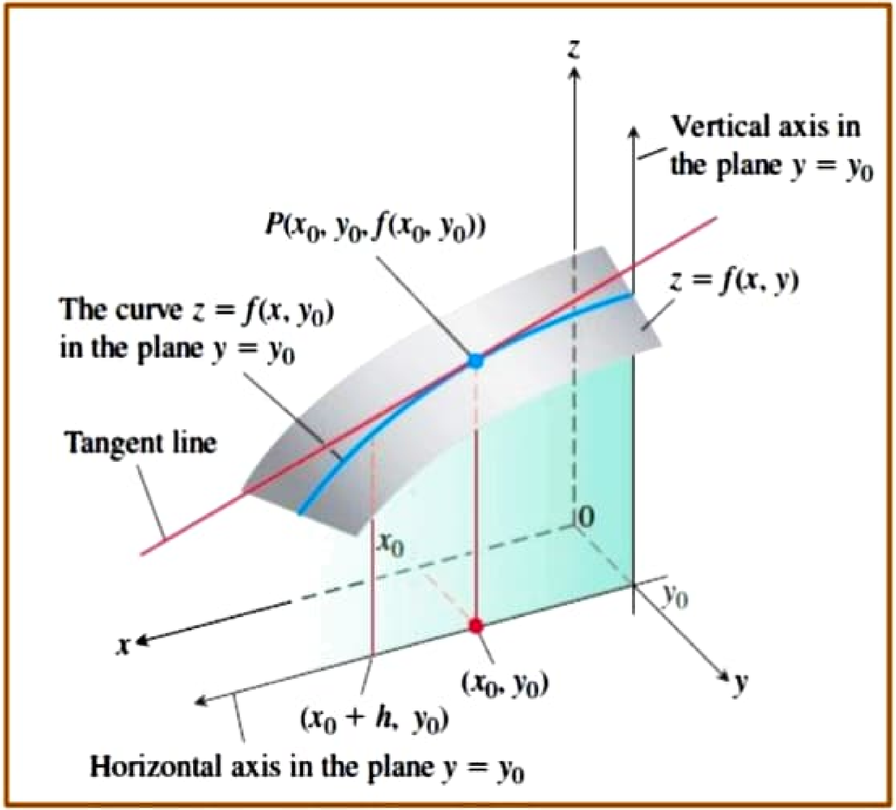
14.3 Partial Derivatives

DEFINITION **The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is**

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

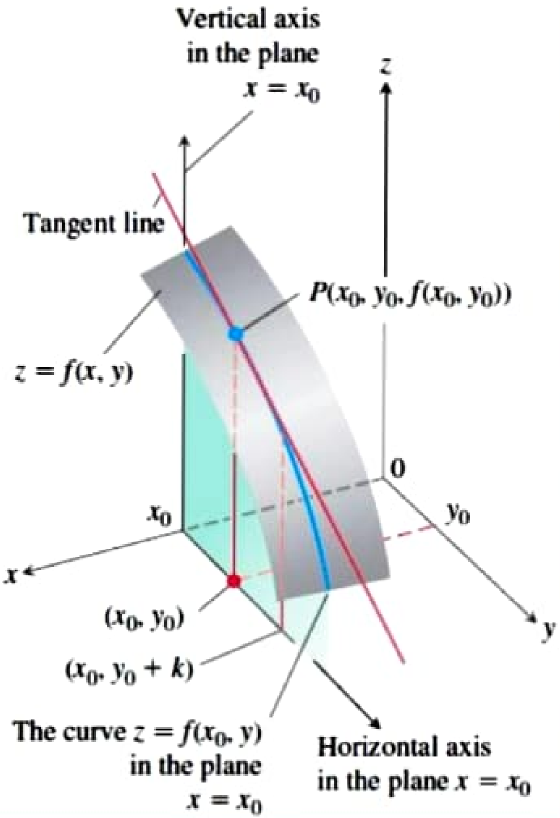
provided the limit exists.


z
Vertical axis in the plane y = y0
P(x0, y0, f(x0, y0))
z = f(x, y)
The curve z = f(x, y0) in the plane y = y0
Tangent line
0
x0
y0
x
y
(x0, y0)
(x0 + h, y0)
Horizontal axis in the plane y = y0

DEFINITION **The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is**

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.



