Marking Scheme

Module 2 (Algebra and Calculus)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with $\boxed{\text{rectangles}}$. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

Solution	Marks	Remarks
1. $\dfrac{d}{d\theta}\sec 6\theta$		
$=\lim\limits_{h\to 0}\dfrac{\sec 6(\theta+h)-\sec 6\theta}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{\cos 6\theta-\cos 6(\theta+h)}{h\cos 6(\theta+h)\cos 6\theta}$		
$=\lim\limits_{h\to 0}\dfrac{2\sin(6\theta+3h)\sin 3h}{h\cos 6(\theta+h)\cos 6\theta}$	1M	
$=6\left(\lim\limits_{h\to 0}\dfrac{\sin 3h}{3h}\right)\left(\lim\limits_{h\to 0}\dfrac{\sin(6\theta+3h)}{\cos 6(\theta+h)\cos 6\theta}\right)$	1M	
$=6(1)\left(\dfrac{\sin 6\theta}{\cos^2 6\theta}\right)$	1M	withhold 1M if the step is skipped
$=6\sec 6\theta\tan 6\theta$	1A	
	----------(5)	
2. Note that $(1+ax)^8=1+C_1^8ax+C_2^8(ax)^2+\cdots+(ax)^8$ and	1M	
$(b+x)^9=b^9+C_1^9b^8x+C_2^9b^7x^2+\cdots+C_7^9b^2x^7+C_8^9bx^8+x^9$.	1M	
Also note that $\lambda_2:\mu_7=7:4$ and $\lambda_1+\mu_8+6=0$.		
Therefore, we have $\dfrac{C_2^8a^2}{C_7^9b^2}=\dfrac{7}{4}$ and $8a+9b+6=0$.	1M	for either one
So, we have $4a^2=9b^2$ and $8a+9b+6=0$.		
Hence, we have $4a^2-9\left(\dfrac{-8a-6}{9}\right)^2=0$.		
Simplifying, we have $7a^2+24a+9=0$.	1M	for $pa^2+qa+r=0$
Thus, we have $a=-3$ or $a=\dfrac{-3}{7}$.	1A	for both correct
	----------(5)	

Solution	Marks	Remarks
3. (a) $\overrightarrow{OP}$		
$=\frac{2}{2+3}\mathbf{a}+\frac{3}{2+3}\mathbf{b}$		
$=\frac{2}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}$	1A	
(b) (i) $\mathbf{a}\cdot\mathbf{b}$		
$=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$	1M	
$=(45)(20)\left(\frac{1}{4}\right)$		
$=225$	1A	
(ii) $\left\lvert\overrightarrow{OP}\right\rvert^2$		
$=\left(\frac{2}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}\right)\cdot\left(\frac{2}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}\right)$		
$=\frac{4}{25}\lvert\mathbf{a}\rvert^2+2\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\mathbf{a}\cdot\mathbf{b}+\frac{9}{25}\lvert\mathbf{b}\rvert^2$		
$=324+108+144$	1M	for using (b)(i)
$=576$		
$\left\lvert\overrightarrow{OP}\right\rvert$		
$=\sqrt{576}$		
$=24$	1A	
	----------(5)	
4. (a) $\int x^2e^{-x}\mathrm{d}x$		
$=-\int x^2\mathrm{d}e^{-x}$		
$=-x^2e^{-x}+\int e^{-x}\mathrm{d}x^2$	1M	for integration by parts
$=-x^2e^{-x}+2\int xe^{-x}\mathrm{d}x$	1A	
$=-x^2e^{-x}-2\int x\mathrm{d}e^{-x}$		
$=-x^2e^{-x}-2\left(xe^{-x}-\int e^{-x}\mathrm{d}x\right)$		
$=-x^2e^{-x}-2xe^{-x}-2e^{-x}+\text{constant}$	1A	
$=-e^{-x}(x^2+2x+2)+\text{constant}$		
(b) The required area		
$=\int_0^6 x^2e^{-x}\mathrm{d}x$	1M	
$=\left[-e^{-x}(x^2+2x+2)\right]_0^6$ (by (a))	1M	for using the result of (a)
$=2-\frac{50}{e^6}$	1A	
	----------(6)	

