01_ACFandPACF

October 29, 2021

1 ACF and PACF

2 Autocorrelation Function / Partial Autocorrelation Function

Before we can investigate autoregression as a modeling tool, we need to look at covariance and correlation as they relate to lagged (shifted) samples of a time series.

2.0.1 Goals

- Be able to create ACF and PACF charts
- Create these charts for multiple times series, one with seasonality and another without
- Be able to calculate Orders PQD terms for ARIMA off these charts (highlight where they cross the x axis)

Related Functions:

stattools.acovf(x[, unbiased, demean, fft, …]) Autocovariance for 1D stattools.acf(x[, unbiased, nlags, qstat, …]) Autocorrelation function for 1d arrays stattools.pacf(x[, nlags, method, alpha]) Partial autocorrelation estimated stattools.pacf_yw(x[, nlags, method]) Partial autocorrelation estimated with non-recursive yule_walker stattools.pacf_ols(x[, nlags]) Calculate partial autocorrelations

Related Plot Methods:

tsaplots.plot_acf(x) Plot the autocorrelation function tsaplots.plot_pacf(x) Plot the partial autocorrelation function

For Further Reading:

Wikipedia: Autocovariance Forecasting: Principles and Practice Autocorrelation NIST Statistics Handbook Partial Autocorrelation Plot

[2]:
```
import pandas as pd
import numpy as np
%matplotlib inline
import statsmodels.api as sm
from statsmodels.tsa.stattools import acovf,acf,pacf,pacf_yw,pacf_ols
import warnings
warnings.filterwarnings("ignore")
```

[3]:
```python
# Load a non-stationary dataset
df1 = pd.read_csv('../Data/airline_passengers.
↪csv',index_col='Month',parse_dates=True)
df1.index.freq = 'MS'

# Load a stationary dataset
df2 = pd.read_csv('../Data/DailyTotalFemaleBirths.
↪csv',index_col='Date',parse_dates=True)
df2.index.freq = 'D'
```

[4]:
```python
df1
```

[4]:
```
            Thousands of Passengers
Month
1949-01-01                      112
1949-02-01                      118
1949-03-01                      132
1949-04-01                      129
1949-05-01                      121
…                                 …
1960-08-01                      606
1960-09-01                      508
1960-10-01                      461
1960-11-01                      390
1960-12-01                      432

[144 rows x 1 columns]
```

[5]:
```python
df2
```

[5]:
```
            Births
Date
1959-01-01      35
1959-01-02      32
1959-01-03      30
1959-01-04      31
1959-01-05      44
…                …
1959-12-27      37
1959-12-28      52
1959-12-29      48
1959-12-30      55
1959-12-31      50

[365 rows x 1 columns]
```

2.1 Autocovariance for 1D

In a deterministic process, like $y = sin(x)$, we always know the value of y for a given value of x. However, in a stochastic process there is always some randomness that prevents us from knowing the value of y. Instead, we analyze the past (or lagged) behavior of the system to derive a probabilistic estimate for $\hat{y}$.

One useful descriptor is covariance. When talking about dependent and independent x and y variables, covariance describes how the variance in x relates to the variance in y. Here the size of the covariance isn't really important, as x and y may have very different scales. However, if the covariance is positive it means that x and y are changing in the same direction, and may be related.

With a time series, x is a fixed interval. Here we want to look at the variance of y_t against lagged or shifted values of y_{t+k}

For a stationary time series, the autocovariance function for γ (gamma) is given as:

$\gamma_{XX}(t_1, t_2) = \mathrm{Cov}\left[X_{t_1}, X_{t_2}\right] = \mathrm{E}[(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})]$

