Arithmetic and Geometric Sequences

and their Summation

等差數列和等比數列及其求和法

Exercises(練習)

1 (a) Find the general term of the arithmetic sequence 12, 7, 2, –3, ...

(b) If the *k*th term of the sequence is –38, find *k*.

(a) 求等差數列 12, 7, 2, –3, ... 的通項。

(b) 若該數列的第 *k* 項是–38，求 *k* 的值。

(a) Let *a* and *d* be the first term and the common difference of the arithmetic sequence respectively.

$\because\ a = 12$

and $d = 7 - 12 = -5$

$$\begin{aligned}\therefore\ T(n) &= a + (n-1)d \\ &= 12 + (n-1)(-5) \\ &= \underline{\underline{17 - 5n}}\end{aligned}$$

(b) $\because\ T(k) = -38$

$$\begin{aligned}\therefore\ 17 - 5k &= -38 \\ -5k &= -55 \\ k &= \underline{\underline{11}}\end{aligned}$$

2. Consider the following sequence

1, 8, 27, 64, ...

(a) Determine the general term *T*(*n*) of the sequence.

(b) Write down the 5th term and the 7th term of the sequence.

考慮以下的數列：

1, 8, 27, 64, ...

(a) 寫出該數列的通項 *T*(*n*)。

(b) 求該數列的第 5 項和第 7 項。

(a) $\because\ T(1) = 1 = 1^3$

$T(2) = 8 = 2^3$

$T(3) = 27 = 3^3$

$T(4) = 64 = 4^3$

$\therefore\ T(n)=n^3$

(b) $\because\ T(n)=n^3$

$\therefore\ T(5)=5^3=125$

$T(7)=7^3=343$

$\therefore$ The 5th term and 7th term of the sequence are 125 and 343 respectively.

3. Consider the following sequence

log 1, log 4, log 9, log 16, ...

(a) Determine the general term $T(n)$ of the sequence.

(b) Write down the 6th term and the 9th term of the sequence.

考慮以下的數列：

log 1, log 4, log 9, log 16, ...

(a) 寫出該數列的通項 $T(n)$。

(b) 求該數列的第 6 項和第 9 項。

(a) $\because\ T(1)=\log 1=\log 1^2$

$T(2)=\log 4=\log 2^2$

$T(3)=\log 9=\log 3^2$

$T(4)=16=\log 4^2$

$\therefore\ T(n)=\log n^2$

$=2\log n$

(b) $\because\ T(n)=2\log n$

$\therefore$
$T(6)=2\log 6$

$T(9)=2\log 9$

$=2\log 3^2$

$=4\log 3$

$\therefore$ The 6th term and the 9th term of the sequence are $2\log 6$ and $4\log 3$ respectively.

4. It is given that the first term a and the common difference d of an arithmetic sequence are 5 and 3 respectively.

(a) Find the general term $T(n)$ of the sequence.

(b) Hence, find the 12th term and the18th term of the sequence.

已知一個等差數列的首項 a 是 5 而公差 d 是 3。

(a) 求該數列的通項 $T(n)$。

(b) 由此，求該數列的第 12 項和第 18 項。.

(a) $\because\ a=5, d=3$

and $T(n)=a+(n-1)d$

$\therefore \quad T(n) = 5 + (n-1)(3)$
$= \underline{\underline{3n+2}}$

(b) $\because \quad T(n) = 3n + 2$

$\therefore \quad T(12) = 3(12) + 2$
$= 38$
$T(18) = 3(18) + 2$
$= 56$

$\therefore$ The 12th term and 18th term of the sequence are 38 and 56 respectively.

5. If 16 is the arithmetic mean between *x* and *y*, while *y* is the arithmetic mean between *x* and 1, find the values of *x* and *y*.
若 16 是 *x* 與 *y* 的等差中項，而 *y* 是 *x* 與 1 的等差中項，求 *x* 和 *y* 的值。

$\because$ 16 is the arithmetic mean between *x* and *y*

and *y* is the arithmetic mean between *x* and 1.

$$\therefore \begin{cases} 16 = \dfrac{x+y}{2} & \ldots\ldots(1) \\ y = \dfrac{x+1}{2} & \ldots\ldots(2) \end{cases}$$

From (1), we have

$x + y = 32$
$y = 32 - x \quad \ldots\ldots(3)$

By substituting (3) into (2), we have

$$32 - x = \frac{x+1}{2}$$
$$64 - 2x = x + 1$$
$$3x = 63$$
$$x = \underline{\underline{21}}$$

By substituting $x = 21$ into (3), we have

$y = 32 - 21$
$= \underline{\underline{11}}$

6. For an arithmetic sequence, it is given that *T*(21) = –16 and *T*(25) = 20. Find

(a) the first term and the common difference of the sequence,

(b) the general term of the sequence.

在一個等差數列中，*T*(21) = –16 及 *T*(25) = 20。求

(a) 首項和公差；

(b) 通項。

(a) Let *a* and *d* be the first term and the common difference of the arithmetic sequence respectively.

$T(21) = a + 20d = -16 \ldots\ldots(1)$
$T(25) = a + 24d = 20 \quad \ldots\ldots(2)$

$(2)-(1),\quad 4d=36$

$\therefore \qquad d=9$

By substituting $d=9$ into (1), we have

$$\begin{aligned} a+20(9)&=-16 \\ a&=-196 \end{aligned}$$

$\therefore$ The first term and the common difference are -196 and 9 respectively.

(b) $$\begin{aligned} T(n)&=a+(n-1)d \\ &=-196+(n-1)(9) \\ &=\underline{\underline{9n-205}} \end{aligned}$$

7. Given an arithmetic sequence –57, –51, –45, ..., 207.

(a) How many terms are there in the sequence?

(b) Find the value of *m* such that the *m*th term is the first positive term.

已知一個等差數列–57, –51, –45, ..., 207。

(a) 問該數列有多少項？

(b) 若第 *m* 項為該數列中的首個正數項，求 *m* 的值。

(a) Let *a* and *d* be the first term and the common difference of the arithmetic sequence respectively.

$\because\ a=-57$

and $d=-51-(-57)=6$

$$\begin{aligned} \therefore\ T(n)&=-57+(n-1)(6) \\ &=6n-63 \end{aligned}$$

Let 207 be the *k*th term.

$$\begin{aligned} \therefore\qquad T(k)&=207 \\ 6k-63&=207 \\ 6k&=270 \\ k&=45 \end{aligned}$$

$\therefore$ There are 45 terms in the sequence.

(b) The *m*th term is the first positive term of the sequence.

$$\begin{aligned} \therefore\qquad T(m)&>0 \\ 6m-63&>0 \\ 6m&>63 \\ m&>\frac{63}{6} \end{aligned}$$

$\therefore\ m=\underline{\underline{11}}$

8. Given two numbers 15 and 63,

(a) insert two arithmetic means between these two numbers,

(b) insert three arithmetic means between these two numbers.

在 15 與 63 兩個數之間插入

(a) 兩個等差中項；

(b) 三個等差中項。

(a) Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$15, 15+d, 15+2d, 63$

$$\begin{aligned} \therefore\ 15+3d &= 63 \\ 3d &= 48 \\ d &= 16 \end{aligned}$$

$\therefore$ The two required arithmetic means are 31 and 47.

(b) Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$15, 15+d, 15+2d, 15+3d, 63$

$$\begin{aligned} \therefore\ 15+4d &= 63 \\ 4d &= 48 \\ d &= 12 \end{aligned}$$

$\therefore$ The three required arithmetic means are 27, 39 and 51.

9. For a geometric sequence, it is given that $T(3) = 5$ and $T(6) = 135$. Find

(a) the first term and the common ratio of the sequence.

(b) the general term of the sequence,

(c) the value of $\dfrac{T(2)+T(3)}{T(5)}$.

在一個等比數列中，$T(3) = 5$ 及 $T(6) = 135$。求

(a) 該數列的首項和公比；

(b) 該數列的通項；

(c) $\dfrac{T(2)+T(3)}{T(5)}$的值。

(a) Let a and R be the first term and the common ratio of the sequence.

$$T(3) = aR^2 = 5 \quad \ldots\ldots(1)$$
$$T(6) = aR^5 = 135 \ldots\ldots(2)$$

$$(2) \div (1),\quad R^3 = 27$$
$$R = 3$$

By substituting $R = 3$ into (1), we have

$$a(3)^2 = 5$$
$$a = \frac{5}{9}$$

$\therefore$ The first term and the common ratio are $\frac{5}{9}$ and 3 respectively.

(c) $\because\ T(n) = \frac{5}{9}(3)^{n-1}$

$\therefore\ T(2) = \frac{5}{9}(3) = \frac{5}{3}$

$T(3) = \frac{5}{9}(3)^2 = 5$

$T(5) = \frac{5}{9}(3)^4 = 45$

$$\therefore\ \frac{T(2)+T(3)}{T(5)} = \frac{\frac{5}{3}+5}{45} = \underline{\underline{\frac{4}{27}}}$$

10. It is given that the first term a and the common ratio R of a geometric sequence are 3 and 2 respectively.

(a) Find the general term $T(n)$ of the sequence.

(b) If $T(k) = 768$, find the value of k.

已知一個等比數列的首項 a 是 3 而公比 R 是 2。

(a) 求該數列的通項 $T(n)$。

(b) 若 $T(k) = 768$，求 k 的值。

(a) $\because\ a = 3, R = 2$

and $T(n) = aR^{n-1}$

$\therefore\ T(n) = \underline{\underline{3(2)^{n-1}}}$

(b) $\because\ T(k) = 768$

$$\therefore\ 3(2)^{k-1} = 768$$
$$2^{k-1} = 256$$
$$2^{k-1} = 2^8$$
$$k - 1 = 8$$
$$k = \underline{\underline{9}}$$

11. It is given that the first term *a* and the common ratio *R* of a geometric sequence are 2 and –4 respectively.

(a) Find the general term $T(n)$ of the sequence.

(b) Hence, find the 4th term and the 7th term of the sequence.

已知一個等比數列的首項 *a* 是 2 而公比 *R* 是–4。

(a) 求該數列的通項 $T(n)$。

(b) 由此，求該數列的第 4 項和第 7 項。

(a) $\because\ a = 2, R = -4$

and $T(n) = aR^{n-1}$

$\therefore\ T(n) = \underline{\underline{2(-4)^{n-1}}}$

(b) $\because\ T(n) = 2(-4)^{n-1}$

$$\begin{aligned}\therefore\ T(4) &= 2(-4)^{4-1}\\ &= 2(-64)\\ &= \underline{\underline{-128}}\end{aligned}$$

$$\begin{aligned}T(7) &= 2(-4)^{7-1}\\ &= 2(4096)\\ &= \underline{\underline{8192}}\end{aligned}$$

12. Find the general term and the 12th term of the geometric sequence 27, –9, 3, –1, …

求等比數列 27, –9, 3, –1, ... 的通項和第 12 項。

Let *a* and *R* be the first term and the common ratio of the geometric sequence respectively.

$\because\ a = 27$ and $R = \dfrac{-9}{27} = -\dfrac{1}{3}$

$\therefore\ T(n) = aR^{n-1} = \underline{\underline{27\left(-\dfrac{1}{3}\right)^{n-1}}}$

$$\begin{aligned}T(12) &= 27\left(-\frac{1}{3}\right)^{12-1}\\ &= 3^3(-3)^{-11}\\ &= -3^3(3)^{-11}\\ &= \underline{\underline{-3^{-8}}}\end{aligned}$$

13. (a) Find the sum of the first 10 terms of an arithmetic series if its first term is 5 and the 10th term is 95.

(b) Find the sum of the first 12 terms of an arithmetic series if its first term is –31 and the common difference is 3.

(a) 若一個等差級數的首項是 5 而第 10 項是 95，求該等差級數首 10 項之和。

(b) 若一個等差級數的首項是–31 而公差是 3，求該等差級數首 12 項之和。

(a) $\because\ a=5, \ell=95$ and $n=10$

$$\therefore\ S(10)=\frac{10(5+95)}{2}$$
$$=\underline{\underline{500}}$$

(b) $a=-31, d=3$ and $n=12$

$$\therefore\ S(12)=\frac{12}{2}[2(-31)+(12-1)(3)]$$
$$=\underline{\underline{-174}}$$

14. (a) Insert two geometric means between 54 and –128.

(b) Insert three geometric means between $\frac{1}{2}$ and 8.

(a) 在 54 與–128 之間插入兩個等比中項。

(b) 在$\frac{1}{2}$與 8 之間插入三個等比中項。

(a) Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence formed is:

$54, 54R, 54R^2, -128$

$$\therefore\ 54R^3=-128$$
$$R^3=-\frac{64}{27}$$
$$R=-\frac{4}{3}$$

The two required geometric means are –72 and 96.

(b) Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence formed is:

$\frac{1}{2}, \frac{1}{2}R, \frac{1}{2}R^2, \frac{1}{2}R^3, 8$

$$\therefore\ \frac{1}{2}R^4=8$$
$$R^4=16$$
$$R=2 \text{ or } R=-2$$

The required geometric means are 1, 2, 4 or –1, 2, –4.

15. If the geometric mean between k and $3k+2$ is $2k+3$, find the value(s) of k.

若 $2k + 3$ 是 k 與 $3k + 2$ 的等比中項，求 k 的值。

$\because$ $2k+3$ is the geometric mean between k and $3k+2$.

$$\therefore \quad (2k+3)^2 = k(3k+2)$$
$$4k^2+12k+9 = 3k^2+2k$$
$$k^2+10k+9=0$$
$$(k+1)(k+9)=0$$
$$k = \underline{\underline{-1}} \text{ or } \underline{\underline{-9}}$$

16. Given an arithmetic series (–17) + (–13) + (–9) + ...,

(a) find the 15th terms,

(b) find the sum of the first 15 terms,

(c) find the sum of the first 30 terms,

(d) hence, find the sum from the 16th term to the 30th term.

已知一個等差級數：

(–17) + (–13) + (–9) + ...

(a) 求第 15 項。

(b) 求首 15 項之和。

(c) 求首 30 項之和。

(d) 由此，求第 16 項至第 30 項之和。

(a) $\because$ $a=-17$ and $d=-13-(-17)=4$

$$\therefore \quad T(15) = -17+(15-1)(4)$$
$$= \underline{\underline{39}}$$

(b)
$$S(15) = \frac{15}{2}(-17+39)$$
$$= \underline{\underline{165}}$$

(c)
$$S(30) = \frac{30[2(-17)+(30-1)(4)]}{2}$$
$$= \underline{\underline{1230}}$$

(d)
$$T(16)+T(17)+\ldots+T(30) = S(30)-S(15)$$
$$= 1230-165$$
$$= \underline{\underline{1065}}$$

17. (a) Find the sum of the arithmetic series 3 + 12 + 21 + ... + 147.

(b) How many terms of the arithmetic series 1 + 3 + 5 + … must be taken to give a sum of 2025?

(a) 求等差級數 3 + 12 + 21 + ... + 147 各項之和。

(b) 對於等差級數 1 + 3 + 5 + ...，問需要取多少項才可使級數之和等於 2025？

(a) Let n be the number of terms of the given series.

$a = 3, d = 12 - 3 = 9, \ell = T(n) = 147$

and $T(n) = a + (n-1)d$

$$\begin{aligned} \therefore\quad 147 &= 3 + (n-1)(9) \\ 144 &= 9(n-1) \\ n-1 &= 16 \\ n &= 17 \end{aligned}$$

$$\begin{aligned} \therefore\quad S(17) &= \frac{17(3+147)}{2} \\ &= \underline{\underline{1275}} \end{aligned}$$

(b) Let N be the number of terms that must be taken.

$a = 1, d = 3 - 1 = 2, S(N) = 2025$

and $S(N) = \frac{N}{2}[2a + (N-1)d]$

$$\begin{aligned} \therefore\quad 2025 &= \frac{N}{2}[2(1) + (N-1)(2)] \\ 2025 &= N[1 + (N-1)] \\ N^2 &= 2025 \\ N &= 45 \end{aligned}$$

$\therefore$ 45 terms of arithmetic series must be taken.

18. Find the least number of terms of the geometric series 3 + 15 + 75 + ... that can give a sum greater than 5000.

對於等比級數 3 + 15 + 75 + ...，問需要取多少項才可使級數之和大於 5000？

Let n be the number of terms required.

$\because\quad a = 3$ and $R = \frac{15}{3} = 5$

$\therefore\quad S(n) = \frac{3(5^n - 1)}{5-1} > 5000$

$$\frac{3(5^n-1)}{4} > 5000$$
$$5^n > \frac{20003}{3}$$
$$\log 5^n > \log\left(\frac{20003}{3}\right)$$
$$n\log 5 > \log\left(\frac{20003}{3}\right)$$
$$n > \frac{\log\left(\frac{20003}{3}\right)}{\log 5}$$
$$n > 5.47 \text{ (cor. to 2 d.p.)}$$

$\therefore$ At least 6 terms is required.

19. **(a)** Find the sum of the first 6 terms of a geometric series if its first term is 27 and the common ratio is $\frac{1}{3}$.

(b) Find the sum of the first 8 terms of a geometric series if its first term is 3 and the common ratio is 2.

(a) 若一個等比級數的首項是 27 而公比是$\frac{1}{3}$，求該級數首 6 項之和。

(b) 若一個等比級數的首項是 3 而公比是 2，求該級數首 8 項之和。

(a) $\because\ a = 27, R = \frac{1}{3}$ and $n = 6$

$$\therefore\ S(6) = \frac{27\left[1-\left(\frac{1}{3}\right)^6\right]}{1-\frac{1}{3}}$$
$$= \frac{81}{2}\left(1-\frac{1}{729}\right)$$
$$= \underline{\underline{\frac{364}{9}}}$$

(b) $\because\ a = 3, R = 2$ and $n = 8$

$$S(8) = \frac{3(2^8-1)}{2-1}$$
$$= 3(256-1)$$
$$= \underline{\underline{765}}$$

20. (a) Find the sum of the terms of the geometric series 2 + (–4) + 8 + (–16) + ... + 128.

(b) How many terms of the series in (a) must be taken to give a sum of 342?

(a) 求等比級數 2 + (–4) + 8 + (–16) + ... + 128 各項之和。

(b) 對於(a) 的等比級數，問需要取多少項才可使級數之和等於 342？

(a) Let n be the number of terms of the given series.

$\because\ a = 2, R = \dfrac{-4}{2} = -2$ and $T(n) = aR^{n-1}$

$$\begin{aligned}\therefore\quad 128 &= 2(-2)^{n-1}\\ (-2)^{n-1} &= 64\\ (-2)^{n-1} &= 2^6\\ n-1 &= 6\\ n &= 7\end{aligned}$$

$\therefore\ S(7) = \dfrac{2[1-(-2)^7]}{1-(-2)} = \underline{\underline{86}}$

(b) Let N be the number of terms that must be taken.

$\because\ a = 2, R = -2$ and $S(N) = \dfrac{a(1-R^n)}{1-R} = 342$

$$\begin{aligned}\therefore\quad 342 &= \frac{2[1-(-2)^n]}{1-(-2)}\\ 342 &= \frac{2[1-(-2)^n]}{3}\\ 513 &= 1-(-2)^n\\ (-2)^n &= -512\\ (-2)^n &= (-2)^9\\ n &= 9\end{aligned}$$

$\therefore$ 9 terms of the geometric series must be taken.

21. Find the sum to infinity of the following geometric series.
求下列等比級數的無限項之和。

(a) $4 + 2 + 1 + \dfrac{1}{2} + \ldots$

(b) $9 - 3 + 1 - \dfrac{1}{3} + \ldots$

(a) $\because\ a = 4$ and $R = \dfrac{2}{4} = \dfrac{1}{2}$

$$\therefore\ S(\infty) = \frac{4}{1-\frac{1}{2}} = \underline{\underline{8}}$$

(b) $\because\ a = 9$ and $R = \dfrac{-3}{9} = -\dfrac{1}{3}$

$$\therefore\ S(\infty) = \frac{9}{1+\frac{1}{3}} = \underline{\underline{\frac{27}{4}}}$$

22. In 15 years' time, Mr. So would like to save \$1 500 000 for his retirement. If he deposits a fixed amount of money at the beginning of each year at a fixed interest rate of 6% p.a., compounded yearly, how much should he deposit at the beginning of the year?
(Give your answer correct to the nearest \$10.)
蘇先生打算在 15 年內儲蓄\$1 500 000 作退休之用。若他在每年年初把一筆固定的款項存入銀行，年利率為 6 %，且每年計算複利息一次，問他每年年初應把多少款項存入銀行？
(答案須準確至最接近的\$10。)

Let \$$x$ be the amount of money required to deposit at the beginning of each year.

Total amount of deposit together with the compound interest in 15 years' time

$=\$[x(1.06)^{15}+x(1.06)^{14}+\ldots x(1.06)]$

$$\begin{aligned}\therefore\quad 1500000 &= x(1.06)^{15}+x(1.06)^{14}+\ldots x(1.06)\\ 1500000 &= x(1.06)+\ldots+x(1.06)^{14}+x(1.06)^{15}\\ 1500000 &= \frac{x(1.06)[(1.06)^{15}-1]}{1.06-1}\\ x &= \frac{1500000\times(1.06-1)}{1.06[(1.06)^{15}-1]}\\ &= 60\,800 \text{ (cor. to the nearest ten)}\end{aligned}$$

$\therefore$ Mr. So should deposit \$60 800 at the beginning of each year.

23. Express the following recurring decimals as fractions.
把下列各循環小數化為分數。

(a) $0.\dot{4}$

(b) $0.3\dot{7}\dot{5}$

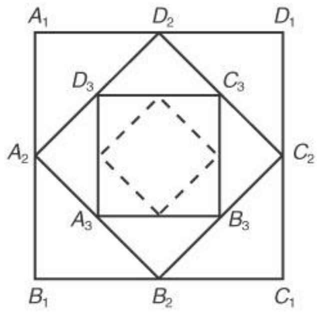
(c) $4.\dot{1}2\dot{5}$

(a)
$$\begin{aligned}0.\dot{4} &= 0.4444\ldots\\ &= 0.4+0.04+0.004+0.0004+\ldots\\ &= \frac{0.4}{1-0.1}\\ &= \underline{\underline{\frac{4}{9}}}\end{aligned}$$

(b)
$$\begin{aligned}0.3\dot{7}\dot{5} &= 0.3+0.075+0.00075+0.0000075\\ &= 0.3+0.075+0.00075+0.0000075+\ldots\\ &= 0.3+\frac{0.075}{1-0.01}\\ &= \underline{\underline{\frac{62}{165}}}\end{aligned}$$

(c)
$$\begin{aligned}4.\dot{1}2\dot{5} &= 4.125125125\\ &= 4+0.125+0.000125+0.000000125\\ &= 4+\frac{0.125}{1-0.001}\\ &= \underline{\underline{4\frac{125}{999}}}\end{aligned}$$

24. In the figure, $A_1B_1C_1D_1$ is a square of side 8 cm. A second square $A_2B_2C_2D_2$ is formed by joining the mid-points of the sides of $A_1B_1C_1D_1$ and this process is continued to form an infinite number of squares.



(a) Find
- **(i)** the perimeters,
- **(ii)** the areas

of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$.

(b) Find the sum of
- **(i)** the perimeters,
- **(ii)** the areas

of all the squares formed (i.e. $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3, \ldots$).