Solution	Marks	Remarks
5. (a) (i) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & -11 \\ 2 & 3 & h \end{vmatrix} \neq 0$	1M	
$8h-44-9+16+33-6h \neq 0$		
$2h-4 \neq 0$		
$h \neq 2$	1A	
$h<2$ or $h>2$		
(ii) z		
$= \dfrac{\begin{vmatrix} 1 & 2 & 11 \\ 3 & 8 & 49 \\ 2 & 3 & k \end{vmatrix}}{2h-4}$	1M	
$= \dfrac{k-14}{h-2}$	1A	
(b) When $h=2$, the augmented matrix of (E) is		
$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & 2 & k \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right)$.	1M	
Since (E) has infinitely many solutions, we have $h=2$ and $k=14$.		
Thus, the solution set of (E) is $\{(-7t-5, 4t+8, t) : t \in \mathbf{R}\}$.	1A	
	----------(6)	

Solution	Marks	Remarks
6. (a) Let r cm be the radius of the water surface in the container.		
Since $\frac{r}{h}=\frac{15}{20}$, we have $\frac{r}{h}=\frac{3}{4}$.	1M	
So, we have $r=\frac{3h}{4}$.		
A		
$=\pi\left(\frac{3h}{4}\right)\sqrt{h^2+\left(\frac{3h}{4}\right)^2}$	1M	
$=\pi\left(\frac{3h}{4}\right)\sqrt{\frac{25h^2}{16}}$		
$=\frac{15}{16}\pi h^2$	1	
(b) Let d cm be the depth of water when the volume of water in the container is $96\pi\ \text{cm}^3$.		
Note that $\frac{\pi d}{3}\left(\frac{3d}{4}\right)^2=96\pi$.	1M	
So, we have $d=8$.	1A	
By (a), we have $A=\frac{15}{16}\pi h^2$.		
At time t s , we have $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{15}{8}\pi h\frac{\mathrm{d}h}{\mathrm{d}t}$.	1M	
Also note that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{3}{\pi}$.		
Therefore, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=\frac{15}{8}\pi(8)\left(\frac{3}{\pi}\right)$.		
Hence, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=45$.	1A	
Thus, the required rate of change is $45\ \text{cm}^2/\text{s}$.		
	----------(7)	

Solution	Marks	Remarks
7. (a) $\sin 3x$		
$= \sin(x+2x)$		
$= \sin x\cos 2x + \cos x \sin 2x$	1M	
$= \sin x(\cos^2 x - \sin^2 x) + 2\sin x\cos^2 x$		
$= \sin x(1-2\sin^2 x) + 2\sin x(1-\sin^2 x)$		
$= 3\sin x - 4\sin^3 x$	1	
(b) (i) $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$= \dfrac{\sin\left(3x-\frac{3\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$= \dfrac{\sin 3x\cos\frac{3\pi}{4} - \cos 3x\sin\frac{3\pi}{4}}{\sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4}}$	1M	
$= \dfrac{\frac{-1}{\sqrt{2}}(\sin 3x + \cos 3x)}{\frac{1}{\sqrt{2}}(\sin x - \cos x)}$		
$= \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$	1	
(ii) $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$		
$\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)} = 2$ (by (b)(i))	1M	for using (b)(i)
Note that $\sin\left(x-\frac{\pi}{4}\right) \neq 0$.		
$3 - 4\sin^2\left(x-\frac{\pi}{4}\right) = 2$ (by (a))	1M	for using (a)
$1 - 4\sin^2\left(x-\frac{\pi}{4}\right) = 0$		
$\left(1-2\sin\left(x-\frac{\pi}{4}\right)\right)\left(1+2\sin\left(x-\frac{\pi}{4}\right)\right) = 0$	1M	
Since $\frac{\pi}{4} < x < \frac{\pi}{2}$, we have $\sin\left(x-\frac{\pi}{4}\right) = \frac{1}{2}$.		
Therefore, we have $x - \frac{\pi}{4} = \frac{\pi}{6}$.		
Thus, we have $x = \frac{5\pi}{12}$.	1A	
	----------(8)	