Vertical axis in the plane $x = x_0$
z
Tangent line
$P(x_0, y_0, f(x_0, y_0))$
$z = f(x, y)$
0
x_0
y_0
x
y
(x_0, y_0)
$(x_0, y_0 + k)$
The curve $z = f(x_0, y)$ in the plane $x = x_0$
Horizontal axis in the plane $x = x_0$

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we use the quotient rule to get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With x held constant and again applying the quotient rule, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

EXAMPLE 4 Find $\partial z/\partial x$ assuming that the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \qquad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}.$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

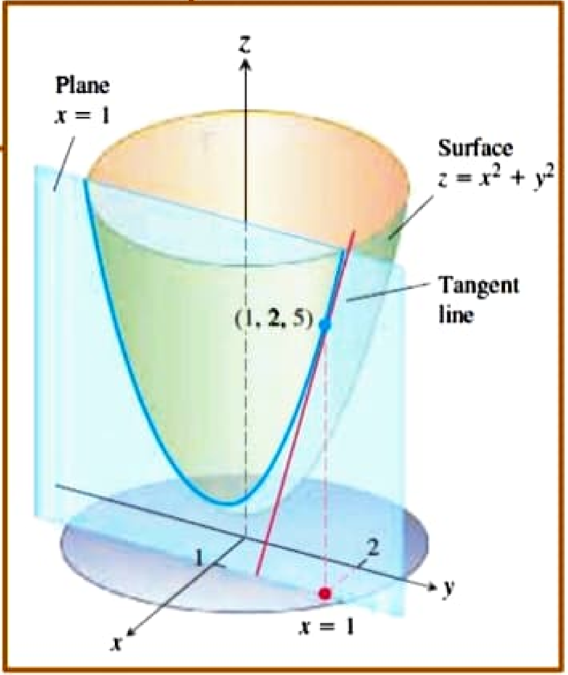
$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

■

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.19).

Solution The parabola lies in a plane parallel to the yz-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at $(1, 2)$:

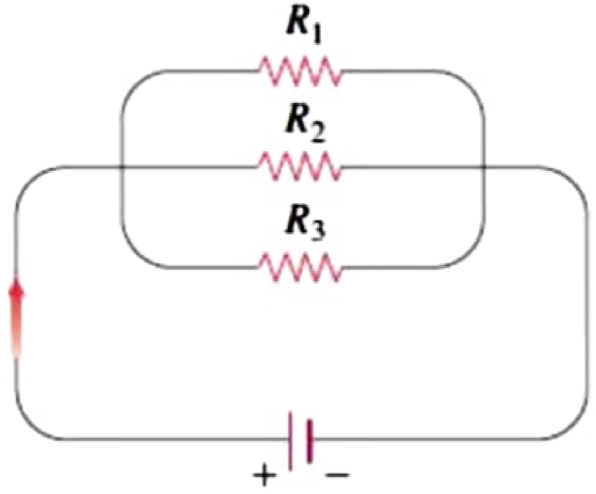
$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$



EXAMPLE 7 If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(Figure 14.20). Find the value of $\partial R / \partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.



Solution To find $\partial R/\partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2{}^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Partial Derivatives and Continuity

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Prove that f is not continuous at the origin.