在圖中，$A_1B_1C_1D_1$ 是一個邊長為 8 cm 的正方形。連接 $A_1B_1C_1D_1$ 各邊的中點後形成了第二個正方形 $A_2B_2C_2D_2$。在不斷重複這個過程下，會形成無限多個正方形。

(a) 求 $A_1B_1C_1D_1$ 和 $A_2B_2C_2D_2$ 的
- **(i)** 周界；
- **(ii)** 面積。

(b) 求所有正方形 (即 $A_1B_1C_1D_1$、$A_2B_2C_2D_2$、$A_3B_3C_3D_3$、...) 的
- **(i)** 周界之和；
- **(ii)** 面積之和。

(a)(i) Perimeter of $A_1B_1C_1D = 4 \times 8 \text{ cm} = \underline{\underline{32 \text{ cm}}}$

In $\triangle A_2B_1B_2$,

$$A_2B_2 = \sqrt{(A_2B_1)^2 + (B_1B_2)^2} \quad \text{(Pyth theorem)}$$
$$= \sqrt{4^2 + 4^2} \text{ cm}$$
$$= 4\sqrt{2} \text{ cm}$$

Perimeter of $A_1B_1C_1D_2 = 4 \times 4\sqrt{2} \text{ cm} = \underline{\underline{16\sqrt{2} \text{ cm}}}$

(ii) Area of $A_1B_1C_1D = 8^2 \text{ cm} = \underline{\underline{64 \text{ cm}^2}}$

Area of $A_1B_1C_1D_2 = (4\sqrt{2})^2 \text{ cm}^2 = \underline{\underline{32 \text{ cm}}}$

(b)(i) The perimeters of the squares $A_1B_1C_1D_1, A_2B_2C_2D_2, \ldots$ are in geometric sequence with common ratio $\frac{\sqrt{2}}{2}$.

$$\begin{aligned}\therefore\quad \text{Sum of perimeters} &= (32+16\sqrt{2}+\dots)\text{ cm}\\ &= \frac{32}{1-\frac{\sqrt{2}}{2}}\text{ cm}\\ &= 32\times\frac{2}{2-\sqrt{2}}\text{ cm}\\ &= 32\times\frac{2(2+\sqrt{2})}{4-2}\text{ cm}\\ &= \underline{\underline{32(2+\sqrt{2})\text{ cm}}}\end{aligned}$$

(ii) The areas of the squares $A_1B_1C_1D_1, A_2B_2C_2D_2, \dots$ are in geometric sequences with common ratio $\frac{1}{2}$.

$$\begin{aligned}\text{Sum of areas} &= (64+32+\dots)\text{ cm}^2\\ &= \frac{64}{1-\frac{1}{2}}\text{ cm}^2\\ &= \underline{\underline{128\text{cm}^2}}\end{aligned}$$

25. How many positive terms are there in the arithmetic sequence $178, 175, 172, \dots$?
問等差數列 178, 175, 172, ... 有多少個正數項？

Let a and d be the first term and the common difference respectively.

$\because\ a = 178$ and $d = 175 - 178 = -3$

$$\begin{aligned}\therefore\ T(n) &= 178 + (n-1)(-3)\\ &= 181 - 3n\end{aligned}$$

Let kth term is the last positive term of the sequence.

$$\begin{aligned}\therefore\quad T(k) &> 0\\ 181 - 3k &> 0\\ k &< 60\frac{1}{3}\end{aligned}$$

$\therefore\ k = 60$

There are 60 positive terms in the sequence.

26. (a) Find the sum of the terms of the arithmetic series $59 + 54 + 49 + \dots + 4$.

(b) What is the maximum number of terms of the arithmetic series $59 + 54 + 49 + \dots$ that can give a positive sum?

(a) 求等差級數 59 + 54 + 49 + ... + 4 各項之和。

(b) 對於等差級數 59 + 54 + 49 + ...，問最多可取多少項，使級數之和仍是正數？

(a) Let n be the number of terms of the given series.

$a = 59, d = 54 - 59 = -5, \ell = T(n) = 4$

and $T(n) = a + (n-1)d$

$$\therefore \quad 4 = 59 + (n-1)(-5)$$
$$-55 = (-5)(n-1)$$
$$n-1 = 11$$
$$n = 12$$

$$\therefore \quad S(12) = \frac{12}{2}[2(59)+(12-1)(-5)]$$
$$= \underline{\underline{378}}$$

(b) Let N be the maximum number of terms for the given series to have a positive sum.

The sum of the series to N terms is given by:

$$S(N) = \frac{N}{2}[(2(59)+(N-1)(-5)]$$
$$= \frac{N}{2}(123-5N)$$

$$\therefore \quad S(N) > 0$$
$$\frac{N}{2}(123-5N) > 0$$
$$123-5N > 0$$
$$N < 24\frac{3}{5}$$

$\therefore$ Maximum number of terms $= \underline{\underline{24}}$

27. It is given that the first term of a geometric sequence is 270 and its common ratio is positive. If $T(3):T(7)=3:5$, find $T(13)$.

已知一個等比數列的首項是 270，而公比是一個正數。若 $T(3) : T(7) = 3 : 5$，求 $T(13)$。

Let R be the common ratio of the geometric sequence.

$\therefore$ $T(3) = 270R^2$ and $T(7) = 270R^6$

$\because$ $T(3):T(7) = 3:5$

$$\therefore \quad \frac{T(3)}{T(7)} = \frac{3}{5}$$
$$\frac{270R^2}{270R^5} = \frac{3}{5}$$
$$R^4 = \frac{5}{3}$$

$$T(13) = 270R^{12}$$
$$= 270(R^4)^3$$
$$= 270\left(\frac{5}{3}\right)^3$$
$$= \underline{\underline{1250}}$$

28. If p and q are the two geometric means between 2 and 50, find the value of $\log p + \log q$.
若 p 和 q 是 2 與 50 之間的兩個等比中項，求 $\log p + \log q$ 的值。

Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence formed is:

$2, 2R, 2R^2, 50$

$$\begin{aligned}\therefore\quad pq &= (2R)(2R^2)\\ &= 2(2R^3)\\ &= 2(50)\\ &= 100\end{aligned}$$

$$\begin{aligned}\therefore\quad \log p + \log q &= \log(pq)\\ &= \log 100\\ &= \underline{\underline{2}}\end{aligned}$$

29. If x is the geometric mean between $\frac{a}{b}$ and $\frac{b}{a}$, find the common ratio of the geometric sequence formed.

若 x 是 $\frac{a}{b}$ 與 $\frac{b}{a}$ 的等比中項，求所形成的等比數列的公比。

Let R be the common ratio of te geometric sequence to be formed.

The geometric sequence formed is: $\frac{a}{b}, \frac{a}{b}R, \frac{b}{a}$

$$\begin{aligned}\therefore\quad \frac{a}{b}R^2 &= \frac{b}{a}\\ R^2 &= \frac{b^2}{a^2}\end{aligned}$$

$R = -\frac{b}{a}$ or $\frac{b}{a}$

$\therefore$ The common ratio is $-\frac{b}{a}$ or $\frac{b}{a}$.

30. A student is asked to find the sum to infinity of the geometric series $6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \dots$, but he just adds up the first 6 terms to find the answer. Find the percentage error in his answer.

某學生需要求出 $6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \dots$的無限項之和，但他只把該級數的首 6 項加起來作為答案。求他的答案的百分誤差。

$\because\quad a = 6, R = \frac{3}{6} = \frac{1}{2}, n = 6$

$$\therefore \quad S(6)=\frac{6\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}}=\frac{189}{16}$$

$$S(\infty)=\frac{6}{1-\frac{1}{2}}=12$$

$$\text{Pecentage error}=\frac{12-\frac{189}{16}}{12}\times 100\%=\underline{\underline{1.5625\%}}$$

31. It is given that the sum of the first 3 terms and the sum of the first 6 terms of a geometric series are 6 and –42 respectively. Find the sum of the first 10 terms of the series.
已知一個等比級數首 3 項之和是 6，而首 6 項之和是–42。求該級數首 10 項之和。

Let a and R be the first term and common ratio respectively.

$\because \quad S(3)=6$ and $S(6)=-42$

$$\therefore \quad \frac{a(1-R^3)}{(1-R)}=6 \quad \ldots\ldots(1)$$

$$\frac{a(1-R^6)}{(1-R)}=-42 \quad \ldots\ldots(2)$$

(2) ÷ (1),

$$\frac{1-R^6}{1-R^3}=-7$$

$$\frac{(1-R^3)(1+R^3)}{1-R^3}=-7$$

$$1+R^3=-7$$

$$R=-2$$

By substituting $R=-2$ into (1), we have

$$\frac{a[1-(-2)^3]}{1-(-2)}=6$$

$$\frac{9a}{3}=6$$

$$a=2$$

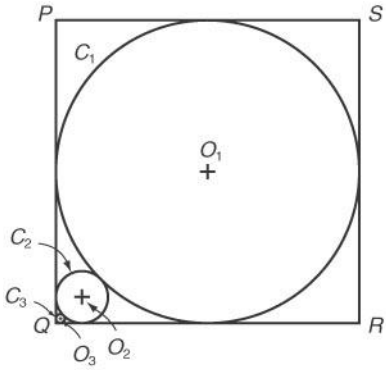
$$\therefore \quad S(10)=\frac{2[1-(-2)^{10}]}{1-(-2)}$$

$$=\frac{2(-1023)}{3}$$

$$=\underline{\underline{-682}}$$

31. In the figure, C_1 is a circle with centre O_1 inscribed in the square *PQRS* of side 4 cm; C_2 is a circle with centre O_2 which touches C_1 externally and touches two sides of the square. C_3 is a circle with centre O_3 which touches C_2 externally and touches two sides of the square. Following the pattern, an infinite number of circles $C_4, C_5, C_6, \ldots$ with centres $O_4, O_5, O_6, \ldots$ are formed. Let r_i be the radius of the circle C_i where $i=1,2,3,4,\ldots$

在圖中，C_1是一個圓心為 O_1的圓，它內切於一個邊長 4 cm 的正方形 $PQRS$；$C2$ 是一個以 O_2為圓心的圓，它與 C_1外切，且與正方形的兩邊相切；C_3是一個以 O_3為圓心的圓，它與 C_2外切，且與正方形的兩邊相切。在不斷重複這個過程下，形成了無限個圓 C_4，C_5，C_6，...，它們的圓心分別為 O_4，O_5，O_6，...。設圓 C_i的半徑為 r_i，其中 i= 1，2，3，4，...。



(a) **(i)** Find r_1 and r_2

(ii) Hence, find the ratio $\frac{r_2}{r_1}$.

(b) Find the sum of the areas of all the circles formed (i.e. $C_1, C_2, C_3, C_4, \ldots$).

(Give your answer correct to 4 significant figures.)

(a) **(i)** 求 r_1和 r_2。

(ii) 由此，求$\frac{r_2}{r_1}$這個比。

(b) 求所有圓 (即圓 C_1、C_2、C_3、C_4、...) 的面積之和。
(答案須準確至四位有效數字。)

(a)(i) $r_1 = \frac{SR}{2} = \underline{\underline{2\text{ cm}}}$

In $\triangle O_1MN$,

$O_1M = r_1 + r_2$,

$O_1N = r_1 - r_2$,

$MN = r_1 - r_2$

$$\begin{aligned} O_1M^2 &= O_1N^2 + MN^2 \quad \text{(Pyth. theorem)} \\ (r_1 + r_2)^2 &= 2(r_1 - r_2)^2 \\ r_1^2 + 2r_1r_2 + r_2^2 &= 2r_1^2 - 4r_1r_2 + 2r_2^2 \\ 6r_1r_2 &= r_1^2 + r_2^2 \\ 12r_3 &= 4 + r_2^2 \\ r_2^2 - 12r_3 + 4 &= 0 \end{aligned}$$

$$r_2 = \frac{12 - \sqrt{12^2 - 4(1)(4)}}{2} \quad \text{or} \quad \frac{12 + \sqrt{12^2 - 4(1)(4)}}{2} \ \text{(rejected)}$$

$$r_2 = \underline{\underline{2(3 - 2\sqrt{2})\text{ cm}}}$$

(ii) $\frac{r_2}{r_1} = \frac{2(3-2\sqrt{2})}{2} = \underline{\underline{3-2\sqrt{2}}}$

(b) The areas of circles are in geometric sequence with common ratio $(3-2\sqrt{2})^2$.

Sum of the areas of all circles

$= \frac{\pi(2)^2}{1-(3-2\sqrt{2})^2}\text{ cm}^2$

$= \underline{\underline{12.95\text{ cm}^2}}$ (cor. to 4 sig. fig.)

Pre-requisite Questions

預備測驗

1. If $f(x) = x^2 - 2x - 1$, find the values of the function when

若 $f(x) = x^2 - 2x - 1$，求在下列各情況中 $f(x)$ 的值。

(a) $x = -3$,

(b) $x = 2$.

. **(a)** $\because\ f(x) = x^2 - 2x - 1$

$$\begin{aligned}\therefore\ f(-3) &= (-3)^2 - 2(-3) - 1\\ &= 9 + 6 - 1\\ &= \underline{\underline{14}}\end{aligned}$$

(b)
$$\begin{aligned}f(2) &= 2^2 - 2(2) - 1\\ &= 4 - 4 - 1\\ &= \underline{\underline{-1}}\end{aligned}$$

2. If $h(x) = (x + k)(2x - 3)$ and $h(2) = -3$**,** find the values of

若 $h(x) = (x + k)(2x - 3)$ 及 $h(2) = -3$，求下列各題的值。

(a) k,

(b) $h(-5)$.

(a)
$$\begin{aligned}\because\quad h(2) &= -3\\ \therefore\quad (2 + k)[2(2) - 3] &= -3\\ k + 2 &= -3\\ k &= \underline{\underline{-5}}\end{aligned}$$

(b) $\because\ h(x) = (x - 5)(2x - 3)$

$$\begin{aligned}\therefore\ h(-5) &= (-5 - 5)[2(-5) - 3]\\ &= (-10)(-13)\\ &= \underline{\underline{130}}\end{aligned}$$

3. If $g(t) = \dfrac{t}{1 + t^2}$, find the values of the function when

若 $g(t) = \dfrac{t}{1 + t^2}$，求在下列各情況中 $g(t)$ 的值。

(a) $t = 2$,

(b) $t = \dfrac{1}{2}$.

(a) $\because\ g(t) = \dfrac{t}{1 + t^2}$

$$\therefore \quad g(2) = \frac{2}{1+2^2} = \underline{\underline{\frac{2}{5}}}$$

(b) $$g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{1+\frac{1}{4}} = \underline{\underline{\frac{2}{5}}}$$

4. If $f(x) = 2x - 3$ and $g(x) = 3x^2 + 1$, find the values of

若 $f(x) = 2x - 3$ 及 $g(x) = 3x^2 + 1$，求下列各題的值。

(a) $f(0) \bullet g(-1)$

(b) $3f(4) - g(1)$

(a) $f(0) = 2(0) - 3 = -3$

$g(-1) = 3(-1)^2 + 1 = 4$

$f(0) \bullet g(-1) = (-3)(4) = \underline{\underline{-12}}$

(b) $f(4) = 2(4) - 3 = 5$

$g(1) = 3(1)^2 + 1 = 4$

$3f(4) - g(1) = 3(5) - 4 = \underline{\underline{11}}$

5 If $f(x) = \sin x$ and $g(x) = \cos x$, find the values of

若 $f(x) = \sin x$ 及 $g(x) = \cos x$，求下列各題的值。

(a) $g(30°) + f(45°)$

(b) $\dfrac{[f(30°)]^2}{g(60°)}$

(a) $g(30°) = \cos 30° = \dfrac{\sqrt{3}}{2}$

$f(45°) = \sin 45° = \dfrac{\sqrt{2}}{2}$

$$g(30°) + f(45°) = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \underline{\underline{\frac{\sqrt{3}+\sqrt{2}}{2}}}$$

(b) $\dfrac{[f(30°)]^2}{g(60°)} = \dfrac{(\sin 30°)^2}{\cos 60°}$

$= \dfrac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)}$

$= \underline{\underline{\frac{1}{2}}}$

6. It is given that $h(x) = x^2 - 3x + 1$. If $g(x) = h(x+2)$, write down the symbolic representation of $g(x)$.

已知 $h(x) = x^2 - 3x + 1$。若 $g(x) = h(x+2)$，試以符號形式表示 $g(x)$。

$g(x) = h(x+2)$
$= (x+2)^2 - 3(x+2) + 1$
$= x^2 + 4x + 4 - 3x - 6 + 1$
$= \underline{\underline{x^2 + x - 1}}$

7. If $f(t) = 2t^2 - mt + 9$ and $f(-3) = 0$, find the values of

若 $f(t) = 2t^2 - mt + 9$ 及 $f(-3) = 0$，求下列各題的值。

(a) m,

(b) $f(1)$.

(a) $\because$ $f(-3) = 0$
$\therefore$ $2(-3)^2 - m(-3) + 9 = 0$
$18 + 3m + 9 = 0$
$m = \underline{\underline{-9}}$

(b) $\because$ $f(x) = 2t^2 - 9t + 9$
$\therefore$ $f(1) = 2(1)^2 - 9(1) + 9$
$= \underline{\underline{2}}$

8. It is given that $f(x) = a(x-3) + b$. If $f(0) = 1$ and $f(1) = 0$, find the values of a and b.

已知 $f(x) = a(x-3) + b$。若 $f(0) = 1$ 及 $f(1) = 0$，求 a 和 b 的值。

$\because$ $f(0) = 1$

$\therefore$ $a(0-3) + b = 1$
$-3a + b = 1$ ……(1)

$\because$ $f(1) = 0$

$\therefore$ $a(1-3) + b = 0$
$-2a + b = 0$ ……(2)

(1) – (2),

$-3a+b-(-2a+b)=1-0$

$a=\underline{\underline{-1}}$

By substituting $a=-1$ into (1), we have

$-3(-1)+b=1$

$b=\underline{\underline{-2}}$

9. It is given that $h(x)=(x-1)(x+2)-kx$ and $h(k)=3k$.

已知 $h(x)=(x-1)(x+2)-kx$ 及 $h(k)=3k$。

(a) Find the value of *k*.

(b) Solve for *x* if $h(x)=2x$.

(a) $\because\ h(k)=3k$

$\therefore\ (k-1)(k+2)-k(k)=3k$

$k^2+k-2-k^2=3k$

$-2k=2$

$k=\underline{\underline{-1}}$

(b) $\because\ h(x)=2x$

$\therefore\ (x-1)(x+2)+x=2x$

$x^2+x-2+x=2x$

$x^2-2=0$

$x^2=2$

$x=\underline{\underline{\sqrt{2}}}$ or $\underline{\underline{-\sqrt{2}}}$

10. It is given that $H(s)=s^2+2s$. If $V(s)=H(2s-1)$, write down the symbolic representation of $V(s)$.

已知 $H(s)=s^2+2s$。若$V(s)=H(2s-1)$，試以符號形式表示 $V(s)$。

$V(s)=H(2s-1)$

$=(2s-1)^2+2(2s-1)$

$=4s^2-4s+1+4s-2$

$=\underline{\underline{4s^2-1}}$

11. If $s(t)=(t-4)^2+2(t+5)$, find $s(2t-3)$.

若 $s(t)=(t-4)^2+2(t+5)$，求 $s(2t-3)$ 的值。

$s(2t-3)=[(2t-3)-4]^2+2[(2t-3)+5]$

$=(2t-7)^2+2(2t+2)$

$=4t^2-28t+49+4t+4$

$=\underline{\underline{4t^2-24t+53}}$

12. If $f(x)=3x-1$ and $g(x)=-2x+5$, find

若 $f(x)=3x-1$ 及 $g(x)=-2x+5$，求

(a) $f(x)+g(x)$,

(b) $[f(x)+g(x)]^2-g(x)$.

(a) $f(x)+g(x)=3x-1+(-2x+5)$

$=3x-2x-1+5$

$=\underline{\underline{x+4}}$

(b) $[f(x)+g(x)]^2-g(x)$

$=(x+4)^2-(-2x+5)$

$=x^2+8x+16+2x-5$

$=\underline{\underline{x^2+10x+11}}$

Level 1 Questions

程度 1 題目

1. Consider the sequence:

$\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$

(a) Write down the next two terms of the sequence.

(b) Determine the general term $T(n)$ of the sequence.

(c) Find the 9th term and 10th term of the sequence.

考慮以下的數列:

$\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$

(a) 寫出接著的兩項。

(b) 寫出通項 $T(n)$。

(c) 求該數列的第 9 項和第 10 項。

(a) The next two terms are $\frac{1}{6}$ and $\frac{1}{7}$.

(b) $\because\ T(1)=\frac{1}{2}=\frac{1}{1+1}$

$T(2)=\frac{1}{3}=\frac{1}{2+1}$

$T(3)=\frac{1}{4}=\frac{1}{3+1}$

$\therefore\ T(n)=\underline{\underline{\frac{1}{n+1}}}$

(c) $\because\ T(n)=\frac{1}{n+1}$

$\therefore\ T(9)=\frac{1}{9+1}=\frac{1}{10}$

and $T(10)=\frac{1}{10+1}=\frac{1}{11}$

$\therefore$ The 9th term and the 10th term of the sequence are $\frac{1}{10}$ and $\frac{1}{11}$ respectively.

2. Consider the sequence:

$$2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots$$

(a) Determine the general term $T(n)$ of the sequence,

(b) Find the 6th term and 7th term of the sequence.

考慮以下的數列:

$$2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots$$

(a) 寫出通項 $T(n)$;

(b) 求第 6 項和第 7 項。

(a) $\because \; T(1) = 2 = \frac{n+1}{1^2}$

$$T(2) = \frac{3}{4} = \frac{2+1}{2^2}$$
$$T(3) = \frac{4}{9} = \frac{3+1}{3^2}$$

$\therefore \; T(n) = \underline{\underline{\frac{n+1}{n^2}}}$

(b) $\because \; T(n) = \frac{n+1}{n^2}$

$\therefore \; T(6) = \frac{6+1}{6^2} = \frac{7}{36}$ and $T(7) = \frac{7+1}{7^2} = \frac{8}{49}$

$\therefore$ The 6th term and the 7th term are $\frac{7}{36}$ and $\frac{8}{49}$ respectively.

3. Consider the sequence:

$1, 4, 9, 16, 25, \ldots$

(a) Write down the next two terms of the sequence.

(b) Determine the general term $T(n)$ of the sequence.

(c) Find the 7th term and 9th term of the sequence.

考慮以下的數列:

$1, 4, 9, 16, 25, \ldots$

(a) 寫出接著的兩項。

(b) 寫出通項 $T(n)$。

(c) 求該數列的第 7 項和第 9 項。

(a) The next two terms are 36 and 49.

(b) $\because\ T(1)=1=1^2$

$T(2)=4=2^2$

$T(3)=9=3^2$

$\therefore\ T(n)=\underline{\underline{n^2}}$

(c) $\because\ T(n)=n^2$

$\therefore\ T(7)=7^2=49$ and $T(9)=9^2=81$

$\therefore$ The 7th term and the 9th term of the sequence are 49 and 81 respectively.

4. Consider the following sequence:

$2,6,12,20,\ldots.$

(a) Determine the general term $T(n)$ of the sequence.

