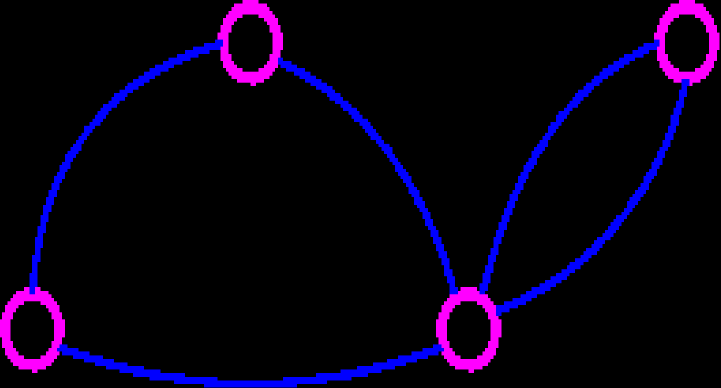
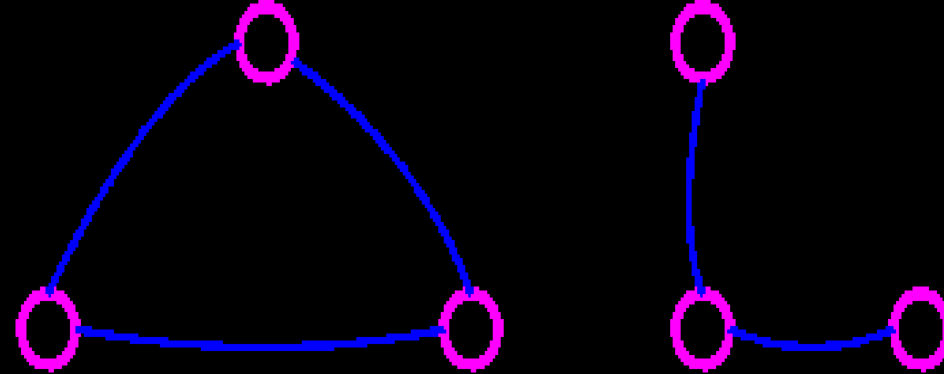
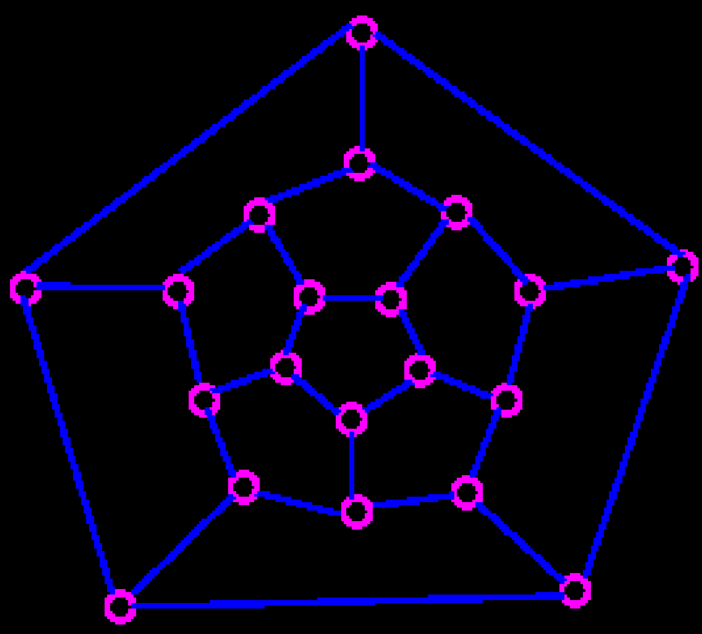
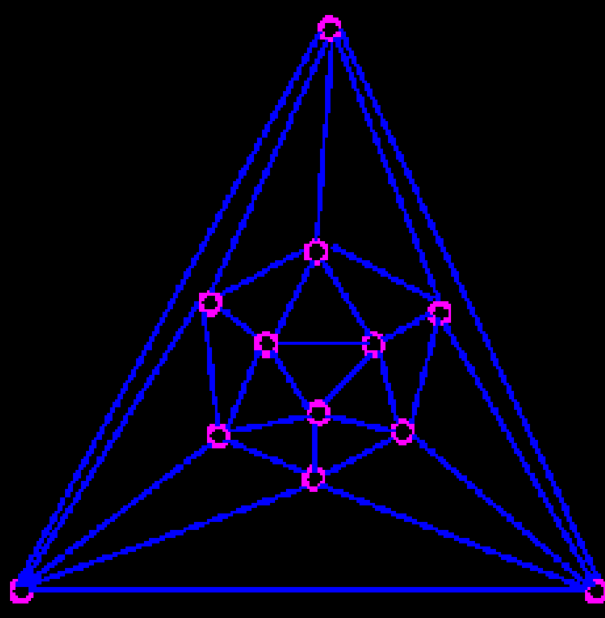
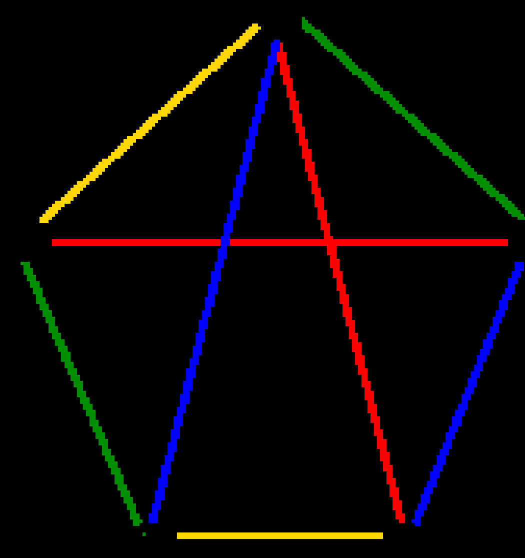




CONNECTIVITY OF GRAPHS







CONNECTIVITY OF A GRAPH

A graph is said to be **connected**, if there is a path between any two vertices.

Some graphs are "more connected" than others.

Two numerical parameters :-

edge connectivity &***vertex connectivity***

are useful in measuring a graph's connectedness.

EDGE CONNECTIVITY

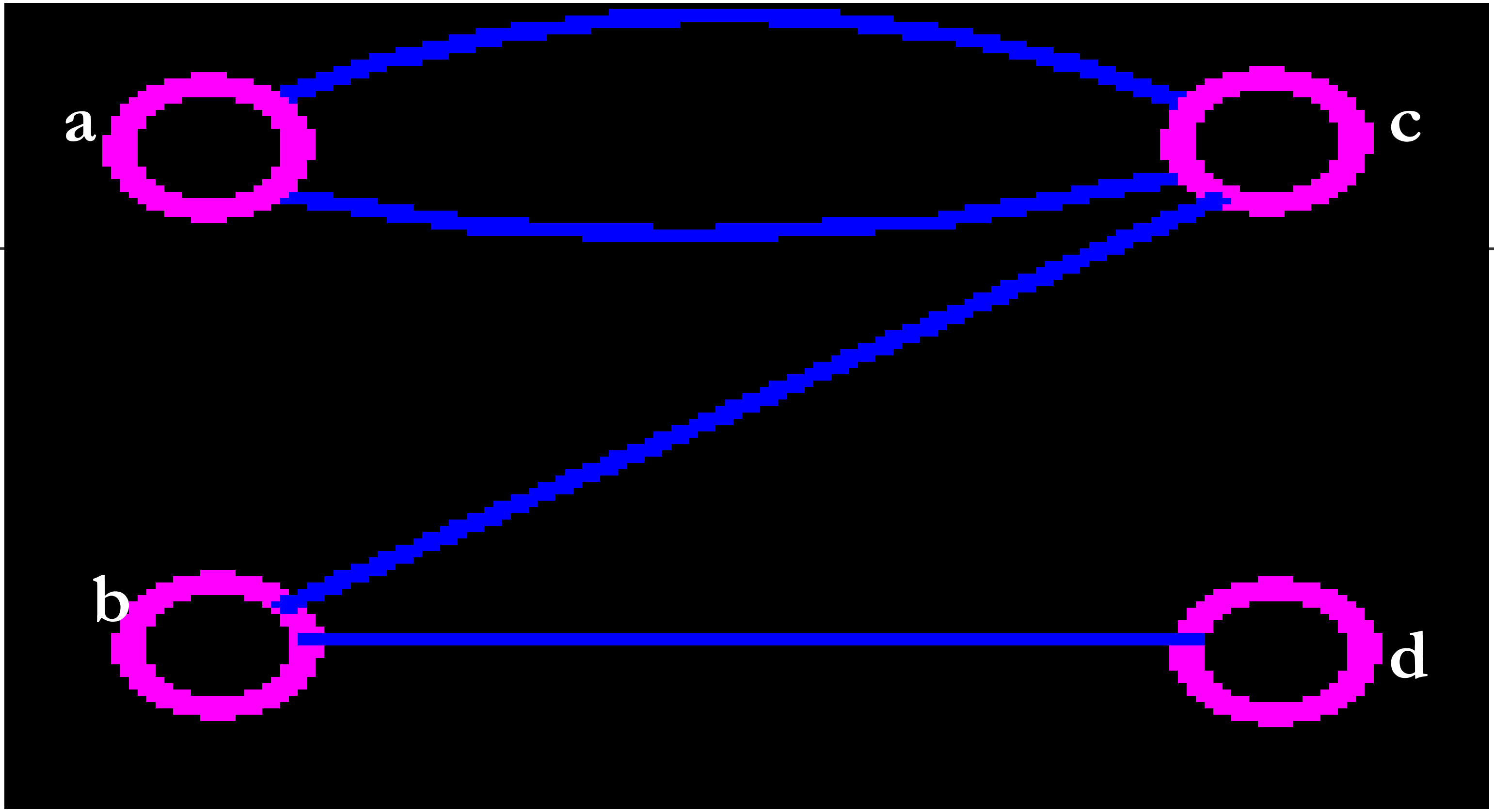
DEFINITION

$$\boldsymbol{\lambda(G) = \min\{k \mid k = |S|,\ G-S\ \text{disconnected},\ S \subseteq E_G\}}$$

[redacted]

The edge-connectivity $\boldsymbol{\lambda}(\boldsymbol{G})$ of a connected graph G is the smallest number of edges whose removal disconnects G.

