Calculus 1.

NoteBook #2.

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A Proof that Differentiable functions are Continous.

Theorem. If a function is differentiable at $x=a$, then it is continuse at $x=a$.

Proof:

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

$$\lim_{x \to a} (x-a) \cdot \lim_{x \to a} \frac{f(x)\cancel{-}f(a)}{x-a} = f'(a) \cdot \lim_{x \to a} \cancel{\ } x-a.$$

$$\lim_{x \to a} f(x)-f(a) = 0.$$

$$\lim_{x \to a} f(x)-f(a) + \lim_{x \to a} \cancel{f(a} f(a) = \lim_{x \to a} f(a).$$

$$\lim_{x \to a} f(x)-f(a)+f(a) = f(a).$$

$$\lim_{x \to a} f(x) = f(a).$$

(the defination of continuse at point a.).

The Power Rule and Derivative of Sums, Differences, and Constant Multiples.

1. Derivative of a constant c

$f(x) = c$:

$\frac{df}{dx} = 0$.

2. Derivative of x

$f(x) = x$.

$f'(x) = 1$.

3. Power rule:

$y = x^n$ where $n \in \mathbb{R}$.

$\frac{dy}{dx} = n \cdot x^{n-1}$

Find the derivative of these functions.

①. $y = x^{15}$:

$\cancel{dy}$ $\frac{dy}{dx} = 15x^{14}$

②. $f(x) = \sqrt[3]{x}$. $= x^{\frac{1}{3}}$

$\frac{df}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

③. $g(x) = \frac{1}{x^{3.7}} = x^{-3.7}$

$g'(x) = \Rightarrow -3.7x^{-4.7}$

Derivative of a costant multiple

If c is a real numbre and f is a differentiable function, then

$$\frac{d}{dx}\left[f(x)\cdot c\right] = c\frac{d}{dx}f(x)$$

E.g. Find the ~~deriv~~ ~~derivable~~ derivative of $f(x) = 5x^3$

Ans:

$$f'(x) = \frac{d}{dx}5x^3$$

$$= 5\frac{d}{dx}x^3$$

$$= 5 \times \cancel{2x^2}\ 3x^2$$

$$= \cancel{10x^2}\ 15x^2$$

Derivative of a sum.

If f and g are differenable functions,

$$\frac{d}{dx}\left(f(x)+g(x) \right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

$$\frac{d}{dx}\left(f(x)\cancel{+}-g(x) \right) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

E.g. Find the derivative of $y = 7x^3 - 5x^2 + 4x - 2$.

$$\frac{dy}{dx} = 7 \times 3x^2 - \cancel{5x}\ 5 \times 2x + 4 - 0.$$

$$= 21x^2 - 10x + 4$$

Introduce to Trig Identitles.

a) : $x^2-6x=7.$

$x^2-6x-7=0.$

$(x-7)(x+1)=0$

$$\begin{cases} x=7 \\ x=-1. \end{cases}$$

b) :

$x^2-6x = 7+(x-7)(x+1).$

$x^2-6x = 7+x^2-6x-7.$

$x^2-6x = x^2-6x.$

solution: all real number.

the second equation is a identitle, because it holds all values.

Defination.

~~Q)~~ ~~Q~~ Decide which of the following is a identitles.

a) $\sin(2x) = 2\sin(x)$.

solutions : $0, \pi, 2\pi \ldots\ldots$

not a identitle.

b). $\cos(\theta+\pi) = -\cos(\theta)$.

works for all value of θ, where $\theta \in R$.

is a identiete.

c) $\sec(x) - \sin(x)\tan(x) = \cos(x)$.

$$\frac{1}{\cos(x)} - \sin(x)\cdot\frac{\sin(x)}{\cos(x)} = \cos(x).$$

$$\frac{1}{\cos(x)}\left(1 - \sin^2(x)\right) = \cos(x).$$

$$1 - \sin^2(x) = \cos^2(x).$$

$$1 - \left(\frac{-\cos(2x)}{2} + 0.5\right) = \frac{\cos(2x)}{2} + 0.5$$

$$0.5 + \frac{\cos(2x)}{2} = \frac{\cos(2x)}{2} + 0.5$$

$$\frac{\cos(2x)}{2} = \frac{\cos(2x)}{2}.$$

is a identitie.

The Pythagorean Identities.

① $\cos^2(\theta) + \sin^2(\theta) = 1$

② $\tan^2(\theta) + 1 = \sec^2(\theta)$.

③ $\cot^2(\theta) + 1 = \csc^2(\theta)$.

!!!

Poorf ①: $\cos^2(\theta) + \sin^2(\theta) = 1$.

By Pythagorean Threom

$$(\cos(\theta))^2 + (\sin \theta)^2 = 1^2$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

Proof ② : $\tan^2\theta + 1 = \sec^2(\theta)$.

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{1}{\cos\theta}\right)^2.$$

$$1 + \tan^2\theta = \sec^2\theta$$

Proof ③ : $\cot^2\theta + 1 = \csc^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}.$$

$$1 + \cot^2\theta = \csc^2\theta$$

SUM And Diffrence Formulas.

