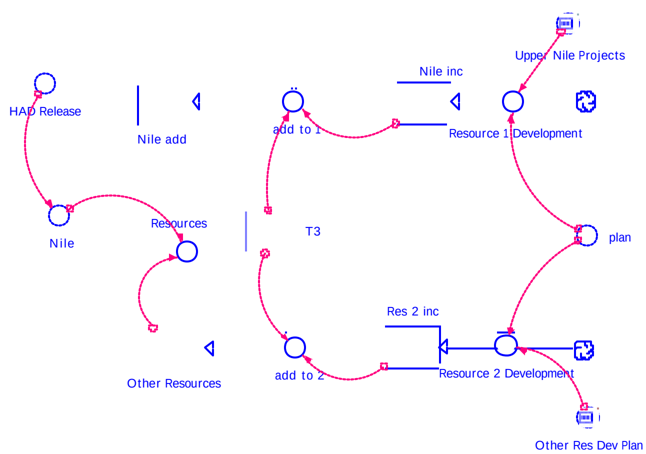
The University of Western Ontario
Department of Civil and Environmental Engineering

Civil Engineering Systems
CEE218b

Course notes



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London, December 2003

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PREFACE

This updated set of notes is prepared to serve as a supporting text for an introductory systems course for civil engineers. The text is not intended to be an independent document. Rather, it is to be used as a supplement to the instructor's lectures.

This text, and the course upon which it is based, is designed to provide the students with the basic concepts of systems analysis and design using simulation and optimization. The course is aimed at the second year Civil Engineering students.

During the past four decades, application of the systems approach to engineering systems design and management has been established as one of the most important advances made in the field of Civil Engineering. A primary emphasis of systems analysis is on providing an improved basis for decision making. It has been concluded that a gap still exists between research studies and the application of systems approach in practice. The objective of this course is to help in providing better communication links between different sub-disciplines of Civil Engineering and reduce the existing gap as much as possible.

While some theoretical aspects are addressed, the course focuses primarily on analytical topics: various problems in civil engineering are presented; the problems are formulated as systems; and techniques are introduced with which to solve the systems. In an effort to provide the students with a complete tutorial package that aids in the formulation of the model, in the selection of the most appropriate technique, and in the implementation of the algorithm, exposure to two software packages, for system simulation and optimization, is planned.

The general objectives of the course are for students to be able to:

- Use systems thinking in addressing engineering problems by understanding: system structure; links and interrelationships between different elements of the structure; feedback; and behaviour of systems over time.
- Understand and use the mathematical model as a device for formalization, standardization and treatment of engineering design and management problems.
- Improve communication skills by formulating, selecting appropriate solution methodology for, solving and presenting engineering design and management decisions based on the implementation of systems analysis.
- Develop an awareness of the potential utility of systems approach to Civil Engineering.
- Recognize the need for life-long learning, interdisciplinarity and use of the systems approach in Civil Engineering as one way for addressing complexity.

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1. INTRODUCTION

Human beings are quick problem solvers. From an evolutionary standpoint, quick problem solvers were the ones who survived. We quickly determine a cause for any event that we think is a problem. Usually we conclude that the cause is another event. For example, if sales are poor (the event that is a problem), then we may conclude that this is because the sales force is insufficiently motivated (the event that is the cause of the problem). This approach works well for simple problems, but it works less well as the problems get more complex.

1.1 Principles of Systems Thinking

The methods of systems thinking provide us with tools for better understanding these difficult management problems. The methods have been used for over thirty years (Forrester 1961 - 1990) and are now well established. However, these approaches require a shift in the way we think about the performance of an organization. In particular, they require that we move away from looking at isolated events and their causes (usually assumed to be some other events), and start to look at the organization as a system made up of interacting parts.

We use the term system to mean an interdependent group of items forming a unified pattern. Since our interest here is in engineering, we will focus on systems of people and technology intended to plan, design, construct and operate engineering infrastructure. Almost everything that goes on in engineering is part of one or more systems. As noted above, when we face a management problem we tend to assume that some external event caused it. With a systems approach, we take an alternative viewpoint - namely that the internal structure of the system is often more important than external events in generating the problem. This is illustrated by the diagram in Figure 1.1. Many people try to explain certain performance by showing how one set of events causes another or, when they study a problem in depth, by showing how a particular set of events is part of a longer term pattern of behavior. The difficulty with this “events causes events" orientation is that it doesn't lead to very powerful ways to alter the undesirable performance. You can continue this process almost forever, and thus it is difficult to determine what to do to improve performance.

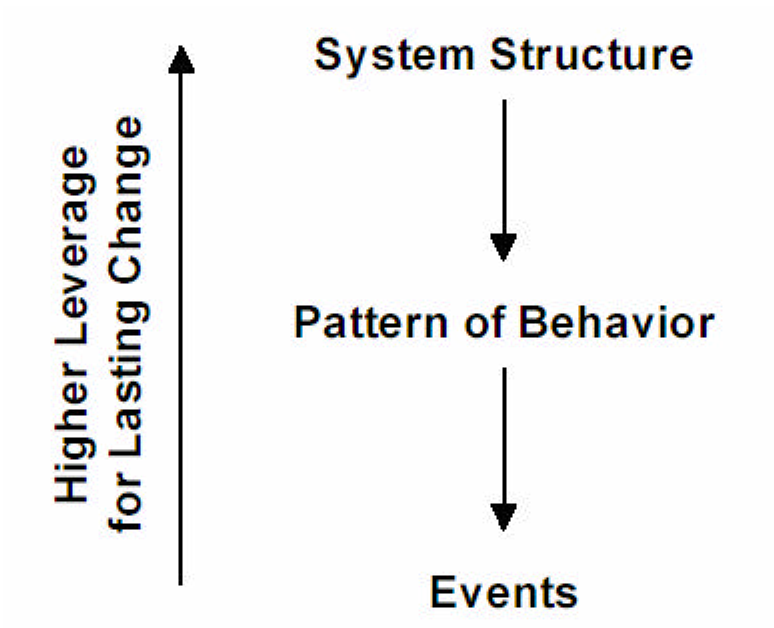


Figure 1.1 Looking for problem solution (high leverage)

If you shift from this event orientation to focusing on the internal system structure, you improve your possibility of finding the problem. This is because system structure is often the underlying source of the difficulty. Unless you correct system structure deficiencies, it is likely that the problem will resurface, or be replaced by an even more difficult problem.

Class exercise 1:

- *Automobile is a __ of components that work together to provide transportation.*
- *Autopilot is a __ for flying an airplane at a specific altitude.*
- *Loading platform is a __ for loading goods into trucks.*

- Management is a __ of people for allocating resources and regulating the activity of a business.

- Family is a __ for living and raising children.

1.2 Open and Feedback System

An **open system** is one characterized by outputs that respond to inputs but where the outputs are isolated from and have no influence on the inputs (Figure 1.2). An open system is not aware of its own performance. In an open system, past action does not control future action (Forester, 1990).

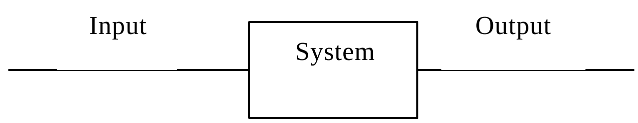


Figure 1.2 Graphical presentation of an open system

A **feedback system** (sometimes called a closed system) is influenced by its own past behavior. A feedback system has a closed loop structure that brings results from past action of the system back to control future action (Figure 1.3). One class of feedback system -*negative feedback*- seeks a goal and responds as a consequence of failing to achieve the goal. A second class of feedback system - *positive feedback*- generates growth process wherein action builds a result that generates still greater action.

A broad purpose may imply a feedback system having many components. But each component can itself be a feedback system in terms of some subordinate purpose. One must then recognize a hierarchy of feedback structures where the broadest purpose of interest determines the scope of the pertinent system. It is in the positive feedback form of system structure that one finds the forces of growth. It is in the negative feedback, or goal seeking, structure of systems that one finds the causes of fluctuation and instability.

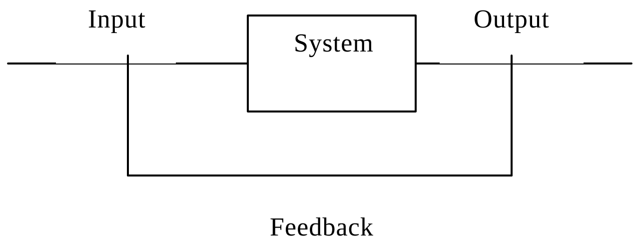


Figure 1.3 Graphical presentation of a feedback system

Another basic concept is the **feedback loop.** The feedback loop is a closed path connecting in sequence a decision that controls action, the level (a state or condition) of the system, and information about the level of the system (Figure 1.4). The single loop structure is the simplest form of feedback system. There may be additional delays and distortions appearing sequentially in the loop. There may be many loops that interconnect. When reading a feedback loop diagram, the main skill is to see the 'story' that the diagram tells: how the structure creates a particular pattern of behavior and how that pattern might be influenced.

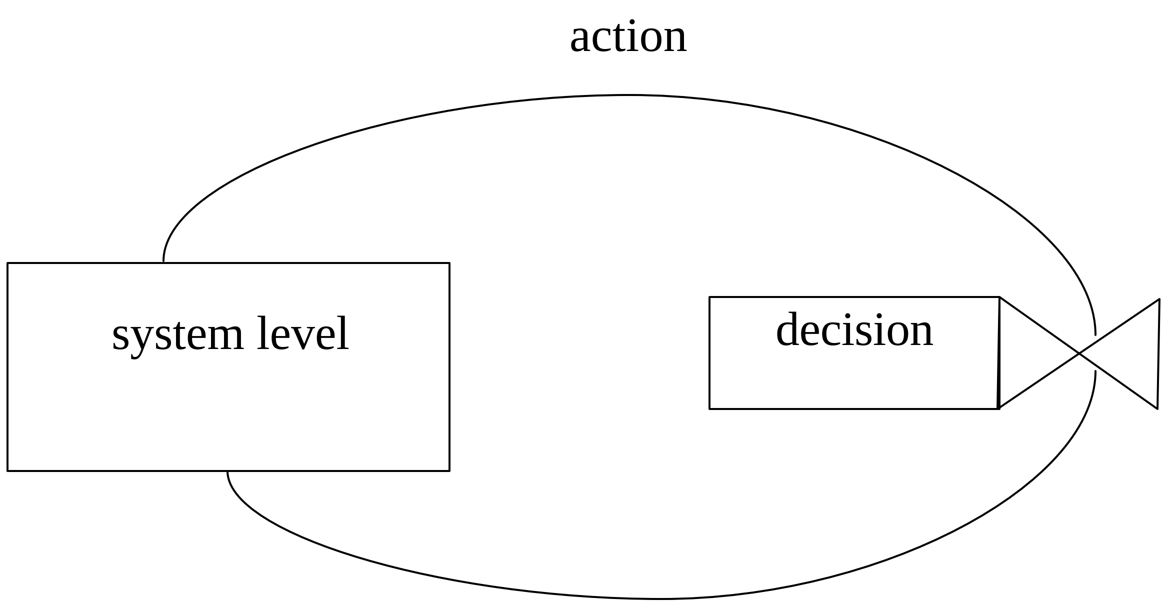


Figure 1.4 A feedback loop

Class exercise 2:

-The parts of a feedback system form a structure shaped as a (chain/loop)__.

-The 'action' represents the flow of something that is controlled by the __ .

-The __ alters the __ of the system.

- The __ of the system is the true condition of the system and is the source of information about the system.

-The recurring __ governing the release of water from the reservoir changes as our available __ about demand, inflow, and storage changes.

1.3 Patterns of Behavior

To start to consider system structure, you first generalize from the specific events associated with your problem to considering patterns of behavior that characterize the situation. Usually this requires that you investigate how one or more variables of interest change over time (for example: flow of water; load on the bridge; wind load; etc.). That is, what patterns of behavior do these variables display. The systems approach gains much of its power as a problem solving method from the fact that similar patterns of behavior show up in a variety of different situations, and the underlying system structures that cause these characteristic patterns are known. Thus, once you have identified a pattern of behavior that is a problem, you can look for the system structure that is know to cause that pattern. By finding and modifying this system structure, you have the possibility of permanently eliminating the problem pattern of behavior.

The four patterns of behavior shown in Figure 1.5 often show up, either individually or in combinations, in systems. In this figure, "Performance" refers to some variable of interest.

With exponential growth (Figure 1.5a), an initial quantity of something starts to grow, and the rate of growth increases. The term **exponential growth** comes from a mathematical model for this increasing growth process where the growth follows a particular functional form called the exponential.

With **goal-seeking behavior** (Figure 1.5b), the quantity of interest starts either above or below a goal level and over time moves toward the goal. Figure 1.5b shows two possible cases, one where the initial value of the quantity is above the goal, and one where the initial value is below the goal.

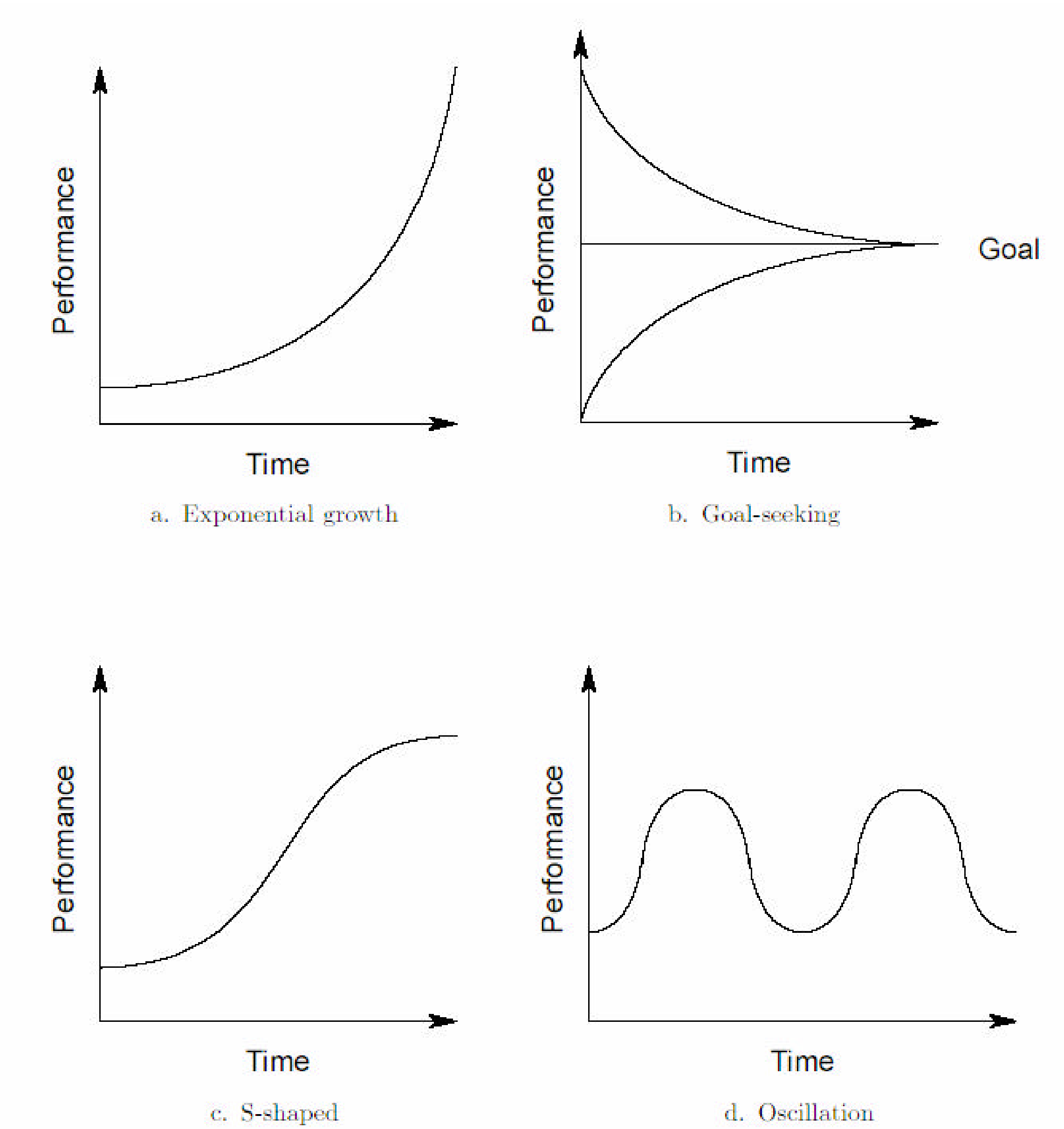


Figure 1.5 Patterns of system behavior

With **s-shaped growth** (Figure 1.5c), initial exponential growth is followed by goal-seeking behavior which results in the variable leveling of.

With **oscillation** (Figure 1.5d), the quantity of interest fluctuates around some level. Note that oscillation initially appears to be exponential growth, and then it appears to be s-shaped growth before reversing direction.

Common combinations of these four patterns include:

- Exponential growth combined with oscillation. With this pattern, the general trend is upward, but there can be declining portions, also.
- Goal-seeking behavior combined with an oscillation whose amplitude gradually declines over time. With this behavior, the quantity of interest will overshoot the goal on first one side and then the other. The amplitude of these overshoots declines until the quantity finally stabilizes at the goal.
- S-shaped growth combined with an oscillation whose amplitude gradually declines over time.

1.4 Causal Loop Diagrams

To better understand the system structures which cause the patterns of behavior discussed in the preceding section, we introduce a notation for representing system structures. When an element of a system indirectly influences itself in the way discussed in section 1.2 the portion of the system involved is called a **feedback loop** or a **causal loop**.

The essence of the discipline of systems thinking lies in a shift of mind:

- seeing interrelationships rather than linear cause-effect chains; and
- seeing processes of change rather than snapshots.

To illustrate the shift consider a very simple system - filling a glass of water. You can think that is not a system, it is too simple. From the linear point of view:

<u>I am filling a glass of water.</u>

However, as we fill the glass:

We are watching the water level.

We monitor the gap between the level and our goal, the desired water level.

We are adjusting the flow of water.

In fact, when we fill the glass we operate in a 'water regulation' system involving five variables: desired water level; current water level; the gap between the two; the faucet position, and the water flow. These variables are organized in a circle or loop of cause-effect relationships, know as a **feedback process**.

To complete our presentation of terminology for describing system structure, note that a linear chain of causes and effects which does not close back on itself is called an **open loop**. A map of the feedback structure of an engineering system, such as that shown in Figure 1.6, is a starting point for analyzing what is causing a particular pattern of behavior. This Figure is an **annotated causal loop diagram** for a simple process, filling a glass of water. This diagram includes elements and arrows (which are called causal links) linking these elements together, but it also includes a sign (either + or -) on each link. These signs have the following meanings:

1 A causal link from one element A to another element B is positive (that is, +) if either (a) A adds to B or (b) a change in A produces a change in B in the same direction.

2 A causal link from one element A to another element B is negative (that is, -) if either (a) A subtracts from B or (b) a change in A produces a change in B in the opposite direction.

Start from the element *Faucet Position* at the top of the diagram. If the faucet position is increased (that is, the faucet is opened further) then the *Water Flow* increases. Therefore, the sign on the link from *Faucet Position* to *Water Flow* is positive. Similarly, if the *Water Flow* increases, then the *Water Level* in the glass will increase. Therefore, the sign on the link between these two elements is positive.

The next element along the chain of causal influences is the *Gap* which is the difference between the *Desired Water Level* and the (actual) *Water Level*. (That is, Gap = Desired Water Level - Water Level.) From this definition, it follows that an increase in *Water Level* decreases *Gap*, and therefore the sign on the link between these two elements is negative. Finally, to close the causal loop back to *Faucet Position*, a greater value for *Gap* presumably leads to an increase in *Faucet Position* (as you attempt to fill the glass) and therefore the sign on the link between these two elements is positive. There is one additional link in this diagram, from *Desired Water Level* to *Gap*. From the definition of *Gap* given above, the in influence is in the same direction along this link, and therefore the sign on the link is positive.

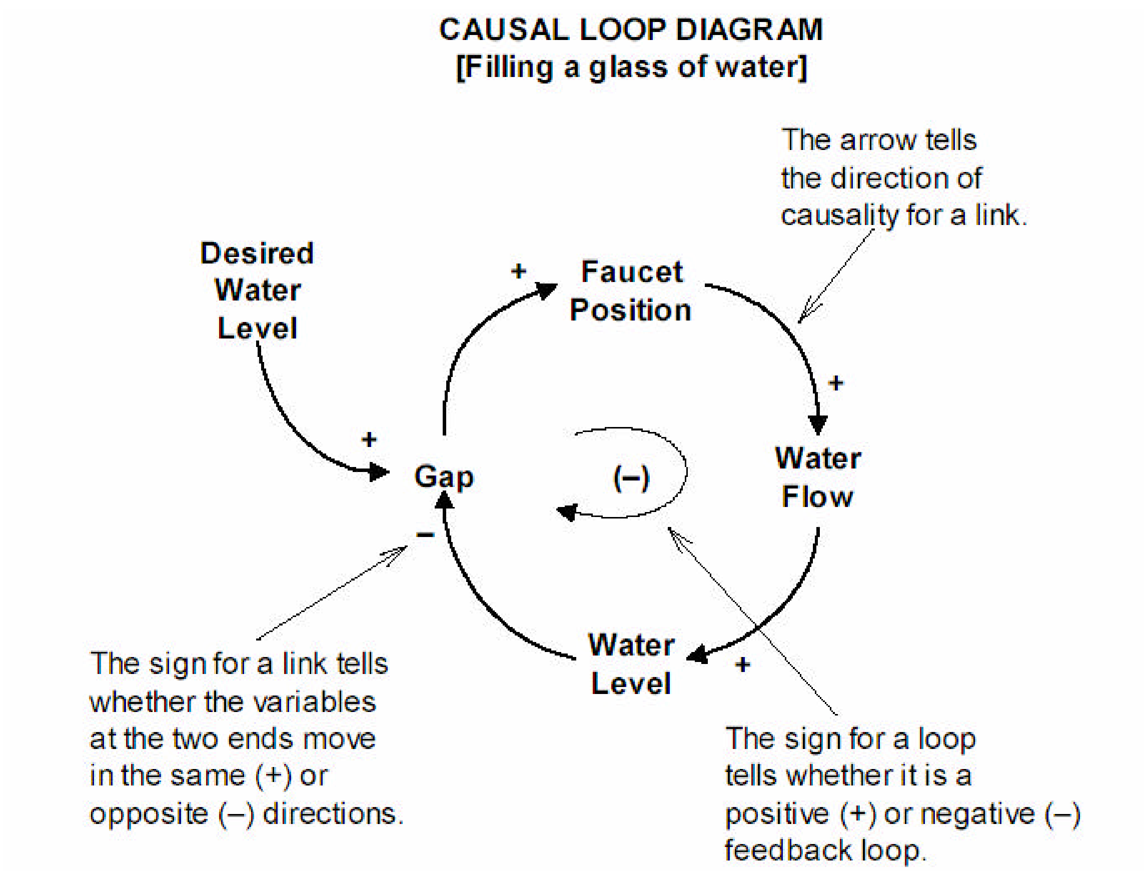


Figure 1.6 Causal loop diagram for 'filling a glass of water'

In addition to the signs on each link, a complete loop also is given a sign. The sign for a particular loop is determined by counting the number of minus (-) signs on all the links that make up the loop. Specifically, (i) A feedback loop is called **positive**, indicated by a + sign in parentheses, if it contains an even number of negative causal links; and (ii) A feedback loop is called **negative,** indicated by a sign in parentheses, if it contains an odd number of negative causal links.

Thus, the sign of a loop is the algebraic product of the signs of its links. Often a small looping arrow is drawn around the feedback loop sign to more clearly indicate that the sign refers to the loop, as is done in Figure 1.6. Note that in this diagram there is a single feedback (causal) loop, and that this loop has one negative sign on its links. Since one is an odd number, the entire loop is negative.

1.5 Positive – Reinforcing Feedback Loop

A positive, or reinforcing, feedback loop reinforces change with even more change. This can lead to rapid growth at an ever-increasing rate. This type of growth pattern is often referred to as exponential growth. Note that in the early stages of the growth, it seems to be slow, but then it speeds up. Thus, the nature of the growth in an engineering system that has a positive feedback loop can be deceptive. If you are in the early stages of an exponential growth process, something that is going to be a major problem can seem minor because it is growing slowly. By the time the growth speeds up, it may be too late to solve whatever problem this growth is creating. Examples that some people believe fit this category include pollution and population growth. Figure 1.7 shows a well know example of a positive feedback loop: Growth of a bank balance when interest is left to accumulate.

Sometimes positive feedback loops are called vicious or virtuous cycles, depending on the nature of the change that is occurring. Other terms used to describe this type of behavior include bandwagon effects or snowballing.

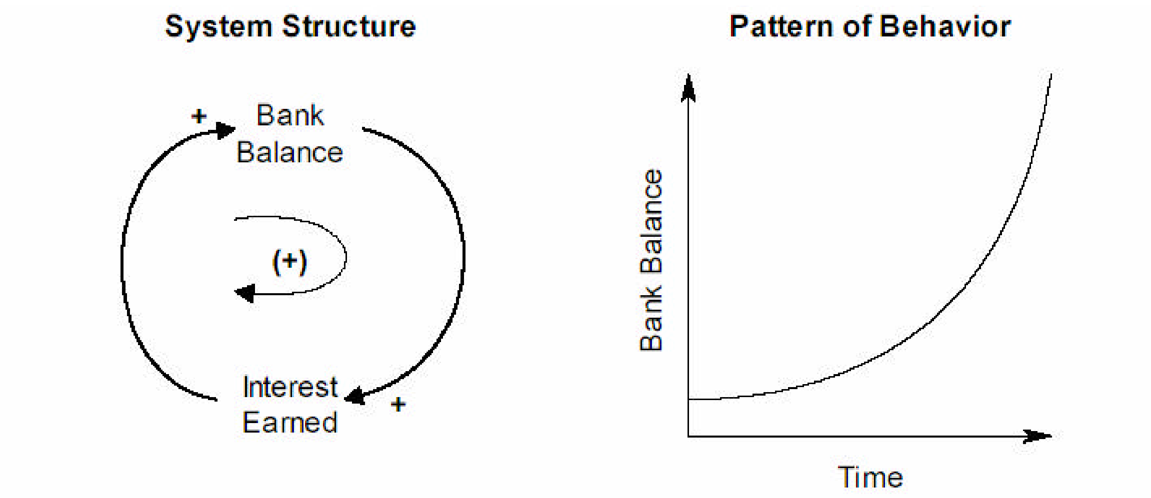


Figure 1.7 Positive (reinforcing) feedback loop: Growth of bank balance

1.6 Negative – Balancing Feedback Loop

A negative, or balancing, feedback loop seeks a goal. If the current level of the variable of interest is above the goal, then the loop structure pushes its value down, while if the current level is below the goal, the loop structure pushes its value up. Many engineering processes contain negative feedback loops which provide useful stability, but which can also resist needed changes. In the face of an external environment which dictates that an organization needs to change, it continues on with similar behavior. Figure 1.7 shows a negative feedback loop diagram for the regulation of a room temperature.

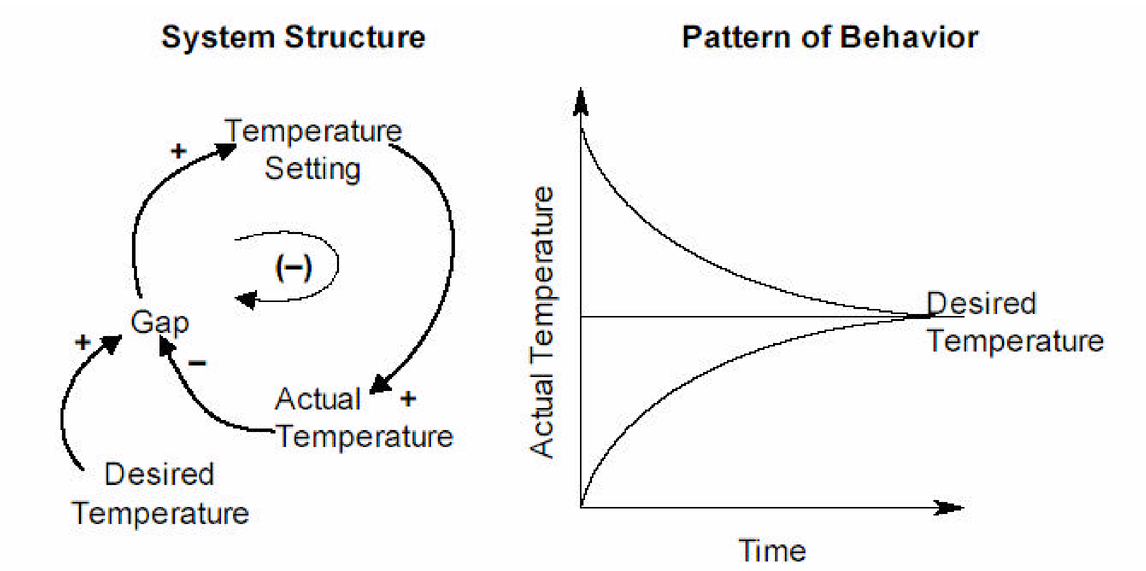


Figure 1.7 Negative (balancing) feedback loop: Room temperature control

1.7 Complex System Behavior

When positive and negative loops are combined, a variety of patterns are possible. Figure 1.8 illustrates four typical system behavior patters.

CURVE A. Typical feedback system in which variable rises at a decreasing rate toward a final value (here a value of 3). The curve might represent information that gives the apparent level of a system as understanding increases toward the true value. Or the curve might represent the way the water is release from a reservoir into an irrigation canal. The change toward the final value is more rapid at first and approaches more and more slowly as the discrepancy decreases between present and final value.

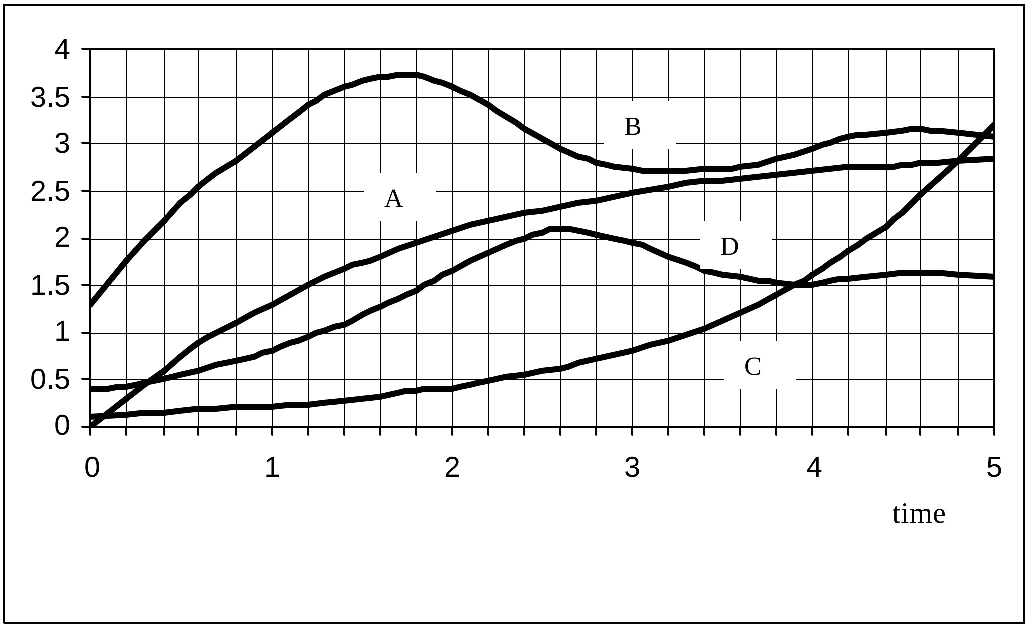


Figure 1.8 Dynamic behavior

CURVE B. It represents a more complicated approach to the final value where the system overshoots the final value, then falls below in trying to recover from the earlier overshoot. Such behavior can result from excessive time delays in the feedback loop or from too violent an effort to correct a discrepancy between apparent system level and the system goal (shower water temperature adjustment).

CURVE C. It shows growth where there is, in each succeeding time interval, the same fractional increase in the variable. Such growth is seen in cell division, in the sales growth of a product, etc.

CURVE D. It shows an initial section of exponential growth followed by a leveling out. It is a composite of an early section similar to curve C that leads to a later section having the characteristics of curve B. This kind of growth can be seen with a product that stagnates because market demand has been satisfied, or because production capacity has been reached, or because quality may have declined.

Class exercise 3:

-Which curve in the Figure 1.8 would best describe:

1. How the temperature of a thermometer changes with time after it is immersed in a hot liquid? __.

2. The position of a pendulum, which is displaced and allowed to swing, is best described by curve __ .

3. Which curve describes the learning process? __

4. Industrialization: capital equipment used to produce more capital equipment. Which curve describes the amount of capital equipment versus time? __

Class exercise 4:

Negative feedback loop-Irrigation canal intake (Figure 1.9): As the float drops it turns the switch on which opens the weir to admit the water. The rising water level causes the float to rise and this in turn gradually shuts off the weir.

