

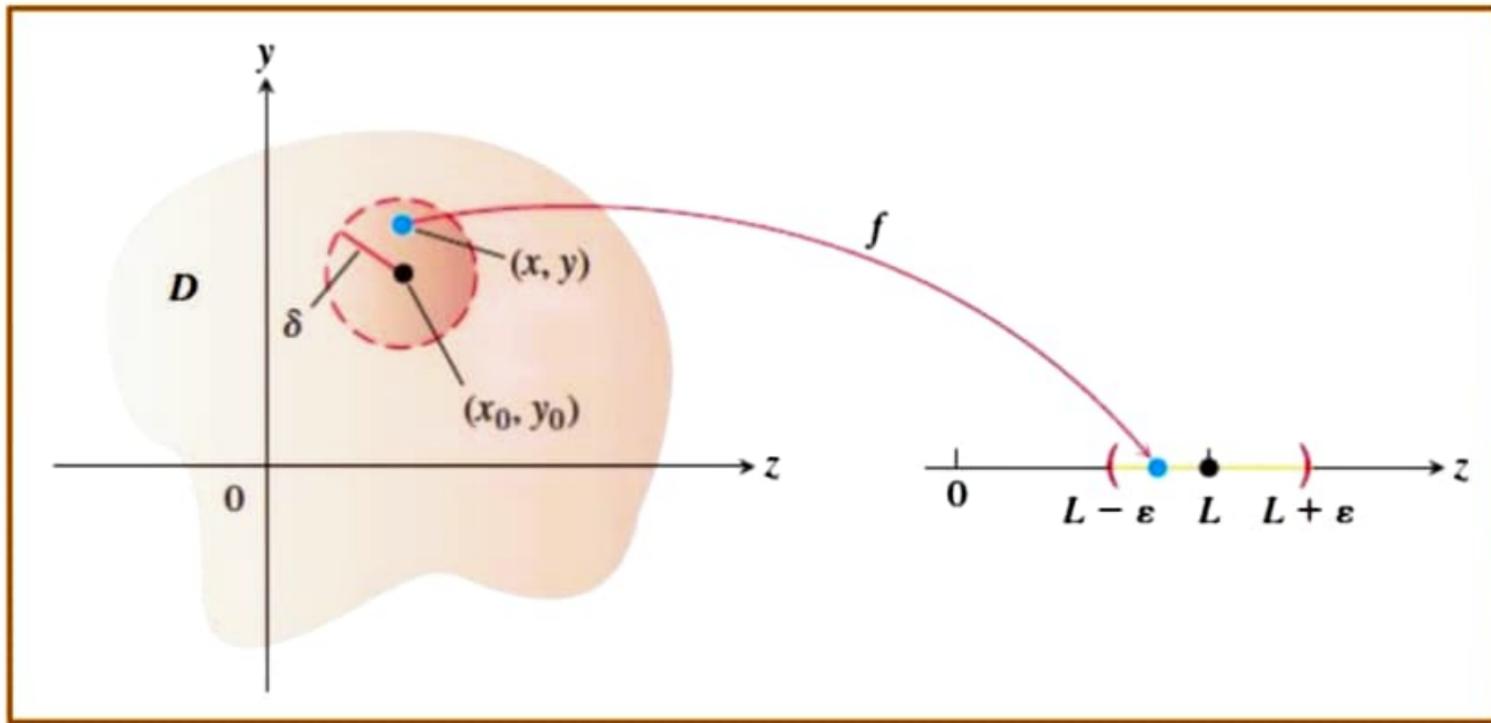
14.2 Limits and continuity in higher dimensions

DEFINITION We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) , and write

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f ,

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



THEOREM 1—Properties of Limits of Functions of Two Variables

The following rules hold if L , M , and k are real numbers and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y) = M.$$

1. Sum Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

2. Difference Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

3. Constant Multiple Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} kf(x, y) = kL \quad (\text{any number } k)$$

4. Product Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

5. Quotient Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}, \quad M \neq 0$$

6. Power Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} [f(x, y)]^n = L^n, \quad n \text{ a positive integer}$$

7. Root Rule:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \sqrt[n]{f(x, y)} = \sqrt[n]{L} = L^{1/n},$$

n a positive integer, and if n is even,
we assume that $L > 0$.