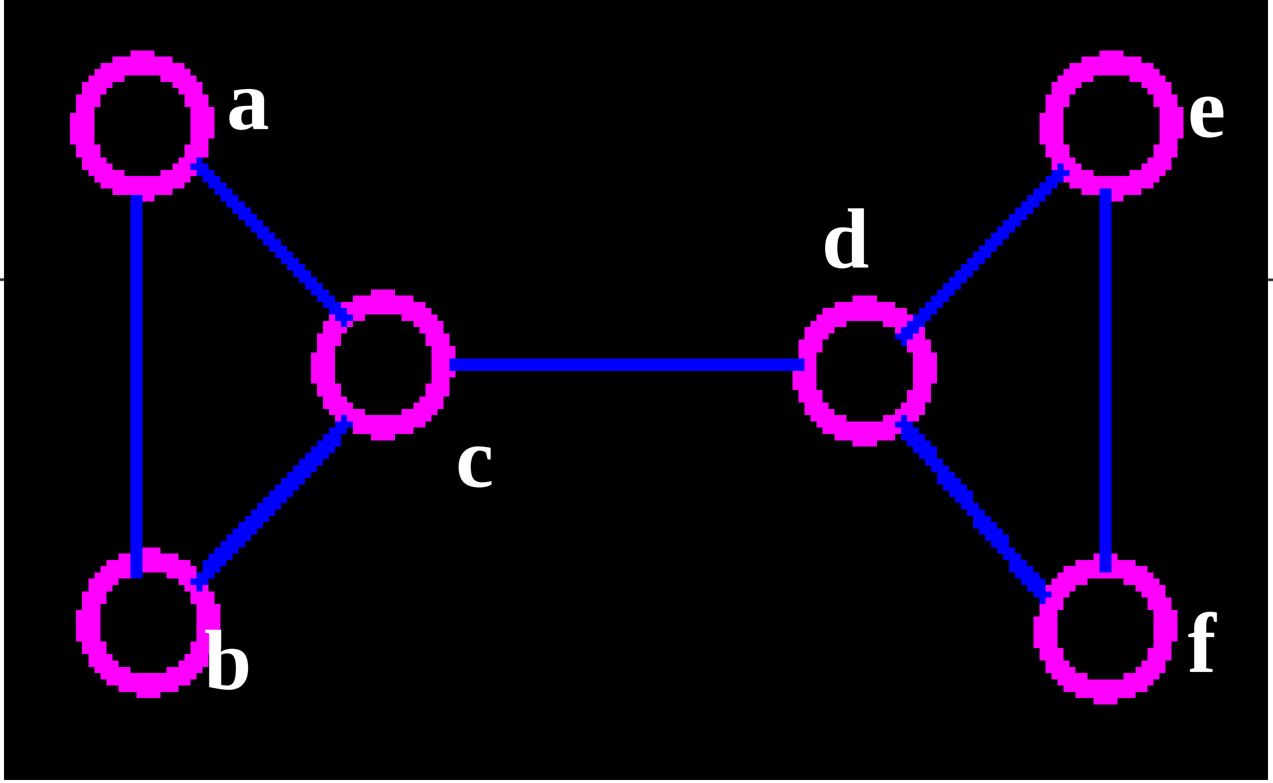
When $\boldsymbol{\lambda(G)} \geq k$, the graph G is said to be k-edge-connected.



The above graph G1 can be split up into two components by removing one of the edges bc or bd.

G1 has edge-connectivity 1.

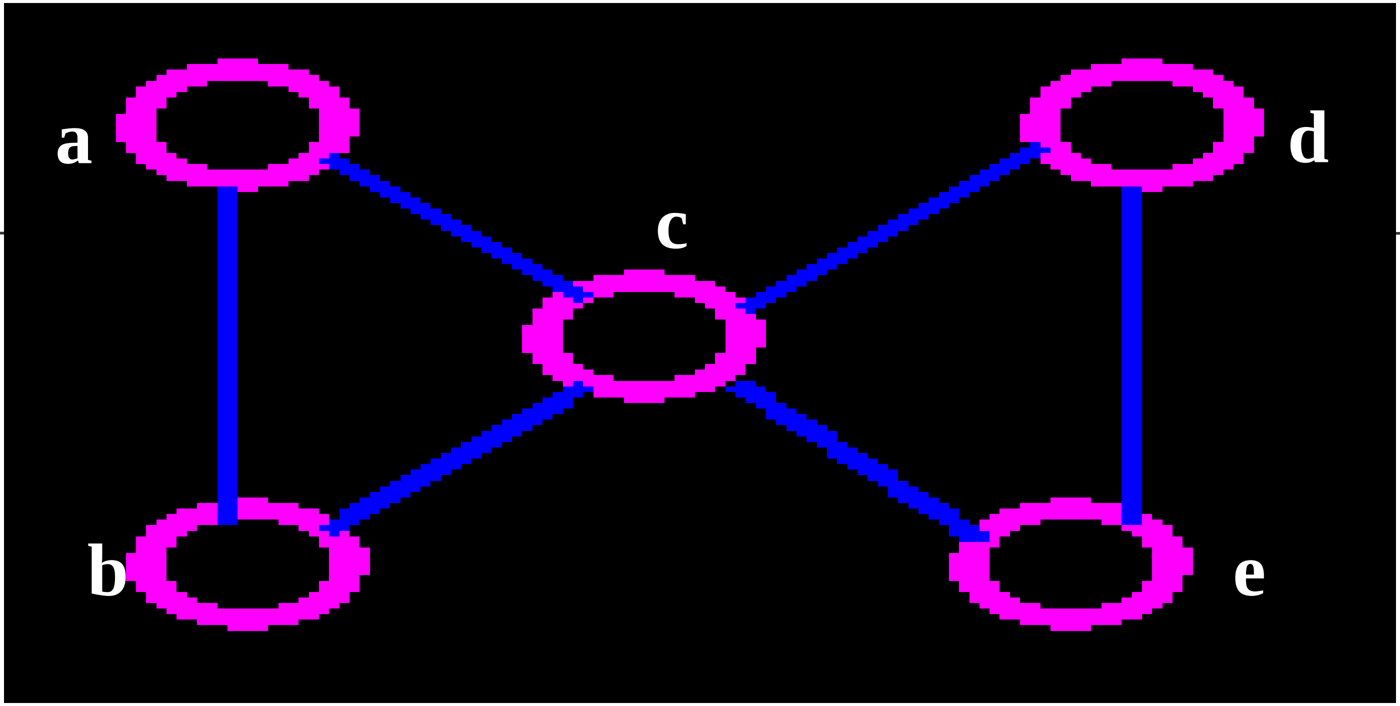
λ(*G1*) =1



The above graph G2 can be disconnected by removing a single edge, cd

G2 has edge-connectivity 1.

$\lambda(G2) = 1$



The above graph G3 cannot be disconnected by removing a single edge, but the removal of two edges (such as ac and bc) disconnects it.

G3 has edge-connectivity 2.

$$\boldsymbol{\lambda(G3) = 2}$$

CUT SET

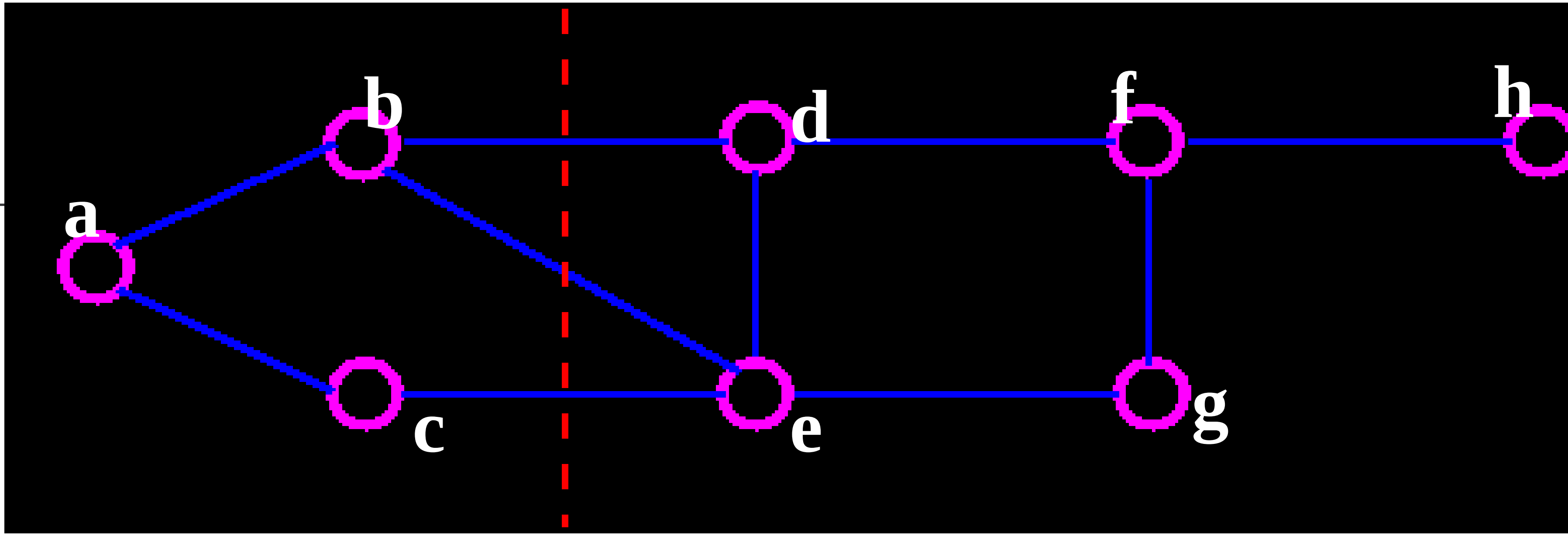
A **cut set** of a connected graph *G* is a set S of edges with the following properties:

1) The removal of all edges in S disconnects *G*.
2) The removal of some (but not all) of edges in S does not disconnects *G*

Cut set is also known as **edge-cut**

-OR-

An **edge- cut** S of G consists of edges so that $G - S$ is disconnected.