$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B).$

$\cos(A+B) = \cancel{so}\ \cos(A)\ \cancel{+}\ \cos(B) - \sin(A)\sin(B).$

memorise them using the "ABCDEFG" song

$\sin(A-B) = \sin(A)\cos(-B) + \cos(A)(-\sin(B)).$

$= \sin(A) + \cos(B) - \cos(A)\ \cancel{+}\ \sin(B).$

$\cos(A-B) = \cos(A)\cos(-B) - \sin(A)\sin(-B).$

$= \cos(A)\cos(B) + \sin(A)\sin(B).$

find an exact value of $\sin(105^\circ)$.

$$\sin(105^\circ) = \sin(60+45)$$
$$= \sin(60)\cdot\cos(\overset{45}{\cancel{60}}) + \cos(60)\sin(45).$$
$$= \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}.$$

If $\cos(v) = 0.9$, $\cos(w) = 0.7$ find $\cos(v+w)$.
Assume v and w are in the first quadrant

$$\cos(v+w) = \cos(v)\cos(w) - \sin(v)\sin(w)$$
$$= 0.63 - \sin(v)\sin(w).$$

$\because \sin^2(v) + \cos^2(v) = 1.$

$$\sin^2(v) = 1 - 0.81$$

$\therefore \quad \sin(v) = \sqrt{0.19}.$

$\because \sin^2(w) + \cos^2(w) = 1$

$$\sin^2(w) = 0.51$$

$$\sin(w) = \sqrt{0.51}$$

$0.63 - \sin(v)\sin(w)$

$= 0.63 - \sqrt{0.19}\,\sqrt{0.51}$

$= 0.63 - \sqrt{0.0969}$

Double Angle Formula.

* $\sin(2\theta) = \sin(\theta + \theta)$

$= \sin\theta + \cos\theta + \cos\theta + \sin\theta$

$= 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos(\theta + \theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

E.g. Find $\cos(2\theta)$ if $\cos(\theta) = -\frac{1}{\sqrt{10}}$ and θ is in quadrant III.

$\cos(2\theta) = \cancel{\cos^2\theta - \sin^2\theta}\ 2\cos^2\theta - 1$

$= 2 \cdot \frac{1}{10} - 1$

$= \frac{1}{5} - 1$

$= -\frac{4}{5}$

E.g. Solve the equation $2\cos(x) + \sin(2x) = 0$.

$$2\cos(x) + 2\cos(x)\sin(x) = 0.$$

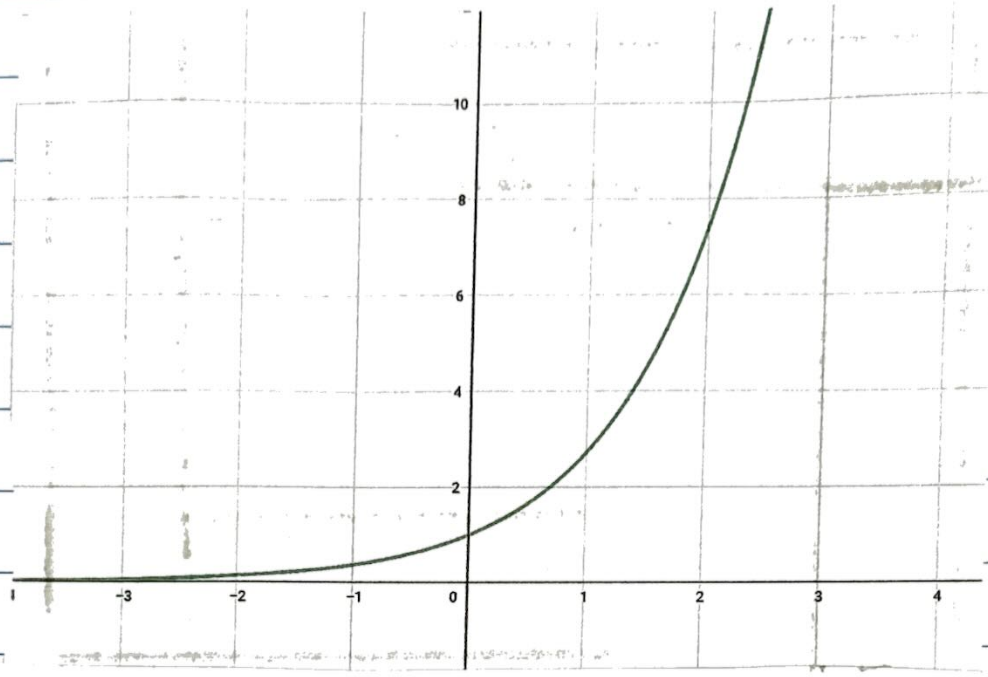
$$2\cos x\Big(1 + \sin(x)\Big) = 0.$$

$$\begin{cases} 2\cos x = 0. \\ \sin x = -1 \end{cases}$$

$$\begin{cases} \cos x = 0. \\ \sin x = -1 \end{cases}$$

$$\begin{cases} x = \frac{\pi}{2} + 2k\pi. \\ x = \frac{3\pi}{2} + 2k\pi. \end{cases}$$

Derivative of e^x.



$y = e^x$.

Three facts about e.

①. $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. definition of "$e$".

②. $\lim_{h \to 0} \frac{e^h - e^0}{h} = 1$ $\qquad$ $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

③. $\frac{d}{dx} e^x = e^x$.

As we know $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

By the derivative of e^x

$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}.$$

$$= \lim_{h \to 0} \frac{e^h - 1}{h} \cdot e^x$$

$$= e^x.$$

E.g. Find the derivative of $g(x) = ex^2 + 2e^x + xe^2 + x^{e^2}$.

$$g'(x) = e \cdot 2x + 2 \cdot e^x + e^2 + e^2 x^{e^2-1}$$

$$= 2ex + 2e^x + e^2 + e^2 x^{e^2-1}$$

Proof or the Sum , Constant , C-Mutiple Rules.