Solution	Marks	Remarks
8. (a) The slope of the tangent to Γ at P		
$=\mathrm{f}'(e^3)$		
$=\dfrac{1}{e^3}\ln(e^3)^2$	1M	
$=\dfrac{6}{e^3}$		
The equation of the tangent to Γ at P is		
$y-7=\dfrac{6}{e^3}(x-e^3)$		
$6x-e^3y+e^3=0$	1A	
(b) $\mathrm{f}(x)$		
$=\displaystyle\int\frac{1}{x}\ln x^2\,\mathrm{d}x$		
$=2\displaystyle\int \ln x\,\mathrm{d}\ln x$	1M	
$=(\ln x)^2+C$		
Since Γ passes through P, we have $7=(\ln e^3)^2+C$.	1M	
Solving, we have $C=-2$.		
Thus, the equation of Γ is $y=(\ln x)^2-2$.	1A	
(c) Note that $\mathrm{f}''(x)=\dfrac{2-2\ln x}{x^2}$.	1A	
Therefore, we have $\mathrm{f}''(x)=0 \iff x=e$.		
[table below]	1M	
Thus, the point of inflexion of Γ is $(e,-1)$.	1A	
	----------(8)	

x	$(0,e)$	e	(e,∞)
$\mathrm{f}''(x)$	+	0	-

Solution	Marks	Remarks
9. (a) The equation of the vertical asymptote is $x+4=0$.	1A	
Note that $f(x)=x-9+\dfrac{36}{x+4}$.	1M	
Thus, the equation of the oblique asymptote is $y=x-9$.	1A	
	----------(3)	
(b) $f'(x)$		
$=\dfrac{d}{dx}\left(x-9+\dfrac{36}{x+4}\right)$		
$=1+36(-1)(x+4)^{-2}$	1M	
$=1-\dfrac{36}{(x+4)^2}$	1A	
$f'(x)$		
$=\dfrac{d}{dx}\left(\dfrac{x^2-5x}{x+4}\right)$		
$=\dfrac{(x+4)(2x-5)-(x^2-5x)}{(x+4)^2}$	1M	
$=\dfrac{x^2+8x-20}{(x+4)^2}$	1A	
	----------(2)	
(c) Note that $f'(x)=\dfrac{(x+10)(x-2)}{(x+4)^2}$.		
So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$.	1A	

x	$(-\infty,-10)$	-10	$(-10,-4)$	$(-4,2)$	2	$(2,\infty)$
$f'(x)$	+	0	−	−	0	+
$f(x)$	↗	−25	↘	↘	−1	↗

(Marks: 1M)

Solution	Marks	Remarks
Thus, the maximum point and the minimum point of G are $(-10,-25)$	1A	
and $(2,-1)$ respectively.	1A	
Note that $f'(x)=\dfrac{(x+10)(x-2)}{(x+4)^2}$ and $f''(x)=\dfrac{72}{(x+4)^3}$.		
So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$.	1A	
Also note that $f''(-10)=\dfrac{-1}{3}<0$ and $f''(2)=\dfrac{1}{3}>0$.	1M	
Further note that $f(-10)=-25$ and $f(2)=-1$.		
Thus, the maximum point and the minimum point of G are $(-10,-25)$	1A	
and $(2,-1)$ respectively.	1A	
	----------(4)	