We can calculate a specific γ_k with:

$$\gamma_k = \frac{1}{n}\sum_{t=1}^{n-k}(y_t - \bar{y})(y_{t+k} - \bar{y})$$

2.1.1 Autocovariance Example:

Say we have a time series with five observations: $\{13, 5, 11, 12, 9\}$. We can quickly see that $n = 5$, the mean $\bar{y} = 10$, and we'll see that the variance $\sigma^2 = 8$. The following calculations give us our covariance values: $\gamma_0 = \frac{(13-10)(13-10)+(5-10)(5-10)+(11-10)(11-10)+(12-10)(12-10)+(9-10)(9-10)}{5} = \frac{40}{5} = 8.0$
$\gamma_1 = \frac{(13-10)(5-10)+(5-10)(11-10)+(11-10)(12-10)+(12-10)(9-10)}{5} = \frac{-20}{5} = -4.0$
$\gamma_2 = \frac{(13-10)(11-10)+(5-10)(12-10)+(11-10)(9-10)}{5} = \frac{-8}{5} = -1.6$
$\gamma_3 = \frac{(13-10)(12-10)+(5-10)(9-10)}{5} = \frac{11}{5} = 2.2$
$\gamma_4 = \frac{(13-10)(9-10)}{5} = \frac{-3}{5} = -0.6$ Note that γ_0 is just the population variance σ^2

Let's see if statsmodels gives us the same results! For this we'll create a fake dataset:

[9]:
```
df = pd.DataFrame({'a':[13, 5, 11, 12, 9]})
arr = acovf(df['a'])
arr
```

[9]:
```
array([ 8. , -4. , -1.6,  2.2, -0.6])
```

2.2 Autocorrelation for 1D

The correlation ρ (rho) between two variables y_1, y_2 is given as:

2.2.1 $\rho = \frac{\mathrm{E}[(y_1-\mu_1)(y_2-\mu_2)]}{\sigma_1\sigma_2} = \frac{\mathrm{Cov}(y_1,y_2)}{\sigma_1\sigma_2}$,

where E is the expectation operator, μ_1, σ_1 and μ_2, σ_2 are the means and standard deviations of y_1 and y_2.

When working with a single variable (i.e. autocorrelation) we would consider y_1 to be the original series and y_2 a lagged version of it. Note that with autocorrelation we work with $\bar{y}$, that is, the full population mean, and not the means of the reduced set of lagged factors (see note below).

Thus, the formula for ρ_k for a time series at lag k is:

$$\rho_k = \frac{\sum\limits_{t=1}^{n-k} (y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum\limits_{t=1}^{n} (y_t - \bar{y})^2}$$

This can be written in terms of the covariance constant γ_k as:

$$\rho_k = \frac{\gamma_k n}{\gamma_0 n} = \frac{\gamma_k}{\sigma^2}$$

For example, $\rho_4 = \frac{\gamma_4}{\sigma^2} = \frac{-0.6}{8} = -0.075$

Note that ACF values are bound by -1 and 1. That is, $-1 \leq \rho_k \leq 1$

[7]:
```
df=pd.DataFrame({'a':[13,5,11,12,9]})
df
```

[7]:

	a
0	13
1	5
2	11
3	12
4	9

[8]:
```
'''
So now that I have this data frame, what I'm going to do is calculate the␣
↪autocorrelation for it in
one dimension.

Remember, it's just the correlation between the lag that time series.
'''
acf(df)
```

[8]:
```
array([ 1.   , -0.5  , -0.2  ,  0.275, -0.075])
```

2.3 Partial Autocorrelation

Partial autocorrelations measure the linear dependence of one variable after removing the effect of other variable(s) that affect both variables. That is, the partial autocorrelation at lag k is the autocorrelation between y_t and y_{t+k} that is not accounted for by lags 1 through k−1.

A common method employs the non-recursive Yule-Walker Equations:

$\phi_0 = 1$
$\phi_1 = \rho_1 = -0.50$
$\phi_2 = \frac{\rho_2 - {\rho_1}^2}{1 - {\rho_1}^2} = \frac{(-0.20)-(-0.50)^2}{1-(-0.50)^2} = \frac{-0.45}{0.75} = -0.60$

As k increases, we can solve for ϕ_k using matrix algebra and the Levinson–Durbin recursion algorithm which maps the sample autocorrelations ρ to a Toeplitz diagonal-constant matrix. The full solution is beyond the scope of this course, but the setup is as follows:

$$\begin{pmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & \rho_0 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_0 \end{pmatrix} \begin{pmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{pmatrix}$$