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x - and y -values of the point being approached into the functional expression to find the limiting value.

(a) $\lim_{(x, y) \rightarrow (0, 1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$

(b) $\lim_{(x, y) \rightarrow (3, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$ ■

EXAMPLE 2 Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \rightarrow (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} && \text{Multiply by a form equal to 1.} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\ &= 0(\sqrt{0} + \sqrt{0}) = 0 && \text{Known limit values}\end{aligned}$$

Continuity

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0) ,
2. $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

$$1. \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$$

$$2. \lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}}$$

$$3. \lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1}$$

$$4. \lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2$$

$$5. \lim_{(x,y) \rightarrow (0,\pi/4)} \sec x \tan y$$

$$6. \lim_{(x,y) \rightarrow (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$$

Answers

$$1. \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - 0^2 + 5}{0^2 + 0^2 + 2} = \frac{5}{2}$$

$$2. \lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$$

$$3. \lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1} = \sqrt{24} = 2\sqrt{6}$$

$$4. \lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2 = \left[\frac{1}{2} + \left(-\frac{1}{3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

$$5. \lim_{(x,y) \rightarrow (0,\frac{\pi}{4})} \sec x \tan y = (\sec 0) \left(\tan \frac{\pi}{4}\right) = (1)(1) = 1$$

$$6. \lim_{(x,y) \rightarrow (0,0)} \cos \left(\frac{x^2 + y^2}{x + y + 1} \right) = \cos \left(\frac{0^2 + 0^2}{0 + 0 + 1} \right) = \cos 0 = 1$$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

$$13. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y}$$

$$14. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$$

$$15. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1}$$

Answers

$$13. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x-y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x-y) = (1-1) = 0$$

$$14. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x-y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x+y) = (1+1) = 2$$

$$15. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x-1} = \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim_{(x,y) \rightarrow (1,1)} (y-2) = (1-2) = -1$$

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0) , then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ does not exist.

EXAMPLE 3 Find $\lim_{(x, y) \rightarrow (0, 0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

EXAMPLE 4 If $f(x, y) = \frac{y}{x}$, does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

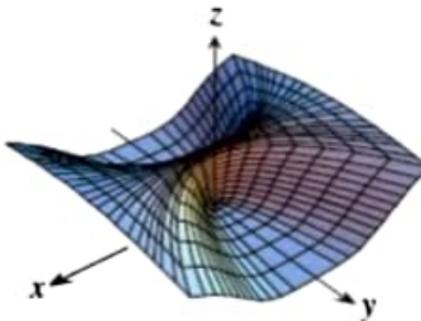
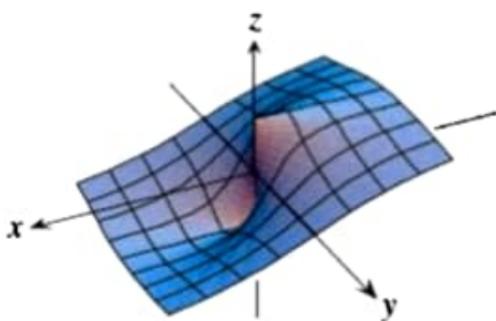
(Figure 14.15) has no limit as (x, y) approaches $(0, 0)$.

No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \rightarrow (0, 0)$.

$$41. f(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$$

$$42. f(x, y) = \frac{x^4}{x^4 + y^2}$$



$$43. f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$$

$$44. f(x, y) = \frac{xy}{|xy|}$$

$$45. g(x, y) = \frac{x - y}{x + y}$$

$$46. g(x, y) = \frac{x^2 - y}{x - y}$$

$$47. h(x, y) = \frac{x^2 + y}{y}$$

$$48. h(x, y) = \frac{x^2y}{x^4 + y^2}$$