(c) Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

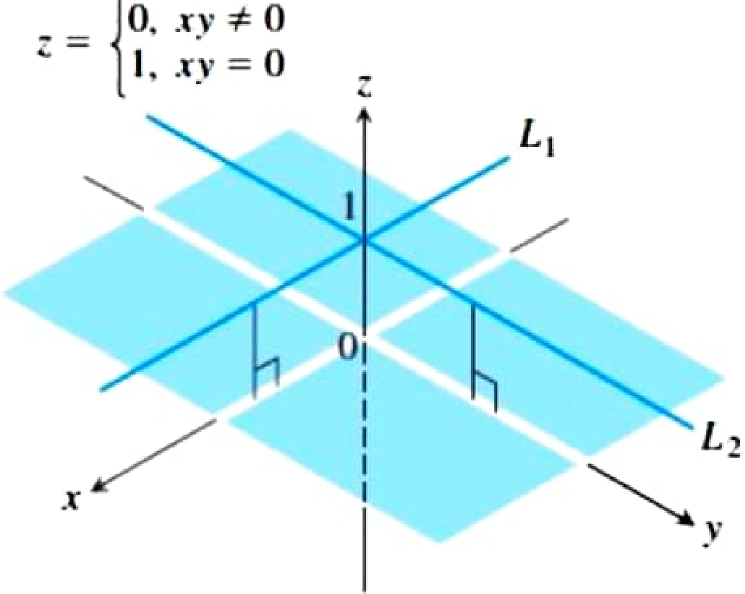


FIGURE 14.21 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since $f(0, 0) = 1$, the limit in part (a) is not equal to $f(0, 0)$, which proves that f is not continuous at $(0, 0)$.

(c) To find $\partial f/\partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x, and the graph of f is the line L_1 in Figure 14.21. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y, so $\partial f/\partial y = 0$ at $(0, 0)$. ■

Second Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x\, \partial y} \text{ or } f_{yx}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y\, \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$\dfrac{\partial^2 f}{\partial x\, \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial y\, \partial x}, \qquad \frac{\partial^2 f}{\partial y^2}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x\, \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) \qquad\qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x)$$

$$= \cos y + ye^x \qquad\qquad = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad\qquad \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad\qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

EXAMPLE 10 Find $\dfrac{\partial^2 w}{\partial x\, \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

THEOREM 4—Differentiability Implies Continuity

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Exercises 14.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$
2. $f(x, y) = x^2 - xy + y^2$
3. $f(x, y) = (x^2 - 1)(y + 2)$
4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
5. $f(x, y) = (xy - 1)^2$
6. $f(x, y) = (2x - 3y)^3$
7. $f(x, y) = \sqrt{x^2 + y^2}$
8. $f(x, y) = (x^3 + (y/2))^{2/3}$
9. $f(x, y) = 1/(x + y)$
10. $f(x, y) = x/(x^2 + y^2)$
11. $f(x, y) = (x + y)/(xy - 1)$
12. $f(x, y) = \tan^{-1}(y/x)$
13. $f(x, y) = e^{(x+y+1)}$
14. $f(x, y) = e^{-x}\sin(x + y)$
15. $f(x, y) = \ln(x + y)$
16. $f(x, y) = e^{xy}\ln y$
17. $f(x, y) = \sin^2(x - 3y)$
18. $f(x, y) = \cos^2(3x - y^2)$

19. $f(x, y) = x^y$

20. $f(x, y) = \log_y x$

21. $f(x, y) = \int_x^y g(t)\, dt$ (g continuous for all t)

22. $f(x, y) = \sum_{n=0}^{\infty} (xy)^n$ ($|xy| < 1$)

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

41. $f(x, y) = x + y + xy$

42. $f(x, y) = \sin xy$

43. $g(x, y) = x^2y + \cos y + y \sin x$

44. $h(x, y) = xe^y + y + 1$

45. $r(x, y) = \ln (x + y)$

Answers

41. $\frac{\partial f}{\partial x} = 1 + y,\ \frac{\partial f}{\partial y} = 1 + x,\ \frac{\partial^2 f}{\partial x^2} = 0,\ \frac{\partial^2 f}{\partial y^2} = 0,\ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

42. $\frac{\partial f}{\partial x} = y \cos xy,\ \frac{\partial f}{\partial y} = x \cos xy,\ \frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy,\ \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy,\ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$

43. $\frac{\partial g}{\partial x} = 2xy + y \cos x,\ \frac{\partial g}{\partial y} = x^2 - \sin y + \sin x,\ \frac{\partial^2 g}{\partial x^2} = 2y - y \sin x,\ \frac{\partial^2 g}{\partial y^2} = -\cos y,\ \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$

44. $\frac{\partial h}{\partial x} = e^y,\ \frac{\partial h}{\partial y} = xe^y + 1,\ \frac{\partial^2 h}{\partial x^2} = 0,\ \frac{\partial^2 h}{\partial y^2} = xe^y,\ \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$

45. $\frac{\partial r}{\partial x} = \frac{1}{x+y},\ \frac{\partial r}{\partial y} = \frac{1}{x+y},\ \frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2},\ \frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2},\ \frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$

Mixed Partial Derivatives

In Exercises 55–60, verify that $w_{xy} = w_{yx}$.