(b) Write down the 7th term and the 10th term of the sequence.

考慮以下的數列:

$2,6,12,20,\ldots.$

(a) 寫出通項 $T(n)$;

(b) 求第 7 項和第 10 項。

(a) $\because\ T(1)=2=1\times 2$

$T(2)=6=2\times 3$

$T(3)=12=3\times 4$

$\therefore\ T(n)=\underline{\underline{n(n+1)}}$

(b) $\because\ T(n)=n(n+1)$

$\therefore\ T(7)=7(7+1)=56$

and $T(10)=10(10+1)=110$

$\therefore$ The 7th term and the 10th term are 56 and 110 respectively.

5. Consider the sequence:

$0,\log 3,\log 9,\log 27,\ldots.$

(a) Determine the general term $T(n)$ of the sequence,

(b) Find the 6th term and 7th term of the sequence.

考慮以下的數列:

$0,\log 3,\log 9,\log 27,\ldots.$

(a) 寫出通項 $T(n)$;

(b) 求第 6 項和第 7 項。

(a) $\because\ T(1)=0=(1-1)\log 3$

$T(2)=\log 3=(2-1)\log 3$

$T(3)=\log 9=\log 3^2=2\log 3=(3-1)\log 3$

$\therefore\ T(n)=\underline{\underline{(n-1)\log 3}}$

(b) $\because\ T(n) = (n-1)\log 3$

$\therefore\ T(6) = (6-1)\log 3$
$= 5\log 3$
$= \log 3^5$
$= \log 243$

$T(7) = (7-1)\log 3$
$= 6\log 3$
$= \log 3^6$
$= \log 729$

$\therefore$ The 6th term and 7th term are log 23 and log 729 respectively.

(P01C14L03Q006)

6. It is given that the first term and the common difference of an arithmetic sequence are 3 and –4 respectively. Find

(a) the general term $T(n)$ of the sequence,

(b) the 6th term and the 36th term of the sequence.

已知一個等差數列的首項是 3 而公差是–4。求

(a) 通項 $T(n)$;

(b) 第 6 項和第 36 項。

(a) $\because\ a = 3, d = -4$ and $T(n) = a + (n-1)d$

$\therefore\ T(n) = 3 + (n-1)(-4)$
$= 3 - 4n + 4$
$= \underline{\underline{7 - 4n}}$

(b) $\because\ T(n) = 7 - 4n$

$\therefore\ T(6) = 7 - 4(6) = -17$
$T(36) = 7 - 4(36) = -137$

$\therefore$ The 6th term and the 36th term are –17 and –137 respectively.

7. Consider the arithmetic sequence:

$5, 13, 21, 29, \ldots$

(a) Find the general term of the sequence.

(b) If the kth term of the sequence is 101, find k.

考慮下列的等差數列:

$5, 13, 21, 29, \ldots$

(a) 求該數列的通項。

(b) 若該數列的第 k 項是 101，求 k 的值。

(a) Let a and d be the first term and the common difference respectively.

$\because\ a = 5$ and $d = 13 - 5 = 8$

$$\begin{aligned}\therefore\ T(n) &= 5 + (n-1)(8)\\ &= 5 + 8n - 8\\ &= \underline{\underline{8n-3}}\end{aligned}$$

(b) $\because\ T(k) = 8k - 3$

$$\begin{aligned}\therefore\ 8k - 3 &= 101\\ 8k &= 104\\ k &= \underline{\underline{13}}\end{aligned}$$

8. It is given that the first term and the common difference of an arithmetic sequence are −5 and 4 respectively. Find

(a) the general term $T(n)$ of the sequence,

(b) the 5th term and the 20th term of the sequence.

已知一個等差數列的首項是–5 而公差是 4。求

(a) 通項 $T(n)$;

(b) 第 5 項和第 20 項。

(a) $\because\ a = -5, d = 4$ and $T(n) = a + (n-1)d$

$$\begin{aligned}\therefore\ T(n) &= -5 + (n-1)(4)\\ &= -5 + 4n - 4\\ &= \underline{\underline{4n-9}}\end{aligned}$$

(b) $\because\ T(n) = 4n - 9$

$$\begin{aligned}\therefore\ T(5) &= 4(5) - 9 = 21\\ T(20) &= 4(20) - 9 = 71\end{aligned}$$

$\therefore$ The 5th term and the 20 term are 21 and 71 respectively.

9. It is given that the first term and the common difference of an arithmetic sequence are 15 and –3 respectively. Find

(a) the general term $T(n)$ of the sequence,

(b) the 6th term and the 33rd term of the sequence.

已知一個等差數列的首項是 15 而公差是–3。求

(a) 通項 $T(n)$;

(b) 第 6 項和第 33 項。

(a) $\because\ a = 15, d = -3$ and $T(n) = a + (n-1)d$

$\therefore \quad T(n) = 15 + (n-1)(-3)$
$= 15 - 3n + 3$
$= \underline{\underline{18 - 3n}}$

(b) $\because \quad T(n) = 18 - 3n$

$\therefore \quad T(6) = 18 - 3(6) = 0$
$T(33) = 18 - 3(33) = -81$

$\therefore$ The 6th term and the 33rd term are 0 and –81 respectively.

10. Consider the arithmetic sequence:
$-39, -31, -23, -15, \ldots$

(a) Find the general term of the sequence.

(b) If the *k*th term of the sequence is 25, find *k*.

考慮下列的等差數列:
$-39, -31, -23, -15, \ldots$

(a) 求該數列的通項。

(b) 若該數列的第 k 項是 25，求 k 的值。

(a) Let *a* and *d* be the first term and the common difference respectively.

$\because \quad a = -39$ and $d = -31 - (-39) = 8$

$\therefore \quad T(n) = -39 + (n-1)(8)$
$= -39 + 8n - 8$
$= \underline{\underline{8n - 47}}$

(b) $\because \quad T(k) = 8k - 47$
$\therefore \quad 8k - 47 = 25$
$8k = 72$
$k = \underline{\underline{9}}$

11. For a arithmetic sequence, it is given that $T(11) = 35$ and $T(18) = 56$, find

(a) the general term of the sequence,

(b) the third term of the sequence.

在一個等差數列中，已知 $T(11) = 35$ 及 $T(18) = 56$，求該數列的

(a) 通項;

(b) 第三項。

(a) Let *a* and *d* be the first term and the common difference respectively.

$T(11) = a + 10d = 35 \ldots\ldots(1)$
$T(18) = a + 17d = 56 \ldots\ldots(2)$

$(2) - (1), \quad 7d = 21$
$d = 3$

By substituting $d = 3$ into (1), we have

$$a+10(3)=35$$
$$a=5$$

$$\begin{aligned}T(n)&=5+(n-1)(3)\\&=5+3n-3\\&=\underline{\underline{3n+2}}\end{aligned}$$

(b) $\because\ T(n)=3n+2$

$\therefore\ T(3)=3(3)+2=\underline{\underline{11}}$

12. Find the number of terms in each of the following arithmetic sequences.
求以下各等差數列的項數。

(a) $108, 101, 94, \ldots, -25$

(b) $33, 37, 41, \ldots, 373$

(a) Let a and d be the first term and the common difference respectively.

$\because\ a=108$ and $d=101-108=-7$

$$\therefore\ \begin{aligned}T(n)&=108+(n-1)(-7)\\&=108-7n+7\\&=115-7n\end{aligned}$$

Let -25 be the kth term.

$$\therefore\ \begin{aligned}T(k)&=-25\\115-7k&=-25\\7k&=140\\k&=20\end{aligned}$$

$\therefore$ There are 20 terms in the sequence.

(b) Let a and d be the first term and the common difference respectively.

$\because\ a=33$ and $d=37-33=4$

$$\therefore\ \begin{aligned}T(n)&=33+(n-1)(4)\\&=33+4n-4\\&=4n+29\end{aligned}$$

Let 373 be the kth term.

$$\therefore\ \begin{aligned}T(k)&=373\\4k+29&=373\\4k&=344\\k&=86\end{aligned}$$

$\therefore$ There are 86 terms in the sequence.

13. For an arithmetic sequence, it is given that $T(2)=8$ and $T(5)=23$, find the general term of the sequence.
在一個等差數列中，已知 $T(2)=8$ 及 $T(5)=23$，求該數列的通項。

Let a and d be the first term and the common difference respectively.

$T(2) = a + d = 8 \ldots\ldots(1)$
$T(5) = a + 4d = 23 \ldots\ldots(2)$

$(2) - (1),\quad 3d = 15$
$d = 5$

By substituting $d = 5$ into (1), we have

$a + 5 = 8$
$a = 3$

$$\begin{aligned} T(n) &= 3 + (n-1)(5) \\ &= 3 + 5n - 5 \\ &= \underline{\underline{5n - 2}} \end{aligned}$$

14. If the arithmetic mean between 9 and *p* is 16, find the value of *p*.
若 9 與 *p* 的等差中項是 16，求 *p* 的值。

$\because$ The arithmetic mean between 9 and *p* is 16.

$$\therefore \quad \frac{9+p}{2} = 16$$
$$9 + p = 32$$
$$p = \underline{\underline{23}}$$

15. If $x+1, 3x-2, 2x+4$ are in arithmetic sequence, find the value of *x*.
若 $x+1, 3x-2, 2x+4$ 是一個等差數列，求 *x* 的值。

$\because$ $x+1, 3x-2, 2x+4$ are in arithmetic sequence.

$$\begin{aligned} \therefore \quad (x+1) + (2x+4) &= 2(3x-2) \\ 3x + 5 &= 6x - 4 \\ 3x &= 9 \\ x &= \underline{\underline{3}} \end{aligned}$$

16. Given two numbers 8 and 50.

(a) insert two arithmetic means between these two numbers,

(b) insert six arithmetic means between these two numbers.

. 在 8 與 50 兩個數之間插入

(a) 兩個等差中項;

(b) 六個等差中項。

(a) Let *d* be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$8, 8+d, 8+2d, 50$

$$\begin{aligned} \therefore \quad 8 + 3d &= 50 \\ 3d &= 42 \\ d &= 14 \end{aligned}$$

$\therefore$ The two required arithmetic means are 22 and 36.

(b) Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$8, 8+d, 8+2d, 8+3d, 8+4d, 8+5d, 8+6d, 50$

$$\begin{aligned} \therefore\quad 8+7d &= 50 \\ 7d &= 42 \\ d &= 6 \end{aligned}$$

$\therefore$ The six required arithmetic means are:

17. Given two numbers –84 and –14,

(a) insert four arithmetic means between these two numbers,

(b) insert six arithmetic means between these two numbers.

在–84 與–14 兩個數之間插入

(a) 四個等差中項;

(b) 六個等差中項。

(a) Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$-84, -84+d, -84+2d, -84+3d, -84+4d, -14$

$$\begin{aligned} \therefore\quad -84+5d &= -14 \\ 5d &= 70 \\ d &= 14 \end{aligned}$$

$\therefore$ The four required arithmetic means are:

$-70, -56, -42, -28$

(b) Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$-84, -84+d, -84+2d, -84+3d, -84+4d, -84+5d, -84+6d\ -14$

$$\begin{aligned} \therefore\quad -84+7d &= -14 \\ 7d &= 70 \\ d &= 10 \end{aligned}$$

$\therefore$ The six required arithmetic means are:

18. It is given that the first term and the common ratio of a geometric sequence are 2 and $\frac{2}{3}$ respectively.

Find

(a) the general term $T(n)$ of the sequence,

(b) the 3rd term and the 5th term of the sequence.

已知一個等比數列的首項是 2 而公比是$\frac{2}{3}$。求該數列的

(a) 通項 $T(n)$;

(b) 第 3 項和第 5 項。

(a) $\because\ a=2, R=\frac{2}{3}$ and $T(n)=aR^{n-1}$

$\therefore\ T(n)=\underline{\underline{2\left(\frac{2}{3}\right)^{n-1}}}$ (or $\frac{2^n}{3^{n-1}}$)

(b) $\because\ T(n)=2\left(\frac{2}{3}\right)^{n-1}$

$\therefore\ T(3)=2\left(\frac{2}{3}\right)^{3-1}$

$=\underline{\underline{\frac{8}{9}}}$

$T(5)=2\left(\frac{2}{3}\right)^{5-1}$

$=\underline{\underline{\frac{32}{81}}}$

19. If $4, p, q, r, 16$ are in arithmetic sequence, find the values of *p*, *q* and *r*.
若$4, p, q, r, 16$是一個等差數列，求 *p*、*q* 和 *r* 的值。

Let *d* be the common difference.

The arithmetic sequence formed is:

$4, 4+d, 4+2d, 4+3d, 16$

$\therefore\ 4+4d=16$

$4d=12$

$d=3$

$\therefore\ p=4+3=\underline{\underline{7}}$

$q=4+2(3)=\underline{\underline{10}}$

$r=4+3(3)=\underline{\underline{13}}$

20. It is given that the first term and the common ratio of a geometric sequence are $\frac{3}{4}$ and $-\frac{1}{3}$ respectively. Find

(a) the general term $T(n)$ of the sequence,

(b) the 4th term and the 5th term of the sequence.

已知一個等比數列的首項是$\frac{3}{4}$而公比是$-\frac{1}{3}$。求該數列的

(a) 通項 $T(n)$;

(b) 第 4 項和第 5 項。

(a) $\because\ a = \dfrac{3}{4}, R = -\dfrac{1}{3}$ and $T(n) = aR^{n-1}$

$$T(n) = \underline{\underline{\frac{3}{4}\left(-\frac{1}{3}\right)^{n-1}}}$$

(b) $\because\ T(n) = \dfrac{3}{4}\left(-\dfrac{1}{3}\right)^{n-1}$

$$\therefore\ T(4) = \frac{3}{4}\left(-\frac{1}{3}\right)^{4-1}$$
$$= \frac{3}{4}\left(-\frac{1}{27}\right)$$
$$= -\frac{1}{36}$$

$$T(5) = \frac{3}{4}\left(-\frac{1}{3}\right)^{4}$$
$$= \frac{3}{4}\left(\frac{1}{81}\right)$$
$$= \frac{1}{108}$$

$\therefore$ The 4th term and the 5th term are $-\dfrac{1}{36}$ and $\dfrac{1}{108}$ respectively

21. It is given that the first term and the common ratio of a geometric sequence are 8 and 2 respectively. Find

(a) the general term $T(n)$ of the sequence,

(b) the 4th term and the 7th term of the sequence.

已知一個等比數列的首項是 8 而公比是 2。求該數列的

(a) 通項 $T(n)$;

(b) 第 4 項和第 7 項。

(a) $\because\ a = 8, R = 2$ and $T(n) = aR^{n-1}$

$\therefore\ T(n) = 8(2)^{n-1} = 2^3(2)^{n-1} = \underline{\underline{2^{n+2}}}$

(b) $\because\ T(n) = 2^{n+2}$

$\therefore\ T(4) = 2^{4+2} = 64$

$T(7) = 2^{7+2} = 512$

$\therefore$ The 4th term and the 7th term are 64 and 512 respectively.

22. It is given that the first term and the common ratio of a geometric sequence are 5 and –3 respectively.

(a) Find the general term $T(n)$ of the sequence.

(b) If $T(k) = -135$, find the value of k.

已知一個等比數列的首項是 5 而公比是–3。求該數列的

(a) 通項 $T(n)$;

(b) 若 $T(k) = -135$，求 k 的值。

(a) $\because\ a = 5, R = -3$ and $T(n) = aR^{n-1}$

$\therefore\ T(n) = \underline{\underline{5(-3)^{n-1}}}$

(b)
$$\begin{aligned} T(k) &= -135 \\ 5(-3)^{k-1} &= -135 \\ (-3)^{k-1} &= -27 \\ (-3)^{k-1} &= (-3)^3 \\ k-1 &= 3 \\ k &= \underline{\underline{4}} \end{aligned}$$

23. For a geometric sequence, it is given that $T(3) = \dfrac{1}{6}$ and $T(6) = \dfrac{9}{16}$. Find

(a) the first term and the common ratio,

(b) the general term of the sequence.

在一個等比數列中，已知 $T(3) = \dfrac{1}{6}$ 及 $T(6) = \dfrac{9}{16}$。求

(a) 首項和公比;

(b) 該數列的通項。

(a) Let a and R be the first term and the common ratio respectively.

$T(3) = aR^2 = \dfrac{1}{6}$ ……(1)

$T(6) = aR^5 = \dfrac{9}{16}$ ……(2)

(2)÷(1), $R^3 = \dfrac{27}{8}$

$R = \underline{\underline{\dfrac{3}{2}}}$

By substituting $R = \dfrac{3}{2}$ into (1), we have

$$\begin{aligned} a\left(\frac{3}{2}\right)^2 &= \frac{1}{6} \\ a &= \frac{1}{6} \times \frac{4}{9} \\ &= \underline{\underline{\frac{2}{27}}} \end{aligned}$$

(b) $\because\ a = \dfrac{2}{27}, R = \dfrac{3}{2}$

$$\therefore\ T(n) = \frac{2}{27}\left(\frac{3}{2}\right)^{n-1}$$
$$= \frac{2}{3^3}\left(\frac{3}{2}\right)^{n-1}$$
$$= \underline{\underline{\frac{3^{n-4}}{2^{n-2}}}}$$

24. Find the number of terms in each of the following geometric sequences.
. 求以下各等比數列的項數。

(a) $3, 6, 12, \ldots, 768$

(b) $\frac{1}{5}, -\frac{3}{20}, \frac{9}{80}, \ldots, \frac{729}{20480}$

(a) Let a and R be the first term and the common ratio respectively.

$\because\ a = 3, R = \frac{6}{3} = 2$ and $T(n) = aR^{n-1}$

$\therefore\ T(n) = 3(2)^{n-1}$

Let 768 be the kth term.

$$3(2)^{k-1} = 768$$
$$2^{k-1} = 256$$
$$2^{k-1} = 2^8$$
$$k - 1 = 8$$
$$k = 9$$

$\therefore$ There are 9 terms in the sequence.

(b) Let a and R be the first term and the common ratio respectively.

$\because\ a = \frac{1}{5}, R = -\frac{3}{20} \times 5 = -\frac{3}{4}$

and $T(n) = aR^{n-1}$

$\therefore\ T(n) = \frac{1}{5}\left(-\frac{3}{4}\right)^{n-1}$

Let $\frac{729}{20480}$ be the kth term.

$$\frac{1}{5}\left(-\frac{3}{4}\right)^{k-1} = \frac{729}{20480}$$
$$\left(-\frac{3}{4}\right)^{k-1} = \frac{729}{4096}$$
$$\left(-\frac{3}{4}\right)^{k-1} = \left(-\frac{3}{4}\right)^{6}$$
$$k - 1 = 6$$
$$k = 7$$

25 Consider the geometric sequence,

$$6, -\frac{9}{2}, \frac{27}{8}, -\frac{81}{32}, \ldots$$

(a) Find the general term of the sequence.

(b) Find the 8th term of the sequence.

. 考慮以下的等比數列:

$$6, -\frac{9}{2}, \frac{27}{8}, -\frac{81}{32}, \ldots$$

(a) 求該數列的通項。

(b) 求該數列的第 8 項。

(a) Let R be the common ratio.

$$R = -\frac{9}{2} \times \frac{1}{6} = -\frac{3}{4}$$

$$\therefore \; T(n) = \underline{\underline{6\left(-\frac{3}{4}\right)^{n-1}}}$$

(b) $\because \; T(n) = 6\left(-\frac{3}{4}\right)^{n-1}$

$$\therefore \; T(8) = 6\left(-\frac{3}{4}\right)^{7} = \underline{\underline{-\frac{6561}{8192}}}$$

26 For a geometric sequence with positive common ratio, it is given that $T(4) = 5$ and $T(6) = 245$. Find

(a) the first term and the common ratio,

(b) the general term of the sequence,

(c) the value of $\dfrac{T(5) - T(4)}{T(3)}$.

在一個公比為正數的等比數列中，已知 $T(4) = 5$ 及 $T(6) = 245$。求

(a) 首項和公比;

(b) 該數列的通項;

(c) $\dfrac{T(5) - T(4)}{T(3)}$ 的值。

(a) Let a and R be the first term and the common ratio respectively.

$T(4) = aR^3 = 5$ ……(1)
$T(6) = aR^5 = 245$……(2)

(2) ÷ (1), $R^2 = 49$
$R = \underline{\underline{7}}$ or -7 (rejected)

By substituting $R = 7$ into (1), we have

$a(7)^3 = 5$

$a = \underline{\underline{\dfrac{5}{343}}}$

(b) $\because\ a = \dfrac{5}{343}, R = 7$ and $T(n) = aR^{n-1}$

$\therefore\ T(n) = \dfrac{5}{343}(7)^{n-1}$

$= \dfrac{5}{7^3}(7)^{n-1}$

$= \underline{\underline{5(7)^{n-4}}}$

(c) $T(3) = 5(7)^{3-4} = 5(7)^{-1}$

$T(4) = 5(7)^{4-4} = 5$

$T(5) = 5(7)^{5-4} = 5(7)$

$\dfrac{T(5) - T(4)}{T(3)} = \dfrac{5(7) - 5}{5(7)^{-1}}$

$= \dfrac{7-1}{7^{-1}}$

$= \underline{\underline{42}}$

27. It is given that the first term of a geometric sequence is 102 and its common ratio is positive. If $T(6):T(11) = 3:2$, find $T(11)$.

已知一個公比為正數的等比數列的首項是 102。若$T(6):T(11) = 3:2$，求 $T(11)$。

Let R be the common ratio of the sequence.

$\therefore\ T(6) = 102R^5$

$T(11) = 102R^{10}$

$\because\ T(6):T(11) = 3:2$

$\therefore\ \dfrac{T(6)}{T(11)} = \dfrac{3}{2}$

$\dfrac{102R^5}{102R^{10}} = \dfrac{3}{2}$

$\dfrac{1}{R^5} = \dfrac{3}{2}$

$R^5 = \dfrac{2}{3}$

$T(11) = 102R^{10}$

$= 102(R^5)^2$

$= 102\left(\dfrac{2}{3}\right)^2$

$= \underline{\underline{\dfrac{136}{3}}}$

28. If $27, a, b, \frac{1}{27}$ are in geometric sequence, find the values of *a* and *b*.

. 若 $27, a, b, \frac{1}{27}$ 是一個等比數列，求 a 和 b 的值。

$\because$ $27, a, b, \frac{1}{27}$ are in geometric sequence

$\therefore$ $a^2 = 27b$ ……(1)

$\frac{a}{27} = b^2$ ……(2)

(1) × (2), $\frac{a^3}{27} = 27b^3$

$a^3 = 27^2 b^3$

$a^3 = (3^2 b)^3$

$a = 9b$

By substituting $a = 9b$ into (1), we have

$(9b)^2 = 27b$

$81b^2 - 27b = 0$

$27b(3b-1) = 0$

$b = \underline{\underline{\frac{1}{3}}}$ or 0 (rejected)

By substituting $b = \frac{1}{3}$ into $a = 9b$, we have

$a = 9\left(\frac{1}{3}\right) = \underline{\underline{3}}$

29. If the geometric mean between $\frac{x-1}{2}$ and $3x+2$ is *x*, find the values of *x*.

若 x 是 $\frac{x-1}{2}$ 與 $3x+2$ 的等比中項，求 x 的值。

$\because$ The geometric mean between $\frac{x-1}{2}$ and $3x+2$ is *x*.

