Calcus 1.

Note Book #3.

IDEAS FLY
WORKSTYLE

For work or life, always can feel free and happy through writing.

180°

A5

50

M&G晨光®

Inverse Trig Functions.

①: Inverse sine function

Restricted $\sin(x)$ has:
Domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
Range $[-1, 1]$.

~~arct~~ $\arcsin(x)$ ~~is~~ has
Domain $[-1, 1]$
Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

②: Inverse Cosine function:

Restricted $\cos(x)$ has
Domain $[0, \pi]$.
Range $[-1, 1]$.

$\arccos(x)$ has:
Domain $[-1, 1]$.
Range $[0, \pi]$.

No. 2.
Date

Inverse tangent function.

⇒ Restricted $\tan(x)$ has.
Domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Range $(-\infty, \infty)$.

$\arctan(x)$ has.
Domain $(-\infty, \infty)$.
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Derivative of Inverse Trig Functions.

No. 3

Date

Because: $y = \sin^{-1}(x)$ means y is the angle whose sine is x.

Then ~~we~~ $x = \sin(y)$ is roughly the same. all we need to do is restrict the range.

So: $\frac{d}{dx} \sin^{-1}(x)$.

$$x = \sin(y).$$

$$\frac{d}{dx} x = \frac{d}{dx} \sin(y).$$

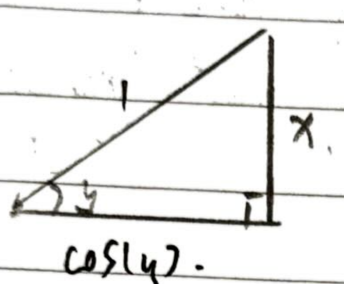
$$1 = \cos(y) \cdot \frac{d}{dx}y.$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}.$$

$$\frac{dy}{dx} = \frac{1}{\cos(\sin^{-1}(x))}.$$

Method ②:

If we look at a right trangle.



$\sin y = x$.

use pythagorean threom: $\sqrt{1-x^2} = \cos(y)$.

if we substitute $\cos(y) = \sqrt{1-x^2}$

to the result from method ①:

we get: $\frac{dy}{dx} = \frac{1}{\cos(y)}$ (recall).

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

for $\cos^{-1}(x)$:

$$y = \cos^{-1}(x)$$

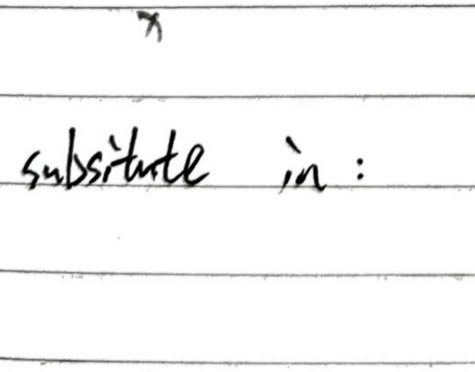
$$\cos(y) = x.$$

$$\frac{d}{dx}\cos y = \frac{d}{dx} x$$

$$-\sin(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}.$$

from right triangle and pythagorean threom :



$$\sin(y) = \sqrt{1-x^2}.$$

subsitute in : $\frac{dy}{dx} = -\frac{1}{\sin(y)}$ (recall).

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}.$$

No. 6

Date

For $\tan^{-1}(x)$:

$y = \tan^{-1}(x)$.

$x = \frac{\sin y}{\cos y}$.

$\frac{d}{dx} x = \frac{1}{\cos^2(y)} \cdot \left(\cos(y) \cdot \frac{d}{dx}\{\sin(y) - \sin(y) \cdot \frac{d}{dx}\cos(y)\right)$.

$1 = \frac{1}{\cos^2(y)} \cdot \left(\cos(y) \cdot \cos(y) \cdot \frac{dy}{dx} + \sin(y) \cdot \sin(y) \cdot \frac{dy}{dx}\right)$.

$1 = \left(\frac{1}{\cos^2(y)} \cdot \cos^2(y)\right)\frac{dy}{dx} + \left(\frac{1}{\cos^2(y)} \cdot \sin^2(y)\right)\frac{dy}{dx}$.

$1 = \frac{dy}{dx}\left(1 + \tan^2(y)\right)$.

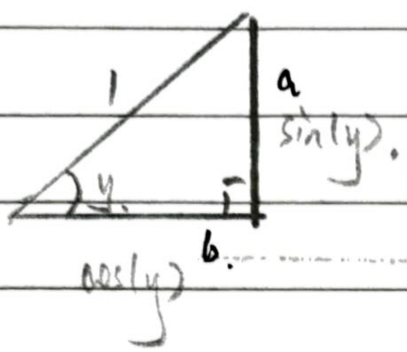
$\frac{dy}{dx} = \frac{1}{1+\tan^2(y)}$.

use pythagoren Identities :

$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$.

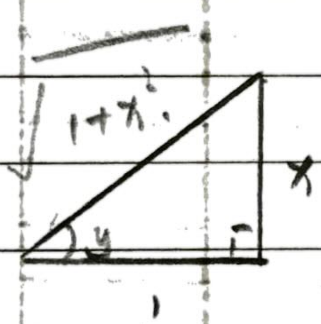
$= \cos^2(y)$.

from right trigianl :



$$x = \frac{a}{b}$$

we can transfer the triangle into



$$\sec y = \frac{\sqrt{1+x^2}}{1}$$
$$= \sqrt{1+x^2}$$

$$\sec^2 y = 1+x^2$$

substitute in :

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

No. 8

Date

Using the riciperical rule :

we can know the expression for $\sec^{-1}$, $\csc^{-1}$ and $\cot^{-1}$!

①: $\frac{d}{dx} \sec^{-1}(x)$.

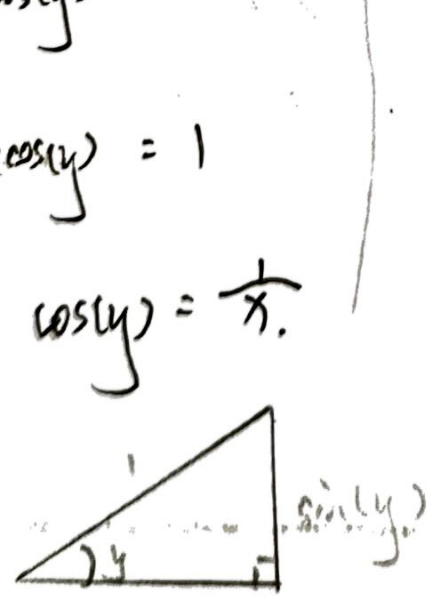
$y = \sec^{-1}(x)$.

$x = \sec(y)$.

$\frac{1}{\cos(y)} = x$.

$x\cos(y) = 1$

$\cos(y) = \frac{1}{x}$.



because $\cos^2\theta + \sin^2\theta = 1$

$\therefore \sin^2(y) = 1 - \cos^2\theta(y)$.

$= 1 - \frac{1}{x^2}$

$\sin(y) = \sqrt{1 - \frac{1}{x^2}}$.

negitive is not considered

$y = \sec^{-1}(x)$.

$x = \sec(y)$.

$\frac{dy}{dx} = \frac{\cos^2(y)}{\sin(y)}$.

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)^2 \div \sqrt{1-\frac{1}{x^2}}.$$

$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{\cancel{x}^{2-1}\sqrt{x^2-1}\cdot\frac{1}{\cancel{\sqrt{x^2}}}}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$

② : $\frac{d}{dx}\csc^{-1}(x)$.

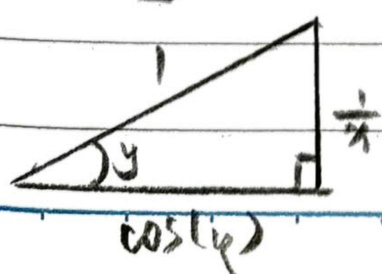
$y = \csc^{-1}(x)$.

$x = \csc(y)$.

$\frac{1}{\sin(y)} = x$.

$x\sin(y) = 1$

$\sin(y) = \frac{1}{x}$.



because $\cos^2\theta + \sin^2\theta = 1$

$\therefore \Rightarrow \cos^2(y) = 1 - \sin^2\theta$

$= 1 - \frac{1}{x^2}$

$\cos(y) = \sqrt{1-\frac{1}{x^2}}$

negitive result is not considered.

$$\begin{cases} \sin(y) = \frac{1}{x} \\ \cos(y) = \sqrt{1-\frac{1}{x^2}} \end{cases}$$

No. 10

Date

$$x = \csc(y).$$

$$1 = \frac{d}{dx}\frac{1}{\sin(y)} \cdot \frac{dy}{dx}.$$

$$1 = -\frac{\cos(x)}{\sin^2(x)} \cdot \frac{dy}{dx}.$$

$$\frac{dy}{dx} = -\frac{\sin^2(x)}{\cos(x)}.$$

$$= -\frac{1}{|x|\sqrt{x^2-1}}$$

③ : $\cot^{-1}(x)$.

$$\frac{d}{dx}\cot^{-1}(x) = -\tan^{-1}(x)$$

$$= -\frac{1}{1+x^2}.$$

As conclution :

$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

$\frac{d}{dx}\cos^{-1}(x) = -\frac{d}{dx}\sin^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$.