Proof: $\frac{d}{dx} c = 0.$

$$\frac{d}{dx} c = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h.}$$
$$= \lim_{h \to 0} \frac{c - c}{h.}$$
$$= \frac{0}{h}$$
$$= 0.$$

Proof: $\frac{d}{dx} x = 1.$

$$\frac{d}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h.}$$
$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h.}$$
$$= \lim_{h \to 0} 1$$
$$= 1$$

Proof : $\frac{d}{dx} x^n = nx^{n-1}$

$$\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n \cancel{(-x^n)}}{h}$$

$$= \lim_{h \to 0} \frac{h(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1})}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}$$

$$= \lim_{h \to 0} n\left(x^{n-1} + \frac{n-1}{2}x^{n-2}h + \cdots + xh^{n-2} + h^{n-1}\right).$$

$$= n\left(x^{n-1} + 0 + 0 + \cdots + 0 + 0\right).$$

$$= nx^{n-1}$$

Proof:

$$\frac{d}{dx} cf(x) = c\frac{df}{dx}$$

$\cancel{\lim}$ $\cancel{d}$

$$\frac{d}{dx} cf(x) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}.$$

$$= \lim_{h \to 0} c \cdot \frac{f(x+h) - f(x)}{h}.$$

$$= \lim_{h \to 0} c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$= c\frac{df}{dx}.$$

The Product Rule and Quotient Rule.

$$\frac{d}{dx}\, f(x)\, g(x) = f(x)\cdot\frac{dg}{dx} + g(x)\cdot\frac{df}{dx}.$$

E.g: find the derivative of $\sqrt{t}\cdot e^t$.

$$\frac{d}{dt}\cdot\sqrt{t}\cdot e^t = \sqrt{t}\cdot \quad + \quad e^t\cdot\frac{1}{2}t^{-\frac{1}{2}}$$

$$= \sqrt{t}\,e^t + \frac{1}{2}e^t t^{-\frac{1}{2}}$$

$$\frac{d}{dx}\,\frac{f(x)}{g(x)} = \frac{g(x)\cdot\frac{df}{dx} - f(x)\cdot\frac{dg}{dx}}{g^2(x)}.$$

E.g: find $\frac{d}{dz}\left(\frac{z^2}{z^3+1}\right)$.

$$\frac{d}{dz}\left(\frac{z^2}{z^3+1}\right) = \frac{(z^3+1)2z - z^2(3z^2+0)}{(z^3+1)^2}.$$

$$= \frac{2z^4+2z-3z^4}{(z^3+1)^2}$$

$$= \frac{2z - z^4}{(z^3+1)^2}$$

Proof of the Product and Quotient Rule.

①: $\frac{d}{dx} f(x) \cdot g(x) = g(x) \cdot \frac{df}{dx} + f(x) \cdot \frac{dg}{dx}.$

Proof: $\frac{d}{dx} f(x) \cdot g(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x)g(x)}{h}.$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]}{h}.$$

$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \frac{f(x)[g(x+h) - g(x)]}{h}.$$

$$= \lim_{h \to 0} g(x+h) \frac{f(x+h) - f(x)}{h} + f(x) \frac{g(x+h) - g(x)}{h}.$$

$$= g(x) \cdot \frac{df}{dx} + f(x) \cdot \frac{dg}{dx}.$$

Proofed.

② : $\frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$ reciprocal rule.

Proof: $\frac{d}{dx}\frac{1}{f(x)} = \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$

$$= \lim_{h\to 0}\frac{\frac{f(x)}{f(x)f(x+h)}-\frac{f(x+h)}{f(x)f(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(x)-f(x+h)}{f(x)f(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x)-f(x+h)}{h\cdot f(x)\cdot f(x+h)}$$

$$= \lim_{h\to 0} -\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)f(x+h)}$$

$$= -\frac{df}{dx}\cdot\frac{1}{f^2(x)}$$

$$= \frac{-f'(x)}{f^2(x)}$$

Proofed.

③. $\frac{d}{dx} \cdot \frac{f(x)}{g(x)} = \frac{gf'(x) - fg'(x)}{g^2(x)}$.

$$\frac{d}{dx} \cdot \frac{f(x)}{g(x)} = \frac{d}{dx}\left(\frac{1}{g(x)} \cdot f(x)\right).$$

$$= -\frac{g'(x)}{g^2(x)} \cdot f(x) + f'(x) \cdot \frac{1}{g(x)}$$

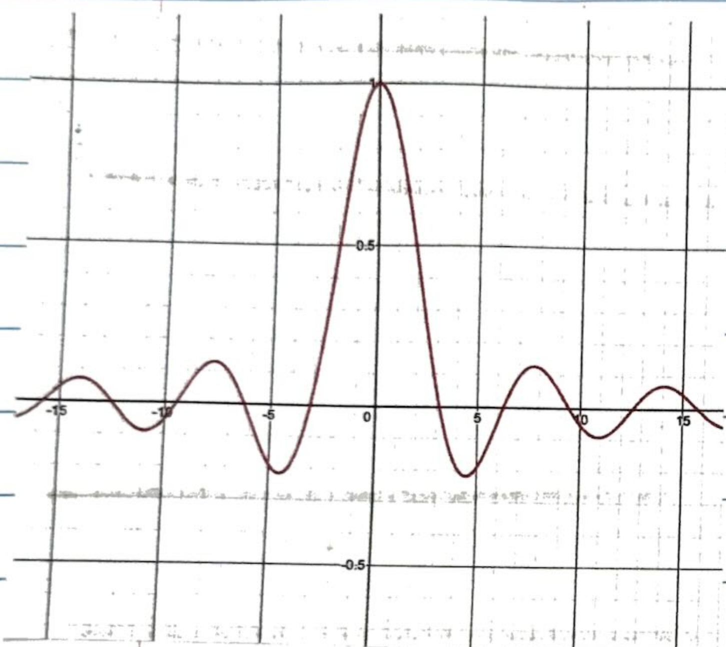
$$= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{g^2(x)}.$$