$\therefore$ $\left(\frac{x-1}{2}\right)(3x+2) = x^2$

$(x-1)(3x+2) = 2x^2$

$3x^2 - x - 2 = 2x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = \underline{\underline{2}}$ or $\underline{\underline{-1}}$

30 If $2x-1, x, x-2$ are in geometric sequence, find the values of x

(Leave your answer in surd form).

若 $2x-1, x, x-2$ *是一個等比數列，求* x *的值。 (答案以根式表示。)*

$\because$ $2x-1, x, x-2$ are in geometric sequence.

$$\therefore \quad (2x-1)(x-2) = x^2$$
$$2x^2 - 5x + 2 = x^2$$
$$x^2 - 5x + 2 = 0$$

$$x = \frac{5+\sqrt{5^2-4(1)(2)}}{2} \text{ or } \frac{5-\sqrt{5^2-4(1)(2)}}{2}$$

$$= \underline{\underline{\frac{5+\sqrt{17}}{2}}} \text{ or } \underline{\underline{\frac{5-\sqrt{17}}{2}}}$$

31. Insert three geometric means between 1875 and 48.

在 1875 與 48 之間插入三個等比中項。

Let R be the common ratio of the geometric sequence to be formed

The geometric sequence formed is:

$1875, 1875R, 1875R^2, 1875R^3, 48$

$$1875R^4 = 48$$
$$R^4 = \frac{16}{625}$$
$$R = \frac{2}{5} \text{ or } -\frac{2}{5}$$

When $R = \frac{2}{5}$, the three required geometric means are

$750, 300, 120$.

When $R = -\frac{2}{5}$, the three required geometric means are

$-750, 300, -120$.

32**.** Insert four geometric means between 6 and $-\frac{2}{81}$.

在 6 與$-\frac{2}{81}$之間插入四個等比中項。

Let R be the common ratio of the geometric sequence to be formed

The geometric sequence formed is:

$6, 6R, 6R^2, 6R^3, 6R^4, -\frac{2}{81}$

$$\therefore \quad 6R^5 = -\frac{2}{81}$$
$$R^5 = -\frac{1}{243}$$
$$R = -\frac{1}{3}$$

The four required geometric means are:

$$-2,\frac{2}{3},-\frac{2}{9},\frac{2}{27}$$

33. Find the sum of the first 16 terms of an arithmetic series if its second term is –8 and the 8th term is 40.
若一個等差級數的第二項是–8 而第 8 項是 40，求該等差級數首 16 項之和。.

Let a and d be the first term and the common difference of the arithmetic series.

$$a+d=-8\ldots\ldots(1)$$
$$a+7d=40\ldots\ldots(2)$$

$$(2)-(1),\quad 6d=48$$
$$d=8$$

By substituting $d=8$ into (1), we have

$$a+8=-8$$
$$a=-16$$

$$\therefore\quad S(16)=\frac{16}{2}[2(-16)+(16-1)(8)]$$
$$=\underline{\underline{704}}$$

34. Find the sum of the first 15 terms of an arithmetic series if its first term is 4 and the 15th term is 26.
若一個等差級數的首項是 4 而第 15 項是 26，求該等差級數首 15 項之和。

$$\because\quad a=4,\ell=26,n=15$$

$$\therefore\quad S(15)=\frac{15(4+26)}{2}$$
$$=\underline{\underline{225}}$$

35. Find the sum of the first 16 terms of an arithmetic series if its third term is 28 and the common difference is -9.
若一個等差級數的第三項是 28 而公差是–9，求該等差級數首 16 項之和。

Let a be the first term of the series

$$\because\quad d=-9$$

$$\therefore\quad a+2(-9)=28$$
$$a=46$$

$$\therefore\quad S(16)=\frac{16}{2}[2(46)+(16-1)(-9)]$$
$$=\underline{\underline{-344}}$$

36. Find the sum of the 29 arithmetic means between –30 and 90.
求在–30 與 90 之間的 29 個等差中項之和。

Let d be the common difference of the arithmetic sequence formed.

$$\begin{aligned}-30+30d&=90\\30d&=120\\d&=4\end{aligned}$$

The first arithmetic mean added $=-30+4=-26$

$$\begin{aligned}\text{Sum of arithmetic means}&=\frac{29}{2}[2(-26)+(29-1)(4)]\\&=\underline{\underline{870}}\end{aligned}$$

37. **(a)** Find the sum of the integers between 1 and 150 inclusive.

(b) Find the sum of the integers between 1 and 150 that are multiples of 7.

(c) Find the sum of the integers between 1 and 150 that are not multiples of 7.

(a) 求在 1 與 150 之間(包括 1 和 150)所有整數之和。

(b) 在 1 與 150 之間所有 7 的倍數之和。

(c) 求在 1 與 150 之間(包括 1 和 150)所有不能被 7 整除的整數之和。

(a) $\because$ There are 150 terms in $1+2+\ldots+150$

$\therefore$ Sum of integers between 1 and 150

$$\begin{aligned}&=\frac{150}{2}(1+150)\\&=\underline{\underline{11325}}\end{aligned}$$

(b) The smallest multiple of 7 between 1 and 150

$=1\times7=7$

The largest multiple of 7 between 1 and 150

$=21\times7=147$

$\therefore$ There are 21 multiples of 7 between 1 and 150.

Sum of multiples of 7 between 1 and 150

$$\begin{aligned}&=\frac{21}{2}(7+147)\\&=\underline{\underline{1617}}\end{aligned}$$

(c) Required sum $=11325-1617=\underline{\underline{9708}}$

38. Consider the arithmetic series $-123-134-145-\ldots-343$.

(a) Find the number of terms of the series.

(b) Find the sum of the series.

考慮等差級數$-123-134-145-\ldots-343$。

(a) 求該級數的項數。

(b) 求該級數之和。

(a) Let d be the common difference of the series and n be the number of terms of the series.

$\because \quad d = -134 - (-123) = -11$

$$\therefore \quad -123 + (n-1)(-11) = -343$$
$$-11(n-1) = -220$$
$$n - 1 = 20$$
$$n = 21$$

$\therefore$ There are 21 terms in the series.

(b) $\because \quad a = -123, \ell = -343, n = 21$

$$\therefore \quad S(21) = \frac{21}{2}[-123 + (-343)]$$
$$= \underline{\underline{-4893}}$$

39. The sum of the first 8 terms of an arithmetic series is 212. If the 21st term is −23, find

(a) the first term and the common difference of the series

(b) the sum of the first 15 terms of the series.

已知一個等差級數首 8 項之和是 212。若第 21 項是–23，求

(a) 該級數的首項和公差;

(b) 該級數首 15 項之和。

(a) Let a and d be the first term and the common difference of the series.

$$S(8) = 212$$
$$\frac{8}{2}[2a + (8-1)d] = 212$$
$$2a + 7d = 53 \ldots\ldots(1)$$

$$T(21) = -23$$
$$a + (21-1)d = -23$$
$$a + 20d = -23 \ldots\ldots(2)$$

$(1) - 2 \times (2)$,

$$-33d = 99$$
$$d = -3$$

By substituting $d = -3$ into (2), we have

$$a + 20(-3) = -23$$
$$a = 37$$

$\therefore$ The first term and the common difference are 37 and −3 respectively.

(b) $\because \quad a = 37, d = -3, n = 15$

$$\therefore \quad S(15) = \frac{15}{2}[2(37) + (15-1)(-3)]$$
$$= \underline{\underline{240}}$$

40. Find the sum of the geometric series $-\frac{1}{81}+\frac{1}{27}-\frac{1}{9}+\ldots+243$.

. 求等比級數 $-\frac{1}{81}+\frac{1}{27}-\frac{1}{9}+\ldots+243$ 各項之和。

Let R and n be the common ratio and the number of terms of the geometric series.

$$R=\frac{1}{27}\div\left(-\frac{1}{81}\right)=-3$$

$$\begin{aligned}T(n)&=243\\-\frac{1}{81}(-3)^{n-1}&=243\\(-3)^{n-1}&=-19683\\(-3)^{n-1}&=(-3)^9\\n&=10\end{aligned}$$

$$\therefore\quad S(10)=\frac{-\frac{1}{81}\left[1-\left(-\frac{1}{3}\right)^{10}\right]}{1-\left(-\frac{1}{3}\right)}=\underline{\underline{-\frac{14762}{1594323}}}$$

41. Find the sum of the first 3 terms of a geometric series if its first term is 49 and the common ratio is $-\frac{1}{7}$.

若一個等比級數的首項是 49 而公比是 $-\frac{1}{7}$，求該級數首 3 項之和。

$$\because\quad a=49, R=-\frac{1}{7}, n=3$$

$$\begin{aligned}\therefore\quad S(3)&=\frac{49\left(1-\left(-\frac{1}{7}\right)^3\right)}{1-\left(-\frac{1}{7}\right)}\\&=49\times\frac{7}{8}\times\left(\frac{344}{343}\right)\\&=\underline{\underline{43}}\end{aligned}$$

42. Find the sum of the first 10 terms of a geometric series if its first term is –5 and the common ratio is 3.

. 若一個等比級數的首項是–5 而公比是 3，求該級數首 10 項之和。

$$\because\quad a=-5, R=3, n=10$$

$$\therefore\quad S(10)=\frac{-5(3^{10}-1)}{3-1}=\underline{\underline{-147620}}$$

43. Find the sum of first 5 terms of a geometric series if its second term is 6 and the common ratio is $\frac{2}{3}$.

若一個等比級數的第二項是 6 而公比是$\frac{2}{3}$，求該級數首 5 項之和。

Let a be the first term.

$$T(2) = 6$$
$$a\left(\frac{2}{3}\right) = 6$$
$$a = 9$$
$$\therefore \quad S(5) = \frac{9\left[1-\left(\frac{2}{3}\right)^5\right]}{1-\frac{2}{3}} = \underline{\underline{\frac{211}{9}}}$$

44. The sum of the first two terms of a geometric series is 8, and the sum from the 3rd term to 4th term is $\frac{8}{9}$. Find the first term and the common ratio of the series.

*已知一個等比級數首兩項之和是 **8**，而第 **3** 項至第 **4** 項之和是*$\frac{8}{9}$*。求該級數的首項和公比。*

Let a and R be the first term and the common ratio respectively.

$$T(1)+T(2) = 8$$
$$a + aR = 8$$
$$a(1+R) = 8 \ldots\ldots(1)$$

$$T(3)+T(4) = \frac{8}{9}$$
$$aR^2 + aR^3 = \frac{8}{9}$$
$$aR^2(1+R) = \frac{8}{9} \ldots\ldots(2)$$

$(2) \div (1)$, $R^2 = \frac{1}{9}$

$$R = \frac{1}{3} \text{ or } -\frac{1}{3}$$

By substituting $R = \frac{1}{3}$ into (1), we have

$$a(1+\frac{1}{3}) = 8$$
$$a = 6$$

By substituting $R = -\frac{1}{3}$ into (1), we have

$$a(1-\frac{1}{3}) = 8$$
$$a = 12$$

$$\therefore \quad \begin{cases} a = 6 \\ R = \frac{1}{3} \end{cases} \text{ or } \begin{cases} a = 12 \\ R = -\frac{1}{3} \end{cases}$$

***45*.** Find the sum of the geometric series $8+4+2+\ldots+\dfrac{1}{16}$.

求等比級數 $8+4+2+\ldots+\dfrac{1}{16}$ 各項之和。

Let R and n be the common ratio and the number of terms of the geometric series.

$$R=\frac{4}{8}=\frac{1}{2}$$

$$\begin{aligned} T(n)&=\frac{1}{16}\\ 8\left(\frac{1}{2}\right)^{n-1}&=\frac{1}{16}\\ \left(\frac{1}{2}\right)^{n-1}&=\left(\frac{1}{2}\right)^{7}\\ n&=8 \end{aligned}$$

$$\therefore\quad S(8)=\frac{8\left[1-\left(\frac{1}{2}\right)^{8}\right]}{1-\frac{1}{2}}=\underline{\underline{\frac{255}{16}}}$$

46. Express the following recurring decimals as fractions.

把下列各循環小數化為分數。

(a) $0.\dot{5}$

(b) $0.0\dot{1}\dot{6}$

(c) $2.\dot{2}2\dot{5}$

(a)

$$\begin{aligned} 0.\dot{5}&=0.5555\ldots\\ &=0.5+0.05+0.005+0.0005+\ldots\\ &=\frac{0.5}{1-0.1}\\ &=\underline{\underline{\frac{5}{9}}} \end{aligned}$$

(b)

$$\begin{aligned} 0.0\dot{1}\dot{6}&=0.016+0.00016+0.0000016\\ &=\frac{0.016}{1-0.01}\\ &=\underline{\underline{\frac{8}{495}}} \end{aligned}$$

(c)

$$\begin{aligned} &2.\dot{2}2\dot{5}\\ &=2+0.225+0.000225+0.000000225+\ldots\\ &=2+\frac{0.225}{1-0.001}\\ &=\frac{247}{111} \end{aligned}$$

47. Find the least number of terms of the geometric series $6+18+54+\ldots$ that can give a sum greater than 5000.

對於等比級數 $6+18+54+\ldots$，問最少需要取多少項才可使級數之和大於 5000？

Let n be the number of terms required and R be the common ratio.

$$R=\frac{18}{6}=3$$

$$\because \quad S(n)=\frac{6(3^n-1)}{3-1}=3(3^n-1)$$

$$3(3^n-1)>5000$$
$$3^n>\frac{5000}{3}+1$$
$$n\log 3>\log\left(\frac{5000}{3}+1\right)$$
$$n>\frac{\log\left(\frac{5000}{3}+1\right)}{\log 3}$$
$$n>6.75 \text{ (cor. to 2 d.p.)}$$

$\therefore$ At least 7 terms are needed.

48 Find the sum to infinity of the following geometric series.

求下列等比級數的無限項之和。

(a) $27+18+12+8+\ldots$

(b) $64-16+4-1+\ldots$

(a) $\because \quad a=27, R=\frac{18}{27}=\frac{2}{3}$

$$\therefore \quad S(\infty)=\frac{27}{1-\frac{2}{3}}=\underline{\underline{81}}$$

(b) $\because \quad a=64, R=-\frac{16}{64}=-\frac{1}{4}$

$$\therefore \quad S(\infty)=\frac{64}{1-\left(-\frac{1}{4}\right)}=\underline{\underline{\frac{256}{5}}}$$

49. Given an geometric sequence $24,-4,\frac{2}{3},-\frac{1}{9},\ldots$, find the sum of all the positive terms of the sequence.

對於等比數列 $24,-4,\frac{2}{3},-\frac{1}{9},\ldots$，求該數列的所有正數項之和。

The positive terms of the sequence $24,-4,\frac{2}{3},-\frac{1}{9},\ldots$ form a geometric sequence.

Let R be the common ratio of the sequence $24, \frac{2}{3}, \ldots$

$$\therefore \quad R = \frac{\frac{2}{3}}{24} = \frac{1}{36}$$

$$\because \quad a = 24, R = \frac{1}{36}$$

$$\text{Sum of all positive terms} = \frac{24}{1 - \frac{1}{36}}$$
$$= \underline{\underline{\frac{864}{35}}}$$

50. Find the first term of an infinite geometric series if the second term is 8 and the sum to infinity is 32.
若一個無限等比級數的第 2 項是 8，而無限項之和是 32，求該無限等比級數的首項。

Let a and R be the first term and the common ratio respectively.

$$\because \quad T(2) = 8$$
$$\therefore \quad aR = 8 \ldots\ldots(1)$$
$$\because \quad S(\infty) = 32$$
$$\therefore \quad \frac{a}{1-R} = 32 \ldots\ldots(2)$$

$$(1) \div (2), \quad R(1-R) = \frac{1}{4}$$
$$R^2 - R + \frac{1}{4} = 0$$
$$(R - \frac{1}{2})^2 = 0$$
$$R = \frac{1}{2}$$

By substituting $R = \frac{1}{2}$ into (1), we have

$$a(\frac{1}{2}) = 8$$
$$a = \underline{\underline{16}}$$

51 If the sum to infinity of a geometric series is 5 and the common ratio of the series is $\frac{1}{3}$. Find

(a) the 5th term of the series,

(b) the sum of the first three terms of the series.

若一個等比級數的無限項之和是 5 而公比是 $\frac{1}{3}$ 。求該級數的

(a) 第 5 項;

(b) 首三項之和。

(a) Let a be the first term.

$\because \quad S(\infty) = 5$

$$\frac{a}{1-\frac{1}{3}} = 5$$

$\therefore \quad a = \frac{15}{2}$

$\therefore \quad T(5) = \frac{15}{2}\left(\frac{1}{3}\right)^4 = \underline{\underline{\frac{5}{54}}}$

(b) $\because \quad a = \frac{15}{2}, R = \frac{1}{3}, n = 3$

$$\therefore \quad S(3) = \frac{\frac{15}{2}\left[1-\left(\frac{1}{3}\right)^3\right]}{1-\frac{1}{3}} = \underline{\underline{\frac{65}{6}}}$$

Level 2 Questions

程度 2 題目

1. How many negative terms are there in the arithmetic sequence $-203, -198, -193, \ldots$?

問等差數列 $-203, -198, -193, \ldots$ 共有多少個負數項?

Common difference = –198 – (–203) = 5

$$\therefore \quad T(n) = -203 + (n-1)(5) = 5n - 208$$

Suppose there are k negative terms in the sequence.

$$T(k) < 0$$
$$5k - 208 < 0$$
$$k < 41\frac{3}{5}$$

$\therefore$ There are 41 negative terms in the sequence.

2. It is given that $\sqrt{3}, \sqrt{12}, \sqrt{p}, \ldots$ are in arithmetic sequence. Find

(a) the value of p,

(b) the sixth term of the sequence.

已知 $\sqrt{3}, \sqrt{12}, \sqrt{p}, \ldots$ 是一個等差數列。求

(a) p 的值;

(b) 該數列的第 6 項。

(a) $\because \quad \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$

$$\therefore \quad \text{Common difference} = \sqrt{12} - \sqrt{3} = 2\sqrt{3} - \sqrt{3} = \sqrt{3}$$

$$\therefore \quad \sqrt{p} = \sqrt{12} + \sqrt{3} = 2\sqrt{3} + \sqrt{3} = 3\sqrt{3} = \sqrt{27}$$

$$\therefore \quad p = \underline{\underline{27}}$$

(b)
$$T(6) = \sqrt{3} + (6-1)\sqrt{3} = \sqrt{3} + 5\sqrt{3} = \underline{\underline{6\sqrt{3}}} \text{ (or } \sqrt{108}\text{)}$$

3. Given an arithmetic sequence $88, 79, 70, \ldots$, if $T(k)$ is the first negative term, find the value of k.

已知一個等差數列 $88, 79, 70, \ldots$。若 $T(k)$ 是該數列的首個負數項，求 k 的值。

$$\text{Common difference} = 79 - 88 = -9$$

$\therefore \quad T(k) = 88 + (k-1)(-9)$
$= 88 - 9k + 9$
$= 97 - 9k$

$T(k) < 0$
$97 - 9k < 0$
$9k > 97$
$k > 10\frac{7}{9}$

$\therefore \quad k = \underline{\underline{11}}$

4. For an arithmetic sequence, the 8th term is 17 times that of the 2nd term, and the 6th term is greater than 9 times that of the 2nd term by 8. Find

(a) the general term of the arithmetic sequence,

(b) the value of k if $T(2k) - 2T(3) = 29$.

已知一個等差數列的第 8 項是第 2 項的 17 倍，而數列的第 6 項又比第 2 項的 9 倍大 8。

(a) 求該等差數列的通項。

(b) 若 $T(2k) - 2T(3) = 29$，求 k 的值。

(a) Let a and d be the first term and the common difference respectively.

$\because \; T(n) = a + (n-1)d$

$\therefore \; T(8) = a + 7d$
$T(2) = a + d$
$T(6) = a + 5d$

$\therefore \quad T(8) = 17T(2)$
$a + 7d = 17(a + d)$
$a + 7d = 17a + 17d$
$16a = -10d$
$8a = -5d \qquad \ldots\ldots(1)$

$T(6) = 9T(2) + 8$
$a + 5d = 9(a + d) + 8$
$a + 5d = 9a + 9d + 8$
$4d = -8a - 8$
$d = -2a - 2 \qquad \ldots\ldots(2)$

By substituting (2) into (1), we have

$8a = -5(-2a - 2)$
$= 10a + 10$
$-2a = 10$
$a = -5$

By substituting $a = -5$ into (2), we have

$d = -2(-5) - 2$
$= 8$

$\therefore \; T(n) = -5 + (n-1)(8)$
$= \underline{\underline{8n - 13}}$

(b) $T(2k) = 8(2k) - 13$
$= 16k - 13$
$T(3) = 8(3) - 13$
$= 11$

$\because$ $T(2k) - 2T(3) = 29$

$\therefore$ $16k - 13 - 2(11) = 29$
$16k - 35 = 29$
$16k = 64$
$k = \underline{\underline{4}}$

5. How many positive terms are there in the arithmetic sequence $230, 212, 194, \dots$?
問等差數列 $230, 212, 194, \dots$ 共有多少個正數項？

Common difference = 212 – 230 = –18

$\therefore$ $T(n) = 230 + (n-1)(-18)$
$= 248 - 18n$

Suppose there are k positive terms in the sequence.

$T(k) < 0$
$248 - 18k > 0$
$k < 13\frac{7}{9}$

$\therefore$ There are 13 positive terms in the sequence.

6. All interior angles of a convex quadrilateral form an arithmetic sequence and the smallest interior angle of the quadrilateral is 60°. Find the other three interior angles of the quadrilateral.
已知一個凸四邊形的所有內角形成一個等差數列，而最小的內角是 $60°$。求該四邊形其餘的內角。

Let d be the common difference.

$\therefore$ The interior angles of the quadrilateral are:

60°, 60° + d, 60° + 2d, 60° + 3d.

$\because$ $60° + (60° + d) + (60° + 2d) + (60° + 3d) = 360°$
$240° + 6d = 360°$
$40° + d = 60°$
$d = 20°$

$\therefore$ The other interior angles of the quadrilateral are 80°, 100° and 120°.

7. For an arithmetic sequence, it is given that $T(1) = 8$ and $T(5) = 3T(2)$. Find

(a) the general term of the sequence,

(b) the 15th term of the sequence.

對於一個等差數列，已知 $T(1) = 8$ 及 $T(5) = 3T(2)$。求該數列的

(a) 通項;

(b) 第 15 項。

Let d be the common difference.

$\therefore$ The interior angles of the quadrilateral are:

$60°$, $60° + d$, $60° + 2d$, $60° + 3d$.

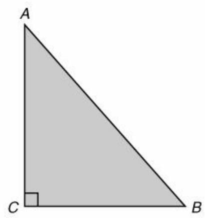
$\because$

$$60°+(60°+d)+(60°+2d)+(60°+3d)=360°$$
$$240°+6d=360°$$
$$40°+d=60°$$
$$d=20°$$

$\therefore$ The other interior angles of the quadrilateral are 80°, 100° and 120°.

8. In the figure, $\triangle ABC$ is a right-angled triangle where $AC > BC$ and $\angle C = 90°$. If the lengths of BC, AC, AB are in arithmetic sequence, find $AB : AC : BC$.

