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1) Given $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$

To find L^{-1}, we use Gauss-Jordan Elimination

We will consider $[L|I]$, where I is identity matrix I_3

$$[L|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

Performing row operation to transform L into Identity matrix

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 4R_1$

$$[L|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - 5R_2$

$$[L|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{array}\right]$$

The left side L become the identity matrix which means the right side now contains the inverse of L

$$\therefore \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 11 & -5 & 1 \end{bmatrix}$$

We can see L^{-1} has 1's on the diagonal.

2) Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

Lets consider $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We will perform Gaussian elimination to transform A into an upper triangular matrix U and will keep track of these operations in L matrix

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Dividing the second row of A by 2

$R_2 \rightarrow R_2/2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 4R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

Now we transformed A into LU

$$A = LU$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

If $A = LDU$, the D is the matrix formed from diagonal elements of U

$$\Rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So the new $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $A^2 - B^2$ is symmetric

4) Given $A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix}$

~~Calculate~~ ~~LU~~ and ~~DU~~

To get U, we need to find

$$E_{21} A = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & -b^2+c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix} = U$$

$$E_{21}^{-1} U = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix}$$

$$E_{21}^{-1} U = A$$

$$\Rightarrow E_{21}^{-1} = L = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix}$$

Converting $A = LU$ to $A = LDU$

$$A = \underset{L}{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}}$$

Here U is the transpose of L

$$\boxed{U = L^T}$$

5) Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}$

Exchange R_1 and R_2 to make 1^{st} pivot non-zero $\Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix} \quad \Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

From $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

$$\Rightarrow PA = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U} = LU$$

$\Rightarrow PA = LU$