HKDSE MATH M2 2019

1. **HKDSE Math M2 2019 Q1**
   Let $f(x)=\dfrac{10x}{7+3x^2}$. Prove that $f(1+h)-f(1)=\dfrac{4h-3h^2}{10+6h+3h^2}$. Hence, find $f'(1)$ from first principles.
   (4 marks)

2. **HKDSE Math M2 2019 Q2**
   Let $P(x)=\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbb{R}$. It is given that the coefficient of x^3 in the expansion of $P(x)$ is 160. Find

   (a) λ,

   (b) $P'(0)$.

   (5 marks)

3. **HKDSE Math M2 2019 Q3**
   A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580cm^3 of liquid X. The researcher finds that during the experiment, $\dfrac{dV}{dt}=-2t$, where $V\text{cm}^3$ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

   (a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

   (b) It is given that $V=h^2+24h$, where h cm is the depth of liquid X in the vessel. Find the value of $\dfrac{dh}{dt}$ when $t=18$.

   (6 marks)

4. **HKDSE Math M2 2019 Q4**
   Define $g(x)=\dfrac{\ln x}{\sqrt{x}}$ for all $x\in(0,99)$. Denote the graph of $y=g(x)$ by G.

   (a) Prove that G has only one maximum point.

   (b) Let R be the region bounded by G, the x-axis and the vertical line passing through the maximum point of G. Find the volume of the solid of revolution generated by revolving R about the x-axis.

   (6 marks)

5. **HKDSE Math M2 2019 Q5**

(a) Using mathematical induction, prove that $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{(n+1)}{n(2n+1)}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(7 marks)

6. **HKDSE Math M2 2019 Q6**

Consider the system of linear equations in real variables x, y, z

$$(E) : \begin{cases} x & - & 2y & - & 2z & = & \beta \\ 5x & + & \alpha y & + & \alpha z & = & 5\beta \\ 7x & + & (\alpha-3)y & + & (2\alpha+1)z & = & 8\beta \end{cases}, \text{ where } \alpha, \beta \in \mathbb{R}.$$

(a) Assume that (E) has a unique solution.

  (i) Find the range of values of α.

  (ii) Express y in terms of α and β.

(b) Assume that $\alpha = -4$. If (E) is inconsistent, find the range of values of β.

(7 marks)

7. **HKDSE Math M2 2019 Q7**

(a) Using integration by parts, find $\int e^x \sin \pi x \, dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x \, dx$.

(7 marks)

8. **HKDSE Math M2 2019 Q8**

Let $h(x)$ be a continuous function defined on $\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \frac{2x^2 - 7x + 8}{x}$ for all $x > 0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = h(x)$ by H. It is given that H passes through the point $(1, 3)$. Find

  (i) the equation of H,

  (ii) the point(s) of inflexion of H.

(8 marks)

9. **HKDSE Math M2 2019 Q9**

Consider the curve $\Gamma : y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$. Denote the tangent of Γ at $x = 3$ by L.

(a) Find the equation of L.
(3 marks)

(b) Let C be the curve $y = \sqrt{4 - x^2}$, where $0 < x < 2$. It is given that L is a tangent to C. Find

(i) the point(s) of contact of L and C;

(ii) the point(s) of intersection of C and Γ;

(iii) the area of region bounded by L, C and Γ.

(9 marks)

10. **HKDSE Math M2 2019 Q10**

(a) Let $0 \leq x \leq \frac{\pi}{4}$. Prove that $\frac{1}{2 + \cos 2x} = \frac{\sec^2 x}{2 + \sec^2 x}$.
(1 mark)

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2 + \cos 2x}\, dx$.
(3 marks)

(c) Let $f(x)$ be a continuous function defined on $\mathbb{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
Prove that $\int_{-a}^{a} f(x) \ln(1 + e^x)\, dx = \int_0^a x f(x)\, dx$ for any $a \in \mathbb{R}$.
(4 marks)

(d) Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2 + \cos 2x)^2} \ln(1 + e^x)\, dx$.
(5 marks)

11. **HKDSE Math M2 2019 Q11**
Let $M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$.
(3 marks)

(b) Prove that $6M^n = (1 - (-5)^n)M + (5 + (-5)^n)I)$ for all positive integers n.
(4 marks)

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1} = A + \frac{1}{(-5)^n} B$ for all positive integers n? Explain your answer.
(5 marks)

12. **HKDSE Math M2 2019 Q12**
Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $|\overrightarrow{AC}| = |\overrightarrow{BC}|$.

(a) Find t.
(3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.
(2 marks)

(c) Find the volume of the pyramid $OABC$.
(2 marks)

(d) Denote the plane which contains A, B and C by Π. It is given that P, Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OQ} = r\mathbf{k}$. Let D be the projection of O on Π.

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(ii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D, E and O. Explain your answer.

(6 marks)