Solution	Marks	Remarks
(d) The required volume		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x+4} \right)^2 dx$	1M	
$= \pi \int_0^5 \left(x - 9 + \frac{36}{x+4} \right)^2 dx$		
$= \pi \int_0^5 \left(x^2 - 18x + 81 + \frac{72(x-9)}{x+4} + \frac{1\,296}{(x+4)^2} \right) dx$		
$= \pi \int_0^5 \left(x^2 - 18x + 153 - \frac{936}{x+4} + \frac{1\,296}{(x+4)^2} \right) dx$	1M	
$= \pi \left[\frac{x^3}{3} - 9x^2 + 153x - 936 \ln\lvert x+4 \rvert - \frac{1\,296}{x+4} \right]_0^5$	1M	
$= \left(\frac{2\,285}{3} - 1\,872 \ln\left(\frac{3}{2}\right) \right) \pi$	1A	
The required volume		
$= \pi \int_0^5 \left(\frac{x^2 - 5x}{x+4} \right)^2 dx$	1M	
$= \pi \int_4^9 \frac{(x-4)^2 (x-9)^2}{x^2} dx$		
$= \pi \int_4^9 \left(\frac{x^4 - 26x^3 + 241x^2 - 936x + 1\,296}{x^2} \right) dx$	1M	
$= \pi \int_4^9 \left(x^2 - 26x + 241 - \frac{936}{x} + \frac{1\,296}{x^2} \right) dx$		
$= \pi \left[\frac{x^3}{3} - 13x^2 + 241x - 936 \ln\lvert x \rvert - \frac{1\,296}{x} \right]_4^9$	1M	
$= \left(\frac{2\,285}{3} - 1\,872 \ln\left(\frac{3}{2}\right) \right) \pi$	1A	
	----------(4)	

Solution	Marks	Remarks
10. (a) Note that $\overrightarrow{AB} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\overrightarrow{AC} = 6\mathbf{i}-6\mathbf{j}$.		
$\overrightarrow{AE}$		
$= \dfrac{1}{1+r}\overrightarrow{AC} + \dfrac{r}{1+r}\overrightarrow{AB}$	1M	any one
$= \dfrac{2r+6}{r+1}\mathbf{i} + \dfrac{r-6}{r+1}\mathbf{j} + \dfrac{r}{r+1}\mathbf{k}$		
Also note that $\overrightarrow{AE} = \dfrac{1}{11}\overrightarrow{AF} + \dfrac{10}{11}\overrightarrow{AD}$ and $\overrightarrow{AC} = 2\overrightarrow{AD}$.		
$\overrightarrow{AF}$		
$= 11\overrightarrow{AE} - 5\overrightarrow{AC}$		
$= \dfrac{-8r+36}{r+1}\mathbf{i} + \dfrac{41r-36}{r+1}\mathbf{j} + \dfrac{11r}{r+1}\mathbf{k}$	1A	for both
Since A , B and F are collinear, we have $\dfrac{2}{-8r+36} = \dfrac{1}{41r-36} = \dfrac{1}{11r}$.	1M	
Solving, we have $r = \dfrac{6}{5}$.	1A	1.2
	----------(4)	
(b) (i) Note that $\overrightarrow{AD} = \dfrac{1}{2}\overrightarrow{AC} = 3\mathbf{i}-3\mathbf{j}$.		
By (a), we have $\overrightarrow{AE} = \dfrac{1}{11}(42\mathbf{i}-24\mathbf{j}+6\mathbf{k})$.	1M	for using (a)
$\overrightarrow{AD}\cdot\overrightarrow{DE}$		
$= \overrightarrow{AD}\cdot(\overrightarrow{AE}-\overrightarrow{AD})$		
$= (3\mathbf{i}-3\mathbf{j})\cdot\left(\dfrac{1}{11}(9\mathbf{i}+9\mathbf{j}+6\mathbf{k})\right)$		
$= 0$	1A	
(ii) $\overrightarrow{AB}\cdot\overrightarrow{BC}$	1M	
$= \overrightarrow{AB}\cdot(\overrightarrow{AC}-\overrightarrow{AB})$		
$= (2\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}-7\mathbf{j}-\mathbf{k})$		
$= 0$		
Therefore, we have $\angle ABC = 90° = \angle ADE$.	1M	
So, we have $\angle CBF = 90° = \angle CDF$.		
Thus, B , D , C and F are concyclic.	1A	f.t.
	----------(5)	
(c) Note that $\overrightarrow{AF} = 12\mathbf{i}+6\mathbf{j}+6\mathbf{k}$ and $\overrightarrow{AP} = \mathbf{i}+7\mathbf{j}-2\mathbf{k}$.		
Since $\angle CBF = 90°$, Q is the mid-point of CF .		
Therefore, we have $\overrightarrow{AQ} = \dfrac{1}{2}(\overrightarrow{AC}+\overrightarrow{AF}) = 9\mathbf{i}+3\mathbf{k}$.	1M	
The volume of the tetrahedron $ABPQ$		
$= \dfrac{1}{6}\left\lvert\overrightarrow{AQ}\cdot(\overrightarrow{AB}\times\overrightarrow{AP})\right\rvert$	1M	
$= \dfrac{1}{6}\left\lvert\begin{vmatrix}9 & 0 & 3\\ 2 & 1 & 1\\ 1 & 7 & -2\end{vmatrix}\right\rvert$		
$= 7$	1A	
	----------(3)	