在圖中，$\triangle ABC$ 是一個直角三角形，其中 $AC > BC$ 及 $\angle C = 90°$。若 BC、AC、AB 的長度是一個等差數列，求 $AB : AC : BC$。



Let d be the common difference and $BC = a$.

$\therefore$ $AC = a + d$

$AB = a + 2d$

$\because$ $AC^2 + BC^2 = AB^2$ (Pyth. theorem)

$$(a+d)^2+a^2=(a+2d)^2$$
$$2a^2+2ad+d^2=a^2+4ad+4d^2$$
$$a^2-2ad-3d^2=0$$
$$(a-3d)(a+d)=0$$
$$a=3d \text{ or } a=-d \text{ (rejected)}$$

$\therefore$ $BC = 3d$, $AC = 4d$, $AB = 5d$

$\therefore$ $AB : AC : BC = \underline{\underline{5:4:3}}$

9. For an arithmetic sequence, it is given that $T(8)=15$ and $T(5)+T(17)=42$. Find

(a) the first term and the common difference of the sequence.

(b) the general term of the sequence.

(c) the smallest term of the sequence that is greater than 1000.

對於一個等差數列，已知 $T(8)=15$ 及 $T(5)+T(17)=42$。求

(a) 該數列的首項和公差;

(b) 該數列的通項;

(c) 該數列中僅大於 1000 的項。

(a) Let a and d be the first term and the common difference respectively.

$\because$ $T(n)=a+(n-1)d$

$\therefore$ $T(8)=a+7d=15$ ……(1)

$T(5)+T(17)=(a+4d)+(a+16d)=42$ ……(2)

From (2),

$$2a+20d=42$$
$$a+10d=21 \quad \ldots\ldots(3)$$

(3) – (1), $3d=6$
$$d=2$$

By substituting d = 2 into (1), we have

$$a+7(2)=15$$
$$a=1$$

∴ The first term is 1 and the common difference is 2.

(b) ∵ a = 1, d = 2

∴ $T(n)=1+(n-1)(2)$
$$=\underline{\underline{2n-1}}$$

(c) Let kth term be the smallest term so that $T(k)$ > 1000.

$$T(k)>1000$$
$$2k-1>1000$$
$$k>500\frac{1}{2}$$

∴ The smallest term $=2\times501-1$
$$=\underline{\underline{1001}}$$

10. Insert three arithmetic means, in terms of a, between $a+3$ and a^2+1,
在 $a+3$ 與 a^2+1 之間插入三個等差中項，答案以 a 表示。

Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is

$a+3$, $a+3+d$, $a+3+2d$, $a+3+3d$, a^2+1

∴ $a+3+4d=a^2+1$
$$4d=a^2-a-2$$
$$d=\frac{a^2-a-2}{4}$$

∴ The required arithmetic means are

$$a+3+\frac{a^2-a-2}{4},\ a+3+2\left(\frac{a^2-a-2}{4}\right),\ a+3+3\left(\frac{a^2-a-2}{4}\right)$$

i.e. $\frac{a^2+3a+10}{4}$, $\frac{a^2+a+4}{2}$, $\frac{3a^2+a+6}{4}$

11. If x is the arithmetic mean between –7 and y, while y is the arithmetic mean between $x+9$ and 14, find the value of x and y.
若 –7 與 y 的等差中項是 x，而 $x+9$ 與 14 的等差中項是 y，求 x 和 y 的值。

∵ x is the arithmetic mean between –7 and y.

∴ $2x=y-7$
$$y=2x+7 \quad \ldots\ldots(1)$$

$\because$ y is the arithmetic mean between $x + 9$ and 14.

$\therefore$ $2y = (x+9)+14$
$2y = x+23$ ……(2)

By substituting (1) into (2), we have

$$\begin{aligned} 2(2x+7) &= x+23 \\ 4x+14 &= x+23 \\ 3x &= 9 \\ x &= \underline{\underline{3}} \end{aligned}$$

By substituting $x = 3$ into (1), we have

$$\begin{aligned} y &= 2(3)+7 \\ &= \underline{\underline{13}} \end{aligned}$$

12. Insert four arithmetic means, in terms of m, between -3 and $2m$.
在 -3 與 $2m$ 之間插入四個等差中項，答案以 m 表示。

Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$-3, -3 + d, -3 + 2d, -3 + 3d, -3 + 4d, 2m$

$$\therefore \quad \begin{aligned} -3+5d &= 2m \\ 5d &= 2m+3 \\ d &= \frac{2m+3}{5} \end{aligned}$$

$\therefore$ The required four arithmetic means are:

$$-3+\frac{2m+3}{5},\ -3+2\left(\frac{2m+3}{5}\right),\ -3+3\left(\frac{2m+3}{5}\right),\ -3+4\left(\frac{2m+3}{5}\right)$$

i.e. $\dfrac{2m-12}{5}, \dfrac{4m-9}{5}, \dfrac{6m-6}{5}, \dfrac{8m-3}{5}$

13. Find the number of terms in the geometric sequence
$2, 10, 50, 250, \ldots$
which is smaller than 5000.
求在等比數列 $2, 10, 50, 250, \ldots$ 中小於 5000 的項數。

. Let R be the common ratio

$$R = \frac{10}{2} = 5$$

$\because$ $a = 2, R = 5$

$\therefore$ $T(n) = 2(5)^{n-1}$

Let $T(k)$ be the greatest term so that $T(k) < 5000$.

$$
\begin{aligned}
T(k) &< 5000 \\
2(5)^{k-1} &< 5000 \\
5^{k-1} &< 2500 \\
\log 5^{k-1} &< \log 2500 \\
(k-1)\log 5 &< \log 2500 \\
k-1 &< \frac{\log 2500}{\log 5} \\
k &< 5.86 \text{ (cor. to 2 d.p.)}
\end{aligned}
$$

∴ There are 5 terms in the sequence which are less than 5000.

14 If $\log(x+2)$ is the arithmetic mean between 1 and $\log(x+3.1)$, find the value of x.

若 1 與 $\log(x + 3.1)$ 的等差中項是 $\log(x + 2)$，求 x 的值。

∵ $\log(x + 2)$ is the arithmetic mean between 1 and $\log(x + 3.1)$.

∴
$$
\begin{aligned}
2\log(x+2) &= 1+\log(x+3.1) \\
\log(x+2)^2 &= \log 10+\log(x+3.1) \\
\log(x^2+4x+4) &= \log[10(x+3.1)] \\
x^2+4x+4 &= 10(x+3.1) \\
x^2-6x-27 &= 0 \\
(x-9)(x+3) &= 0 \\
x &= \underline{\underline{9}} \text{ or } x=-3 \text{ (rejected)}
\end{aligned}
$$

15. If $\sin\theta$ is the arithmetic mean between $3\cos\theta$ and $-\sin\theta$, find θ for $0° \le \theta \le 360°$.

若 $3\cos\theta$ 與 $-\sin\theta$ 的等差中項是 $\sin\theta$，其中$0° \le \theta \le 360°$，求 θ 的值。

∵ $\sin\theta$ is the arithmetic mean between $3\cos\theta$ and $-\sin\theta$.

∴
$$
\begin{aligned}
2\sin\theta &= 3\cos\theta+(-\sin\theta) \\
3\sin\theta &= 3\cos\theta \\
\tan\theta &= 1
\end{aligned}
$$

∴ $\theta = \underline{\underline{45°}}$ or $\underline{\underline{225°}}$

16. It is given that $T(1), T(2), T(3), \ldots, T(n), \ldots$ are in geometric sequence. Show that the sequence $2T(1), 2^2T(2), 2^3T(3), \ldots, 2^nT(n), \ldots$ is a geometric sequence.

. 已知 $T(1),T(2),T(3),...,T(n),...$ 是一個等比數列。證明 $2T(1),2^2T(2),2^3T(3),...,2^nT(n),...$也是一個等比數列。

∵ $T(1)$, $T(2)$, ... are in geometric sequence.

∴ $\frac{T(n)}{T(n-1)} = R$ for some constant R,

$$\frac{2^nT(n)}{2^{n-1}T(n-1)} = 2\times\frac{T(n)}{T(n-1)} = 2R$$

$\therefore$ $2T(1)$, $2^2T(2)$, … are geometric sequence with common ratio $2R$.

17. The lengths of sides of $\triangle ABC$ are in geometric sequence and the perimeter of the triangle is 38 cm. If the shortest side of the triangle is $\frac{4}{9}$ of its longest side, find

(a) the lengths of the three sides of the triangle,

(b) the area of the triangle.

$\triangle ABC$ 的邊長是一個等比數列且其周界為 38 cm。若該三角形最短的邊的邊長是最長的邊的 $\frac{4}{9}$，求

(a) 該三角形三條邊的長度;

(b) 該三角形的面積。

(a) Let the length of shortest sides of the triangle be a cm and R be the common ratio. ($R > 1$)

$\therefore$ The length of sides of the triangle are a cm, aR cm and aR^2 cm.

$\therefore$ $a + aR + aR^2 = 38$ ……(1)

$a = \frac{4}{9}aR^2$ ……(2)

From (2), $R^2 = \frac{9}{4}$

$R = \frac{3}{2}$ or $-\frac{3}{2}$ (rejected)

By substituting $R = \frac{3}{2}$ into (1), we have

$$a + \frac{3a}{2} + \frac{9a}{4} = 38$$
$$\frac{19a}{4} = 38$$
$$a = 8$$

$\therefore$ The lengths of sides of the triangle are 8 cm, 12 cm and 18 cm.

(b) Let $S = \frac{a + aR + aR^2}{2}$ cm

$= \frac{8 + 12 + 18}{2}$ cm

$= 19$ cm

By Heron's formula,

area $= \sqrt{19(19-8)(19-12)(19-18)}$ cm^2

$= \underline{\underline{\sqrt{1463}\text{ cm}^2}}$

18 Find the number of terms in the geometric sequence

$800, 600, 450, \ldots$

which is greater than 60.

求在等比數列$800, 600, 450, \ldots$中大於 60 的項數。

Let R be the common ratio.

$$R = \frac{600}{800} = \frac{3}{4}$$

$\because \quad a = 800, \quad R = \frac{3}{4}$

$\therefore \quad T(n) = 800\left(\frac{3}{4}\right)^{n-1}$

Let $T(k)$ be the greatest term so that $T(k) > 60$.

$$\begin{aligned} T(k) &> 60 \\ 800\left(\frac{3}{4}\right)^{k-1} &> 60 \\ \left(\frac{3}{4}\right)^{k-1} &> \frac{3}{40} \\ \log\left(\frac{3}{4}\right)^{k-1} &> \log\frac{3}{40} \\ (k-1)\log\frac{3}{4} &> \log\frac{3}{40} \\ k-1 &< \frac{\log\frac{3}{40}}{\log\frac{3}{4}} \\ k &< 10.004 \text{ (cor. to 3 d.p.)} \end{aligned}$$

$\therefore$ There are 10 terms in the sequence which are greater than 60.

19. It is given that the three interior angles of a triangle are in geometric sequence. If the smallest interior angle is one-fourth of the largest interior angle, find all the interior angles of the triangle.

(Give your answers correct to 2 decimal places.)

已知一個三角形的三個內角形成一個等比數列。若最小的內角是最大的內角的四分之一，求該三角形的所有內角。

(答案須準確至二位小數。)

Let the smallest interior angle of the triangle is a and the common ratio is R. ($R > 1$)

$\therefore$ The interior angles are a, aR and aR^2.

$\because \quad a = \frac{1}{4}aR^2$

$\therefore \quad R^2 = 4$

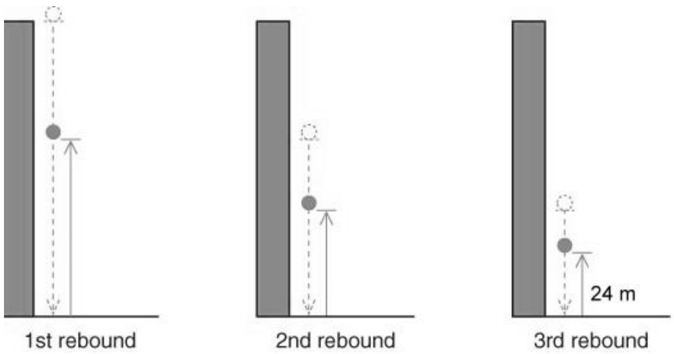
$\quad R = 2$ or -2 (rejected)

$$a + 2a + 2^2 a = 180° \quad (\angle \text{ sum of } \Delta)$$
$$7a = 180°$$
$$a = \frac{180°}{7}$$
$$= 25.7143°$$

$\therefore$ The interior angles are 25.71° (cor. to 2 d.p.), 51.43° (cor. to 2 d.p.) and 102.86° (cor. to 2 d.p.)

20. A ball is dropped vertically from the top of a tower. Each time it reaches the ground, it rebounds to a height which is $\frac{2}{3}$ of its previous height. If the ball reaches a height of 24 m after the third rebound, find the height of the tower.

. 把一個皮球從一幢大廈的頂層垂直掉下來。當皮球到達地面時，它會垂直向上反彈至原來高度的 $\frac{2}{3}$ 。若皮球第 3 次反彈後所達到的高度是 24m，求該大廈的高度。



Let h m be the height of the tower. After third rebounds, the height of the ball reaches

$$= h\left(\frac{2}{3}\right)^3$$
$$= \frac{8h}{27}$$

$$\therefore \quad \frac{8h}{27} = 24$$
$$h = 81$$

$\therefore$ The height of the building is 81 m.

21. The value of a car is $270 000 in the year 2005 and it is known that the value of the car decreases by 9% each year.

(a) Find the value of the car in the year 2017. (Give your answer correct to the nearest dollar.)

(b) When will the value of car half of that in the year 2005?

某私家車在 2005 年的售價是 $270 000。已知該輛私家車的售價每年都減少 9 %。

(a) 求該輛私家車在 2017 年的售價。(答案須準確至最接近的元。)

(b) 問該輛私家車在哪一年的售價將會減至 2005 年時的售價的一半？

(a) Value of car in the year 2017

$= \$270000 \times (1-9\%)^{12}$
$= \underline{\underline{\$87\,068}}$ (cor. to the nearest dollar)

(b) Let n be the number of years.

$$270000 \times (1-9\%)^n = \frac{270000}{2}$$
$$0.91^n = \frac{1}{2}$$
$$\log 0.91^n = \log\frac{1}{2}$$
$$n \log 0.91 = \log\frac{1}{2}$$
$$n = \frac{\log\frac{1}{2}}{\log 0.91}$$
$$= 7.35 \text{ (cor. to 2 d.p.)}$$

∴ The value of the car will be halved in the year 2013.

22. If p, q, r and s are the four geometric means between 2 and 50, find the value of $\log p + \log q + \log r + \log s$.
若 p, q, r 和 s 是 2 與 50 之間的四個等比中項，求 $\log p + \log q + \log r + \log s$ 的值。

Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence to be formed is:

2, $2R$, $2R^2$, $2R^3$, $2R^4$, 50.

∴ $2R^5 = 50$

∴ $\log p + \log q + \log r + \log s$
$= \log(pqrs)$
$= \log[(2R)(2R^2)(2R^3)(2R^4)]$
$= \log(16R^{10})$
$= \log[4(2R^5)^2]$
$= \log 10000$
$= \underline{\underline{4}}$

23. In a certain experiment, a kind of algae is placed in a pool. The area covered by the algae is doubled at every three hours. If it will take 18 hours to cover the pool, how long will it take for the algae to cover $\frac{1}{3}$ of the pool? (Give your answer correct to 2 decimal places.)
在一次實驗中，某類海藻被放置於水池。每隔三小時，水池被海藻覆蓋的面積便增加一倍，而 18 小時後，整個水池的表面被海藻完全覆蓋。問需要多少時間，該海藻才可覆蓋 $\frac{1}{3}$ 個水池？(答案須準確至二位小數)

Let A be the area of the pool and A_0 be the area covered by the algae at the beginning.

∴ $A = A_0(2)^6$

Let t be time required for covering $\frac{1}{3}$ of the pool.

$$\frac{1}{3}A = A_0(2)^{\frac{t}{3}}$$
$$\frac{2^6}{3}A_0 = A_0(2)^{\frac{t}{3}}$$
$$2^{\frac{t}{3}} = \frac{2^6}{3}$$
$$2^t = \frac{2^{18}}{27}$$
$$\log 2^t = \log\frac{2^{18}}{27}$$
$$t\log 2 = \log\frac{2^{18}}{27}$$
$$t = \frac{\log\frac{2^{18}}{27}}{\log 2}$$
$$t = 13.25 \text{ (cor. to 2 d.p.)}$$

$\therefore$ It takes 13.25 hours to cover $\frac{1}{3}$ of the pool.

24. **(a)** For $b > 0$, insert three geometric means , in terms of b, between $\frac{1}{25b^2}$ and $25b^2$.

(b) If the general term of geometric sequence $\ldots, \frac{1}{25b^2}, x, y, z, 25b^2, \ldots$ is 3^{n-3}, find the value of b.

(a) 對於 $b > 0$，在 $\frac{1}{25b^2}$ 與 $25b^2$ 之間插入三個等比中項，答案以 b 表示。

(b) 若等比數列 $\ldots, \frac{1}{25b^2}, x, y, z, 25b^2, \ldots$ 的通項是 3^{n-3}，求 b 的值。

(a) Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence formed is:

$$\frac{1}{25b^2}, \frac{R}{25b^2}, \frac{R^2}{25b^2}, \frac{R^3}{25b^2}, 25b^2$$

$$\therefore \quad \frac{R^4}{25b^2} = 25b^2$$
$$R^4 = 625b^4$$
$$R = 5b \text{ or } -5b$$

When $R = 5b$, the required geometric means are: $\frac{1}{5b}$, 1, $5b$.

When $R = -5b$, the required geometric means are: $-\frac{1}{5b}$, 1, $-5b$.

(b) From the result of (a), the common ratio of the geometric sequence is $5b$ or $-5b$.

$\because$ General term = 3^{n-3}

$\therefore$ Common ratio = 3

$\therefore$ $5b = 3$ or $-5b = 3$

$$b = \underline{\underline{\frac{3}{5}}} \text{ or } \underline{\underline{-\frac{3}{5}}}$$

25. How many terms must be taken from the arithmetic series $47+43+39+\ldots$ to make a sum equals 24?

對於等差級數 $47+43+39+\ldots$ ，問需要取多少項才可使級數之和等於 24？

Let N be the required number of terms.

$$\begin{aligned}\text{Common difference} &= 43-47\\ &= -4\end{aligned}$$

$$\begin{aligned}\therefore \quad \frac{N}{2}[2(47)+(N-1)(-4)] &= 24\\ N(49-2N) &= 24\\ 2N^2-49N+24 &= 0\\ (2N-1)(N-24) &= 0\end{aligned}$$

$$N = \frac{1}{2}\text{(rejected)} \quad \text{or} \quad N = 24$$

$\therefore$ The number of terms is 24.

26. If $\sin\theta$ is the geometric mean between $\cos\theta$ and 1, find the value of $\cos\theta$ for θ is an acute angle.

若 $\sin\theta$ 是 $\cos\theta$ 和 1 的等比中項，其中 θ 是一個銳角，求 $\cos\theta$ 的值。

$\because$ $\sin\theta$ is the geometric mean between $\cos\theta$ and 1.

$$\begin{aligned}\therefore \quad \sin^2\theta &= (\cos\theta)(1)\\ \sin^2\theta &= \cos\theta\\ 1-\cos^2\theta &= \cos\theta\\ \cos^2\theta+\cos\theta-1 &= 0\end{aligned}$$

$$\cos\theta = \frac{-1+\sqrt{1^2-4(1)(-1)}}{2} \quad \text{or} \quad \frac{-1-\sqrt{1^2-4(1)(-1)}}{2}\text{(rejected)}$$

$$\cos\theta = \underline{\underline{\frac{-1+\sqrt{5}}{2}}}$$

27. It is given that $15, x, y$ are in arithmetic sequence and $x, y, 49$ are in geometric sequence. Find the values of x and y.

已知 $15, x, y$ 是一個等差數列，而 $x, y, 49$ 是一個等比數列。求 x 和 y 的值。

$\because$ 15, x, y are in arithmetic sequence.

$\therefore$ $2x = 15 + y$

$y = 2x - 15$ ……(1)

$\because$ x, y, 49 are in geometric sequence.

$\therefore$ $y^2 = 49x$ ……(2)

By substituting (1) into (2), we have

$$(2x-15)^2 = 49x$$
$$4x^2 - 60x + 225 = 49x$$
$$4x^2 - 109x + 225 = 0$$
$$(4x-9)(x-25) = 0$$
$$x = \underline{\underline{\frac{9}{4}}} \quad \text{or} \quad x = \underline{\underline{25}}$$

When $x = \frac{9}{4}$, $y = 2\left(\frac{9}{4}\right) - 15 = \underline{\underline{-\frac{21}{2}}}$

When $x = 25$, $y = 2(15) - 15 = \underline{\underline{15}}$

28. **(a)** Find the sum of the terms of the arithmetic series $-47-40-33-\ldots+9$.

(b) Find the maximum number of terms in the arithmetic series $-47-40-33-\ldots$ that can give a negative sum.

. **(a)** 求等差級數 $-47-40-33-\ldots+9$ 各項之和。

(b) 對於等差級數 $-47-40-33-\ldots$ ，問最多取多少項使到級數之和仍是負數？

(a) Common difference $= -40-(-47)$

$= 7$

Let n be the number of terms in the series.

$$-47+(n-1)(7) = 9$$
$$7(n-1) = 56$$
$$n-1 = 8$$
$$n = 9$$

$\therefore$ Sum of the series $= \frac{9}{2}(-47+9)$

$= \underline{\underline{-171}}$

(b) Let N be the maximum number of terms.

Sum of N terms of the series

$$= \frac{N}{2}[2(-47)+(N-1)(9)]$$
$$= \frac{N(9N-103)}{2}$$

For the sum of the series is negative,

$$\frac{N(9N-103)}{2}<0$$
$$9N-103<0 \quad (\because N>0)$$
$$N<11\frac{4}{9}$$

$\therefore$ The maximum number of term is 11.

29. It is given that $1, k, \frac{1}{5}, \ldots$ is a geometric sequence of positive real numbers.

(a) Find the value of k. (Leave your answer in surd form.)

(b) Find the general term of the sequence.

(c) Find $T(1)\times T(3)\times T(5)\times \ldots T(2003)$.

. 已知 $1, k, \frac{1}{5}, \ldots$ 是一個正實數的等比數列。

(a) 求 k 的值。(答案以根式表示。)

(b) 求該數列的通項。

(c) 求 $T(1)\times T(3)\times T(5)\times \ldots \times T(2003)$ 的值。

(a) $k^2 = 1\cdot\left(\frac{1}{5}\right)$

$$k=\frac{1}{\sqrt{5}} \quad (\because k>0)$$

(b) common ratio $=\frac{k}{1}$

$$=k$$
$$=\frac{1}{\sqrt{5}}$$

$\therefore$ $T(n)=1\cdot\left(\frac{1}{\sqrt{5}}\right)^{n-1}$

$$=\underline{\underline{5^{\frac{1-n}{2}}}}$$

(c) $T(1) \times T(3) \times T(5) \times \ldots \times T(2003)$

$$=5^0\times 5^{-1}\times 5^{-2}\times \ldots \times 5^{-1001}$$
$$=5^{0-1-2-\ldots-1001}$$
$$=5^{-\frac{1001}{2}(1+1001)}$$
$$=\underline{\underline{5^{-501501}}}$$

30. The perimeter of an n-sided polygon is 160 cm. If the lengths of sides are arranged in an ascending order, they form an arithmetic sequence and the common difference is 4 cm. If the longest side of the polygon is 34 cm, find the number of sides of the polygon.