$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\csc^{-1}(x) = -\frac{d}{dx}\sec^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\cot^{-1}(x) = -\frac{d}{dx}\tan^{-1}(x) = -\frac{1}{1+x^2}$.

No. 12

Date

E.g. Find the derivative for $y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$.

$$\frac{dy}{dx} = 1 \div \left[1+\left(\frac{a+x}{a-x}\right)^2\right] \cdot \frac{a-x+a+x}{(a-x)^2}$$

$$= 1 \div \left[1+\left(\frac{a+x}{a-x}\right)^2\right] \cdot \frac{2a}{(a-x)^2}.$$

$$= \frac{2a}{1+\frac{(a+x)^2}{\cancel{(a-x)^2}} \cdot \cancel{(a-x)^2}}.$$

$$= \frac{2a}{a^2-2ax+2x^2+2ax+a^2}.$$

$$= \frac{\cancel{2}a}{\cancel{2}(a^2+x^2)}.$$

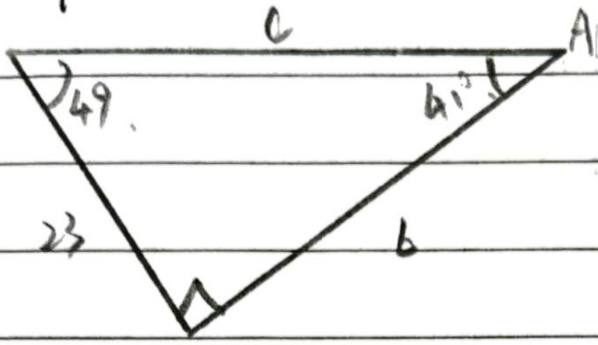
$$= \frac{2}{a^2+x^2}.$$

No. 1.3

Date

Solving Right triangles.

Solve for $\angle A$, b, c.



$$\tan(49^\circ) = \frac{b}{23}$$

$$b = 23 \cdot \tan 49^\circ$$

$$b = 26.45847$$

$$\angle A + 49 + 90^\circ = 180^\circ$$

$$\angle A = 41^\circ$$

$$\cos(41) \cdot c = b$$

$$\cos(41) = \frac{b}{c}$$

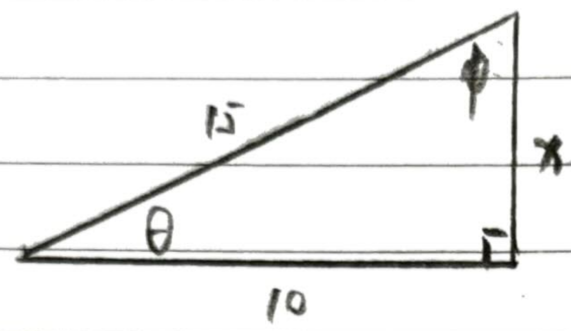
$$c = \frac{(\tan 49) \cdot 23}{\cos(41)}$$

$$c = 35.05782$$

No. 14

Date

E.g.



$\cos\theta \cdot 15 = 10.$

$\cos\theta = \frac{2}{3}$

$\theta = 48.1896851$

$\phi + \theta + 90^\circ = 180.$

$\phi = 90^\circ - \theta$

$\phi = 90^\circ - \cos^{-1}\left(\frac{2}{3}\right).$

$\phi = 41.81031$[illegible]$5.$

$\sin\theta \cdot 15 = x.$

$x = 15 \cdot \sin \circ \cos^{-1}\left(\frac{2}{3}\right).$

$x = 11.1803399.$

No. 15

Date

Maxinum and Minimum Values,

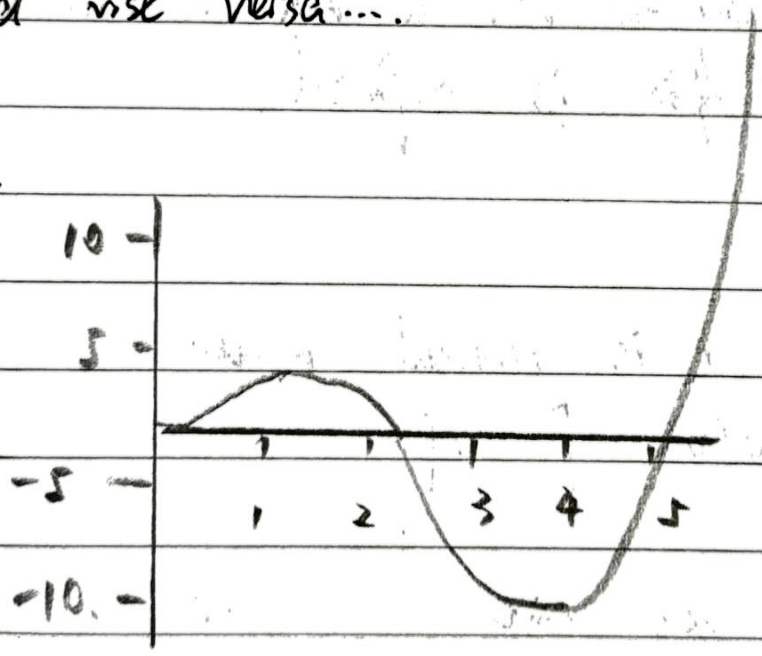
Definition: A function $f(x)$ has an absolute maxium at $x=c$ if $f(c) \geq f(x)$ for all x in the domain of f.

and vise versa.

The point $(c, f(c))$ is called an abslonte maxium $f(c)$ is called the absolnte maxium value.

and vise versa....

E.g.



This graph has a abs. ~~min~~ minimm value of -8

it has a abs. min at $(3.5, f(2.25))$.

No. 16

Date

Defination: Absolute Max and Min can also be called global max, global min.

Defination: A function $f(x)$ has an local maximum at $x=c$ if $f(c) \geq f(x)$ for all points near c.

and vise versa.

For a function $f(x)$ to has a min or max at $x=a$ only if $f(a)$ is defined.

Definition: A number c is critical number for a function f if

$f'(c)$ does not exist or

$f'(c) = 0$.

Note if f has a local max or min at c, then c must be a critical number.

But! A critical number may not be a local min or max. (e.g. x^3)

First Derivative and Second Derivative test.

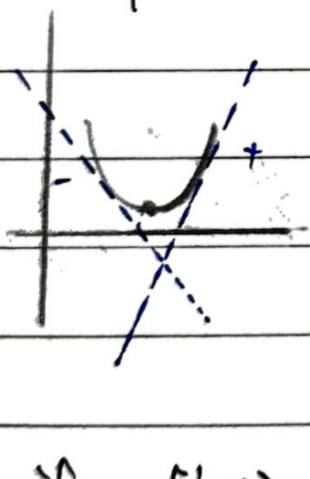
Recall:

$f(x)$ has a local max at $x=c$ if $f(c) \geq f(x)$ near c.

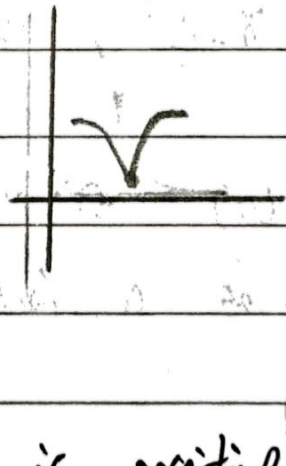
First Derivative test:

if f is a continuse function near $x=c$ and c is a critical number then.

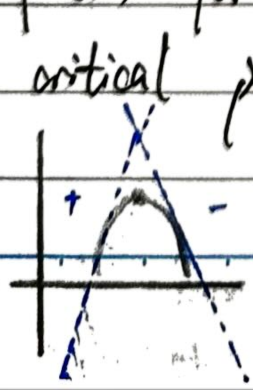
if $f'(x)$ for $x<c$ is negative or $f'(x)$ for $x>c$ is possitive, the the critical point is a local min.