Solution	Marks	Remarks
11. (a) $\displaystyle\int_0^1 \frac{1}{x^2+2x+3}\,\mathrm{d}x$		
$\displaystyle=\int_0^1 \frac{1}{(x+1)^2+2}\,\mathrm{d}x$	1M	
$\displaystyle=\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^1$	1M	
$\displaystyle=\frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$		
$\displaystyle=\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$	1A	
	----------(3)	
(b) (i) $\displaystyle\frac{2\tan\theta}{1+\tan^2\theta}$		
$\displaystyle=\frac{\frac{2\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$		
$=2\sin\theta\cos\theta$		
$=\sin 2\theta$	1	
$\displaystyle\frac{1-\tan^2\theta}{1+\tan^2\theta}$		
$\displaystyle=\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$		
$=\cos^2\theta-\sin^2\theta$		
$=\cos 2\theta$	1	
(ii) Let $t=\tan\theta$. Then, we have $\displaystyle\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{1+t^2}$.	1M	
Note that $\displaystyle\frac{1}{\sin 2\theta+\cos 2\theta+2}=\frac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+2}=\frac{1+t^2}{t^2+2t+3}$.		
$\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$\displaystyle=\int_0^1 \frac{1+t^2}{t^2+2t+3}\left(\frac{1}{1+t^2}\right)\mathrm{d}t$	1M	
$\displaystyle=\int_0^1 \frac{1}{t^2+2t+3}\,\mathrm{d}t$		
$\displaystyle=\int_0^1 \frac{1}{x^2+2x+3}\,\mathrm{d}x$		
$\displaystyle=\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (a))	1M	(a)
	----------(5)	

Solution	Marks	Remarks
(c) Let $y=\frac{\pi}{4}-\theta$. Then, we have $\frac{\mathrm{d}\theta}{\mathrm{d}y}=-1$.	1M	
$\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=-\int_{\frac{\pi}{4}}^{0}\frac{\sin\left(\frac{\pi}{2}-2y\right)+1}{\sin\left(\frac{\pi}{2}-2y\right)+\cos\left(\frac{\pi}{2}-2y\right)+2}\,\mathrm{d}y$		
$=\int_0^{\frac{\pi}{4}}\frac{\cos 2y+1}{\cos 2y+\sin 2y+2}\,\mathrm{d}y$		
$=\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$	1	
	----------(2)	
(d) $\int_0^{\frac{\pi}{4}}\frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=\int_0^{\frac{\pi}{4}}\frac{4(\sin 2\theta+1)+4(\sin 2\theta+1)+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta+4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta+4\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$ (by (c))	1M	for using (c)
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+\cos 2\theta+2}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$	1M	
$=4\int_0^{\frac{\pi}{4}}\mathrm{d}\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	$\pi+$(b)(ii)
Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$ and $J=\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$.		
Note that $I+J=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta=\frac{\pi}{4}$.	1M	
By (c), we have $I=J=\frac{\pi}{8}$.	1M	for using (c)
$\int_0^{\frac{\pi}{4}}\frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=8I+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,\mathrm{d}\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	$\pi+$(b)(ii)
	----------(3)	

