mathbf{i}=\mathbf{j}$ $(\theta=90°)$ and $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$ $(\theta=0°$

(b) For $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ with θ the angle between them,

$\mathbf{a}\times\mathbf{b}=(y_1z_2-y_2z_1)\mathbf{i}+(x_2z_1-x_1z_2)\mathbf{j}+(x_1y_2-x_2y_1)\mathbf{k}$, i.e.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_1&y_1&z_1\\x_2&y_2&z_2\end{vmatrix}$$ [This formula is EXTRA IMPORTANT in DSE!!!!!!]

(3) Vector products are particularly useful in finding areas/ volumes.
Below are some essential formulae:

(a) For a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$,
its area is given by $\text{Area}=|\mathbf{a}\times\mathbf{b}|$

► Area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ = $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

► In a 2-D coordinate plane, the area enclosed by points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ is $\frac{1}{2}\begin{vmatrix}x_1&y_1\\x_2&y_2\\x_3&y_3\\x_1&y_1\end{vmatrix}$. This is an essential formula to find area in Core maths.

Can you prove this by using vectors, say let $\mathbf{a}=x_1\mathbf{i}+y_1\mathbf{j}$, $\mathbf{b}=x_2\mathbf{i}+y_2\mathbf{j}$ ***and*** $\mathbf{c}=x_3\mathbf{i}+y_3\mathbf{j}$ ***?***

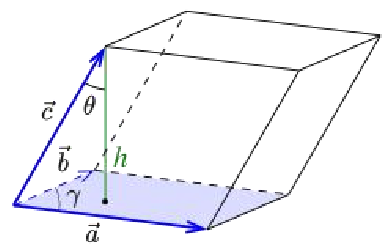
(b) The scalar triple product is defined as $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

Note that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

It can be used to find the volume of a parallelepiped, i.e. a structure with all faces parallelograms.

For a parallelepiped with edges $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$,

its volume = $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$



$$\begin{aligned}\text{Volume} &= (|\mathbf{a}\times\mathbf{b}|)(h)\\ &= (|\mathbf{a}\times\mathbf{b}|)(|\mathbf{c}|)\cos\theta\\ &= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \;=\; \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\end{aligned}$$

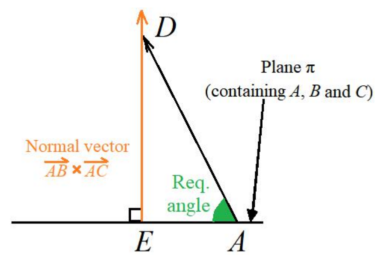
► Volume of a tetrahedron (all sides triangles) $=\dfrac{1}{3}(\dfrac{1}{2}|\mathbf{a}\times\mathbf{b}|)(h) = \dfrac{1}{6}[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]$

If $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$ are vectors such that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=0$, O, A, B and C all **lie on the same plane** and O, A, B and C are said to be **COPLANAR**.

12.3 Supplement and techniques related to vectors

- Finding the height of a tetrahedron/ parallelepiped:
  First find the volume by scalar triple product, then find the area of the base and divide them.

- Finding the angle between two planes/lines (in relation with 3D trigo in Core)
  - We may make use of the dot product $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$. Changing the subject to θ,
    $\theta=\cos^{-1}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$, which allows the angle between two lines to be found.
  - To find the angle between a plane π containing some points A, B, C and a line AD, we **may make use of cross product** to **find a vector normal to π**, and then find the required angle by dot product/ other trigonometric ratios (or say, $90°-\phi$).



- Solving problems related to four centres/ describing geometric relationship
  Sometimes questions in DSE will ask you about the relation of some points/ to prove statements related to four centres of triangle. The following is a summary of the centres and the related expressions you should be considering.

Centres	Relevant lines	Applicable lines/ skills
Centroid	Medians $\overrightarrow{OG}=\dfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$	Mid-point finding ($\dfrac{1}{2}(\mathbf{a}+\mathbf{b})$) $AO:OD=BO:OE=CO:OF=2:1$
Circumcentre	Perpendicular bisectors	Dot product = 0 (perpendicularity) Mid-point finding ($\dfrac{1}{2}(\mathbf{a}+\mathbf{b})$)
Orthocentre	Altitudes [See practice 12!]	Dot product = 0 (perpendicularity) **Scalar triple product = 0** (mutually perpendicular *AND COPLANAR*)

The finding of in-centres involve angle bisectors which are too complicated at this level.