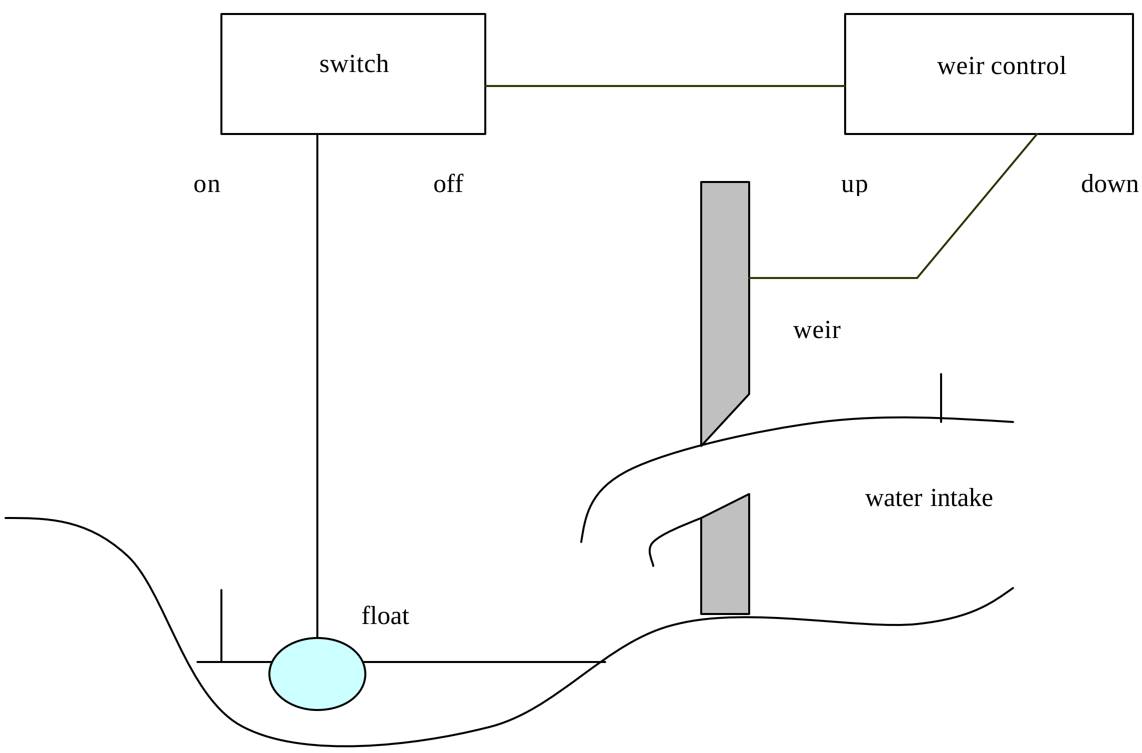


Figure 1.9 Irrigation canal intake

1. The water controls the water level which controls the weir opening, this is a __ system.

2. Sketch and label the flow diagram of the water intake system using symbols from the feedback loop graph.

3. Modify the flow diagram by removing float and introducing manual control of the weir.

4. The figure from question 3 shows an __ system.

5. FR is set at 0.1 m^3/minute. In ten minutes there will be __ of water in the canal.

6. The units of measure must always accompany any numerical value to define the quantity. The units of measure of water flow rate are __ / __ . The water volume is measured in __ .

7. On the following figure, plot the water flow rate of 0.1 m^3/minute.

8. If the FR = 0.04 m^3/minute and if the canal is empty at time 0, the amount of water is

__ at 10 min

__ at 20 min

__ at 40 min

__ at 60 min.

9. Plot the amount of water in the canal for every point in time.

10. On the same graph, plot the water-time relationship showing the volume of water in the canal for a flow rate of 0.2, 0.1 and 0.02 m^3/min.

11. On the following graph use two vertical scales. Using the flow rate scale show FR=0.05 m^3/min. Using the volume scale show the volume of water (m^3) in the canal at all points in time.

12. Using weir with float control (graph in problem 2) suppose that the flow rate is 0.2 m^3/min when the canal is empty and declines proportionally to zero when the canal contains 4 m^3 .

13. Express FR as an equation.

14. Find T when W=0.

15. Replace T in the equation and find the FR when W=2.5m^3. Check the value using the graph from question 12.

1.8 Creating Causal Loop Diagrams

To start drawing a causal loop diagram, decide which events are of interest in developing a better understanding of system structure. From these events, move to showing (perhaps only qualitatively) the pattern of behavior over time for the quantities of interest. Finally, once the pattern of behavior is determined, use the concepts of positive and negative feedback loops, with their associated generic patterns of behavior, to begin constructing a causal loop diagram which will explain the observed pattern of behavior.

The following tutorial for drawing causal loop diagrams are based on guidelines by Forester (1990) and Senge (1990)

1. Think of the elements in a causal loop diagram as variables which can go up or down, but don't worry if you cannot readily think of existing measuring scales for these variables.

- Use nouns or noun phrases to represent the elements, rather than verbs. That is, the actions in a causal loop diagram are represented by the links (arrows), and not by the elements.
- Be sure that the definition of an element makes it clear which direction is *up* for the variable.
- Generally it is clearer if you use an element name for which the positive sense is preferable.
- Causal links should imply a direction of causation, and not simply a time sequence. That is, a positive link from element A to element B does not mean first A occurs and then B occurs. Rather it means, when A increases then B increases.

2. As you construct links in your diagram, think about possible unexpected side effects which might occur in addition to the in influences you are drawing. As you identify these, decide whether links should be added to represent these side effects.

3. For negative feedback loops, there is a goal. It is usually clearer if this goal is explicitly shown along with the *gap* that is driving the loop toward the goal. This is illustrated by the examples in the preceding section on regulating room temperature.

4. A difference between actual and perceived states of a process can often be important in explaining patterns of behavior. Thus, it may be important to include causal loop elements for both the actual value of a variable and the perceived value. In many cases, there is a lag (delay) before the actual state is perceived. For example, when there is a change in concrete quality, it usually takes a while before we perceive this change.

5. There are often differences between short term and long term consequences of actions, and these may need to be distinguished with different loops.

6. If a link between two elements needs a lot of explaining, you probably need to add intermediate elements between the two existing elements that will more clearly specify what is happening.

7. Keep the diagram as simple as possible, subject to the earlier points. The purpose of the diagram is not to describe every detail of the management process, but to show those aspects of the feedback structure which lead to the observed pattern of behavior.

1.9 Problems

1.1 Fill in your answer:

A governor and the engine to which it is coupled form a __ to deliver power at constant engine speed.

Thermostat is a __ system that reacts to the temperature that is being controlled.

The parts of a feedback system form a structure shaped as a (chain/loop) __.

The __ arrangement of a __ system brings the result of past actions back to guide present decisions.

The information about the system changes as a result of past __ governing __ that have changed the levels of the system of which we are a part.

1.2 Using a water level of 0.8 m^3, what is the flow rate from equation FR=1/20 (2-WL)?_ _

How much water will be added during the next four seconds? __

How much water is in the tank after the first 8 seconds? __

Using a table calculate flow rate and water level every 4 seconds. Complete the table for 1 min and plot the curves of water level and flow rate as your calculation *progresses (it is important for learning that you actually do these calculations and the plotting-pay attention to the way in which the variables are changing and why).*

1.3 Test are you a system thinker: A good systems thinker, particularly in an organizational setting, is someone who can see four levels operating simultaneously: events, patterns of behavior, systems and mental models.

Group exercises:

STEP 1 The problem is ...

(The issue should be important to you and your team. Purpose is to lay the groundwork for a systems understanding of your own situation. Choose a chronic problem. Choose a problem limited in scope. Choose a problem whose history is known, and which you can describe. Make sure your description of the problem is as accurate as possible. Do not jump to the conclusions. Be nonjudgemental.)

STEP 2. Telling the story. (This process is known as model building. During this phase, you develop a theory or hypothesis that makes sense and could explain why the system is generating the problems you see. Do not take classical problem solving, precisely defining a problem statement. Try to keep in mind the following question: How did we (through our processes and our practices) contribute to or create the circumstances (good or bad) we face now?

Option A: Make the list. (Identify the key factors that seem likely to capture the problem or are critical to telling the story)

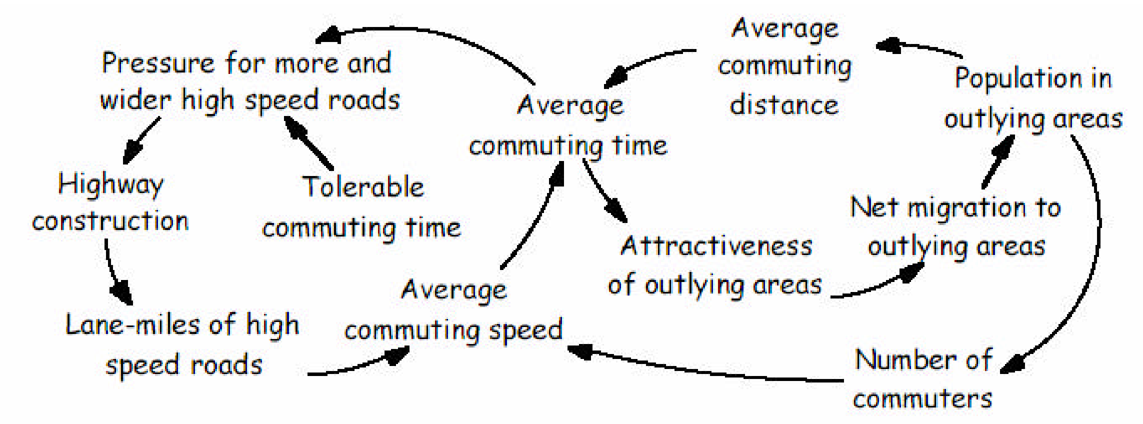
Option B: Draw a picture. (Draw the most important graph about your problem and add a few words.

The five WHYS. First why- pick the symptom where you wish to start. Write it down with plenty of room around.

The successive whys. Repeat the process for every statement you wrote.

1.4 In question below (i) assign polarities to each of the arrows; (ii) assign polarities to each of the feedback loops; (iii) write a brief but insightful paragraph describing the role of the feedback loops in your diagram. [Don't describe every link in your diagram -- assume your figure and its polarities take care of that; talk mainly about the loops.]

Feedback loops in highway construction and traffic density. After assigning polarities, consider what the left-hand loop would do by itself? What do the right-hand loops contribute?



1.5 *Present a loop of your own (from your own thinking or from book of some sort). Tell enough in words so that your picture and the story it tries to tell are clear.*

1.10 References

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2. SYSTEMS ANALYSIS

Systems Analysis is the use of rigorous methods to help determine preferred plans and designs for complex, often large-scale systems. It combines knowledge of the available analytic tools, understanding of when each is more appropriate, and skill in applying them to practical problems. It is both mathematical and intuitive, as is all planning and design (de Neufville, 1990).

Systems Analysis is a relatively new field. Its development parallels that of the computer, the computational power of which enables us to analyse complex relationships, involving many variables, at reasonable cost. Most of its techniques depend on the use of the computer for practical applications. Systems Analysis may be thought of as the set of computer-based methods essential for the planning of major projects. It is thus central to a modern engineering curriculum.

Systems Analysis covers much of the same material as operations research, in particular linear and dynamic programming and decision analysis. The two fields differ substantially in direction, however. Operations research tends to be interested in specific techniques and their mathematical properties. Systems Analysis focuses on the use of the methods.

Systems Analysis includes the topics of engineering economy, but goes far beyond them in depth of concept and scope of coverage. Now that both personal computers and efficient financial calculators are available, there is little need for professionals to spend much time on detailed calculations. It is more appropriate to understand the concepts and their relationship to the range of techniques available to deal with complex problems.

Systems Analysis emphasises the kinds of real problems to be solved; considers the relevant range of useful techniques, including many besides those of operations research; and concentrates on the guidance they can provide toward improving plans and designs.

Use of Systems Analysis instead of the more traditional set of tools generally leads to substantial improvements in design and reductions in cost. Gains of 30% are not uncommon. These translate into an enormous advantage when one is considering projects worth tens and hundreds of millions of dollars.

2.1 Definitions

There are many variations in the definition of what a system is, but all of the definitions share many common traits. Some kind of system is inherent in all but the most trivial Civil Engineering planning and design problems. To understand a problem, the engineer must be able to recognise and understand the system that surrounds and includes it. Some of the reasons for the poor system definition in the former projects include: poor communications, lack of knowledge of inter-relationships, politics, limited objectives, and transportation difficulties.

What, then, is a system? The dictionary definition of the term "system" is a mass of verbiage providing no less than 15 ways to define the word. In the most general sense a **system** may be defined as **a collection of various structural and non-structural elements which are connected and organised in such a way as to achieve some specific objective by the control and distribution of material resources, energy and information.**

A more formal definition of a system can be stated as:

$$S: \mathbf{X} \rightarrow \mathbf{Y} \tag{2.1}$$

where **X** is an input vector and **Y** is an output vector. So, system is a set of operations which transforms input vector **X** into output vector **Y.**

The usual representation of the system definition is presented in Figure 2.1. The system's objects are input, output, process, feedback, and a restriction. **Input** energises the operation of a given process. The final state of the process is known as the **output**. **Feedback** performs a number of operations to compare the actual output with an objective, and identifies the discrepancies that exist between them.

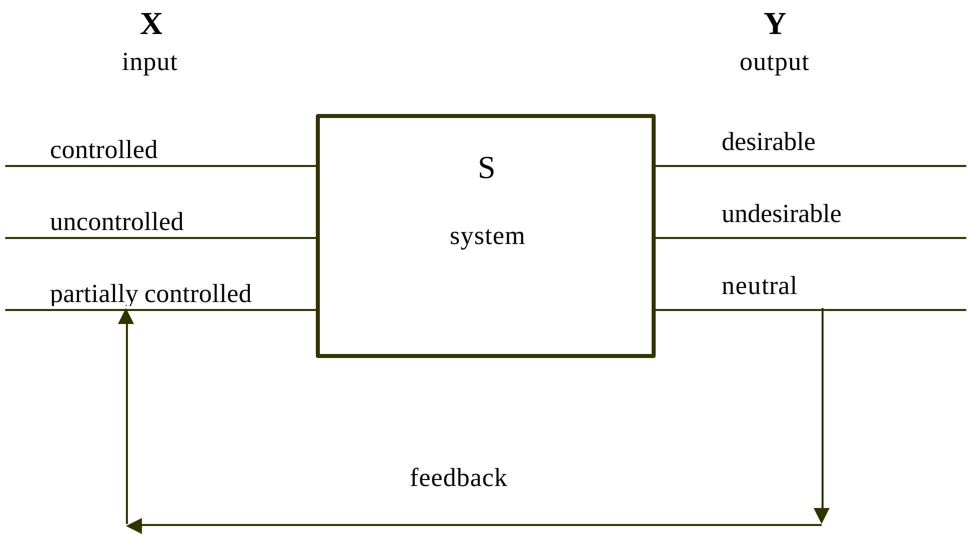

Figure 2.1 System presentation

To avoid any misunderstanding, let's define some terms often in use:

Mathematical model - set of equations that describes and represents the real system;

Decision variables - the controllable and partially controllable inputs;

Policy - resulting set of decision variables, when each decision variable is assigned a particular value;

Objective function - quantity used to measure the effectiveness of a particular policy, and is expressed as a function of the decision variables;

Constraints - physical, economic or any other restrictions applied to the model;

Feasible policy - a policy which does not violate any constraints;

Policy space - the subset consisting of all possible feasible policies.

Class exercise 1:

You are charged with the design of the UWO parking lot Z. Specify in words possible objectives, the unknown decision variables, and the constraints that must be met by any solution of the problem.

2.2 Systems approach

The systems approach is a general problem-solving technique that brings more objectivity to the planning/design processes. It is, in essence, good design; a logical and systematic approach to problem solution in which assumptions, goals, objectives, and criteria are clearly defined and specified. Emphasis is placed on relating system performance to these goals. A hierarchy of systems, which allows handling of a complex system by looking at its component parts or subsystems, is identified. Quantifiable and nonquantifiable aspects of the problem are identified, and immediate and long-range implications of suggested alternatives are evaluated.

The systems approach establishes the proper order of inquiry and helps in the selection of the best course of action that will accomplish a prescribed goal by broadening the information base of the decision maker; by providing a better understanding of the system and the inter-relatedness of the system and its component subsystems; and by facilitating the prediction of the consequences of several alternative courses of action.

The systems approach is a framework for analysis and decision making. It does not solve problems, but does allow the decision maker to undertake resolution of a problem in a logical, rational manner. While there is some art involved in the efficient application of the systems approach, other factors play equally important roles. The magnitude and complexity of decision processes require the most effective use possible of the scientific (quantitative) methods of systems analysis. However, one has to be careful not to rely too heavily on the methods of systems analysis. Outputs from simplified analyses have a tendency to take on a false validity because of their complexity and technical elegance.

The steps in the systems approach include (Jewell, 1986):

- Definition of the problem;
- Gathering of data;
- Development of criteria for evaluating alternatives;
- Formulation of alternatives;
- Evaluation of alternatives;
- Choosing the best alternative; and
- Final design/plan implementation.

Often several steps in the systems approach are considered simultaneously, facilitating feedback and allowing a natural progression in the problem-solving process. The systems approach has several defining characteristics. It is a repetitive process, with feedback allowed from any step to any previous step. Without this feedback ability, the systems approach is not being applied. Frequently, because systems analysis takes such a broad approach to problem solving, interdisciplinary teams must be called in. Coordination and commonality of technique among the disciplines is sometimes hard to achieve. However, if applied with ingenuity and flexibility, the systems approach can provide a common basis for understanding among specialists from seemingly unrelated fields and disciplines. Close communication among the parties involved in applying the systems approach is essential if this understanding is to be achieved.

Definition of the problem. Problem definition may require iteration and careful investigation, because problem symptoms may mask the true cause of the problem. A key step of problem definition requires the identification of any systems and subsystems that are part of the problem, or related in some way to it. This set of systems and interrelationships is called the environment of the problem. This environment sets the limit on factors that will be considered when analysing the problem. Any factors that cannot be included in the problem environment must be included as inputs to, or outputs from, the problem environment.

When defining the problem and its environment, the best approach is to make the definition as general as possible. The largest problem over which there is a reasonable chance of maintaining control should be the problem defined.

Gathering of Data. Gathering of data to assist in planning and design decision making through the systems approach will generally be done in conjunction with several steps. Some background data will have to be gathered at the problem definition stage and data gathering and analysis will continue through the final plan/design and implementation stage. Data that are gathered as the approach continues will help to identify when feedback to a previous step is required.

Data will be required at the problem definition stage to evaluate if a problem really exists; to establish what components, subsystems, and elements can be reasonably included in the delineation of the problem environment; and to define interactions between components and subsystems. Data will be needed during later steps to establish constraints on the problem and systems involved in it, to increase the set of quantifiable variables and parameters (constants) through statistical observation or development of measuring techniques, to suggest what mathematical models might contribute effectively to the analysis, to estimate values for coefficients and parameters used in any mathematical models of the system, and to check the validity of any estimated system outputs. When feedback is required, the data previously acquired can assist in redefining the problem, systems, or system models.

Development of evaluative criteria. Evaluative criteria must be developed to measure the degree of attainment of system objectives. These evaluative criteria will facilitate a rational choice of a particular set of actions (from among a large number of feasible alternatives) which will best accomplish the established objectives. Some evaluative criteria will provide an absolute value of how good the solution is, such as the cost of producing one unit of some product. Other evaluative criteria will only produce relative values that can be compared among the alternatives to rank them in order of preference, as in economic comparisons such as benefit/cost analysis.

In most complex real world problems, more than one objective can be identified. A quantitative or qualitative analysis of the trade-offs between the objectives must be made. It may be possible to specify all but one objective as set levels of performance, which become system constraints. Then the system can be designed to perform optimally in terms of the remaining objective. For many problems, cost effectiveness would be the primary objective. Cost effectiveness can be defined as the lowest possible cost for a set level of control of a system, or the highest level of system control for a set cost.

Formulation of alternatives. Formulation of alternatives is essentially the development of system models that, in conjunction with evaluative criteria, will be used in later analysis and decision making. If at all possible, these models should be mathematical in nature. However, it should not be assumed that mathematical model building and optimisation techniques are either required or sufficient for application of the systems approach. Many problems contain unquantifiable variables and parameters which would render results generated by even the most elegant mathematical model meaningless. If it is not practical to develop mathematical models, subjective models that describe the problem environment and systems included can be constructed. Models allow a more explicit description of the problem and its systems and facilitate the rapid examination of alternatives. The primary emphasis in this text will be on problems that can at least partially be represented by mathematical models. However, limitations of these models and appropriate applications of subjective models will be indicated.

Effective model building is a combination of art and science. The science includes the technical principles of mathematics, physics, and engineering science. The art is the creative application of these principles to describe physical or social phenomena. Practice is the best way to learn the art of model building, but this practice must be predicated on a thorough understanding of the science. Although the art of modelling cannot be taught in a single course, a course that introduces the student to the systems approach will lay the foundation for further development.

Evaluation of alternatives. To evaluate the alternatives that have been developed, some form of analysis procedure must be used. Numerous mathematical techniques are available, including the simplex method for linear programming models, the various methods for solving ordinary and partial differential equations or systems of differential equations, matrix algebra, various economic analyses, and deterministic or stochastic computer simulation. Subjective analysis techniques may be used for multiobjective analysis, or subjective analysis of intangibles.

The appropriate analysis procedures for a particular problem will generate a set of solutions for the alternatives that can be tested according to the established evaluative criteria. In addition, these solution procedures should allow efficient utilisation of manpower and computational resources.

As part of the analysis stage, the importance of each variable should be checked. This is called sensitivity analysis, and it involves testing how much the model output will change for given changes in the values of the decision variable and model parameters.

Choosing the best alternative. Choice of the best alternative from among those analysed must be made in the context of the objectives and evaluative criteria previously established, but also must take into account nonquantifiable aspects of the problem such as aesthetic and political considerations. The chosen alternative will greatly influence the development of the final plan/design and will determine in large part the implementability of the suggested solution.

Preferably the best alternative can be chosen from the mathematical optimisation within feasibility constraints. Frequently, however, a system cannot be completely optimised. Near optimum solutions can still be useful, especially if sensitivity analysis has shown that the solution (and thus the objective function) is not sensitive to changes in the decision variables near the optimum point.

Final plan/design implementation. Actual final planning/design is primarily a technical matter that is conducted within the constraints and specifications developed in earlier stages of the systems approach. One of the end products of final planning or design is a report which describes the recommendations made. To be effective, this report must also include information on what has gone into the application of the systems approach to the problem. The report should be written in the context of the audience for which it is intended. A well-written nontechnical report can go a long way toward developing public support for the recommended problem solution, whereas, a well-written technical report given the same audience may be intimidating and actually reduce support for the recommendations.

2.3 Systems engineering

Systems engineering may be defined as the art and science of selecting from a large number of feasible alternatives, involving substantial engineering content, that particular set of actions which will best accomplish the overall objectives of the decision makers, within the constraints of law, morality, economics, resources, politics, social life, nature, physics etc.

Another definition expresses the systems engineering as a set of methodologies for studying and analysing the various aspects of a system (structural and non-structural) and its environment by using mathematical and/or physical models.

Systems engineering is currently the popular name for the engineering processes of planning and design used in the creating of a `system' or project of considerable complexity.

Design. The design of a system represents a decision about how resources should be transformed to achieve some objectives. The **final design is a choice of a particular combination of resources and a way to use them**; it is selected from other combinations that would accomplish the same objectives. For example, the design of a building to provide 100 apartments represents a selection of the number of floors, the spacing of the columns, the type of materials used, and so on; the same result could be achieved in many different ways.

A design must satisfy a number of technical considerations. It must conform to the laws of the natural sciences; only some things are possible. To continue with the example of the building, there are limits to the available strength of either steel or concrete, and this constrains what can be built. The creation of a good design for a system thus requires solid technical competence in the matter at hand. Engineers may take this fact to be self-evident, but it often needs to be stressed to industrial or political leaders motivated by their hopes for what a proposed system might accomplish.

Economics and values must also be taken into account in the choice of design; the best design cannot be determined by technical considerations alone. Moreover, these issues tend to dominate the final choice between many possible designs, each of which appears equally effective technically. The selection of a design is then determined by the costs and relative values associated with the different possibilities. The choice between constructing a building of steel or concrete is generally a question of cost, as both can be essentially equivalent technically. For more complex systems, political or other values may be more important than costs. In planning an airport for a city for instance, it is usually the case that several sites can be made to perform technically; the final choice hinges on societal decisions about the relative importance of ease of access and the environmental impacts of the airport, in addition to its cost.

Planning. Planning and design are so closely related that it is difficult to separate one from the other. The planning process closely follows the systems approach, and may involve the use of sophisticated analysis and computer tools. However, the scope of problems addressed by planning is different from the scope of design problems. Basically, **planning is the formulation of goals and objectives that are consistent with political, social, environmental, economic, technological, and aesthetic constraints; and the general definition of procedures designed to meet those goals and objectives**. Goals are the desirable end states that are sought. They may be influenced by actions or desires of government bodies, such as legislatures or courts; of special interest groups; or of administrators. Goals may change as the interests of the concerned groups change.

Objectives relate ways in which the goals can be reached. Planning should be involved in all aspects of an engineering project, including preliminary investigations, feasibility studies, detailed analysis and specifications for implementation and/or construction, and monitoring and maintenance. A good plan will bring together diverse ideas, forces, or factors, and combine them into a coherent, consistent structure that when implemented, will improve target conditions without deprecating non-target conditions. Effective use of the systems approach will help ensure that planning studies address the true problem at hand. Planning studies that do not do this could not, if implemented, produce useful and desirable changes.

2.4 Mathematical modelling

In general, to obtain a way to control or manage a physical system we use a mathematical model which closely represents the physical system. Then the mathematical model is solved and its solution is applied to the physical system. Models, or idealised representations, are an integral part of everyday life. Common examples of models include model aeroplanes, portraits, globes, and so on. Similarly, models play an important role in science and business, as illustrated by models of the atom, models of genetic structure, mathematical equations describing physical laws of motion or chemical reactions, graphs, organisation charts, and industrial accounting systems. Such models are invaluable for abstracting the essence of the subject of inquiry, showing inter-relationships, and facilitating analysis (Hillier and Lieberman, 1990).

Mathematical models are also idealised representations, but they are expressed in terms of mathematical symbols and expressions. Such laws of physics as $F = ma$ and $E = mc^2$ are familiar examples. Similarly, the mathematical model of a business problem is the system of equations and related mathematical expressions that describe the essence of the problem. Thus, if there are n related quantifiable decisions to be made, they are represented as **decision variables** (say, $x_1, x_2, ..., x_n$) whose respective values are to be determined. The appropriate measure of performance (e.g. profit) is then expressed as a mathematical function of these decision variables (e.g. $P = 3x_1 + 2x_2 + ... + 5x_n$). This function is called the **objective function**. Any restrictions on the values that can be assigned to these decision variables are also expressed mathematically, typically by means of inequalities or equations (e.g. $x_1 + 3x_1x_2 + 2x_2$ £ 10). Such mathematical expressions for the restrictions often are called **constraints**. The constants (coefficients or right-hand sides) in the constraints and the objective function are called the **parameters** of the model. The mathematical model might then say that the problem is to choose the values of the decision variables so as to maximise the objective function, subject to the specified constraints. Such a model, and minor variations of it, typify the models used in operations research.

Mathematical models have many advantages over a verbal description of the problem. One obvious advantage is that a mathematical model describes a problem much more concisely. This tends to make the overall structure of the problem more comprehensible, and it helps to reveal important cause-and-effect relationships. In this way, it indicates more clearly what additional data are relevant to the analysis. It also facilitates dealing with the problem in its entirety and considering all its inter-relationships simultaneously. Finally, a mathematical model forms a bridge to the use of high-powered mathematical techniques and computers to analyse the problem. Indeed, packaged software for both microcomputers and mainframe computers is becoming widely available for many mathematical models.

The procedure of selecting the set of decision variables which maximises/ minimises the objective function subject to the systems constraints, is called the **optimisation procedure**. The following is a general optimisation problem. Select the set of decision variables $x^*_1, x^*_2, \ldots, x^*_n$ such that

$$\text{Min or Max } f(x_1, x_2, \ldots, x_n)$$

subject to:

$$
\begin{aligned}
g_1(x_1, x_2, \ldots, x_n) &\leq b_1 \\
g_2(x_1, x_2, \ldots, x_n) &\leq b_2 \\
g_m(x_1, x_2, \ldots, x_n) &\leq b_m
\end{aligned}
\tag{2.2}
$$

where b_1, b_2, ..., b_m are known values.

If we use the matrix notation (2.2) can be rewritten as:

$$\text{Min or Max } f(\mathbf{x}) \tag{2.3}$$

subject to:

$$g_j(\mathbf{x}) \leq b_j \qquad j = 1, 2, ..., m$$

When optimisation fails, due to system complexity or computational difficulty, a reasonable attempt at a solution may often be obtained by **simulation**. Apart from facilitating trial and error design, simulation is a valuable technique for studying the sensitivity of system performance to changes in design parameters or operating procedure. Simulation will be presented in the Section 3 of this text.

According to equations (2.2) and (2.3) our main goal is the search for an **optimal**, or best **solution**. Some of the techniques developed for finding such solutions are presented in Sections 4 and 5 of this text. However, it needs to be recognised that these solutions are optimal only with respect to the model being used. Since the model necessarily is an idealised rather than an exact representation of the real problem, there cannot be any Utopian guarantee that the optimal solution for the model will prove to be the best possible solution that could have been implemented for the real problem. There just are too many imponderables and uncertainties associated with real problems. However, if the model is well formulated and tested, the resulting solution should tend to be a good approximation to the ideal course of action for the real problem. Therefore, rather than be deluded into demanding the impossible, the test of the practical success of an operations research study should be whether it provides a better guide for action than can be obtained by other means.

The eminent management scientist and Nobel Laureate in Economics, Herbert Simon, points out that **satisficing** is much more prevalent than optimising in actual practice. In coining the term *satisficing* as a combination of the words *satisfactory* and *optimising*, Simon is describing the tendency of managers to seek a solution that is "good enough" for the problem at hand. Rather than trying to develop various desirable objectives (including well-established criteria for judging the performance of different segments of the organisation), a more pragmatic approach may be used. Goals may be set to establish minimum satisfactory levels of performance in various areas, based perhaps on past levels of performance or on what the competition is achieving. If a solution is found that enables all of these goals to be met, it is likely to be adopted without further ado. Such is the nature of satisficing.

The distinction between optimising and satisficing reflects the difference between theory and the realities frequently faced in trying to implement that theory in practice. In the words of Samuel Eilon, "optimising is the science of the ultimate; satisficing is the art of the feasible."

2.5 Classification of mathematical models and optimisation techniques

Depending on the nature of the objective function and the constraints, the optimisation problem can be classified as:

Linear - the objective function and all the constraints are linear in terms of the decision variables;

Nonlinear - part or all of the constraints and/or the OF are nonlinear;

Deterministic - if each variable and parameter can be assigned a definite fixed number or a series of fixed numbers for any given set of conditions;

Probabilistic/Stochastic - contain variables the value of which are subject to some measure of randomness or uncertainty;

Static - models which do not explicitly take time into account;

Dynamic - models which involve time-dependent interactions;

Distributed Parameters - models which take into account detailed variations in behaviour from point to point throughout the system;

Lumped Parameters - models which ignore the variations and the parameters and dependent variables can be considered to be homogeneous throughout the entire system.

Several techniques of optimisation are available to solve the above optimisation problems, such as:

- calculus;
- linear programming;
- non-linear programming;
  - direct search;
  - gradient search;
  - complex method;
  - geometric programming; and
  - other.
- dynamic programming;
- others.

For dynamic systems:

- queuing theory;
- game theory;
- network theory;
- the calculus of variation;
- the maximum principle;
- quasi-linearisation; and
- others.

All the models and techniques mentioned up to now are dealing with the single objective function (Eqns. 2.2 and 2.3). If we have problems which require more than one objective functions then in mathematical terms we are dealing with "vector optimisation" or multiobjective optimisation presented in Section 6 of this manuscript.

Class exercise 2

Choose the optimization technique for solving the problem from the Exercise 1 according to the classification above. Discuss the characteristics of the parking lot design problem from the model classification point of view.

Class exercise 3:

Tables 2.1, 2.2, and 2.3 provide possible recommendations of model choice for different steps of the water resources planning procedure (Simonovic 1989).

2.6 Problems

2.1 What are the characteristics of civil engineering planning or management problems that are most suitable for analysis using quantitative systems analysis techniques?

2.2 Identify one engineering planning problem and specify in words possible objectives, the unknown decision variables whose values need to be determined, and the constraints or relationships that must be met by any solution of the problem.

2.3 Describe the political, economic, technical, aesthetic, etc. issues involved in one civil engineering problem you are familiar with.

2.4 In the above defined problem (2.2) separate the quantitative variables from nonquantitative. Identify the decision variables in both sets. Identify the objectives, both quantitative and nonquantitative. Of the quantitative objectives, indicate the indices of quantification which most appropriately reflect that objective. Also, indicate those which might be reduced to common terms (e.g., dollars). Identify all possible constraints. Using the classifications presented in class identify appropriate solution techniques for your problem.

2.5 In most general terms, list all the stages of civil engineering planning and describe the task of each.

Inventory, Forecast and Analysis of Available Water Resources

Activities	*Available Groundwater*	*Surface Water Analysis*	*Reservoirs*	*Unconventional Sources*	*Reservoir Yield Determination*
Modeling purpose	*- Determination of groundwater wells yield*	*- Determination of runoff distribution* *- Statistical analysis of mean, low and flood flows* *-Generation of mean flows for ungaged locations*	*- Presentation of existing reservoir data* *- Determination of maximum reservoir storage for potential sites*	*- Recycling* *- Desalinization*	*- Estimation of optimal reservoir yield*
Suggested modeling techniques	*- Simulation*	*- Rainfall-runoff models* *- Simulation* *- Statistical distributions fitting* *- Regression* *- Linear and nonlinear interpolation*	*- Optimization* *- Input-Output diagrams*	*- Parameter estimation*	*- Implicit stochastic optimization*

Table 2.1 *Activities, modeling purpose and suggested modeling techniques at the first planning step.*

Inventory, Forecast and Analysis of Water Demand

Activities	*Municipal and Industrial Water Supply*	*Irrigation*	*Power Production*	*Water Quality Control*	*Other Uses*
Modeling Purpose	*-Estimation of population growth - Prediction of industrial development*	*-Estimation of the water demand - Derivation of crop yield-soil moisture relation*	*- Estimation of power demand - Thermal and hydro power production*	*- Estimation of clean water amount necessary for dilution*	*- Recreation requirements -Fish production - Wildlife*
Suggested modeling techniques	*-Systems dynamics model (diagram of flows, of people, of resources, and products) -Input-output diagrams -Trends estimation*	*- Simulation - Mathematical formula (Blaney Criddle) - Optimization of water allocation (stochastic dynamic programming) - Optimization of crop structure (linear programming)*	*- Input-output modeling - Simulation - Optimization - Interpretive structural modeling*	*- Quality simulation - Optimization of wastewater discharge sites - Optimization of waste load*	*- Water Level computation (simulation) - Water quality simulation*

Table 2.2 Activities, modeling purpose and suggested modeling techniques at the second planning step.