[10]:
```
'''
And then we have to actually pass in a specific number of Lag's we want to do␣
↪this for.
so you can say nlags=4 and we'll go ahead and say 4 because we have 5 rows.
and value of nlags=n-1 so 5-1=4 so nlags=4.

then we'll also specify the method='mle'.
And so we passed Method mle above here in order to use the biased acf␣
↪coefficients and MLE stands for
maximum likelihood estimation.

So when you run this, you'll get back an array of the partial autocorrelation␣
↪function values.

'''

pacf_yw(df['a'],nlags=4,method='mle')
```

[10]:
```
array([ 1.        , -0.5       , -0.6       , -0.38541667, -0.40563273])
```

2.3.1 Partial Autocorrelation with OLS

This provides partial autocorrelations with ordinary least squares (OLS) estimates for each lag instead of Yule-Walker.

[11]:
```
'''
now stats models also provides another partial
autocorrelation function, pacf_ols and this provides partial autocorrelation of␣
↪ordinary
least squares, estimates for each leg instead of the youw Walker equations.

basic premise behind why there's different ways of calculating partial␣
↪autocorrelation
function is because there's actually different ways of calculating correlation␣
↪itself.

In fact, you may be familiar with the term Peerson correlation coefficient, and␣
↪that's because that's
```

```
a actually a particular equation for measuring correlation, the Peerson␣
↪correlation coefficient.

So if you keep expanding on that idea that there's different ways of measuring␣
↪error and measuring correlation,
then as you imagine, there'd be different ways of measuring partial␣
↪autocorrelation function.

'''
pacf_ols(df['a'],nlags=4)
```

[11]:
```
array([ 1.        , -0.49677419, -0.43181818,  0.53082621,  0.25434783])
```

3 Plotting

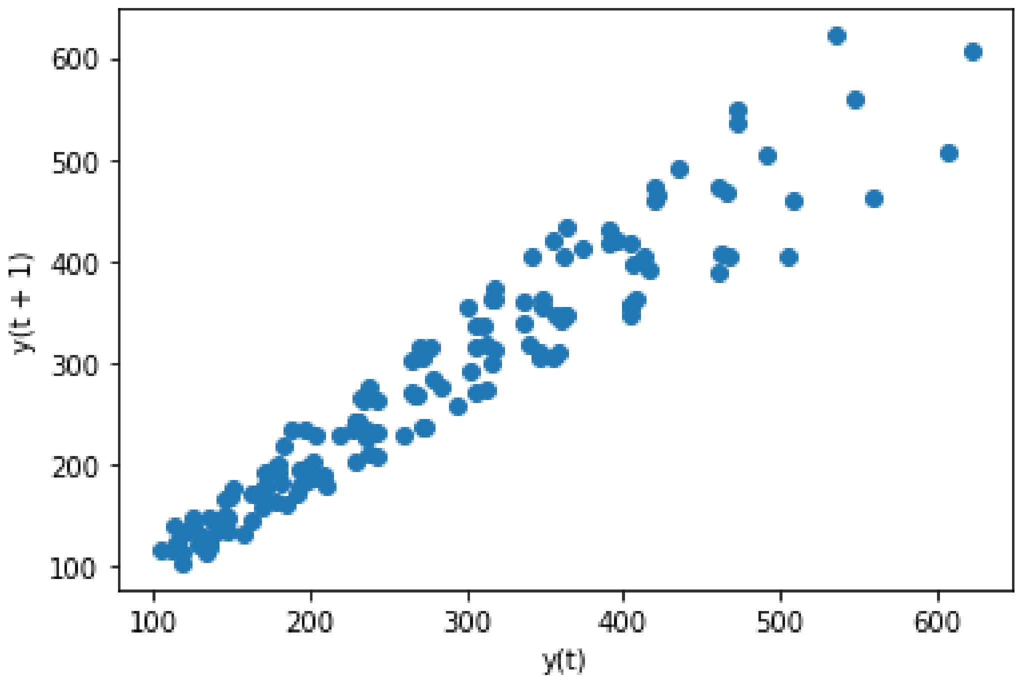
The arrays returned by .acf(df) and .pacf_yw(df) show the magnitude of the autocorrelation for a given y at time t. Before we look at plotting arrays, let's look at the data itself for evidence of autocorrelation.