We can disconnect *G* by removing the three edges **bd**, **be**, and **ce**, but we cannot disconnect it by removing just two of these edges.

So,{ *bd, be, ce* }is a cut set of the graph G **.**

{df,eg} is another cut set

{fh} is another cut set

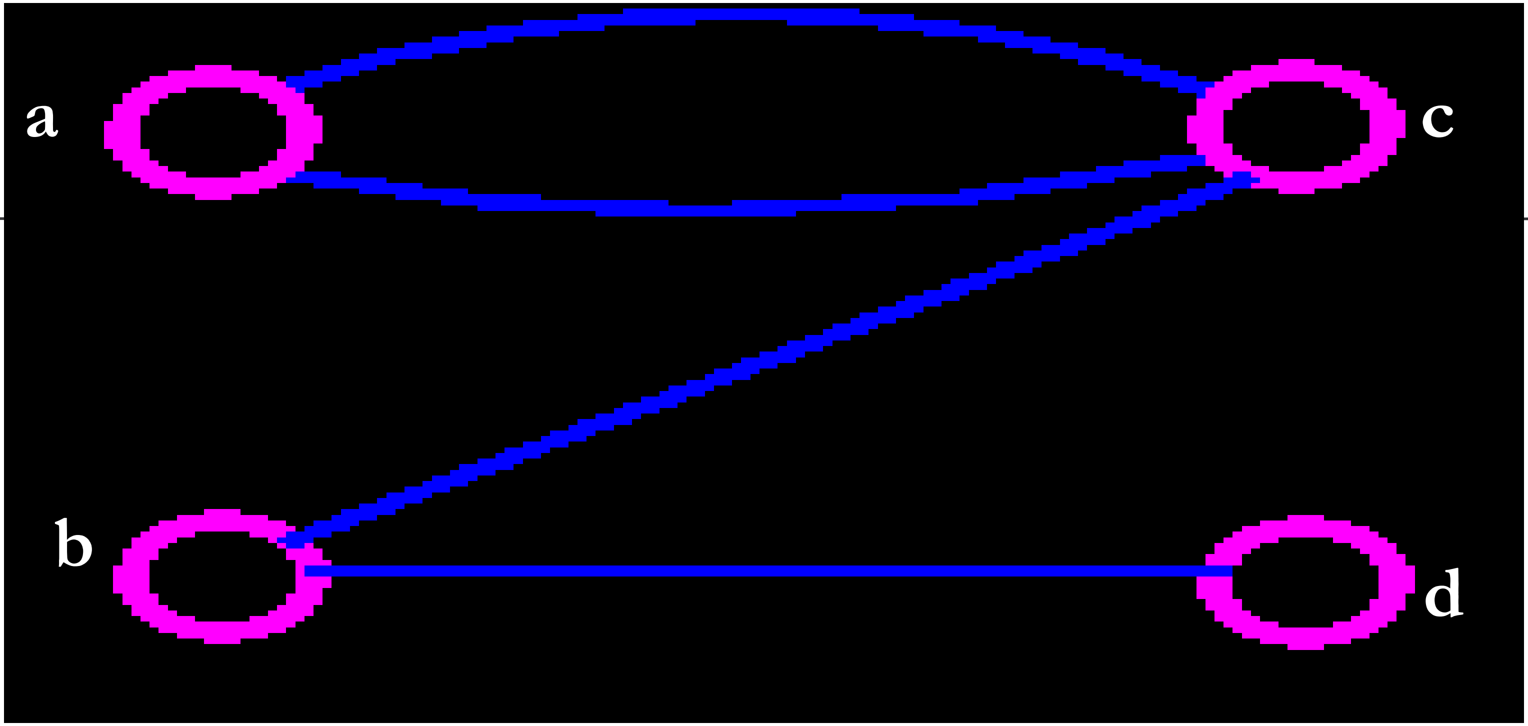
BRIDGE

Or **cut-edge** (*not edge-cut*)

is an edge-cut consisting of a single edge.

Or

A **bridge** is a single edge whose removal disconnects a graph



The above graph G can be split up into two components by removing one of the edges *bc* or *bd*. Therefore, edge bc or bd is a bridge

RELATIONSHIP BETWEEN CUT-SET & EDGE CONNECTIVITY

$$\boldsymbol{\lambda(G) = \min\{k \mid k = |S|,\ G - S \text{ disconnected},\ S \subseteq E_G\}}$$

Where

S is the cut set of graph G(V,E)

THEOREM

“If S is a cut set of the connected graph G, then G − S has two components”

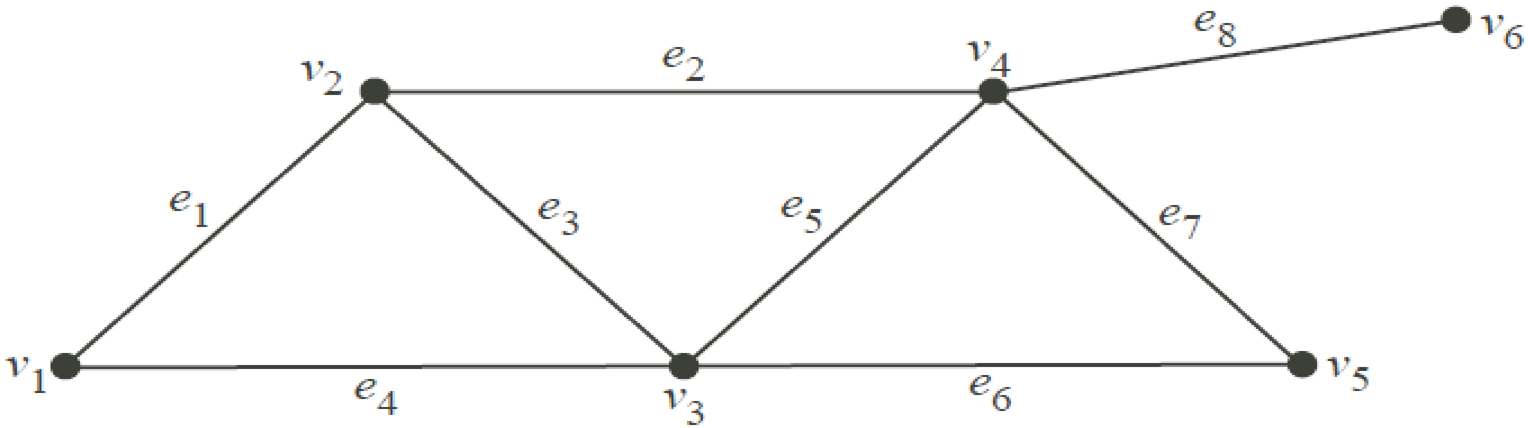
Proof.

Let $S = \{e1, \ldots, ek\}$. The graph $G-\{e1, \ldots, ek-1\}$ is connected (and so is G if $k = 1$) by condition #2.

When we remove the edges from the connected graph, we get at most two

components.

EXAMPLE



{e1, e4}, {e6, e7}, {e1, e2, e3}, {e8}, {e3, e4, e5, e6}, {e2, e5, e7}, {e2, e5, e6} *and* {e2, e3, e4} *are cut sets*.

PROPOSITION

Let G be a graph. Then the edge connectivity $\lambda(G)$ is less than or equal to the minimum vertex degree $\delta_{min}(G)$.

Proof:

Let ***v*** be a vertex of graph ***G***, with degree $k = \delta_{min}(G)$. Then the deletion of the ***k*** edges that are incident on vertex ***v*** separates ***v*** from the other vertices of ***G***

THEOREM

An edge **e** is a bridge of **G** iff **e** lies on no cycle on **G**

Proof:

By definition- A **bridge** is a single edge whose removal disconnects a graph. If it lies on a cycle, its removal will not disconnect the graph