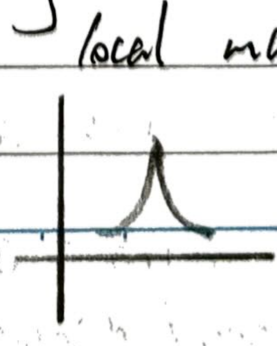
or



if $f'(x)$ for $x<c$ is possitive and $f'(x)$ for $x>c$ is negative, the critical point is a local max.



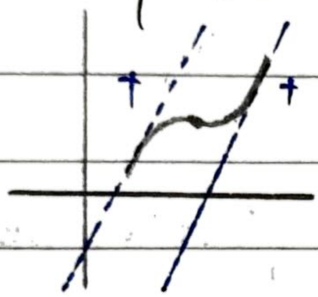
or.



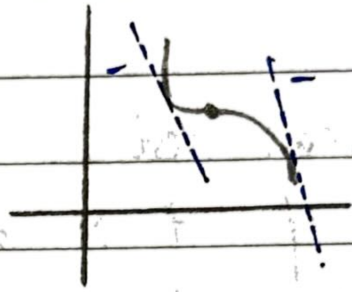
No. 18

Date

if $f'(x)$ for $x > c$ and $f'(x)$ for $x < c$ is both pos. or neg, then the critical point is not a local min or max.



or



Second Derivative test:

if f is a continuous function near $x = c$, then

if $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local min.

if $f'(c) = 0$, and $f''(c) < 0$, then

$f(c)$ is a local max

Note: if $f''(c) = 0$, the the result of the second derivative is inconclusive.

Extrem Value example.

E.g. Find the absolute maximum and minimum values for $g(x) = \frac{x-1}{x^2+x+2}$ on interval $[0, 4]$.

$$g'(x) = \frac{1}{(x^2+x+2)^2} \cdot \left[(x^2+x+2)\cdot 1 - (x-1)\cdot(2x+1+0)\right].$$

$$g'(x) = \frac{1}{(x^2+x+2)^2} \cdot \left[(x^2+x+2) - (2x^2-x-1)\right].$$

$$g'(x) = \frac{1}{(x^2+x+2)^2} \cdot (-x^2+2x+3).$$

$$g'(x) = -\frac{(x-3)(x+1)}{(x^2+x+2)^2}.$$

$$-\frac{(x-3)(x+1)}{(x^2+x+2)^2} = 0.$$

$$\begin{cases} x = 3 \\ x = -1 \text{ (not in the interval.)} \end{cases}$$

No. 20

Date

critical points: $x=3$, $x=1$.

for $g'(x)\big|_{x=3}$.

$$g'(x)\big|_{x=2.9} = -2.2014\times10^{-3} < 0.$$

$$g'(x)\big|_{x=3.1} = 1.894\times10^{-3} > 0.$$

so: $x=3$ is a local min.

No. 4.

Date

E.g. $f(x) = |x| - x^2$.

$f'(x) = (\sqrt{x^2})' - (x^2)'$

$= \frac{1}{2} \cdot x^{-1} \cdot 2x - 2x$

$= \frac{2}{2} \cdot \frac{x}{x} - 2x$

$= 1 - 2x$

$1 - 2x = 0$.

$2x = 1$

$x = \frac{1}{2}$

$f'(x) \Big|_{x = \frac{1}{2} + 0.1} = -\frac{1}{5}$.

< 0.

$f'(x) \Big|_{x = \frac{1}{2} - 0.1} = \frac{1}{5}$.

so $f(\frac{1}{2})$ is a local max.

No. 22.

Date

The Mean Value Theorem.

The Mean Value Theorem:

let $f(x)$ be a function defind on $[a, b]$
such that

①. $f(x)$ is continuse on $[a, b]$.

②. $f(x)$ is defrentiabale on (a, b).

then there must be a number c in $[a, b]$
such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

the derivative of
f at c.

the average rate
of chang of f
on $[a, b]$.

In other words:

for a function $f(x)$, assuming there is a secant line intersecting the graph at point $(a, f(a))$, $(b, f(b))$. Then there must be a tangent line at $(c, f(c))$ (where c is a value on $[a, b]$) that has the same slope with the secant line.

E.g. Verify the ~~ma~~ mean value threoem for $f(x) = 2x^3 - 8x + 1$ on interval $[1, 3]$.

To Verify:

Find c in $[1, 3]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$\begin{aligned} f'(x) &= 6x^2 - 8 + 0. \\ &= 6x^2 - 8. \end{aligned}$$

solve equation:

$$\cancel{*} \quad 6x^2-8 = \frac{f(3)-f(1)}{3-1}$$

$$6x^2-8 = 18$$

$$6x^2 = 24.$$

$$x^2 = 4$$

$$x = \pm 2.$$ (negitive 2 not on [1,3])

$$x = 2.$$

E.g. If f is a differentiable function and $f(1)=7$ and $-3 \leq f'(x) \leq -2$ on ~~the~~ $[1, 6]$
What is the biggist and smallest values that are possible for $f(6)$.

smallest :

$$-3 = \frac{f(6)-f(1)}{6-1}$$

$$-3 = \frac{f(6)-7}{5}$$

$$f(6) = -15+7.$$

$$f(6) = -8.$$

biggest: $-2 = \frac{f(6) - f(1)}{6-1}$

$-2 = \frac{f(6) - 7}{5}$

$f(6) = -10 + 7$

$f(6) = -3.$

Ans: $-8 \le f(6) \le -3.$

Rolle's Theroen: if $f(x)$ is a function defined on $[a, b]$ and.

①: $f(x)$ is continuous on $[a, b]$.

②: $f(x)$ is differentiable on (a, b).

③. $f(a) = f(b)$.

then there is a number c in $[a, b]$. such that

$$f'(c) = 0.$$

No. 26

Date

Derivatives and the Shape of a Graph Consepts.

* The function $f(x)$ is increasing if:

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

The function $f(x)$ is decresing if:

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

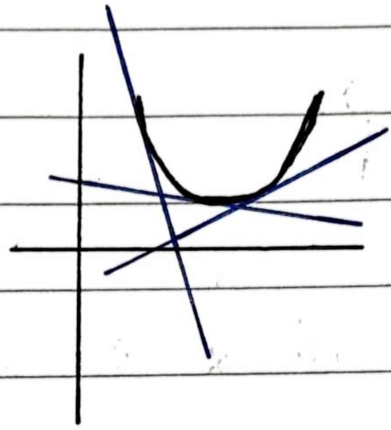
Increasing / Decreasing test:

If $f'(x) > 0$ for all x on an interval, then f is a increasing function.

If $f'(x) < 0$ for all x on the interval, then f is a decresing function.

f(x) is concave up on an interval (a, b) if the tangent lines lie below the graph of the function.

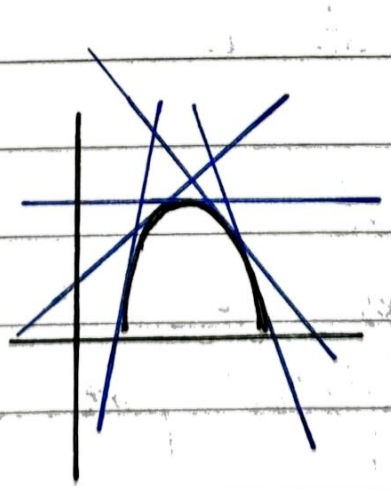
E.g.



the tangent line are below the graph draw in black.

f(x) is concave down on an interval (a,b) if the tangent line lie above the graph of the function.

E.g.



the tangent line are above the graph.

No. 28

Date

The convexity test:

If $f''(x) > 0$ for all x in an interval, then

f is concave up.

if $f''(x) < 0$ for all x in an interval, then

f is concave down.

$f(x)$ has an inflection point at $x = c$ if:

$f(x)$ is continuse at c and it changes

its concavity at c.

Inflection point test.

If $f''(x)$ changes sigh at $x = c$, then

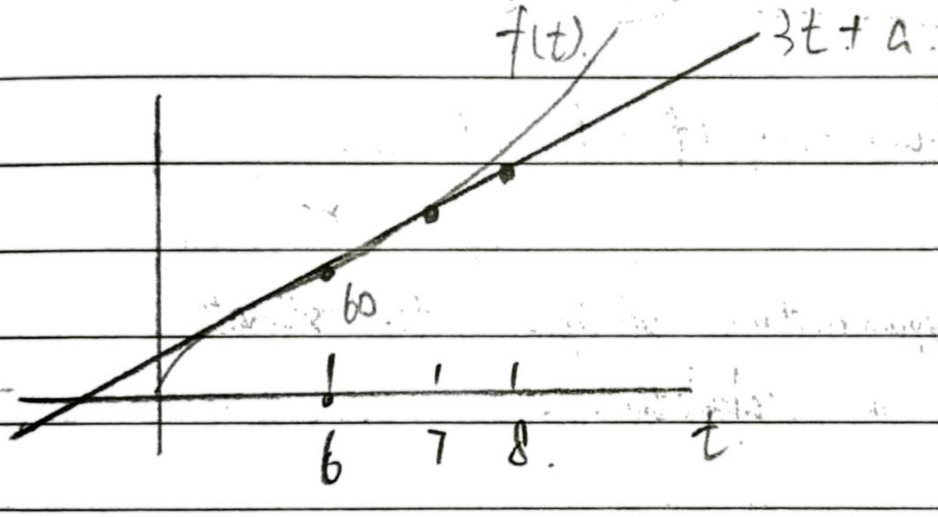
f has an inflection point at $x = c$.