Pandas has a built-in plotting function that plots increasing y_t values on the horizontal axis against lagged versions of the values y_{t+1} on the vertical axis. If a dataset is non-stationary with an upward trend, then neighboring values should trend in the same way. Let's look at the Airline Passengers dataset first.

[12]:
```
'''
So if we call like plot here, so if we take a look at this result, PANDAS has␣
↪that built-In plotting

function that plots increasing Y(T) values on the horizontal axis against␣
↪lagged versions of the values

Y(T+1) vertical axis.

If a data set is non stationary with an upward trend, then neighboring values␣
↪should trend the same
way.

So here we can see the airline passengers.
And visually, this shows evidence of very strong autocorrelation.

So we can see here that clearly there's some sort of correlation between Y(T)␣
↪and then Y(T+1)
, which is indicative of autocorrelation.
'''

from pandas.plotting import lag_plot
```

```
lag_plot(df1["Thousands of Passengers"])
```

[12]: <AxesSubplot:xlabel='y(t)', ylabel='y(t + 1)'>

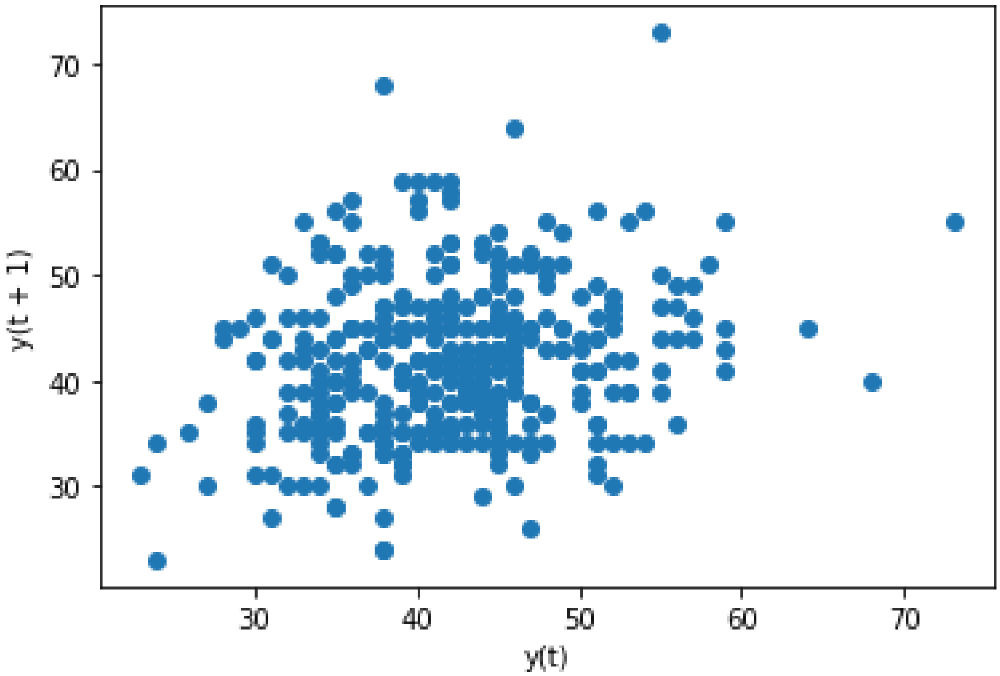


Visually this shows evidence of a very strong autocorrelation; as y_t values increase, nearby (lagged) values also increase.