VERTEX CONNECTIVITY

DEFINITION

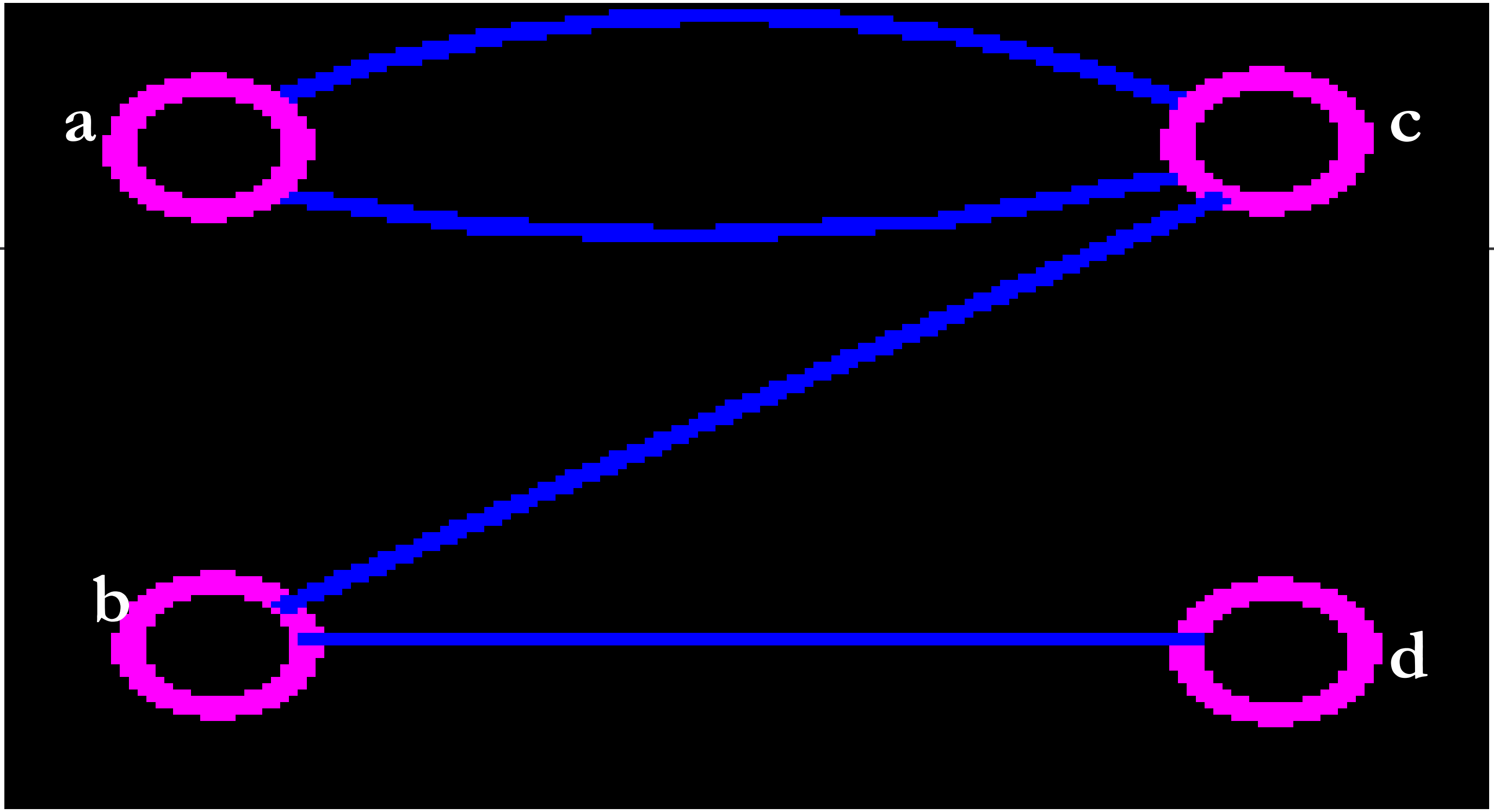
$$\boldsymbol{k(G) = \min\{k \mid k = |S|,\ G - S\ \text{disconnected},\ S \subseteq VG\}}$$

$$0 \leq \kappa(G) \leq n\text{-}1$$

Where n is the number of vertices of the graph.

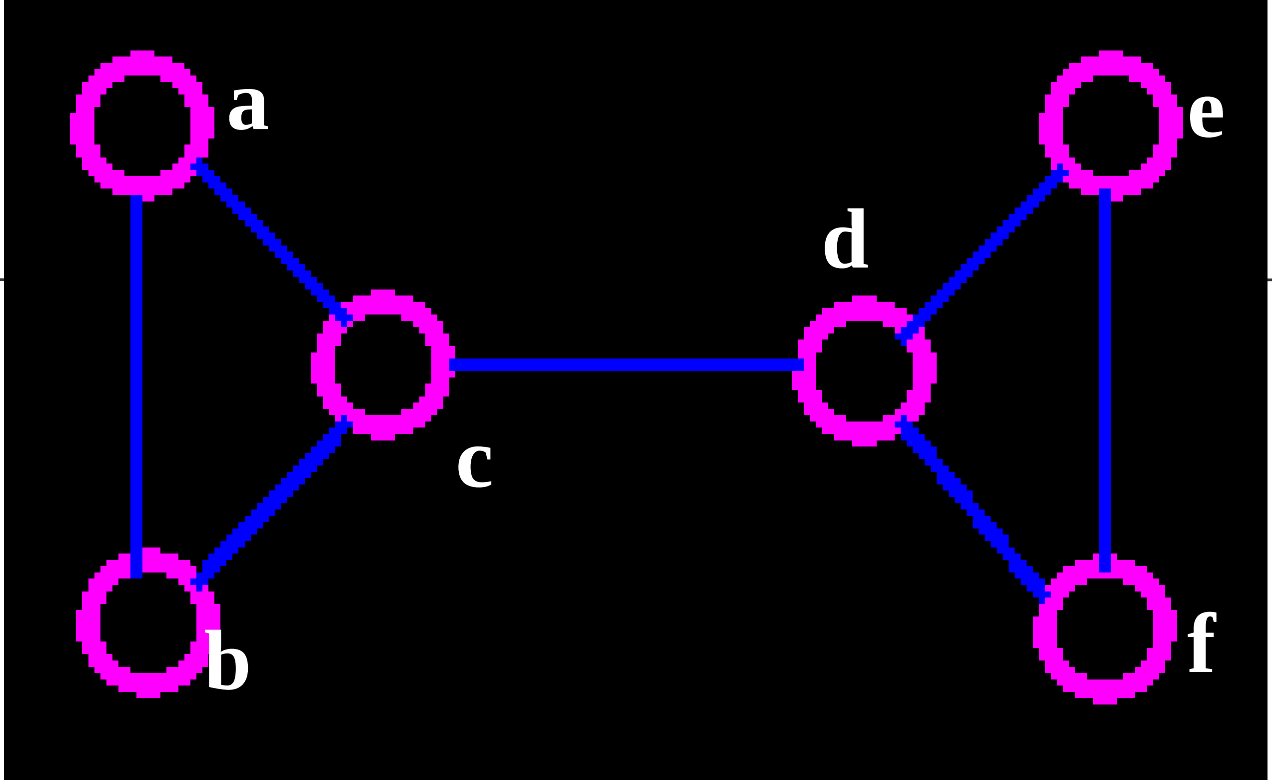
The **connectivity** (or vertex connectivity) **K**(G) of a connected graph G is the minimum number of vertices whose removal disconnects G.

When **K**(G) $\geq k$, the graph is said to be ***k*-connected** (or k-vertex connected).



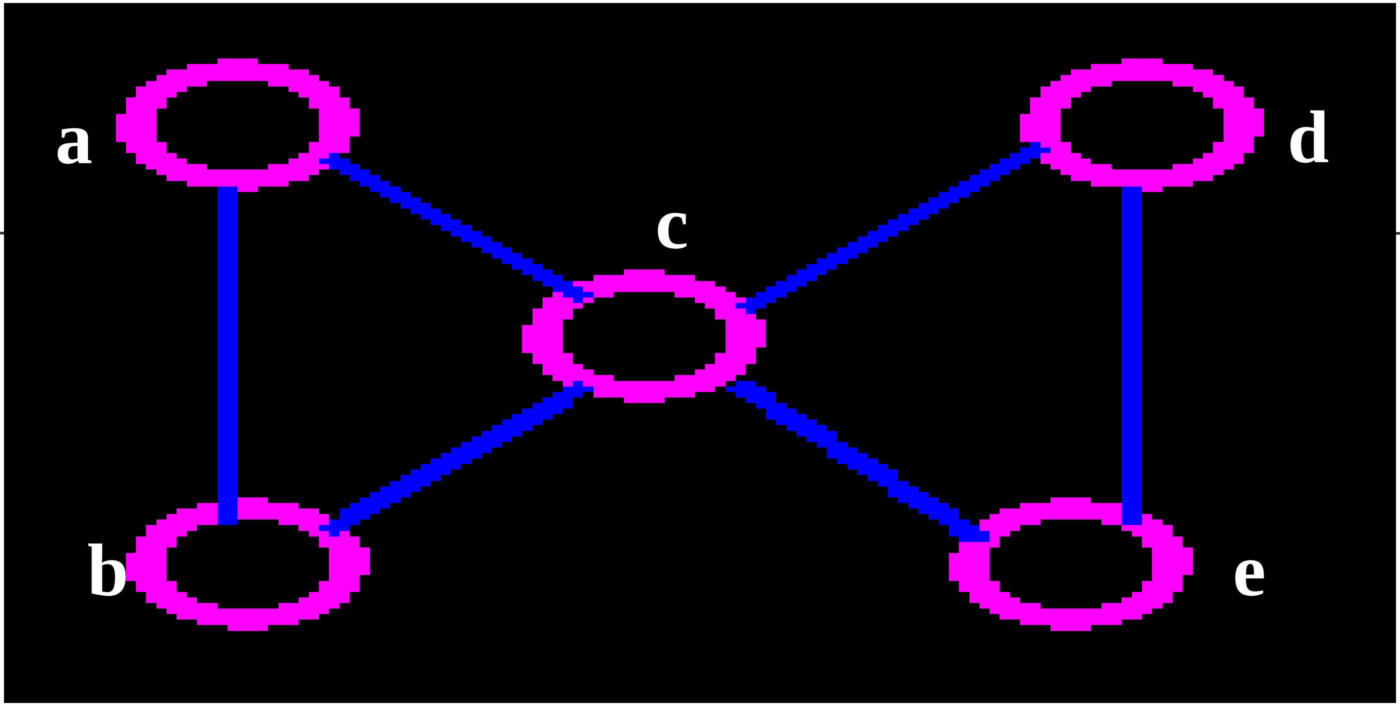
The above graph G can be disconnected by removal of single vertex (either b or c). The G has connectivity 1. That is

G is 1-connected or simply *connected*.