No. 29

Date

Linear Approximation.

E.g. Suppose $f(t)$ is the temperature in degrees Fahrenheit at time t (measured in hours), where $t = 0$ represents midnight. Suppose that $f(6) = 60°$ and $f'(6) = 3°/\text{hr}$. What is your best ~~a~~ estimate for the temperature at 7:00 and 8:00 a.m.



$3t + a = 60$ (where $t = 6$).

$a = 42$.

7:00 : $3t + a = 21 + 42$

$= 63°$.

8:00 : $3t + a = 24 + 42$

$= 66°$.

Linear Approximation Principle:

$$f(a+\Delta x) \approx f(a) + f'(a)\,\Delta x.$$

Δx is a number close to a.

$$f(x) \approx \underbrace{f(a) + f'(a)\cdot(x-a)}$$

linearization of f at a.

E.g. Use the approximation principle to estimate $\sqrt{59}$ with a calculator.

let $f(x)$ be $\sqrt{x}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} x^{\frac{1}{2}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

$f'(x)\Big|_{x=64} = \frac{1}{16}.$

$$\sqrt{59} \approx \sqrt{64} + \frac{1}{16}\cdot(-5).$$

$$\approx 8 - \frac{5}{16}$$

$$\approx 7\frac{11}{16}$$

E.g. Use a linearization of $y = \sin(x)$ to estimate $\sin(33°)$ without a calculator.

$$\sin(33) \approx \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\frac{3}{180}\cdot\pi.$$

$$\approx \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\pi}{60}.$$

No. 32

The Differential.

Recall:

$$f(a+\Delta x) \approx f(a) + f'(a)\cdot \Delta x.$$

subtract $f(a)$ on both sides:

$$f(a+\Delta x) - f(a) = f'(a)\cdot \Delta x.$$

Defination:

The differential $dx = \Delta x$.

the differential $df = f'(x)\Delta x$

$dy = df$.

The change in f: $\Delta f = f(x+\Delta x) - f(x)$.

The linearization of $f(x+\Delta x)$ can be rewritten into:

$$\Delta f \approx df$$

$$\Delta y \approx dy.$$

e.g. For $f(x) = x\ln x$, find :

1) df

$$df = f'(x) \cdot \Delta x$$
$$= (\ln x + 1)\Delta x$$
$$= (\ln x + 1)\,dx.$$

2) : df when $x = 2$ and $\Delta x = -0.3$.

$$df = (\ln x + 1)\,dx.$$
$$= -(\ln 2 + 1) \cdot 0.3$$
$$= -0.3(\ln 2 + 1).$$

3) : Δf when $x = 2$, $\Delta x = -0.3$.

$$\Delta f = f(x + \Delta x) - f(x).$$
$$= 1.7\ln(1.7) - 2\ln 2.$$

No. 34

Date

E.g. Suppose that the radius of a sphere is ~~was~~ measured as 8 cm, with a possible error of 0.5 cm. Use the differential to estimate the result error in computing the volume.

let $f(x) = 4\pi x^2$.

$$\Delta f = f(x+\Delta x) - f(x)$$
$$= 289\pi - 256\pi$$
$$= 33\pi.$$

L'Hospital's Rule.

No. 25
Date

Defination: A limit of form $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called $\frac{0}{0}$ indeterminate form if:

$$\lim_{x\to a} f(x) = 0$$
$$\lim_{x\to a} g(x) = 0.$$

Defination: A limit of form $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called $\frac{\infty}{\infty}$ indeterminated form if:

$$\lim_{x\to a} f(x) = \pm\infty$$
$$\lim_{x\to a} g(x) = \pm\infty$$

Theorem. L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq$ in an open interval aramd a (expet possibly at a). If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminite form, then.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

No. 36

Date

E.g. $\lim_{x \to \infty} \frac{x}{3^x}$

$$\lim_{x \to \infty} \frac{x}{3^x} = \lim_{x \to \infty} \frac{1}{\frac{d}{dx}(3^x)}$$

$$= 0.$$

E.g. $\lim_{x \to 0} \frac{\sin(x) - x}{(\sin x)^3}$

$$= \lim \left(\cos(x) - x \right) \div \left(3\sin^2(x) \cdot \cos(x) \right).$$

$$= \lim_{x \to 0} \frac{1}{3\sin^2(x)} \cdot \frac{\cos(x) - x}{\cos(x)}$$

$$= \lim_{x \to 0} \frac{1}{3\sin^2(x)} \cdot \lim_{x \to 0} \frac{\cos(x) - x}{\cos(x)}.$$

$$= \lim_{x \to 0} \frac{1}{3\sin^2(x)}.$$

$$= \lim_{x \to 0} \frac{0}{3 \cdot 2\cos(x)}.$$

$$= \frac{0}{6}$$

$$= 0.$$

L'Hospital's Rule : Additional Indeterminate Formes

E.g. $\lim_{x \to 0} \sin x \ln x$.

$$\lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{\sin x}\right)}.$$

$$= \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{-\frac{\cos(x)}{(\sin x)^2}}.$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\cot(x)} \cdot \frac{1}{\frac{1}{\sin x}}$$

$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{\sin x}{-\cot(x)}.$$

$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{\sin^2(x)}{-\cos(x)}$$

$$= \lim_{x \to 0^+} - \frac{\sin^2(x)}{x\cos(x)}.$$

$$= \lim_{x \to 0^+} - \frac{\sin^2(x)}{x}.$$

$$= \lim_{x \to 0^+} \frac{\cos(x) \cdot 2\sin(x)}{1}$$

$$= 0.$$

E.g. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

$$= \lim_{x\to\infty} e^{x \cdot \ln\left(1+\frac{1}{x}\right)}.$$

$$= \lim_{x\to\infty} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}.$$

$$= \lim_{x\to\infty} e^{\frac{\frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}}}$$

$$= \lim_{x\to\infty} e^{\frac{1}{1+\frac{1}{x}}}$$

$$= e^{1}$$

$$= e.$$

Summary:

$1^\infty \Rightarrow \infty\ln 1^\infty \Rightarrow \infty \cdot 0.$

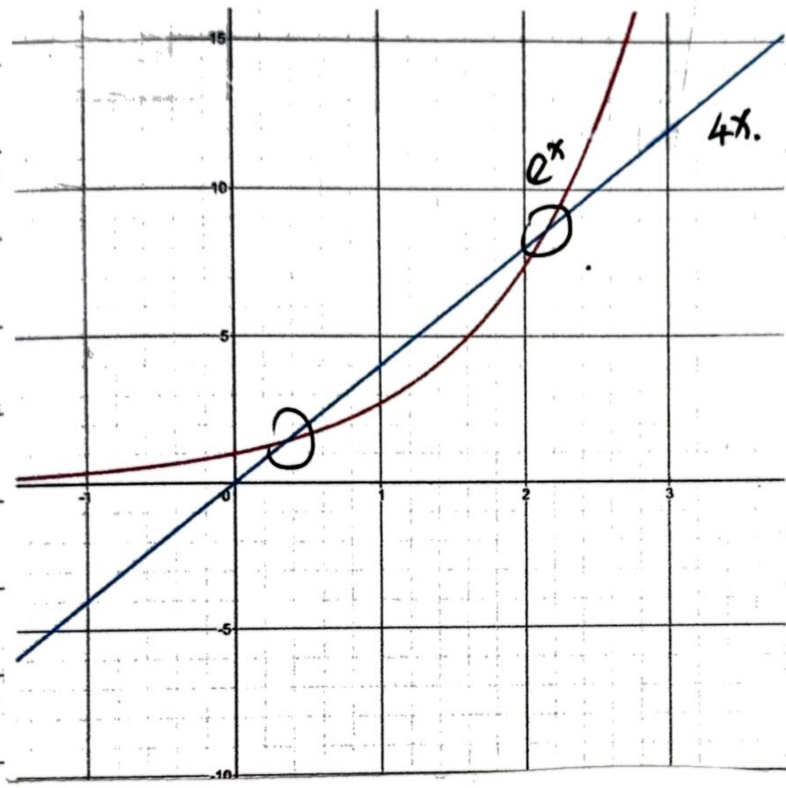
$\infty^0 \Rightarrow 0\ln \infty^0 \Rightarrow \infty \cdot 0$

$0^0 \Rightarrow 0\ln 0 \Rightarrow 0 \cdot (-\infty).$

No.
Date

Newton's Method.