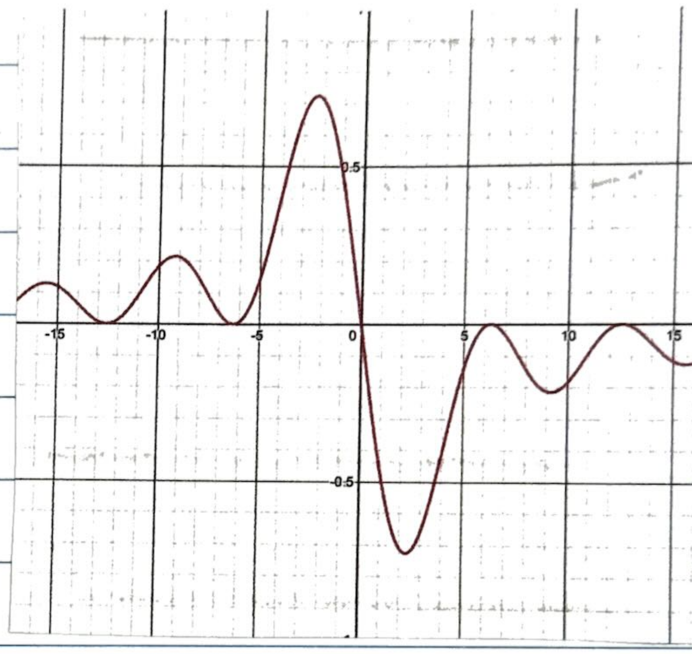
$$= \frac{f'(x) \cdot g(x) - g'(x)f(x)}{g^2(x)}.$$

Some special Trig Limits.

①. ~~lim~~ $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$

②. $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$.





$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$.

E.g. Estimate $\sin 0.01769$

because : $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

so : $\lim_{\theta \to 0} (\sin\theta = \theta)$.

plug in the value: $\sin 0.01769 \approx 0.01769$.

E.g. Find $\lim_{x \to 0} \frac{\tan 7x}{\sin \cancel{7} 4x}$.

$$\lim_{x \to 0} \frac{\tan(7x)}{\sin(4x)} = \frac{\frac{\sin 7x}{\cos 7x}}{\sin 4x}.$$

$$= \frac{\frac{7x}{\cos 7x}}{4x}.$$

$$= \cancel{3} \quad \frac{\frac{7}{4}}{\cos 7x}.$$

$$= \frac{7}{4} \cdot \frac{1}{\cos 7x}$$

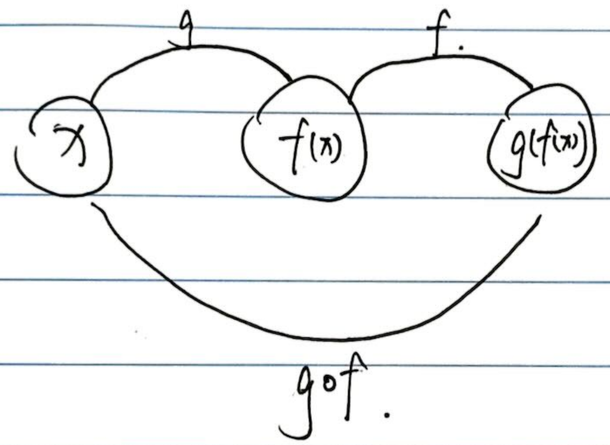
$$= \frac{7}{4} \cdot \frac{1}{1}$$

$$= \frac{7}{4}.$$

Composition of functions.

Defination : $g \circ f(x) : g(f(x))$.

$$\Downarrow$$



$g \circ f \neq f \circ g$

let $p(x) = x^2+x$, $q(x) = -2x$. Find:

a): $q \circ p(1) = \cancel{-2x} \; -2(x^2+x)$
$= -4$.

b): $q \circ p(x) = -2(x^2+x)$
$= -2x^2 - 2x$

c): $p \circ q(x) = (-2x)^2 + (-2x)$
$= 4x^2 - 2x$.

d): $p \circ p(x) = (x^2+x)^2 + (x^2+x)$.
$= x^4 + 2x^3 + x^2 + x^2 + x$.
$= x^4 + 2x^3 + 2x^2 + x$.

$h(x) = \sqrt{x^2+7}$, Find function f, g ~~so~~ so that $h(x) = f \circ g(x)$.

$h(x) = \sqrt{x^2+7}$.

if $f \circ g(x) = h$.

$f(x) = \sqrt{x}$

$g(x) = x^2+7$.

or :

$h(x) = \sqrt{x^2+7}$

if $h(x) = f \circ g(x)$

$f(x) = \sqrt{x+7}$

$g(x) = x^2$.

Derivative of Trig. Functions.

$\Rightarrow \sin'(x) = \cos(x)$

so $\cos'(x) = -\sin(x)$.

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)'$$

$$= \frac{\cos(x) \cdot \sin'(x) - \sin(x) \cdot \cos'(x)}{\cos^2(x)}$$

$$= \frac{\cos(x) \cdot \cos(x) + \sin(x)\sin(x)}{\cos^2(x)}.$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}.$$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x).$$

$\tan'(x) = \sec^2(x)$.

$$\sec'(x) = \frac{d}{dx} \frac{1}{\cos(x)}$$

$$= -\frac{-\sin(x)}{\cos^2(x)}.$$

$$= \frac{\sin(x)}{\cos^2(x)}.$$

$$= \sin(x) \sec^2(x).$$

or :

$$= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)}$$

$$= \sec(x) \cdot \tan(x).$$

$\sec'(x) = \sin(x) \sec^2(x)$.

or:

$\sec'(x) = \sec(x) \cdot \tan(x)$.

$$\cot'(x) = \frac{d}{dx}\frac{1}{\tan(x)}.$$

$$= -\frac{\tan'(x)}{\tan^2(x)}.$$

$$= -\frac{\sec^2(x)}{\tan^2(x)}.$$

$$= -\frac{\frac{1}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}}.$$

$$= -1 \div \cos^2(x) \div \left(\sin^2(x) \div \cos^2(x)\right).$$

$$= -1 \cancel{\div \cos^2(x)} \div \sin^2(x) \cancel{\times \cos^2(x)}.$$

$$= -\frac{1}{\sin^2(x)}.$$

$$= -\csc^2(x).$$

$$\cot'(x) = -\csc^2(x).$$

$$\csc'(x) = \frac{d}{dx}\frac{1}{\sin(x)},$$

$$= -\frac{\cos(x)}{\sin^2(x)}$$

$$= -\frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)}$$

$$= -\cot(x)\csc(x).$$

E.g. for $g(x) = \left(\frac{x\cos(x)}{c+\cot(x)}\right)$, find $g'(x)$.

