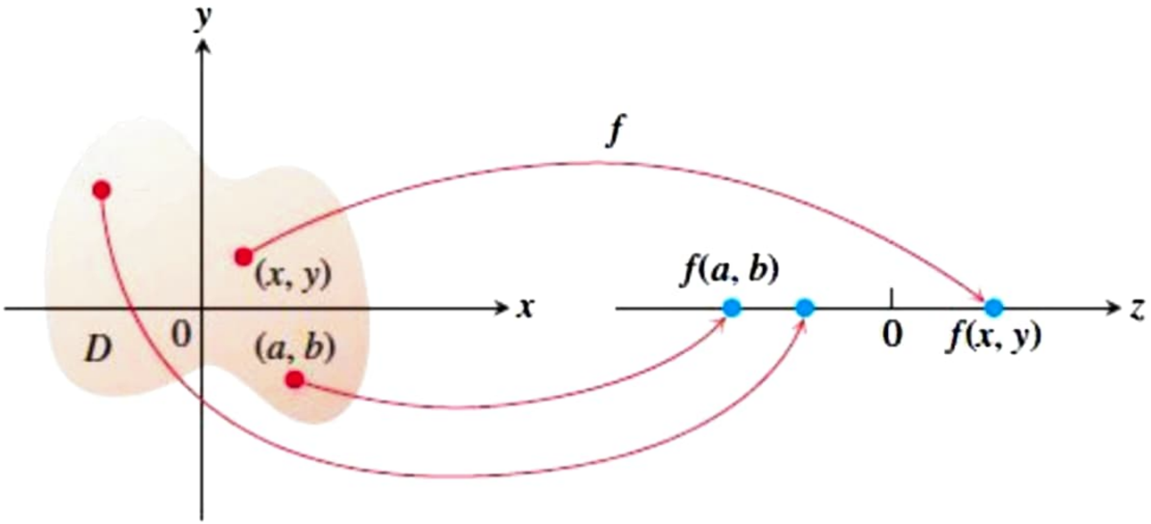
14.1 Functions of Several Variables

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

An Arrow diagram for the function z=f(x,y)



Domain and Range

Domain: The set D of all possible input values is called domain of the function.

Range: The set of all output values of $f(x)$ as x varies throughout D is called the range of the function.

Example 1

Domain and range of functions of two variables:

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

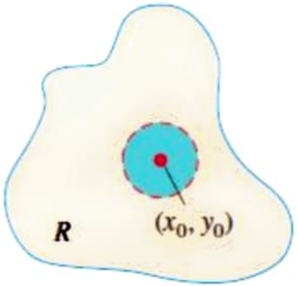
Example 1

Domain and range of functions of three variables:

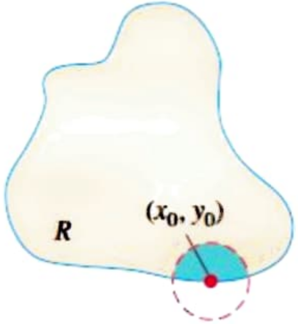
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

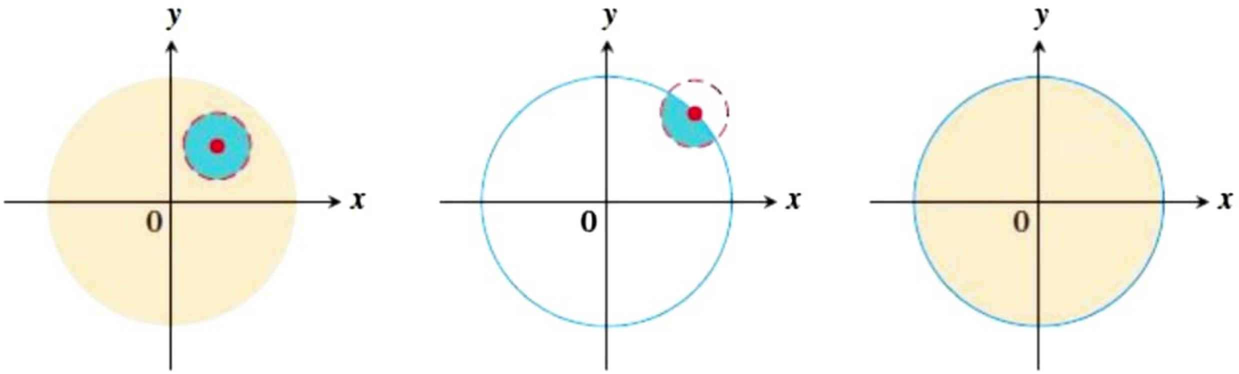
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