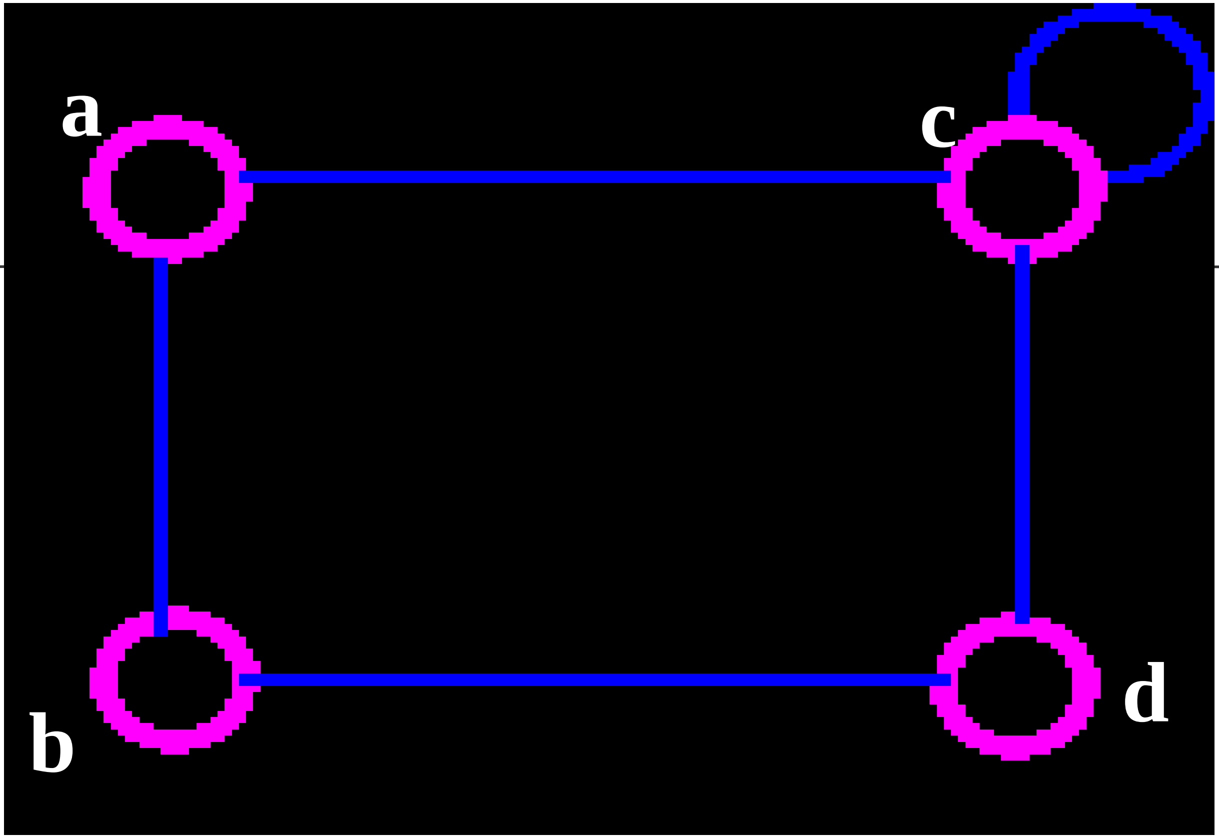


The above graph G can be disconnected by removal of single vertex (either c or d).

The G has connectivity 1



The above G can be disconnected by removing just one vertex i.e., vertex c. The G has connectivity 1



The above G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The G has connectivity 2

VERTEX CUT SET

A vertex-cut set of a connected graph G is a set S of vertices with the following properties.

1) the removal of all the vertices in S disconnects G.
2) the removal of some (but not all) of vertices in S does not disconnects G.

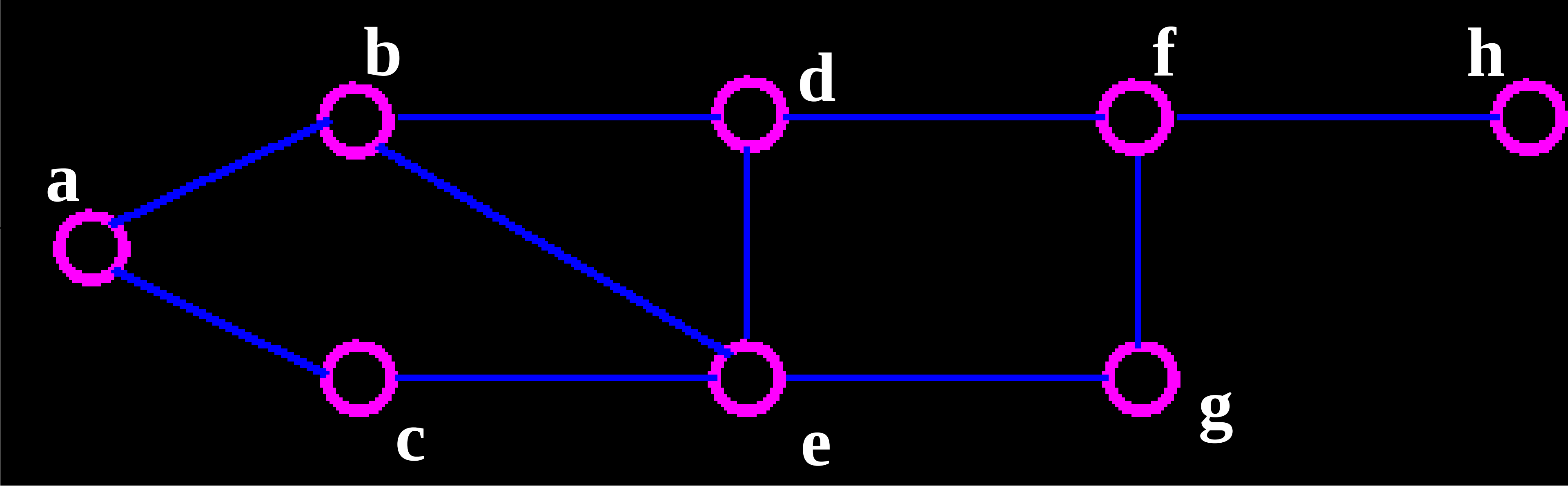
-OR-

A **vertex-cut** in a graph G is a vertex set S such that G-S is disconnected.

Also known as **separating set.**

We also say that S **separates** the vertices u and v and it is a

(u, v)**-separating set**, if u and v belong to different connected components of G−S.



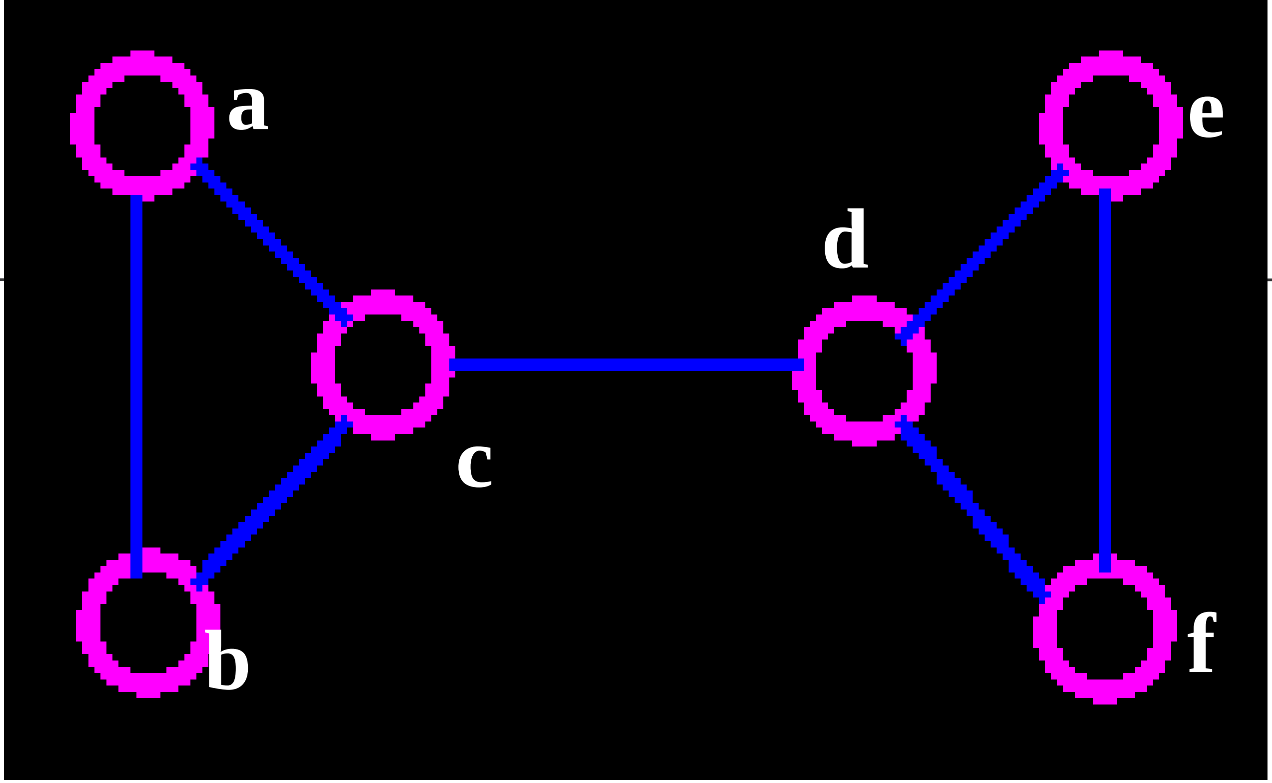
We can disconnects the graph by removing the two vertices b and e, but we cannot disconnect it by removing just one of these vertices.

The vertex-cut set of G is {b, e}.

CUT-VERTEX

A **cut-vertex** (*not vertex-cut*) is a single vertex whose removal disconnects a graph.

