Taylor's Formula for f(x,y) at (0,0)

Taylor's Formula for $f(x, y)$ at the Origin

$$f(x, y) = f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$

$$+ \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right)$$

$$+ \frac{1}{(n+1)!}\left(x^{n+1} \frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1} \frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)}$$

EXAMPLE 1 Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

Solution We take $n = 2$ in Equation (8):

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$

$$+ \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\Big|_{(cx, cy)}.$$

Calculating the values of the partial derivatives,

$$f(0,0) = \sin x \sin y \Big|_{(0,0)} = 0, \qquad f_{xx}(0,0) = -\sin x \sin y \Big|_{(0,0)} = 0,$$

$$f_x(0,0) = \cos x \sin y \Big|_{(0,0)} = 0, \qquad f_{xy}(0,0) = \cos x \cos y \Big|_{(0,0)} = 1,$$

$$f_y(0,0) = \sin x \cos y \Big|_{(0,0)} = 0, \qquad f_{yy}(0,0) = -\sin x \sin y \Big|_{(0,0)} = 0,$$

we have the result

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)), \quad \text{or} \quad \sin x \sin y \approx xy.$$

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

1. $f(x, y) = xe^y$
2. $f(x, y) = e^x \cos y$
3. $f(x, y) = y \sin x$
4. $f(x, y) = \sin x \cos y$
5. $f(x, y) = e^x \ln (1 + y)$
6. $f(x, y) = \ln (2x + y + 1)$
7. $f(x, y) = \sin (x^2 + y^2)$
8. $f(x, y) = \cos (x^2 + y^2)$

Answers

1. $f(x,y) = xe^y \;\Rightarrow\; f_x = e^y,\, f_y = xe^y,\, f_{xx} = 0,\, f_{xy} = e^y,\, f_{yy} = xe^y$

$\Rightarrow\; f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = x + xy$ quadratic approximation;

$f_{xxx} = 0,\, f_{xxy} = 0,\, f_{xyy} = e^y,\, f_{yyy} = xe^y$

$\Rightarrow\; f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= x + xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0\right) = x + xy + \frac{1}{2}xy^2$, cubic approximation

Answers

2. $f(x, y) = e^x \cos y \Rightarrow f_x = e^x \cos y$, $f_y = -e^x \sin y$, $f_{xx} = e^x \cos y$, $f_{xy} = -e^x \sin y$, $f_{yy} = -e^x \cos y$

$\Rightarrow f(x, y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$= 1 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 + x + \frac{1}{2}\left(x^2 - y^2\right)$, quadratic approximation;

$f_{xxx} = e^x \cos y$, $f_{xxy} = -e^x \sin y$, $f_{xyy} = -e^x \cos y$, $f_{yyy} = e^x \sin y$

$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left[x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right]$

$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left(x^3 - 3xy^2\right)$, cubic approximation

Answers

3. $f(x, y) = y \sin x \Rightarrow f_x = y \cos x,\ f_y = \sin x,\ f_{xx} = -y \sin x,\ f_{xy} = \cos x,\ f_{yy} = 0$

$\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$

$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = xy$, quadratic approximation;

$f_{xxx} = -y \cos x,\ f_{xxy} = -\sin x,\ f_{xyy} = 0,\ f_{yyy} = 0$

$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right]$

$= xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = xy$, cubic approximation

Answers

4. $f(x, y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y$, $f_y = -\sin x \sin y$, $f_{xx} = -\sin x \cos y$, $f_{xy} = -\cos x \sin y$, $f_{yy} = -\sin x \cos y \Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0) = x$, quadratic approximation;
$f_{xxx} = -\cos x \cos y$, $f_{xxy} = \sin x \sin y$, $f_{xyy} = -\cos x \cos y$, $f_{yyy} = \sin x \sin y$
$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right]$
$= x + \frac{1}{6}\left[x^3 \cdot (-1) + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right] = x - \frac{1}{6}(x^3 + 3xy^2)$, cubic approximation

Answers

5. $f(x, y) = e^x \ln(1 + y) \Rightarrow f_x = e^x \ln(1 + y), f_y = \frac{e^x}{1+y}, f_{xx} = e^x \ln(1 + y), f_{xy} = \frac{e^x}{1+y}, f_{yy} = -\frac{e^x}{(1+y)^2}$