Comparison and Ranking the Alternative Solutions

Activities	*Setting objectives and developing evaluation criteria*	*Evaluating the alternatives*	*Selecting an alternative*	*Sensitivity analysis*
Modeling purpose	*- Model of purposes for every "water system"*	*- Evaluation of alternative sets according to all used criteria*	*- Decision making models for ranking alternatives*	*- Analyzing the influence of decision makers' preferences and model parameters on alternatives rank*
Suggested modeling Techniques	*- Objectives tree* *- Hierarchical diagram* *- Interaction matrices*	*- Cost estimation model* *- Cost-benefit model* *- Cost-effectiveness model*	*- Discrete multi-objective techniques* *- Decision table (showing ranking of alternatives for each criteria)* *- Criterion function (mathematical expression for establishing the overall ranking of alternative)*	*- Preference relations* *- Discrete multi-objective techniques*

Table 2.3 Activities, modeling purpose and suggested modeling techniques at the fourth planning step

2.7 References

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3. SYSTEM SIMULATION

3.1 Introduction

Simulation models “describe” how a system operates, and are used to determine changes resulting from a specific course of action. Such models are sometimes referred to as “cause-and-effect” models. They describe the state of the system in response to various inputs but give no direct measure of what decisions should be taken to improve the performance of the system. Therefore, the simulation is problem solving technique which contains the following phases :

a) development of a model of the system;
b) operation of the model (i.e. generation of outputs resulting from the application of inputs); and
c) observation and interpretation of the resulting outputs.

The essence of simulation is modeling and experimentation. Simulation does not directly produce “the answer” to a given problem. In the graphical form simulation procedure is represented in Figure 3.1. Simulation includes a wide variety of procedures. In order to choose among, and use them effectively, the potential user must know:

i) how they operate;
ii) how they can be expected to perform; and
iii) how this performance relates to the problem under investigation.

Major components of a simulation model are :

Input : quantities that “drive” the model (in water resources engineering models for example a principal input is the set of streamflows, rainfall sequences, pollution loads, water and power demands, etc.).

Physical Relationships : mathematical expression of the relationship among the physical variables of the system being modeled (continuity, energy conservation reservoir volume and elevation, outflow relations, routing equations, etc.).

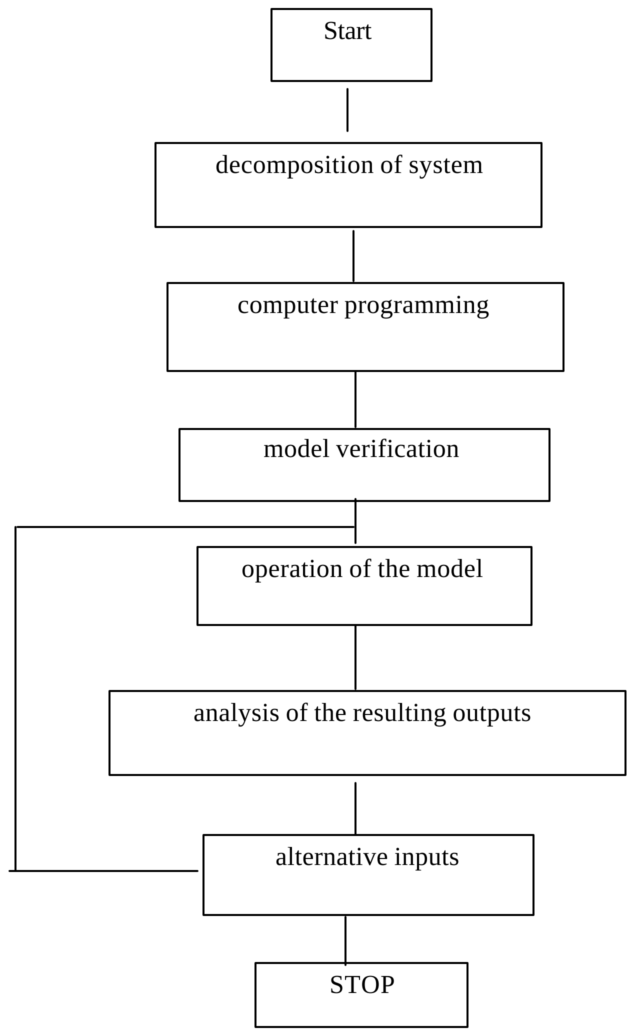


Figure 3.1 The simulation procedure

Nonphysical Relationships : those that define economic variables, political conflicts, public awareness, etc.

Operation Rules : the rules that govern operational control.

Outputs : the final product of operations on inputs by the physical and nonphysical relations in accordance with operating rules.

Engineering domain can benefit from already available computer-based simulation modeling tools. There are numerous tools used for implementing simulation in Civil Engineering planning and design. Complex Civil Engineering problems heavily rely on systems thinking, which is defined as the ability to generate understanding through engaging in the mental model-based processes of construction, comparison, and resolution. Computer software tools like *STELLA*, *DYNAMO*, *VENSIM*, *POWERSIM* (High Performance Systems, 1992; Lyneis et al., 1994; Ventana, 1996; Powersim Corp., 1996) and others help the execution of these processes. One that will be used in this course is *Vensim* environment developed to support a special simulation approach known as System Dynamics.

3.2 System Dynamics simulation approach for Civil Engineering problems

System Dynamics simulation approach relies on understanding complex inter-relationships existing between different elements within a system. This is achieved by developing a model that can simulate and quantify the behavior of the system. Simulation of the model over time is considered essential to understand the dynamics of the system. Understanding of the system and its boundaries, identifying the key variables, representation of the physical processes or variables through mathematical relationships, mapping the structure of the model and simulating the model for understanding its behavior are some of the major steps that are carried out in the development of a system dynamics simulation model. It is interesting to note that the central building blocks of the principles of system dynamics approach are well suited for modeling any physical system.

System dynamics simulation tools are well suited for representing mental models that have been developed using systems thinking paradigm introduced in Section 1 of this text. Practically, these tools are built around a progression of structures. Stocks, flows, converters and connectors are the principal building blocks of structure and they are discussed in details later in this section.

The power and simplicity of use of system dynamics simulation applications is not comparable with those developed in functional algorithmic languages. In a very short period of time, the users of the simulation models developed by system dynamics tools can experience the main advantages of this approach. The power of simulation is the ease of constructing "what if" scenarios and tackling big, messy, real-world problems. In addition, general principles upon which the system dynamics simulation tools are developed apply equally to social, natural, and physical systems. Using these tools in Civil Engineering allows enhancement of models by adding social, economic, and ecological sectors into the model structure.

System dynamics is an academic discipline introduced in the 1960s by the researchers at the Massachusetts Institute of Technology. System dynamics was originally rooted in the management and engineering sciences but has gradually developed into a tool useful in the analysis of social, economic, physical, chemical, biological and ecological systems (Sterman, 2000). In the field of system dynamics, a *system* is defined as a collection of elements which continually interact over time to form a unified whole. The underlying pattern of interactions between the elements of a system is called the *structure* of the system. One familiar water resources engineering example of a system is a reservoir. The structure of a reservoir is defined by the interactions between inflow, storage, outflow, and other variables specific to a particular reservoir location (storage curve, evaporation, infiltration, etc.). The structure of the reservoir includes the variables important in influencing the system. The term *dynamics* refers to change over time. If something is dynamic, it is constantly changing in response to the stimuli influencing it. A dynamic system is thus a system in which the variables interact to stimulate changes over time. *System Dynamics* is a methodology used to understand how systems change over time. The way in which the elements or variables composing a system vary over time is referred to as the *behavior* of the system. In the reservoir example, the behavior is described by the dynamics of reservoir storage growth and decline. This behavior is due to the influences of inflow, outflow, losses and environment, which are elements of the system. One feature which is common to all systems is that a system's structure determines the system's behavior. System Dynamics links the behavior of a system to its underlying structure. System Dynamics can be used to analyze how the structure of a physical, biological, or any other system can lead to the behavior which the system exhibits. By defining the structure of a reservoir, it is possible to use system dynamics analysis to trace out the behavior over time of the reservoir based upon its structure.

The structure-behavior link need not be limited to systems which are well defined historically or analytically. System Dynamics can also be used to analyze how structural changes in one part of a system might affect the behavior of the system as a whole. Perturbing a system allows one to test how the system will respond under varying sets of conditions. Once again referring to a reservoir, someone can test the impact of a drought on the reservoir or analyze the impact of the elimination of a particular user on the behavior of the entire system.

In addition to relating system structure to system behavior and providing users a tool for testing the sensitivity of a system to structural changes, System Dynamics requires a person to partake in the rigorous process of modeling system structure. Modeling a system structure forces a user to consider details typically glossed over within a mental model.

3.3 Basic Building Blocks of System Dynamics Simulation Environments

The stock and flow notation provides a general way to graphically characterize any system process. This may seem ambitious: any process! However, if we consider the characteristics that are generally shared by all engineering processes and the components which make up these processes, it is a remarkable fact that all such processes can be characterized in terms of variables of two types, stocks (levels, accumulations) and flows (rates).

The basic parts of *Vensim* language are: Stocks (Box Variable; Level), Rates (Flows), Auxiliary Variables and Arrows (Connectors). They have a very particular meaning and full understanding of it is essential for accurate System dynamics simulation modeling.

Levels

Levels or stocks are the nouns of the *Vensim* language. Looking at the magnitudes of the stocks at a point in time will tell you how things "are" within a system at that point in time.

Stocks are signified by rectangles. Operationally, stocks function as *accumulators*. Think about it…accumulations are *everywhere*! The food in your stomach. The money in your wallet. The knowledge in your head. The love in your heart. All are accumulations. Stocks can be used to depict both material and non-material accumulations.

The magnitudes of Stocks within a system persist even if the magnitudes of all the activities fall to zero. When you take a snapshot of a system only the accumulations that the activities had filled and drained would appear in the picture. The picture would show the state of the system at that point in time.

Because they accumulate, stocks often act as "buffers" within a system. In this role, stocks enable inflows and outflows to be out of balance with each other – i.e., out of equilibrium. For example, it was stocks which enabled us to move beyond the hunter and gatherer phase of our existence! If we had not figured out how to store food, we'd still be spending all of our time looking for food because we have to eat what we killed or found, right when we killed or found it. By accumulating inventories of food, we could consumer calories when we wanted, and the rate at which we consumed them could at times be far in excess of the rate at which we were adding food to the inventory.

Rates

Levels and rates (stocks and flows) fit together. If stocks are nouns, flows are verbs. You don't have one without the other. If there is an accumulations of something, that accumulation *had to* result from some activity, a flow of something. And if there is a flow of something, there must be an associated build-up or depletion. Give it some thought. There's really no way around it. Activities always leave "tracks", whether you intend them to or not. Tracks, in turn, stimulate activities.

Flows are signified by a pipe with a spigot (Figure 3.2), flow regulator, and one or two arrowheads attached. Stuff flows through the pipe, in the direction indicated by the arrowhead. The flow volume is calculated by the algebraic expression, or number, that you enter into the flow regulator. You can imagine that large volumes cause the spigot to be opened wide, while small volumes cause it to be shut down.

Flows are used to depict activities. (i.e., things in motion). Flows can have several attributes. They can be: 1) conserved or non-conserved; 2) uni-directional or bi-directional; and 3) not unit-converted or unit converted.

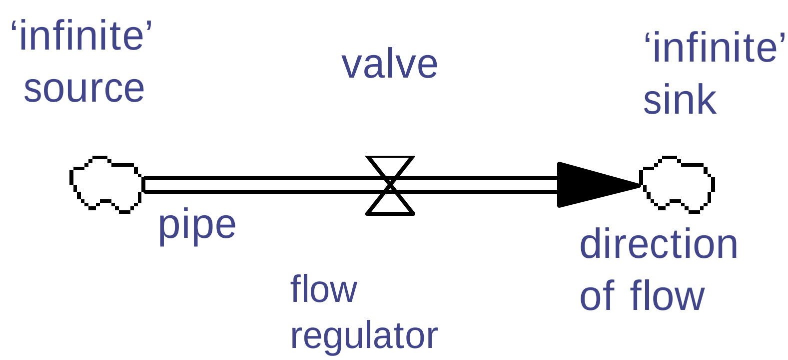


Figure 3.2 Rate

A conserved flow draws down one stock as it fill another. The stuff which is flowing is *conserved*, in the sense that it only changes its location within the system (not its magnitude).

In some cases, however, you might want to use the *same* flow to transport stuff both into and out of an accumulation.

In most cases, flows are dominated in the same units of "stuff" as the stocks to which they are attached. For example, bathtubs hold water. You would *not* expect to have jello, motor oil or pasta flowing through the faucet or the drain. The stock would be denominated in units of water. The flow also would be denominated in units of water, with the suffix *per time.* When the units are not converted, the only difference between stock and flow denomination in the "per time" suffix.

So, stocks and flows are inseparable. You can't have one without the other. Both are necessary for generating change over time, or dynamics. If we want only a static snapshot of reality, stocks *alone* would be sufficient. But without flows, no *change* in the magnitude of the stocks could occur. In order to move from snapshots to movies, we need flows.

Fortunately for us, stocks and flows *do* exist in combination. When stocks de-couple rates of flow, they serve as buffers, or shock-absorbers. Because stocks play this role, we do not need to match inflows to outflows all the time.

Auxiliary variables

Auxiliary variables often function as adverbs, modifying the activities (or verbs) within the system. But they also serve many other functions, and so are really best characterized as "catchalls". Auxiliary variables transform inputs into outputs. They can represent either information or material quantities. They often are used to break out the detail of the logic which otherwise would be buried within a flow regulator. Frequently they represent "score-keeping" variables such as cost, cycle-time, profit, etc. Unlike levels, auxiliary variables do *not* accumulate. The value for an auxiliary variable is re-calculated from scratch in each time step. Auxiliary variables thus have no "memory". Auxiliary variables will play one of four roles: stock-related, flow-related, stock/flow related, and external input-related.

In their stock-related role, auxiliary variables can provide an alternate way to measure the magnitude of a stock, or they sometimes can be used to substitute for a stock concept.

Auxiliary variables can also play flow-related roles in the models. The roles are twofold. First, they can be used to "roll up" the net of several flow processes. Or, they can be used to break out the components of the logic of a flow, so as to avoid diagram clutter.

The final category of use for auxiliary variables is "external inputs". External inputs include some time series inputs (often implemented via the graphical function), as well as various built-in functions.

Arrows

The final building block is the arrow. Arrows link stocks to auxiliary variables, stocks to flow regulators, flow regulators to flow regulators, auxiliary variables to flow regulators, and auxiliary variables to other auxiliary variables. Arrows represent inputs and outputs, *not* inflows and outflows! Arrows do not take on numerical values-they merely transmit values taken on by other building blocks. When you step on a scale and look down at your weight as registered on the dial, the resulting numerical magnitude is transmitted to your eye without having any influence whatsoever on the numerical value. Such transmissions would be represented using an arrow. Arrows cannot connect *into* a stock! Only flows can change stocks.

3.4 System Dynamics Modeling Process

Though there is no universally accepted process for developing and using good quality system dynamics models there are some basic practices that are quite commonly used. The following steps are a useful guideline.

Issue statement. The issue statement is simply a statement of the problem that makes it clear what the purpose of the model will be. Clarity of purpose is essential to effective model development. Developing a model of a system or process without specifying how the system needs to be improved or what specific behavior is problematic is difficult. Having a clear problem in mind makes it easier to develop models with good practical applicability.

Variable Identification. Identify some key quantities that will need to be included in the model for the model to be able to address the issues at hand. Usually a number of these are very obvious. It can sometimes be useful just to write down all of the variables that might be important and try to rank them in order to identify the most important ones.

Reference modes. A reference mode is a pattern of behavior over time. Reference modes are drawn as graphs over time for key variables, but are not necessarily graphs of observed behavior. Rather, reference modes are cartoons that show a particular characteristic of behavior that is interesting. For example, a company's sales history may be growing but bumpy, and the reference mode may be the up and down movement around the growth trend. Reference modes can refer to past behavior, or future behavior. They can represent what you expect to have happen, what you fear will happen and what you hope will happen. Reference modes should be drawn with an explicitly labeled time axis to help refine, clarify and bound a problem statement.

Reality Check. Define some Reality Check statements about how things must interrelate. These include a basic understanding of what actors are involved and how they interact, along with the consequences for some variables of significant changes in other variables. Reality Check information is often simply recorded as notes (often mental notes) about what connections need to exist. It is based on knowledge of the system being modeled.

Dynamic hypotheses. A dynamic hypothesis is a theory about what structure exists that generates the reference modes. A dynamics hypothesis can be stated verbally, as a causal loop diagram, or as a stock and flow diagram. The dynamic hypotheses you generate can be used to determine what will be kept in models, and what will be excluded. Like all hypotheses, dynamic hypotheses are not always right. Refinement and revision is an important part of developing good models.

Simulation Model. A simulation model is the refinement and closure of a set of dynamic hypotheses to an explicit set of mathematical relationships. Simulation models generate behavior through simulation. A simulation model provides a laboratory in which you can experiment to understand how different elements of structure determine behavior.

The above process is iterative and flexible. As you continue to work with a problem you will gain understanding that changes the way you need to think about the things you have done before. Later chapters in this guide go through the process of model development from a number of different perspectives placing emphasis on different parts of the above process.

Vensim provides explicit support for naming variables, writing Reality Check information, developing dynamic hypotheses and building simulation models. Creating good issue statements and developing reference modes can easily be done with pencil and paper or using other technologies. Dynamic hypotheses can be developed as visual models in Vensim, or simply sketched out with pencil and paper. Simulation is the one stage where it would be impractical to dispense with the computer altogether.

3.5 Fundamental Structures and Behaviors

As discussed above the process of building models requires the generation of an hypothesis about what kind of structure might be responsible for the behavior in the reference mode. This can be quite difficult, but it does become easier as you gain experience with model building. One reason it becomes easier is that you will gain experience with different structures and the behavior they generate. Studying the simplest and most fundamental patterns of behavior and the simplest structure that can generate the behavior is a useful way to get started on gaining this experience.

In the following sections we percent growth, decay, adjustment and oscillation and some simple structures that generate them. The models are presented in causal loop form to emphasize feedback which is an essential part of generating a dynamic hypothesis.

Exponential Growth (money.mdl)

Suppose that you deposit $100 in the bank. If the interest rate is 10% per year (compounded daily) and you wait 100 years what will happen? This is an example of a first order positive feedback loop.

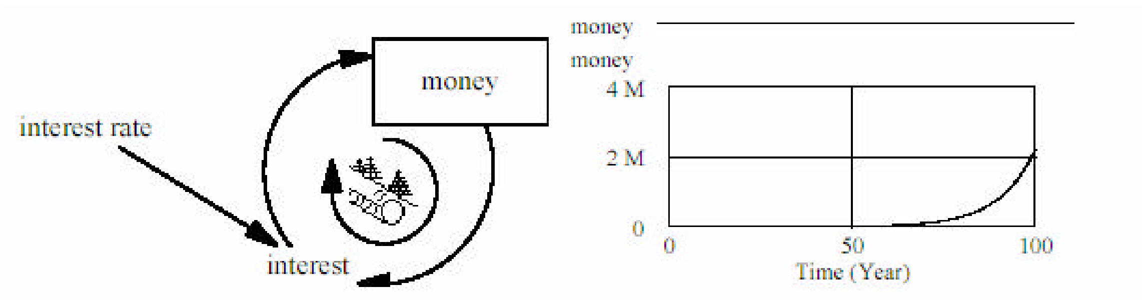


Figure 3.3 Exponential growth

At the end of 100 years there would be over two million dollars in the bank. Exponential growth is interesting because it demonstrates a constant doubling time. If it takes, as it does in this example, about 7 years to go from 100 to 200 dollars, it will also take about 7 years to go from 1 million to 2 million dollars. It is a useful exercise to explore the relationship between the interest rate and the time it takes the money to double.

Note that for this example you can either select TIME STEP to be small enough (about .125) so that it makes no difference or choose Runge-Kutta integration.

Exponential Decay (workers.mdl)

Suppose that you have 100 people working for you and you decide never to hire anyone again. Your average worker hangs around for 10 years. What will happen to your workforce? This is an example of a first order negative feedback loop.

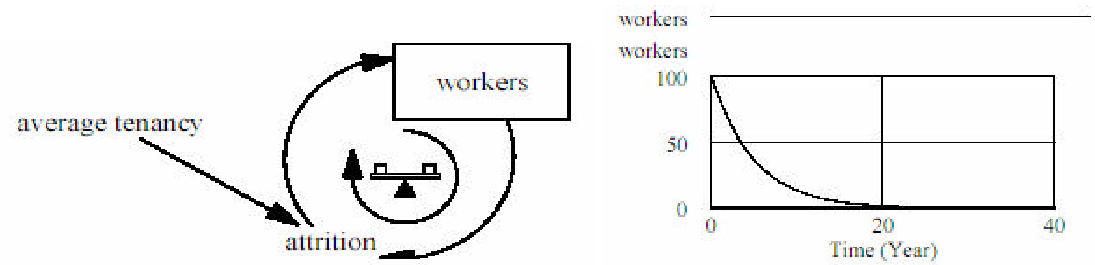


Figure 3.4 Exponential decay

The number of workers will decrease, very quickly at first and then more slowly. This is the opposite of exponential growth where changes was slow at first and then quick and there is a similar constancy in exponential decay. If it takes, as it does here, about seven years for half the workers to leave, it will take another seven for half of those that remain to leave. It is interesting to experiment and see the relationship between *average tenancy* and the time it takes half the workers to leave.

The important thing about exponential decay is that if you move the goal away from zero, you get the same behavior. Suppose you have a house at 10 degrees and you put it into a 40 degree neighborhood, what happens to the house temperature?

S-Shaped Growth (mice.mdl)

Suppose that you let some mice loose in your house and don't try to kill them. What will happen to the mouse population? At first glance this sounds very much like the exponential growth example, and for the first little time it is. Try as they might, though, the mice will never get 2 million of themselves into your house.

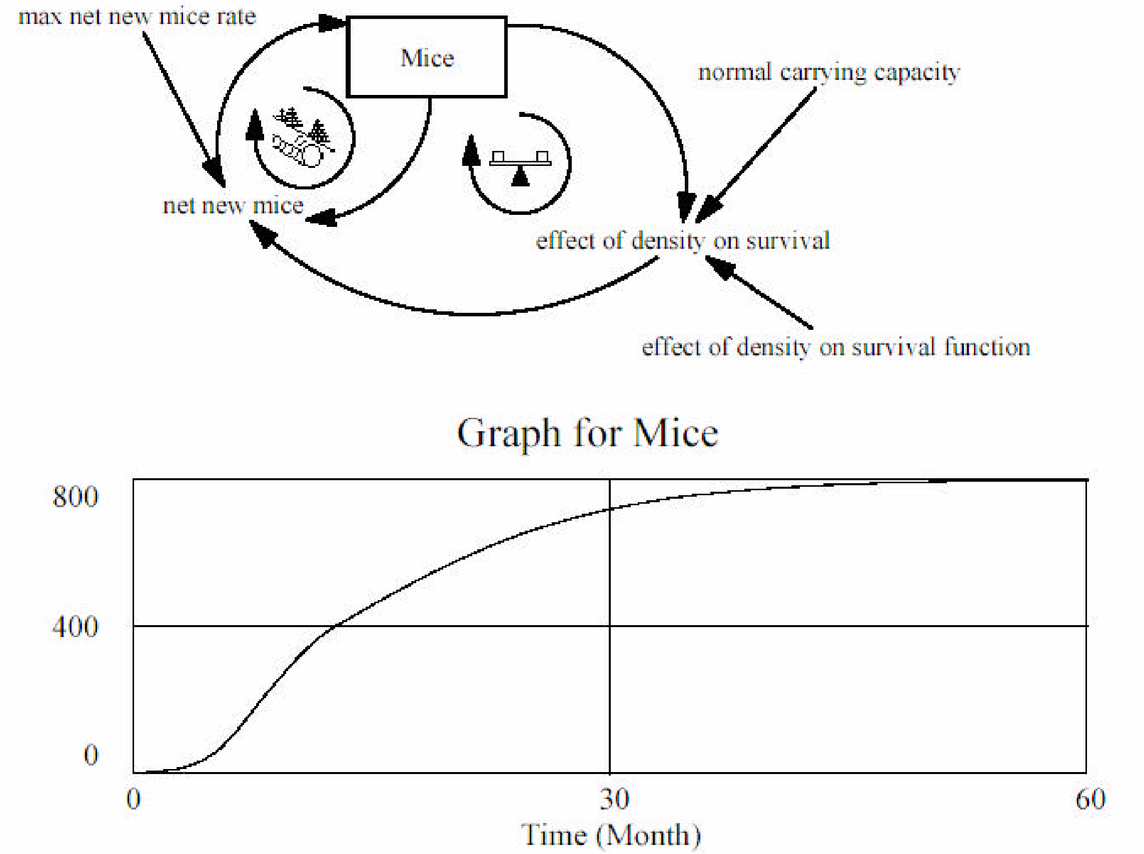


Figure 3.5 S-Shaped growth

The beginning of this does indeed look like exponential growth while the end looks like exponential adjustment. To get this kind of transition there has to be more than a single feedback loop operating. Often, and in the case of this simple model, there are both a positive and negative feedback loop with the positive loop dominate in the beginning and the negative loop dominant in the end. It is interesting to experiment with the Constant *max net new mice rate* to see the effect this has. Watching the way *net new mice* behaves as you change values is especially interesting. The process of adjustment in this model is almost entirely dependent on the shape of the Lookup *effect of density on survival* function. Experimenting with different shapes for this function can also be interesting.

Oscillation (spring.mdl)

Consider the problem of a spring pushing a weight on a frictionless horizontal surface.

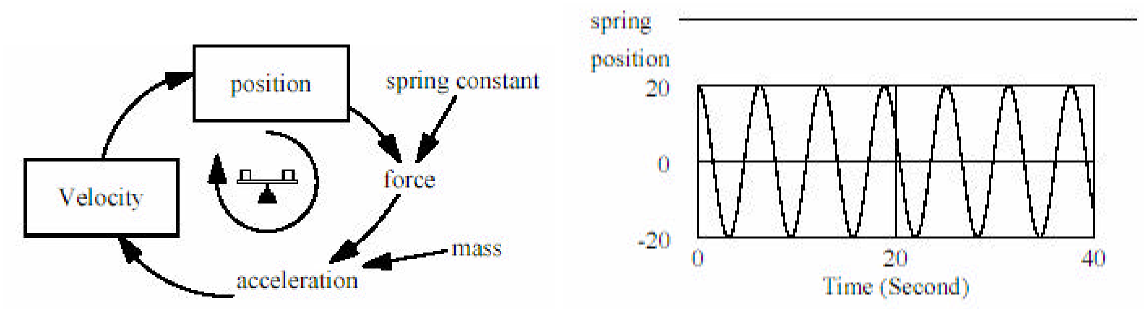


Figure 3.6 Oscillation

You need to be sure to use Runge-Kutta integration when simulating this model to get the correct result. This is the simplest structure that can generate oscillations, and the oscillations are undamped.

3.6 An Engineering System Dynamics Model Example

This section is focused on the conceptual issues in the development, analysis and use of a model. After working through this example you should have a better understanding of how to think about, and work through, a system dynamic simulation model with Vensim.

Background

You are involved in the production and sale of prefabricated window frames. Overall your company is doing quite well, but you often go through periods of low capacity utilization followed by production ramp up and added shifts. While all of this is normally blamed on market demand and the condition of the economy, you have your doubts. Looking back at sales and production over the last 8 years it seems that sales is more stable than production. Your goal is to determine why this might be, and what you can do about it. In attacking this problem you want to simplify as much as possible your current situation. There are a number of reasons for this simplification:

- It is easier to understand a simple model.
- You can get results quickly and decide if you are on the right track.
- It is more effective to start with a simple model and add detail, than to build a complex model and attempt to extract insights from it after it is complete.

- Using a simple model forces you to take an overview which is usually useful in the initial modeling phases.

It is not always true that you want to start building simple models. In many cases the behavior you see is the result of complexity, and Vensim provides a very rich set of tools for dealing with complexity. Until you have had substantial experience, however, simplicity is highly recommended. While it is not uncommon to discover that what you have developed is not rich enough for the problem you wish to address, it is rare not to gain understanding in the process. Conversely, large complex models can become significant resource drains, providing no payoff for a very long time.

Reference modes

A reference mode is a graphical statement about a problem. Verbally, the problem was stated as "production is less stable than sales." Graphically we might draw:

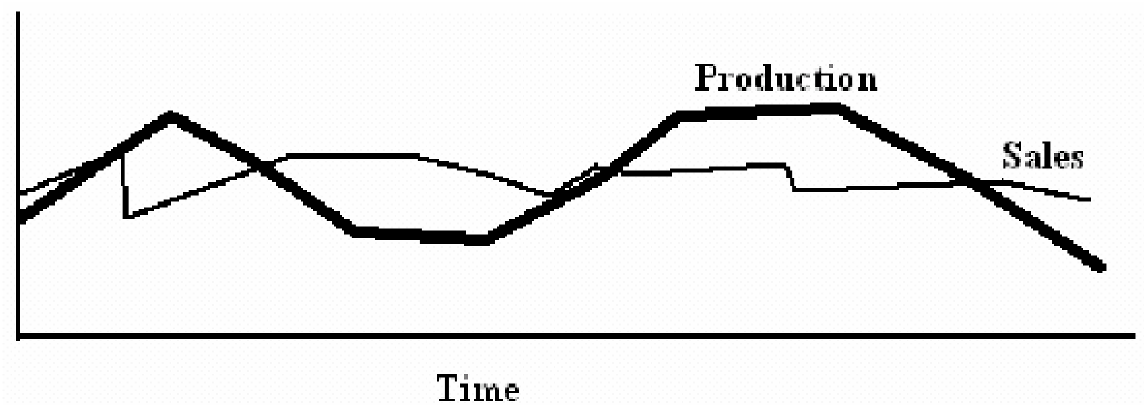


Figure 3.7 Reference modes

This reference mode is a sketch of behavior we might expect a model to produce. It might be real data from your records, or your expectation of what might happen in a new situation. The reference mode is used to focus activity. Having mapped out one or more reference modes the goal is to define the simplest structure that is sensible and capable of generating patterns of behavior that qualitatively resemble the reference modes. If appropriate, such a structure can also be refined in order to develop a model that can be validated quantitatively against the available data.

Reality check

Lets put down some common sense statements about how the business works.

- Without any workers there is no production.
- Without any inventory we can't ship.
- If sales go up for a sustained period we will try to expand production.
- With no production inventory will never go up.

Reality Check information in this form should be kept in the back of your mind as you develop dynamics hypotheses and build a simulation model.

Dynamic hypothesis

A dynamic hypothesis is an idea about what structure might be capable of generating behavior like that in the reference modes. For this example we can formulate a dynamic hypothesis simply by thinking about how the two variables in the reference mode are connected — that is by specifying the set of policies (or rules) that determine *production* given *sales.* The dynamic hypothesis for this firm is that a manager is setting production based on current sales, but is amplifying the amount resulting in higher (or lower) production than is necessary. The reference mode supplies us with two variables — *production* and *sales* — that we will want to include in the model. This is a reasonably good basis on which to begin a sketch, so let us put these variables down to start the model.

Prefabricated windows production model (wfinv1.mdl)

Now the question arises, how are *production* and *sales* related. Clearly there is a close relationship, since it is necessary to produce something before it is sold. Sales and production are related in two ways:

Physical: production is required to produce goods to sell

Information: managers base production decisions on current or recent sales

We will start the model with the physical side. When production occurs, goods are not immediately sold. Instead, they are stored in an inventory until a sale occurs, at which point they are removed from inventory. In general, there will be an inventory, or some combination of inventory and backlog, separating production from sales. If a backlog is used in the model, it is useful to consider orders and shipments instead of simply sales. In this model, we will just use an inventory.

We construct *Inventory* as a Level, then add a rate flowing in and a rate flowing out. Next, we use the variable merge tool to drag our two existing variables, *production* and *sales* onto the valves.

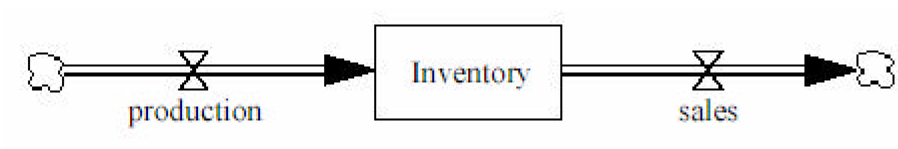


Figure 3.8 Inventory part of the model

It is worth noting at this point that we could have created the same diagram by first entering the level containing *Inventory*, then adding and naming the two rates. The reason we chose to add *Inventory* then attach the existing variables as rates was to work through the problem as it came to light, rather than working out the problem first then putting it on the sketch.

Now we need to figure out how *production* gets determined. Over the long term investment and capacity are clearly important, but these have been extremely stable. In the shorter term more people are hired and, if necessary, an additional production shift added. There is quite a bit of complicated stuff going on when shifts are added: management changes, maintenance scheduling problems arise, and so on. However, as a first approximation, more people make more products, and this is a good starting point, so we add the level *Workforce*.

The things that change workforce are hiring, layoffs, firings and retirements. Again, for simplicity we combine all of these into a composite concept — the *net hire rate.* Note that net hire rate can either increase or decrease the workforce.

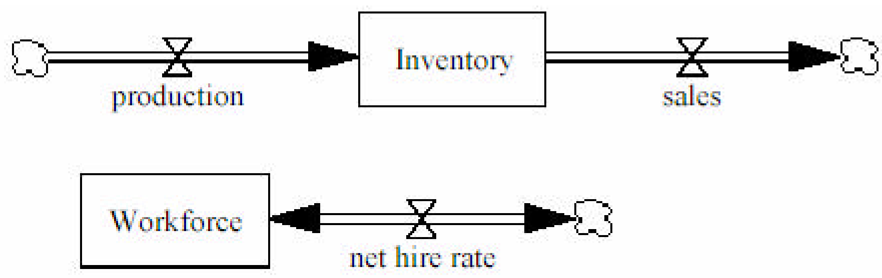


Figure 3. 9 Workforce part of the model

Behavioral relationships

This is the physical part of the problem. Now it is necessary to make some of the behavioral (information) connections. Putting down the important physical stocks and flows is often a good starting point in developing a model. It lets you make part of the system concrete, and this can simplify the conceptualization of other parts of the system. Alternative approaches include building a causal loop diagram and converting that to stock and flow form, or writing equations directly from causal loop diagrams. You might also want to draw a causal loop diagram, then start over again with a stock and flow diagram. What works best varies by individual and by problem, so we try to present some alternative approaches in different chapters in this guide.