E.g. Find the solution to equation $e^x = 4x$.



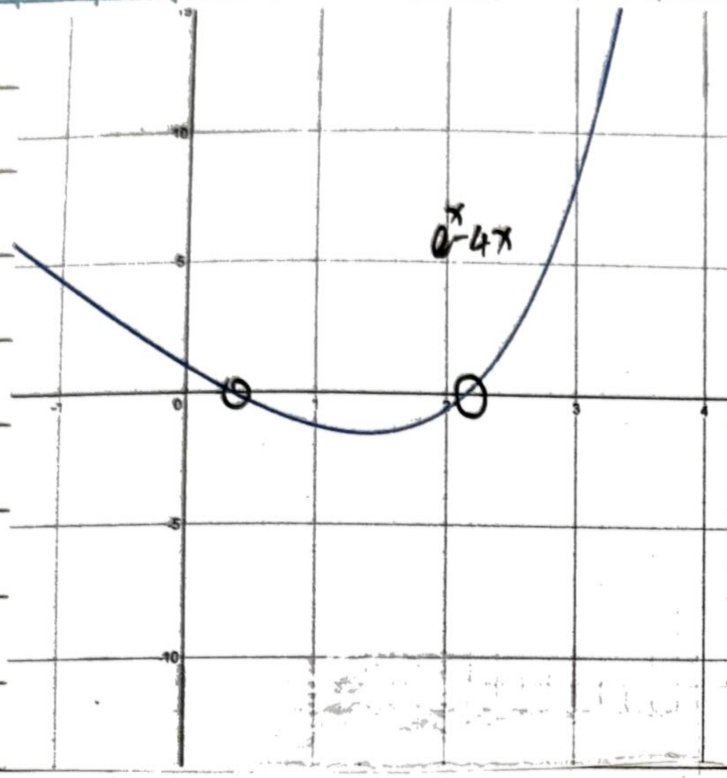
we can transeform the equation into:

$$e^x - 4x = 0$$

then our problem becomes finding zeros of the graph $y = e^x - 4x$

No. 40

Date



let $f(x) = e^x - 4x$.

Just watch 3B1B's video for Newton's Method

$\cancel{x_0} = x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_1)}$

$\vdots$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ the equation for Newton's Method.

Back to the E.g.

$$f(x) = e^x - 4x.$$

$$x_1 = x - \frac{e^x - 4x}{e^x - 4}.$$

let x_0 be 2.

$x_1 = 2.180269634$

$x_2 = 2.153950513$

$x_3 = 2.153292768.$

$x_4 = 2.153292364$

$x_5 = 2.153292364.$

let x_0 be 0.4.

$x_1 = 0.356870964$

$x_2 = 0.357402875$

$x_3 = 0.357402963$

$x_4 = 0.357402962$

No. 42

Date

~~then~~ in conclusion $\begin{cases} x = 2.15329 2364 \\ x = 0.35740 29562 . \end{cases}$

summary:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Antiderivative.

E.g. If $g'(x) = 3x^2$, what could $g(x)$ be.

$g(x) = x^3 + c$.

Defination: A function $F(x)$ is called an antiderivative of $f(x)$ on an interval (a,b) if

$F'(x) = f(x)$ on (a,b).

e.g. x^3+c is an antiderivative of $3x^2$.

If $F(x)$ is an antiderivative for $f(x)$, then all other antiderivative for $f(x)$ can be written in the form $F(x)+c$.

No. 44

Date

Function $f(x)$	Antiderivative $F(x)$	Function $f(x)$	Antiderivative $F(x)$
1	x	$\sec^2(x)$	$\tan(x)$.
x	$\frac{1}{2}x^2$	$\sec(x)\tan(x)$	$\sec(x)$.
x^n $(n \neq -1)$	$\frac{1}{n+1}x^{n+1}$	$\frac{1}{1+x^2}$	$\tan^{-1}(x)$.
x^{-1}	$\ln(x)$.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$.
e^x	e^x	$c \cdot x^n$	$c \cdot \frac{1}{n+1}x^{n+1}$
$\sin(x)$	$-\sin(x)$.	$c \cdot f(x)$	$c \cdot F(x)$.
$\cos(x)$	$\sin(x)$.	$f(x) + g(x)$	$F(x) + G(x)$.

E.g. Find the general antiderivative for ~~f(x)~~

$f(x) = \frac{5}{1+x^2} - \frac{1}{2\sqrt{x}}$

$f(x) = 5 \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

$= 5 \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$F(x) = 5 \cdot \tan^{-1}(x) - \frac{1}{2} \cdot 2x^{\frac{1}{2}} + C$

$= 5\tan^{-1}(x) + \sqrt{x} + C$

No.

Date

Finding antiderivative Using Initial Conditions.

E.g. If $g'(x) = e^x - 3\sin(x)$, and $g(2\pi) = 5$, find $g(x)$.

$$g(x) = e^x + 3\cos(x) + C.$$

$$g(2\pi) = e^{2\pi} + 3 + C$$

$$5 = e^{2\pi} + 3$$

$$C = 2 - e^{2\pi}.$$

$$g(x) = e^x + 3\sin(x) + 2 - e^{2\pi}$$

E.g. If $f''(x) = \sqrt{x}\left(x - \frac{1}{x}\right)$, find a equation for $f(x)$, if $f(1) = 0$ and $f(2) = 0$.

$$f''(x) = \sqrt{x} \cdot x - \sqrt{x} \cdot \frac{1}{x}$$

$$= x^{\frac{1}{2}+1} - x^{\frac{1}{2}+(-1)}$$

$$= x^{\frac{3}{2}} - x^{-\frac{1}{2}}$$

$$f'(x) = \frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C$$

$$f(x) = \frac{2}{5} \cdot \frac{2}{7}x^{\frac{7}{2}} - 2 \cdot \frac{2}{3}x^{\frac{3}{2}} + Cx + D.$$

$$= \frac{4}{35}x^{\frac{7}{2}} - \frac{4}{3}x^{\frac{3}{2}} + Cx + D.$$

No. 46

Date

$$f(1) = \frac{4}{35} - \frac{4}{3} + C + D.$$

$$= -\frac{128}{105} + C + D.$$

$$C + D = \frac{128}{105}.$$

$$f(2) = \frac{4}{35} \times 2^{\frac{7}{2}} - \cancel{8}\ \frac{4}{3} \times 2^{\frac{3}{2}} + 2C + D.$$

$$2C + D = \frac{4}{35} \times 2^{\frac{7}{2}} - \frac{4}{3} \times 2^{\frac{3}{2}}$$

$$2C + D = -2.478240909.$$

$$\begin{cases} C + D = \frac{128}{105} \\ 2C + D = -\left(\frac{4}{35} \times 2^{\frac{7}{2}} - \frac{4}{3} \times 2^{\frac{3}{2}}\right). \end{cases}$$

$$\begin{cases} C = 1.25919329. \\ D = -0.4014567121 \end{cases}$$

$$f(x) = \frac{4}{15}x^{\frac{7}{2}} - \frac{3}{4}x^{\frac{3}{2}} + 1.2591329x - 0.4014567\Big|_{1}^{2}$$

E.g. You are standing on the edge of a cliff at a height of 30 meters. You ~~throug~~ throw a tomato stright up in the air at a speed of 20 meters per second. How long does it take the tomato to reach ~~o~~ the ground. what is the ~~20~~ velocity at impact.

$a(t) = -9.8\,m/s^2.$

$V_0 = 20\,m/s$

$S_0 = 30\,m.$

$s'(0) = 20.$

$s''(0) = 30.$

$s'(t) = -9.8t + C_1$

$s'(0) = -9.8 \cdot 0 + C_1 = 20.$

$C_1 = 20.$

No. 48

Date

$s'(t) = 9.8t + 20.$

$s(t) = -9.8 \cdot \frac{1}{2}t^2 + 20t + C_2.$

$s(0) = 30.$

$C_2 = 30.$

$s(t) = 0.$

$-\frac{49}{10}t^2 + 20t + 30 = 0.$

$\frac{49}{10}t^2 + 20t = -30.$

$t = \cancel{1.16657829} \approx 5.248210948.$

$s'(5.248210948) \approx -31.43246729.$

Proof that Two Antiderivative differ by a constant.