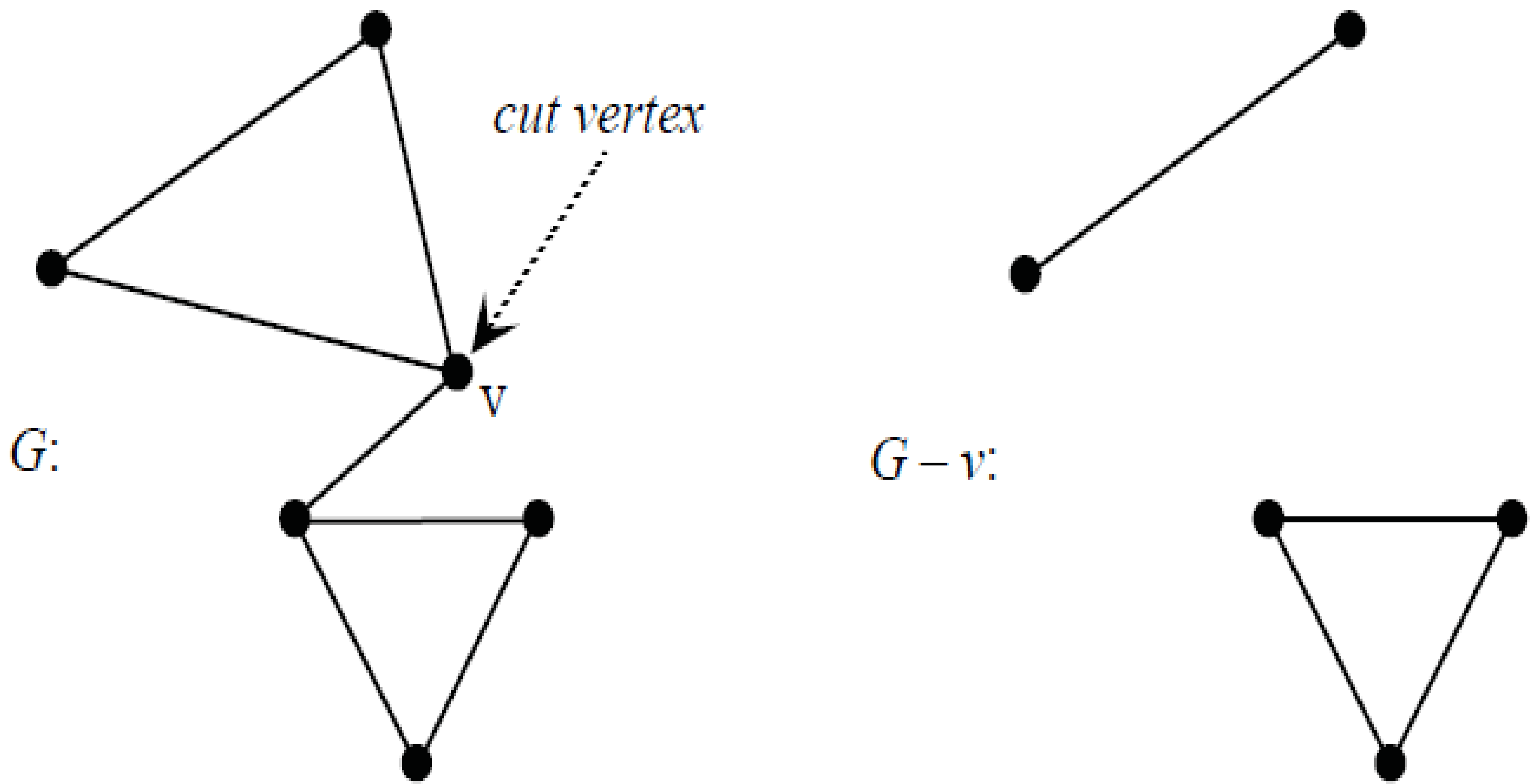
Cut vertex is also known as **cut point**



The above graph G can be disconnected by removal of single vertex (either c or d).

The vertex **c** or **d** is a cut-vertex

[redacted]



RELATIONSHIP BETWEEN VERTEX -CUT & CONNECTIVITY

The **(vertex) connectivity number** $k(G)$ of G is defined as

$$\boldsymbol{k(G) = \min\{k \mid k = |S|, G - S \text{ disconnected } S \subseteq V_G\}}$$

.

A graph G is k-**connected**, if $k(G) \geq k$.

In other words,

- $k(G) = 0$, if G is disconnected,
- $k(G) = V_G - 1$, if G is a complete graph, and
- otherwise $k(G)$ equals the minimum size of a vertex cut of G.

THEOREM

The vertex **v** *is a cut vertex of the connected graph* **G** *if and only if there exist two vertices* **u** *and* **w** *in the graph* **G** *such that*

(i) $v \neq u$, $v \neq w$ *and* $u \neq w$, *but*

(ii) v *is on every* u–w *path.*

PROOF

First, let us consider the case that **v** is a cut-vertex of G. Then, **G − v** is not connected

and there are at least two components

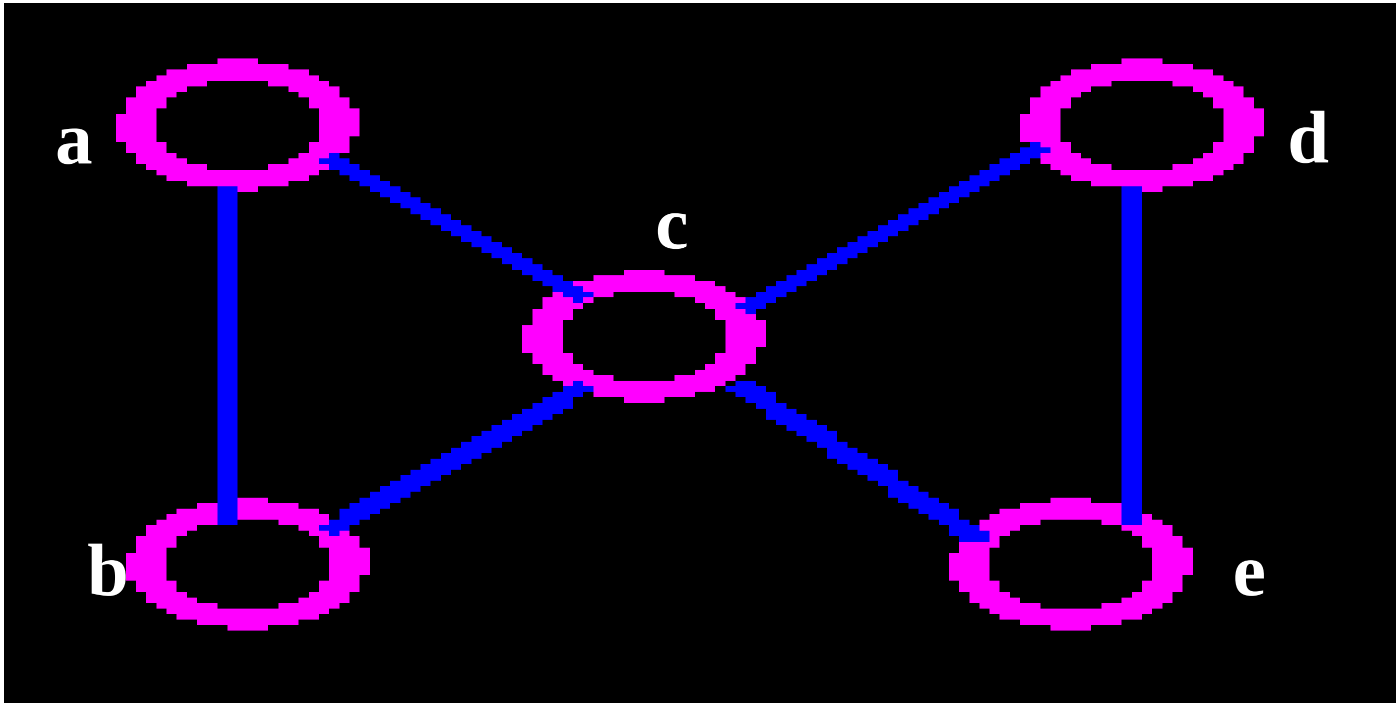
G1 = (V1,E1) and G2 = (V2,E2). We choose **u ∈ V1**

and **w ∈ V2**. The **u–w** path is in G because it is connected. If **v** is not on this path, then the path

is also in **G − v** . The same reasoning can be used for all the **u–w** paths in G.

If **v** is in every **u–w** path, then the vertices **u** and **w** are not connected in **G − v**.

EXAMPLE



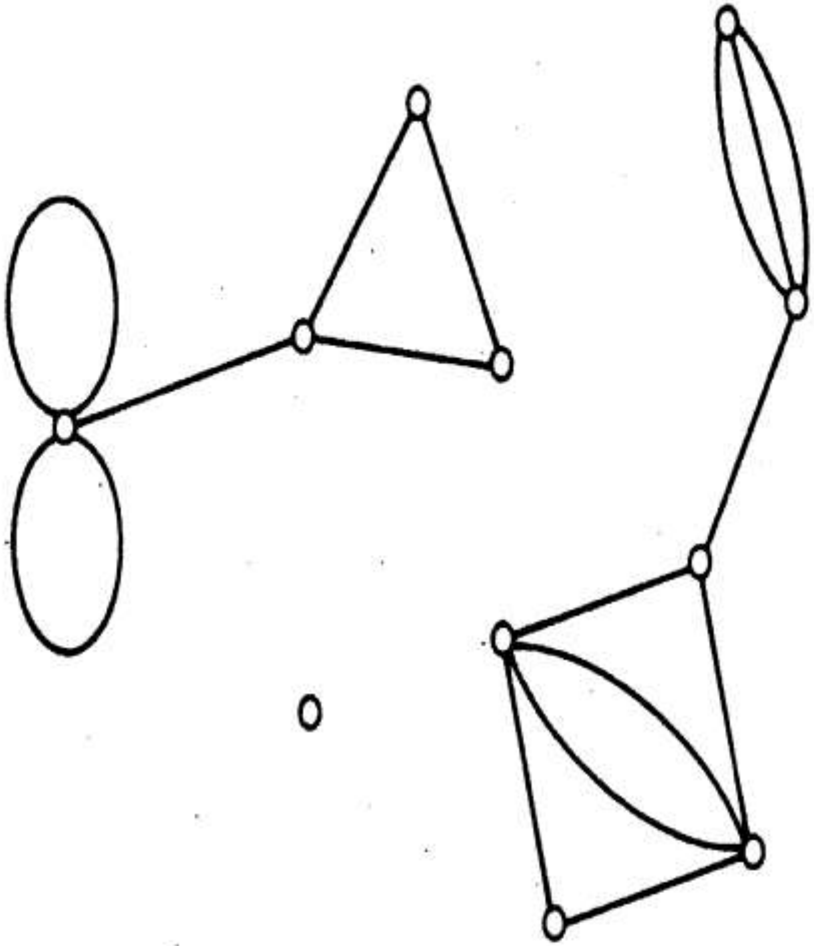
BLOCK

A block is a connected graph which has no cut vertices.