let $x\cos(x)$ be u.

$$\frac{du}{dx} = x\cdot(-\sin(x)) + \cos(x)\cdot 1$$

$$= \cos(x) - x\sin(x).$$

let $c+\cot(x)$ be w:

$$\frac{dw}{dx} = 0 + -\csc^2(x).$$

$$= -\csc^2(x).$$

then : $g'(x) = \frac{wu' - uw'}{w^2}$

$$= \frac{w(\cos(x) - x\sin(x)) - uw'}{(M + \cot(x))^2}$$

$$= \frac{(c + \cot(x))(\cos(x) - x\sin(x)) + x\cos(x)\csc^2(x)}{M^2 + 2M\cot(x) + \cot^2(x)}.$$

Recti-linear Motion.

E.g. A particle moves up and down along a straight line, It's position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$

Find $s'(t)$ and $s''(t)$. and Interpret them.

$$s'(t) = 4t^3 - \frac{16}{3} \cdot 3t^2 + 2 \cdot 6t$$
$$= 4t^3 - 16t^2 + 12t.$$

$$s''(t) = \frac{d}{dx}(s'(t)).$$

$$= 3 \cdot 4t^2 - 2 \cdot 16t + 12$$

$$= 12t^2 - 32t + 12.$$

$s'(t)$ mean's the velocity of the partical at time t.

$s''(t)$ mean's the acceleration of partical at time t.

Lorgorithms.

$\log_a b = c$ means $a^c = b$.

a) $\log_2 16 = 4$

b) $\log_2 2 = 1$

c) $\log_2 \frac{1}{2} = -1$

d) $\log_2 \frac{1}{8} = -3$

e) $\log_2 1 = 0$.

$\log_{10} 1000000 = 6$.

$\log_{10} 0.01 = -2$.

$\log_{10} -100 =$ Does not exist.

$\log_3 \frac{1}{9} = -2$, so $3^{-2} = \frac{1}{9}$.

$\log 13 = 1.11394$, so $10^{1.11394} = 13$.

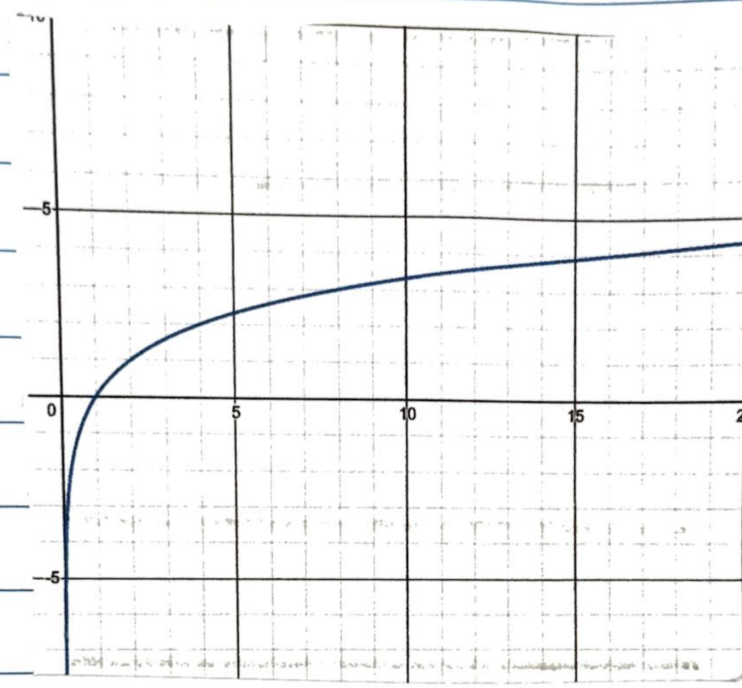
$\ln \frac{1}{e} = -1$, so $e^{-1} = \frac{1}{e}$.

$3^u = 9.78$, so $\log_3 9.78 = u$.

$e^{3x+7} = 4-y$, so $\ln(4-y) = 3x+7$.

Log functions and their graphs.

$y = \log_2(x)$.



Domain : $x > 0$.

Range : All real number.

Vertical asymptope : when $x = 0$.

Rules for Combining Logs and exponents.

$\boxed{\begin{array}{l} \text{Log Rule : for any base } a,\ \log_a a^x = x. \\ \text{for any base } a,\ a^{\log_a x} = x. \end{array}}$

$3^{\log_3 4} = 4.$

$\ln(e^x) = x.$

$10^{\log(3z)} = 3z.$

Bases have to be the same for log and exponets to undo themselves.

Log Rules.

zero rule. ①: $\log_n 1 = 0 \quad n \in R^+.$

product rule. ②: $\log_n(xy) = \log_n(x) + \log_n(y) \quad n \in R^+ \quad (x, y) \in R^+.$

quotient rule. ③: $\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y). \quad n \in R^+, \quad (x, y) \in R^+.$

power rule. ④: $\log_n(x^y) = \log_n(x) \cdot y.$

E.g. Rewright the following as a sum of logs.

a) $\log\left(\frac{x}{yz}\right).$

$= \log(x) - \log(yz).$

$= \log(x) - \log(y) - \log(z).$

b) $\log(5 \cdot 2^t).$

$= \log(5) + t\log(2).$

E.g. Rewrite as a single log:

a) $\log_5 a - \log_5 b + \log_5 c$

$= \log_5 (a \div b \cdot c).$

$= \log_5 \left(\frac{ac}{b}\right).$

b) $\ln(x+1) + \ln(x-1) - 2\ln(x^2-1).$

$= \ln\left[(x+1)(x-1) \div (x^2-1)^2\right].$

$= \ln\left[(x+1)(x-1) \div (x+1)^2(x-1)^2\right].$

$= \ln\left(\frac{\cancel{(x+1)}\cancel{(x-1)}}{(x+1)^{2-1}(x-1)^{2-1}}\right).$

$= \ln\left(\frac{1}{x^2-1}\right).$

The Chain Rule.

