HKDSE MATH M2 2013

1. **HKDSE Math M2 2013 Q1**

   Find $\frac{d}{dx}(\sin 2x)$ from first principles.

   (4 marks)

2. **HKDSE Math M2 2013 Q2**

   Suppose the coefficient of x and x^2 in the expansion of $(1+ax)^n$ are -20 and 180 respectively. Find the values of a and n.

   (4 marks)

3. **HKDSE Math M2 2013 Q3**

   Prove, by mathematical induction, that for all positive integers n,

   $$1+\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1))}=\frac{4n+1}{3n+1}.$$

   (5 marks)

4. **HKDSE Math M2 2013 Q4**

   The slope at any point (x,y) of a curve is given by $\frac{dy}{dx}=e^x-1$. It is given that the curve passes through the point $(1,e)$.

   (a) Find the equation of the curve.

   (b) Find the equation of tangent to the curve at the point where the curve cuts the y-axis.

   (5 marks)

5. **HKDSE Math M2 2013 Q5**

   Consider a continuous function $f(x)=\frac{3-3x^2}{3+x^2}$. It is given that

x	$x<-1$	-1	$-1<x<0$	0	$0<x<1$	1	$x>1$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

   ('$+$' and '$-$' denote 'positive value' and 'negative value' respectively.)

   (a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.

   (b) Find the asymptote(s) of the graph of $y=f(x)$.

   (c) Sketch the graph of $y=f(x)$.

   (6 marks)

6. **HKDSE Math M2 2013 Q6**

Figure 1 shows the shaded region with boundaries $C: y = \dfrac{-x^2}{2} + 2x + 4$, $L_1: y = 4$ and $L_2: x = 5$. It is given that C intersects L_1 at $(0, 4)$ and $(4, 4)$.

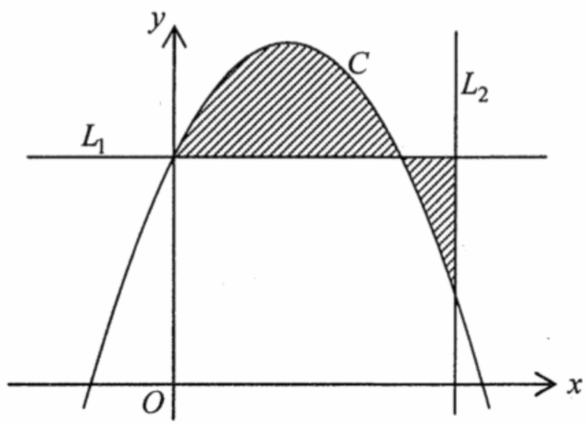

Figure 1

(a) Find the area of the shaded region.

(b) Find the volume of solid of revolution when the shaded region is revolved about L_1.

(6 marks)

7. **HKDSE Math M2 2013 Q7**

(a) Prove the identity $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$.

(b) Using (a), prove the identity $\tan y = \dfrac{\sin 8y \cos 4y \cos 2y}{(1 + \cos 8y)(1 + \cos 4y)(1 + \cos 2y)}$.

(5 marks)

8. **HKDSE Math M2 2013 Q8**

Let M be the matrix $\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}$, where $k \neq 0$.

(a) Find M^{-1}.

(b) If $M \begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find the value of k.

(5 marks)

9. **HKDSE Math M2 2013 Q9**

Consider the following system of linear equations in x, y and z

$$(E) \begin{cases} x - ay + z = 2 \\ 2x + (1-2a)y + (2-b)z = a+4 \\ 3x + (1-3a)y + (3-ab)z = 4 \end{cases}, \text{ where } a \text{ and } b \text{ are real numbers.}$$

It is given that (E) has infinitely many solutions.

(a) Find the values of a and b.

(b) Solve (E).

(5 marks)

10. **HKDSE Math M2 2013 Q10**

Let $\overrightarrow{OA} = 2\mathbf{i}$ and $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN : NA = k : 1$. BM intersects ON at P (see Figure 2).

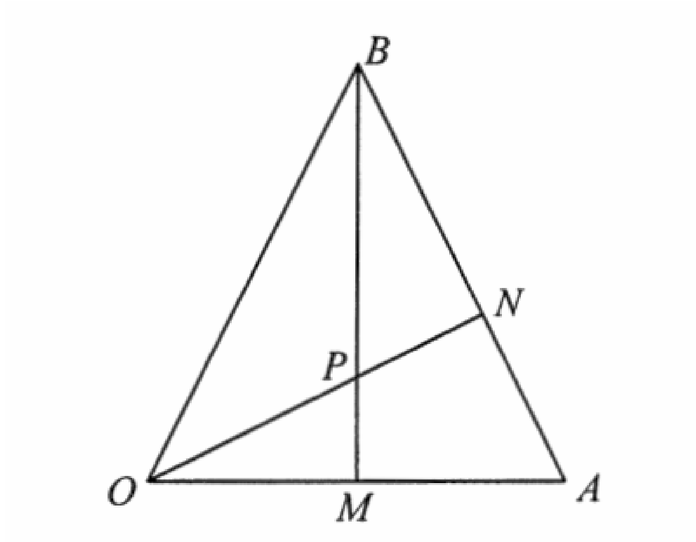

Figure 2

(a) Express $\overrightarrow{ON}$ in terms of k.