(a) Interior point



(b) Boundary point



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all
boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

Remark

- The empty set has no interior points and no boundary points. This implies that the empty set is open (because it does not contain points that are not interior points), and at the same time it is closed (because there are no boundary points that it fails to contain).
- The entire xy-plane is also both open and closed. Since open because every point in the plane is an interior point, and closed because it has no boundary points.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.4. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

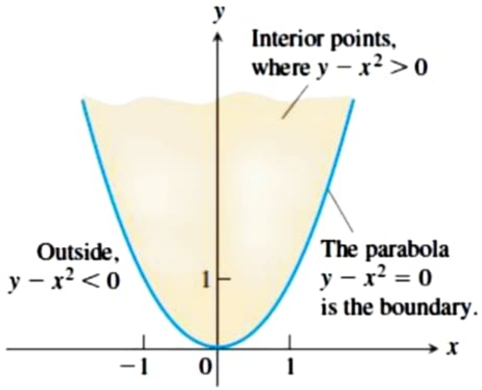


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

Level curve

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

EXAMPLE 3 Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure 14.5.

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.5) are the circles

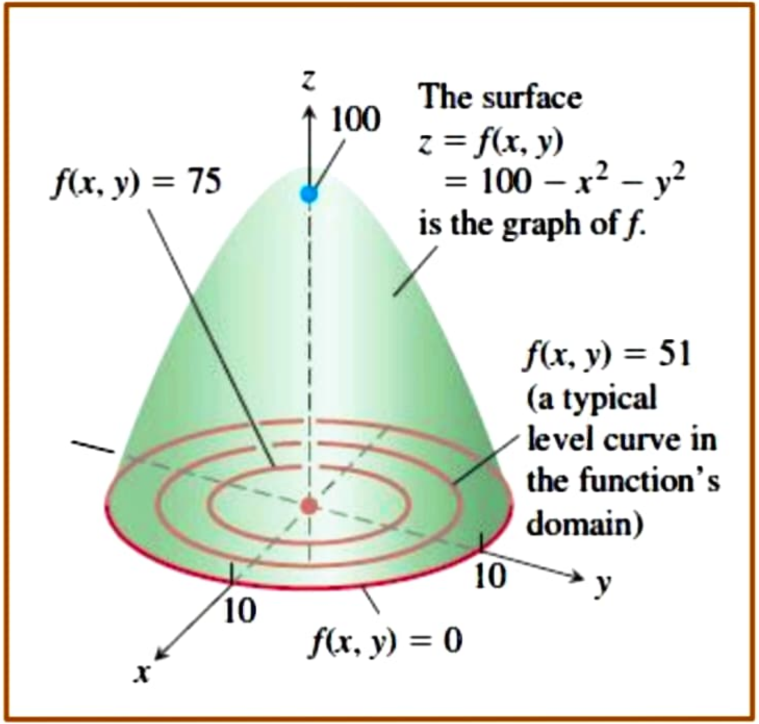
$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$

$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

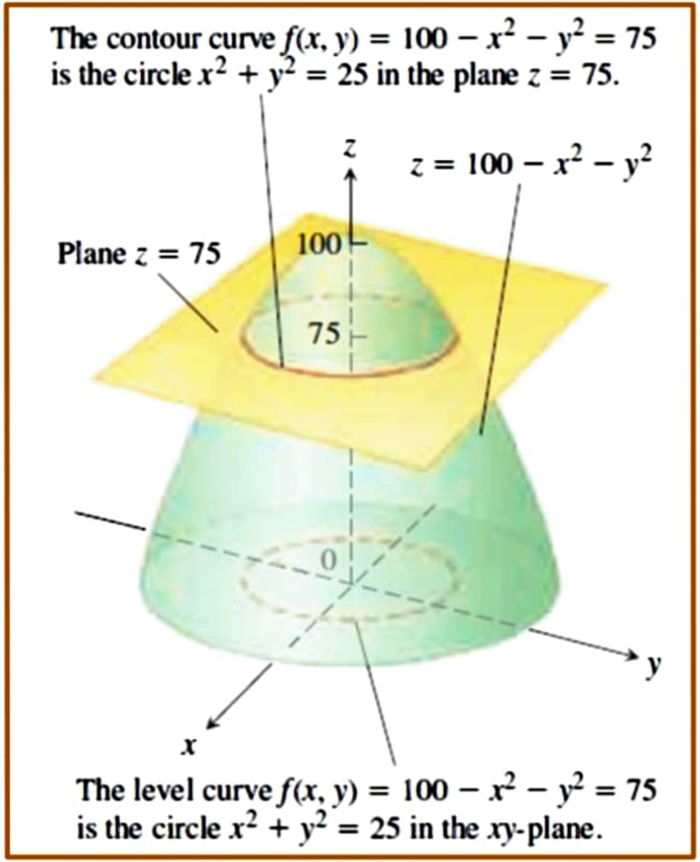
If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■

The graph and selected level curve for the above function



Contour curve

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f. Figure 14.6 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.