- Geometric relationship to be described: Colinear/ mid-pt (3 pts), concyclic (4 pts), **coplanar**

Vector is a abstract topic that requires a lot of drills. As you have learnt all the relevant concepts required, you should try working on some full questions listed below.

[Practice 15] DSE 2014 M2 #8

Let $\overrightarrow{OP} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OR} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OQ}$. Hence find the volume of the tetrahedron $OPQR$.

(b) Find the acute angle between the plane OPQ and the line OR, correct to the nearest $0.1°$. (8 marks)

Answers: $6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$; Volume = 1, angle = 6.8°

[Practice 16] DSE 2020 M2 #12

Let $\overrightarrow{OP} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$, where O is the origin. R is a point lying on PQ such that $PR : RQ = 1 : 3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$. (2 marks)

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$. (2 marks)

(c) Let N be a point such that $\overrightarrow{ON} = \lambda(\overrightarrow{OP} \times \overrightarrow{OR})$, where λ is a real number.

(i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

(ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i} + \mu\mathbf{j} - 10\mathbf{k}$.

~~(1) Find λ and μ. (You have done this in Practice 9!)~~

(2) Denote the angle between ΔOPQ and ΔNPQ by θ. Find $\tan\theta$. (8 marks)

Answer: $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, area = 9, $\overrightarrow{NR}$ IS perpendicular to $\overrightarrow{PQ}$, $\tan\theta = \frac{2}{3}$

[Practice 17] DSE 2019 M2 #12

Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$.

(a) Find t . (3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (2 marks)

(c) Find the volume of the pyramid $OABC$. (2 marks)

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OR} = r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer. (6 marks)

Answer: $t = 2$, $\overrightarrow{AB} \times \overrightarrow{AC} = 48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}$, pyramid volume $= 48$

$\overrightarrow{OD} = \frac{96}{41}\mathbf{i} - \frac{72}{41}\mathbf{j} + \frac{96}{41}\mathbf{k}$; $p = 6$, $q = -8$, $r = 6$ so $\overrightarrow{OE} = \frac{41}{576}\overrightarrow{OD}$ (collinear)

[Practice 18] DSE 2016 M2 #12 (b)

Let $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\overrightarrow{OP} = \mathbf{i} - \mathbf{j}$, $\overrightarrow{OC} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, where O is the origin. Denote the plane containing A, B and C by Π.

(i) Find a unit vector perpendicular to Π.

(ii) Find the angle between CD and Π.

(iii) It is given that E is a point lying on Π such that DE is perpendicular to Π.

Let F be a point such that $\overrightarrow{PF} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$. Describe the geometric relationship between D, E and F. Explain your answer. (10 marks)

Unit vector: $\pm(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})$; angle = $\sin^{-1}\left(\frac{3\sqrt{11}}{11}\right)$; D is the mid-point of EF

[Practice 19] DSE 2018 M2 #12

The position vectors of points A, B, C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A, B, C by Π.
Let E be the projection of D on Π.

(a) Find

(i) $\overrightarrow{AB}\times\overrightarrow{AC}$,

(ii) the volume of the tetrahedron $ABCD$,

(iii) $\overrightarrow{DE}$. (5 marks)

~~(b) Let F be a point lying on BC such that DF is perpendicular to BC~~ .~~(Done in Practice 14)~~

The result $\overrightarrow{DF}=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is obtained and it is proved that $BC\perp EF$.

(c) Find the angle between ΔBCD and Π. (3 marks)

Answer: $32\mathbf{i}+8\mathbf{j}-28\mathbf{k}$, volume = 24, $\overrightarrow{DE}=-\frac{32}{13}\mathbf{i}-\frac{8}{13}\mathbf{j}+\frac{28}{13}\mathbf{k}$; angle = $\cos^{-1}\left(\frac{3\sqrt{13}}{13}\right)$