Mock Exam 6

Section A

1. *Reference: HKDSE Math M2 2015 Q7*

(a) R.H.S.

$$= \frac{1-\cos 2x}{1+\cos 2x}$$

$$= \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x - 1)} \qquad \text{1M}$$

$$= \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tan^2 x$$

$$= \text{L.H.S.}$$

$$\therefore \quad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x} \qquad 1$$

(b) $f(x)$

$$= \sec^4 x - 2\tan^4 x$$

$$= (\sec^2 x)^2 - 2\tan^4 x$$

$$= (1+\tan^2 x)^2 - 2\tan^4 x \qquad \text{1M}$$

$$= 1 + 2\tan^2 x + \tan^4 x - 2\tan^4 x$$

$$= 1 + 2\tan^2 x - \tan^4 x$$

$$= 1 + \frac{2(1-\cos 2x)}{1+\cos 2x} - \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^2 \qquad \text{1M}$$

$$= \frac{(1+\cos 2x)^2 + 2(1-\cos 2x)(1+\cos 2x) - (1-\cos 2x)^2}{(1+\cos 2x)^2}$$

$$= \frac{(2)(2\cos 2x) + 2 - 2\cos^2 2x}{1 + 2\cos 2x + \cos^2 2x}$$

$$= \frac{2 + 4\cos 2x - 2\left(\dfrac{1+\cos 4x}{2}\right)}{1 + 2\cos 2x + \dfrac{1+\cos 4x}{2}}$$

$$= \underline{\underline{\frac{2 + 8\cos 2x - 2\cos 4x}{3 + 4\cos 2x + \cos 4x}}} \qquad \text{1A}$$

(5)

2. *Reference: HKCEE A. Math 2009 Q11*

$$\text{The general term} = C_r^8 (3x)^{8-r}\left(-\frac{2}{x^2}\right)^r$$

$$= C_r^8 (3)^{8-r}(-2)^r x^{8-3r} \qquad \text{1M}$$

For the x^2 term, we have

$$8 - 3r = 2 \qquad \text{1M}$$

$$r = 2$$

$$\therefore \quad \text{The coefficient of } x^2 = C_2^8 (3)^{8-2}(-2)^2$$

$$= \underline{\underline{81\ 648}} \qquad \text{1A}$$

(3)

3. *Reference: HKDSE Math M2 2016 Q2*

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\sqrt{2(1+h)-1}} - \frac{1}{\sqrt{2(1)-1}}\right] \qquad \text{1M}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sqrt{2h+1}} - 1\right)$$

$$= \lim_{h \to 0} \frac{1-\sqrt{2h+1}}{h\sqrt{2h+1}} \qquad \text{1M}$$

$$= \lim_{h \to 0} \frac{(1-\sqrt{2h+1})(1+\sqrt{2h+1})}{h\sqrt{2h+1}(1+\sqrt{2h+1})} \qquad \text{1M}$$

$$= \lim_{h \to 0} \frac{1-(2h+1)}{h\sqrt{2h+1}(1+\sqrt{2h+1})}$$

$$= \lim_{h \to 0} \frac{-2}{\sqrt{2h+1}(1+\sqrt{2h+1})}$$

$$= \frac{-2}{2}$$

$$= \underline{\underline{-1}} \qquad \text{1A}$$

(4)

4. $y^2 = \sin x$

Differentiating both sides of the equation with respect to x, we have

$$2y\frac{dy}{dx} = \cos x \text{................ } (*) \qquad \text{1M}$$

$$\frac{dy}{dx} = \frac{\cos x}{2y}$$

Differentiating both sides of (*) with respect to x, we have

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = -\sin x \qquad \text{1M}$$

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{\cos x}{2y}\right)\left(\frac{\cos x}{2y}\right) = -y^2$$

$$2y\frac{d^2y}{dx^2} + \frac{(1-\sin^2 x)}{2y^2} = -y^2 \qquad \text{1M}$$

$$2y\frac{d^2y}{dx^2} = -y^2 - \frac{1-y^4}{2y^2}$$

$$= \frac{-2y^4 - 1 + y^4}{2y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1+y^4}{4y^3} \qquad \text{1}$$

(4)

5. $\int \frac{\ln x}{x^3}\,dx = \int \ln x\, d\left(\frac{-1}{2x^2}\right)$ 1M

$= \left(\frac{-1}{2x^2}\right)\ln x - \int \frac{-1}{2x^2}\,d(\ln x)$ 1M

$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$ 1M

$= \underline{\underline{-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C}}$, where C is a constant 1A