If $g'(x)$ on the interval (a, b), then $g(x) = C$.

Proof: by mean value threom.

$$g'(x) = \frac{g(b) - g(a)}{b-a}.$$

$$0 = \frac{g(b) - g(a)}{b-a}.$$

$$g(b) = g(a).$$

$g(x) = c$ for some constant c.

If $g_1(x)$ and $g_2(x)$ are two functions defined on $[a, b]$. and $g_1'(x) = g_2'(x)$. on (a, b).

Proof: by the mean value threom and previous statment:

$$g_1'(x) = g_2'(x).$$

$$g_1'(x) - g_2'(x) = 0.$$

No. 70

Date

$$\left(g_1'(x) - g_2'(x)\right)' = 0.$$

$$g_1(x) - g_2(x) = C.$$

$g_1(x) = g_2(x) + C$ for some constant C.

Summation Notation.

E.g. $\sum_{i=1}^{5} 2^i$

$= 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= 2 + 4 + 8 + 16 + 32.$

$= 62.$

could be expressed:

```
sum = 0.
for i in range [1, 6).:
    sum += 2^i
```

E.g. Write in Σ notation: $6+9+12+15+18.$

$$\boxed{\sum_{i=2}^{6} 3i}.$$

E.g. Write in Σ notation: $\frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \frac{31}{32}.$

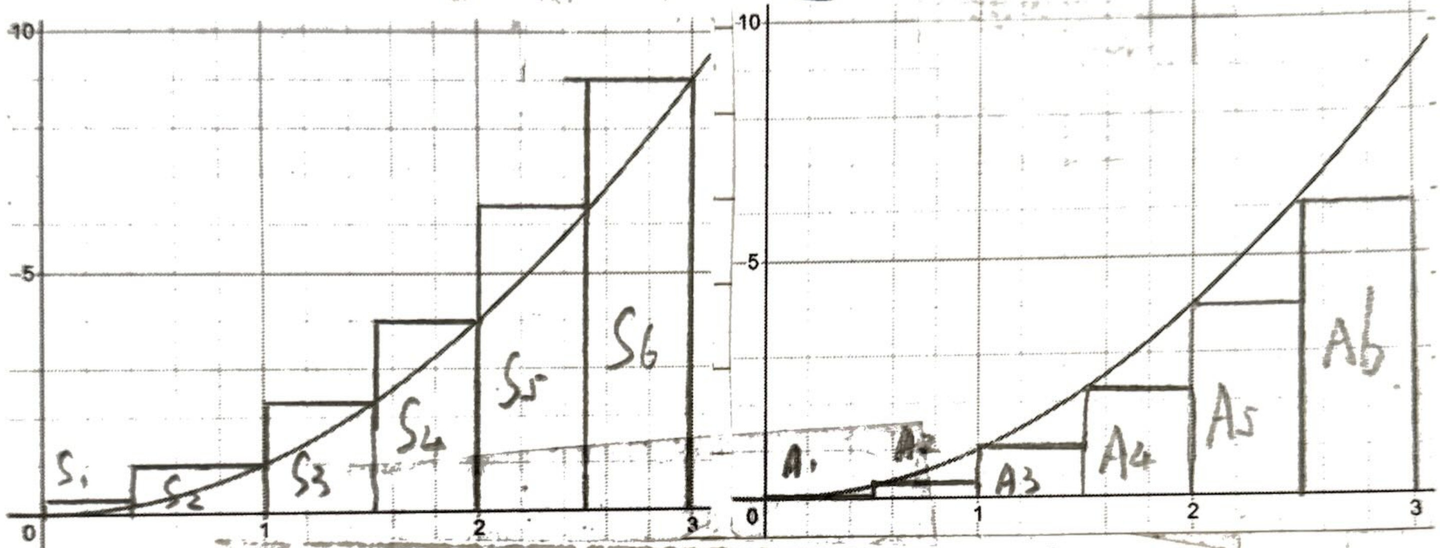
$$\boxed{\sum_{n=2}^{5} \frac{2^n - 1}{2^n}}.$$

No. 52

Date

Approximating Areas

e.g. Estimate the area under the curve $y=x^2$ between $x=0$ and $x=3$ by appximating it with 6 rectengles.



$$S = \frac{6}{3}\cdot\left(\frac{3}{6}n\right)^2.$$

$$= \frac{1}{2}\cdot\left(\frac{1}{2}n\right)^2.$$

$$= \frac{1}{2}\cdot\frac{1}{4}n^2$$

$$= \frac{1}{8}n^2.$$

$$S_1+S_2+S_3+\cdots+S_6 = \sum_{n=1}^{6}\frac{1}{8}n^2$$

$$= \frac{91}{8}. \quad (11.375).$$

$$A = \frac{6}{3}\cdot\left(\frac{3}{6}(n-1)\right)^2.$$

$$= \frac{6}{3}\cdot\left(\frac{3}{6}n-\frac{3}{6}\right)^3.$$

$$= \frac{1}{2}\cdot\left(\frac{1}{4}n^2-\frac{1}{2}n+\frac{1}{4}\right).$$

$$= \frac{1}{8}n^2-\frac{1}{4}n+\frac{1}{8} \qquad A_1+A_2+\cdots+A_6 = \sum_{n=1}^{6}\frac{1}{8}n^2-\frac{1}{4}n+\frac{1}{8} = \frac{55}{8}\ (6.875).$$

Avarge: $\dfrac{11.375 + 6.875}{2} = 9.125$.

e.g. Estimate the area under the curve $y = x^2$ between $x=0$ and $x=3$. by ~~ax~~ approximating it with 12 retangles

$S = \frac{3}{12}\left(x \cdot \frac{3}{12}\right)^2$.

$= \frac{1}{4} \cdot \frac{1}{16}x^2$.

$= \frac{1}{64}x^2$.

$S_1 + S_2 + \cdots + S_{12} = \sum_{x=1}^{12} \frac{1}{64}x^2$.

$= \frac{325}{32}$ (10.15625).

$\dfrac{\frac{2447}{256} + \frac{325}{32}}{2} = \frac{5047}{512}$

(9.857421875).

$A = \frac{3}{12}\left(\cancel{(x-1)}\ \frac{3}{12}\left(x - \frac{3}{12}\right)\right)^2$.

$= \cancel{\frac{3}{4}}\ \frac{1}{4}\left(\frac{1}{4}x - \frac{1}{16}\right)^2$.

$= \frac{1}{4}\left(\frac{1}{16}x^2 - \frac{1}{32}x + \frac{1}{256}\right)$.

$= \frac{1}{64}x^2 - \frac{1}{128}x + \frac{1}{1024}$.

$A_1 + A_2 + A_3 + \cdots + A_4 + A_5 + A_6 + \cdots + A_{12} = \sum_{x=1}^{12} \frac{1}{64}x^2 - \frac{1}{128}x + \frac{1}{1024}$.

$= \frac{2447}{256}$ (9.55859375).

No. 54

Date

Use 100 rectangles:

$$S = \frac{3}{100}\left(\frac{3}{100}x\right)^2.$$

$$A = \frac{3}{100}\left(\frac{3}{100}(x-1)\right)^2.$$

$$S_1 + S_2 + \cdots + S_{100} = \sum_{x=1}^{100} \frac{3}{100}\left(\frac{3}{100}x\right)^2.$$

$$= \frac{182709}{20000} \quad (9.13545).$$

$$A_1 + A_2 + A_3 + \cdots + A_{100} = \sum_{x=1}^{100}$$

$$= 8.86545.$$

$$\frac{\left(\frac{182709}{20000} + 8.86545\right)}{2} = 9.00045$$

Use n rectangles:

$S = \frac{3}{n}\left(\frac{3}{n}x\right)^2$

$= \cancel{\frac{3}{n^2}x}$

$A = \frac{3}{n}\left(\frac{3}{n}(x-1)\right)^2.$

$S_1 + S_2 + \cdots + S_n = \sum_{x=1}^{n} \frac{3}{n}\left(\frac{3}{n}x\right)^2.$

$A_1 + A_2 + \cdots + A_n = \sum_{x=1}^{n} \frac{3}{n}\left(\frac{3}{n}(x-1)\right)^2.$

the exact area is given by the limit:

$\lim_{n\to\infty} \sum_{x=1}^{n} \frac{3}{n}\left(\frac{3}{n}x\right)^2$

or

$\lim_{n\to\infty} \sum_{x=1}^{n} \frac{3}{n}\left(\frac{3}{n}(x-1)\right)^2.$

No. 56

Date

Compute the exact area under the curve $y=x^2$ between $x=0$ and $x=3$.