Recall:

$$f \circ g(x) = f(g(x)).$$

g = inner function.
f = outer function.

E.g. rewrite into composite:

$$h(x) = \sqrt{\sin x}.$$
$$= (\sqrt{x}) \circ (\sin x),$$

$$k(x) = 5(\tan x + \sec x)^3.$$

$$= (5x^3) \circ \left(\tan(x) + \sec(x)\right).$$

$$r(x) = e^{\sin(x^2)}.$$
$$= e^x \circ \left(\sin(x^2)\right).$$

Rule.

The Chain Rule tells: if g is differentiable at x and f is differenctible at $g(x)$, the $f \circ g$ is differenctible at x

Also:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Also: let $u = g(x)$,

$y = f(u)$

$= f(g(x))$.

then $\frac{du}{dx} = g'(x)$.

$\frac{dy}{du} = f'(u)$.

$\frac{dy}{dx} = (f \circ g)'(x)$.

so $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

E.g. Find the derivative of $\sqrt{\sin(x)}$.

$$\frac{d}{dx}\sqrt{\sin x} = (\sin x)^{\frac{1}{2}}$$

$$= 2(\sin x)^{-\frac{1}{2}} \cdot \cos x.$$

E.g. Find the derivative of $k(x) = 5(\tan x + \sec x)^3$

$$k'(x) = 3 \times 5\,(\tan x + \sec x)^2 \cdot \left(\sec^2(x) + \sin x \sec^2(x)\right).$$

$$= 15(\tan x + \sec x)^2 \cdot \left(\sec^2 x + \sin x \sec^2 x\right).$$

$$= 15\sec^2 x\,(\tan x + \sec x)^2 \cdot (x + \sin x).$$

E.g. Find the derivative of $r(x) = e^{\sin(x^2)}$.

$$r'(x) = e^{\sin(x^2)} \cdot \frac{d}{dx}\sin(x^2).$$

$$= e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$$

E.g. Show that $(5^x)' = (\ln 5) \cdot 5^x$

$$5^x = e^{(\ln 5)x}.$$

so: $(5^x)' = \frac{d}{dx} e^{(\ln 5)x}.$

$$= e^{(\ln 5)x} \cdot (\ln 5)x.$$

$$= 5^x \cdot (\ln 5)$$

As a conclution: $a^x = a^x \cdot \ln a \quad a \in R^+$

E.g. Find the derivative of $y = \sin 5x \sqrt{2^{\cos(5x)} + 1}$

let $u = \sin(5x)$

$$\frac{du}{dx} = \cos(5x) \cdot 5.$$

$$\frac{dy}{dx} = u' \cdot \sqrt{2^{\cos(5x)}+1} + \left(\sqrt{2^{\cos(5x)}+1}\right)' u.$$

let $w = \sqrt{2^{\cos(5x)}+1}$

$$\frac{dw}{dx} = \frac{d}{dx}\left(2^{\cos(5x)}+1\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(2^{\cos(5x)}+1\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(2^{\cos(5x)}+1\right)$$

$$= \frac{1}{2}\left(2^{\cos(5x)}+1\right)^{-\frac{1}{2}} \cdot \frac{d}{dx} 2^{\cos(5x)}$$

$$= -\frac{5}{2}\left(2^{\cos(5x)}+1\right)^{-\frac{1}{2}} \cdot 2^{\cos(5x)} \cdot \ln 2 \cdot \sin(5x)$$

$$\frac{dy}{dx} = \cos(5x) \cdot 5\sqrt{2^{\cos(5x)}+1} - \frac{5}{2}\left(2^{\cos(5x)}+1\right)^{-\frac{1}{2}} \cdot 2^{\cos(5x)} \cdot \ln 2 \cdot (\sin(5x))^2$$

$$= 5\cos(5x)\sqrt{2^{\cos(5x)}+1} - \frac{5}{2}\ln 2 \frac{1}{\sqrt{2^{\cos(5x)}+1}} \cdot 2^{\cos(5x)} \sin^2(5x)$$

Justification of the Chain Rule.

$$\frac{d}{dx} f \circ g(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x)}.$$

$$= \lim_{h \to 0.} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x).} \cdot \frac{g(x+h) - g(x).}{h.}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x)).}{g(x+h) - g(x).} \cdot g'(x).$$

let $u = g(x+h)$.

$$= g'(x) \cdot \lim_{h \to 0} \frac{f(u) - f(u+h).}{u - u+h.}$$

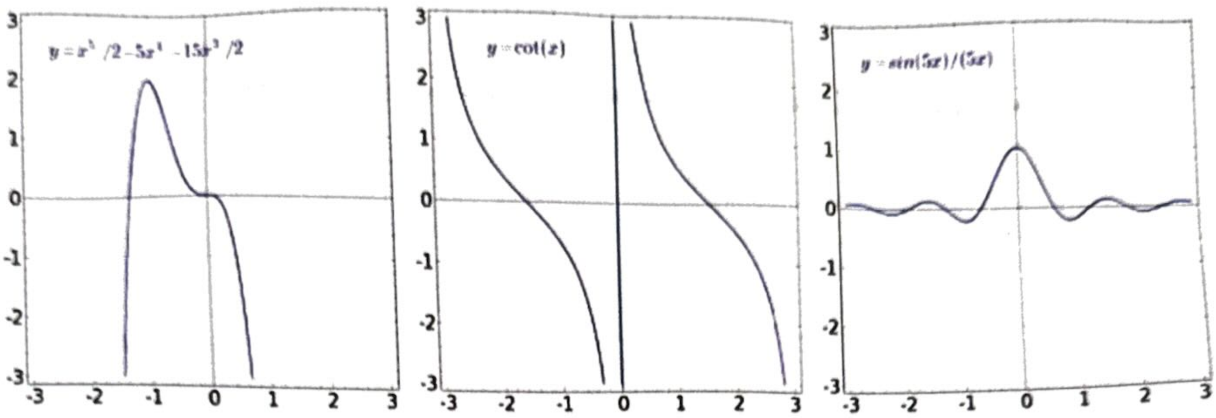
$$= g'(x) \cdot \lim_{h \to 0} \frac{f(u) - f(u+h).}{h.}$$