已知一個有 n 條邊的多邊形的周界是 160 cm。若將邊長由小至大排序，它們形成一個公差是

4 cm 的等差數列。若該多邊形最長的邊的邊長是 34 cm，求該多邊形的邊數。

If the lengths of the sides of the polygon are arranged in a descending order, they form an arithmetic sequence with common difference -4.

$$\frac{n}{2}[2(34)+(n-1)(-4)]=160$$
$$n(36-2n)=160$$
$$n^2-18n+80=0$$
$$(n-10)(n-8)=0$$
$$n=10 \quad \text{or} \quad n=8$$

For n = 10,

$$\text{length of smallest side} = [34-(10-1)(4)]\text{ cm}$$
$$= -2\text{ cm (rejected)}$$

∴ n = 10 is not a reasonable answer.

∴ The number of sides of the polygon is 8.

31. The sum of the first n terms of an arithmetic series is $7n^2-2n$. Find the nth term of the series.

已知一個等差級數的首 n 項之和是 $7n^2-2n$，求該級數的第 n 項。

$$\begin{aligned}T(n) &= S(n)-S(n-1)\\ &= 7n^2-2n-[7(n-1)^2-2(n-1)]\\ &= [7n^2-7(n-1)^2]-[2n-2(n-1)]\\ &= 7[n^2-(n-1)^2]-2[n-(n-1)]\\ &= 7(n^2-n^2+2n-1)-2\\ &= 7(2n-1)-2\\ &= \underline{\underline{14n-9}}\end{aligned}$$

32. A boy saves \$1 on the first day, \$4 on the second day, \$7 on the third day and so on. If the boy saves \$590 in total on nth day, find n.

一名男孩在第一天儲蓄 \$1，第二天儲蓄 \$4，第三天儲蓄 \$7，餘此類推。若該男孩在第 n 天共儲存了 \$590，求 n 的值。

∵ The boy saves \$3 more than that of yesterday.

∴ The money saved form an arithmetic sequence.

∴ $\text{Total money saved} = \frac{n}{2}[2(1)+(n-1)(3)]$

$$= \frac{n}{2}(3n-1)$$

∴ $$\frac{n}{2}(3n-1)=590$$
$$3n^2-n-1180=0$$
$$(3n+59)(n-20)=0$$

∴ $n=\underline{\underline{20}}$ or $n=-\frac{59}{3}$ (rejected)

33. **(a)** Find the sum of the geometric series $1+2+2^2+\ldots+2^n$.

(b) **(i)** By using the result of (a), simplify $2\times 2^2\times 2^{2^3}\times 2^{2^3}\times\ldots\times 2^{2^n}$.

(ii) Find the least number of terms in $2\times 2^2\times 2^{2^3}\times 2^{2^3}\times\ldots\times 2^{2^n}$ such that its product is greater than 100 000 000.

(a) 求等比級數 $1+2+2^2+...+2^n$ 各項之和。

(b) **(i)** 利用從**(a)**所得的結果，化簡 $2\times 2^2\times 2^{2^2}\times 2^{2^3}\times...\times 2^{2^n}$ 。

(ii) 問最少需要多少項才可使到 $2\times 2^2\times 2^{2^2}\times 2^{2^3}\times...\times 2^{2^n}$ 的積大於 100 000 000？

(a) $\because\ a=1, R=2$

$$\therefore\ S(n)=\frac{1(2^{n+1}-1)}{2-1}$$
$$=\underline{\underline{2^{n+1}-1}}$$

(b) **(i)** $2\times 2^2\times 2^{2^2}\times\ldots\times 2^{2^n}$

$$=2^{1+2+2^2+\ldots+2^n}$$
$$=\underline{\underline{2^{2^{n+1}-1}}}\quad(\text{by (a)})$$

(ii) Let N be the least number of term.

$$\because\ 2\times 2^2\times\ldots\times 2^{2^N}>100000000$$
$$2^{2^{N+1}-1}>100000000$$
$$\log 2^{2^{N+1}-1}>\log 100000000$$
$$(2^{N+1}-1)\log 2>8$$
$$2^{N+1}>\frac{8}{\log 2}+1$$
$$(N+1)\log 2>\log\left[\frac{8}{\log 2}+1\right]$$
$$N>\frac{1}{\log 2}\left\{\log\left[\frac{8}{\log 2}+1\right]\right\}-1$$
$$>3.79\quad(\text{cor. to 2 d.p.})$$

$\therefore$ At least 4 terms should be taken.

34. It is given that $a_1, a_2, \ldots$ form a geometric sequence with $a_2-a_1=256$ and the sum to the infinity of the corresponding series is 144. Find the sum of all positive terms of the series.

已知 $a_1, a_2, ...$ 是一個等比數列，而 $a_2-a_1=256$ 及相應級數的無限項之和是 144。求該級數所有正數項之和。

Let r be the common ratio.

$\because\ a_1-a_2=256$

$\therefore\ a_1-a_1r_1=256$

$$a_1(1-r)=256$$
$$a_1=\frac{256}{1-r}$$

$\because \quad S(\infty) = 144$

$$\therefore \quad \frac{a_1}{1-r} = 144$$
$$\frac{256}{(1-r)(1-r)} = 144$$
$$\frac{16}{(1-r)^2} = 9$$
$$(1-r)^2 = \frac{16}{9}$$
$$1-r = \frac{4}{3} \quad \text{or} \quad -\frac{4}{3}$$
$$r = -\frac{1}{3} \quad \text{or} \quad \frac{7}{3} \text{(rejected)}$$

By substituting $r = -\frac{1}{3}$ into $a_1(1 - r) = 256$, we have

$$a_1\left[1-\left(-\frac{1}{3}\right)\right] = 256$$
$$a_1 = 256\left(\frac{3}{4}\right)$$
$$= 192$$

$\therefore$ Sum of all positive terms $= \dfrac{192}{1-\left(-\frac{1}{3}\right)^2}$

$= \underline{\underline{216}}$

35. It is given that the sum of the first 4 terms and the sum of the first 8 terms of a geometric series are 3 and 51 respectively. If the common ratio of the series is positive, find

(a) the common ratio,

(b) the sum of the first 10 terms of the series.

已知一個公比為正數的等比級數的首 4 項之和是 3，而首 8 項之和是 51。求

(a) 公比;

(b) 該級數首 10 項之和。

(a) Let a and R be the first term and the common ratio respectively.

$$\because \quad S(4) = \frac{a(1-R^4)}{1-R}$$

$$\therefore \quad \frac{a(1-R^4)}{1-R} = 3 \ldots (1)$$

$$\because \quad S(8) = \frac{a(1-R^4)}{1-R}$$

$$\therefore \quad \frac{a(1-R^8)}{1-R} = 51 \ldots (2)$$

(2) ÷ (1), $\dfrac{1-R^8}{1-R^4}=17$

$$\frac{(1-R^4)(1+R^4)}{1-R^4}=17$$

$$1+R^4=17$$

$$R^4=16$$

$R=\underline{\underline{2}}$ or -2 (rejected)

(b) By substituting R = 2 into (1), we have

$$\frac{a(1-2^4)}{1-2}=3$$

$$15a=3$$

$$a=\frac{1}{5}$$

$$\therefore \quad S(10)=\frac{\frac{1}{5}(2^{10}-1)}{2-1}$$

$$=\underline{\underline{\frac{1023}{5}}}$$

36 A car is now decelerating from 100 m/s to rest. It reduces half of its speed for every second. Find

(a) the speed of the car in the 10th second,

(b) the distance travelled in the 10th second,

(c) the distance travelled before it comes to rest.

一輛汽車從 100 m/s 開始減速直至完全停下來。已知該輛汽車的速度每秒減少一半。

求該輛汽車

(a) 在第 10 秒的速度;

(b) 在首 10 秒所行走的距離;

(c) 在停下前所行走的距離。

(a) Speed of car in 10th second

$$=100\times\left(\frac{1}{2}\right)^{10}\text{ m/s}$$

$$=\underline{\underline{\frac{25}{256}\text{ m/s}}}$$

(b) Distance travelled

$$= \left[100 + 100\left(\frac{1}{2}\right) + 100\left(\frac{1}{2}\right)^2 + \ldots + 100\left(\frac{1}{2}\right)^9\right] \text{m}$$

$$= 100\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^9}\right] \text{m}$$

$$= \left\{100 \cdot \frac{1\left[1 - \frac{1}{2^{10}}\right]}{1 - \frac{1}{2}}\right\} \text{m}$$

$$= \underline{\underline{\frac{25575}{128} \text{m}}}$$

(c) Total distance travelled

$$= \left[100 + 100\left(\frac{1}{2}\right) + 100\left(\frac{1}{2}\right)^2 + \ldots\right] \text{m}$$

$$= 100\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right] \text{m}$$

$$= 100 \cdot \left[\frac{1}{1 - \frac{1}{2}}\right] \text{m}$$

$$= \underline{\underline{200 \text{m}}}$$

37. The figure shows a right-angled triangle *ABC* with $\angle ABC = 90°$. $h_1, h_3, h_5, \ldots$ are perpendiculars to *AC*, $h_2, h_4, h_6, \ldots$ are perpendiculars to *BC*.

(a) Show that $h_1, h_2, h_3, \ldots$ are in geometric sequence and express the common ratio in terms of θ.

(b) If $AB = 20 \text{cm}$ and $h_1 + h_2 + h_3 + \ldots = 20(2\sqrt{3} + 3)$ cm, find the value of θ.

圖中所示為一個直角三角形 *ABC*，其中 $\angle ABC = 90°$，$h_1, h_3, h_5, \ldots$ 是垂直於 *AC* 的線，$h_2, h_4, h_6, \ldots$ 是垂直於 *BC* 的線。

(a) 證明 $h_1, h_2, h_3, \ldots$ 形成一個等比數列，並求該數列的公比，答案以 θ 表示。.

(b) 若 *AB* = 20 cm 及 $h_1 + h_2 + h_3 + \ldots = 20(2\sqrt{3} + 3)$ cm，求 θ 的值。

(a) In $\triangle ABD$,

$h_1 = AB \cos \theta$

In $\triangle DBE$,

$\because \angle BDE = \theta$ (alt.$\angle$s, $AB // DE$)

$$\therefore h_2 = h_1 \cos\theta$$
$$= AB\cos^2\theta$$

In $\triangle DEF$,

$\because \angle DEF = \theta$ (alt.$\angle$s, $BD // EF$)

$$\therefore h_3 = h_2 \cos\theta$$
$$= AB\cos^3\theta$$
$$\vdots$$

$h_1, h_2, h_3, \ldots$ are in geometric sequence with

$\cos\theta$ as common ratio..

(b) $\because$ $h_1 + h_2 + h_3 + \ldots$

$$= AB\cos\theta + AB\cos^2\theta + A\cos^3\theta + \ldots$$
$$= AB(\cos\theta + \cos^2\theta + \cos^3\theta + \ldots)$$
$$= \frac{AB\cos\theta}{1-\cos\theta}$$

$$\therefore \quad \frac{20\cos\theta}{1-\cos\theta} = 20(2\sqrt{3}+3)$$
$$\frac{\cos\theta}{1-\cos\theta} = (2\sqrt{3}+3)$$
$$\cos\theta = (2\sqrt{3}+3)(1-\cos\theta)$$
$$\cos\theta = (2\sqrt{3}+3) - (2\sqrt{3}+3)\cos\theta$$

$$(1+3+2\sqrt{3})\cos\theta = 2\sqrt{3}+3$$
$$\cos\theta = \frac{2\sqrt{3}+3}{4+2\sqrt{3}}$$
$$= \frac{\sqrt{3}(2+\sqrt{3})}{2(2+\sqrt{3})}$$
$$= \frac{\sqrt{3}}{2}$$
$$\theta = \underline{\underline{30^\circ}}$$

38. Given an geometric series $-64+16+(-4)+\ldots$

(a) Find the sum to infinity of the series.

(b) Find the sum of the first n terms of the series.

(c) A student estimates the sum to infinity of the series by the answer in (b). If the relative error of the estimation is less than 0.0001, find the smallest value of n.

已知$-64+16+(-4)+\ldots$ 為一個等比級數。

(a) 求該級數的無限項之和。

(b) 求該級數的首 n 項之和。

(c) 某學生利用 (b) 的結果來估算上列的等比級數的無限項之和。求 n 的最小值使估計值的相對誤差小於 0.0001。

(a) Common ratio $= \dfrac{16}{-64} = -\dfrac{1}{4}$

$$\therefore \quad S(\infty) = \frac{-64}{1-\left(-\frac{1}{4}\right)} = \underline{\underline{-\frac{256}{5}}}$$

(b) $$S(n) = \frac{-64\left[1-\left(-\frac{1}{4}\right)^n\right]}{1-\left(-\frac{1}{4}\right)} = \underline{\underline{-\frac{256}{5}\left[1-\left(-\frac{1}{4}\right)^n\right]}}$$

(c) relative error $= \dfrac{S(n) - S(\infty)}{S(\infty)}$

$$= \frac{-\frac{256}{5}\left[1-\left(-\frac{1}{4}\right)^n\right]+\frac{256}{5}}{-\frac{256}{5}}$$

$$= -1 + 1 - \left(-\frac{1}{4}\right)^n$$

$$= -\left(-\frac{1}{4}\right)^n$$

$$\therefore \quad \frac{1}{4^n} < 0.0001$$

$$4^{-n} < 0.0001$$

$$-n\log 4 < -4$$

$$n > \frac{4}{\log 4}$$

$$= 6.64 \text{ (cor. to 2 d.p.)}$$

***39*.** A student is asked to find the sum to infinity of the geometric series $12 + (-4) + \frac{4}{3} + (-\frac{4}{9}) + \ldots$. If he just adds up the first 6 terms to find the answer, find the percentage error in his answer.
(Give your answer correct to 3 significant figures.)

. 某學生需要求出級數$12 + (-4) + \frac{4}{3} + \left(-\frac{4}{9}\right) + \ldots$ 的無限項之和，但他只把該級數的首 6 項加起來作為答案。求他的答案的百分誤差。(答案須準確至三位有效數字。)

$\because \quad$ common ratio $= -\dfrac{4}{12} = -\dfrac{1}{3}$

$$\therefore \quad S(6) = \frac{12\left[1-\left(-\frac{1}{3}\right)^6\right]}{1-\left(-\frac{1}{3}\right)} = \frac{728}{81}$$

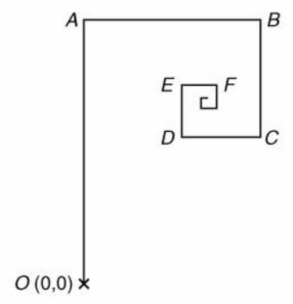
$$S(\infty) = \frac{12}{1-\left(-\frac{1}{3}\right)} = 9$$

$\therefore \quad$ Percentage error $= \dfrac{9 - \frac{728}{81}}{9} \times 100\%$

$$= \underline{\underline{0.137\%}} \text{ (cor. to 3 sig. fig.)}$$

Level 2+ Questions

程度 2+題目



1. In the figure, an ant starts from $O(0,0)$ and moves as follows:

(i) Moving northwards and travels to $A(0,3)$.

(ii) Moving eastwards and travels at a distance $\frac{2}{3}$ of his northbound trip.

(iii) Moving southwards and travels at a distance $\frac{2}{3}$ of his eastbound trip.

(iv) Moving westwards and travels at a distance $\frac{2}{3}$ of his southbound trip.

(v) Moving northwards and travels at a distance $\frac{2}{3}$ of his westbound trip.

The ant repeats (ii) to (v) infinitely.

(a) Find the total distance that the ant travels.

(b) The ant finally reaches his nest. Find the coordinates of the nest.

在圖中，一隻螞蟻從 $O(0, 0)$出發，並根據下列方法移動。

(i) 向北移動，抵達 $A(0, 3)$。

(ii) 向東移動，行走距離為牠在向北移動時的$\frac{2}{3}$。

(iii) 向南移動，行走距離為牠在向東移動時的$\frac{2}{3}$。

(iv) 向西移動，行走距離為牠在向南移動時的$\frac{2}{3}$。

(v) 向北移動，行走距離為牠在向西移動時的$\frac{2}{3}$。

該隻螞蟻不斷重複(ii)至(v)。

(a) 求牠所行走的總距離。

(b) 該隻螞蟻最終回到牠的巢穴。求牠的巢穴的坐標。

(a) $\because$ The distance travelled by the ant is $\frac{2}{3}$ of the distance travelled in previous path.

$\therefore$ Total distance $= 3 + 3\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \ldots$

$$= 3\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right]$$

$$= 3\left(\frac{1}{1 - \frac{2}{3}}\right)$$

$$= \underline{\underline{9}}$$

(b) Consider the vertical movement of the ants,
Distance travelled

$$= 3 - 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^4 + \ldots$$

$$= 3 + 3\left(-\frac{4}{9}\right) + 3\left(-\frac{4}{9}\right)^2 + \ldots$$

$$= \frac{3}{1 - \left(-\frac{4}{9}\right)}$$

$$= \frac{27}{13}$$

Consider the horizontal movement of the ants, Distance travelled

$$= 3 \times \frac{2}{3} - 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^2 + 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^4 + \ldots$$

$$= 2 + 2\left(-\frac{4}{9}\right) + 2\left(-\frac{4}{9}\right)^2$$

$$= \frac{2}{1 - \left(-\frac{4}{9}\right)}$$

$$= \frac{18}{13}$$

$\therefore$ The coordinates of the nest are $\left(\frac{27}{13}, \frac{18}{13}\right)$.

2. Given an sequence $1, 11, 111, 1111, \ldots$

(a) Find the general term of the sequence.

(Note: $\underbrace{111\ldots1}_{n}$ is not accepted as an answer.)

(b) Using the result in (a), show that the integer $\underbrace{111\ldots1}_{2005}\underbrace{222\ldots2}_{2005}$ is product of two consecutive integers.

(c) Find the general term of the sequence $1, 12, 123, 1234, \ldots$

已知一個數列$1, 11, 111, 1111, \ldots$

(a) 求該數列的通項。

(注意: $\underbrace{111...1}_{n}$ 不被接受為答案。)

(b) 利用從 (a) 所得的結果，證明整數 $\underbrace{111...1}_{2005}\underbrace{222...2}_{2005}$ 是兩個連續整數的積。

(c) 求數列 1,12,123,1234,... 的通項。

(a)

$$11=10+1$$
$$111=100+10+1=10^2+10+1$$
$$1111=1000+100+10+1=10^3+10^2+10+1$$
$$\vdots$$
$$\therefore\quad \underbrace{111\ldots1}_{n}=10^{n-1}+10^{n-2}+\cdots+10+1$$
$$=\frac{1(10^n-1)}{10-1}$$
$$=\underline{\underline{\frac{10^n-1}{9}}}$$

(b)

$$\underbrace{111\ldots1}_{2005}\underbrace{222\ldots2}_{2005}$$
$$=\underbrace{111\ldots1}_{2005}\times10^{2005}+\underbrace{222\ldots2}_{2005}$$
$$=\underbrace{111\ldots1}_{2005}\times10^{2005}+2\times\underbrace{111\ldots1}_{2005}$$
$$=\underbrace{111\ldots1}_{2005}\times(10^{2005}+2)$$
$$=\underbrace{111\ldots1}_{2005}\times3\left[\frac{10^{2005}-1}{3}+1\right]$$
$$=\underbrace{333\ldots3}_{2005}\times\left[\frac{10^{2005}-1}{3}+1\right]$$
$$=\underbrace{333\ldots3}_{2005}\times\left[3\left(\frac{10^{2005}-1}{9}\right)+1\right]$$
$$=\underbrace{333\ldots3}_{2005}\times\left(3\times\underbrace{111\ldots1}_{2005}+1\right)$$
$$=\underbrace{333\ldots3}_{2005}\times\underbrace{33\ldots34}_{2005}$$

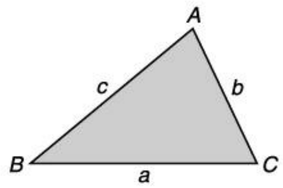
(c) $\because$

$$12=11+1$$
$$123=111+11+1$$
$$1234=1111+111+11+1$$

$\therefore$ General term

$$=\frac{10^n-1}{9}+\frac{10^{n-1}-1}{9}+\ldots+1$$
$$=\frac{1}{9}\left[(10^n+10^{n-1}+\ldots+10^1)-1-1\ldots-1\right]$$
$$=\frac{1}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$
$$=\frac{1}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$

3. In the figure, $\triangle ABC$ is an acute-angled triangle. Let $BC=a, CA=b, AB=c$ and area of $\triangle ABC=S$.



(a) Express $\tan A$ in terms of a, b, c and S.

(b) If a^2, b^2, c^2 are in arithmetic sequence, show that $\frac{1}{\tan A}, \frac{1}{\tan B}, \frac{1}{\tan C}$ are also in arithmetic sequence.

在圖中，$\triangle ABC$ 是一個銳角三角形。設 $BC=a, CA=b, AB=c$ 及$\triangle ABC$ 的面積 $=S$。

(a) 以 a,b,c 和 S 表示 $\tan A$。

(b) 若 a^2, b^2, c^2 是一個等差數列，證明 $\frac{1}{\tan A}, \frac{1}{\tan B}, \frac{1}{\tan C}$ 也是一個等差數列。

(a) By the cosine formula,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}. \quad \ldots (1)$$

Moreover,

$$S=\frac{1}{2}bc\sin A$$
$$\sin A=\frac{2S}{bc} \quad \ldots (2)$$

(2) ÷ (1)

$$\tan A=\frac{4S}{b^2+c^2-a^2}$$

(b) $\because$ a^2, b^2, c^2 are in arithmetic sequence.

$\therefore$ $b^2-a^2=c^2-b^2$ … (*)

From (a), $\frac{1}{\tan A}=\frac{b^2+c^2-a^2}{4S}$

Similarly, $\frac{1}{\tan B}=\frac{a^2+c^2-b^2}{4S}$,

$$\frac{1}{\tan C}=\frac{a^2+b^2-c^2}{4S}$$

$$\frac{1}{\tan B}-\frac{1}{\tan A}$$
$$=\frac{a^2+c^2-b^2}{4S}-\frac{b^2+c^2-a^2}{4S}$$
$$=\frac{2(a^2-b^2)}{4S}$$

$$\frac{1}{\tan C}-\frac{1}{\tan B}$$
$$=\frac{a^2+b^2-c^2}{4S}-\frac{a^2+c^2-b^2}{4S}=\frac{2(b^2-c^2)}{4S}$$

$\therefore\quad \dfrac{1}{\tan C}-\dfrac{1}{\tan B}=\dfrac{1}{\tan B}-\dfrac{1}{\tan A}$ (by (*))

$\therefore\quad \dfrac{1}{\tan A}, \dfrac{1}{\tan B}, \dfrac{1}{\tan C}$ are in arithmetic sequence

4. The terms of the geometric sequence $1,3,3^2,3^3,\ldots$ are divided into the groups $G_1,G_2,G_3,G_4,\ldots$

$G_1: 1$
$G_2: 3,3^2$
$G_3: 3^3,3^4,3^5$
$G_4: 3^6,3^7,3^8,3^9$
$\vdots$

(a) Find the total number of elements from groups G_1 to G_n.