Now let's look at the stationary Daily Total Female Births dataset:

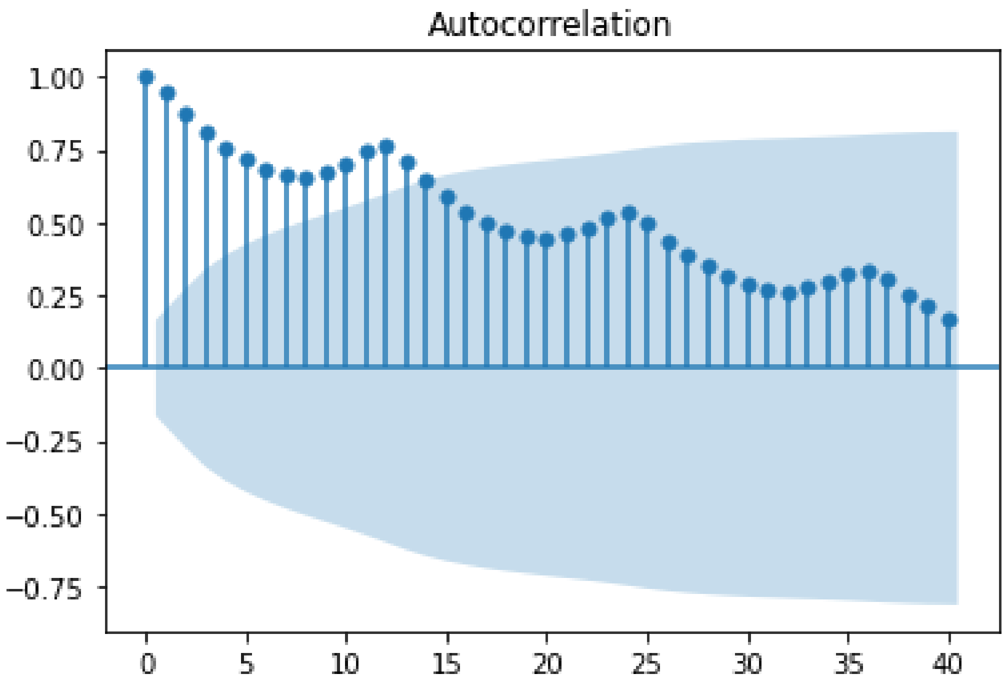
[14]:
```
'''
Let's look now at the stationary data set.

And run that notice now we don't get something with nearly as much correlation␣
↪between Y(T) and Y(T+1)


So then there's little evidence here indicating that there's strong auto␣
↪correlation for many legs for
the births data set, which in part makes sense.
You wouldn't expect the number of births on one day to be highly correlated␣
↪with the day before or the
day after.
'''

lag_plot(df2['Births'])
```