$$= g'(x) \cdot f'(u).$$

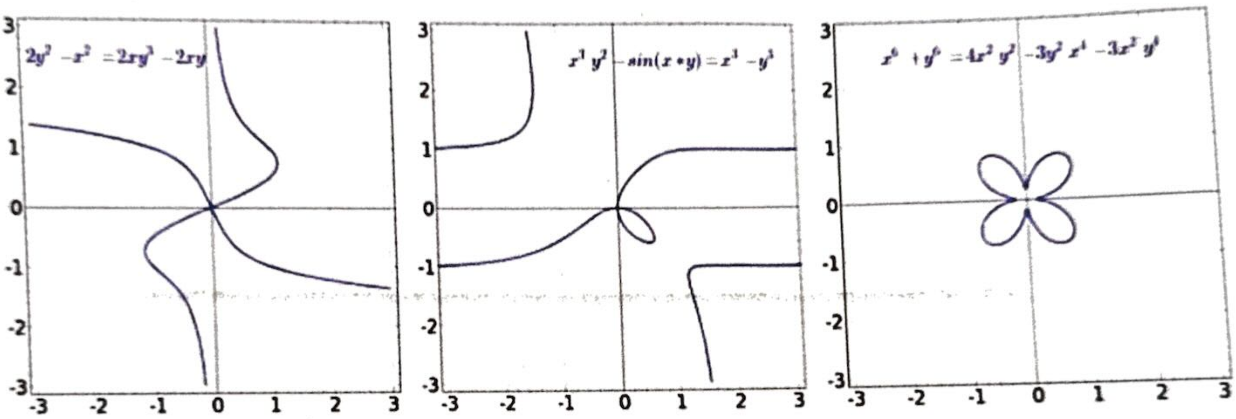
$$= f'(g(x)) \cdot g'(x).$$

Implicit Diffrentiation.

Explicitly defined functions



Implicitly defined curves



E.g. Find the equation of the tangent line for $9x^2 + 4y^2 = 25$ at point $(1, 2)$.

Method ①: solve for y:

$4y^2 = 25 - 9x^2$.

$4y^2 = (5+3x)(5-3x)$.

$y^2 = \frac{(5+3x)(5-3x)}{4}$

$$y = \pm\sqrt{\frac{(5-3x)(5+3x)}{4}}.$$

$$y = \pm \; \pm\frac{1}{2}\cdot\sqrt{25 - 9x^2}$$

because (1 , 2) is in the I quatranm :

$$y = \frac{1}{2}\sqrt{25-9x^2}.$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2}\cdot\frac{1}{2}(25-9x^2)^{-\frac{1}{2}}\cdot(0-2\cdot9\cdot x).\Big|_{x=1}$$

$$= -\frac{1}{4}(25-9x^2)^{-\frac{1}{2}}\cdot 18x.\Big|_{x=1}$$

$$= -\frac{9}{2}(25-9x)^{-\frac{1}{2}}x \Big|_{x=1}$$

$$= -\frac{9}{8}.$$

tangent line equation : $\frac{\text{rise}}{\text{run}}$ + offset .

$$y = -\frac{9}{8}\cdot(x-1)+2 .$$

Method 2: Implicit diffrentiation.

original equation: $9x^2 + 4y^2 = 25$

take derivative on both side: $\frac{d}{dx}9x^2 + \frac{d}{dx}4y^2 = \frac{d}{dx}25$

$$9 \cdot 2x + \frac{d}{dx}4y^2 = 0.$$

$$18x + 4\frac{d}{dx}y^2 = 0.$$

Use the chain Rule:

let y be a function of x.

$$18x + \overset{f'(g(x))}{4 \cdot 2y}\,\overset{g'(x).}{\frac{dy}{dx}} = 0.$$

$\Longrightarrow$ $\frac{dy}{dx}$ is also the slope of the tangent line.

solve for $\frac{dy}{dx}$: $\frac{dy}{dx} \cdot 8y = -18x.$

$$\frac{dy}{dx} = \frac{-18x}{8y}.$$

$$\frac{dy}{dx} = \frac{-9}{4}\frac{x}{y}.$$

plug in the values $(1, 2)$.

$$\frac{dy}{dx} = \frac{-9}{4} \cdot \frac{x}{y} \Big|_{x=1,\ y=2.}$$

$$\frac{dy}{dx} = \frac{-9}{4} \cdot \frac{1}{2}$$
$$= -\frac{9}{8}.$$

find the equation ~~of~~ for the tangent line:

$$\frac{rise}{run} + offset = \underbrace{-\frac{9}{8}(x-1)+2}_{\text{The answer}}.$$

E.g. Find y' for equation: $x^3y^2 - \sin(xy) = x^3 - y^3$.