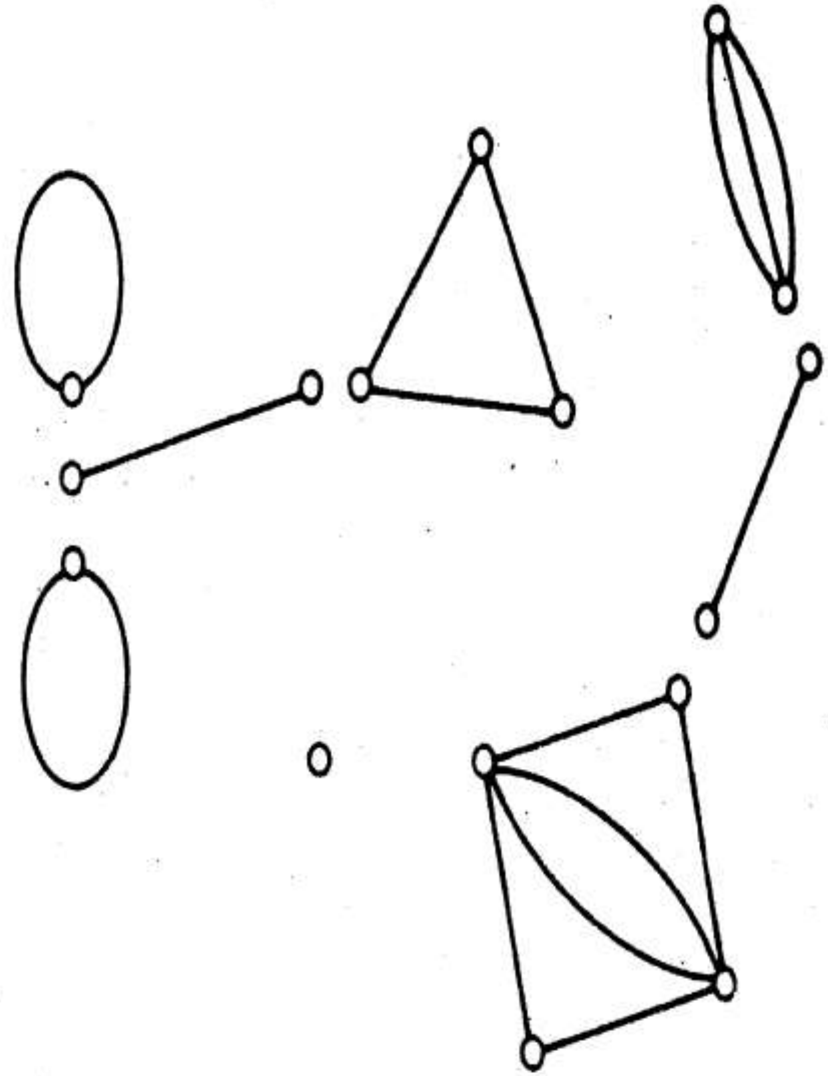
A block of a graph is a maximal sub graph with no cut vertices.

GRAPH

BLOCKS



(a)



(b)

MENGER'S THEOREM

Menger's theorem states that “If u and v are non-adjacent vertices in a graph G, then the maximum number of internally disjoint u-v paths equals the minimum number of vertices in a u-v separating set.”

PROOF BY INDUCTION

<u>Basis: </u>$m = 2$.

<u>Inductive step:</u>

Assume true for all graphs of size $\leq m$

- Let U be a minimum u-v separating set.
- Clearly, the number of u-v disjoint paths is at most $|U| = k$.

PROOF (CONT)

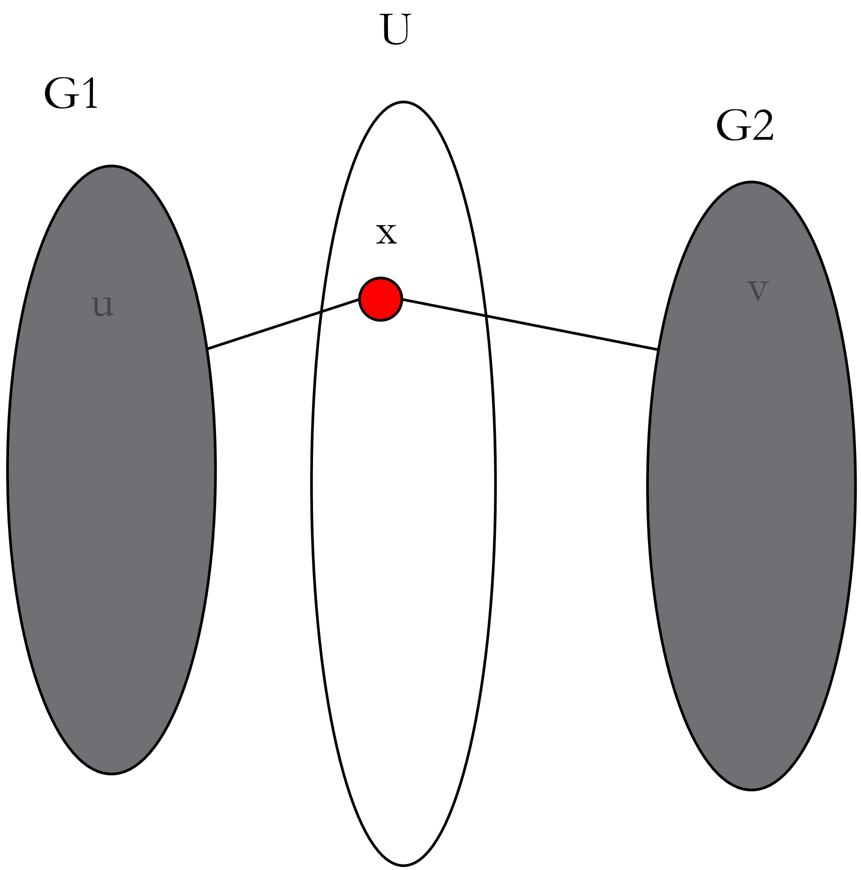
We look at all minimum u-v separating sets. There are three cases

Case 1: There is a u-v separating set U that contains a vertex that is adjacent to both u and v

Case 2: There is u-v separating set W with a vertex not adjacent to u and a vertex not adjacent to v

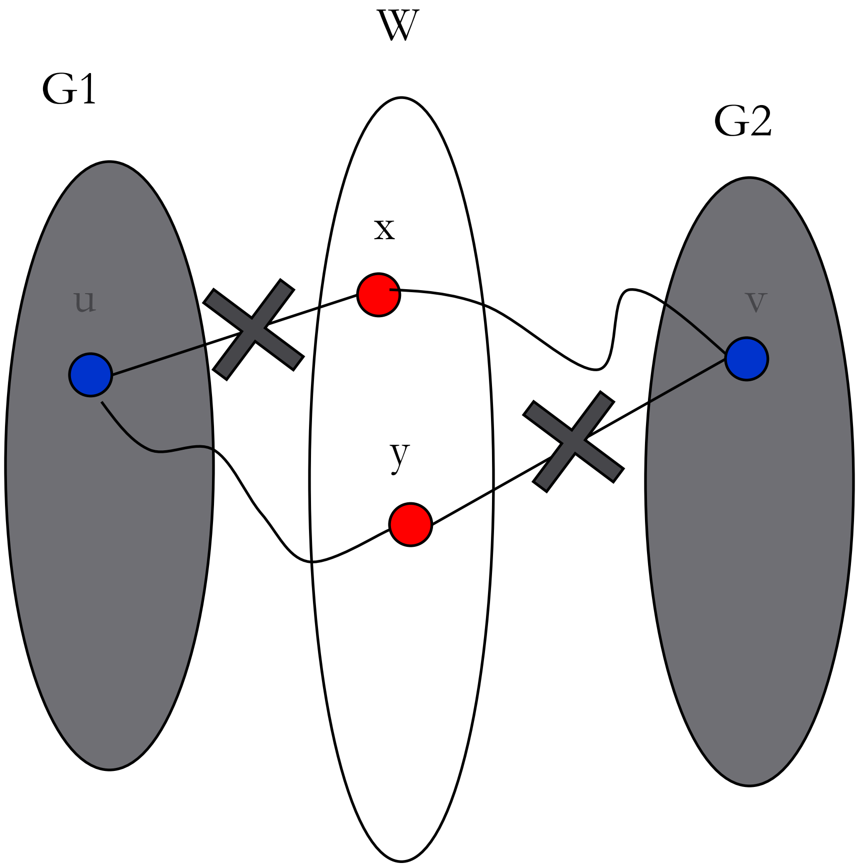
Case 3: For each min. u-v separating set S, either (every vertex in S is adjacent to u but not to v) or (every vertex in S is adjacent to v but not to u)

CASE 1



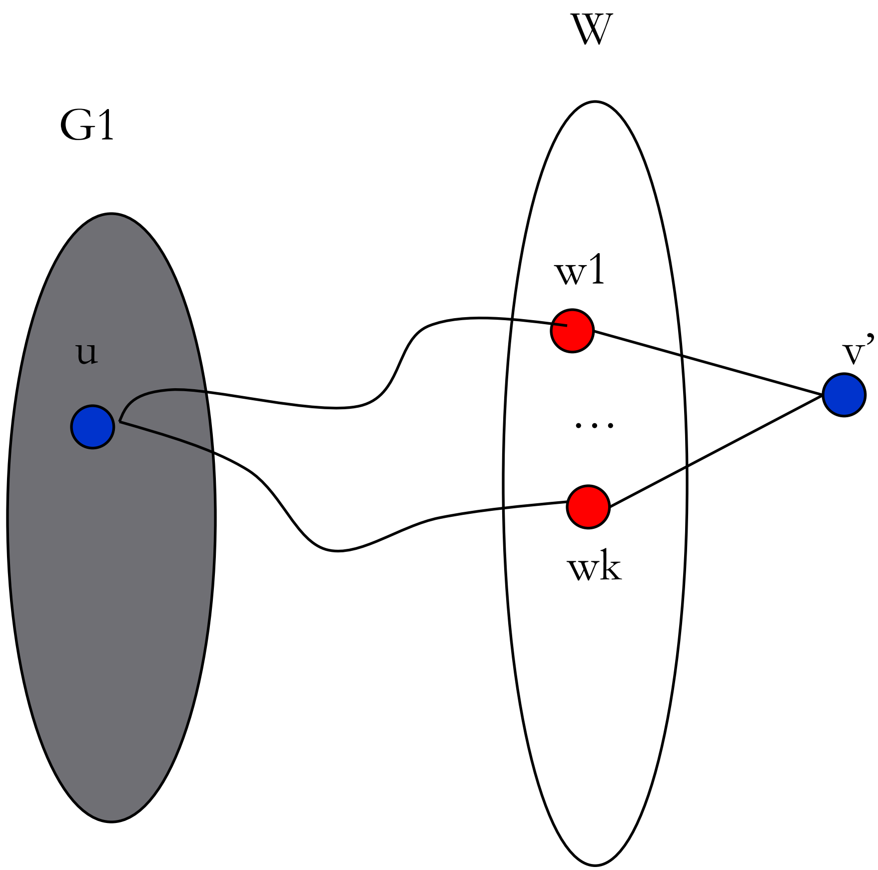
- Consider G-{x}:
  - It's size is less than m
  - U-{x} is a min. separating set for G-{x}.
- Since |U-{x}| = k -1, by the induction hypothesis, there are k-1 internally disjoint u-v paths in G-{x}
- So in G, we have these paths plus u-x-v
  - Done.