In completing the information connection, we will try to keep things as simple as possible. Starting with production we want to remove all the complexities of adding shifts and mothballing equipment and simply state that *production* is proportional to *Workforce*. We add the proportionality constant *productivity.* Also, *net hire rate* is dependent on the value of *Workforce*. This gives us the model in Figure 3.10.

The net hire rate is the net number of people hired. The most straightforward way to formulate this is as a stock adjustment process. In a stock adjustment process you take an existing value of a variable (usually a stock) and compare it to some target or desired level, then take an action based on the difference between the two. For example if you are driving a car at 40 MPH and wish to be going 50 MPH you would depress the accelerator. Your car's speed (a stock) will increase at a rate that depends on how far you depress the accelerator and the car you are driving.

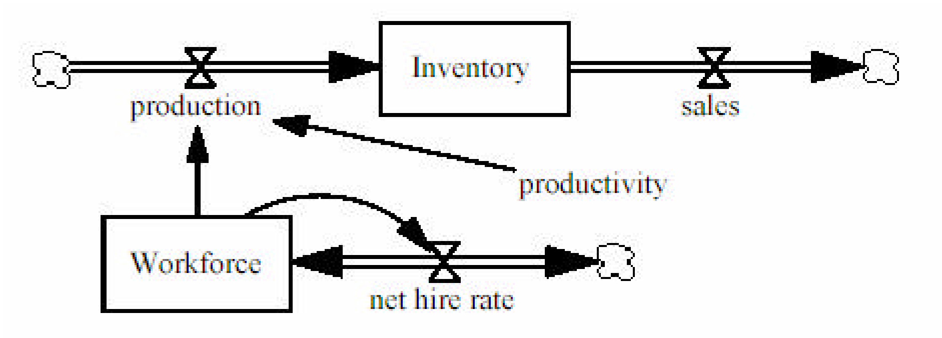


Figure 3.10 Connected model

To capture this stock adjustment process we add in the variables *target workforce* and *time to adjust workforce* and connect them as shown :

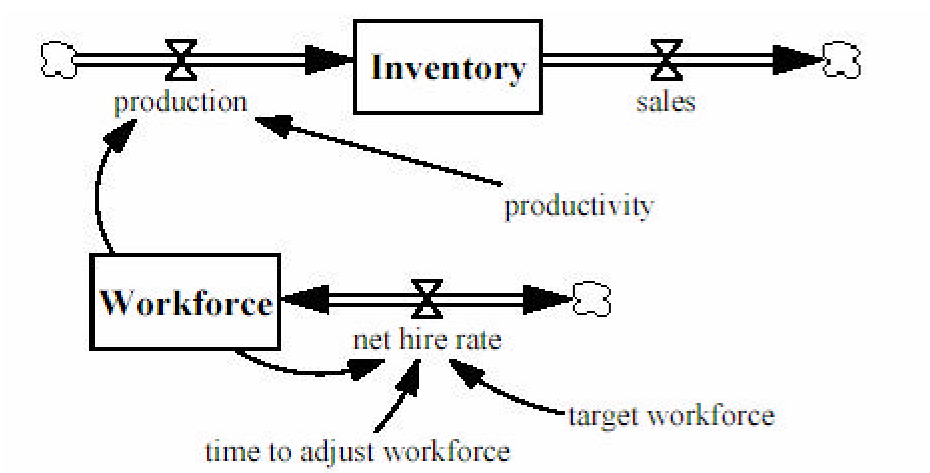


Figure 3.11 Modified model

time to adjust workforce represents the time required for management to agree on a change in the workforce level and screen potential applicants or notify workers to be laid off. *target workforce* is the number of people you need to produce the amount you want to produce. The Level *Workforce* is initialized at this value. Now we add the concept of *target production*, and connect it *to target workforce*. We will *set target production* on the basis of *sales*.

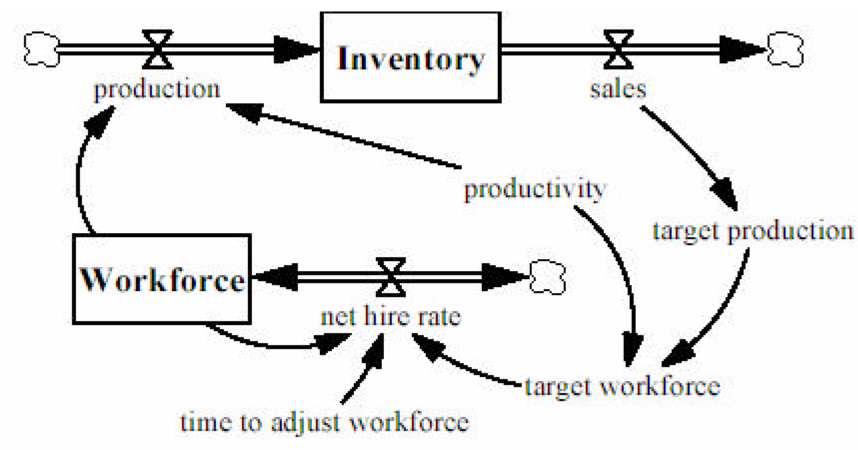


Figure 3.12 Complete model

This is a complete model, though it does have a critical error of omission which will be brought to light during simulation. The next step is to take the conceptual model and turn it into a simulation model.

Equation set (wfinv1.vmf)

The full equation set for this model follows:

```
FINAL   TIME   =
100      Units:
Month

INITIAL TIME =
0 Units: Month

TIME   STEP   =
0.25    Units:
Month
```

```
SAVEPER  =  TIME
STEP      Units:
Month

Inventory      =
INTEG(production
-sales, 300)

Units: Frame

net hire rate = (target workforce-Workforce)/time to adjust
  workforce
Units:
Person/Month

production                =
Workforce*productivity Units:
Frame/Month

productivity = 1
Units: Frame/Month/Person

sales   =   100   +
STEP(50,20)    Units:
Frame/Month

target  production  =
sales         Units:
Frame/Month

target     workforce      =      target
production/productivity Units: Person

time     to     adjust
workforce  =  3  Units:
Month

Workforce   =   INTEG(
net hire rate, target
```

```
workforce)      Units:
Person
```

Each equation is consistent with discussion during the initial conceptualization. The equation for *sales* has *sales* steady at 100 until time 20, when *sales* step up to, and thereafter remain at, 150. This input pattern is used to test that the system is indeed at an equilibrium, and then check the adjustment to a new operating condition. This step input pattern is the cleanest pattern available for looking at the internally generated dynamics of a system such as this. It allows you to observe how a single shock propagates through the system without further external disturbance. The model is complete. A simulation is performed with the name WF1.

Analysis

A Causes Strip graph for *production* and causes strip of *Workforce* show the dynamics.

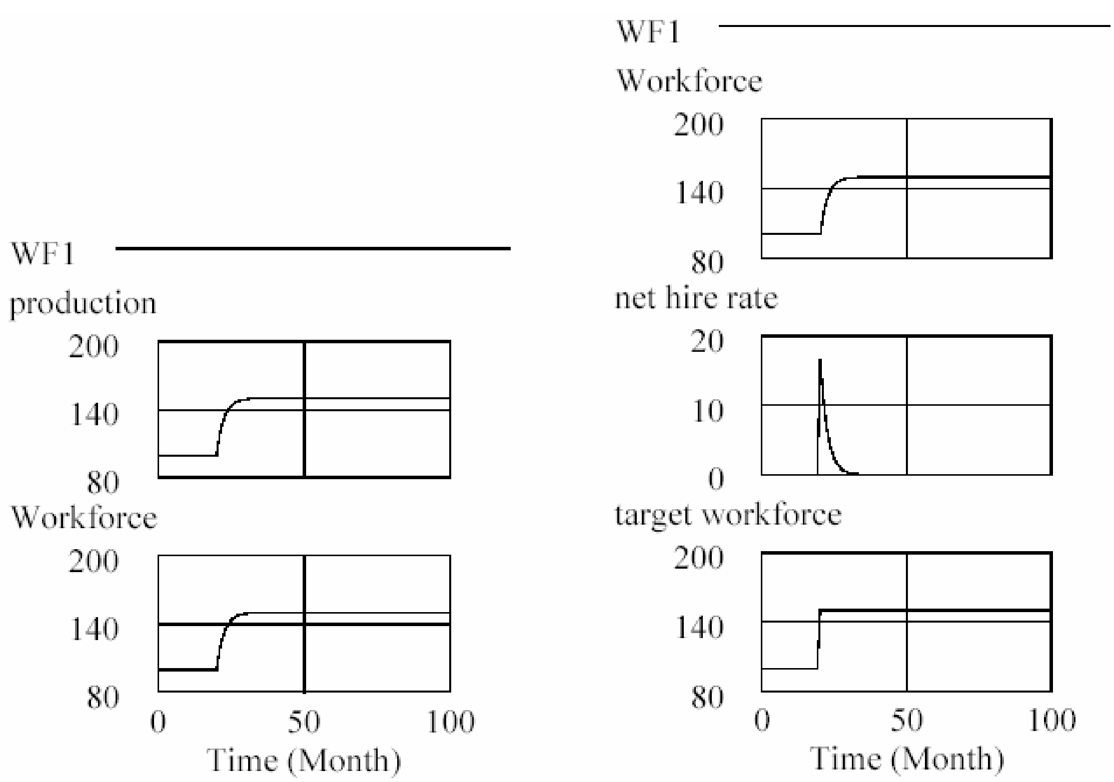


Figure 3.13 A causes strip graph for *production* and *Workforce*

It is immediately obvious that there is not a lot of variability in *production* or *Workforce*. There is very smooth adjustment from the initial 100 Widgets/Month to 150 Widgets/Month. Unlike our reference mode, the model does not appear to generate more variability in production than sales. At this point it is worth taking a look at *Inventory*.

Inventory falls smoothly from its initial value of 300, to about 150. Since the purpose of holding an inventory is to be sure that the right product configuration is available for customers, there is clearly something wrong. Any discrepancy in *Inventory* from the level necessary to meet product mix requirements and have a comfortable safety stock needs to be corrected, and this model does not make that correction.

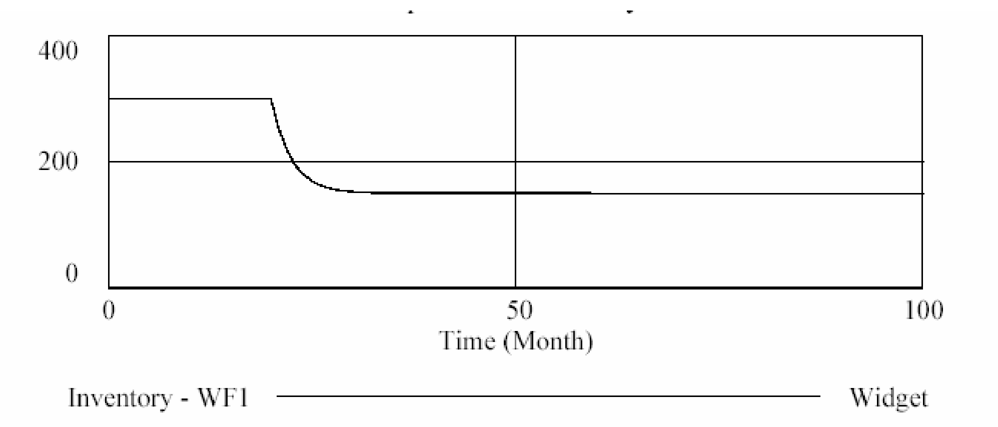


Figure 3.14 Graph for *Inventory*

Model refinement (wfinv2.vmf)

In order to refine the model we introduce *target inventory*, *inventory correction* and two additional Constants. The idea is simple — *target inventory* is the amount of stock that should be held based on expectations about sales. The *inventory correction* is the correction for a deviation of *Inventory* from its target. A new loop has been introduced and is highlighted.

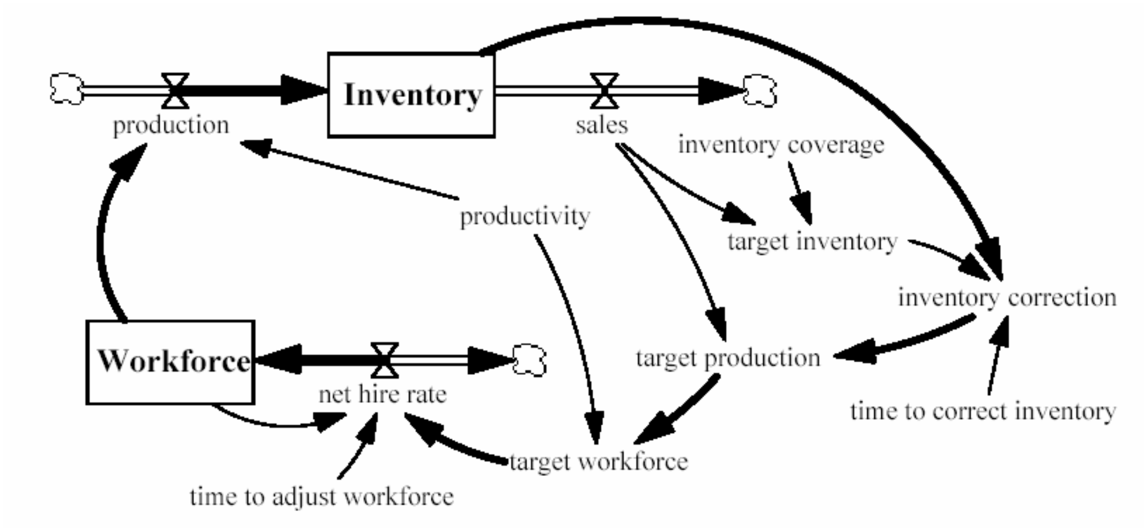


Figure 3.15 Refined model

Additional equations

```
target production = sales + inventory correction Units:
Frame/Month
inventory correction = (target inventory - Inventory)/ TIME
TO CORRECT INVENTORY
Units: Frame/Month

TIME TO CORRECT INVENTORY = 2 Units: Month
target inventory = sales * INVENTORY COVERAGE Units: Frame
INVENTORY COVERAGE = 3 Units: Month
```

inventory correction is a stock adjustment formulation, just as *net hire rate* was. The *time to correct inventory* represents the time required to notice significant changes in inventory and schedule corrections in production.

The important difference between this formulation and that of *net hire rate* is that the *net hire rate* directly influences the stock it is attempting to adjust (*Workforce*) whereas *inventory correction* influences *target production, net hire rate, Workforce, production* and finally *inventory*. This connection has an intervening level, *Workforce*, which has important dynamic consequences.

Refined model behavior

The model is simulated and the run named WF2. First generate a graph of behavior for *Workforce* with datasets from both runs loaded WF1 and WF2.

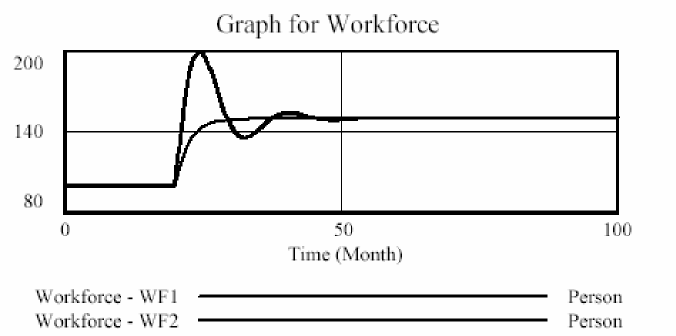


Figure 3.16 Refined model behavior

The behavior is dramatically different from the earlier model. *Workforce* is less stable and there is oscillation. Causal tracing is performed on *Workforce*, and on *target workforce* (graph not shown) and *target production*, and the output shown in Figure 3.17 below:

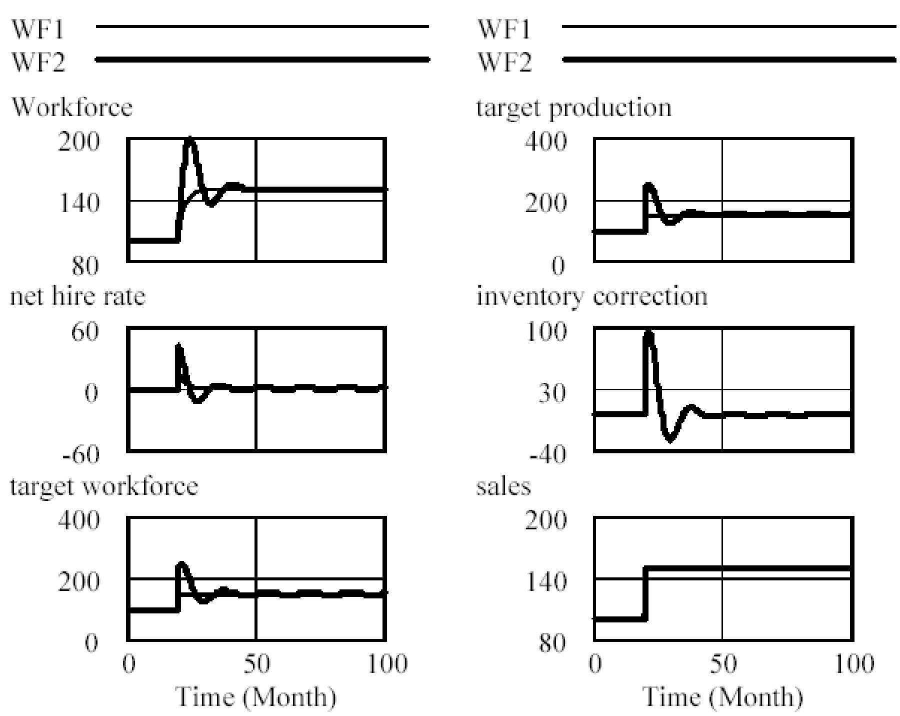


Figure 3.17 Causal tracing for the refined model

inventory correction was not in the first model. Therefore there is no plot from WF1 for *inventory correction*. All the oscillatory behavior in *target production* is due to *inventory correction*. A graph of *Inventory* shows similar oscillation, but a different final value from the first run. When sales increase, inventory now eventually adjusts to a new, higher desired level, rather than falling as before.

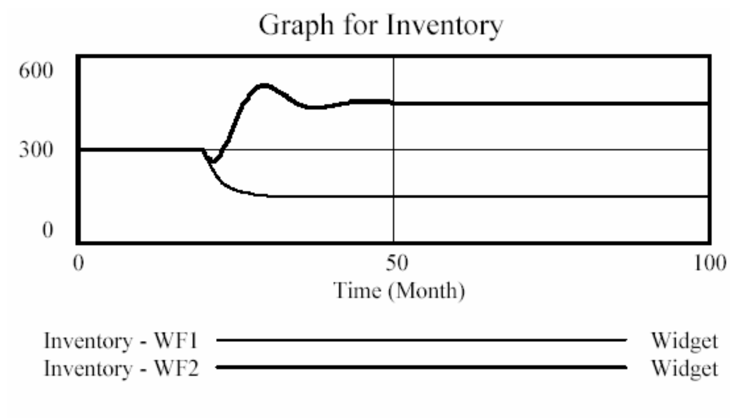


Figure 3.18 Graph for *Inventory* for the refined model

Phasing and oscillation

To get some useful insight into this model, and oscillations in general, use the Graphs tab in the Control Panel to create a graph containing *Inventory*, *target inventory*, *production* and *sales*. Set the scales to zero minimum and 600 maximum values for all variables. You will also need to go the Datasets tab of the Control Panel and put WF2 first.

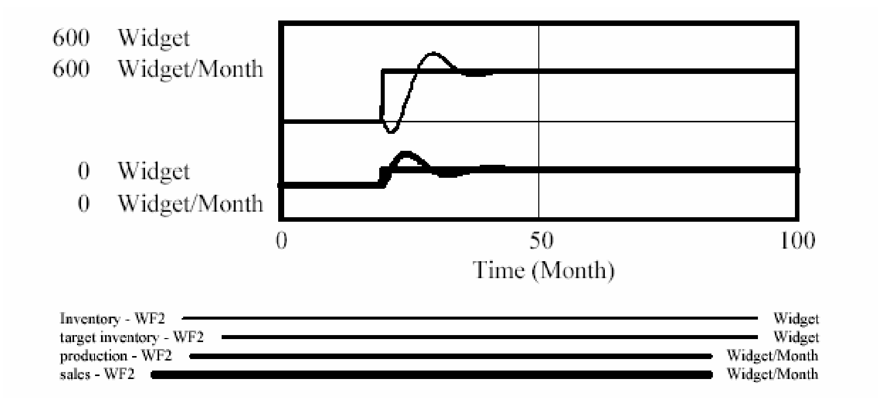


Figure 3.19 Phasing and oscillation

We want to focus in on the timing of changes just after the jump in sales. Hold down the shift key and dragging across the portion of the graph of interest, or go to the Time Axis tab of the Control panel, to set the time focus from approximately 18 months to 36 months. Now the custom graph generates this output:

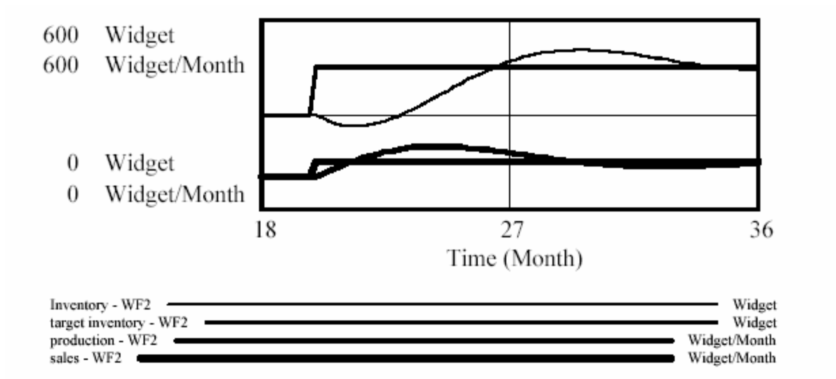


Figure 3.20 Modified phasing and oscillation

Initially, when *sales* step up, inventory begins to fall because *sales* exceed *production*. *production* gradually increases because *target production* increases with *sales* and *inventory correction* increases as inventory falls. However, there is a delay in this process due to the time required to hire new workers. By month 21.3, *production* is equal to *sales*, so *inventory* stops falling and then begins to grow.

While *Inventory* is increasing, so is *production*. As long as *inventory* is below *target inventory*, there is pressure to increase *production*, even when *production* is already above *sales*. In fact, *production* needs to get sufficiently above *sales* so that the ongoing difference balances the pressure from *inventory correction* before *net hire rate* will go negative. Note that when *Inventory* is equal to *target inventory*, *production* is still higher than *sales* because the workforce is excessive, causing an ongoing increase in *Inventory*. The variability in *production* is now greater than that in *sales*; the model is exhibiting the kind of amplification we identified in the reference mode.

In fact, the amplification of variability from sales to production is physically inevitable. The change in production must exceed the change in sales for some time in order to replace inventory lost before production can adjust and to adjust inventory to the new, higher target level.

Sensitivity

It is interesting to experiment with the model by changing different parameters. One possible policy to improve the company's performance would be to correct deviations in inventory more aggressively. We can simulate this by using a decreased value for *time to correct inventory*. Reducing *time to correct inventory* from 2 months to 1 month, we get the simulation output for *production* is shown in Figure 3.21.

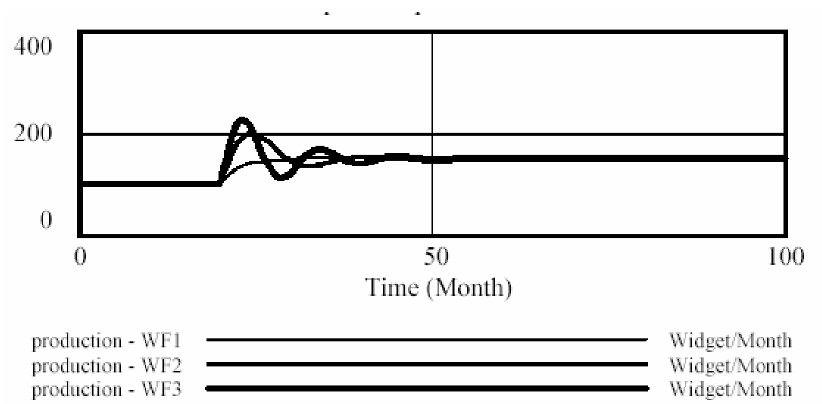


Figure 3.21 Graph for *production*

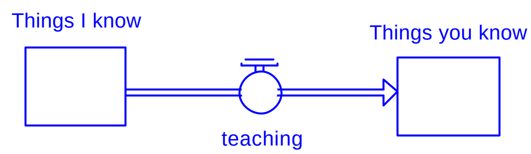
3.7 Problems

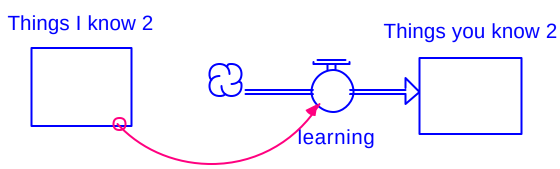
3.1 Identify at least one accumulation that exists within you.

3.2 Identify at least one stock within the engineering activity which your actions help to fill or deplete.

3.3 You have learnt about the basic building blocks (objects) of modeling tool Vensim which are used to develop System Dynamics simulation models. With the knowledge you have gained so far, define the objects " Auxiliary variabler" and "Arrowr" in your own words. Comment on their usage in Vensim models.

3.4 I am teaching you what I know. Each of the following is a map of that process. For each explain what you like and do not like about the map as a representation of the process.





3.5 Create a simple Vensim sketch to represent each of the key feedback loop processes described below:

(a) The activity of construction causes city to grow. As city grows and more people come, construction also increases since there is more people.

(b) At the height of the Cold War, the nuclear arms race was a cause of great public concern. Since neither the US nor the USSR had perfect knowledge of the size of the other's actual arsenal, neither side could be quite sure whether they had 'enough' weapons relative to the other. When the US perceived that the USSR had increased their arsenal the US stepped up its production of weapons in order to keep up. Similar behavior is exhibited by the USSR.

(c) When John performs well in the class, his self-confidence grows. With this increasing self-confidence comes even better performance.

3.6 Develop two simple Vensim models similar to the ones you have built in the first lab session. Describe the problem you have chosen, stocks and flows, etc. Run the models and generate results (just the graphs). Comment on the results based on the changes you observe when you change the flow rates.

3.7 Use one of generic flow templates to represent each of the activities described below:

(a) Generation of noise pollution at the construction site;

(b) Regulation of room temperature with a thermostat; and

(c) Generation of solid waste (optional: show the impact of recycling).

3.8 If you look at the model structure for the window manufacturing problem we have developed in Section 3.6 you will see that it fails the Reality Check "Without Inventory we can't ship." Try to develop a structure to correct this flaw. You will need to create an explicit shipment variable and use a nonlinear function (Lookups are often used here) to constrain shipments.

3.8 References

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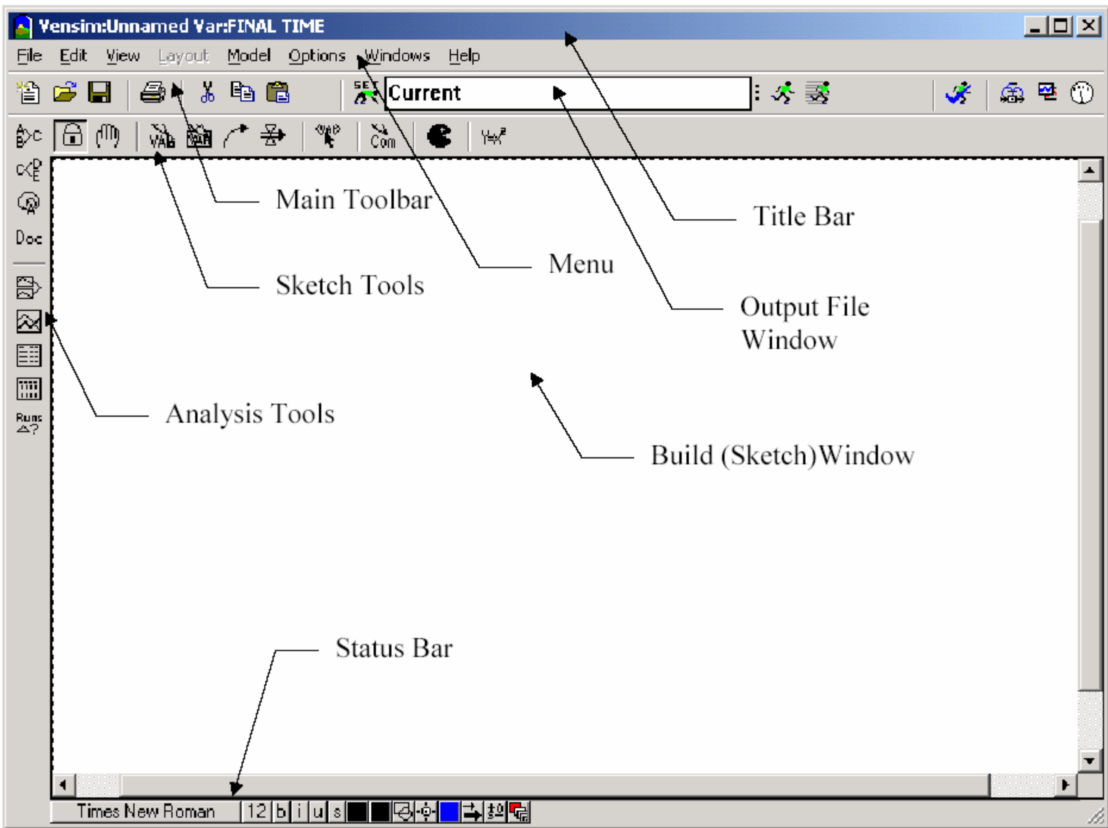
Appendix A.

Vensim® PLE Quick Reference and Tutorial (Kirkwood, 2002)



General Points

1. File operations and cutting/pasting work in the standard manner for Windows programs.
2. Most of the operations you need to routinely do to use Vensim can be carried out from the various toolbars. Many of the menu items are duplicates of toolbar buttons.
3. Nomenclature: The term "click" means to press and release the left mouse button. The term "drag" means to press and hold the left mouse button, and then move the mouse. The term "right-click" means to press and release the right mouse button. An alternative to right-clicking is to "control-click," which means to hold down the control key, and then press and release the left mouse button.
4. Vensim works using a "workbench" metaphor. That is, you create and modify a model by using the various "tools" on the toolbars. At all times there is a "Workbench Variable" which is the model variable that some tools automatically apply to. This variable is shown in the center of the Title Bar. For example, in the diagram above, "FINAL TIME" is the Workbench Variable.

Main Toolbar

Button	Definition
	**New Model:** Creates a new Vensim model.
	**Open Model:** Opens an existing Vensim Model.
	**Save:** Saves the Vensim model under its current name. (The name can be changed using the "Save As" option on the "File" menu.)
	**Print:** Prints the current selection in the Build Window (or the entire sketch, if there is no selection). A "Print Options" dialog will be displayed to make custom settings. A "selection" is made by dragging the mouse to select a rectangular area.
	**Cut:** Cuts the current selection into the Windows Clipboard.
	**Copy:** Copies the current selection into the Windows Clipboard.
	**Paste:** Pastes the current contents of the Windows Clipboard into the sketch.
	**Set up a Simulation:** Highlights the constants and lookups on the sketch in the Build Window. Clicking on a highlighted name allows you to temporarily change it for only this simulation run.
	**Name the Simulation to be Made:** The selected dataset is shown in the box. To change this, click on the vertical bar to the right of the box.
	**Run a Simulation:** If the dataset shown in the box to the right of this button already exists, you will be asked if you want to overwrite it.
	**Automatically simulate on change (SyntheSim):** A visual sensitivity analysis simulation approach vs. the change, compute, and review approach.
	**Run Reality Checks:** Allows user to make statements thought to be true about a model for it to be useful, and provides the machinery to automatically test a model for conformance with those statements
	**Build Windows - show/circulate:** Makes the Build (Sketch) Window visible.
	**Output Windows - show/circulate:** Makes the Output Windows visible. If they are visible, circulate the active Output Window.
	**Control Panel:** Shows the Control Panel. This is used to select the Workbench Variable, adjust the time axis for graphs, set the type of scaling for graphs, manage datasets, and create/manage custom graphs.

Sketch Tools

Button	Definition
	**Lock Sketch:** Sketch is locked. Move/Size pointer can select sketch objects and the Workbench Variable but cannot move sketch objects.
	**Move/Size Words and Arrows (Pointer):** Moves, sizes, and selects sketch objects in the Build Window. Move an object by dragging it with the Hand Move/Size pointer. Select an object by clicking on it with the Hand Move/Size pointer. Select multiple objects by dragging across them with the Hand Move/Size pointer. Add or subtract an object to/from the currently selected objects by shift-clicking on it. (Hint: As a shortcut, many of the other tools will act like the Move/Size pointer for purposes of moving objects.)
	**Variable - Auxiliary/constant:** Creates variables (that is, Constants, Auxiliaries, etc.) in the Build Window. Click on the spot in the sketch where you want to insert the variable, and a box will open up for you to enter the name of the variable. To edit the name of an existing variable, click on it with the Variable tool. To enter the variable into the sketch, press the "Enter" key. Right-click on the variable name with the Move/Size pointer or Variable tool to change the way the variable is displayed (for example, to change the font or to put a Clear Box around it so that the name can be displayed on multiple lines.)
	**Box Variable - Level:** Create variables with a box shape in the Build Window (used for Levels/Stocks). Works in a similar manner to the Variable tool. The size of the box can be adjusted by dragging the box handle (small circle) with the Move/Size pointer tool.
	**Arrow:** Creates straight or curved arrows in the Build Window. Click on the origin variable, and then move the arrow pointer to the destination variable and click again to create a straight-line arrow. (Note: Do NOT drag from the origin to the destination.) You can make the line into a curve by dragging on its "handle" (the small circle) with the Move/Size pointer or Arrow tool. (Hint: As a shortcut, you can directly create a curved arrow by clicking on the origin variable, then clicking on a blank part of the sketch, and finally clicking on the destination variable.)
	**Rate:** Creates Rate (Flow) constructs in the Build Window, consisting of perpendicular "pipe" sections, a valve and, if necessary, sources or sinks (clouds). As with the Arrow tool, first click on the origin variable, and then click on the destination variable. If you make one of these clicks on a blank part of the Sketch, then a cloud will be created for the appropriate origin or destination. You can make right-angle bends in the pipe by Shift-clicking on blank parts of the Sketch with the Rate tool where you want the bends to appear. The formatting of the valve can be changed by right-clicking on it.
	**Shadow Variable:** Adds an existing model variable to the sketch in the Build Window as a shadow (ghost) variable (without adding its causes).
	**Sketch Comment:** Adds comments and pictures to the sketch in the Build Window. Click where you wish to place the Sketch Comment, and a dialog will open up with many options for the form of the comment. Note that with some types of comments you may have difficulties accessing some model variables after you have created the comment because the comment is overlaid on the model variable. If this happens, push the comment to the background using the rightmost button on the Status Bar.
	**Delete:** Deletes structure, variables, or comments from a sketch in the Build Window.
	**Equations:** Creates and edits model equations using the Equation Editor. After the Equation tool is selected, variable without equations will be highlighted in the Build Window. Click on a variable to start the Equation Editor for that variable.

