HKDSE MATH M2 2020

1. **HKDSE Math M2 2020 Q1**

   (a) Expand $(1-x)^4$.

   (b) Find the constant k such that the coefficient of x^2 in the expansion of $(1+kx)^9(1-x)^4$ is -3.

   (4 marks)

2. **HKDSE Math M2 2020 Q2**
   Define $f(x) = \dfrac{x}{\sqrt{2+x}}$, for all $x > -2$. Find $f'(2)$ from first principles.
   (4 marks)

3. **HKDSE Math M2 2020 Q3**

   (a) Let x be an angle which is not a multiple of $30°$. Prove that

   (i) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$,

   (ii) $\tan x \tan(60° - x)\tan(60° + x) = \tan 3x$.

   (b) Using (a)(ii), prove that $\tan 55° \tan 65° \tan 75° = \tan 85°$.

   (6 marks)

4. **HKDSE Math M2 2020 Q4**

   (a) Find $\displaystyle\int \sin^2\theta \, d\theta$.

   (b) Define $f(x) = 4x(1-x^2)^{\frac{1}{4}}$ for all $x \in [0,1]$. Denote the graph of $y = f(x)$ by G. Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

   (6 marks)

5. **HKDSE Math M2 2020 Q5**

   (a) Using mathematical induction, prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all positive integers n.

   (b) Using (a), evaluate $\displaystyle\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$.

   (7 marks)

6. **HKDSE Math M2 2020 Q6**
   Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the region by O. Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

(a) Define $S = u^2 + v^2$. Does S increase at a constant rate? Explain your answer.

(b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q. Find the rate of change of the area of $\triangle OPQ$ when $u = 2$.

(7 marks)

7. **HKDSE Math M2 2020 Q7**

Let $f(x)$ be a continuous function defined on $\mathbb{R}$. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $(1, 2)$ and $f'(x) = -2x + 8$ for all $x \in \mathbb{R}$.

(a) Find the equation of Γ.

(b) Let L be a tangent to Γ such that L passes through the point $(5, 14)$ and the slope of L is negative. Denote the point of contact of Γ and L by P. Find

(i) the coordinates of P,

(ii) the equation of the normal to Γ at P.

(8 marks)

8. **HKDSE Math M2 2020 Q8**

Define $P = \begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $M = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix}$ such that $|M| = 1$ and $PM = MQ$, where a, b and c are real numbers.

(a) Find a, b and c.

(b) Define $R = \begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b)(i), prove that $(\alpha P + \beta R)^{99} = \alpha^{99}P + \beta^{99}R$ for any real numbers α and β.

(8 marks)

9. **HKDSE Math M2 2020 Q9**

Let $f(x) = \dfrac{(x+4)^3}{(x-4)^2}$ for all real numbers $x \neq 4$. Denote the graph of $y = f(x)$ by H.

(a) Find the asymptote(s) of H.
(3 marks)

(b) Find $f''(x)$.
(2 marks)

(c) Someone claims that there are two turning points of H. Do you agree? Explain your answer.
(2 marks)

(d) Find the point(s) of inflexion of H.
(2 marks)

(e) Find the area of the region bounded by H, the x-axis and the y-axis.
(3 marks)

10. **HKDSE Math M2 2020 Q10**

(a) Using integration by substitution, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right) dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\, dx$.

(3 marks)

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\, dx$.

(3 marks)

(c) (i) Using $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot\dfrac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \dfrac{x\csc^2 x}{\cot x - 1}\, dx = \dfrac{\pi}{8}\ln(2+\sqrt{3})$

(7 marks)

11. **HKDSE Math M2 2020 Q11**

(a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = k \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h, k \in \mathbb{R}.$$

(i) Assume that (E) has a unique solution.

(1) Prove that $h \neq -3$.

(2) Solve (E).

(ii) Assume that $h = -3$ and (E) is consistent.

(1) Prove that $k = -2$.

(2) Solve (E).

(9 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = -2 \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h \in \mathbb{R}.$$

Someone claims that there are at least two values of h such that (F) has a real solution (x, y, z) satisfying $3x^2 + 4y^2 - 7z^2 = 1$. Do you agree? Explain your answer.
(4 marks)

12. **HKDSE Math M2 2020 Q12**
Let $\overrightarrow{OP} = \mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i}-7\mathbf{j}-4\mathbf{k}$, where O is the origin. R is a point lying on PQ such that $PR : RQ = 1 : 3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$.

(2 marks)

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$.

(2 marks)

(c) Let N be a point such that $\overrightarrow{ON} = \lambda(\overrightarrow{OP} \times \overrightarrow{OR})$, where λ is a real number.

  (i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

  (ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i} + \mu\mathbf{j} - 10\mathbf{k}$.

    (1) Find λ and μ.

    (2) Denote the angle between $\triangle OPQ$ and $\triangle NPQ$ by θ. Find $\tan\theta$.

(8 marks)