(b) If A, N, P and M are concyclic, find the value of k.

(5 marks)

11. **HKDSE Math M2 2013 Q11**

(a) Let $0 < \theta < \frac{\pi}{2}$. By finding $\frac{d}{d\theta}\ln(\sec\theta + \tan\theta)$, or otherwise, show that $\int \sec\theta \, d\theta = \ln(\sec\theta + \tan\theta) + C$, where C is any constant.
(2 marks)

(b) (i) Using (a) and a suitable substitution, show that $\int \frac{du}{\sqrt{u^2-1}} = \ln(u + \sqrt{u^2-1}) + C$ for $u > 1$.

(ii) Using (b)(i), show that $\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\, dx = \ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$.

(5 marks)

(c) Let $t = \tan\phi$. Show that $\frac{d\phi}{dt} = \frac{1}{1+t^2}$.

Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1+2\cos^2\phi}}\, d\phi$.

(5 marks)

12. **HKDSE Math M2 2013 Q12**

In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m. The width of the river is always 30 m. In the beginning, Mike stands at the starting point P in the opposite bank which is 30 m from A. Mike's wife, situated at A, is watching him running along the bank for x m at a constant speed of 7 m s^{-1} to point Q and then swimming at a constant speed of 1.4 m s^{-1} along a straight path to reach B.

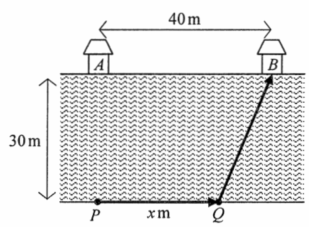

Figure 3

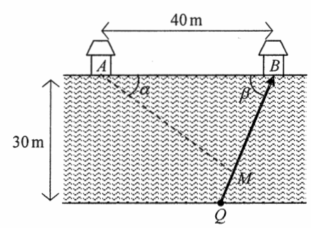

Figure 4

(a) Let T seconds be the time that Mike travels from P to B.

(i) Express T in terms of x.

(ii) When T is minimum, show that x satisfies the equation $2x^2 - 160x + 3125 = 0$.

Hence show that $QB = \dfrac{25\sqrt{6}}{2}$ m.

(6 marks)

(b) In Figure 4, Mike is swimming from Q to B with QB equal to the value mentioned in (a)(ii). Let $\angle MAB = \alpha$ and $\angle ABM = \beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that $MB = \dfrac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}}$.

(ii) Find the rate of change of α when $\alpha = 0.2$ radian. Correct your answer to 4 decimal places.

(7 marks)

13. **HKDSE Math M2 2013 Q13**

For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\text{tr}(M) = a + d$.

Let A and B be 2×2 matrices such that $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\text{tr}(MN) = \text{tr}(NM)$.

(ii) Show that $\text{tr}(A) = 4$.

(iii) Find the value of $|A|$.

(6 marks)

(b) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

(i) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - \mathrm{tr}(C) \cdot \lambda + |C| = 0$.

(5 marks)

(c) Find the two values of λ such that $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$.

(2 marks)

14. **HKDSE Math M2 2013 Q14**

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = 90°$. P is a variable point such that $\overrightarrow{AP} \cdot \overrightarrow{BP} + \overrightarrow{BP} \cdot \overrightarrow{CP} + \overrightarrow{CP} \cdot \overrightarrow{AP} = 0$. Let D be the fixed point such that $\overrightarrow{OD} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OD} = \mathbf{d}$.

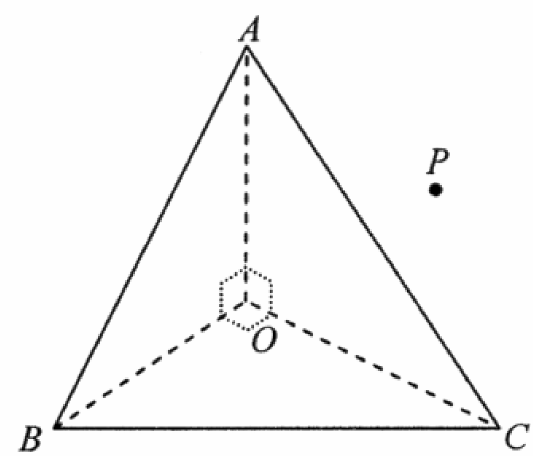

Figure 5

(a) (i) Show that $\overrightarrow{AP} \cdot \overrightarrow{BP} = \mathbf{p} \cdot \mathbf{p} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p}$.

(ii) Using (a)(i), show that $\mathbf{p} \cdot \mathbf{p} = 2\mathbf{p} \cdot \mathbf{d}$.

(iii) Show that $|\mathbf{p} - \mathbf{d}| = |\mathbf{d}|$.

Hence show that P lies on the sphere centred at D with fixed radius.

(8 marks)

(b) (i) Alice claims that O lies on the sphere mentioned in (a)(iii). Do you agree? Explain your answer.

(ii) Suppose P_1, P_2 and P_3 are three distinct points on the sphere in (a)(iii) such that $\overrightarrow{DP_1} \times \overrightarrow{DP_2} = \overrightarrow{DP_2} \times \overrightarrow{DP_3}$. Alice claims that the radius of the circle passing through P_1, P_2 and P_3 is OD. Do you agree? Explain your answer.

(4 marks)