Analysis Tools

Button	Definition
	**Causes Tree:** Creates a tree-type graphical representation showing the causes of the Workbench Variable.
	**Uses Tree:** Create a tree-type graphical representation showing the uses of the Workbench Variable.
	**Loops:** Displays a list of all feedback loops passing through the Workbench Variable.
	**Document:** Presents equations, definitions, and units of measure for the model.
	**Causes Strip:** Displays graphs in a strip for the Workbench Variable and its causes, which allows you to trace causality by showing the direct causes of the Workbench Variable. (This can also be used to display the graph for a lookup function.)
	**Graph:** Displays a graph for the Workbench Variable in a larger graph than the Causes Strip. Note that a custom graph showing any desired variables can be developed using the Graphs tab on the Control Panel.
	**Table:** Generates a table of values for the Workbench Variable that is displayed in a row.
	**Table Time Down:** Generates a table of values for the Workbench Variable that is displayed in a column.
	**Runs Compare:** Compares all Lookups and Constants in the first loaded dataset to those in the second loaded dataset. You manage datasets using the Datasets tab on the Control Panel: To change the order of the loaded datasets, click on a loaded dataset that is currently not at the top of the loaded dataset list, and this dataset will be moved to the top of the loaded dataset list. The order of the loaded datasets impacts the order in which datasets are displayed by various graph tools.

Notes on the analysis tools:

1. The output of the analysis tools is “dead” in the sense that it does not change if you make more simulation runs. Therefore, if you are doing experimentation, be careful that you keep track of what simulation run generated the output displayed in each output window.
2. Most output windows display information related to the Workbench Variable. You can select a variable to be the Workbench Variable by double-clicking on it in the Sketch. You can also select the Workbench Variable by double-clicking on a variable name in an output window.
3. The icons in the upper left corner of the output windows control various useful things. The horizontal bar at the left-most end deletes the window. The small padlock next to this is used to lock-out the delete function so that you do not accidentally delete the window. (Click this again to reactivate the delete function.) The small printer icon is used to print the contents of the window, and the clipboard icon is used to copy it into the Windows Clipboard. The floppy disk icon is used to save the window contents to a file.
4. For graph output when there are multiple curves on the same graph, the curves are displayed in different colors on the computer screen. With some black-and-white printers, it may be difficult to tell the curves apart in printed output. The curves can be marked with distinct numbers using the “Show Line Markers on Graph Lines” option on the “Options” menu.

Status Bar

Button	Definition
Times New Roman	**Set fonts on selected vars:** Select a variable in the Build Window by clicking on it before clicking this button. Select multiple variables by dragging across them with the Move/Size pointer tool. Add or remove a variable from the selected set by shift-clicking on it with the Move/Size pointer. If no variables are selected, then you can change the defaults used in the model.
12	**Set size on selected vars:** Select the variable(s) before clicking on this button.
b	**Set bold on selected vars:** Select the variable(s) before clicking on this button.
i	**Set italic on selected vars:** Select the variable(s) before clicking on this button.
u	**Set underline on selected vars:** Select the variable(s) before clicking on this button.
s	**Set strikethrough on selected vars:** Select the variable(s) before clicking on this button.
	**Set color on selected vars:** Select the variable(s) before clicking on this button.
	**Set box color on selected vars:** Select the variable(s) before clicking on this button.
	**Set surround shape on selected vars:** Select the variable(s) before clicking on this button.
	**Set text position on selected vars:** This is useful for variables that have an associated graphic, such as a level box or a rate valve.
	**Set color on selected arrows:** Select an arrow by clicking on its handle (small circle). Select multiple arrows by dragging across them with the Move/Size pointer tool. Add or remove an arrow from the selected set by shift-clicking on its handle with the Move/Size pointer.
	**Set arrow width on selected arrows:** This is used to set width. More detailed control of the appearance of an arrow can be obtained by right-clicking with the Move/Size pointer tool on the handle (small circle) of the arrow. In particular, a delay mark can be put on the arrow in that way.
	**Set polarity on selected arrows:** Use this to mark arrows in a causal loop diagram. More control of this and other aspects of arrows can be obtained by right-clicking with the Move/Size pointer on the handle (small circle) of the arrow.
	**Push the highlighted words to the background:** This is useful if you have created a large comment with the Sketch Comment tool that overlaps one or more model variables (for example, a large box around some portion of the model). In this case, you may not be able to select the variables because they are "behind" the comment. Select the comment and push it to the background to fix this problem.

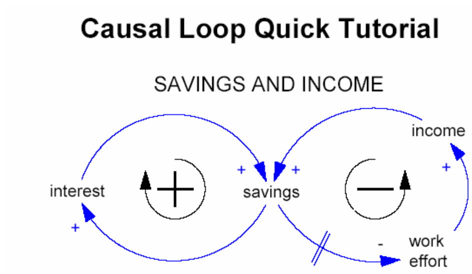
Highlights of the Menus

Certain aspects of Vensim PLE can only be controlled by using menu items. This list shows some useful menu options:

1. **Edit:** There are "Find" options on this menu that can be used to track down all instances of a variable in a model sketch. This is useful when there are shadow (ghost) variables so that a variable can show up in more than one place in the model sketch.
2. **View:** The "Refresh" option repaints the computer screen. Certain computer video drivers have difficulty correctly responding to Windows graphics commands. This can result in "trash" on the screen (for example, an arrow that is not attached to anything, and that you cannot delete). Use the Refresh option to remove this trash. If this happens often, see item 4 below for a more permanent fix.
3. **Model:** The "Settings>Time Bounds tab" redisplays the "Time Bounds for Model" dialog that is displayed when you initially create a model. Use this dialog to change the time bounds variables (INITIAL TIME, FINAL TIME, TIME STEP, SAVEPER, and units for time).
4. **Options:** If you are continually having trash displayed on your computer screen (see item 2 above), then select the option "Continually Refresh Sketches." This should eliminate the problem, but it may slow down working with sketches somewhat. If you cannot tell the different curves apart when you print more than one curve on the same graph, then select "Show Line Markers on Graph Lines" to have each curve marked with a different number.
5. **Help:** There is extensive online help for Vensim PLE

Miscellaneous Useful Items

1. With some printers, you may find that the curves for one or more arrows are not shown in printed output of the model sketch. This can happen when the curve is almost, but not quite, a straight line. (It will be so close to a straight line that you may not be able to see the curvature on the computer screen.) Make sure that the curve is either truly a straight line or that it has visible curvature on the computer screen, and it should print all right.
2. Vensim can check units for your models. While it takes a little time to enter units for your equations, doing this can catch certain types of common errors (for example, using units of "days" one place and "weeks" somewhere else).
3. The Equation Editor is generally self-explanatory, but there is one useful point that is not immediately obvious; namely, how to enter a lookup function using the graphical dialog. When you start the Equation Editor on a variable that you wish to make into a lookup function, the "Type" of the variable will probably be shown as "Constant" in the left-center of the Equation Editor dialog. Drop down the type selection box and change the type to "Lookup." After you do this, a new button labeled "As Graph" will appear in the dialog next to the "Help" button a couple of lines below the Type box. Click this to start the Graph Lookup definition dialog.



In this tutorial, you will create the causal loop diagram shown above. Use the following steps:

1. Start Vensim, and click the New Model button on the Main Toolbar. The "Model Settings Time Bounds" dialog will display. You only need to worry about this for simulation models, so just click the "OK" button. A blank Build Window will appear. This is where you create your causal loop diagram. (A diagram in the Build Window is referred to as a "sketch.")
2. Set the default font for this sketch to Arial 10 point, as follows: Click on the font name at the left end of the Status Bar. Since you do not have any items selected, the dialog will ask if you want to change the font and color defaults. Click the "Yes" button. The "View Defaults" dialog will be displayed. Change the "Face" to Arial and the "Size" to 10 points, and then click the "OK" button.
3. Select the "Variable - Auxiliary/Constant" tool from the sketch tools by clicking on it. Then click on the spot in the Build Window where you want to place the variable "interest." An editing box will open. Enter "interest", and press the "Enter" key. This word "interest" will appear on the sketch in Arial 10 point type. Repeat this process to create the variables "savings" and "income," as shown in the diagram above. (Hints: If you discover that you have incorrectly spelled a variable name, select the "Variable - Auxiliary/Constant" tool, and click on the incorrectly spelled name. This will open the editing box for the variable. If you wish to completely remove a variable, or other element, from a sketch, select the "Delete" tool from the sketch tools, and click on the element to remove it.)
4. Repeat this process again to create the variable "work effort" shown in the diagram above. When you do this, both "work" and "effort" will appear on the same line. To put them on separate lines, as shown in the sketch above, place the tip of the mouse arrow on the small circle that is to the right and below the words "work effort," and drag this circle with the mouse until "work" and "effort" are on separate lines. To change other characteristics of how "work effort" is displayed, do the following: Right-click (or, equivalently, control-click) on "work effort" in the sketch. A dialog box will open that provides a variety of options for how this variable is displayed. In the upper left section of this dialog, which is labeled "Shape," note that the option "Clear Box" is selected. The small circle that you dragged to rearrange "work effort" is a "handle" that you use to change the shape of the Clear Box around "work effort." Note that after you finish working on a variable with the "Variable – Auxiliary/Constant" tool the handle will disappear. When this happens, you can make the handle reappear by selecting the "Move/Size Words and Arrows" pointer tool from the sketch tools.

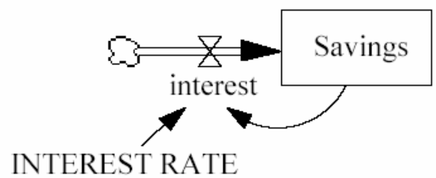
5. Select the “Arrow” tool from the sketch tools by clicking on it. Then click on the variable “interest” in the Build Window, release the mouse button, and then click on the variable “savings.” A straight arrow will appear from “interest” to “savings.” Select the Move/Size pointer tool, and drag the handle (small circle) on this arrow to curve it as shown above. (Hint: As a shortcut, you can also drag using the Arrow tool. Put the tip of the curved arrow mouse pointer over the handle of the arrow in the sketch, and the curved arrow mouse pointer will change to a small hand. Then press the mouse button to drag.)
6. Repeat this process to draw arrows from “savings” to “interest,” “savings” to “work effort,” “work effort” to “income,” and “income” to “savings,” as shown in the sketch above. (Hints: You can speed up the process of creating a curved arrow by clicking with the Arrow tool at an intermediate point on the sketch between the variable names as you create the arrow. This will create a curved arrow in a single step. To remove an arrow, place the tip of the pointer on the “Delete” tool over the arrow’s head and click.)
7. You can move a variable by dragging it using the Move/Size tool. Note that as you move a variable, the arrows connected to that variable rearrange themselves to remain connected to the variable. (Hint: You can also drag a variable with the “Variable - Auxiliary/Constant” tool or the “Arrow” tool. Place the pointer of the tool over the variable name, hold down the left mouse button, and drag.)
8. Place a delay symbol on the arrow from “savings” to “work effort,” as shown in the sketch at the top of this tutorial, by right-clicking with the Move/Size tool on the arrow’s handle (small circle) in the Build Window, and then checking the “Delay marking” option in the upper center part of the dialog box that is displayed. (Hints: As a shortcut, you can also display this dialog box by right-clicking with the tip of the pointer for the “Arrow” tool cursor on the arrow’s handle. You can select more than one arrow at the same time by dragging the Move/Size pointer tool across their handles. You can add an additional arrow to the set of selected arrows by shift-clicking on its handle. When you select an arrow, its handle fills in.)
9. Add the plus signs shown in the sketch above to the arrows by selecting the arrows in the sketch as discussed in Step 8, and then clicking on the “Set polarity on selected arrows” button on the Status Bar. Select the plus sign from the displayed options. Follow an analogous procedure to add the minus sign to the arrow from “savings” to “work effort.”
10. You can change the positions of polarity signs by right-clicking on each arrow’s head or handle in turn and selecting the desired location from the “Position polarity mark at the” buttons at the bottom of the dialog. If you wish, you can also change the size of a plus or minus sign by changing the “Font” for the arrow from this same dialog.
11. Place the positive feedback loop symbol inside the positive feedback loop as follows: Select the “Sketch Comment” tool from the sketch tools by clicking it. Then click inside the positive feedback loop on the sketch. From the Comment Description dialog that opens, select a “Shape” of “Loop Clkwse.” Select “Image” under “Graphics,” and drop down the list of images to the right of this. Select the plus sign, and click on the “OK” button. The negative feedback loop symbol can be added to the sketch in an analogous manner with a “Shape” of “Loop Counter.”

12. Finally, add the title “SAVINGS AND INCOME” by selecting the “Sketch Comment” tool, clicking on the sketch above the feedback loops, and entering “SAVINGS AND INCOME” into the “Comment” box. Change the font size to 12 points, and then click the “OK” button. Drag the handle of the Clear Box around this title to position all of the words on the same line.
13. Experiment with printing the diagram, and copying it into the Clipboard and then pasting it into a word processing document. Note that the printed or copied diagram does not have the handles (small circles) that may be shown on the sketch. Note that you must select the diagram to copy it into the Clipboard. You can do this by selecting “Select All” from the Edit menu, or by dragging across the entire diagram with the Move/Size pointer tool. (Hint: I f you use “Select All,” portions of curved arrows near the edge of the sketch may be chopped off in the copied diagram.)
14. If you wish, save your diagram using the “Save” button on the Main Toolbar.
15. Vensim PLE provides three analysis tools for analyzing the logical structure of a causal loop diagram: “Causes Tree,” “Uses Tree,” and “Loops.” Specifically, the “Causes Tree” and “Uses Tree” tools show the variables that are causally connected to the current Workbench Variable in tree diagrams, while the “Loops” tool shows the causal (feedback) loops that contain the Workbench Variable.
16. Make sure that “savings” is selected as the Workbench Variable. The name of the Workbench Variable is shown following the rightmost colon on the “Title Bar.” If “savings” is not selected as the Workbench Variable, select it by clicking on it in the sketch with the Move/Size tool. It will now be shown as the Workbench Variable on the Title Bar.
17. Click on the “Causes Tree” analysis tool. A window will open that displays the diagram shown in the left box below. Click on the “Uses Tree” analysis tool, and a window will open that displays the diagram shown in the right box below. These two tree diagrams trace backward and forward, respectively, through the causal link structure of the causal loop diagram starting at the current Workbench Variable, which is “savings.” Note that when a loop is completed back to the Workbench Variable, that variable’s name is shown in parentheses.

18. Make sure that “savings” is still selected as the Workbench Variable, and click on the “Loops” analysis tool. The information shown below will be displayed. This is a list of all the causal (feedback) loops through the Workbench Variable.

Loop Number 1 of length 1
savings
interest
Loop Number 2 of length 2
savings
work effort
income

Simulation Model Quick Tutorial



(1) FINAL TIME = 100
Units: Year
The final time for the simulation.

(2) INITIAL TIME = 0
Units: Year
The initial time for the simulation.

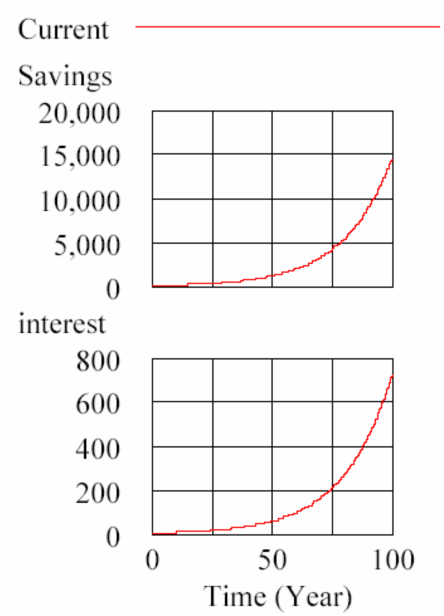
(3) interest=
Savings*INTEREST RATE
Units: **undefined**

(4) INTEREST RATE=
0.05
Units: **undefined**

(5) SAVEPER =
TIME STEP
Units: Year
The frequency with which output is stored.

(6) Savings= INTEG (
interest,
100)
Units: **undefined**

(7) TIME STEP = 0.25 [0,?]
Units: Year
The time step for the simulation.



Before starting this tutorial, complete the preceding “Causal Loop Quick Tutorial.” The three figures above show a stock-and-flow diagram, simulation model equations, and an output strip graph for a model of a savings account. Create these as follows:

1. Start Vensim and click the “New Model” button on the Main Toolbar. The “Time Bounds for Model” dialog will display. To make the periods quarters, change the “TIME STEP” to “0.25” (without the quotes) and the “Units for Time” to “Year.” (This can be done by selecting the “Year” option from the drop down menu. Then click the “OK” button.)

2. Select “Box Variable -L evel” from the sketch tools, and click on the spot in the Build Window where you want “Savings” to appear. Enter “Savings” into the editing box that opens, and then press the “Enter” key. The “Savings” box will appear on the sketch.
3. Select the “Rate” tool from the sketch tools. Click to the left of the “Savings” box on the sketch in the Build Window, and release the mouse button. A cloud will appear on the sketch where you clicked. Move the mouse pointer to the “Savings” box and click. An editing box will appear. Enter “interest,” and press “Enter.” A flow pipe will now connect the cloud to the “Savings” box, and there will be a valve on this pipe with “interest” underneath, as shown in the stock-and-flow diagram at the top of this tutorial.
4. Select the “Variable - Auxiliary/Constant” sketch tool, and click on the spot in the Build Window where you want to place “INTEREST RATE.” Enter “INTEREST RATE into the editing box that opens, and press the “Enter” key. This variable now appears in the sketch.
5. Select the “Arrow” tool from the sketch tools, and draw arrows from “Savings” to “interest,” and from “INTEREST RATE” to “interest.” This completes the stock and flow diagram shown above.

Entering equations

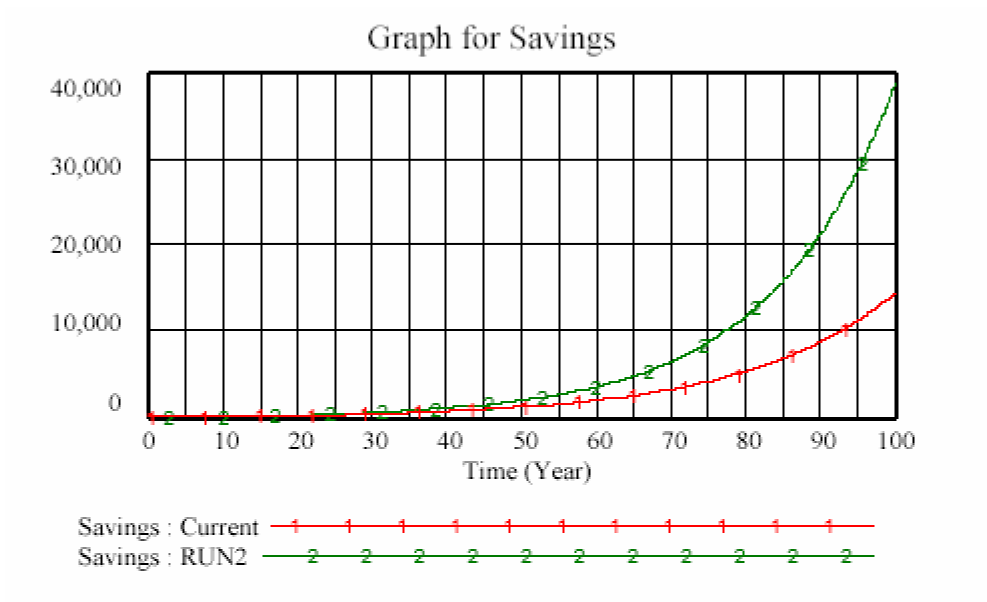
6. Select the “Equations” tool from the sketch tools. The variables “interest,” “INTEREST RATE,” and “Savings” will be highlighted in reverse video on the sketch. This highlighting signifies that these variables do not have valid equations assigned to them.
7. Click on “interest” in the sketch, and the “Editing equation for” dialog will open. Enter “Savings*INTEREST RATE” (without the quotes) into the box near the top of the dialog to the right of the equals sign. (Hint: You can “click in” this entire equation with the mouse using the keypad in the center of the dialog and the list of variable in the right-center of the dialog. For a simple equation like this one, this does not save much time over typing in the equation. However, it is a good idea to click in variable names so that you do not misspell them.) Click the “OK” button after you finish entering the equation.
8. If you have entered the equation correctly, the “Editing equation” dialog will close, and the “interest” variable will no longer show in reverse video on the sketch. In order to see how Vensim indicates an error in an equation, you may wish to spell “Savings” incorrectly in the equation. (Hint: You can click on a variable name in the sketch at any time the “Equations” tool is selected to edit the equation for that variable, even if you have already entered an equation for the variable.)
9. Click on “Savings” in the sketch, and the “Editing equation” dialog will open for “Savings.” Since Vensim is able to determine from the model sketch that the level of “Savings” is equal to the integral of “interest,” this equation is already shown next to the equal sign near the top of the dialog. Enter “100” (without the quotes) into the “Initial Value” box near the top of the dialog. As the name of this box indicates, it is for setting the initial value for “Savings.” Click the “OK” button to close the “Editing equation” dialog.

10. Click on “INTEREST RATE” in the sketch, and enter “0.05” (without the quotes) into the box near the top of the dialog to the right of the equals sign. Click the “OK” button. You have now finished creating a model for a savings account that has an initial balance of 100 and an interest rate of 0.05 (that is, 5 per cent) per year. To show the equations for this model, select the “Document” tool from the analysis tools. The set of equations shown at the top of this tutorial will appear in a window.

Running a traditional simulation and displaying the results

11. Run the simulation model by clicking on the “Run a Simulation” button on the Main Toolbar. I f a dataset named “Current” already exists on your computer, you will be shown a message
    asking you whether you want to write over this. Answer “Yes” if this message appears. You will then momentarily see a dialog counting off the time as the simulation executes.
12. Make sure that “Savings” is selected as the Workbench Variable. The name of the Workbench Variable is shown following the rightmost colon on the “Title Bar.” If “Savings” is not selected as the Workbench Variable, select it by clicking on it in the sketch with the Pointer sketch tool. It will now be shown as the Workbench Variable on the Title Bar. Now select the “Causes Strip” tool from the analysis tools. The graphs for “Savings” and “interest” shown at the top of this tutorial will appear in a window.
13. You may wish to save your model using the “Save” button on the Main Toolbar. Also, you will find it useful to experiment with the “Build Windows - show/circulate” and “Output Windows - show/circulate” buttons on the Main Toolbar. Note that the Build Window containing the sketch of the stock-and-flow diagram is a standard window just like the Output Windows that show the model equations and the curves for the simulation output. All these windows act like normal Windows windows. In particular, if you click on an exposed part of the sketch in the Build Window while Output Windows also show on the computer screen, these Output Windows will vanish. What has happened is that the Build Window has been brought to the top of the stack of windows, and it now covers the Output Windows. You can bring the Output Windows back on top of the Build Window by clicking “Output Windows - show/circulate” on the Main Toolbar.
14. Conduct a sensitivity analysis for “INTEREST RATE” as follows: Click on the “Set up a Simulation” button on the Main Toolbar. When you do this, “INTEREST RATE” will be highlighted on the stock and flow diagram. This means that it is a constant which can be changed for a particular simulation run. Click on the highlighted “INTEREST RATE,” and an editing box will open showing the “INTEREST RATE” value of 0.05. Change this to 0.06, and press “Enter” to close the editing box. (Hint: Note that this change only applies to the next simulation run. After that, “INTEREST RATE” reverts to 0.05.)
15. Click the “Run a Simulation” button on the Main Toolbar. A dialog box will open saying “Dataset Current already exists. Do you want to overwrite it?” Answer “No.” Another dialog will open showing the datasets that are stored in the current directory. Enter “run2” for the “File name,” and press the OK button. The simulation now runs, and the results are stored in a new dataset file called “run2.vdf.”

16. Make sure that "Savings" is selected as the Workbench Variable, and click the "Graph" button from the analysis tools. The graph shown on the next page will be displayed, except that the two curves will be shown in different colors, rather than marked by "1" and "2." To add these markers, select "Options" on the "Menu," and check the option "Show Line Markers on Graph Lines." (Hint: If you print your graphs on a black and white printer, you will want to add these markers to better distinguish the curves in the printed output.)
17. Click on the "Runs Compare" analysis tool. This will display a window that shows the values of "INTEREST RATE" for the two simulation runs that have been completed.



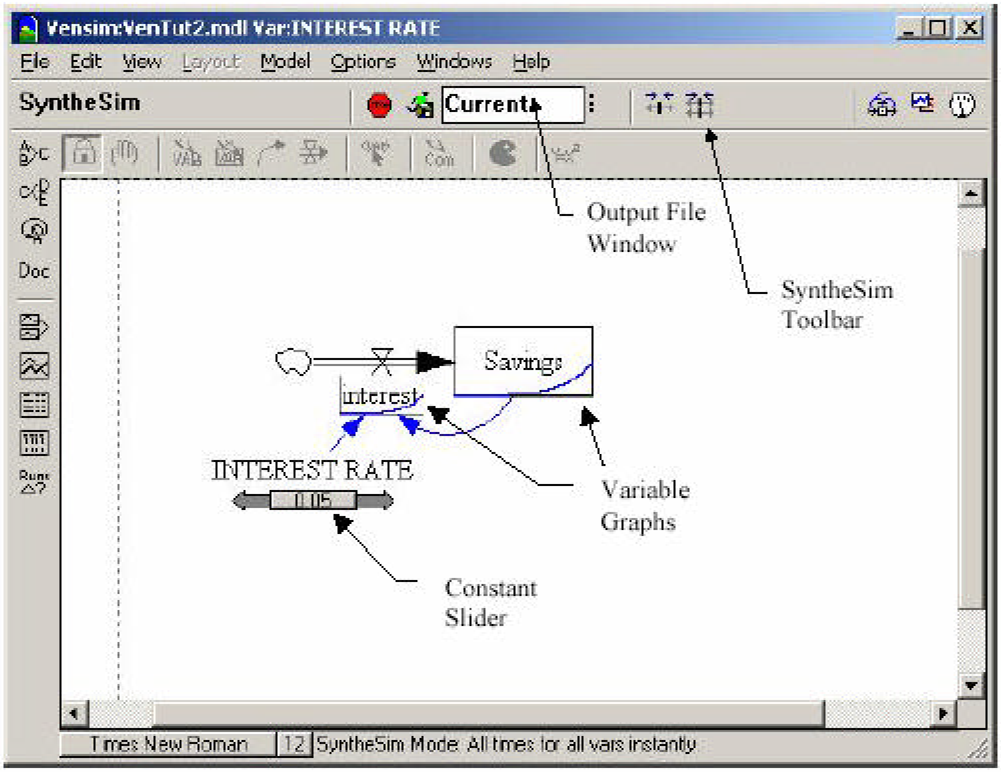
18. You can change the display order of the curves on your graph, as well as add or delete curves from those shown on the graph. Do this by selecting "Control Panel" from the "Main Toolbar," and then clicking on the "Datasets" tab. This shows the datasets that are currently "loaded" (that is, being used to create graphs) as well as other datasets that are available. A dataset can be "unloaded" by clicking on it in the list of "Loaded" datasets, and then clicking the double left-pointing arrow (<<). Once a dataset is unloaded, it will no longer be displayed when you create a graph until you reload it from the list of "Available" datasets by clicking it, and then clicking the double right-pointing arrow (>>). (Hints: The datasets are displayed on a graph in the order that they are listed in the "Loaded" list. If you click on a dataset that is not currently at the top of the "Loaded" list, this dataset will be moved to the top of the list. Note that any changes you make using the Control Panel options apply only to graphs that are created after you make these changes. All previously created graphs are "dead" in the sense that they do not change after they are created.)
19. You can create a table of values for the Workbench Variable by selecting the "Table" analysis tool. The "Table" window is "live," unlike the graphical output windows.

That is, as long as you leave this window open, each time you select the "Table" tool, the data for the current Workbench Variable is added to the data already in the "Table" window. (Hint: The "Table" tool is useful for creating data tables to be copied into a spreadsheet when you wish to create more complex graphical displays than is possible with Vensim. The "Table Time Down" analysis tool also creates a table for the Workbench Variable, but this table is arranged with values for different times in separate rows rather than separate columns.)

20. The causal loop analysis tools ("Causes Tree," "Uses Tree," and "Loops") can be used to analyze the causal loop structure of a stock and flow diagram. Also, you can enter units for your variables in the equation editor. If you do this, then Vensim will check units for you.
21. You can control the time period used for simulation runs by selecting the "Time Bounds…" option from "Model>Settings" on the Menu. On the tab that is displayed for "Time Bounds…," the "INITIAL TIME" entry sets the starting time for the simulation, and the "FINAL TIME" entry sets the ending time for the simulation. "TIME STEP" is the time interval at which new values are calculated for the model variables, and "SAVEPER" is the time interval at which the values are saved for the model variables. (Hints: The value of SAVEPER only impacts the interval at which the simulation results are saved for display in graphs or tables, and not the accuracy of the simulation calculations. SAVEPER should be set to an integer multiple of TIME STEP. If you set TIME STEP to a value less than one, then it is usually a good idea to set it to a power of 0.5, that is 0.5, 0.25, 0.125, 0.0625, etc. If TIME STEP is set to other values, then there may be roundoff error due to the way that computers store fractional numbers. The graphical display of simulation results can be controlled from the "Time Axis" tab of the "Control Panel" item on the Main Toolbar. Consult the online help for further information about this.)

Working with graphical sensitivity analysis (SyntheSim)

22. You can graphically view sensitivity of a model to the values of constants using a tool called "SyntheSim." When using SyntheSim, the results of the simulation are graphically overlaid on the model. Scales with bar sliders are used to represent changes that can be made to constants. Graphs are used to represent the output or impact on model variables.
23. Start SyntheSim by clicking on the "Automatically simulate on change" button on the Main Toolbar. If a dataset named "Current" already exists, you will be shown a message asking you whether you want to write over this. Answer "Yes" if this message appears. (See Step 28 for a full explanation.) You will then see the SyntheSim screen as shown below.

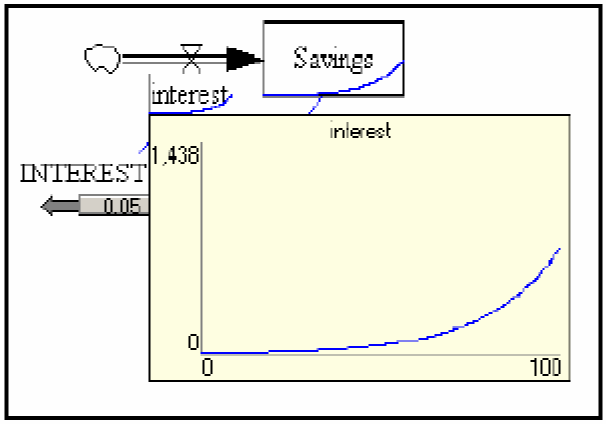


24. New buttons appear on the Main Toolbar when using SyntheSim. The table below describes each SyntheSim button.

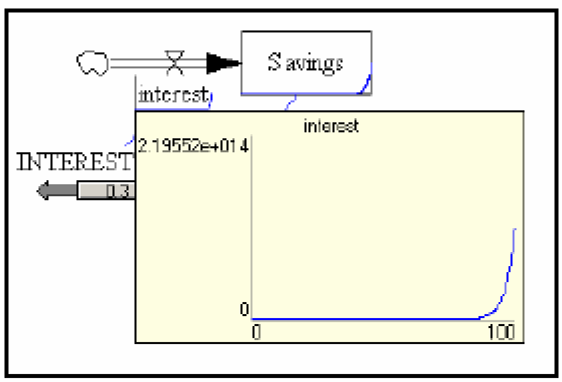
SyntheSim Toolbar

Button	Definition
	**Automatically simulate on change:** Starts the simulation
	**Stop simulating:** Stops the simulation and returns to the model definition view
	**Save this run to…:** Save the current values for the variables to a dataset
	**Reset Current Slider to base model val (Home):** Reset the selected constant slider to the value in the base model
	**Reset all Constants/Lookups to base model vals (Ctrl+Home):** Reset all constant sliders to the values in the base model

25. To see a variable graph in larger detail, hover the mouse pointer over the variable. For instance, hover the mouse pointer over the "interest" variable. The larger graph is superimposed on the variable as shown below.



26. To view the sensitivity of the model, drag the slider for INTEREST RATE right and left and watch how the variable graphs change in real time. Thus, using SyntheSim you can view the sensitivity of a model to changes in the values of constants without saving the results for each model run, rerunning, and comparing results. This graphical response is illustrated in the next figure. Notice that the INTEREST RATE slider was moved from 0.05 to 0.30, and the "interest" and "Savings" variables now rise at steeper rates to much higher final values.



27. To compare a SyntheSim sensitivity run to a previous run, click on the "Save this Run to…" button. Then change the dataset file name and click Save. Vensim will stop saving the SyntheSim results to the previous dataset file and will start saving the results using the current values for any constants that are changed to this new dataset file. Note that this does not change the base model. For example, the Document tool will still show the constant values as established before SyntheSim was started. However, the analysis tools (such as Causes Strip, Graph, and Table) will show the values for sliders and dependent variables as they were last changed in SyntheSim.

