

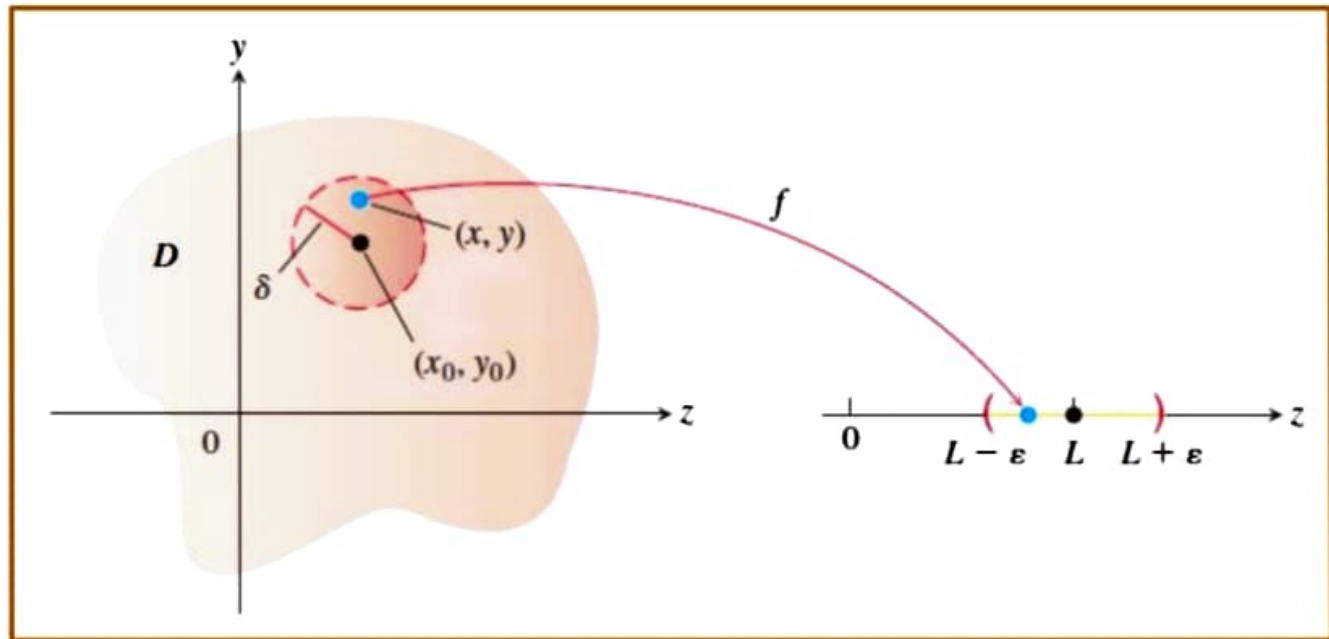
## 14.2 Limits and continuity in higher dimensions

**DEFINITION** We say that a function  $f(x, y)$  approaches the **limit**  $L$  as  $(x, y)$  approaches  $(x_0, y_0)$ , and write

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$$

if, for every number  $\varepsilon > 0$ , there exists a corresponding number  $\delta > 0$  such that for all  $(x, y)$  in the domain of  $f$ ,

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



## THEOREM 1—Properties of Limits of Functions of Two Variables

The following rules hold if  $L$ ,  $M$ , and  $k$  are real numbers and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

2. *Difference Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

3. *Constant Multiple Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} kf(x, y) = kL \quad (\text{any number } k)$$

4. *Product Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

5. *Quotient Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$$

6. *Power Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} [f(x, y)]^n = L^n, \quad n \text{ a positive integer}$$


7. *Root Rule:*

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$

$n$  a positive integer, and if  $n$  is even,  
we assume that  $L > 0$ .

**EXAMPLE 1** In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the  $x$ - and  $y$ -values of the point being approached into the functional expression to find the limiting value.

$$(a) \quad \lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

$$(b) \quad \lim_{(x,y) \rightarrow (3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$


**EXAMPLE 2** Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ .

**Solution** Since the denominator  $\sqrt{x} - \sqrt{y}$  approaches 0 as  $(x, y) \rightarrow (0, 0)$ , we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by  $\sqrt{x} + \sqrt{y}$ , however, we produce an equivalent fraction whose limit we *can* find:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}$$

Multiply by a form equal to 1.