(4)

6. *Reference: HKDSE Math M2 2013 Q3*

For $n = 1$,

L.H.S. $= 1 - \frac{1}{2!} = \frac{1}{2}$

R.H.S. $= \frac{1}{(1+1)!} = \frac{1}{2}$

$\therefore$ The proposition is true for $n = 1$. 1

Next, assume the proposition is true for $n = k$, where k is a positive integer, that is,

$1 - \frac{1}{2!} - \frac{2}{3!} - \frac{3}{4!} - \cdots - \frac{k}{(k+1)!} = \frac{1}{(k+1)!}$, 1

when $n = k + 1$,

L.H.S.

$= 1 - \frac{1}{2!} - \frac{2}{3!} - \frac{3}{4!} - \cdots - \frac{k}{(k+1)!} - \frac{k+1}{(k+2)!}$

$= \frac{1}{(k+1)!} - \frac{k+1}{(k+2)!}$ (by the assumption) 1

$= \frac{k+2}{(k+2)[(k+1)!]} - \frac{k+1}{(k+2)!}$

$= \frac{(k+2)-(k+1)}{(k+2)!}$

$= \frac{1}{[(k+1)+1]!}$ 1

$=$ R.H.S.

$\therefore$ The proposition is true for $n = k + 1$.

By the principle of mathematical induction, the proposition is true for all positive integers n. 1

(5)

Analysis

Inspect the L.H.S. term by term. $-\frac{n}{(n+1)!}$ is the general term of the sequence $-\frac{1}{2!}, -\frac{2}{3!}, \ldots$, while '1' is not a term in the sequence.

7. $\because$ The graph has the vertical asymptote $x = 2$.

$\therefore$ $B = -2$ 1A

Substituting $(1, 0)$ into $y = Ax + \frac{1}{x-2}$,

$0 = A(1) + \frac{1}{1-2}$

$A = 1$ 1A

$\therefore$ $y = x + \frac{1}{x-2}$

When $x \to \pm\infty$, $\dfrac{1}{x-2} \to 0$.

$\therefore$ $y = x$ is an oblique asymptote. 1A

$$\frac{dy}{dx} = 1 - \frac{1}{(x-2)^2}$$

When $\dfrac{dy}{dx} = 0$,

$$\begin{aligned} 1 - \frac{1}{(x-2)^2} &= 0 \\ (x-2)^2 &= 1 \\ x - 2 &= \pm 1 \\ x &= 1 \text{ or } 3 \end{aligned}$$ 1M

x	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$\dfrac{dy}{dx}$	+	0	−	Undefined	−	0	+

1M

When $x = 1, y = 0$

When $x = 3, y = 3 + \dfrac{1}{3-2} = 4$

$\therefore$ $(1, 0)$ is the maximum point. 1A

$(3, 4)$ is the minimum point. 1A

(7)

8. (a) Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$. 1M

$$\begin{aligned} \int \sqrt{4-x^2}\,dx &= \int \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta\, d\theta \\ &= \int 4\cos^2\theta\, d\theta \\ &= \int (2 + 2\cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C \\ &= 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\frac{x}{2} + 2\left(\frac{x}{2}\right)\sqrt{1 - \left(\frac{x}{2}\right)^2} + C \\ &= 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + C, \text{ where } C \text{ is a constant} \end{aligned}$$

1M (for the line $= \int (2 + 2\cos 2\theta)\, d\theta$); 1 (for the last line)

(b) Area $= \int_{-2}^{2} 2\sqrt{4-x^2}\,dx - \int_0^2 (2x - x^2)\,dx$ 1M

$$= \left[4\sin^{-1}\frac{x}{2} + x\sqrt{4-x^2}\right]_{-2}^{2} - \left[x^2 - \frac{x^3}{3}\right]_0^2$$ 1M

$$= \underline{\underline{4\pi - \frac{4}{3}}}$$ 1A

(6)

9. *Reference: HKCEE A. Math 2001 Q8*

(a) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= (13)(10)\left(\dfrac{16}{65}\right)$ 1M

$= \underline{\underline{32}}$ 1A

(b) Let $\mathbf{u}$ be the projection of $(5\mathbf{a}+k\mathbf{b})$ on $\mathbf{b}$. Then $\mathbf{u} = \left[\dfrac{(5\mathbf{a}+k\mathbf{b})\cdot\mathbf{b}}{|\mathbf{b}|}\right]\dfrac{\mathbf{b}}{|\mathbf{b}|}$.