$\lim_{n\to\infty} \sum_{x=1}^{n} \frac{3}{n}\left(\frac{3}{n}x\right)^2.$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{x=1}^{1} \left(\frac{3}{n}x\right)^2.$

$= \lim_{n\to\infty} \frac{3}{n} \cdot \sum_{x=1}^{n} \frac{9}{n^2} x^2.$

$= \lim_{n\to\infty} \frac{3}{n} \cdot \frac{9}{n^2} \sum_{x=1}^{n} x^2.$

$= \lim_{n\to\infty} \frac{\cancel{27}\,9}{n^3} \cdot \frac{n(n+1)(2n+1)}{\cancel{6}\,2}$

$= \lim_{n\to\infty} \frac{9}{2} \frac{\cancel{n}(n+1)(2n+1)}{\cancel{n^3}\,n^2}.$

$= \lim_{n\to\infty} \frac{9}{2} \cdot \frac{(n+1)(2n+1)}{n^2}.$

$= \lim_{n\to\infty} \frac{9}{2} \cdot \frac{2n^2+3n+1}{n^2}.$

$= \frac{9}{2} \lim_{n\to\infty} \frac{4n+3}{2n}.$

$= \frac{9}{2} \lim_{n\to\infty} \frac{4}{2}.$

$= \frac{9}{2} - 2.$

$\boxed{= 9.}$

~~Reimon~~

Definition: A Rieman is normaly experssed as below:

$$\lim_{n \to \infty} \sum_{x=a}^{n} \frac{b}{n} f\left(\frac{b}{n} x\right).$$

or

$$\lim_{n \to \infty} \sum_{x=a}^{n} \frac{b}{n} f\left(\frac{b}{n}(x-1)\right)$$

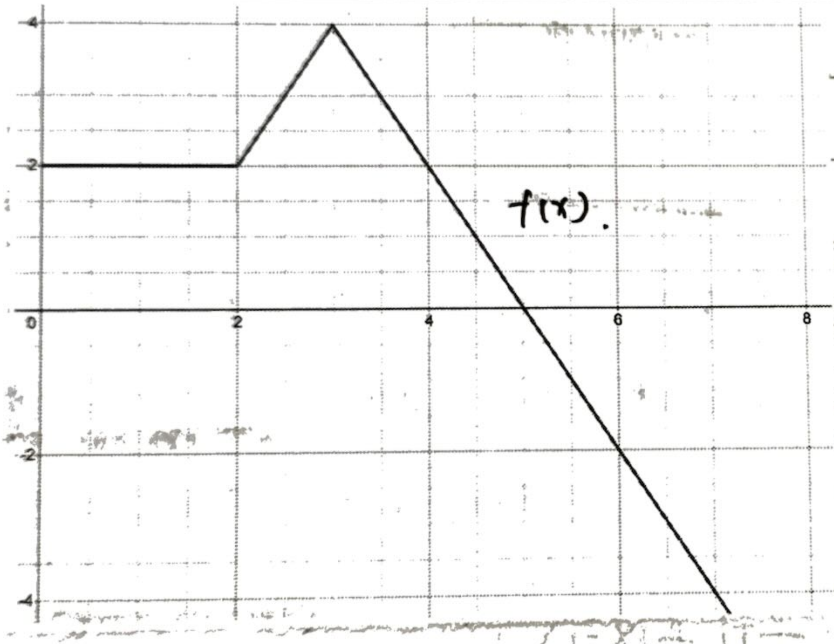
No. 58

Date

The Fundamental Theorem of Calculus, Part ①

E.g. Suppose $f(x)$ has the graph shown, and let $g(x)$ be

$$g(x) = \int_0^x f(t)\,dt.$$



Find $g(1) = 1 \times 2$

$= 2.$

$g(2) = 2 \times 2$

$= 4.$

No. 59

Date

$g(3) = g(2) + \frac{1}{2}\cdot 1 \cdot (4+2).$

$= 4 + \frac{1}{2}\cdot 6.$

$= 7.$

$g(4). = g(2) + 2(g(3) - g(2))$

$= 4 + 6.$

$= 10.$

$g(5) = g(4) + \frac{1}{2}\cdot 2\cdot 1$

$= 10 + 1$

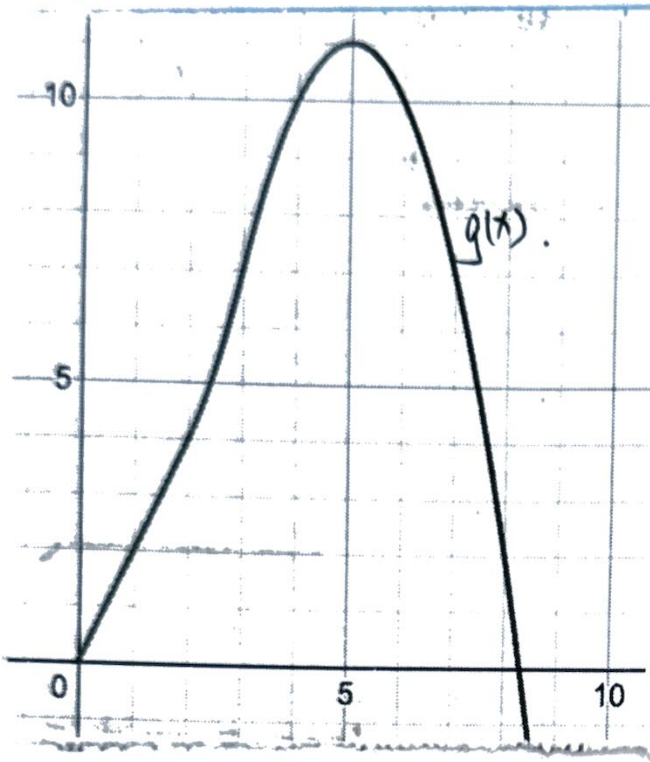
$= 11$

$g(6) = g(4)$

$= 10.$

$g(7) = g(3)$

$= 7$



$g'(x) > 0$ when $f(x) > 0$.
$g'(x) < 0$ when $f(x) < 0$.
$g'(x) = 0$ where $f(x) = 0$.

In fact:

$$\boxed{g'(x) = f(x).}$$

Theorem. (Fundamental Theorem of Calculus, Part I):
If $f(x)$ is continuse on $[a, b]$ then for $a \le x \le b$ the function

$$g(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) = f(x).$$

E.g. Find:

$$\frac{d}{dx}\int_5^x \sqrt{t^2+3}\ dt = \boxed{\sqrt{x^2+3}}.$$

$$\frac{d}{dx}\int_4^x \sqrt{t^2+3}\ dt = \boxed{\sqrt{x^2+3}}.$$

$$\frac{d}{dx}\int_x^4 \sqrt{t^2+3}\ dt$$
$$= \frac{d}{dx}\underline{-\int_4^x} \sqrt{t^2+3}\ dt$$
$$= \boxed{-\sqrt{x^2+3}}.$$

$$\frac{d}{dx}\int_4^{\sin(x)} :$$

$$g(\sin x) = \frac{d}{dx}\int_4^{\sin(x)} \sqrt{t^2+3}\ dt.$$

$$g(\sin(x))' = g'(\sin(x)) \cdot \cos(x).$$

$$f(x) = \sqrt{\sin^2(x)+3} \cdot \cos(x).$$

$$\boxed{\frac{d}{dx}\int_4^{\sin(x)} \sqrt{t^2+3}\ dt = \sqrt{\sin^2(x)+3} \cdot \cos(x).}$$

No. 62

Date

Threome Summery:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

The Fundamental Theorem of Calculus, Part 2.