Solution	Marks	Remarks
12. (a) A $=\begin{pmatrix}3&1\\0&3\end{pmatrix}$ $=3^1\begin{pmatrix}1&0\\0&1\end{pmatrix}+3^0(1)\begin{pmatrix}0&1\\0&0\end{pmatrix}$		
So, the statement is true for $n=1$.	1	
Assume that $A^k=3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}$, where k is a positive integer.	1M	
A^{k+1} $=A^kA$ $=\left(3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}3&1\\0&3\end{pmatrix}$ (by induction assumption)	1M	for using induction assumption
$=\left(3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\left(3\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)$ $=3^{k+1}I+3^kk\begin{pmatrix}0&1\\0&0\end{pmatrix}+3^k\begin{pmatrix}0&1\\0&0\end{pmatrix}+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}^2$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}0&1\\0&0\end{pmatrix}$		
Therefore, the statement is true for $n=k+1$ if it is true for $n=k$. By mathematical induction, the statement is true for all positive integers n .	1 ----------(4)	
(b) (i) Note that $P^{-1}=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$.	1A	
$P^{-1}BP$ $=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}\begin{pmatrix}5&1\\-4&1\end{pmatrix}\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$ $=\begin{pmatrix}3&1\\0&3\end{pmatrix}$ $=A$	1A	
(ii) By (b)(i), we have $P^{-1}BP=A$. So, we have $(P^{-1}BP)^n=A^n$. Therefore, we have $P^{-1}B^nP=A^n$. Hence, we have $B^n=PA^nP^{-1}$.	1M	
B^n $=\begin{pmatrix}-1&0\\2&-1\end{pmatrix}\left(3^nI+3^{n-1}n\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$	1M	
$=3^nI+3^{n-1}n\begin{pmatrix}-1&0\\2&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$ $=3^nI+3^{n-1}n\begin{pmatrix}2&1\\-4&-2\end{pmatrix}$	1	

Solution	Marks	Remarks
B $=\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$ $=3^1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+3^0(1)\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$		
So, the statement is true for $n=1$.		
Assume that $B^k=3^kI+3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, where k is a positive integer.		
B^{k+1} $=B^kB$		
$=\left(3^kI+3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right)\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$ (by induction assumption)	1M	for using induction assumption
$=\left(3^kI+3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right)\left(3I+\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right)$	1M	
$=3^{k+1}I+3^kk\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}+3^k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}+3^{k-1}k\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}^2$		
$=3^{k+1}I+3^k(k+1)\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		
$=3^{k+1}I+3^k(k+1)\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$		
Therefore, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(iii) $\lvert A^m-B^m\rvert=4m^2$		
$\left\lvert 3^{m-1}m\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}-3^{m-1}m\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\right\rvert=4m^2$		
$(3^{m-1})^2m^2\begin{vmatrix} -2 & 0 \\ 4 & 2 \end{vmatrix}=4m^2$	1M	
$-4m^2(3^{2(m-1)})=4m^2$		
$3^{2(m-1)}=-1$	1M	
Note that $-1<0<3^{2(m-1)}$.		
Thus, there does not exist a positive integer m such that $\lvert A^m-B^m\rvert=4m^2$.	1A	f.t.
	----------(8)	