Since $\mathbf{u}$ is a unit vector, we have

$\dfrac{(5\mathbf{a}+k\mathbf{b})\cdot\mathbf{b}}{|\mathbf{b}|} = 1$ 1M

$\dfrac{5\mathbf{a}\cdot\mathbf{b}+k|\mathbf{b}|^2}{|\mathbf{b}|} = 1$

$\dfrac{5(32)+k(10)^2}{10} = 1$ 1M

$k = \underline{\underline{-\dfrac{3}{2}}}$ 1A

(5)

10. *Reference: HKDSE Math M2 2014 Q9*

(a) According to the situation, we have

$$\begin{cases} x+y+z=8 \\ x+2y+5z=14 \end{cases}.$$

The augmented matrix is

$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 2 & 5 & 14 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 4 & 6 \end{array}\right) \quad (R_2 - R_1 \to R_2)$ 1M

Let $z=t$, where t is any real number, then $y = 6-4t$ and $x = 2+3t$. 1A

Since x, y and z are non-negative integers, we have $\begin{cases} t \ge 0 \\ 6-4t \ge 0 \\ 2+3t \ge 0 \end{cases}$.

$\therefore \quad 0 \le t \le \dfrac{3}{2}$

$\therefore \quad t = 0$ or 1 1M

$\therefore$ There are two sets of combinations of x, y and z.

$\therefore$ The claim is disagreed. 1A

(b) According to the situation, we have $\begin{cases} x+y+z=8 \\ x+2y+5z=4 \\ 100x+150y+300z=1100 \end{cases}$,

i.e. $\begin{cases} x+y+z=8 \ldots\ldots\ldots\ldots\ldots\ (1) \\ x+2y+5z=14 \ldots\ldots\ldots\ (2) \\ 2x+3y+6z=22 \ldots\ldots\ (3) \end{cases}$ 1M

Note that $(1)+(2)=(3)$, so the above system is equivalent to the system in (a). 1M

$\therefore$ There are two sets of combinations of x, y and z. 1A

(7)

Section B

11. (a) (i) $\overrightarrow{AB} = (2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) - (2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ -1 & -2 & -2 \end{vmatrix}$$

$= -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ 1A

Analysis

We can find the area of ΔABC by $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ first.

Then find the area of ΔACD by subtracting the area of ΔABC from the area of $ABCD$.

$$\left.\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\ &= \frac{1}{2}\sqrt{(-2)^2 + (-1)^2 + 2^2} \\ &= \frac{3}{2} \end{aligned}\right\} \text{1M}$$

$\therefore$ Area of $\Delta ACD = 6 - \frac{3}{2}$ 1M

$= \frac{9}{2}$

$= 3 \times$ Area of ΔABC

By treating AB and DC as the bases of ΔABC and ΔACD respectively, the two triangles have the same height.

$\therefore$ $AB : DC =$ Area of ΔABC : Area of $\Delta ACD = \underline{\underline{1 : 3}}$ 1A

(ii) $\because$ $AB \parallel DC$ and $AB : DC = 1 : 3$

$\therefore$ $\overrightarrow{DC} = 3(2\mathbf{j} + \mathbf{k})$ 1A

$\therefore$ Position vector of $D = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - 3(2\mathbf{j} + \mathbf{k})$

$= \underline{\underline{\mathbf{i} - 7\mathbf{j} - \mathbf{k}}}$ 1A

(6)

Smart Tips

Position vector of D
= Position vector of C – $\overrightarrow{DC}$

(b) (i) $\overrightarrow{AF} = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= 2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$

Volume of the prism $ABCHFG$

$= \frac{1}{2}\left|\overrightarrow{AF} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right|$

$= \frac{1}{2}\left|(2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}) \cdot (-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right|$

$= \frac{1}{2}\left|(2)(-2) + (-3)(-1) + (-3)(2)\right|$

$= \frac{7}{2}$ 1M

Since the prisms $ABCHFG$ and $ACDEFH$ have the same height,

volume of $ABCHFG$: volume of $ACDEFH$

= area of ΔABC : area of ΔACD

= 1 : 3 1M

Volume of the prism $ABCDEFGH$

= Volume of the prism $ABCHFG$ + Volume of the prism $ACDEFH$

= 4 × Volume of the prism $ABCHFG$

$= 4 \times \frac{7}{2}$

$= \underline{\underline{14}}$ 1A

(ii) Let h be the height of the prism $ABCDEFGH$ with respect to the base $ABCD$.