[14]: <AxesSubplot:xlabel='y(t)', ylabel='y(t + 1)'>



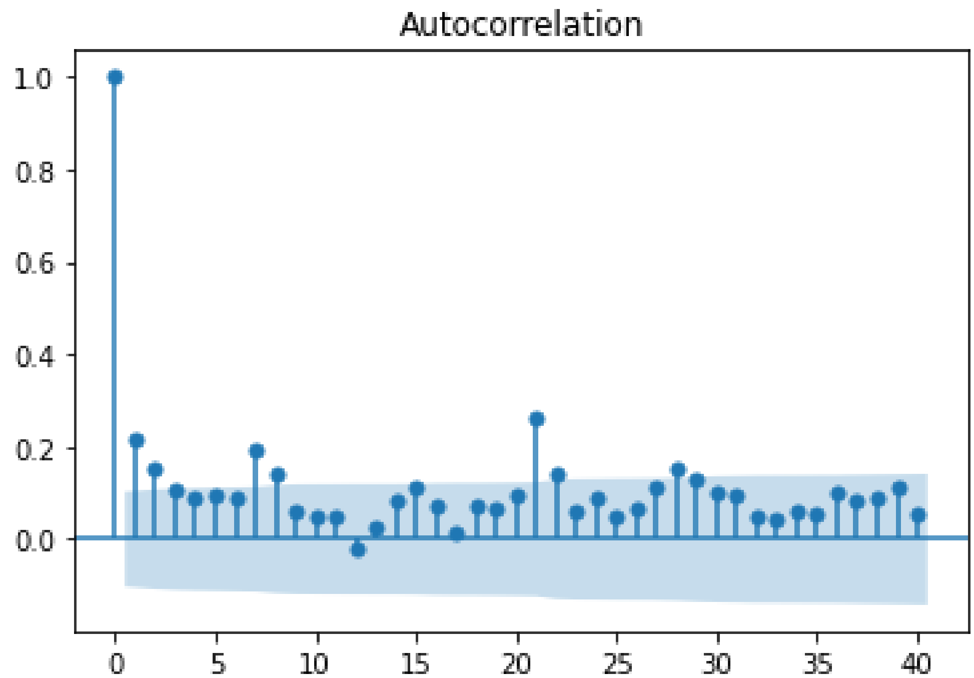
[17]:
```
'''
And notice again, there's clear indication of seasonality in the data because␣
↪there's these general
peaks and the autocorrelation.

So there's these little local maximums which indicates that at a certain point␣
↪the auto correlation
starts to increase, which makes sense because they're happening around a yearly␣
↪basis.


So what we may also be interested in is that we have this shaded region.

Well, by default, this is a 95 percent confidence interval.
And basically all that means is it's suggesting that correlation values outside␣
↪of this confidence interval
are very highly likely to be a correlation.

And notice your shaded region gets larger and larger as your legs gets larger␣
↪and larger as well, which
kind of makes sense.
```

```
You're going to be more sure of the smaller lags steps for autocorrelation than
 ↪the larger lags steps.

'''

from statsmodels.graphics.tsaplots import plot_acf,plot_pacf

plot_acf(df1,lags=40);
```



This plot indicates non-stationary data, as there are a large number of lags before ACF values drop off.

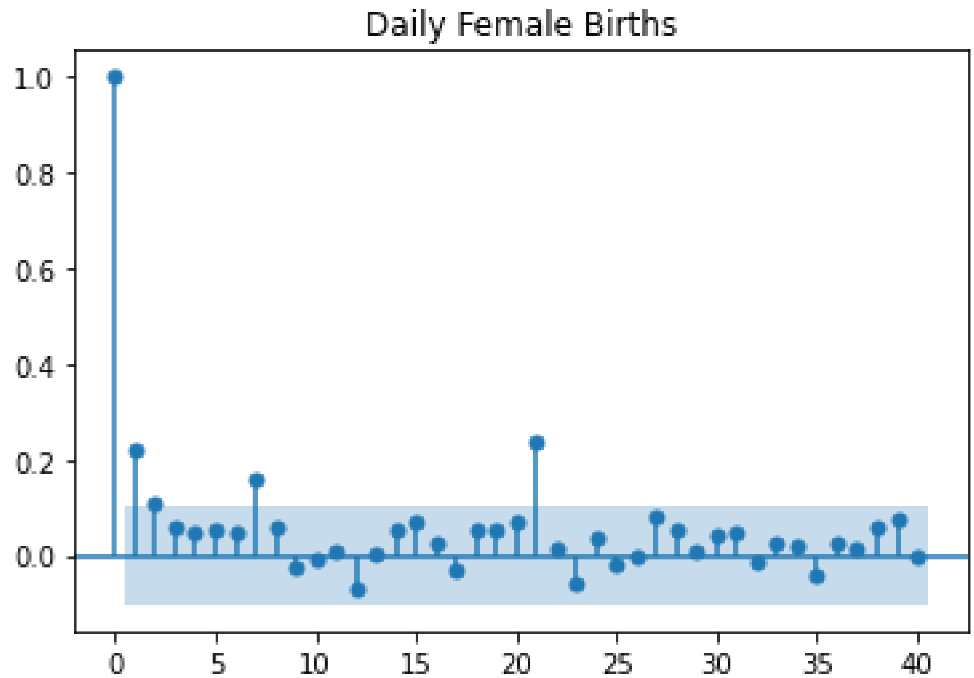
[18]:
```
'''
So this is the autocorrelation plot for the stationary data.

There's a very sharp drop off because it's stationary.
You don't see this sort of behavior indicating anything of seasonality.

'''
plot_acf(df2,lags=40);
```



This is a typical ACF plot for stationary data, with lags on the horizontal axis and correlations on the vertical axis. The first value y_0 is always 1. A sharp dropoff indicates that there is no AR component in the ARIMA model.