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y}$$

Algebra

$$= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y})$$

Cancel the nonzero factor  $(x - y)$ .

$$= 0(\sqrt{0} + \sqrt{0}) = 0$$

Known limit values

# Continuity

**DEFINITION** A function  $f(x, y)$  is **continuous** at the point  $(x_0, y_0)$  if

1.  $f$  is defined at  $(x_0, y_0)$ ,
2.  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$  exists,
3.  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$ .

A function is **continuous** if it is continuous at every point of its domain.

## Exercises 14.2

### Limits with Two Variables

Find the limits in Exercises 1–12.

1.  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2.  $\lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}}$

3.  $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1}$

4.  $\lim_{(x,y) \rightarrow (2,-3)} \left( \frac{1}{x} + \frac{1}{y} \right)^2$

5.  $\lim_{(x,y) \rightarrow (0,\pi/4)} \sec x \tan y$

6.  $\lim_{(x,y) \rightarrow (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

# Answers

$$1. \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - 0^2 + 5}{0^2 + 0^2 + 2} = \frac{5}{2}$$

$$2. \lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$$

$$3. \lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1} = \sqrt{24} = 2\sqrt{6}$$

$$4. \lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2 = \left[\frac{1}{2} + \left(-\frac{1}{3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

$$5. \lim_{(x,y) \rightarrow (0, \frac{\pi}{4})} \sec x \tan y = (\sec 0) \left(\tan \frac{\pi}{4}\right) = (1)(1) = 1$$

$$6. \lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^2 + y^2}{x + y + 1}\right) = \cos\left(\frac{0^2 + 0^2}{0 + 0 + 1}\right) = \cos 0 = 1$$

### Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

$$13. \lim_{\substack{(x, y) \rightarrow (1, 1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y}$$

$$14. \lim_{\substack{(x, y) \rightarrow (1, 1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$$

$$15. \lim_{\substack{(x, y) \rightarrow (1, 1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1}$$

# Answers

$$13. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x - y) = (1 - 1) = 0$$

$$14. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x + y) = (1 + 1) = 2$$

$$15. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1} = \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim_{(x,y) \rightarrow (1,1)} (y - 2) = (1 - 2) = -1$$

### Two-Path Test for Nonexistence of a Limit

If a function  $f(x, y)$  has different limits along two different paths in the domain of  $f$  as  $(x, y)$  approaches  $(x_0, y_0)$ , then  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$  does not exist.

**EXAMPLE 3** Find  $\lim_{(x, y) \rightarrow (0, 0)} \frac{4xy^2}{x^2 + y^2}$  if it exists.

**EXAMPLE 4** If  $f(x, y) = \frac{y}{x}$ , does  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$  exist?

**EXAMPLE 5** Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

**EXAMPLE 6** Show that the function

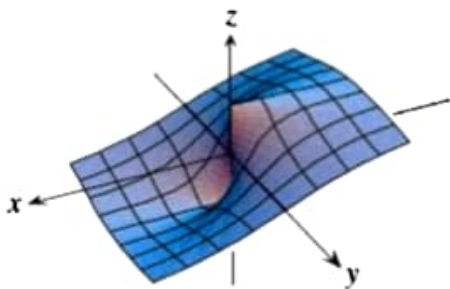
$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as  $(x, y)$  approaches  $(0, 0)$ .

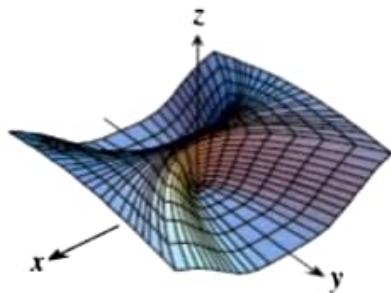
### No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as  $(x, y) \rightarrow (0, 0)$ .

41.  $f(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$



42.  $f(x, y) = \frac{x^4}{x^4 + y^2}$



43.  $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$

44.  $f(x, y) = \frac{xy}{|xy|}$

45.  $g(x, y) = \frac{x - y}{x + y}$

46.  $g(x, y) = \frac{x^2 - y}{x - y}$

47.  $h(x, y) = \frac{x^2 + y}{y}$

48.  $h(x, y) = \frac{x^2 y}{x^4 + y^2}$