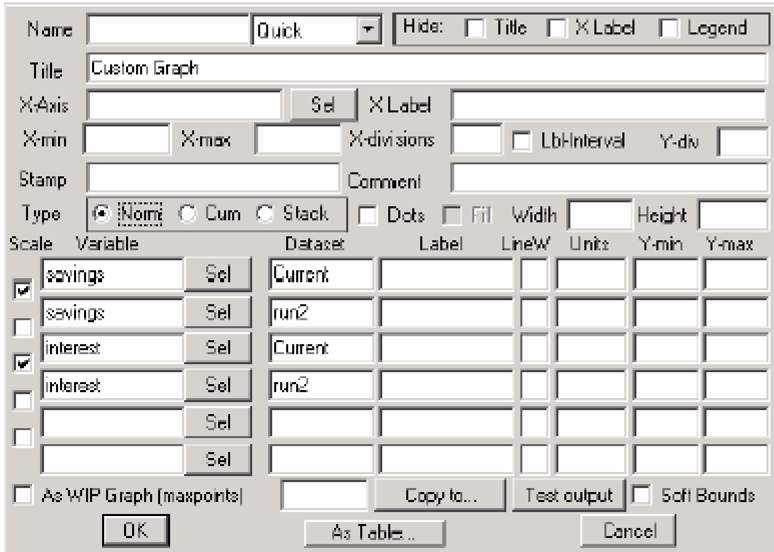
28. **Warning:** Any changes made in SyntheSim are automatically saved to the dataset file shown in the "Output File" window even if you do not explicitly save the dataset. Therefore, use caution when setting the "Output File" for SytheSim, or you may overwrite a dataset file that you wish to save. Ensure that the dataset in the "Output File" window is a dataset that you wish to change when using SyntheSim. (Recall

that when the SyntheSim simulation starts the prompt asks if it is ok to overwrite the current dataset. You can say no and change the dataset file name there—refer to Step 23).

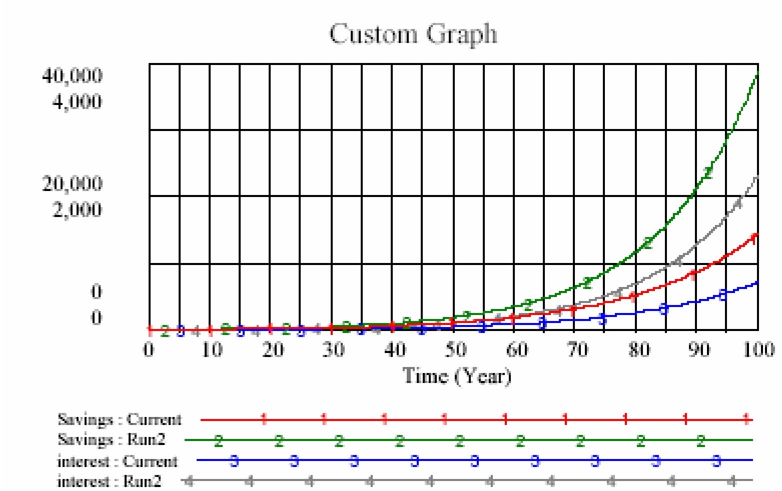
29. To reset the constants in the model to their base model values, use either of the reset buttons. "Reset Current Slider to base model val" will reset only the selected constant slider to the value in the base model. "Reset all Constants/Lookups to base model vals" will reset all constant sliders to the values in the base model. 30. To stop SyntheSim, click the "Stop simulating" button. This will return you to the model definition view.

Working with custom graphs

31. You can create a custom graph by selecting "New…" on the "Graphs" tab of the Control Panel, which is on the Main Toolbar. For example, suppose that you wish to display graphs of "Savings" and "interest" for both simulation runs "Current and "run2," which you created earlier, on a single graph. There is no way to do this using the "Causes Strip" or "Graph" tools from the analysis tools." However, you can create a custom graph that shows all four of these curves on a single graph.
32. To create this custom graph, select the "Control Panel" from the Main Toolbar, and then select the "Graphs" tab from the dialog that is displayed. To create a new custom graph, click the "New…" button in the lower right portion of the dialog, and the dialog shown at the top of the next page (but not yet filled in as shown in that figure) will be displayed. Using that figure as a guide, fill in the "Title," and the entries shown on the "Variable" and "Dataset" columns in the lower portion of the dialog. The four entries in the "Variables" and "Datasets" columns specify which variables, from which simulation datasets, are to be displayed.
33. Finally, check the two boxes as shown under the entry "Scale" in the lower left portion of the dialog. Each checked box specifies that the variable immediately above the checked box and the variable immediately below the checked box are both to be displayed using the same scale on the vertical axis of the graph. Thus, with the boxes checked as shown in the figure on the next page, the two graphs for "Savings" are to be shown with the same vertical scale, and the two graphs for "interest" are to be shown with the same vertical scale (although the scales for the two pairs of graphs do not have to be the same). Click the "OK" button to close the custom graph definition dialog.



34. You can display the custom graph that you have just defined by clicking the "CUSTOM_GRAPH" entry on the Control Panel "Graphs" tab, and then clicking the "Display" button. When you do this, the graph below is displayed. Note that the vertical axis scale 40,000, 20,000, and 0 applies to the two graphs for "Savings," while the vertical axis scale 4,000, 2,000, and 0 applies to the two graphs for "interest." Further information about defining custom graphs is available in the online help.

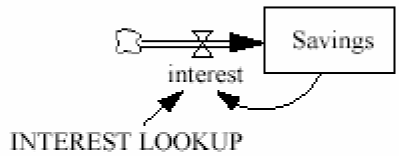


Working with lookup functions

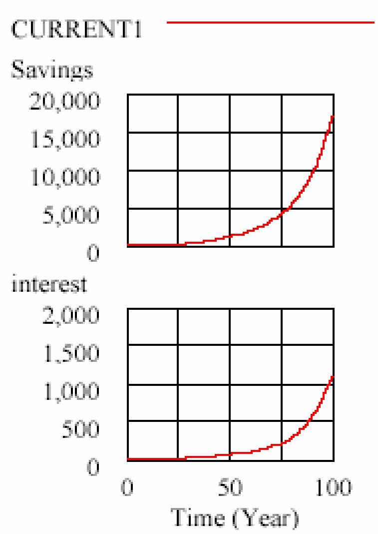
35. You can specify an arbitrary functional relationship between two variables in a simulation model using a lookup function. To do this, you specify a table of pairs of values for the two variables, and Vensim uses linear interpolation to determine a value of the dependent variable for any value of the independent variable that is not specified in the table. To illustrate this approach, we will modify the savings account model developed above to consider a situation where there is a five percent interest

rate on savings amounts less than $5,000, and a seven percent interest rate on savings amounts of $5,000 or greater.

36. Modify the stock and flow diagram for the savings account that you created earlier to create the stock and flow diagram shown below. To do this, change the constant name "INTEREST RATE" to "INTEREST LOOKUP."



37. Use the "Equations" tool on the Main Toolbar to specify the lookup function equation for "INTEREST LOOKUP." To do this, open the equation editor on INTEREST LOOKUP. On the left side of the equation editing dialog, about two-thirds of the way up the dialog, there is an entry titled "Type" which shows "Constant" as the type of INTEREST LOOKUP. Click the arrow just to the right of the "Constant" to display a dropdown list of possible variable types. From this list, select "Lookup." After you do this, a new button labeled "As Graph" will appear to the left of the "Help" button on the left side of the dialog below the "Type" entry.
38. Click on this "As Graph" button. When you do this, the Graph Lookup definition dialog will be displayed. You can enter a lookup function either by drawing it on the graph in the center of this dialog, or by entering pairs of numbers in the two column table on the left side of this dialog labeled "Input" and "Output". Enter the lookup function using the Input and Output table as follows: In the first row of this table, enter 0 for Input and 0 for Output. In the



second row of this table, enter 5000 for Input and 250 for Output. In the third row of this table, enter 20000 for Input and 1300 for Output. Then click the "OK" button on the Graph Lookup definition dialog, and click the "OK" button on the equation editor. [What you have specified by these entries is that for an input (that is, "Savings") of 0, the output (that is, "interest") is 0, for an input of 5000, the output is 250, and for an input of 20000, the output is 1300. These are the correct values for the interest procedure specified in Step 35 above.]

39. Use the equation editor to specify the following equation for "interest:" INTEREST LOOKUP(Savings) This specifies that the value of "interest" for any value of "Savings" should be obtained by doing a linear interpolation between the values specified in INTEREST LOOKUP.

40. Run this simulation model, and display the Causes Strip graph for "Savings. You will obtain the results shown at the top of this page. As a check, you may wish to use the Table tool to obtain values for "Savings" and "interest." The final value of "Savings" after 100 years should be 17,057, and the final value for "interest" should be 1,094. Comparing these strip graphs with the ones at the beginning of this section shows that the savings balance is somewhat higher after 100 years with the modified interest procedure, but the difference is not dramatic. This is because the higher interest rate does not start until the savings balance reaches $5,000, and that does not happen until year 78.

41. For further information about using lookup functions, consult the online help. While this example illustrates the use of lookup functions, it should be noted that the Vensim IF THEN ELSE function could also be used to calculate "interest" for this example. Consult the online help for information about IF THEN ELSE.

42. Finally, note that the Vensim sensitivity analysis feature can be used to temporarily change the shape of a lookup function for a particular simulation run.

Using "Time" as a variable

43. Some Vensim functions are explicit functions of time, and in order to use these functions you must enter "Time" as a variable in your model. As an example, suppose that you wish to use a sine function with an amplitude of 100 units and a period of 12 months as a simple model of the seasonal variation component in demand for some product within a Vensim simulation model. Then a Vensim equation to represent this is

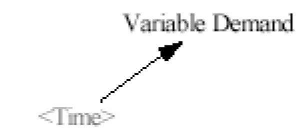
    Variable Demand = 100 * sin(2 * 3.14159 * Time / 12)

    where Time is measured in months.
44. Press the New Model button to create a new Vensim simulation model. Make TIME STEP equal to 0.25, and leave the other time bounds for the model at their default values.

45. Use the "Variable – Auxiliary/Constant" tool to enter a variable called "Variable Demand" (without the quotes) into your sketch for this model, and then attempt to use this tool to enter another variable called "Time" (also without the quotes) into this sketch. When you attempt to enter the variable Time, you will receive the following error message: "The variable Time already exists." Click the OK button to clear this error message, and then press the Esc key to clear the variable entry box.
46. If you select the Document tool to examine the equations for your model, there will not be any variable called Time listed. However, even though it does not appear in

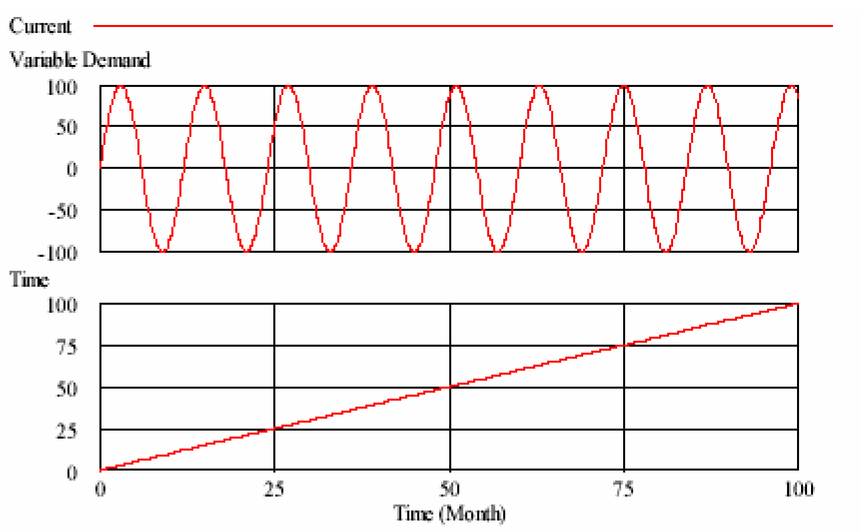
the Document tool output, the variable Time is a built-in variable that exists in every Vensim simulation model.

47. Since Time already exists in the model, you enter it into the model sketch by using the “Shadow Variable” tool, rather than the “Variable – Auxiliary/Constant” tool. Select the Shadow Variable tool, click on the sketch where you want the variable Time to appear, and then select Time from the list of variables that is displayed. Then use the Arrow tool to draw an arrow from the Time variable to the Variable Demand variable. You will now have a sketch that looks something like the following:



(Note that Time appears in angle brackets to indicate that it is a shadow variable.)

48. Complete the model by using the Equations tool to enter the equation for Variable Demand shown above in item 34. Since Time is a shadow variable, do not enter an equation for it.
49. Press the “Run a Simulation” button to run the simulation model, make “Variable Demand” the Workbench Variable, and press the Causes Strip button. The following graph will be displayed:



4. OPTIMIZATION BY CALCULUS

4.1 Introduction

In practice, Civil Engineers always have to make their designs conform to a variety of constraints (De Neufville, 1990). They generally have to work within a budget. They will have to satisfy engineering standards designed to ensure the quality, the performance, and the safety of the system. They will also have to meet legal restrictions the community has imposed to achieve its health and other objectives. The design of the primary water supply system for New York City, for example, had to provide water at a minimum pressure of 40 psi (pounds per square inch) as measured at the curb, with less than a legal maximum for parts per million of various chemicals.

Constraints generally express some objectives for the system which could not be dealt with analytically in an obvious way. Setting constraints on a system is an alternative to the difficult problem of dealing explicitly with the multiple outputs of a system, which has been very difficult to do. Thus New York City's standard of 40 psi at the curb represents an effort to ensure that the water pressure will be adequate for all residents of the city.

The analysis will often demonstrate that many of these restrictions do not make much sense in detail. They may easily cost far more to meet than is worth spending on the objective they represent. New York's standard of 40 psi for water pressure this entailed enormous extra costs for the construction of its huge underground aqueducts. Tens of millions of dollars could have been saved in the design if even slightly lower pressure had been permitted in a few areas. While agreeing with the principle behind the restrictions, in this case the idea that water pressure should be "adequate," there is generally much scope for improvements at the level of details (perhaps 38 psi would have been a better standard than 40 psi).

The reason that many restrictions on design do not make sense in details is that they were never established on the basis of an analysis of the system. New York's standard of 40 psi was certainly not based on a demonstration that it was the optimum level, or that it was functionally preferable to 39 psi or 41 psi; it was almost certainly agreed to because it was somewhere in the right region and because it was a nice round number.

Part of the Civil Engineer's job in the future should be to help define the appropriate levels of restrictions or, even better, to optimize the multiple output production process directly. Meanwhile, however, the designer often will simply have to meet the restrictions.

The difficulty in meeting restrictions on the design is that they complicate the optimization process. To see this consider some function of many variables:

$$g(\mathbf{X}) = g(X_1, \ldots, X_i, \ldots, X_n) \tag{4.1}$$

which we assume to be continuous and analytic in that it has first and second derivatives. If this function is unconstrained, the necessary conditions for its optimum (a maximum or a minimum) is that all the first derivatives with respect to its arguments equal zero:

$$?g \, / \, ?X_i = 0 \qquad \text{for all} \quad i \tag{4.2}$$

On the other hand, if the function is constrained, these conditions for optimality may or may not apply. The optimum may then be at the intersections of the function with the constraint. Figure 4.1 makes the point. In a large scale system with hundred of variable and constraints, the optimum could be located at an infinity of points.

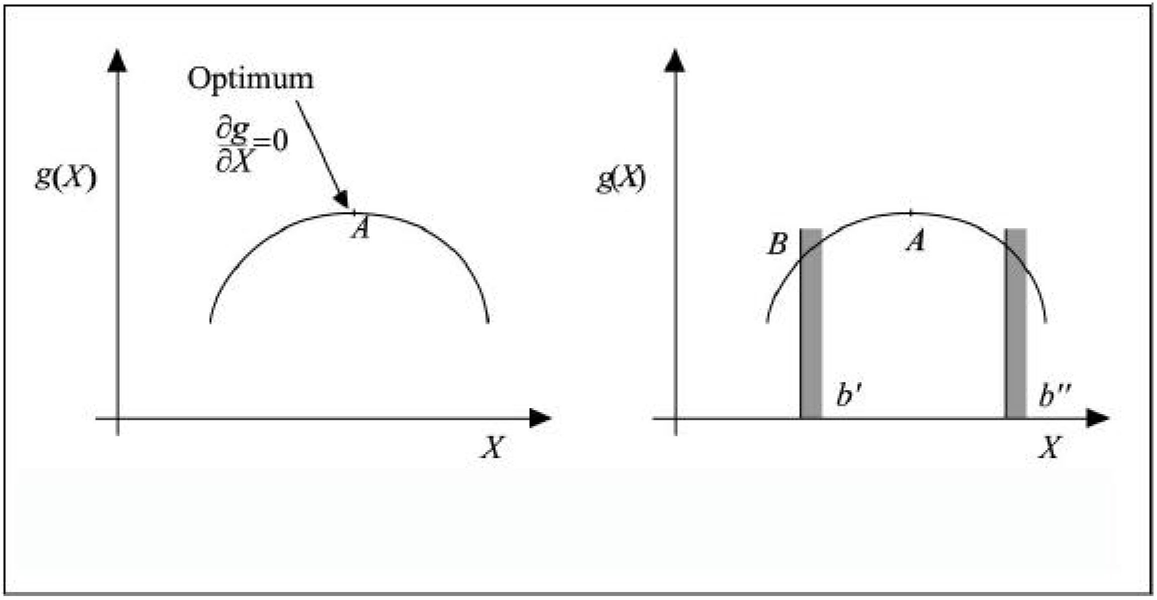


Figure 4.1 In unconstrained situation (left) the maximum is at A; with constraints (right) it is either at B, if X $\leq$ b′, or at A again if X $\leq$ b″.

Class exercise 1

$$\underset{\{x_1^*, x_2^*\}}{Minimize}\ z = x_1^2 + 2x_2^2 \tag{4.4}$$

subject to

$$-x_1^2 + x_2 \geq 1 \tag{4.5}$$

$$x_1 + x_2 \geq 3 \tag{4.6}$$

Since only 2 variables are involved, the problem may be displayed graphically as in Figure 4.2(a). The inequality constraints (4.5) and (4.6) define respectively the areas above the lines $x_2 = x_1^2 + 1$ and $x_2 = - x_1 + 3$ the intersection of which is shown shaded in Figure 4.2(b). Selection from this feasible space is guided by the objective function which for this case may be graphed as a series of elliptical contours centred on the origin. By inspection the point C ($x_1 = 1, x_2 = 2$) is seen to yield the minimum value of $z = 9$.

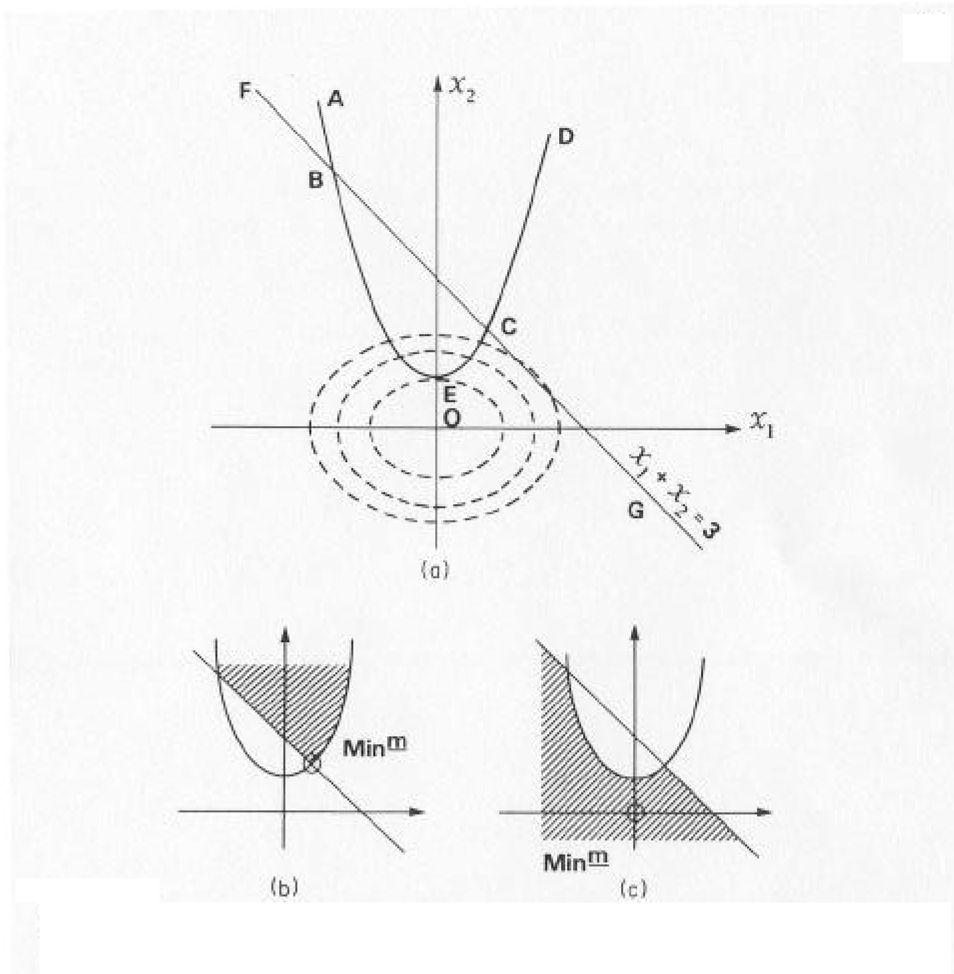


Figure 4.2 (a) Non-linear minimization in 2 variables. (b) Feasible space of Example 1. (c) Feasible space of Example 2

Class exercise 2

Consider an example in which the signs of the constraints (4.5) and (4.6) are reversed, i.e.

$$\underset{\{x_1^*, x_2^*\}}{\text{Minimize}}\ z = x_1^2 + 2x_2^2$$

subject to

$$-x_1^2 + x_2 \le 1 \qquad (4.7)$$

$$x_1 + x_2 \le 3 \tag{4.8}$$

By comparison with the previous example it may be visualized that the feasible space defined by (4.7) and (4.8) is the space below the line FBECG (Figure 4.2c). This space includes the origin, which now forms the optimum (minimum) solution.

Class exercise 3

If the sign of the x_2 term in (4.5) is changed, the feasible space is now defined by the constraints:

$$x_1^2 + x_2 \le -1 \tag{4.9}$$

$$x_1 + x_2 \ge 3.$$

A brief examination shows that this feasible space is non-existent and no solution exists to the minimization problem.

Therefore, it is clear that when a mathematical model includes inequality constraints:

(a) solution exists only if the constraints intersect to define a finite feasible space,

(b) the solution may lie on the edge of the feasible space, thus implying that at least one of the constraints is active – i.e. affects the solution,

(c) the solution may lie in the interior of the feasible space, representing an unconstrained optimization of the non-linear objective function.

Consider now some variation of the previous examples, to examine the significance of equality constraints.

Class exercise 4

Assume that (4.5) is changed to an equality constraint. The problem now becomes

$$\underset{\{x_1^*,\, x_2^*\}}{\text{Minimize}}\ z = x_1^2 + 2x_2^2$$

subject to

$$x_1^2 - x_2 + 1 = 0 \tag{4.10}$$

$$x_1 + x_2 \geq 3.$$

With reference to Figure 4.2 it may be seen that the feasible 'space' is now reduced to the two arcs of AB and CD. Although only line segments, there is still an infinite set of feasible solutions, from which an optimal point may be selected. By inspection, it is apparent that point C is again the minimum.

Class exercise 5

Let both (4.5) and (4.6) be converted to equality constraints, to make the problem:

$$\underset{\{x_1^*,\, x_2^*\}}{\text{Minimize}}\ z = = x_1^2 + 2x_2^2$$

subject to

$$x_1^2 - x_2 + 1 = 0$$

$$x_1 + x_2 - 3 = 0. \tag{4.11}$$

The feasible space is now further reduced to the points B and C and the solution is obtained by evaluating the objective functions at these two roots of the simultaneous equations. Optimization in the usual sense is not possible.

Note that if a further equality constraint is added to the problem, e.g. $x_1 = 0$*, then, in general, the feasible space is empty and no solution is possible. Only in coincidental cases, e.g.* $x_1 = 1$*, will a feasible point be defined.*

It can be seen that the nature of the solution depends on the number of variables 'n' and the number of equality constraints' l'.

For $l > n$ *no solution exists in general.*

For $l = n$ one or more discrete points (depending on the degree of the constraint equations) are uniquely defined at which the objective function may be evaluated and a solution obtained by inspection.

For $l < n$ the feasible space (if it exists) may contain an infinite number of solutions and an optimal solution, either on the edge or in the interior, must be determined by search or other techniques.

In this connection it may be noted that any inequality constraint may be converted into an equality constraint by the introduction of a slack variable. For example

$$- x_1^2 + x_2 \geq 1$$

is equivalent to

$$- x_1^2 + x_2 - x_3 = 1. \qquad (4.12)$$

Thus a problem involving n decision variables, k inequality constraints, and l equality constraints may be converted into a problem with $(n + k)$ variables and $(l + k) = m$ equality constraints. The significance of the relative values of l and n discussed above is therefore unaffected by the value of k.

4.2 Unconstrained functions

A single variable function

A function of a single variable may be examined by means of differential calculus (as pointed in 4.1) to see if an extreme value (i.e. a maximum of minimum) exists. For this to be possible the function must be: (i) continuous within the feasible domain of the variable; (ii) differentiable once to locate critical points; and (iii) differentiable twice to determine the sense of the critical points.

In general if a function $z = F(x)$ exists then the values of x at which critical points may exist are defined by setting the first derivative to zero, i.e.

$$dz / dx = 0 \tag{4.13}$$

Let one such solution of (4.13) be x*. Then the nature of the critical point is given by evaluating the second derivative of z at x = x*. Thus for:

$d^2z / dx^2 (x^*) < 0$ the critical point is a maximum
$d^2z / dx^2 (x^*) > 0$ the critical point is a minimum.

Temporary lapses of memory in recalling this rule may be aided by the picture of Figure 4.3 in which the contents of an inverted or upright wine glass are considered as analogous to the sign of the second derivative.

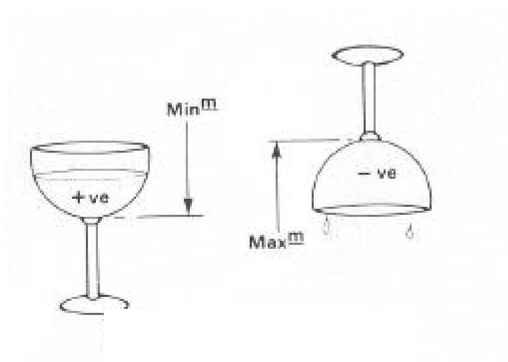


Figure 4.3 Wine-glass analogy

For the case in which the second derivative is zero at the stationary point some further analysis is necessary. Find the lowest derivative which is non-zero at the stationary point, i.e. $d^nz/dx^n(x^*)$? 0. If n is odd (n = 1, 3, 5,…) the solution found is a point of inflection or a saddle point. If n is even (n = 2, 4,…) the rules of the wine-glass analogy may be used as before.

Class exercise 6

Traffic passing through a road tunnel is causing a bottleneck and some form of access control or speed control is considered as a means of maximizing the capacity. A survey indicates that the headway (H miles) between vehicles is related to the speed (V mph) by the empirical relation:

$$H = 0.4/(50 - V) \tag{4.14}$$

i.e.

$$\text{density } D \text{ (vehicles/mile)} = 1/H = (50 - V)/0.4 \tag{4.15}$$

or

$$D = 125 - 2.5V.$$

The traffic flow Q (vehicles/hour) is then given by:

$$Q = DV = 125V - 2.5V^2 \tag{4.16}$$

The maximum flow is thus given by the critical point defined by

$$dQ/dV = 125 - 5V = 0 \tag{4.17}$$

or

$$V^* = 25 \text{ mph.}$$

Since

$$d^2Q/dV^2 = -5 < 0$$

the flow is a maximum when

$$Q = 125 * 25 - 2.5 * 25^2 = 1562.5 \text{ vehicles/hour.}$$

Steps might be taken to maintain traffic speed as close to 25 mph as possible (by controlling access). On the other hand, the headway at this speed (84.5 ft) seems somewhat excessive and may be attributable to other factors such as poor illumination or a greasy surface, which could be the real reason for the bottleneck.

A multiple variable function

Given a function

$$F(\mathbf{x}) \quad \text{where } \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \tag{4.18}$$

for which all the first partial derivatives exist at all points within the feasible domain of the function, a necessary condition for a stationary point is:

$$\frac{\partial F}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n. \tag{4.19}$$

Such a stationary point may be a minimum, a maximum, or a point of inflection. Moreover, the possibility must be considered that two or more such stationary points may exist.

In order to test the nature of the stationary point it is necessary to evaluate all the second derivatives at the point, i.e.

$$?^2F \ / \ ?x_i ?x_j \qquad i, j, = 1, 2, \ldots, n. \tag{4.20}$$

These second derivatives may be assembled in a Hessian matrix, thus:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x_1^2} & \dfrac{\partial^2 F}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 F}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 F}{\partial x_2 \partial x_1} & \dfrac{\partial^2 F}{\partial x_2^2} & \cdots & \dfrac{\partial^2 F}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 F}{\partial x_n \partial x_1} & \dfrac{\partial^2 F}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 F}{\partial x_n^2} \end{bmatrix} \tag{4.21}$$

It will be noted that **H** is symmetric since:

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i} \tag{4.22}$$

A *sufficient* condition for the stationary point to be a minimum is that all the second derivatives exist at the point, and that all the principal minors are positive. (This implies that **H** is positive-definite. The principal minors of a matrix **A** are defined as follows:

$$p_1 = |a_{11}|$$

$$p_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

…..

$$p_n = |\mathbf{A}| \tag{4.23}$$

where the symbol │ │means ‘the determinant of’.

The corresponding condition that the stationary point is a maximum is that the even-numbered principal minors (p_2, p_4,…) are positive and the odd-numbered principal minors (p_1, p_3, p_5,…) are negative. (This implies that **H** is negative-definite.) If stationarity exists but the sufficient condition is not satisfied, then the nature of the point is indeterminate and must be investigated by other means (e.g. numerical).

Another method of describing the sufficient condition is in terms of the eigen values of the Hessian matrix evaluated at the stationary point. If all eigen values of the Hessian are negative at **x*** then the stationary point is a local maximum. Clearly if all the eigen values are negative for all possible values of **x** the point **x*** is a global maximum. If all eigen values of the Hessian are positive at **x*** then the stationary point is a local minimum. Similarly, the point is a global minimum if the eigen values are positive for all possible values of **x**.

4.3 Constrained optimization

The practical problem is then to find ways to define optimality conditions which can pinpoint the optimum when there are constraints. For analytic functions there are two principal means of doing this, Lagrangeans and the Kuhn-Tucker conditions. The first is explained in detail in following sections and the second one is out of the scope of this course.

The purpose of this chapter is to provide insight into the nature of the optimal design of constrained systems. It develops the key concept of 'shadow price' that is essential in the practical application of optimization, particularly in sensitivity analysis. Optimization problem with *equality constraints* solved by Lagrangeans, brings out the concept of shadow prices.

Lagrangeans

Lagrangeans constitute the easiest solution to optimization subject to constraints. Historically they are also the first. They may be considered an intermediate step to the general optimality conditions published by Kuhn and Tucker in 1951.

Lagrangeans solve the problem of optimizing a function subject to equality constraints, ones that must be met exactly. These routinely occur in practice. A standard example is the requirement to use one's budget entirely – otherwise it might be cut in the following period. Formally, the Lagrangean addresses the problem of optimizing $g(\mathbf{X})$ subject to a series of constraints $h_j(\mathbf{X}) = b_j$. Putting the matter into the standard vocabulary used with optimization, we call the function to be optimized the objective function.

We then write the problem as:

$$\begin{aligned} &\text{Optimize:} && g(\mathbf{X}) \\ &\text{Subject to:} && h_j(\mathbf{X}) = b_j \quad \text{and} \quad \mathbf{X} \geq 0 \end{aligned} \tag{4.24}$$

The condition that all **X** be positive is a standard assumption that in no way limits practical applications: variables can always be redefined to meet this requirement. The purpose of this condition is to simplify the second order, the sufficiency conditions of optimality. In linear programming, discussed in Chapter 5, this sign convention is also exploited to facilitate the optimization procedure.

The problem is solved by constructing a new function to be optimized. This is called the *Lagrangean* and is:

$$L = g(\mathbf{X}) - \sum \lambda_j [h_j(\mathbf{X}) - b_j] \qquad (4.25)$$

Each parameter λ_j is a constant associated with a particular equation, and is to be solved for. These parameters are known as the *Lagrangean multipliers*.

The marvelous aspect of the Lagrangean is that it simultaneously permits us to optimize $g(\mathbf{X})$ and to satisfy the constraints $h_j(\mathbf{X}) = b_j$. All we do is apply the conventional process for unconstrained functions to the Lagrangean. The difference is that whereas in considering $g(\mathbf{X})$ we had i variables, one for each X_i, we now have (i + j) variables, including one for each of the λ_j multipliers associated with the j constraints.

Applying the standard method of optimization to the Lagrangean we set all first derivatives equal to zero and obtain:

$$?L / ?X_i = ?g / ?X_i - \sum_j \lambda_j ?h / ?X_i = 0 \qquad (4.26)$$

$$?L / ?\lambda_j = h_j(\mathbf{X}) - b_j = 0 \qquad (4.27)$$

The (i + j) set of X_i and λ_j that satisfy these (i + j) equations will determine the optimum. This solution will be designated (X^*, λ^*) where the superscript * indicates optimality.

Notice that the optimal solution to the Lagrangean clearly satisfies the constraints placed on the original problem; they are identical to the $?L/?\lambda_j$ equations that help define the optimal solution. Likewise, this solution also maximizes g(**X**) since, at optimality where the constraints are met, the second term in the Lagrangean equals zero and optimizing L is then equivalent to optimizing g(**X**) (see Class exercise 7).