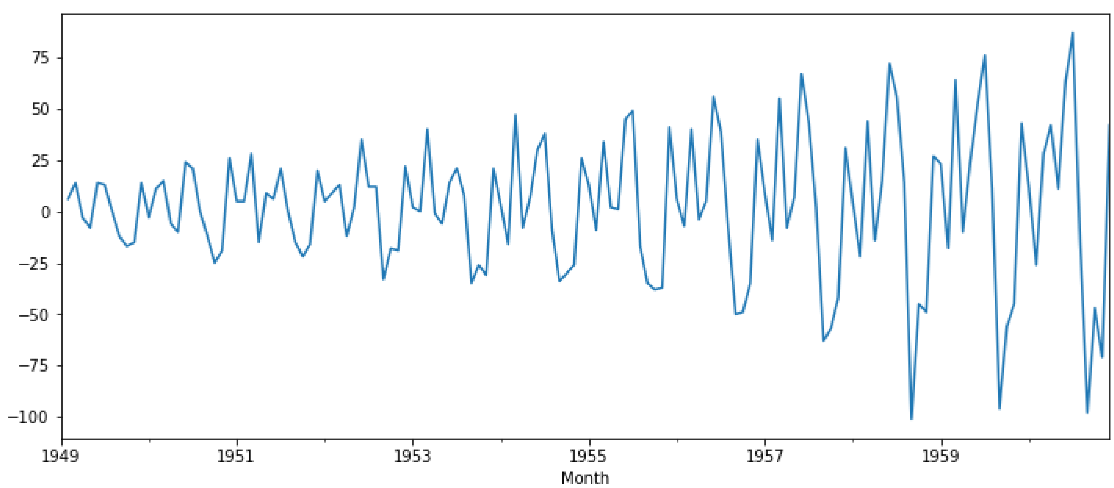
Next we'll look at non-stationary data with the Airline Passengers dataset:

[21]:
```
'''
So in general, partial autocorrelation plots work best with data that's already␣
↪stationary.
'''

plot_pacf(df2,lags=40,title="Daily Female Births");
```

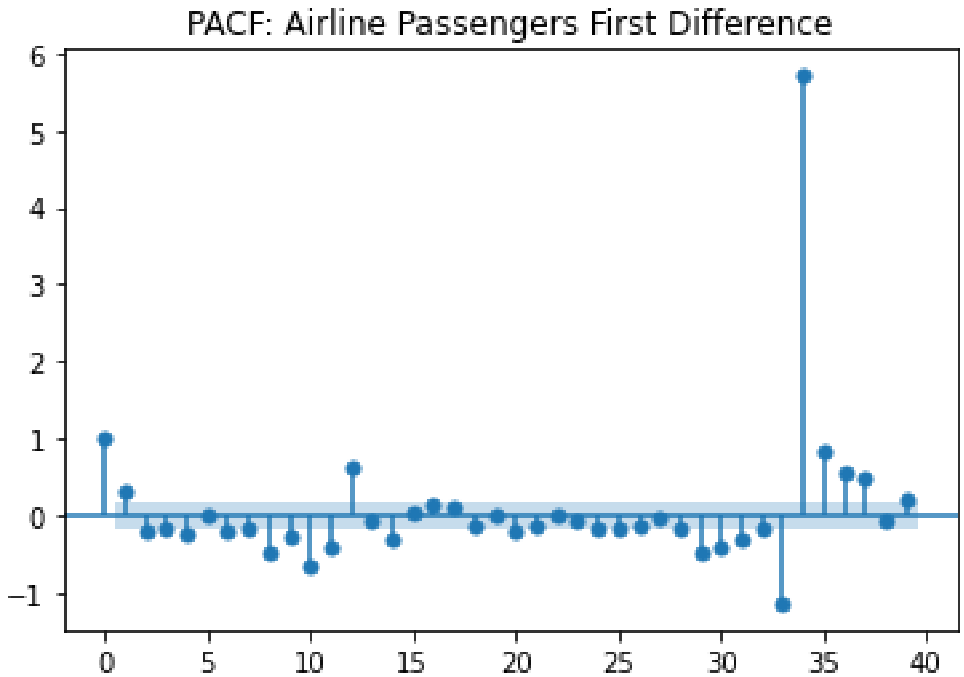


To make the Airline Passengers data stationary, we'll first apply differencing:

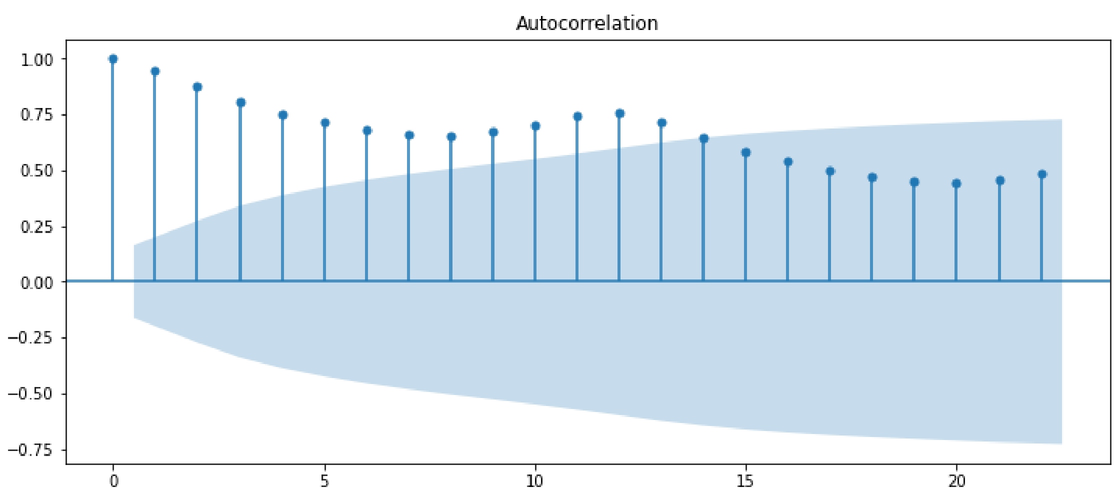
[22]:
```
from statsmodels.tsa.statespace.tools import diff

df1['d1'] = diff(df1['Thousands of Passengers'],k_diff=1)
df1['d1'].plot(figsize=(12,5));
```



[23]:
```
title='PACF: Airline Passengers First Difference'
lags=40
plot_pacf(df1['d1'].dropna(),title=title,lags=np.arange(lags));  # be sure to␣
↪add .dropna() here!
```



[25]:
```
import matplotlib.pyplot as plt
fig, ax = plt.subplots(figsize=(12,5))

plot_acf(df1['Thousands of Passengers'],ax=ax);
```



A NOTE ABOUT AUTOCORRELATION: Some texts compute lagged correlations using the Pearson Correlation Coefficient given by: $r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$

These are easily calculated in numpy with numpy.corrcoef(x,y) and in Excel with =CORREL(x,y). Using our example, r_0 is still 1, but to solve for r_1:

$x_1 = [13, 5, 11, 12], \bar{x_1} = 10.25$ $y_1 = [5, 11, 12, 9]$, $\bar{y_1} = 9.25$ $r_{x_1 y_1} = \frac{(13-10.25)(5-9.25)+(5-10.25)(11-9.25)+(11-10.25)(12-9.25)+(12-10.25)(9-9.25)}{\sqrt{((13-10.25)^2+(5-10.25)^2+(11-10.25)^2+(12-10.25)^2)}\sqrt{((5-9.25)^2+(11-9.25)^2+(12-9.25)^2+(9-9.25)^2)}} = \frac{-19.25}{33.38} = -0.577$ However, there are some shortcomings. Using the Pearson method, the second-to-last term r_{k-1} will always be 1 and the last term r_k will always be undefined.

```
[ ]:
```