$$\text{Area of } ABCD \times h = \text{Volume of } ABCDEFGH$$
$$6h = 14$$
$$h = \frac{7}{3} \qquad \text{1A}$$

$$\overrightarrow{FC} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$
$$= -3\mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$FC = \sqrt{(-3)^2 + 1^2 + 1^2}$$
$$= \sqrt{11}$$

Let θ be the acute angle between the line FC and the plane $ABCD$.

$$\sin\theta = \frac{h}{FC}$$
$$= \frac{7}{3\sqrt{11}} \qquad \text{1M}$$
$$\theta = 44.7° \quad \textit{(cor. to the nearest 0.1°)} \qquad \text{1A}$$

$\therefore$ The acute angle between the line FC and the plane $ABCD$ is $44.7°$.

(6)

12. *Reference: HKDSE Math M2 2013 Q11*

(a) $$\text{R.H.S.} = \frac{1-t^2}{1+t^2}$$
$$= \frac{1-\tan^2 x}{1+\tan^2 x}$$
$$= \frac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \qquad \text{1M}$$
$$= \cos 2x$$
$$= \text{L.H.S.}$$

$$\therefore \quad \cos 2x = \frac{1-t^2}{1+t^2} \qquad 1$$

(2)

(b) (i) Let $t = \tan x$. Then

$$dt = \sec^2 x dx$$
$$= (1+\tan^2 x)dx$$
$$dx = \frac{dt}{1+t^2} \qquad \text{1M}$$

$$\int \frac{dx}{3+\cos 2x} = \int \frac{1}{3+\dfrac{1-t^2}{1+t^2}} \times \frac{1}{1+t^2}\,dt \qquad \text{1M}$$
$$= \int \frac{1+t^2}{3+3t^2+1-t^2} \times \frac{1}{1+t^2}\,dt$$
$$= \int \frac{dt}{2t^2+4} \qquad 1$$

(ii) Let $t = \sqrt{2}\tan\theta$. Then $dt = \sqrt{2}\sec^2\theta d\theta$. 1M

$$\int \frac{dt}{t^2+2} = \int \frac{\sqrt{2}\sec^2\theta d\theta}{2\tan^2\theta + 2}$$
$$= \frac{\sqrt{2}}{2}\int d\theta$$
$$= \frac{\sqrt{2}}{2}\theta + C$$
$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \text{, where } C \text{ is a constant}$$ 1

(5)

(c) (i)
$$\frac{d}{dx}\left(\frac{\sin 2x}{3+\cos 2x}\right) = \frac{(3+\cos 2x)(2\cos 2x) - \sin 2x(-2\sin 2x)}{(3+\cos 2x)^2}$$ 1M
$$= \frac{6\cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(3+\cos 2x)^2}$$
$$= \frac{6\cos 2x + 2}{(3+\cos 2x)^2}$$
$$= \frac{6(3+\cos 2x) - 18 + 2}{(3+\cos 2x)^2}$$
$$= \underline{\underline{\frac{6}{3+\cos 2x} - \frac{16}{(3+\cos 2x)^2}}}$$ 1A

(ii) By (c)(i), we have

$$\left[\frac{\sin 2x}{3+\cos 2x}\right]_0^{\frac{\pi}{4}} = 6\int_0^{\frac{\pi}{4}}\frac{dx}{3+\cos 2x} - 16\int_0^{\frac{\pi}{4}}\frac{dx}{(3+\cos 2x)^2}$$ 1M

$$\therefore \int_0^{\frac{\pi}{4}}\frac{dx}{(3+\cos 2x)^2} = \frac{3}{8}\int_0^{\frac{\pi}{4}}\frac{dx}{3+\cos 2x} - \frac{1}{16}\left[\frac{\sin 2x}{3+\cos 2x}\right]_0^{\frac{\pi}{4}}$$ 1A
$$= \frac{3}{16}\int_0^1\frac{dt}{t^2+2} - \frac{1}{16}\left(\frac{1}{3} - 0\right) \quad \text{(by (b)(i))}$$
$$= \frac{3}{16}\left[\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 - \frac{1}{48} \quad \text{(by (b)(ii))}$$
$$= \frac{3\sqrt{2}}{32}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{48}$$ 1