(b) Find the sum of elements in group G_n.

把等比數列 $1,3,3^2,3^3,\ldots$ 的項放置於組別 $G_1,G_2,G_3,G_4,\ldots$中。

$G_1: 1$
$G_2: 3,3^2$
$G_3: 3^3,3^4,3^5$
$G_4: 3^6,3^7,3^8,3^9$
$\vdots$

(a) 問組別 G_1 至 G_n 共有多少項？

(b) 求組別 G_n 內各項的總和。

(a) $\because$ Number of elements in group $G_n = n$

$\therefore$ Total number of elements

$$=1+2+3+\ldots+n$$
$$=\underline{\underline{\frac{n(n+1)}{2}}}$$

(b) By the result of (a),

the last element of group $G_{n-1}=3^{\frac{(n-1)n}{2}-1}$,

the last element of group $G_n=3^{\frac{n(n+1)}{2}-1}$.

$\therefore$ The first element of group $G_n = 3^{\frac{(n-1)n}{2}}$

$\therefore$ Sum of elements in group G_n

$$= 3^{\frac{(n-1)n}{2}} + 3^{\frac{(n-1)n}{2}+1} + \ldots + 3^{\frac{n(n+1)}{2}-1}$$
$$= 3^{\frac{(n-1)n}{2}} [1 + 3 + 3^2 + \ldots + 3^{n-1}]$$
$$= 3^{\frac{(n-1)n}{2}} \cdot \left[\frac{1 \cdot (3^n - 1)}{3 - 1}\right]$$
$$= \underline{\underline{\frac{3^{\frac{(n-1)n}{2}} \cdot (3^n - 1)}{2}}}$$

5. Let n be an positive integer such that $2^n - 1$ is a prime number.

(a) Find all the factors of

(i) 2^{n-1},

(ii) $2^{n-1}(2^n - 1)$.

(b) If $m = 2^{n-1}(2^n - 1)$, find the sum of all the factors of m.

設 n 為一個正整數，使到 $2^n - 1$ 是一個質數。

(a) 求下列各數的所有因數。

(i) 2^{n-1}

(ii) $2^{n-1}(2^n - 1)$

(b) 若 $m = 2^{n-1}(2^n - 1)$，求 m 的所有因數之和。

(a) (i) The factors of 2^n are: 1, 2, 2^2, …, 2^{n-1}

(ii) $\because$ $2^n - 1$ is a prime number

$\therefore$ 1 and $2^n - 1$ are only factors of $2^n - 1$

$\therefore$ Factor of $2^n(2^n - 1)$ are:

1, 2, 2^2, …, 2^{n-1}; $2^n - 1$, $2(2^n - 1)$, $2^2(2^n - 1)$, …, $2^{n-1}(2^n - 1)$

(b) Sum of factors

$$= 1 + 2 + 2^2 + \ldots + 2^{n-1} + (2^n - 1) + 2(2^n - 1) + \ldots$$
$$+ \ldots + 2^{n-1}(2^n - 1)$$
$$= (1 + 2 + \ldots + 2^{n-1}) + (2^n - 1)(1 + 2 + \ldots + 2^{n-1})$$
$$= (1 + 2 + \ldots + 2^{n-1})(1 + 2^n - 1)$$
$$= 2^n(1 + 2 + \ldots + 2^{n-1})$$
$$= 2^n \cdot \frac{1(2^n - 1)}{2 - 1}$$
$$= 2^n(2^n - 1)$$
$$= 2[2^{n-1}(2^n - 1)]$$
$$= \underline{\underline{2m}}$$

6. In the figure, OAB is a right-angled triangle with $\angle OAB = 90°$. $B_1, B_2, B_3, \ldots$ are points on OB such that $ACB_1A_1, A_1C_1B_2A_2, A_2C_2B_3A_3, \ldots$ are squares. Let $S_1, S_2, S_3, \ldots$ denote the area of $ACB_1A_1, A_1C_1B_2A_2, A_2C_2B_3A_3, \ldots$ respectively.

(a) Show that S_1, S_2, S_3 are in geometric sequence.

(b) Express the sum of areas of all the squares in terms of S_1 and S_2.

(c) If $S_1 = 1$ and the sum of areas of all the squares is 2, find $\angle AOB$.

在圖中，OAB 是一個直角三角形，且$\angle OAB = 90°$。$B_1, B_2, B_3, \ldots$是 OB 上的點，使到 $ACB_1A_1, A_1C_1B_2A_2, A_2C_2B_3A_3, \ldots$是正方形。設 $ACB_1A_1, A_1C_1B_2A_2, A_2C_2B_3A_3, \ldots$的面積分別為 $S_1, S_2, S_3, \ldots$。

(a) 證明 S_1, S_2, S_3 是一個等比數列。

(b) 以 S_1 和 S_2 表示所有正方形的面積之和。

(c) 若 $S_1 = 1$ 及所有正方形的面積之和是 2，求$\angle AOB$。

(a) Let l_1, l_2, … be the lengths of squares ACB_1A_1, $A_1C_1B_2A_2$,… respectively.

$\because \triangle B_1B_2C_1 \sim \triangle B_2B_3C_2$ (AAA)

$$\therefore \quad \frac{l_1 - l_2}{l_2 - l_3} = \frac{l_2}{l_3}$$

$$l_3l_1 - l_2l_3 = l_2^2 - l_2l_3$$

$$l_3l_1 = l_2^2$$

$$\frac{l_2}{l} = \frac{l_3}{l_2}$$

$$\frac{l_2^2}{l_1^2} = \frac{l_3^2}{l_2^2}$$

$$\frac{S_2}{S_1} = \frac{S_3}{S_2}$$

$\therefore S_1$, S_2, S_3, are in geometric sequence.

(b) Common ratio $= \dfrac{S_2}{S_1}$

$$\therefore \quad \text{Sum ofareas of all squares} = \frac{S_1}{1 - \frac{S_2}{S_1}}$$

$$= \frac{S_1}{\frac{S_1 - S_2}{S_1}}$$

$$= \underline{\underline{\frac{S_1^2}{S_1 - S_2}}}$$

(c) $\because$ $S_1 = 1$

$\therefore$

$$l_1 = \sqrt{S_1} = 1$$

$$\frac{1^2}{1-S_2} = 2$$

$$1 = 2(1-S_2)$$

$$S_2 = \frac{1}{2}$$

$$l_2 = \frac{1}{\sqrt{2}}$$

$$l_1 = \sqrt{S_1}$$

$\because \triangle B_1B_2C_1 \sim \triangle BOA$ (AAA)

$\therefore \angle C_1B_2B_1 = \angle AOB$

In $\triangle B_1B_2C_1$

$$\tan\angle AOB = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}-1$$

$$\angle AOB = \underline{\underline{22.5^\circ}}$$

7. In the figure, C_1 is a unit circle with centre O_1 and touches the horizontal line *AB*. Circle C_2 of radius r_2 is constructed so that it touches both C_1 and *AB* and its centre lies on O_1A. Circle $C_3, C_4, C_5, \ldots$ are constructed in a similar way. Denote $\angle O_1AB$ by θ.

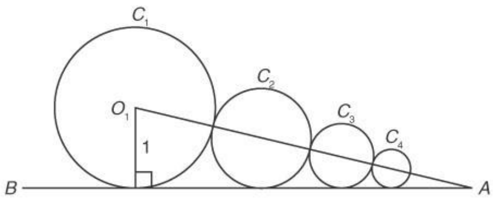
(a) Express r_2 in terms of θ.

(b) Find the sum of areas of circles $C_1, C_2, C_3, C_4, C_5, \ldots$

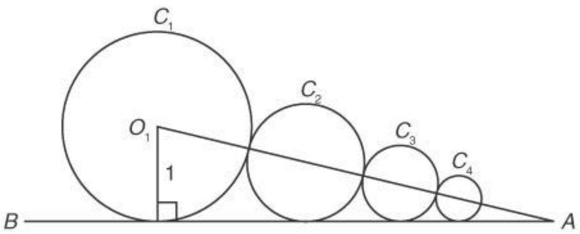
(c) If the length of O_1A doubles the radius of C_1, find the smallest values of *n* such that

(i) the radius of C_n is less than 0.01,

(ii) the sum of areas of circles $C_n, C_{n+1}, C_{n+2}, \ldots$ is less than 0.01π.



在圖中，C_1是一個圓心為 O_1 的單位圓，並與水平線 *AB* 相切。作半徑為 r_2 的圓 C_2，使到它的圓心在 O_1A 上，並與圓 C_1 和 *AB* 相切。以同樣方法作 $C_3, C_4, C_5, \ldots$。以 θ 表示 $\angle O_1AB$。



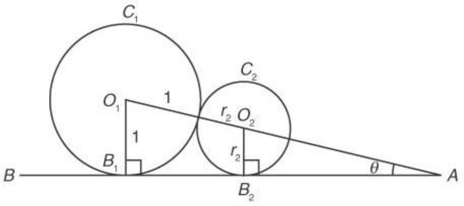
(a) 以 θ 表示 r_2。

(b) 求圓 $C_1, C_2, C_3, C_4, C_5, \ldots$ 的面積之和。

(c) 若 O_1A 的長度是 C_1 的半徑的一倍，求 n 的最小值，使得

(i) C_n 的半徑小於 0.01;

(ii) $C_n, C_{n+1}, C_{n+2}, \ldots$ 的面積之和小於 0.01π。



(a)

In $\triangle O_1B_1A$,

$$\sin\theta = \frac{1}{O_1A}$$

$$O_1A = \frac{1}{\sin\theta}$$

$\because \triangle O_1B_1A \sim \triangle O_2B_2A$ (AAA)

$$\therefore \quad \frac{O_1B_1}{O_2B_2} = \frac{O_1A}{O_2A}$$

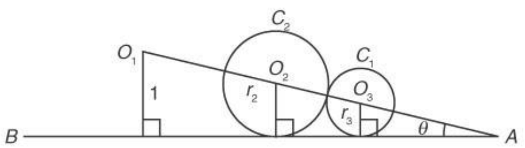
$$\frac{1}{r_2} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - (r_2 + 1)}$$

$$\frac{1}{r_2} = \frac{1}{1 - (r_2 + 1)\sin\theta}$$

$$1 - (r_2 + 1)\sin\theta = r_2$$

$$r_2(1 + \sin\theta) = 1 - \sin\theta$$

$$r_2 = \underline{\underline{\frac{1 - \sin\theta}{1 - \sin\theta}}}$$



(b)

$\because \triangle O_2B_2A \sim \triangle O_3B_3A$ (AAA)

$$\therefore \frac{r_2}{r_3} = \frac{\frac{r_2}{\sin\theta}}{\frac{r_2}{\sin\theta} - (r_2 + r_3)}$$

$$\frac{r_2}{r_3} = \frac{r_2}{r_2 - (r_2 + r_3)\sin\theta}$$

$$\begin{aligned} r_2^2 - r_2^2\sin\theta - r_2r_3\sin\theta &= r_2r_3 \\ r_2^2(1-\sin\theta) &= r_2r_3(1+\sin\theta) \\ r_3 &= r_2\frac{1-\sin\theta}{1+\sin\theta} \\ &= r_2^2 \end{aligned}$$

Similarly, $r_4 = r_2^3,\ \ r_5 = r_2^4,\ \ldots$

$$\begin{aligned} \therefore\ \text{Sum of areas} &= \pi(1^2) + \pi r_2^2 + \pi r_3^2 + \ldots \\ &= \pi(1 + r_2^2 + r_2^4 + \ldots) \\ &= \frac{\pi}{1-r_2^2} \\ &= \frac{\pi}{1-\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^2} \\ &= \underline{\underline{\frac{\pi(1+\sin\theta)^2}{4\sin\theta}}} \end{aligned}$$

(c) **(i)** $\because$ length of $OA = 2 \times$ radius of C_1

$\therefore \sin\theta = \frac{1}{2}$

$$\therefore \quad r_2 = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$$

$$\because \quad r_n = r_2^{n-1} = \left(\frac{1}{3}\right)^{n-1}$$

$$\begin{aligned} \therefore \quad r_n &< 0.01 \\ \left(\frac{1}{3}\right)^{n-1} &< 0.01 \\ \log\left(\frac{1}{3}\right)^{n-1} &< -2 \\ (n-1)\log\frac{1}{3} &< -2 \\ n-1 &> \frac{-2}{\log\frac{1}{3}} \\ n &> 5.19\ (\text{cor. to 2 d.p.}) \end{aligned}$$

$\therefore \quad n = \underline{\underline{6}}$

(ii) Sum of areas of C_n, C_{n+1} …

$$= \pi(r_2^{n-1})^2 + \pi(r_2^n)^2 + \ldots$$
$$= \pi(r_2^{2n-2} + r_2^{2n} + \ldots)$$
$$= \frac{\pi\, r_2^{2n-2}}{1 - r_2^2}$$
$$= \frac{\pi\left(\frac{1}{3}\right)^{2n-2}}{1-\left(\frac{1}{3}\right)^2}$$
$$= \frac{\pi}{8*(3^{2n-4})}$$

For the sum of area of *Cn*, C_{n+1}, is less than 0.01 π,

$$\frac{\pi}{8(3^{2n-4})} < 0.01\pi$$
$$8(3^{2n-4)} > 100$$
$$3^{2n-4} > \frac{25}{2}$$
$$\log 3^{2n-4} > \log\frac{25}{2}$$
$$(2n-4)\log 3 > \log\frac{25}{2}$$
$$2n-4 > \frac{\log\frac{25}{2}}{\log 3}$$
$$n > 3.15 \text{(cor. to 2 d.p.)}$$

Multiple Choice Questions

多項選擇題

1 If the general term of a sequence is $\dfrac{n(n-2)}{(n+3)^2}$, find the 4th term of the sequence.

若一個數列的通項是 $\dfrac{n(n-2)}{(n+3)^2}$，求該數列的第 4 項。

A. $\dfrac{8}{49}$

B. $\dfrac{4}{49}$

C. $\dfrac{1}{2}$

D. $\dfrac{6}{49}$

$\boxed{\text{A}}$

$\because \quad T(n)=\dfrac{n(n-2)}{(n+3)^2}$

$\therefore \quad T(4)=\dfrac{4(4-2)}{(4+3)^2}$

$\quad\quad =\underline{\underline{\dfrac{8}{49}}}$

2. Find the next term of the sequence $6, 13, 27, 55, \ldots$

求數列 6,13,27,55,... 接著的一項。

A. 100

B. 109

C. 111

D. 119

$\boxed{\text{C}}$

$\because \quad T(2)=13=2\times 6+1$
$T(3)=27=2\times 13+1$
$T(4)=55=2\times 27+1$

$\therefore \quad T(5)=2\times 55+1=\underline{\underline{111}}$

3. Which of the following is the general term of the sequence $5, 8, 11, 14, \ldots$?

下列何者是數列 5,8,11,14,... 的通項?

A. $5n+3$

B. $3n+2$

C. $5n$

D. $4n+1$

$\boxed{\text{B}}$

$\because \quad T(2)=8=5+(2-1)(3)$
$T(3)=11=5+(3-1)(3)$
$T(4)=14=5+(4-1)(3)$

$\therefore \quad T(n)=5+(n-1)(3)$
$\quad\quad =\underline{\underline{3n+2}}$

4. Find the next term of the sequence $\dfrac{2}{3}, \dfrac{8}{9}, \dfrac{32}{27}, \ldots$

求數列 $\dfrac{2}{3}, \dfrac{8}{9}, \dfrac{32}{27}$ 接著的一項。

A. $\dfrac{128}{81}$

B. $\dfrac{128}{192}$

C. $\dfrac{54}{81}$

D. $\dfrac{54}{192}$

$\boxed{\text{A}}$

$\because \quad T(2)=\dfrac{8}{9}=\dfrac{2}{3}\times\dfrac{4}{3}$
$T(3)=\dfrac{32}{27}=\dfrac{8}{9}\times\dfrac{4}{3}$

$\therefore \quad T(4) = \dfrac{32}{27} \times \dfrac{4}{3}$

$= \underline{\underline{\dfrac{128}{81}}}$

(P01C14L06Q005)

5. Which of the following is/are in arithmetic sequence?

下列何者是等差數列?

I. $-1, 0, 1, 2, \ldots$

II. $\log 2, \log 4, \log 8, \log 16, \ldots$

III. $\dfrac{3}{4}, \dfrac{9}{8}, \dfrac{27}{16}, \dfrac{81}{32}, \ldots$

A. I only

B. I and II only

C. II and III only

D. I, II and III

$\boxed{\text{B}}$

For I:

$\because \quad T(2) - T(1) = 0 - (-1) = 1$

$T(3) - T(2) = 1 - 0 = 1$

$T(4) - T(3) = 2 - 1 = 1$

4 −1, 0, 1, 2, … is an arithmetic sequence.

For II:

$\because \quad T(2) - T(1) = \log 4 - \log 2$

$= \log \dfrac{4}{2}$

$= \log 2$

$T(3) - T(2) = \log 8 - \log 4$

$= \log \dfrac{8}{4}$

$= \log 2$

$T(4) - T(3) = \log 16 - \log 8$

$= \log \dfrac{16}{8}$

$= \log 2$

$\therefore$ log 2, log 4, log 8, log 16, … is an arithmetic sequence.

For III:

$\because \quad T(2) - T(1) = \dfrac{9}{8} - \dfrac{3}{4}$

$= \dfrac{3}{8}$

$T(3) - T(2) = \dfrac{27}{16} - \dfrac{9}{8}$

$= \dfrac{9}{16}$

$\therefore \quad \dfrac{3}{4}, \dfrac{9}{8}, \dfrac{27}{16}, \dfrac{81}{32}$ ***is not an arithmetic sequence.***

6. Which of the following about the arithmetic sequence $T(1), T(2), \ldots, T(n)$ is/are correct?

下列何者關於等差數列 $T(1), T(2), \ldots, T(n)$ 的敍述是正確的?

I. $\dfrac{T(2)}{T(1)} = \dfrac{T(3)}{T(2)}$

II. $T(1) + T(2) + \ldots + T(n) = \dfrac{T(1) + T(n)}{2}$

III. $T(n) = (n-1)T(2) + (2-n)T(1)$

A. I only

B. II only

C. III only

D. None of them are correct

$\boxed{\text{C}}$

For I:

1, 2, 3, 4, … is an arithmetic sequence.

$\therefore \quad \dfrac{T(2)}{T(1)} = \dfrac{2}{1} = 2$

$\dfrac{T(3)}{T(2)} = \dfrac{3}{2}$

$\therefore \quad \dfrac{T(2)}{T(1)} \neq \dfrac{T(3)}{T(2)}$

$\therefore$ I is not correct.

For II:

$\because \quad T(1) + T(2) + \ldots + T(n)$

$= \dfrac{n}{2}[T(1) + T(n)]$

$\therefore$ II is not correct.

For III:

$\because$ Common difference $= T(2) - T(1)$

$\therefore$ $T(n) = T(1) + (n-1)[T(2) - T(1)]$
$= T(1) + (n-1)T(2) - (n-1)T(1)$
$= (2-n)T(1) + (n-1)T(2)$

$\therefore$ III is correct.

7. If the sum of the first n terms of an arithmetic sequence is $2n^2 + 3n$, find the 5th term.
若一個等差數列的首 n 項之和是 $2n^2 + 3n$，求第 5 項。

A. 21
B. 65
C. 25
D. 44

$\boxed{\text{A}}$

$T(5) = S(5) - S(4)$
$= [2(5)^2 + 3(5)] - [2(4)^2 + 3(4)]$
$= (50 + 15) - (32 + 12)$
$= \underline{\underline{21}}$

8. The general term of the arithmetic sequence $48, 40, 32, \dots$ is
等差數列 48,40,32,... 的通項是

A. $8n$.
B. $8(n+5)$.
C. $48 - n$.
D. $8(7-n)$.

$\boxed{\text{D}}$

$\because$ Common difference $= 40 - 48$
$= -8$

$\therefore$ $T(n) = 48 + (n-1)(-8)$
$= 56 - 8n$
$= \underline{\underline{8(7-n)}}$

9. Find the general term of the arithmetic sequence $7, 13, 19, \dots$
求等差數列 7,13,19,... 的通項。

A. $6n + 13$
B. $7n - 1$
C. $6n + 1$
D. $7n + 1$

$\boxed{\text{C}}$

$\because$ Common difference $= 13 - 7$
$= 6$

$\therefore$ $T(n) = 7 + (n-1)(6)$
$= \underline{\underline{6n + 1}}$

10. Find the 8th term of the arithmetic sequence $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \dots$

求等差數列 $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \dots$ 的第 8 項。

A. $\frac{25}{6}$
B. $\frac{13}{3}$
C. $\frac{23}{6}$
D. $\frac{2}{3}$

$\boxed{\text{C}}$

$\because$ Common difference $= \frac{5}{6} - \frac{1}{3}$
$= \frac{1}{2}$

$\therefore$ $T(8) = \frac{1}{3} + (8-1)\left(\frac{1}{2}\right)$
$= \frac{1}{3} + \frac{7}{2}$
$= \underline{\underline{\frac{23}{6}}}$

11 Let $T(n)$ be the nth term of an arithmetic sequence. If $T(3) = 26$ and $T(2) : T(12) = 1 : 4$, find $T(20)$.
設一個等差數列的第 n 項為 $T(n)$。若 $T(3) = 26$ 及 $T(2) : T(12) = 1 : 4$，求 $T(20)$ 的值。

A. 128
B. 112

C. 156

D. 80

$\boxed{\text{A}}$

Let a and d be the first term and the common difference respectively.

$$T(3) = a + 2d = 26 \quad \text{......(1)}$$
$$T(2) = a + d$$
$$T(12) = a + 11d$$

$\because \quad T(2) : T(12) = 1 : 4$

$\therefore \quad T(12) = 4T(2)$

$$a + 11d = 4(a + d)$$
$$a + 11d = 4a + 4d$$
$$3a = 7d$$
$$a = \frac{7}{3}d \qquad \text{......(2)}$$

By substituting (2) into (1), we have,

$$\frac{7}{3}d + 2d = 26$$
$$\frac{13d}{3} = 26$$
$$d = 6$$

By substituting $d = 6$ into (2), we have

$$a = \frac{7}{3}(6)$$
$$= 14$$

$$\therefore \quad T(20) = 14 + (20 - 1)(6)$$
$$= \underline{\underline{128}}$$

12. Find the number of terms in the arithmetic sequence $9, 3, -9, \ldots -87$.
求等差數列 9,3,–9,...,–87 的項數。

A. 15

B. 16

C. 17

D. 18

$\boxed{\text{C}}$

Let n be the number of terms.

$$\text{Common difference} = 3 - 9$$
$$= -6$$

$$\therefore \quad 9 + (n-1)(-6) = -87$$
$$-6(n-1) = -96$$
$$n - 1 = 16$$
$$n = 17$$

$\therefore$ There are 17 terms in the sequence.