Theorem: If f is continuse on $[a, b]$, then

$$\int_a^b f(x)\,dx$$

$$\boxed{= F(b) - F(a).}$$

Proof: from part 1:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$\Downarrow$$

$$\int_a^x f(t)\,dt = F(x).$$

$$\frac{d}{dx}\int_x^b f(t)\,dt = -\frac{d}{dx}\int_b^x f(t)\,dt$$

$$\Downarrow$$

$$\int_x^b f(t)\,dt = -F(x)$$

No. 64

Date

In conclusion :

$$\int_{n}^{b} f(t)\,dt = F(b)$$

$$\int_{a}^{n} f(t)\,dt = -F(a)$$

(where n is any Real number).

$$\Downarrow$$

$$\boxed{\int_{a}^{b} f(x)\,dx = F(b) - F(a)}$$

E.g. Find $\int_{-1}^{-5} 3x^2 - \frac{4}{x} dx$.

$$= \int_{-1}^{-5} (3 \cdot x^2 - 4 \cdot \frac{1}{x}). dx.$$

$$= 3 \cdot \frac{1}{3} x^3 + 4 \cdot \ln(x) \Big|_{-1}^{-5}$$

$$= x^3 + 4 \ln|x| \Big|_{-1}^{-5}$$

$$= (-5)^3 + 4 \ln(5) - (-1)^3 + 4 \ln(1).$$

$$= -125 + 4 \ln(5) + 1 + 0$$

$$= -124 + 4 \ln(5).$$

Date

E.g. Find $\int_1^4 \frac{y^2-y+1}{\sqrt{y}}\, dy$

$$= \int_1^4 (y^2-y+1)\cdot y^{-\frac{1}{2}}$$

$$= \int_1^4 y^{\frac{3}{2}} - y^{\frac{1}{2}} + y^{-\frac{1}{2}}$$

$$= \cancel{\int} \frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} \Big|_1^4$$

$$= \frac{146}{15}$$

Theroen Summery:

$$\boxed{\int_a^b F'(x)\, dx = F(b) - F(a).}$$

The Substitution Method.

E.g. Find $\int 2x \sin(x^2)\, dx$.

let u be (x^2), du be $(2x\, dx)$.

$$\int 2x \sin(x^2)\, dx$$
$$= \int du \sin(\cancel{x^2}\, u)$$
$$= \int \sin(u)\, du.$$
$$= -\cos(u) + c$$
$$= -\cos(x^2) + c.$$

E.g. Find $\int \frac{x}{1+3x^2}\, dx$

let u be $(1+3x^2)$, du be $(6x\, dx)$

$$\int \frac{x}{1+\cancel{x}3x^2}\, dx$$
$$= \int \frac{x}{u}\, dx$$
$$= \int \frac{1}{6} \cdot \frac{1}{u} \cdot du.$$
$$= \frac{1}{6} \ln|1+3x^2| + C$$

No. 68

Date

E.g. $\int e^{7x} \, dx$.

let $\cancel{e^x}$ u be $7x$ du be $(7+c)$

$\int e^{7x} \, dx$

$= \int e^{u} \, dx$.

$= \int e^{u} \frac{1}{7} du$.

$= \frac{1}{7} \int e^{u} \, du$

$= \frac{1}{7} e^{7x}$

E.g $\int_{e}^{e^2} \frac{\ln(x)}{x} \, dx$.

let u be $\ln(x)$, du be $\frac{1}{x} dx$

$\int_{e}^{e^2} \frac{\ln(x)}{x} \, dx$

$= \int_{e}^{e^2} u \cdot du$

$= \frac{1}{2} u^2 \Big|_{e}^{e^2}$

$= \frac{1}{2} \ln^2(x) \Big|_{e}^{e^2}$

$= \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot 1$

$= \frac{3}{2}$

Why U-substitution Works.

Proof of U-substitution.

from chain Rule:

$$\frac{d}{dx}(F(g(x))) = F'(g(x)) \cdot g'(x).$$

$$F'(g(x)) \cdot g'(x) = \frac{d}{dx}F(g(x)).$$

$$\int F'(g(x)) \cdot g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx.$$

$$\int F'(g(x)) \cdot g'(x)\,dx = F(g(x)) + c.$$

let $u = g(x)$, $du = g'(x)\,dx$.

from that.

$$\int F'(g(x))\,g'(x)\,dx = \int F'(u)\,du. \quad ①.$$

solve for ①, we get.

$$\int F'(u)\,du = F(u) + c.$$

plug in $u = g(x)$.

we get $\int F'(u)\, du$.

$= F(u) + c$

$= F(g(x)) + c$.

which is exactly $\int F'(g(x))\, g'(x)\, dx$,
the integral we want to solve at the begining.

No. 7/11
Date

Average Value of a Function.

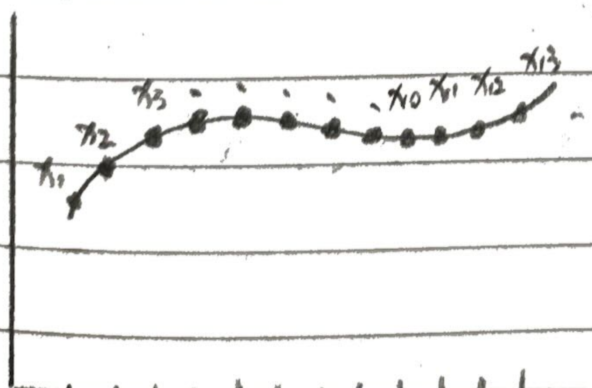
The average of a number list $q_1, q_2, q_3, q_4, q_5 \cdots\cdots q_n$.

we simply add up all the elements and divide the sum by the number of elements in a list.

If we expree is in Σ notation, we get.

$$\text{Average} = \left(\sum_{a=1}^{n} q_a\right) \div n.$$

For a continuous function $f(x)$ on an interval $[a, b]$, we could estimate ~~this~~ the average value of the function by sampling it at a bunch of x-values.



$$\cdots\cdots \frac{f(x_1) + f(x_2) + f(x_3) + \cdots f(x_n)}{n}.$$

$$= \left(\sum_{i=1}^{n} f(x_i)\right) \div n.$$

No. 72

Date

The apporamation gets better as the number of sample points gets larger, so we can define

$$\text{average} = \left(\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\right) \div n.$$

if we write the average in form of Reimann sum.

$$\text{average} = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\right) \div n. \cdot \frac{\Delta x}{\Delta x}$$

since : $n \cdot \Delta x = b-a$

$$\text{average} = \lim_{\Delta x\to 0} \left(\sum_{i=1}^{n} f(x_i)\,\Delta x\right) \div (b-a).$$

$$\Downarrow$$

~~which is~~

$$\boxed{\text{average} = \left(\int_a^b f(x)\,dx\right) \div (b-a).}$$

No. ~~4~~ 13

Date

E.g. Find the average value of function $g(x) = \frac{1}{1-5x}$ on the interval $[2, 5]$.

Question: Is there a ~~numbers~~ $f(c)$ in the interval $[2, 5]$ for wich $g(c)$ is the average of $g(x)$ on interval $[2, 5]$.

Ans: Yes there is that number c according to intermedia value threom.

$$f(c) = \left(\int_2^5 g(x)\,dx\right) \div (5-2).$$

$$= \left(\int_2^5 \frac{1}{1-5x}\,dx\right) \div 3.$$

let $u = 1-5x$, $du = -4\,dx$.

$$f(c) = \left(\int_2^5 -\frac{1}{4}\cdot\frac{1}{u}\,du\right) \div 3.$$

$$= \left(-\frac{1}{5}\ln|u|\,\Big|_2^5\right) \div 3.$$

$$= -\frac{1}{5}\ln(5x-1)\Big|_2^5 \div 3.$$

$$f(c) = -\frac{1}{15}\ln(24) + \frac{1}{15}\ln(9).$$

$$\frac{1}{1-5c} = -\frac{1}{15}\ln(24) + \frac{1}{15}\ln(9).$$

$$\frac{1}{1-5c} = \frac{1}{15}\ln\left(\frac{3}{8}\right).$$

$$1 = \frac{1}{15}\ln\left(\frac{3}{8}\right) - \frac{1}{3}\ln\left(\frac{3}{8}\right)c.$$

$$c = \frac{3}{\ln\left(\frac{8}{3}\right)} + \frac{1}{5}$$

$$\approx 3.25863624347.$$

Theorem : Mean Value Theorem for Integral.

$$f_{average} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

No. 76

Date

Proof of the Mean Value Theroem for Integral.

Mean value Theroem for integral stated that:

$$f_{average} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

Proof:

The intermediate value theorm stated:

For continuse function $f(x)$ defined on $[x_1, x_2]$ if L between $f(x_1)$ and $f(x_2)$ then for some value c between x_1, x_2, $f(c) = L$.

If $f(x)$ is constant on $[a, b]$, then $f_{average} = f(c)$ for all c values

Suppose f is not constant on $[a, b]$, then f has a min value "M" and a max value "m".

$$m \leq f_{average} \leq M$$

because $m \leq f(x) \leq M$.

$$\frac{\int_a^b m \, dx}{b-a} \leq \frac{\int_a^b f(x) \, dx}{b-a} \leq \frac{\int_a^b M \, dx}{b-a}.$$

$$\frac{m(b-a)}{b-a} \leq f_{average} \leq \frac{M(b-a)}{b-a}.$$

$$m \leq f_{average} \leq M.$$

By Intermediate value theroen:

$f_{averge} = f(c)$ for some c in $[a, b]$.

2022.4.11