The contour curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the plane $z = 75$.
z
$z = 100 - x^2 - y^2$
Plane $z = 75$
100
75
0
y
x
The level curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the xy-plane.

Exercises 14.1

Domain, Range, and Level Curves

In Exercises 1–4, find the specific function values.

1. $f(x, y) = x^2 + xy^3$

   a. $f(0, 0)$ b. $f(-1, 1)$

   c. $f(2, 3)$ d. $f(-3, -2)$

2. $f(x, y) = \sin(xy)$

   a. $f\left(2, \frac{\pi}{6}\right)$ b. $f\left(-3, \frac{\pi}{12}\right)$

   c. $f\left(\pi, \frac{1}{4}\right)$ d. $f\left(-\frac{\pi}{2}, -7\right)$

Answers

1. (a) $f(0,0) = 0$ (b) $f(-1,1) = 0$ (c) $f(2,3) = 58$
   (d) $f(-3,-2) = 33$

2. (a) $f\left(2, \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (b) $f\left(-3, \frac{\pi}{12}\right) = -\frac{1}{\sqrt{2}}$ (c) $f\left(\pi, \frac{1}{4}\right) = \frac{1}{\sqrt{2}}$
   (d) $f\left(-\frac{\pi}{2}, -7\right) = -1$

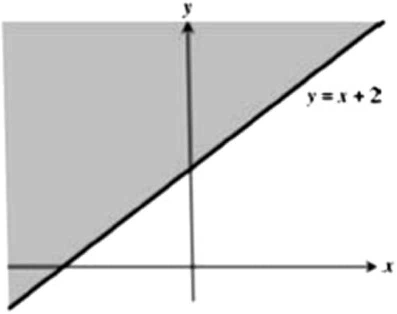
In Exercises 5–12, find and sketch the domain for each function.

5. $f(x, y) = \sqrt{y - x - 2}$

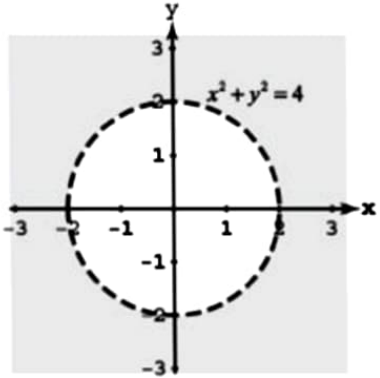
6. $f(x, y) = \ln(x^2 + y^2 - 4)$

Answers

5. Domain: all points (x, y) on or above the line $y = x + 2$



6. Domain: all points (x, y) outside the circle $x^2 + y^2 = 4$



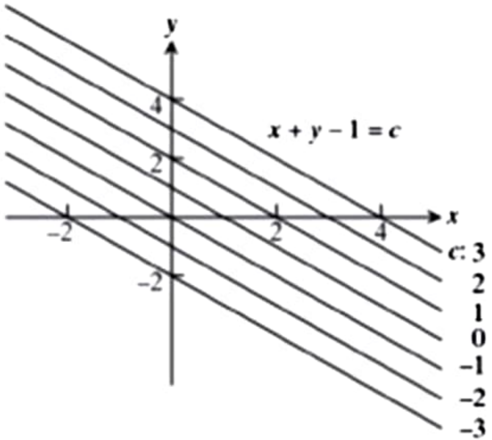
In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$

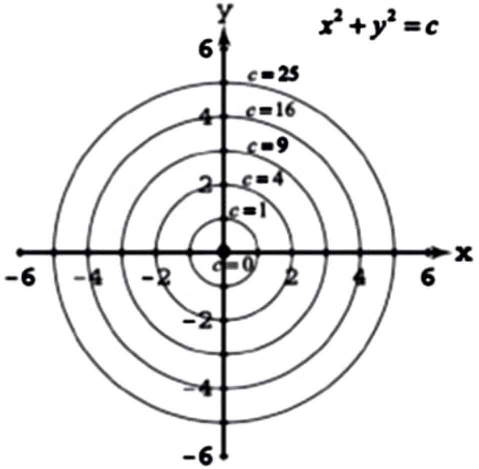
14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$

Answers

13.



14.



Problem No. 13