55. $w = \ln(2x + 3y)$

56. $w = e^x + x \ln y + y \ln x$

57. $w = xy^2 + x^2y^3 + x^3y^4$

58. $w = x \sin y + y \sin x + xy$

59. $\omega = \dfrac{x^2}{y^3}$

60. $\omega = \dfrac{3x - y}{x + y}$

Using the Partial Derivative Definition

In Exercises 63–66, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

63. $f(x, y) = 1 - x + y - 3x^2y, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(1, 2)$

64. $f(x, y) = 4 + 2x - 3y - xy^2, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 1)$

65. $f(x, y) = \sqrt{2x + 3y - 1}, \quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 3)$

Answers

63. . $f_x(1,2) = \lim\limits_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim\limits_{h \to 0} \frac{[1 - (1+h) + 2 - 6(1+h)^2] - (2-6)}{h} = \lim\limits_{h \to 0} \frac{-h - 6(1 + 2h + h^2) + 6}{h}$

$= \lim\limits_{h \to 0} \frac{-13h - 6h^2}{h} = \lim\limits_{h \to 0} (-13 - 6h) = -13,$

$f_y(1,2) = \lim\limits_{h \to 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim\limits_{h \to 0} \frac{[1 - 1 + (2+h) - 3(2+h)] - (2-6)}{h} = \lim\limits_{h \to 0} \frac{(2 - 6 - 2h) - (2-6)}{h}$

$= \lim\limits_{h \to 0} (-2) = -2$

Answers

64. $f_x(-2,1) = \lim_{h \to 0} \frac{f(-2+h,1) - f(-2,1)}{h} = \lim_{h \to 0} \frac{[4 + 2(-2+h) - 3 - (-2+h)] - (-3+2)}{h}$

$= \lim_{h \to 0} \frac{(2h - 1 - h) + 1}{h} = \lim_{h \to 0} 1 = 1,$

$f_y(-2,1) = \lim_{h \to 0} \frac{f(-2,1+h) - f(-2,1)}{h} = \lim_{h \to 0} \frac{[4 - 4 - 3(1+h) + 2(1+h)^2] - (-3+2)}{h}$

$= \lim_{h \to 0} \frac{(-3 - 3h + 2 + 4h + 2h^2) + 1}{h} = \lim_{h \to 0} \frac{h + 2h^2}{h} = \lim_{h \to 0} (1 + 2h) = 1$

Answers

65. $$f_x(-2,3) = \lim_{h \to 0} \frac{f(-2+h,3) - f(-2,3)}{h} = \lim_{h \to 0} \frac{\sqrt{2(-2+h)+9-1} - \sqrt{-4+9-1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2h+4} - 2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{2h+4} - 2}{h} \frac{\sqrt{2h+4} + 2}{\sqrt{2h+4} + 2} \right) = \lim_{h \to 0} \frac{2}{\sqrt{2h+4} + 2} = \frac{1}{2},$$

$$f_y(-2,3) = \lim_{h \to 0} \frac{f(-2,3+h) - f(-2,3)}{h} = \lim_{h \to 0} \frac{\sqrt{-4+3(3+h)-1} - \sqrt{-4+9-1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3h+4} - 2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{3h+4} - 2}{h} \frac{\sqrt{3h+4} + 2}{\sqrt{3h+4} + 2} \right) = \lim_{h \to 0} \frac{3}{\sqrt{2h+4} + 2} = \frac{3}{4}$$

66. $f(x, y) = \begin{cases} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, 0)$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(h^3+0)}{h^2+0} - 0}{h} = \lim_{h \to 0} \frac{\sin h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(0+h^4)}{0+h^2} - 0}{h} = \lim_{h \to 0} \frac{\sin h^4}{h^3} = \lim_{h \to 0} \left(h \cdot \frac{\sin h^4}{h^4}\right) = 0 \cdot 1 = 0$$

67. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in the a. plane $x = 2$ b. plane $y = -1$.

68. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the a. plane $x = -1$ b. plane $y = 1$.