CASE 2



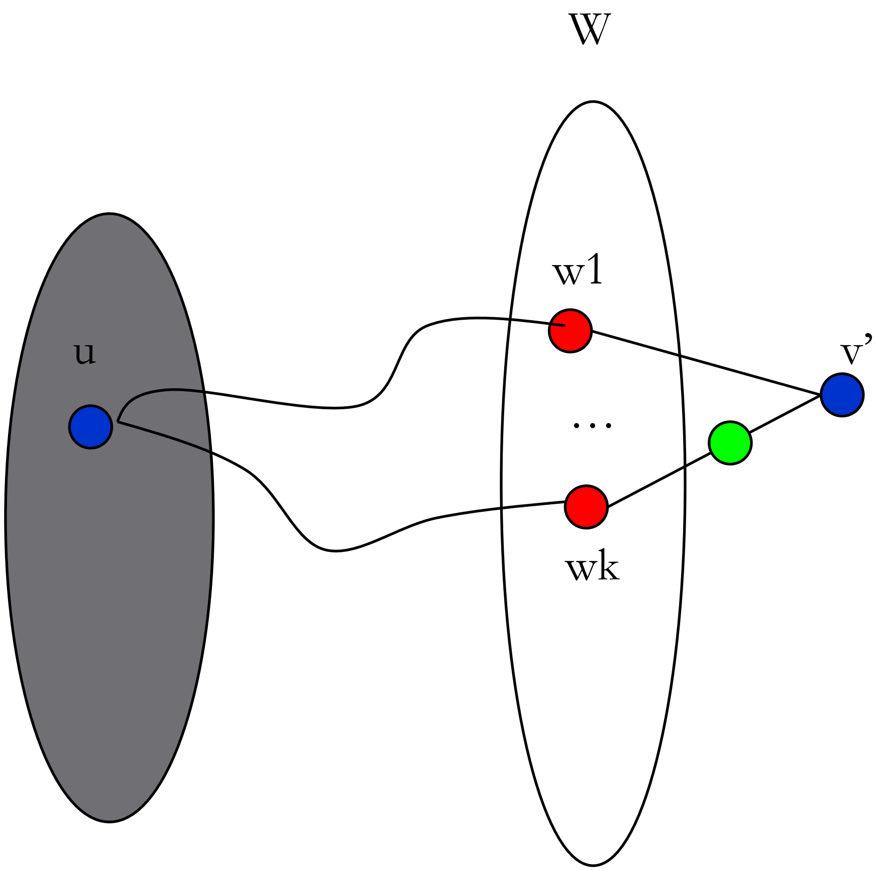
- Note: x and y can be the same vertex

CASE 2



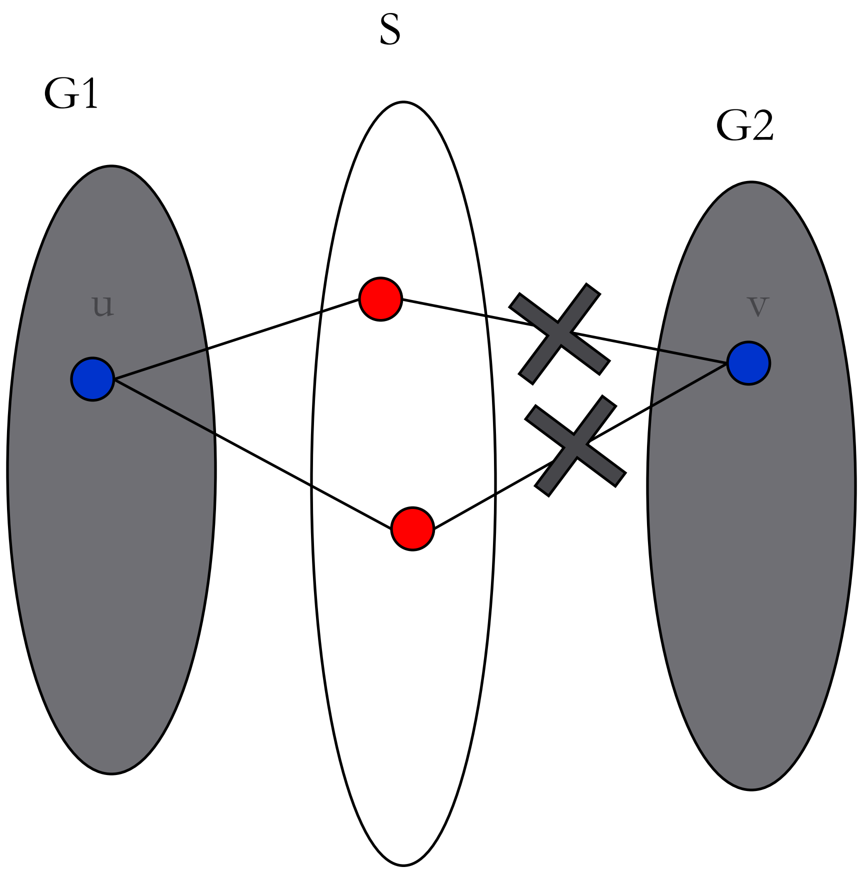
- $W = \{w1, \ldots, wk\}$
- First let's construct G(u) which contains all u-wi paths for all wi $\in$ W in G1 + W
- Make a new graph G'(u) by adding a new vertex v' to G(u) and connecting it to all wi
- Construct G(v) and G'(v) similarly

CASE 2



- size G’(u) < m
- W is a min u-v’ seperating set of size k.
- By the ind. hyp., there are k disjoint u-v’ paths.
- We take these paths and delete v’ from them. Call the resulting paths P1
- With similar reasoning, we conclude that G’(v) has k disjoint v-u’ paths. Generate paths P2 in a similar fashion.
- Combine P1 and P2 using the vertices wi.
- We obtained k internally disjoint paths for G

CASE 3



- We have either the situation on the left or the symmetric case (where v is connected to all in S)

CASE 3

- Let P = {u,x,y, … , v} be a u-v geodesic in G
- Let e = (x,y) and consider G-e
- Claim: The size k' of any minimum u-v separating set in G-e is also k.

- Clearly, k' $\geq$ k-1.
- Suppose, for contradiction, that k' = k-1 (i.e. the claim is false).
- Let Z be a min u-v separating set in G-e
- Z + {x} is a min u-v separating set in G
- So all vertices in Z are adjacent to u (we are in case 3)
- Z + {y} is a min u-v separating set in G
- So y is adjacent to v

CASE 3 (CONT)

- G-{e} has a min. u-v separating set of size k.
- By ind. hyp. It has k internally disjoint u-v paths.

- So does G!

[redacted]

The **edge-connectivity** version of Menger's theorem is as follows:

Let G be a finite undirected graph and x and y two distinct vertices. Then the theorem states that the size of the minimum edge cut for x and y is equal to the maximum number of pairwise edge-independent paths from x to y.

The **vertex-connectivity** statement of Menger's theorem is as follows:

Let G be a finite undirected graph and x and y two nonadjacent vertices. Then the theorem states that the size of the minimum vertex cut for x and y is equal to the maximum number of pairwise vertex-independent paths from x to y.

APPLICATION

<u>**Network survivability**</u>

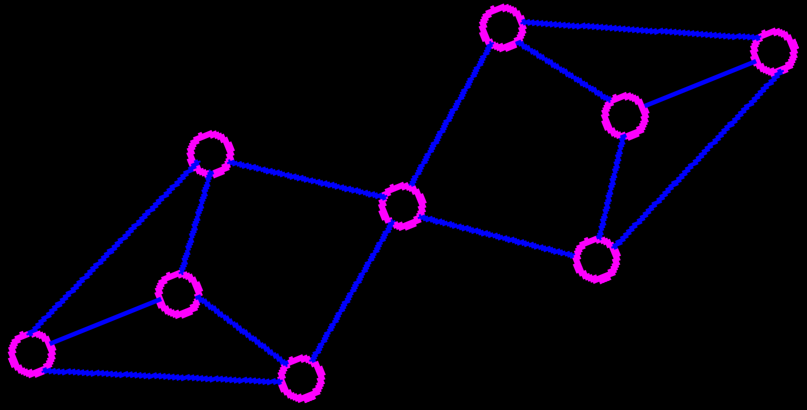
The connectivity measures **K**(G) and **λ(*G*)** are used in a quantified model of network survivability, which is the capacity of a network to retain connections among its nodes after some edges or nodes are removed

AND NOTE ONE MORE THING…

For every connected graph,

$$K(G) \leq \lambda(G) \leq \delta_{min}(G)$$

Thank you for your support….



SHAMEER P H
DEPT OF FUTURES STUDIES
KERALA UNIVERSITY
phshameer@gmail.com