Class exercise 7

Consider the problem:

Maximize:

$$g(\mathbf{X}) = 3X_1X_2^2$$

Subject to:

$$h(\mathbf{X}) = X_1 + X_2 = 3$$

$$X_1, X_2 > 0$$

The Lagrangean is

$$L = 3X_1X_2^2 - l(X_1 + X_2 - 3)$$

There is only one l because there is only one constraint. There are (i + j) = 3 equations defining the optimal solution. These are:

$$?L / ?X_1 = 3X_2^2 - l = 0$$

$$?L / ?X_2 = 6X_1X_2 - l = 0$$

$$?L / ?l = X_1 + X_2 - 3 = 0$$

The first two equations imply that $X_2^* = 2X_1^*$. *The third then leads to*

$$\mathbf{X}^* = (1,2) \quad \text{and}$$

$$l^* = 12$$

The optimum is

$$g^*(\mathbf{X}) = 12.$$

The Lagrangean multiplier has a most important interpretation. It gives the systems designer valuable information about the sensitivity of the results to changes in the problems. At the optimum, the Lagrangean multiplier λ_j equals the rate of the change of the optimum objective function $g^*(\mathbf{X})$ as the constraint $b_j = h_j(\mathbf{X})$ varies. To see this, take the partial derivative of the Lagrangean with respect to this constraint and obtain:

$$?L / ?b_j = \lambda_j \quad \text{all} \quad j \tag{4.27}$$

Now since the constraints are identically equal to zero at the optimum,

$$g^*(\mathbf{X}) = L \tag{4.28}$$

and

$$?g^* / ?b_j = ?L / ?b_j = \lambda_j \quad \text{all} \quad j \tag{4.29}$$

In the vocabulary of economics and systems analysis, the Lagrangean multiplier is known as the *shadow price* of the constraint. This neatly describes the parameter: *it is the implicit price to be paid, in terms of changes of g(**X**), per unit change of the constraint b_j.* Being a derivative, the value of the shadow price in general varies with the value of the constraint; the values calculated are thus instantaneous, at the margin, for any given value of the constraint.

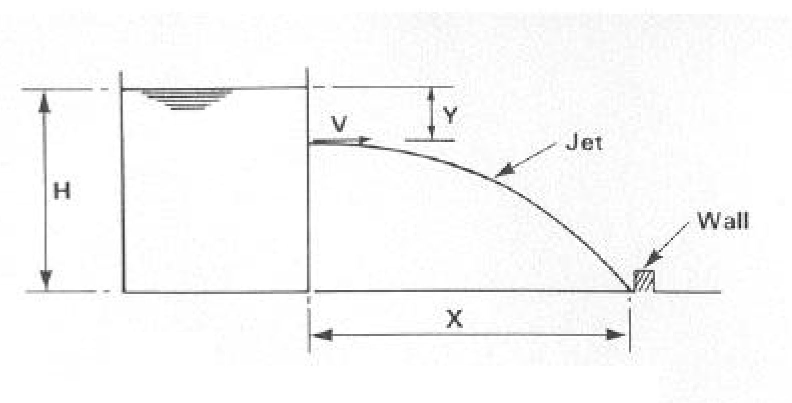
Semantic caution: Although the Lagrangean multiplier is called the shadow price, it has no necessary connection with money. Its units are always in terms of those of the function to be maximized, $g(\mathbf{X})$, divided by those of the relevant constraint. Thus for the design of New York city's water supply system: when the objective was to maximize the capacity to deliver water subject to constraints on the pressure, the shadow price of the pressure standard was in terms of gallons per psi. The shadow price is expressed in terms of money only when g(**X**) is some monetary amount such as dollars of benefit from a system.

The shadow price on a constraint is the quantitative indication of whether it is worthwhile to alter this requirement. It is the information that allows the designer to judge to what extent each of the constraints on a system make sense in detail. For New York City's water supply system, for example, the shadow price on the requirement that water be delivered throughout the city at a minimum pressure of 40 psi at the curb was extremely high; unless the community for some reason thought this was an absolutely vital threshold, it would be reasonable to lower this standard for the areas of highest evaluation so that the resulting improvements to the overall system could be shared by the whole community.

This issue of sensitivity of the optimum to changes in the definition of the problem is one of the central topics of all practical applications of systems analysis.

4.4 Problems

4.1 A rather old riveted, steel plate storage tank has an unfortunate habit of 'popping' corroded rivets, issuing a jet of water on the adjacent work area. A small wall is to be located to retain the ponding. If the total depth is H determine the maximum distance x from the tank wall where the jet may strike the ground.



4.2 A rectangular, open-topped reservoir is to be proportioned. The flowrate, and thus the cost of supplying water to the tank, is inversely proportional to the storage volume provided. Typical figures are as follows:

Volume (m^3)	Flow (m^3/s)	Supply cost ($)
100	0.65	10 000
500	0.50	2000
1000	0.40	1000
2000	0.35	500

The cost of constructing the tank is based on the following rates: Base $2/m^2; Sides $4/m^2; and Ends $6/m^2. Find the dimensions for least cost. The supply cost may be approximated by the function

$$c_1 = 10^6 / \text{Volume in } m^3.$$

4.3 Given the problem:

Maximize: $Z = 2X + 3XY$

Subject to: $XY = X^2/9 = 13$

$X, Y \geq 0$

(a) Formulate the problem, state the optimality criteria and solve.

(b) What is the practical significance of the Lagrangean multiplier l ?

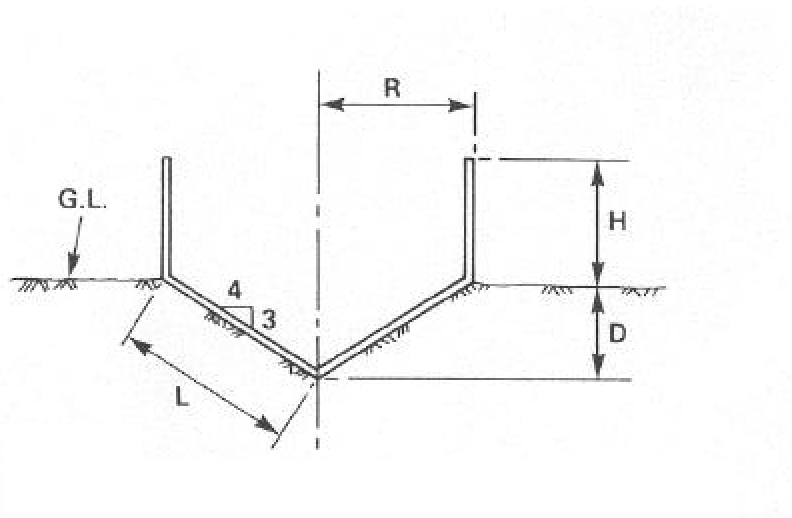
4.4 Assume that the number of highway miles that can be graded, H, is a function of both the hours of labor, L, and of machines, M:

$$H = 0.5L^{0.2} M^{0.8}$$

(a) Minimize the cost of grading 20 miles, given that the hourly rates for labor and machines are: C_L = \$20; C_M = \$160.

(b) Interpret the significance of the Lagrangean miltiplier.

4.5 A sedimentation tank is circular in plan with vertical sides above ground and a conical hopper bottom below ground, the slope of the conical part being 3 vertically to 4 horizontally. Determine the proportions to hold a volume of 4070m^3 for minimum area of the bottom and sides.



4.6 Find the optimal reservoir size for irrigation. The total benefit (TB) derived from irrigation water in a region has been estimated as

$$TB = 100\,q - 0.0005\,q^2 \qquad (1)$$

where TB is expressed as dollars per year (\$/yr) and q represents dependable water supply, expressed in acre-feet per year (AF/yr).

Estimates of the annual total cost (TC) of a reservoir at different sizes results in the expression

$$TC = 44.42\, q^{0.90} + 0.098\, q^{1.45} \tag{2}$$

where TC has units of dollars per year.

The agency responsible for reservoir construction wishes to maximize net benefit (NB), defined as total benefit minus total cost. Algebraically,

$$\text{Maximize} \quad NB(q) = TB(q) - TC(q). \tag{3}$$

4.5 References

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5. LINEAR PROGRAMMING (LP)

As indicated in Section 1 of this text, LP has been considered one of the most widely used techniques in Civil Engineering. This section will introduce basic concepts of the linear programming optimization technique (de Neufville, 1990; Hillier and Lieberman, 1990; Jewell, 1986).

5.1 What is Linear Programming?

In many engineering decision problems, the aim is to maximize or minimize some objective, but there are certain constraints on what can be done in optimising this objective. The term *linear programming* (LP) is used to mean a way of modelling many of these problems so that they have a special structure, and it also denotes the way of solving problems with such a structure. It is a technique which can be applied in many different problem domains. The first step in formulating a model, as discussed earlier, of the problem is to decide which are the *decision variables*. These are the quantities which can be varied, and their variations affect the value of the objective. The second step in the formulation is to express the *objective* in terms of the decision variables. Lastly, in formulating the model, we must write down the *constraints* that restrict the choices of our decision variables. The common sense constraints are that we cannot make negative our decision variables. We can summarize the mathematical representation of the LP model in the following way. Letting x_j be the level of Activity j, for j = 1, 2 ..., n, you want to select a value for each x_j such that:

$$C_1x_1 + C_2x_2 + ... + C_nx_n$$

is maximized, or minimized, depending on the context of the problem. The x_j are constrained by a number of relations, each of which is one of the following types:

$$a_1x_1 + a_2x_2 + ... + a_nx_n \,?\!=\, a$$
$$b_1x_1 + b_2x_2 + ... + b_nx_n = b$$
$$c_1x_1 + c_2x_2 + ... + c_nx_n =\!?\, c.$$

The first relation includes the possible restriction $x_j \geq 0$. Such a constrained optimization problem may have:

(i) no feasible solution, that is, there may be no values of all the x_j, for j = 1, 2,..., n, that satisfy every constraint;

(ii) a unique optimal feasible solution;

(iii) more than one optimal feasible solution; or

(iv) a feasible solution such that the objective function is unbounded; that is, the value of the function can be made as large as desired in a maximization problem, or as small in a minimization problem, by selecting an appropriate feasible solution.

A simple reference problem

Let us consider the problem currently being analyzed by the management of Simon, Inc., a small manufacturer of bathroom tiles. Simon has been convinced by its distributor that there is an existing market for both a medium and a high priced tiles. In fact, the distributor is so confident of the market that if Simon can make the tiles at a competitive price, the distributor has agreed to purchase everything Simon can manufacture over the next three months.

After a thorough investigation of the steps involved to manufacture a tile, Simon has determined that each tile produced will require the following operations:

1. preparing the material,
2. glazing,
3. finishing,
4. inspection and packaging.

The head of manufacturing has analyzed each of the operations and has concluded that if the company produces a medium priced Standard model, each tile produced will require $\frac{7}{10}$ of an hour in the preparation department, $\frac{1}{2}$ of an hour in the glazing department, 1 hour in the finishing department, and $\frac{1}{10}$ of an hour in the inspection and packaging department. Similarly, the more expensive Deluxe model will require 1 hour of preparation time, $\frac{5}{6}$ of an hour of glazing time, $\frac{2}{3}$ of an hour of finishing time, and $\frac{1}{4}$ of an hour of inspection and packaging time. This production information is summarized in Table 5.1.

The accounting department has analyzed these production figures, assigned all relevant variable costs, and has arrived at process for both types of tiles that will result in a profit of \$10 for every Standard tile produced and \$9 for every Deluxe tile produced.

	DEPARTMENTS			
Product	Preparation	Glazing	Finishing	Inspection & Packaging
Standard Tile	$\frac{7}{10}$ hr.	$\frac{1}{2}$ hr.	1 hr.	$\frac{1}{10}$ hr.
Deluxe Tile	1 hr.	$\frac{5}{6}$ hr.	$\frac{2}{3}$ hr.	$\frac{1}{4}$ hr.

Table 5.1 Production Operations and Production Requirements Per Tile

In addition, the head of manufacturing has studied his work load for the next three months and estimates that he should have available a maximum of 630 hours of preparation time, 600 hours of glazing time, 708 hours of finishing time, and 135 hours of inspection and packaging time. Simon's problem then is to determine how many Standard and how many Deluxe tiles should be produced in order to *maximize the profit*. If you were in charge of production scheduling for Simon, Inc., what decision would you make given the above information? That is, how many Standard and how many Deluxe tiles would you produce in the next three months?

Changing the sense of the optimization

Any linear maximization model can be viewed as an equivalent linear minimization model, and vice versa, by accompanying the change in the optimization sense with a change in the signs of the objective function coefficients. Specifically,

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j \text{ can be treated as minimize } \sum_{j=1}^{n} (-c_j)\, x_j$$

and vice versa.

Changing the sense of an inequality

All inequalities in a linear programming model can be represented with the same directioned inequality since:

$$\sum_{j=1}^{n} a_j x_j \le b \text{ can be written as } \sum_{j=1}^{n} (-a_j)\, x_j \ge -b,$$

and vice versa.

Converting an inequality to an equality

An inequality in a linear model can be represented as an equality by introducing a non-negative variable as follows:

$$\sum_{j=1}^{n} a_j x_j \le b \text{ can be written as } \sum_{j=1}^{n} a_j x_j + 1s = b \text{ where } s \ge 0$$
$$\sum_{j=1}^{n} a_j x_j \ge b \text{ can be written as } \sum_{j=1}^{n} a_j x_j - 1t = b \text{ where } t \ge 0.$$

It is common to refer to a variable such as s as a *slack variable*, and t as a *surplus variable*.

Converting equalities to inequalities

Any linear equality or set of linear equalities can be represented as a set of like-directioned linear inequalities by imposing one additional constraint. The idea generalizes as follows:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i = 1, 2, ..., m \text{ can be written as}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, ..., m \text{ and } \sum_{j=1}^{n} \boldsymbol{a}_j x_j \leq \boldsymbol{b}.$$

where:

$$\boldsymbol{a}_j = -\sum_{i=1}^{m} a_{ij} \text{ and } \boldsymbol{b} = -\sum_{i=1}^{m} b_i.$$

5.2 Canonical forms for linear optimization models

Sometimes it is convenient to be able to write *any* linear optimization model in a compact and unambiguous form. The various transformations presented in the previous section allow you to meet this objective, although it is now apparent that there is considerable latitude in the selection of a particular canonical form to employ. We illustrate two such representations here.

Any linear optimization model can be viewed as:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, ..., m$$

$$x_j \geq 0 \text{ for } j = 1, 2, ..., n.$$

Similarly, any linear optimization model can be written as:

$$\text{minimize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i = 1, 2, ..., m \; (b_i \geq 0)$$

$$x_j \geq 0 \text{ for } j = 1, 2, ..., n.$$

It is typical, although not required, that $n > m$.

5.3 Geometric interpretation

There are two geometric representations of a linear optimization model. One is called the *solution space representation* and is treated here. The other is called the *requirements space representation*; it is less important for a beginner to understand.

Solution space representation - two dimensions (variables)

Here we proceed directly to the consideration of a numerical example:

$$\text{maximize } 12x_1 + 15x_2 \qquad (5.1)$$

subject to

$$4x_1 + 3x_2 \leq 12 \quad (5.2)$$

$$2x_1 + 5x_2 \leq 10 \quad (5.3)$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0. \quad (5.4)$$

The problem is graphed in Figure 5.1. Observe that both (5.2) and (5.3) are drawn as equations. Then each inequality is indicated by an arrow on the side of the line representing permissible values of x_1 and x_2. Since the two variables must be non-negative, the region of permissible values is bounded also by the two coordinate axes.

Accordingly, the polygon $\overline{oabc}$ represents the region of values for x_1 and x_2 that satisfy all the constraints. This polygon is called the *solution set*. The set points described by the polygon is *convex*. The vertices 0, a, b, and c are referred to as the *extreme points* of the polygon in that they are not on the interior of any line segment connecting two distinct points of the polygon.

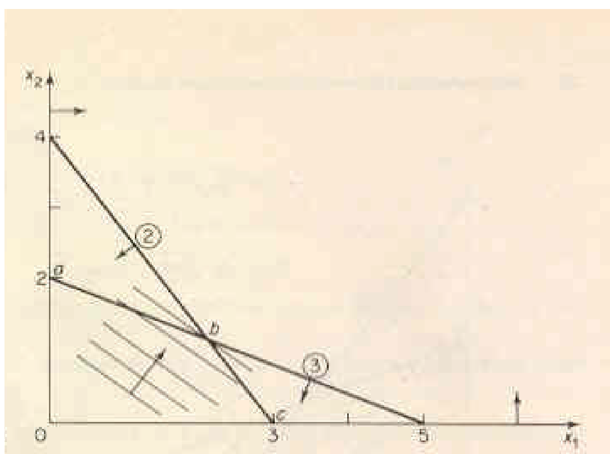


Figure 5.1 Solution space

The parallel lines in the figure represent various values of the objective function. The arrow points in the direction of increasing values of the objective function. The optimal solution is at the extreme point b, where $x_1 = 15/7$, $x_2 = 8/7$, and $12x_1 + 15x_2 = 300/7$.

Alternative optimal solutions

If the coefficients of the objective function are changed so as to alter the direction of the parallel lines in Figure 5.1, it is clear that the optimal solution may change, but in any case there is always an extreme-point optimal solution. Consider, in Figure 5.2 a problem for one specific rotation of the parallel lines:

$$\text{maximize } 4x_1 + 10x_2 \tag{5.5}$$

again subject to the same constraints (5.2), (5.3), and (5.4).

Now, all the points (an infinite number) on the segment ab are optimal. Thus $x_1 = 15/7$ and $x_2 = 8/7$ are still optimal. But so are $x_1 = 0$ and $x_2 = 2$, as well as any positive-weighted average of these two optimal solutions. The optimal value of the objective function is 20.

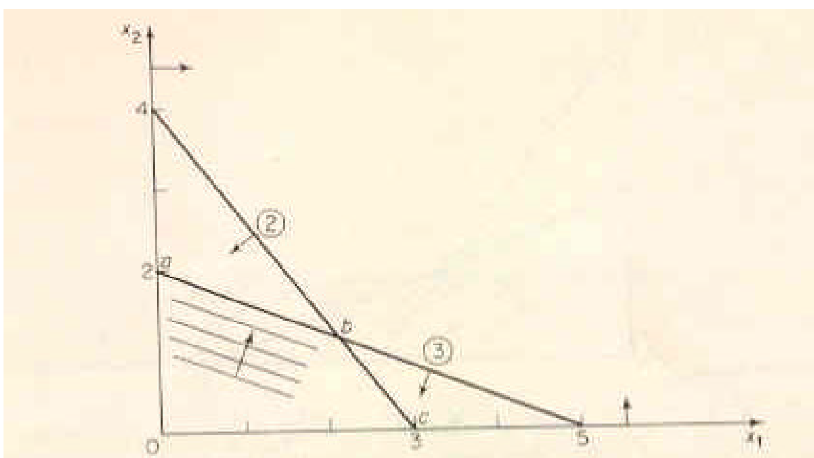


Figure 5.2 Alternative optimal solutions

Unbounded optimal solutions

The third illustration, shown in Figure 5.3 is based on the model:

$$\text{maximize } -2x_1 + 6x_2 \tag{5.6}$$

subject to:

$$-1x_1 - 1x_2 \leq -2 \tag{5.7}$$

$$-1x_1 + 1x_2 \leq 1 \tag{5.8}$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0. \tag{5.9}$$

The solution set for this problem is unbounded. The objective function for the problem can be made arbitrarily large. Given any value for the objective function, there always is a solution point having an even greater objective function value. And there always is such a point satisfying (5.8) with equality.

Infeasible problem

The fourth illustration, shown in Figure 5.4 is based on the problem:

$$\text{maximize } 1x_1 + 1x_2 \tag{5.10}$$

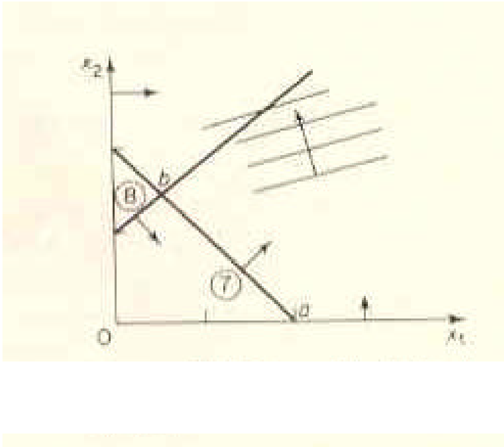
subject to:

$$-1x_1 + 1x_2 \leq -1 \tag{5.11}$$

$$1x_1 - 1x_2 \leq -1 \tag{5.12}$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0. \tag{5.13}$$

Figure 5.4 illustrates that the problem does not have a feasible solution.



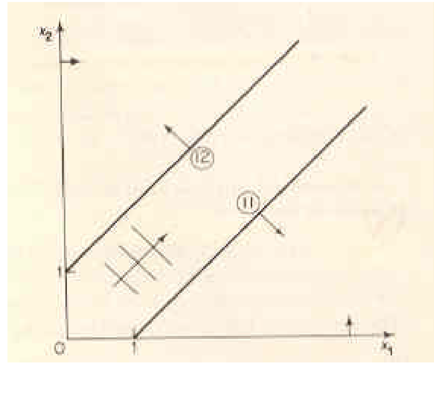


Figure 5.3 Unbounded solution

Figure 5.4 Infeasible solution

Conclusion

We will summarize the results of the several geometric illustrations you have studied.

(i) If the solution set is nonempty, it is convex and may be either bounded or unbounded.

(ii) If the solution set is nonempty, the optimal value of the objective function may be finite or unbounded. If finite, then an optimal solution exists at an extreme point.

5.4 Simplex method of solution

You will probably never have to calculate manually the solution of a linear programming model in a real application, since a computer will do the work for you. You may legitimately ask, then, "Why do I need to know the underlying theory of linear optimization models? In the light of considerable experience in applying linear programming to engineering problems, we are persuaded that an engineer must understand the principles explained here in order to make truly effective and sustained use of this management tool.

Many different algorithms have been proposed to solve linear programming problems, but the one below has proved to be the most effective in general. The procedure appears to be the following:

Step 1. Select a set of m variables that yields a feasible starting trial solution. Eliminate the selected m variables from the objective function.

Step 2. Check the objective function to see whether there is a variable that is equal to zero in the trial solution but would improve the objective function if made positive. If such a variable exists, go to Step 3. Otherwise, stop.

Step 3. Determine how large the variable found in the previous step can be made until one of the m variables in the trial solution becomes zero. Eliminate the latter variable and let the next trial set contain the newly found variable instead.

Step 4. Solve for these m variables, and set the remaining variables equal to zero in the next trial solution. Return to Step 2.

Interestingly, the resulting algorithm in fact does find an optimal solution to a general linear programming model in a finite number of iterations. Often this method is termed *Dantzig's simplex algorithm*, in honour of the mathematician who devised the approach.

We first examine a "well-behaved" problem and explain the *simplex method* by means of this example. Afterwards we complete the presentation of the details.

Class exercise

Consider the model:

$$\text{maximize } 4x_1 + 5x_2 + 9x_3 + 11x_4$$

subject to:

$$1x_1 + 1x_2 + 1x_3 + 1x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_1 \geq 0 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 \geq 0.$$

Let x_0 be the value of the objective function and add slack variables. Then write the system as:

$$
\begin{array}{llll}
1x_0 - 4x_1 - 5x_2 - 9x_3 - 11x_4 & = 0 & \underline{\text{Row 0}} & \\
1x_1 + 1x_2 + 1x_3 + 1x_4 + 1x_5 & = 15 & \underline{\text{Row 1}} & \\
7x_1 + 5x_2 + 3x_3 + 2x_4 \quad + 1x_6 & = 120 & \underline{\text{Row 2}} & (5.14) \\
3x_1 + 4x_2 + 10x_3 + 15x_4 \qquad + 1x_7 & = 100 & \underline{\text{Row 3}}, &
\end{array}
$$

where all the variables must be nonnegative. Notice how the introduction of the variable x_0 in Row 0 permits us to express the objective function in equation form.

The task of Step 1 is to find a starting feasible solution to (5.14). There are a large number of such solutions, but it is certainly most convenient to begin with $x_0 = 0$, $x_5 = 15$, $x_6 = 120$, $x_7 = 100$, and all other variables equal to 0. In other words, start with an all-slack solution. We term this an *initial feasible basic solution*, and x_0, x_5, x_6, and x_7 are known as the *basic variables*, sometimes shortened to the *basis*. The remaining variables we call *nonbasic*.

Interpretation of coefficients in Row 0

Each coefficient represents the increase (for negative coefficients) or decrease (for positive coefficients) in x_0 with a unit increase of the associated nonbasic variable.

Iteration 1. For Step 2 the simplex method adopts the following easy-to-apply rule for deciding the variable to enter the next trial basis.

> **Simplex Criterion I (maximization).** If there are nonbasic variables having a negative coefficient in Row 0, select one such variable with the most negative coefficient, that is, the best per-unit potential gain (say x_j). If all nonbasic variables have positive or zero coefficients in Row 0, an optimal solution has been obtained.

To decide which variable should leave the basic we will apply the following rule for Step 3.

> **Simplex Criterion II**. (a) Take the ratios of the current right-hand side to the coefficients of the entering variable x_j (ignore ratios with zero or negative numbers in the denominator). (b) Select the minimum ratio - that ratio will equal the value of x_j in the next trial solution. The minimum ratio occurs for a variable x_k in the present solution; set $x_k = 0$ in the solution.

The process of applying Criterion II is known as a *change-of-basis* calculation, or a *pivot operation*. A detailed calculation is presented in Table 5.1.

Iteration 2. At this point the first iteration of the simplex method has been completed. On returning to Step 2, you are ready to determine whether an optimal solution has been obtained or if another simplex iteration is required. Criterion I, which examines the nonbasic variables, indicates that a still better solution seems to exist. You might profitably enter into the basis either x_1, or x_2, or x_3. Criterion I selects x_1, since it promises the greatest gain per unit increase. Next perform the Step 3 calculations, using Criterion II. From Table 5.2, notice that x_1 will replace x_5 in the next trial solution.

Iteration 3. Having completed the second simplex iteration, once more examine the coefficients in Row 0 to ascertain whether you have discovered an optimal solution. It now appears favourable to enter x_3 and remove x_4 which was entered at the first iteration.

At this iteration you have just seen another aspect to the computational rule in Criterion II. To sum up, Criterion II ensures that each new basic solution results only in zero or positive values for the trial values of the basis. Consequently, the solution remains feasible at every iteration.

Iteration 4. All the coefficients in Row 0 are nonnegative, and consequently Criterion I asserts you have found an optimal solution. Thus the calculations are terminated in Step 2.

Iteration	Basis	Current Values	x_1	x_2	x_3	x_4	x_5	x_6	X_7	Row
	x_0	0	-4	-5	-9	- 11				0
	x_5	15	1	1	1	1	1			1
1	x_6	120	7	5	3	2		1		2
	x_7	100	3	5	10	15			1	3
	x_0	$\frac{220}{3}$	$-\frac{9}{5}$	$-\frac{4}{3}$	$-\frac{5}{3}$				$\frac{11}{15}$	0
	x_5	$\frac{25}{3}$	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{1}{3}$		1		$-\frac{1}{15}$	1
2	x_6	$\frac{320}{3}$	$\frac{33}{5}$	$\frac{13}{3}$	$\frac{5}{3}$			1	$-\frac{2}{15}$	2
	x_4	$\frac{20}{3}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{2}{3}$	1			$\frac{1}{15}$	3
	x_0	$\frac{1105}{12}$		$\frac{1}{6}$	$-\frac{11}{12}$		$\frac{9}{4}$		$\frac{7}{12}$	0
	x_1	$\frac{125}{12}$	1	$\frac{5}{6}$	$\frac{5}{12}$		$\frac{5}{4}$		$-\frac{1}{12}$	1
3	x_6	$\frac{455}{12}$		$-\frac{7}{6}$	$-\frac{13}{12}$		$-\frac{33}{4}$	1	$\frac{5}{12}$	2
	x_4	$\frac{55}{12}$		$\frac{1}{6}$	$\frac{7}{12}$	1	$-\frac{1}{4}$		$\frac{1}{12}$	3
	x_0	$\frac{695}{7}$		$\frac{3}{7}$		$\frac{11}{7}$	$\frac{13}{7}$		$\frac{5}{7}$	0
	x_1	$\frac{50}{7}$	1	$\frac{5}{7}$		$-\frac{5}{7}$	$\frac{10}{7}$		$-\frac{1}{7}$	1
4	x_6	$\frac{325}{7}$		$-\frac{6}{7}$		$\frac{13}{7}$	$-\frac{61}{7}$	1	$\frac{4}{7}$	2
	x_3	$\frac{55}{7}$		$\frac{2}{7}$	1	$\frac{12}{7}$	$-\frac{3}{7}$		$\frac{1}{7}$	3

Table 5.1 Simplex Tableau for the Case Study Problem

Summary

In brief, the simplex method consists of:

Step 1. Select an initial basis.

Step 2. Simplex Criterion I. If the solution is not optimal go to Step 3. Otherwise, stop.

Step 3. Apply Simplex Criterion II.

Step 4. Make a change of basis, and return to Step 2.

You can easily interpret the progress of the simplex method in the geometry of the solutions space. Each basis corresponds to a cortex of the convex polyhedral set of feasible solutions. Going from one basis to the next represents going from one extreme point to an adjacent one. Thus the simplex method can be said to seek an optimal solution by *climbing along the edges, from one vertex of the convex polyhedral solution set to a neighbouring one*.Once you have understood the straightforward logic of the simplex iterations, you can save yourself considerable writing effort by organizing the computations in a convenient tabular form called a *simplex tableau* (Table 5.2).

Another way of organizing a *simplex tableau* is shown in Table 5.3. Please note that slightly modified Simplex Criteria will apply to this presentation. Simplex Criterion I will state: If there are nonbasic variables having a negative z_j value choose the variable with the largest $(c_j - z_j)$ to introduce into solution. If all $(c_j - z_j)$ are less or equal to zero we have the optimal solution. Simplex Criterion II remains the same.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
BASIS	c_j	-4	-5	-9	-11	0	0	0	Current values
x_1	-4	1	5/7	0	-5/7	10/7	0	-1/7	50/7
x_6	0	0	-6/7	0	13/7	-61/7	1	4/7	325/7
x_3	-9	0	2/7	1	12/7	-3/7	0	1/7	55/7
	z_j	0	3/7	0	11/7	13/7	0	5/7	695/7
	$c_j - z_j$	-4	-38/7	-9	-88/7	-13/7	0	-5/7	

Table 5.3 The final iteration of alternative Simplex Tableau for the Case Study Problem

5.5 Completeness of the Simplex algorithm

In Criterion I, when two or more variables appear equally promising, as indicated by the values of their coefficients in Row 0, an arbitrary rule may be adopted for selecting one of these. For example, use the lowest-numbered variable, or one suspected to be in the final basis.

In Criterion II, when two or more variables in the current basis are to fall simultaneously to the level zero upon introducing the new variable, only one of these is to be removed from the basis. The others remain in the basis at zero level. The resultant basis is termed *degenerate*. A delicate question of theory is involved in the selection of a tie-breaking rule. Unless some care is given to the method of deciding which variable is to be removed from the basis, you cannot *prove* the method always converges. However, long experience with simplex computations has led to the conclusion that for all *practical* purposes, the selection can be arbitrary and the associated danger of nonconvergence is negligible.

If you find at some iteration in applying Criterion II that there is no positive coefficient in any row for the entering variable, then there exists an unbounded optimal solution. In this event, the entering variable can be made arbitrarily large, the value of x_0 thereby increases without bound, and the current basis variables remain nonnegative. Thus we now may drop the earlier assumption that the optimal value of the objective function is finite. The simplex algorithm provides an indication of when an unbounded optimal solution occurs. Criterion II is easily reworded to cover this case.

Starting basis

Here we turn to the selection of an initial basis to begin the algorithm. Because each constraint in the example of the preceding section was of the form:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ where } b_i \geq 0, \tag{5.15}$$

adding a slack variable to each relation and starting with an all-slack basic solution provided a simple way of initiating the simplex algorithm. The constraints in any linear programming model can be written as:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad \text{for} \quad i = 1, 2, ..., m \quad \text{where } b_i \geq 0. \tag{5.16}$$

In this form, if a variable appears only in constraining relation i and has a coefficient of 1, as would be the case for a slack variable, it can be used as part of the initial basis. But relation i may not have such a variable. (This would occur, for example, if the ith equation were linearly dependent on one or more of the other equations, such as being a sum of two equations.) Then we may employ the following approach.

Write the constraints as:

$$\sum_{j=1}^{n} a_{ij} x_j + 1y_i = b_i \qquad \text{for } i = 1, 2, ..., m \quad \text{where } b_i \geq 0, \tag{5.17}$$

and where $y_i \geq 0$; then use y_i as the basic variable for relation i. (We have assumed, for simplicity, that every constraint requires the addition of a y_i.) The name *artificial variable* is given to y_i because it is added as an artifice in order to obtain an initial-trial solution. Is this approach legitimate? The answer is yes, provided you satisfy Condition A.

> **Condition A**. To ensure that the final solution is meaningful, every y_i must equal 0 at the terminal iteration of the simplex method.

The Big M method

There are a number of computational techniques for guaranteeing Condition A. One approach is to add to the maximizing objective function each y_i with a large penalty-cost coefficient:

$$x_0 - \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} My_i = 0, \tag{5.18}$$

where M is relatively large. Thus each y_i variable is very costly as compared to any of the x_j variables. To initiate the algorithm, you first eliminate each y_i from (5.18) by using (5.17). This gives:

$$x_0 - \sum_{j=1}^{n} c_j x_j - M \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j = -M \sum_{i=1}^{m} b_i, \tag{5.19}$$

which simplifies to:

$$x_0 - \sum_{j=1}^{n} (c_j + M \sum_{i=1}^{m} a_{ij}) x_j = -M \sum_{i=1}^{m} b_i. \tag{5.20}$$

Because the y_i variables are so expensive, the very technique of optimizing drives the y_i variables to zero, *provided* there exists a feasible solution. (Incidentally, whenever a y_i drops from a basis at some iteration, you need never consider using it again, and can eliminate it from further computations.) An example will clarify the approach.