13. Find the number of terms in the arithmetic sequence $15, 22, 29, \ldots, 169$.
求等差數列 15,22,29,...,169 的項數。

A. 22

B. 23

C. 24

D. 25

$\boxed{\text{B}}$

Let n be the number of terms.

$$\text{Common difference} = 22 - 15$$
$$= 7$$

$$\therefore \quad 15 + (n-1)(7) = 169$$
$$7(n-1) = 154$$
$$n - 1 = 22$$
$$n = 23$$

$\therefore$ There are 23 terms in the sequence.

14. How many positive integers less than 300 are not divisible by 17?
問有多少個小於 300 的正整數不能被 17 整除？

A. 283

B. 282

C. 281

D. 280

$\boxed{\text{A}}$

Let n be the number of integer which is divisible by 17.

$$\therefore \quad 17 + (n-1)(17) < 300$$
$$17n < 300$$
$$n < 17\frac{11}{17}$$

$\therefore$ Number of integers that are not divisible by 17

$$= 300 - 17$$
$$= \underline{\underline{283}}$$

15 The 6th term and the 13th term of an arithmetic sequence are 53 and 109 respectively.

Find the general term of the sequence.

已知一個等差數列的第 6 項是 53，而第 13 項是 109。求該數列的通項。

A. $7n+11$

B. $9n-8$

C. $16n+31$

D. $8n+5$

$\boxed{\text{D}}$

Let a and d be the first term and the common difference respectively.

$$T(6)=a+5d=53 \quad \ldots\ldots(1)$$

$$T(13)=a+12d=109 \quad \ldots\ldots(2)$$

$$(2)-(1),\quad 7d=56$$
$$d=8$$

By substituting $d=8$ into (1), we have,

$$a+5(8)=53$$
$$a=13$$

$$\therefore\quad T(n)=13+(n-1)(8)$$
$$=\underline{\underline{8n+5}}$$

***16*.** Which of the following are geometric sequences?

下列何者是等比數列？

I. $\frac{4}{5},\frac{8}{15},\frac{16}{45},\frac{32}{135},\ldots$

II. $3,6,9,12,\ldots$

III. $2^{x},2^{x+2}.2^{x+4},2^{x+6},\ldots$

A. I and II only

B. I and III only

C. II and III only

D. I, II and III

$\boxed{\text{B}}$

For I:

$$\frac{8}{15}\div\frac{4}{5}=\frac{2}{3}$$
$$\frac{16}{45}\div\frac{8}{15}=\frac{2}{3}$$
$$\frac{32}{135}\div\frac{16}{45}=\frac{2}{3}$$

$\therefore$ I is a geometric sequence.

For II:

$$\frac{6}{3}=2$$
$$\frac{9}{6}=\frac{3}{2}$$

$\therefore$ 3, 6, 9, 12, … is not a geometric sequence.

For III:

$$\frac{2^{x+2}}{2^{x}}=2^{2}$$
$$\frac{2^{x+4}}{2^{x+2}}=2^{2}$$
$$\frac{2^{x+6}}{2^{x+4}}=2^{2}$$

$\therefore$ $2^{x}, 2^{x+2}, 2^{x+4}, 2^{x+6}, \ldots$ is a geometric sequence.

17. If $3x-2,4x,2x-7$ are three consecutive terms of an arithmetic sequence, find x.

若 $3x-2,4x,2x-7$ 是一個等差數列的三個連續項，求 x 的值。

A. 9

B. –9

C. 3

D. –3

$\boxed{\text{D}}$

$\because$ $3x-2,\ 4x,\ 2x-7$ are in arithmetic sequence.

$$\therefore\quad (3x-2)+(2x-7)=2(4x)$$
$$5x-9=8x$$
$$3x=-9$$
$$x=\underline{\underline{-3}}$$

18. If $6p+3, 2p+1, 4p-5$ are three consecutive terms of an arithmetic sequence, find p.
若 $6p+3, 2p+1, 4p-5$ 是一個等差數列的三個連續項，求 p 的值。

A. $\frac{3}{8}$

B. 0

C. $\frac{2}{3}$

D. 1

$\boxed{\text{C}}$

$\because$ $6p+3,\ 2p+1,\ 4p-5$ are in arithmetic sequence.

$$\therefore \quad (6p+3)+(4p-5)=2(2p+1)$$
$$10p-2=4p+2$$
$$6p=4$$
$$p=\underline{\underline{\frac{2}{3}}}$$

19. If four arithmetic means are inserted between m and $3m$, find the second arithmetic mean inserted.
若在 m 與 $3m$ 之間插入四個等差中項，求插入的第二個等差中項。

A. $\frac{7m}{5}$

B. $2m$

C. $\frac{9m}{5}$

D. $\frac{2m}{5}$

$\boxed{\text{C}}$

Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$$m,\ m+d,\ m+2d,\ m+3d,\ m+4d,\ 3m$$

$$\therefore \quad m+5d=3m$$
$$d=\frac{2m}{5}$$

$\therefore$ The second arithmetic mean

$$=m+2\left(\frac{2m}{5}\right)$$
$$=\underline{\underline{\frac{9m}{5}}}$$

20. If five arithmetic means are inserted between p and $3q$, find the sum of the arithmetic means inserted.
若在 p 與 $3q$ 之間插入五個等差中項，求插入的等差中項之和。.

A. $5(p+3q)$

B. $\frac{3q-p}{6}$

C. $2p+9q$

D. $\frac{5(p+3q)}{2}$

$\boxed{\text{D}}$

Let d be the common difference of the arithmetic sequence to be formed.

The arithmetic sequence formed is:

$$p,\ p+d,\ p+2d,\ p+3d,\ p+4d,\ p+5d,\ 3q$$

$$\therefore \quad p+6d=3q$$
$$d=\frac{3q-p}{6}$$

Sum of the arithmetic means

$$=(p+d)+(p+2d)+(p+3d)+(p+4d)+(p+5d)$$
$$=5p+15d$$
$$=5(p+3d)$$
$$=5\left[p+3\left(\frac{3q-p}{6}\right)\right]$$
$$=\underline{\underline{\frac{5(p+3q)}{2}}}$$

(P01C14L06Q021)

21. If the first term and the common ratio of a geometric sequence are a and R respectively, which of the following about the sequence are true?
若一個等比數列的首項是 a 而公比是 R，下

列何者關於該數列的敍述是正確的？

I. General term $= aR^n$

II. Sum of first n terms $= \dfrac{a(1-R^n)}{1-R}$

III. Sum to infinity $= \dfrac{a}{1-R}$ $(-1<R<1)$

I. 通項 $= aR^n$

II. 首 n 項之和 $= \dfrac{a(1-R^n)}{1-R}$

III. 無限項之和 $= \dfrac{a}{1-R}$ $(-1<R<1)$

A. I and II only
B. I and III only
C. II and III only
D. I, II and III

$\boxed{\text{C}}$

For I:

General term $= aR^{n-1}$

$\therefore$ I is not true.

For II:

Sum of first n terms

$= \dfrac{a(1-R^n)}{1-R}$

$\therefore$ II is true.

For III:

Sum to infinity $= \dfrac{a}{1-R}$

$\therefore$ III is true.

22. Find the general term of the geometric sequence $\dfrac{2}{3}, -\dfrac{4}{15}, \dfrac{8}{75}, \ldots$

求等比數列 $\dfrac{2}{3}, -\dfrac{4}{15}, \dfrac{8}{75}, \ldots$ 的通項。

A. $-\dfrac{2}{3}\left(\dfrac{2}{5}\right)^{n-1}$

B. $\dfrac{2}{3}\left(-\dfrac{2}{5}\right)^{n-1}$

C. $\left(-\dfrac{4}{15}\right)^{n-1}$

D. $\left(-\dfrac{4}{15}\right)^{n-1}$

$\boxed{\text{B}}$

$T(2) = -\dfrac{4}{15} = \dfrac{2}{3}\left(-\dfrac{2}{5}\right)^{2-1}$

$T(3) = \dfrac{8}{75} = \dfrac{2}{3}\left(-\dfrac{2}{5}\right)^{3-1}$

$\therefore \quad T(n) = \underline{\underline{\dfrac{2}{3}\left(-\dfrac{2}{5}\right)^{n-1}}}$

23. Find the general term of the geometric sequence $\dfrac{1}{3}, 1, 3, 9, \ldots$

求等比數列 $\dfrac{1}{3}, 1, 3, 9, \ldots$ 的通項。

A. 3^n
B. 3^{n-1}
C. 3^{n-2}
D. 3^{n-3}

$\boxed{\text{C}}$

$T(2) = 1 = \dfrac{1}{3} \times 3^{2-1}$

$T(3) = 3 = \dfrac{1}{3} \times 3^{3-1}$

$T(4) = 9 = \dfrac{1}{3} \times 3^{4-1}$

$\therefore \quad T(n) = \dfrac{1}{3} \times 3^{n-1}$

$= \underline{\underline{3^{n-2}}}$

24. Find the 8th term of the geometric sequence $8, -\dfrac{16}{3}, \dfrac{32}{9}, \ldots$

求等比數列 $8, -\dfrac{16}{3}, \dfrac{32}{9}, \ldots$ 的第 8 項。

A. $\frac{1024}{2187}$

B. $-\frac{1024}{2187}$

C. $\frac{2048}{6561}$

D. $-\frac{2048}{6561}$

$\boxed{B}$

$$\begin{aligned}\text{Common ratio} &= -\frac{16}{3} \div 8 \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\therefore \quad T(8) &= 8\left(-\frac{2}{3}\right)^{8-1} \\ &= \underline{\underline{-\frac{1024}{2187}}}\end{aligned}$$

25. It is given that $3, 12, 48, \ldots$ are in geometric sequence. If $T(k)$ is the first term in the sequence greater than 1200, $k =$

已知 $3, 12, 48, \ldots$ 是一個等比數列。若 $T(k)$ 是該數列中首個使得 $T(k) > 1200$ 的項，則 $k =$

A. 5.

B. 6.

C. 7.

D. 8.

$\boxed{B}$

$$\begin{aligned}\text{Common ratio} &= \frac{12}{3} \\ &= 4\end{aligned}$$

$\therefore \quad T(n) = 3(4)^{n-1}$

$\because \quad T(k) > 1200$

$$\begin{aligned}\therefore \quad 3(4)^{k-1} &> 1200 \\ 4^{k-1} &> 400 \\ \log 4^{k-1} &> \log 400 \\ (k-1)\log 4 &> \log 400 \\ k &> \frac{\log 400}{\log 4} + 1 \\ &= 5.32 \quad (\text{cor. to 2 d.p.}) \\ k &= \underline{\underline{6}}\end{aligned}$$

***26*.** The sum of first n terms of a geometric sequence is $-2(1-4^n)$, find the third term of the sequence.

已知一個等比數列首 n 項之和是 $-2(1-4^n)$，求該數列的第 3 項。

A. 96

B. 384

C. -96

D. -384

$\boxed{A}$

$$\begin{aligned}T(3) &= S(3) - S(2) \\ &= -2(1-4^3) - [-2(1-4^2)] \\ &= 126 - 30 \\ &= \underline{\underline{96}}\end{aligned}$$

27 Find the number of terms in the geometric sequence $1, a^2, a^4, \ldots, a^{4m+2}$.

求等比數列 $1, a^2, a^4, \ldots, a^{4m+2}$ 的項數。

A. $2m+1$

B. $2m+2$

C. $4m+2$

D. $4m+3$

$\boxed{B}$

$$\begin{aligned}\text{Common ratio} &= \frac{a^2}{1} \\ &= a^2\end{aligned}$$

Let n be the number of terms.

$$\begin{aligned}1 \bullet (a^2)^{n-1} &= a^{4m+2} \\ a^{2n-2} &= a^{4m+2} \\ 2n-2 &= 4m+2 \\ n-1 &= 2m+1 \\ n &= 2m+2\end{aligned}$$

$\therefore$ There are $2m+2$ terms in the sequence.

28. Find the number of terms in the geometric sequence $\frac{2}{3}, -\frac{1}{2}, \frac{3}{8}, -\frac{9}{32}, \ldots, \frac{2187}{32768}$.

求等比數列 $\frac{2}{3}, -\frac{1}{2}, \frac{3}{8}, -\frac{9}{32}, \ldots, \frac{2187}{32768}$ 的項數。

A. 6

B. 7

C. 8

D. 9

$\boxed{\text{D}}$

Common ratio $= \dfrac{-\frac{1}{2}}{\frac{2}{3}} = -\dfrac{3}{4}$

Let n be the number of terms.

$$\begin{aligned}
\frac{2}{3}\left(-\frac{3}{4}\right)^{n-1} &= \frac{2187}{32768} \\
\left(-\frac{3}{4}\right)^{n-1} &= \frac{6561}{65536} \\
\left(-\frac{3}{4}\right)^{n-1} &= \left(-\frac{3}{4}\right)^{8} \\
n-1 &= 8 \\
n &= 9
\end{aligned}$$

$\therefore$ There are 9 terms in the sequence.

29. If $x+1, 3x+1, 5x+3$ are three consecutive terms of a geometric sequence, find the value(s) of x.
若 $x+1, 3x+1, 5x+3$ 是一個等比數列的三個連續項，求 x 的值。

A. -1

B. 1

C. -1 or $\dfrac{1}{2}$

D. 1 or $-\dfrac{1}{2}$

$\boxed{\text{D}}$

$\because$ $x+1,\ 3x+1,\ 5x+3$ are in geometric sequence.

$\therefore$
$$\begin{aligned}
(3x+1)^2 &= (x+1)(5x+3) \\
9x^2+6x+1 &= 5x^2+8x+3 \\
4x^2-2x-2 &= 0 \\
2x^2-x-1 &= 0 \\
(2x+1)(x-1) &= 0 \\
x &= \underline{\underline{-\frac{1}{2}}} \text{ or } \underline{\underline{1}}
\end{aligned}$$

30. The third term and the sixth term of a geometric sequence are 18 and 486 respectively. Find the first term of the sequence.
已知一個等比數列的第 3 項是 18，而第 6 項是 486。求該數列的首項。

A. 3

B. 2

C. 6

D. 8

$\boxed{\text{B}}$

Let a and R be the first term and the common ratio respectively.

$\therefore$
$$\begin{aligned}
T(n) &= aR^{n-1} \\
T(3) &= aR^2 = 18 \quad \text{......(1)} \\
T(6) &= aR^5 = 486 \quad \text{......(2)}
\end{aligned}$$

$(2)\div(1)$, $R^3 = 27$
$R = 3$

By substituting $R=3$ into (1), we have

$$\begin{aligned}
a(3^2) &= 18 \\
a &= 2
\end{aligned}$$

$\therefore$ The first term of the sequence is 2.

31. The second term and the fourth term of a geometric sequence of positive numbers are $\dfrac{4}{3}$ and $\dfrac{4}{27}$, find the fifth term of the sequence.
已知一個由正數組成的等比數列的第 2 項是 $\dfrac{4}{3}$，而第 4 項是 $\dfrac{4}{27}$，求該數列的第 5 項。

A. $\dfrac{1}{81}$

B. $\dfrac{4}{243}$

C. $\dfrac{4}{81}$

D. $\dfrac{3}{245}$

$\boxed{\text{C}}$

Let a and R be the first term and the common ratio respectively.

$\therefore \quad T(n) = aR^{n-1}$

$T(2) = aR = \dfrac{4}{3} \quad ......(1)$

$T(4) = aR^5 = \dfrac{4}{27} \quad ......(2)$

$(2) \div (1)$,

$R^2 = \dfrac{1}{9}$

$R = \dfrac{1}{3}$ or $-\dfrac{1}{3}$ (rejected)

By substituting $R = \dfrac{1}{3}$ into (1), we have

$a\left(\dfrac{1}{3}\right) = \dfrac{4}{3}$

$a = 4$

$\therefore \quad T(5) = 4\left(\dfrac{1}{3}\right)^4$

$= \underline{\underline{\dfrac{4}{81}}}$

32. If x and y are the geometric means between a and b, s is the geometric mean between x and y, which of the following is/are **NOT** true?
若 x 和 y 是 a 與 b 之間的等比中項，而 s 是 x 與 y 的等比中項，下列何者並**不正確**？

I. $xy = ab$

II. $s^2 = ab$

III. $x^2 = ab$

A. I only

B. II only

C. III only

D. I, II and III

$\boxed{\text{C}}$

The geometric sequence formed is:

a, x, s, y, b

$\therefore \quad xy = s^2$

$\because \quad a, s, b$ is also in geometric sequence.

$\therefore \quad s^2 = ab$

$\therefore \quad$ II is correct.

$\therefore \quad xy = ab$

$\therefore \quad$ I is correct.

33 It is given that x and y are positive numbers. If x and y are the geometric means between 1 and 100, find $\log x + \log y$.
已知 x 和 y 是正數。若 x 和 y 是 1 與 100 之間的等比中項，求 $\log x + \log y$。

A. 1

B. 2

C. 4

D. 8

$\boxed{\text{B}}$

Let R be the common ratio of the geometric sequence to be formed.

The geometric sequence formed is 1, R, R^2, 100

$\therefore \quad R^3 = 100$

$\log x + \log y = \log xy$

$= \log(R \bullet R^2)$

$= \log R^3$

$= \log 100$

$= \underline{\underline{2}}$

34. Find the sum of first 20 terms of the arithmetic series $18 + 14 + 10 + 6 + \ldots$
求等差級數 $18+14+10+6+...$ 首 20 項之和。

A. 1120

B. –800

C. –400

D. –580

$\boxed{\text{C}}$

Common difference $= 14 - 18$

$= -4$

$\therefore \quad \text{Sum} = \frac{20}{2}[2(18)+(20-1)(-4)]$
$= \underline{\underline{-400}}$

35. Find the sum of the arithmetic series
求等差級數 $3+7+11+...+63$ 各項之和。
$3+7+11+\ldots+63$.

A. 528
B. 495
C. 561
D. 1056

$\boxed{\text{A}}$

$$\begin{aligned}\text{Common difference} &= 7-3\\ &= 4\end{aligned}$$

Let n be the number of terms of the series.

$$\begin{aligned}3+(n-1)(4) &= 63\\ 4(n-1) &= 60\\ n &= 16\end{aligned}$$

$\therefore \quad \text{Sum} = \frac{16}{2}(3+63)$
$= \underline{\underline{528}}$

36. If the second term and the fifth term of a geometric series are 8 and 1 respectively, find the sum of the first 6 terms of the series.
若一個等比級數的第 2 項是 8，而第 5 項是 1，求該級數首 6 項之和。

A. 15
B. 63
C. $\frac{63}{2}$
D. 31

$\boxed{\text{C}}$

Let a and R be the first term and the common ratio respectively.

$\because \quad T(n) = aR^{n-1}$

$\therefore \quad T(2) = aR = 8 \quad ......(1)$
$T(5) = aR^4 = 1 \quad ......(2)$

$(2) \div (1), \quad R^3 = \frac{1}{8}$

$R = \frac{1}{2}$

By substituting $R = \frac{1}{2}$ into (1), we have

$$a\left(\frac{1}{2}\right) = 8$$
$$a = 16$$

$$\therefore \quad S(6) = \frac{16\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}} = 32\left(\frac{63}{64}\right) = \underline{\underline{\frac{63}{2}}}$$

37. If the general term $T(n)$ of an arithmetic sequence is $4+3n$, find $T(4)+\ldots+T(20)$.
若一個等差數列的通項 $T(n)$ 是 $4+3n$，求 $T(4)+...+T(20)$。

A. 640
B. 680
C. 800
D. 1000

$\boxed{\text{B}}$

$\because \quad T(n) = 4+3n$

$\therefore \quad T(4) = 16$
$T(20) = 64$

$T(4)+...+T(20)$
$= \frac{17}{2}(16+64)$
$= \underline{\underline{680}}$

38. $1+2+4+8+\ldots+2^{3n} =$

A. $2^{3n}-1$
B. $2^{3n-1}-1$
C. $2^{3n+1}-1$
D. $2^{3n}+1$

$\boxed{\text{C}}$

$\because$ The series is a geometric series with common ratio 2.

$\therefore$ $1+2+4+...+2^{3n}$
$=\dfrac{1(2^{3n+1}-1)}{2-1}$
$=\underline{\underline{2^{3n+1}-1}}$

39. The 6th term of a geometric series is 486 and the common ratio is 3. Find the sum of the first 5 terms of the series.
已知一個等比級數的第 6 項是 486 而公比是 3。求該級數首 5 項之和。

A. 242
B. 486
C. 162
D. 121

$\boxed{A}$

Let a be the first term.

$\because$ $T(6)=486$

$\therefore$ $a(3)^5=486$
$a=2$

$\therefore$ $S(5)=\dfrac{2(3^5-1)}{3-1}$
$=\underline{\underline{242}}$

40. In a geometric series, if the sum of the first three terms is $\dfrac{9}{4}$ and the sum to infinity is 2, find the first term of the series.
若一個等比級數的首 3 項之和是 $\dfrac{9}{4}$，而無限項之和是 2，求該級數的首項。

A. $\dfrac{4}{3}$
B. 3
C. 4
D. $-\dfrac{1}{2}$

$\boxed{A}$

Let a and R be the first term and the common ratio respectively.

$\therefore$ $\dfrac{a(1-R^3)}{1-R}=\dfrac{9}{4}$ ......(1)
$\dfrac{a}{1-R}=2$ ......(2)

$(1)\div(2), 1-R^3=\dfrac{9}{8}$
$R^3=-\dfrac{1}{8}$
$R=-\dfrac{1}{2}$

By substituting $R=-\dfrac{1}{2}$ into (2), we have

$\dfrac{a}{1-\left(-\dfrac{1}{2}\right)}=2$
$a=\underline{\underline{\dfrac{4}{3}}}$

41. Find the sum to infinity of the geometric series $1+\dfrac{3}{10}+\dfrac{9}{100}+\dfrac{27}{1000}+\dots$
求等比級數 $1+\dfrac{3}{10}+\dfrac{9}{100}+\dfrac{27}{100}+\dots$ 的無限項之和。

A. $\dfrac{7}{10}$
B. $\dfrac{8}{7}$
C. $\dfrac{10}{7}$
D. $\dfrac{10}{3}$

$\boxed{C}$

Common ratio $=\dfrac{3}{10}\div 1$
$=\dfrac{3}{10}$

$\therefore$ Sum to infinity $=\dfrac{1}{1-\dfrac{3}{10}}$
$=\underline{\underline{\dfrac{10}{7}}}$

42. Find the sum of all positive terms in the geometric sequence $\frac{2}{3}, -\frac{4}{15}, \frac{8}{75}, \ldots$

求等比數列 $\frac{2}{3}, -\frac{4}{15}, \frac{8}{75}, \ldots$ 中所有正數項之和。

A. $\frac{5}{21}$

B. $\frac{10}{21}$

C. $\frac{50}{63}$

D. $\frac{50}{87}$

$\boxed{\text{C}}$

Common ratio $= -\frac{4}{15} \div \frac{2}{3} = -\frac{2}{5}$

$\because$ All positive terms of the sequence formed another geometric sequence with common ratio $\left(-\frac{2}{5}\right)^2$.

$\therefore$ Sum of all positive terms

$$= \frac{\frac{2}{3}}{1-\left(-\frac{2}{5}\right)^2}$$

$$= \underline{\underline{\frac{50}{63}}}$$