$\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$

$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)] = y + \frac{1}{2}(2xy - y^2)$, quadratic approximation;

$f_{xxx} = e^x \ln(1 + y), f_{xxy} = \frac{e^x}{1+y}, f_{xyy} = -\frac{e^x}{(1+y)^2}, f_{yyy} = \frac{2e^x}{(1+y)^3}$

$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)]$

$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2]$

$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2 y - 3xy^2 + 2y^3)$, cubic approximation

Answers

6. $f(x, y) = \ln(2x + y + 1) \Rightarrow f_x = \frac{2}{2x + y + 1}, f_y = \frac{1}{2x + y + 1}, f_{xx} = \frac{-4}{(2x + y + 1)^2}, f_{xy} = \frac{-2}{(2x + y + 1)^2},$

$f_{yy} = \frac{-1}{(2x + y + 1)^2} \Rightarrow f(x, y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)]$

$= 0 + x \cdot 2 + y \cdot 1 + \frac{1}{2}[x^2 \cdot (-4) + 2xy \cdot (-2) + y^2 \cdot (-1)] = 2x + y + \frac{1}{2}(-4x^2 - 4xy - y^2)$

$= (2x + y) - \frac{1}{2}(2x + y)^2$, quadratic approximation;

$f_{xxx} = \frac{16}{(2x + y + 1)^3}, f_{xxy} = \frac{8}{(2x + y + 1)^3}, f_{xyy} = \frac{4}{(2x + y + 1)^3}, f_{yyy} = \frac{2}{(2x + y + 1)^3}$

$\Rightarrow f(x, y) \approx$ quadratic $+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$

$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{6}(x^3 \cdot 16 + 3x^2 y \cdot 8 + 3xy^2 \cdot 4 + y^3 \cdot 2)$

$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(8x^3 + 12x^2 y + 6xy^2 + y^2)$

$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(2x + y)^3$, cubic approximation

Answers

7. $f(x, y) = \sin(x^2 + y^2) \Rightarrow f_x = 2x\cos(x^2 + y^2)$, $f_y = 2y\cos(x^2 + y^2)$, $f_{xx} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)$, $f_{xy} = -4xy\sin(x^2 + y^2)$, $f_{yy} = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2)$

$\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$

$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 2) = x^2 + y^2$, quadratic approximation;

$f_{xxx} = -12x\sin(x^2 + y^2) - 8x^3\cos(x^2 + y^2)$, $f_{xxy} = -4y\sin(x^2 + y^2) - 8x^2y\cos(x^2 + y^2)$, $f_{xyy} = -4x\sin(x^2 + y^2) - 8xy^2\cos(x^2 + y^2)$, $f_{yyy} = -12y\sin(x^2 + y^2) - 8y^3\cos(x^2 + y^2)$

$\Rightarrow f(x, y) \approx \text{quadratic} + \frac{1}{6}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)]$

$= x^2 + y^2 + \frac{1}{6}(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = x^2 + y^2$, cubic approximation

Answers

8. $f(x, y) = \cos(x^2 + y^2) \Rightarrow f_x = -2x \sin(x^2 + y^2)$, $f_y = -2y \sin(x^2 + y^2)$,

$f_{xx} = -2 \sin(x^2 + y^2) - 4x^2 \cos(x^2 + y^2)$, $f_{xy} = -4xy \cos(x^2 + y^2)$, $f_{yy} = -2 \sin(x^2 + y^2) - 4y^2 \cos(x^2 + y^2)$

$\Rightarrow f(x, y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)]$

$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0] = 1$, quadratic approximation;

$f_{xxx} = -12x \cos(x^2 + y^2) + 8x^3 \sin(x^2 + y^2)$, $f_{xxy} = -4y \cos(x^2 + y^2) + 8x^2 y \sin(x^2 + y^2)$,

$f_{xyy} = -4x \cos(x^2 + y^2) + 8xy^2 \sin(x^2 + y^2)$, $f_{yyy} = -12y \cos(x^2 + y^2) + 8y^3 \sin(x^2 + y^2)$

$\Rightarrow f(x, y) \approx$ quadratic $+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$

$= 1 + \frac{1}{6}(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = 1$, cubic approximation