(5)

13. (a) (i)
$$QQ^TP = \begin{pmatrix} a \\ b \\ c \end{pmatrix}(a \quad b \quad c)\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$ 1A
$$= \begin{pmatrix} abc - abc & -a^2c + a^2c & a^2b - a^2b \\ b^2c - b^2c & -abc + abc & ab^2 - ab^2 \\ bc^2 - bc^2 & -ac^2 + ac^2 & abc - abc \end{pmatrix}$$ 1A
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ QQ^TP is the 3×3 zero matrix. 1

WatchOut

$$(a \ b \ c)\begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \begin{pmatrix} a \\ b \\ c \end{pmatrix}(a \ b \ c)$$

(ii) $I + P = \begin{pmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{pmatrix}$

$$\begin{aligned} |I + P| &= 1 - abc + abc + b^2 + c^2 + a^2 \\ &= 1 + a^2 + b^2 + c^2 \\ &= 2 \end{aligned}$$

$(I + P)^{-1}$

$$= \frac{1}{2}\begin{pmatrix} \begin{vmatrix} 1 & -a \\ a & 1 \end{vmatrix} & -\begin{vmatrix} c & -a \\ -b & 1 \end{vmatrix} & \begin{vmatrix} c & 1 \\ -b & a \end{vmatrix} \\ -\begin{vmatrix} -c & b \\ a & 1 \end{vmatrix} & \begin{vmatrix} 1 & b \\ -b & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -c \\ -b & a \end{vmatrix} \\ \begin{vmatrix} -c & b \\ 1 & -a \end{vmatrix} & -\begin{vmatrix} 1 & b \\ c & -a \end{vmatrix} & \begin{vmatrix} 1 & -c \\ c & 1 \end{vmatrix} \end{pmatrix}^T \qquad \text{1M}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + a^2 & ab + c & ac - b \\ ab - c & 1 + b^2 & bc + a \\ ac + b & bc - a & 1 + c^2 \end{pmatrix} \qquad \text{1A}$$

$\frac{1}{2}(I - P + QQ^T)$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}\right] \qquad \text{1M}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + a^2 & ab + c & ac - b \\ ab - c & 1 + b^2 & bc + a \\ ac + b & -a + bc & 1 + c^2 \end{pmatrix}$$

$$\therefore \quad (I + P)^{-1} = \frac{1}{2}(I - P + QQ^T) \qquad 1$$

$$\begin{aligned} \frac{1}{2}(I - P + QQ^T) &= (I + P)^{-1} \\ \frac{1}{2}(I - P + QQ^T)(I + P) &= I \\ \frac{1}{2}[(I - P)(I + P) + QQ^T(I + P)] &= I \qquad \text{1M} \\ \frac{1}{2}(I^2 - P^2 + QQ^T + QQ^T P) &= I \\ I - P^2 + QQ^T &= 2I \qquad \text{1M} \\ QQ^T &= I + P^2 \qquad 1 \end{aligned}$$

(10)

(b) Let $P = \frac{1}{\sqrt{6}}M = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}$. 1M

Take $a = \frac{1}{\sqrt{6}}$, $b = \frac{1}{\sqrt{2}}$ and $c = \frac{1}{\sqrt{3}}$, then $a^2+b^2+c^2=1$.

By (a),

$$I+P^2=QQ^T$$
$$(I+P^2)P=QQ^TP$$
$$P+P^3=0$$

Similarly, $P^3+P^5=0, P^5+P^7=0, \dots, P^{2015}+P^{2017}=0$. 1M

$$\begin{aligned}
&M+\frac{1}{6}M^3+\frac{1}{6^2}M^5+\cdots+\frac{1}{6^{1008}}M^{2017}\\
&=(\sqrt{6}P)+\frac{1}{6}(\sqrt{6}P)^3+\frac{1}{6^2}(\sqrt{6}P)^5+\cdots+\frac{1}{6^{1008}}(\sqrt{6}P)^{2017}\\
&=\sqrt{6}P+\sqrt{6}P^3+\sqrt{6}P^5+\cdots+\sqrt{6}P^{2017}\\
&=\sqrt{6}[P+(P^3+P^5)+(P^7+P^9)+\cdots+(P^{2015}+P^{2017})] \qquad \text{1M}\\
&=\sqrt{6}P\\
&=\underline{\underline{\begin{pmatrix} 0 & -\sqrt{2} & \sqrt{3} \\ \sqrt{2} & 0 & -1 \\ -\sqrt{3} & 1 & 0 \end{pmatrix}}} \qquad \text{1A}
\end{aligned}$$