$$\frac{d}{dx}\left(x^3y^2 - \sin(xy)\right) = \frac{d}{dx}(x^3 - y^3).$$

$$x^3 \cdot \frac{d}{dx}y^2 + 3x^2 \cdot y^2 - \cos(xy) \cdot \frac{d}{dx}xy = 3x^2 - 3y^2 \cdot \frac{dy}{dx}.$$

$$x^3 \cdot 2y \cdot \frac{dy}{dx} + 3x^2y^2 - \cos(xy) \cdot x\frac{dy}{dx} + y = 3x^2 - 3y^2 \cdot \frac{dy}{dx}.$$

$$2x^3y\frac{dy}{dx} + x\cos(xy)\frac{dy}{dx} + 3x^2y^2 + y = 3x^2 - 3y^2 \cdot \frac{dy}{dx}.$$

$$\frac{dy}{dx}\left(2x^3y + x\cos(xy)\right) + \frac{dy}{dx}3y^2 = 3x^2 - 3x^2y^2 - y.$$

$$\frac{dy}{dx}\left(2x^3y + x\cos(xy) + 3y^2\right) = 3x^2 - 3x^2y^2 - y.$$

$$\frac{dy}{dx} = \frac{3x^2 - 3x^2y^2 - y}{2x^3y + x\cos(xy) + 3y^2}.$$

Derivative of Exponential Functions.

Recall: $e^{\ln x} = x$.

E.g: $\frac{d}{dx} 5^x = \frac{d}{dx} (e^{\ln 5})^x$

$= \frac{d}{dx}(e^{\ln 5 \cdot x})$.

$= e^{\ln 5 \cdot x} \cdot \ln 5$

$= 5^{x} \cdot \ln 5$.

Note: $\boxed{\frac{d}{dx}(a^x) = a^x \cdot \ln a.}$

$\frac{d}{dx}(x^a) = a x^{(a-1)}$.

Derivative of Log Functions.

Find $\frac{d}{dx}\log_a(x)$.

let $y = \log_a(x)$.

so: $\cancel{y}$ $a^y = x$.

take the derivative of the both sides:

$$\frac{d}{dx} a^y = \frac{d}{dx} x.$$

$$\ln a \cdot a^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a^y \cdot \ln a}.$$

since: $a^y = x$

$$\frac{dy}{dx} = \frac{1}{\ln a \cdot x}.$$

so: $$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln a}.$$

if we let a be e.

then $\frac{d}{dx} \ln x = \frac{1}{\ln(e) \cdot x}$

$= \frac{1}{x}$.

Rule we get: $\frac{d}{dx} (\log_a(x)) = \frac{1}{x \ln a}$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$.

SJ

Logarithmetic Differentiation.

E.g. find $\frac{d}{dx}(x^x)$.

let $y = x^x$

$\ln(y) = x\ln(x)$.

$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln(x)$.

$\frac{dy}{dx} = (1 + \ln(x))\cdot y$.

$\frac{d}{dx}x^x = (1 + \ln(x))\cdot x^x$

$\frac{d}{dx}x^x = x^x(1+\ln x)$.

e.g. find $\frac{d}{dx}(\tan x)^{\frac{1}{x}}$

let $y = (\tan x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(\tan x)$.

$$\frac{dy}{dx}\cdot\frac{1}{y} = \frac{1}{x}\cdot\frac{d}{dx}\ln(\tan x) + (-x^{-2}).\ln(\tan(x)).$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\cdot\frac{1}{\tan(x)}\cdot\sec^2(x) - x^{-2}\ln(\tan x)\right)y$$

$$= \left(\frac{1}{x}\cdot\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\cos^2(x)} - x^{-2}\ln(\tan x)\right)y$$

$$= \left(\frac{1}{x}\cdot\sec(x)\csc(x) - x^{-2}\ln(\tan x)\right)y.$$

$$\frac{d}{dx}(\tan x)^{\frac{1}{x}} = (\tan x)^{\frac{1}{x}}\left(\frac{1}{x}\sec(x)\csc(x) - x^{-2}\ln(\tan x)\right).$$

E.g. Find the derivative of $y = \frac{x\cos(x)}{(x^2+x)^5}$

$$y = \frac{x\cos(x)}{x^5(x+1)^5}$$

$$y = \frac{\cos(x)}{x^4(x+1)^5}$$

$$\ln y = \ln(\cos x) - \ln(x^4) - \ln((x+1)^5).$$

$$\ln y = \ln(\cos x) - 4\ln x - 5\ln(x+1).$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\sec x\cdot\sin x - 4\frac{1}{x} - 5\cdot\frac{1}{x+1}\cdot 1$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\tan x - \frac{4}{x} - \frac{5}{x+1}$$

$$\frac{dy}{dx} = -\left(\tan x + \frac{4}{x} + \frac{5}{x+1}\right) \cdot \frac{x\cos(x)}{(x^2+x)^5}$$

Inverse functions.

Definition : The inverse function for f, written $f^{-1}(x)$, undoes what f does.

Fact ①: Inverse functions reverse the role y and ~~y~~ x.

Fact ②. The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line $y = x$.

or by : turing right for 90° dregrees.

Fact ③: $f \circ f^{-1}(x) = x$, $f^{-1} \circ f(x) = x$.

E.g: find the ~~of~~ inverse of $f(x)$.

$$f(x) = \frac{5-x}{3x}.$$

let $y = \frac{5-x}{3x}.$

$x = \frac{5-y}{3y}$: reverse the role of x and y.

solve for y:

$$x = \frac{5-y}{3y}.$$

$$\frac{3xy}{3y} = \frac{5-y}{3y}.$$

$$3xy = 5-y.$$

$$3xy + y = 5$$

$$y(3x+1) = 5$$

$$y = \frac{5}{3x+1}$$

$$f^{-1}(x) = \frac{5}{3x+1}$$

☆. Fact 4: For ~~A~~ a function f to has it's inverse function, it must ~~t~~ satisfies the horizontal line test. (ie for every horizontal line, the ~~that~~ line only intersect the graph of $f(x)$ at ~~one~~ a single point.)

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