(4)

14. (a) $\because$ $AB = CB$

$\therefore$ $\angle BAC = \angle BCA = \theta$ *(base ∠s, isos. Δ)*

In ΔABC, $\angle ABC = \pi - \theta - \theta$ *(∠ sum of Δ)*

$= \pi - 2\theta$ 1A

$\because$ $BD = ED$ *(radii)*

$\therefore$ $\angle DEB = \angle DBE = \pi - 2\theta$ *(base ∠s, isos. Δ)*

In ΔBDE, $\angle BDE = \pi - (\pi - 2\theta) - (\pi - 2\theta)$ *(∠ sum of Δ)*

$= 4\theta - \pi$ 1M

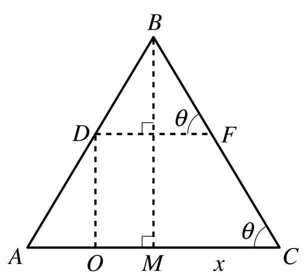
$$\begin{aligned}
\therefore\ S &= \frac{1}{2}(1)^2(4\theta-\pi)-\frac{1}{2}(1)(1)\sin(4\theta-\pi) \qquad \text{1M}\\
&= 2\theta-\frac{\pi}{2}-\frac{1}{2}(-\sin 4\theta)\\
&= 2\theta-\frac{\pi}{2}+\frac{1}{2}\sin 4\theta \qquad \text{1}
\end{aligned}$$

(4)

Analysis

Consider the areas of ΔBDE and the sector BDE.

(b) (i)



Construct a horizontal line DF such that F lies on BC.

$\angle BFD = \theta$

Vertical distance between B and the x-axis $= \sin\theta + 1$ 1M

Consider ΔBCM.

$BM = \sin\theta + 1$

$\tan\theta = \dfrac{\sin\theta + 1}{x}$ 1M

$x = \cos\theta + \cot\theta$ 1

Smart Tips

Vertical distance between B and the x-axis
= Vertical distance between B and $D + OD$

(ii) $x = \cos\theta + \cot\theta$

Differentiating both sides with respect to time t,

$\dfrac{dx}{dt} = -\sin\theta\dfrac{d\theta}{dt} - \csc^2\theta\dfrac{d\theta}{dt}$ 1M

Since $\dfrac{dx}{dt} = -1$,

$-1 = -\sin\theta\dfrac{d\theta}{dt} - \csc^2\theta\dfrac{d\theta}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{1}{\sin\theta + \csc^2\theta}$

Smart Tips

Since C is moving towards O, the sign of $\dfrac{dx}{dt}$ is negative.

When ΔABC is equilateral, $\theta = \dfrac{\pi}{3}$. We have

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{1}{\sin\left(\frac{\pi}{3}\right) + \csc^2\left(\frac{\pi}{3}\right)} \\ &= \frac{1}{\frac{\sqrt{3}}{2} + \left(\frac{2}{\sqrt{3}}\right)^2} \\ &= \frac{6}{3\sqrt{3} + 8} \\ &= \frac{48 - 18\sqrt{3}}{37}\end{aligned}$$ 1A

$S = 2\theta - \dfrac{\pi}{2} + \dfrac{1}{2}\sin 4\theta$

Differentiating both sides with respect to time t,

$\dfrac{dS}{dt} = 2\dfrac{d\theta}{dt} + 2\cos 4\theta\dfrac{d\theta}{dt}$

$= 2(1 + \cos 4\theta)\dfrac{d\theta}{dt}$ 1A

When $\theta = \dfrac{\pi}{3}$,

$\dfrac{dS}{dt} = 2\left[1 + \cos\left(4 \times \dfrac{\pi}{3}\right)\right]\left(\dfrac{48 - 18\sqrt{3}}{37}\right)$ 1M

$= \dfrac{48 - 18\sqrt{3}}{37}$

$\therefore$ The rate of increase of S is $\dfrac{48 - 18\sqrt{3}}{37}$ square units per second. 1A

(8)