Class exercise

Consider the problem:

$$\text{maximize } -3x_1 - 2x_2 \tag{5.21}$$

subject to:

$$1x_1 + 1x_2 = 10 \tag{5.22}$$

$$1\,x_1 \qquad = 4 \tag{5.23}$$

$$x_1 \geq 0 \qquad x_2 \geq 0 \tag{5.24}$$

Then, after adding a surplus variable x_3 in (5.23), you can write the model as:

$$\begin{array}{llll}
x_0 + 3x_1 + 2x_2 & & = \ 0 & \text{Row 0} \\
1x_1 + 1x_2 & & = 10 & \text{Row 1} \\
1x_1 & - 1x_3 & = \ 4 & \text{Row 2.}
\end{array} \tag{5.25}$$

Next, introduce artificial variables y_1 and y_2, and let M = 10, giving:

$$\begin{array}{llllll}
x_0 + 3x_1 + 2x_2 & & + 10y_1 & + 10y_2 & = \ 0 & \text{Row 0} \\
\quad 1x_1 + 1x_2 & & + 1y_1 & & = 10 & \text{Row 1} \\
\quad 1x_1 & - 1x_3 & & + 1y_2 & = \ 4 & \text{Row 2.}
\end{array} \tag{5.26}$$

To initiate the algorithm, you have to subtract (M = 10) times Row 1 and (M = 10) times Row 2 from Row 0 to eliminate y_1 and y_2:

$$\begin{array}{llllll}
x_0 - 17x_1 - 8x_2 + 10x_3 & & & & = -140 & \text{Row 0} \\
\quad 1x_1 + 1x_2 & & + 1y_1 & & = \ \ 10 & \text{Row 1} \\
\quad \ 1x_1 & - 1x_3 & & + 1y_2 & = \ \ \ 4 & \text{Row 2.}
\end{array} \tag{5.27}$$

Verify that $x_1 = 4$ and $x_2 = 6$ are optimal.

5.6 Duality in LP

There is a unifying concept, namely *duality*, that establishes the inter-connections for all of the sensitivity analysis techniques.

Primal and dual problems

Consider the pair of linear programming models:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j \tag{5.28}$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \text{ for } i = 1, 2, ..., m \tag{5.29}$$

$$x_j \ge 0 \text{ for } j = 1, 2, ..., n. \tag{5.30}$$

and

$$\text{minimize} \quad \sum_{i=1}^{m} b_i y_i \tag{5.31}$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j \text{ for } j = 1, 2, ..., n \tag{5.32}$$

$$y_i \ge 0 \text{ for } i = 1, 2, ..., m. \tag{5.33}$$

For the sake of definiteness, we arbitrarily call (5.28), (5.29), and (5.30) the *primal problem* and (5.31), (5.32), and (5.33) its *dual problem*.

Class exercise

As an illustration, consider the following pair of problems:

$$\text{maximize } 4x_1 + 5x_2 + 9x_3 \tag{5.34}$$

subject to:

$$1x_1 + 1x_2 + 2x_3 \;?\!=16$$

$$7x_1 + 5x_2 + 3x_3 \;?\!= 25 \qquad (5.35)$$

$$x_1 \;?\!= 0 \quad x_2 =?0 \quad x_3 =0$$

<u>Primal</u>;

and

$$\text{minimize } 16y_1 + 25y_2 \qquad (5.36)$$

subject to:

$$1y_1 + 7y_2 =4$$

$$1y_1 + 5y_2 =5 \qquad (5.37)$$

$$2y_1 + 3y_2 \;?\!= 9$$

$$y_1 =0 \quad y_2 \;?\!= 0$$

<u>Dual</u>.

Loosely put, the dual problem can be viewed as the primal model flipped on its side:

(i) The jth column of coefficients in the primal is the same as the jth row of coefficients in the dual.

(ii) The row of coefficients of the primal objective function is the same as the column of constants on the right-hand side of the dual.

(iii) The column of constants on the right-hand side of the primal is the same as the row of coefficients of the dual objective function.

(iv) The direction of the inequalities and sense of optimization are reversed in the pair of problems.

The proposition of significance is the following:

Dual Theorem

(a) In the event that both the primal and dual problems possess feasible solutions, then the primal problem has an optimal solution x^*_j, for j = 1, 2, ..., n, the dual problem has an optimal solution y^*_i, for i = 1, 2, ..., m, and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* . \tag{5.38}$$

(b) If either the primal or dual problem possesses a feasible solution with a finite optimal objective function value, then the other problem possesses a feasible solution with the same optimal objective-function value.

Primal (Maximize)	Dual (Minimize)
Objective function	Right-hand side
Right-hand side	Objective function
jth column of coefficients	jth row of coefficients
ith row of coefficients	ith column of coefficients
jth variable nonnegative	jth relation an inequality
jth variable unrestricted in sign	jth relation an equality
ith relation an inequality	ith variable nonnegative
ith relation an equality	ith variable unrestricted in sign

Table 5.4 Relationship between primal and dual problems

5.7 Sensitivity Analysis

Sensitivity analysis is the study of how the optimal solution and the value of the optimal solution to a linear program change given changes in the various coefficients of the problem. That is, we are interested in answering questions such as the following:

(1) what effect will a change in the coefficients in the objective function (c_j's) have?;

(2) what effect will a change in the right-hand side values (b_i's) have?; and

(3) what effect will a change in the coefficients in the constraining equations (a_{ij}'s) have? Since sensitivity analysis is concerned with how these changes affect the optimal solution, the analysis begins only after the optimal solution to the original linear programming has been obtained. Hence, sensitivity analysis can be referred to as postoptimality analysis.

There are several reasons why sensitivity analysis is considered so important from an engineering point of view. First, consider the fact that engineering businesses operate in a dynamic environment. Prices of raw material change over time; companies purchase new machinery to replace old; stock process fluctuate; employee turnovers occur; etc. If a linear programming model has been used in a decision-making situation and later we find changes in the situation cause changes in some of the coefficients associated with the initial linear programming formulation, we would like to determine how these changes affect the optimal solution to our original linear programming problem. Sensitivity analysis provides us with this information without requiring us to completely solve a new linear program. For example, if the profit for the Simon, Inc. Standard tiles were reduced from \$10 to \$7 per tile, sensitivity analysis call tell the manager whether the production schedule of 540 Standard tiles and 252 Deluxe tiles is still the best decision or not. If it is, we will not have to solve a revised linear program with $7x_1 + 9x_2$ as the objective function.

Sensitivity analysis can also be used to determine how critical estimates of coefficients are in the solution to a linear programming problem. For example, suppose the management of Simon, Inc. realizes the \$10 profit coefficient for Standard tiles is a good, but rough, estimate of the profit the tiles will actually provide. If sensitivity analysis shows Simon, Inc. should produce 540 Standard tiles and 252 Deluxe tiles as long as the actual profit for Standard tiles remains between \$6.00 and \$14.00, management can feel comfortable that the recommended production quantities are optimal. However, if the range for the profit of Standard tiles is \$9.90 to \$12.00, management may want to reevaluate the accuracy of the \$10.00 profit estimate. Management would especially want to consider what revisions would have to be made in the optimal production quantities if the profit for Standard tiles dropped below the $9.90 limit.

As another phase of postoptimality analysis, management may want to investigate the possibility of adding resources to relax the binding constraints. In the Simon, Inc. problem, management would possibly like to consider providing additional hours (e.g., overtime) for the preparations and finishing operations. Sensitivity analysis can help answer the important questions of how much will each added hour be worth in terms of increasing profits and what is the maximum number of hours that can be added before a different basic solution becomes optimal.

Thus, you can see that through sensitivity analysis we will be able to provide valuable information for the decision maker. We begin our study of sensitivity analysis with the coefficients of the objective function.

5.7.1 Sensitivity Analysis – The Coefficients of the Objective Function

In this phase of sensitivity analysis, we will be interested in placing ranges on the values of the objective function coefficients such that as long as the actual value of the coefficient is within this range, the optimal solution will remain unchanged. As stated in the previous section, this information will tell us if we have to alter the optimal solution when a coefficient actually changes, and will provide us with an indication of how critical the estimates of the coefficients are in arriving at the optimal solution.

In the following sensitivity analysis procedures, we will be assuming that only one coefficient changes at a time and that all other objective function coefficients remain at the values defined in the initial linear programming model. To illustrate the analysis for the coefficients of the objective function, let us again consider the final Simplex tableau for the Simon, Inc. problem.

		x_1	x_2	s_1	s_2	s_3	s_4	
BASIS	c_j	10	9	0	0	0	0	*Current value*
x_2	9	0	1	30/16	0	-210/160	0	252
s_2	0	0	0	-15/16	1	25/160	0	120
x_1	10	1	0	-20/16	0	300/160	0	540
s_4	0	0	0	-11/32	0	45/320	1	18
z_j		10	9	70/16	0	111/16	0	7668
$c_j - z_j$		0	0	-70/16	0	-111/16	0	

Coefficients of the Nonbasic Variables

The sensitivity analysis procedure for coefficients of the objective function depends upon whether we are considering the coefficient of a basic or nonbasic variable. For now, let us consider only the case of nonbasic variables.

Since the nonbasic variables are not in solution, we are interested in the question of how much the objective function coefficient would have to change before it would be profitable to bring the associated variable into solution. Recall that it is only profitable to bring a variable into solution if its $(c_j - z_j)$ entry in the net evaluation row is greater than or equal to zero.

Suppose we denote a change in the objective function coefficient of variable x_j by Δc_j. Thus,

$$\Delta c_j = c_j' - c_j \qquad (5.39)$$

where: c_j = the value of the coefficient of x_j in the original linear program; and c_j' = the new value of the coefficient of x_j.

Using this notation, we can write the new objective function coefficient as

$$c_j^{'} = c_j + \Delta c_j \tag{5.40}$$

It will be desirable to bring the nonbasic variable, x_j, into solution if the new objective function coefficient is such that $c_j^{'}$ - $z_j > 0$ (i.e. if it will increase the value of the objective function). On the other hand, we will not want to bring the variable x_j into solution, and thus, will not change our current optimal solution as long as $c_j^{'}$ - $z_j \leq 0$. Our goal in this phase of sensitivity analysis is to determine the range of values $c_j^{'}$ can take on and still not affect the optimal solution.

Recall that z_j is computed by multiplying the coefficients of the <u>basic variables</u> (c_j column of the Simplex tableau) by the corresponding elements in the *j*th column of the A portion of the tableau. Thus, a change in the objective function coefficient for a nonbasic variable cannot affect the value of the z_j's. Therefore, the values of $c_j^{'}$ that do not require us to change the optimal solution are given by

$$c_j^{'} - z_j \leq 0 .$$

Since z_j will be known in the final Simplex tableau, any new coefficient $c_j^{'}$ for a nonbasic variable such that

$$c_j^{'} \leq z_j$$

will not cause a change in the current optimal solution.

Note that there is no lower limit on the new coefficient $c_j^{'}$. This is certainly as expected, since we have a maximization objective function and thus lower and lower $c_j^{'}$ values will make the nonbasic variables even less desirable.

Thus, for nonbasic variables, we can now establish a range of $c_j^{'}$ values which will not affect the current optimal solution. We call this range the *range of insignificance* for the nonbasic variables. It is given by:

$$-\infty < c_j^{'} \leq z_j \, .$$

As long as the objective function coefficients for nonbasic variables remain within their respective ranges of insignificance, the nonbasic variables will remain at a zero value in the optimal solution. Thus, the current optimal solution and the value of the objective function at the optimal solution will not change.

Coefficients of the Basic Variables

Let us start by asking the question of how much would the objective function coefficient of a basic variable have to change before it would be profitable to change the current optimal solution. Again, realize that we will only change the current optimal solution if one or more of the net evaluation row values ($c_j^{'}$ - z_j) becomes greater than zero.

Let us consider a change in the objective function coefficient for the basic variable x_1 in the Simon, Inc. problem. Let the new coefficient value be $c_1^{'}$. Using equation (5.40), we can write $c_1^{'} = c_1 + \Delta c_1$ where c_1 is the original coefficient 10 and Δc_1 is the change in the coefficient. Thus,

$$c_1^{'} = 10 + \Delta c_1 \tag{5.41}$$

Let us now see what happens to the final Simplex tableau of the Simon, Inc. problem where the objective function coefficient for x_1 becomes $10+\Delta c_1$. This tableau is as follows:

		x_1	x_2	s_1	s_2	s_3	s_4	
BASIS	c_j	$10+\Delta c_1$	9	0	0	0	0	Current value
x_2	9	0	1	$30/16$	0	$-210/160$	0	252
s_2	0	0	0	$-15/16$	1	$25/160$	0	120
x_1	$10+\Delta c_1$	1	0	$-20/16$	0	$300/160$	0	540
s_4	0	0	0	$-11/32$	0	$45/320$	1	18
z_j		$10+\Delta c_1$	9	$\frac{70}{16}-\frac{20}{16}\Delta c_1$	0	$\frac{111}{16}+\frac{30}{16}\Delta c_1$	0	7668+540 Δc_1
c_j - z_j		0	0	$-\frac{70}{16}+\frac{20}{16}\Delta c_1$	0	$-\frac{111}{16}-\frac{30}{16}\Delta c_1$	0	

How does the change of Δc_1 affect our final tableau? First, note that since x_1 is a basic variable, the new objective function coefficient $c_1' = 10+\Delta c_1$ appears in the c_j column of the Simplex tableau. This means that the $10+\Delta c_1$ value will affect the z_j values for several of the variables. By looking at the z_j row, you can see that the new coefficient affects the z_j values of the basic variable x_1, both nonbasic variables (s_1 and s_3), and the objective function.

Recall that a decision to change the current optimal solution must be based on values in the net evaluation row. What variables have experienced a change in (c_j - z_j) values because of the change Δc_1? As you can see, the change in the objective function coefficient for basic variable x_1 has caused changes in the $(c_j - z_j)$ values for both of the nonbasic variables. The $(c_j - z_j)$ values for all the basic variables remained unchanged; $(c_j - z_j) = 0$.

We have just identified the primary difference between the objective function sensitivity analysis procedure for basic and nonbasic variables. That is, a change in the objective function coefficient for a nonbasic variable only affects the $c_j - z_j$ value for that variable; however, a change in the objective function coefficient for a basic variable can affect the $c_j - z_j$ values for *all* nonbasic variables.

Returning to the Simon , Inc. problem with the coefficient for x_1 changed to $10 + \Delta c_1$, we know that our current solution will remain optimal as long as all $(c_j - z_j) \leq 0$. Since the basic variables all still have $(c_j - z_j) = 0$, we will have to determine what range of values for Δc_1 will keep the $(c_j - z_j)$ values for all nonbasic values less than or equal to zero.

For nonbasic variables s_1, we must have

$$-{}^{70}\!/_{16} + {}^{20}\!/_{16}\,\Delta c_1 \leq 0\,. \tag{5.42}$$

Solving for Δc_1, we see it will not be profitable to introduce s_1 as long as

$${}^{20}\!/_{16}\,\Delta c_1 \leq {}^{70}\!/_{16}$$

$$\Delta c_1 \leq {}^{16}\!/_{20}\,({}^{70}\!/_{16}) = {}^{7}\!/_{2}$$

$$\Delta c_1 \leq 3.5.$$

For nonbasic variables s_3, we must have

$$-\frac{111}{16} - \frac{30}{16}\Delta c_1 \le 0. \tag{5.43}$$

Solving for Δc_1, we see it will not be profitable to introduce s_3 as long as

$$-\frac{30}{16}\,\Delta c_1 \le \frac{111}{16}$$

$$\frac{30}{16}\,\Delta c_1 \ge -\frac{111}{16}$$

$$\Delta c_1 \quad \ge \frac{16}{30}\left(-\frac{111}{16}\right) = -\frac{111}{30}$$

$$\Delta c_1 \ge -3.7.$$

Thus, in order to keep the net evaluation row values of the nonbasic variables less than or equal to zero, and keep the current optimal solution, changes in c_1 cannot exceed a 3.5 increase ($\Delta c_1 \le 3.5$) or a 3.7 decrease ($\Delta c_1 \ge -3.7$). Hence, our current solution will remain optimal as long as:

$$-3.7 \le \Delta c_1 \le 3.5. \tag{5.44}$$

From equation (5.41), we know that $\Delta c_1 = c_1' - 10$ where c_1' is the new value of the coefficient for x_1 in the objective function. Thus we can use equation (5.44) to define a range for the coefficient values of x_1 that will not cause a change in the optimal solution. This is done as follows:

$$-3.7 \le c_1' - 10 \le 3.5;$$

therefore,

$$6.30 \leq c_1^{'} \leq 13.50.$$

The above result indicates to the decision maker that as long as the profit for one Standard tile is between \$6.30 and \$13.50, the current production quantities of 540 Standard tiles and 252 Deluxe tiles will be optimal. We refer to the above range of values for the objective function coefficient of x_1 as the *range of optimality for* c_1.

To see how the management of Simon , Inc. can make use of the above sensitivity analysis information, suppose that because of an increase in raw material prices, the profit of the Standard tile is reduced to \$7 per unit. The range of optimality for c_1 indicates that the current solution $x_1 = 540$, $x_2 = 252$, $s_1 = 0$, $s_2 = 120$, $s_3 = 0$, $s_4 = 18$ will still be optimal. To see the effect of this change, let us calculate the final Simplex tableau for the Simon , Inc. problem after c_1 has been reduced to \$7.

		x_1	x_2	s_1	s_2	s_3	s_4	Current vlaue
BASIS	c_j	7	9	0	0	0	0	
x_2	9	0	1	30/16	0	-210/160	0	252
s_2	0	0	0	-15/16	1	25/160	0	120
x_1	7	1	0	-20/16	0	300/160	0	540
s_4	0	0	0	-11/32	0	45/320	1	18
z_j		7	9	130/16	0	21/16	0	6048
$c_j - z_j$		0	0	-130/16	0	-21/16	0	

Since all of the (c_j - z_j) values are less than or equal to zero, the solution is optimal. As you can see, this solution is the same as our previous optimal solution. Note, however, that because of the decrease in profit for the Standard tiles, the total profit has been reduced to \$6048 = 7668 + 540 Δc_1 = 7668 + 540 (- 3).

What would happen if the profit per Standard tile were reduced to \$5? Again, we refer to the range of optimality for c_1. Since c_1 = 5 is outside the range, we know that a change this large will cause a new solution to be optimal. Consider the following Simplex tableau containing the same basic feasible solution but with the value of c_1 = 5.

		x_1	x_2	s_1	s_2	s_3	s_4	Current value
BASIS	c_j	5	9	0	0	0	0	
x_2	9	0	1	30/16	0	-210/160	0	252
s_2	0	0	0	-15/16	1	25/160	0	120
x_1	5	1	0	-20/16	0	300/160	0	540
s_4	0	0	0	-11/32	0	45/320	1	18
z_j		5	9	170/16	0	-39/16	0	4968
$c_j - z_j$		0	0	-170/16	0	39/16	0	

As expected, the solution x_1 = 540, x_2 = 252, s_1 = 0, s_2 = 120, s_3 = 0, s_4 = 18 is no longer optimal! The coefficient for s_3 in the net evaluation row is now greater than zero. This implies that at least one more iteration must be performed to reach the optimal solution. Check for yourself to see that the new optimal solution will require production of 300 Standard tiles and 420 Deluxe tiles.

Thus, we see how the range of optimality can be used to quickly determine whether or not a change in the objective function coefficient of a basic variable will cause a change in the optimal solution. Note that by using the range of optimality to determine whether or not the change in profit coefficient for a basic variable is large enough to cause a change in the optimal solution, we can avoid the time-consuming process of reformulating and resolving the entire linear programming problem.

The general procedure for determining the range of optimality for the basic variable associated with column j and row i of the Simplex tableau is to first find the range of values for Δc_1 that satisfy

$$(c_k - z_k) - \bar{a}_{ik}\,\Delta c_j \le 0 \tag{5.45}$$

for each nonbasic variable x_k or s_k. In this process, we will obtain a limit on Δc_j for each nonbasic variable. The most restrictive upper and lower limits on Δc_j will be used to define the range of optimality. For example, if one of these inequalities requires $\Delta c_j \le 3$ and another requires $\Delta c_j \le 1$, the $\Delta c_j \le 1$ is the most restrictive limit and will be the upper limit for Δc_j.

After considering every nonbasic variable, we will have found upper and lower limits on Δc_j in the following form:

$$a \le \Delta c_j \le b$$

where: a = lower limit, and b = upper limit (a is $-\infty$ if inequalities (5.45) do not provide a lower limit on Δc_j and b is $+\infty$ if inequalities (5.45) do not provide an upper limit).

Using the relationship $(\Delta c_j = c_j' - c_j)$, the range of optimality is then given by

$$a \le c_j^{'} - c_j \le b$$

or

$$c_j + a \le c_j^{'} \le c_j + b \tag{5.46}$$

Applying the above procedure to the objective function coefficient c_2 in our Simon, Inc. problem (the profit per unit for Deluxe tiles), we see that in order to satisfy inequalities (5.45), Δc_2 must satisfy

$$(-70/16) - (30/16)\Delta c_2 \le 0 \tag{5.47}$$

and

$$(-111/16) - (21/16)\Delta c_2 \le 0. \tag{5.48}$$

From (5.47), we get

$$\Delta c_2 \ge -7/3 .$$

And from (5.48), we get

$$\Delta c_2 \le 111/21 .$$

Thus we have

$$-7/3 \le \Delta c_2 \le 111/21 .$$

Using equation (5.46) and our original profit coefficient of $c_2 = 9$, we have the following range of optimality for c_2:

$$9 - 7/3 \le c_2^{'} \le 9 + 111/21$$

or

$$6.67 \le c_2^{'} \le 14.29.$$

Thus we see that as long as the profit of Deluxe tiles is between \$6.67 per unit and \$14.29 per unit, the production quantities of 540 Standard tiles and 252 Deluxe tiles will remain optimal.

As a summary, we present the following managerial interpretation of sensitivity analysis for the objective function coefficients. Think of the basic variables as corresponding to our current product line and the nonbasic variables as representing other products we might produce. Within bounds, changes in the profit associated with one of the products in our current product line would not cause us to change our product mix or the amounts produced, but the changes would have an effect on our total profit. Of course, if the profit associated with one of our products changed drastically, we would change our product line (i.e., move to a different basic solution). For products we are not currently producing (nonbasic variables), it is obvious that a decrease in per unit profit would not make us want to produce them; however, if the per unit profit for one of these products became large enough, we would want to consider adding it to our product line.

5.7.2 Sensitivity Analysis – The Right-Hand Sides

A very important phase of sensitivity analysis, both from a theoretical and practical point of view, is the study of the effect of changes of the right-hand sides on the optimal solution and the value of the optimal solution. By changes of the right-hand sides, we mean simply changing the values of one of the elements in the **b** vector in the matrix representation of a linear program. We shall see that the Dual plays a role in this analysis.

Quite often in linear programming problems we can interpret the $b_i's$ as the resources available. For example, in the Simon, Inc. problem, the right-hand side values represented the number of man-hours available in each of four departments. Thus, valuable management information could be provided if we knew how much it would be worth to the company if one or more of these production time resources were increased. Sensitivity analysis of the right-hand sides can help provide this information.

The Dual Variables as Shadow Prices

As we saw earlier in the section 5.6, in our study of the relationships between the Primal and the Dual problems, there is a Dual variable associated with every constraint of the Primal problem. It turns out that the values of these dual variables indicate how much the value of the optimal solution to the Primal problem will change given a change in one of the right-hand side values. This is another property if the relationships between the Primal and Dual problems: If b_i (the right-hand side of the ith primal constraint) is increased by one unit, the value of the optimal solution to the Primal problem will change by u_i units, where the u_i is the value of the dual variable corresponding to the ith primal constraint. We assume here that the same basis will remain optimal after making this increase in b_i. The reason for this assumption will be made clear later.

From the Primal-Dual relationship, we know that we can read the values of the optimal dual variables for the Simon, Inc. problem from the final Simplex tableau of the Primal problem.

BASIS	c_j	x_1 10	x_2 9	s_1 0	s_2 0	s_3 0	s_4 0	Current value
x_2	9	0	1	30/16	0	-210/160	0	252
s_2	0	0	0	-15/16	1	25/160	0	120
x_1	10	1	0	-20/16	0	300/160	0	540
s_4	0	0	0	-11/32	0	45/320	1	18
z_j		10	9	70/16	0	111/16	0	7668
$c_j - z_j$		0	0	-70/16	0	-111/16	0	

Recall that the values of the dual variables are given by the negative of the $(c_j - z_j)$ entries for the slack variables in the Simplex tableau corresponding to the optimal solution of the Primal problem. Thus, you can see from the above tableau that the values of the dual variables are $u_1 = 70/16$, $u_2 = 0$, $u_3 = 111/16$, $u_4 = 0$.

According to the new above introduced property, an increase in b_2 or b_4 of one unit should have no effect on the objective function. Clearly this is true for Simon, Inc. problem since the glazing and inspection and packaging constraints are satisfied as strict *inequalities.* That is, the slack variable associated with each of these constraints is in the basic solution. Since there is slack time in both of these departments, we could not expect the value of the optimal solution to change when we increased the amount of slack time. In other words, we already have too much of each of these resources.

Notice that the dual variables corresponding to the first and third constraints are positive. This is to be expected since both of these constraints are binding in the optimal solution. Thus, if we could increase the amount of preparation and/or finishing time available, we could expect the value of the optimal solution to increase. The dual variable $u_1 = 70/16$ indicates that the value of the optimal solution will increase by $70/16$ if b_1 - the maximum available preparation time – is increased by one unit. Similarly, the dual variable $u_3 = 111/16$ indicates that the value of the optimal solution will increase by $111/16$ if b_3 - the maximum available finishing time - is increased by one unit.

Let us consider the preparation operation in detail. Suppose we increase the original 630 hours available to 631 hours and then compute the revised final Simplex tableau. From this tableau, which is shown below, we see that the objective function has increased by $70/16$ as expected.

		x_1	x_2	s_1	s_2	s_3	s_4	Current value
BASIS	c_j	10	9	0	0	0	0	
x_2	9	0	1	30/16	0	-210/160	0	$253\,14/16$
s_2	0	0	0	-15/16	1	25/160	0	$119\,1/16$
x_1	10	1	0	-20/16	0	300/160	0	$538\,12/16$
s_4	0	0	0	-11/32	0	45/320	1	$17\,21/32$
z_j		10	9	70/16	0	111/16	0	$7672\,6/16$
$c_j - z_j$		0	0	-70/16	0	-111/16	0	

You are probably wondering how this final tableau was computed. Certainly, we did not go through all of the Simplex iterations again after changing b_1 from 630 to 631. You can see that the only changes in the tableau are the differences in the values of the basic variables (i.e., the last column). The entries in this last column of the Simplex tableau have been obtained by merely adding the first five entries on the third column to the last column in the previous tableau. That is,

$$\begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \\ 70/16 \end{bmatrix} + \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} = \begin{bmatrix} 253\,14/16 \\ 119\,1/16 \\ 538\,12/16 \\ 17\,21/32 \\ 7672\,6/16 \end{bmatrix}$$

The reason for this is as follows. The entries in column s_1 tell us how many units of the basic variables will be driven out of solution if one unit of variable s_1 is introduced into solution. Increasing b_1 by one unit has just the reverse effect. That is, it is just the same as taking one unit of s_1 out of solution. If b_1 is increased by one unit, then the entries in the s_1 column tell us how many units of each of the basic variables may be added to solution for every unit of s_1 added. The increase in the value of the objective function corresponding to adding one unit of b_1 is equal to the negative of $(c_3 - z_3)$, or equivalently, the value of the dual variable u_1.

Since this is such an important property, let us state it again. That is, associated with every constraint of the Primal problem is a dual variable. The value of the dual variable indicates how much the objective function of the Primal problem will increase as the value of the right-hand side of the associated primal constraint is increased by one unit.

In other words, the value of the dual variable indicates the value of one additional unit of a particular resource. Hence, this value can be interpreted as the maximum value or price we would be willing to pay to obtain the one additional unit of the resource. Because of this interpretation, the value of one additional unit of a resource is often called the *shadow price* of the resource. Thus, the values of the dual variables are often referred to as shadow prices.

Range of Feasibility for the Right-Hand Side Values

The dual variables can be used to predict the change in the value of the objective function corresponding to a unit change in one of the b_i's. But here is the "catch". The dual variables can be used in this fashion only as long as the change in the b_i's is not large enough to make the current basis infeasible.

Thus, we will be interested in determining how much a particular b_i can be changed without causing a change in the current optimal basis. In effect we will do this by calculating a range of values over which a particular b_i can vary without any of the current basic variables becoming infeasible. This range of values will be referred to as the *range of feasibility*.

To demonstrate the effect of increasing a resource by several units, consider increasing the available preparation time for the Simon, Inc. problem by 10 hours. Will the new basic solution be feasible? If so, we can expect an increase in the objective function of $10u_1 = 10\ (^{70}\!/_{16}) = \$43.75$. As before we can calculate the new values of the basic variables by adding to the old values the change in b_i times the coefficients in the s_1 column.

$$\text{New solution} = \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \end{bmatrix} + 10 \begin{bmatrix} ^{30}\!/_{16} \\ -^{15}\!/_{16} \\ -^{20}\!/_{16} \\ -^{11}\!/_{32} \end{bmatrix} = \begin{bmatrix} 270\,^{12}\!/_{16} \\ 110\,^{10}\!/_{16} \\ 527\,^{8}\!/_{16} \\ 14\,^{9}\!/_{16} \end{bmatrix}$$

Since the new solution is still feasible (i.e. all of the basic variable are ≥ 0), the prediction made by the dual variable of a \$43.75 change in the objective function resulting from a 10-unit increase in b_i is correct.

How do we know when a change in b_i is so large that the current basis will become infeasible? We shall first answer this question specifically for the Simon, Inc. problem.

But, you should realize that the following procedure applies for only the less-than-or-equal-to constraints of a linear program. The procedures for the cases of greater-than-or-equal-to and equality constraints are discussed later in this section.

We begin by showing how to calculate upper and lower bounds for the maximum amount b_1 can be changed before the current basis becomes feasible. Given a change in b_1 of Δb_1, the *new basic solution* is given by:

$$\text{New solution} = \begin{bmatrix} x_2 \\ s_2 \\ x_1 \\ s_4 \end{bmatrix} = \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \end{bmatrix} = \begin{bmatrix} 252 = 30/16\,\Delta b_1 \\ 120 - 15/16\,\Delta b_1 \\ 540 - 20/16\,\Delta b_1 \\ 18 - 11/32\,\Delta b_1 \end{bmatrix} \tag{5.48}$$

As long as the value of each variable in the new basic solution remains non-negative, the new basic solution will remain feasible and, therefore, optimal. We can keep the variables non-negative by limiting the change in b_1 (i.e., Δb_1) so that we satisfy each of the following conditions:

$$252 + 30/16\,\Delta b_1 \geq 0 \tag{5.49}$$

$$120 - 15/16\,\Delta b_1 \geq 0 \tag{5.50}$$

$$540 - 20/16\,\Delta b_1 \geq 0 \tag{5.51}$$

$$18 - 11/32\,\Delta b_1 \geq 0. \tag{5.52}$$

Note that the left-hand sides of the above inequalities represent the values of the basic variables after b_1 has been changed by Δb_1.

Working algebraically with the four inequalities (5.49 – 5.52), we get:

$$\Delta b_1 \geq (16/30)(-252) = -134.4$$
$$\Delta b_1 \leq (-16/15)(-120) = 128$$

$$\Delta b_1 \le (-16/20)(-540) = 432$$
$$\Delta b_1 \le (-32/11)(-18) = 52\,4/11.$$

Since we must satisfy all four inequalities, the most restrictive limits on Δb_1 must be used; therefore, we have

$$-134.4 \le \Delta b_1 \le 52\,4/11.$$

Using $\Delta b_1 = b_1^{'} - b_1$, where $b_1^{'}$ is the new number of hours available in the preparation department and b_1 is the original number of hours available (b_1=630), we have the following

$$-134.4 \le b_1^{'} - 630 \le 52\,4/11$$

or

$$495.6 \le b_1^{'} \le 682\,4/11.$$

The above range of values for $b_1^{'}$ indicates that as long as the time available in preparations department is between 495.6 and 682 $4/11$ hours, the current basis will remain feasible and optimal. Thus, this is the range of feasibility for the right-hand side value of the preparations constraint.

Since $u_1 = 70/16$, we know we can improve profit by $70/16$ for an additional hour of preparation time. Suppose we increase b_1 by $52\,4/11$ hours to the upper limit of its range of feasibility, $682\,4/11$. The new profit with this change will be 7668 + ($70/16$) ($52\,4/11$) = 7897 $1/11$, and the new optimal solution (using (5.48)) is

$$x_2 = 252 + 30/16\,(52\,4/11) = 350\,2/11$$
$$s_2 = 120 - 15/16\,(52\,4/11) = 70\,10/11$$

$$x_1 = 540 - {}^{20}\!/_{16}(52\,{}^{4}\!/_{11}) = 474\,{}^{6}\!/_{11}$$

$$s_4 = 18 + {}^{11}\!/_{32}(52\,{}^{4}\!/_{11}) = 0$$

with the nonbasic variables s_1 and s_3 still equal to zero.

What has happened to our solution in the above process? You can see that the increased preparation time has caused us to revise the production plan so that Simon will produce a greater number of Deluxe tiles and a slightly smaller number of Standard tiles. Overall, the profit will be increased by $({}^{70}\!/_{16})\,(52\,{}^{4}\!/_{11}) = \$229\,{}^{1}\!/_{11}$.

Our procedure for determining the range of feasibility has involved only the preparation constraint. The procedure for calculating the range of feasibility for the right-hand side value of any less-than-or-equal-to constraint is the same. The first step (paralleling (5.48)) for a general constraint i is to calculate the range of values for Δb_i that satisfy the conditions shown below.

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ . \\ . \\ . \\ . \\ \bar{b}_m \end{bmatrix} + \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ . \\ . \\ . \\ . \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \end{bmatrix} \qquad (5.53)$$

Current solution (last column of the final Simplex tableau)

Column of the final Simplex tableau corresponding to the <u>slack</u> variable associated with constraint i

This determines a lower and an upper limit on Δb_i. The range of feasibility can then be established.

Similar arguments to the ones presented in this section can be used to develop a procedure for determining the range of feasibility for the right-hand side value of a greater-than-or-equal-to constraint. Essentially, the procedure is the same with the column corresponding to the surplus variable associated with the constraint playing the central role. For a general greater-than-or-equal-to constraint i, we first calculate the range of values for Δb_i that satisfy the conditions shown in (5.54).

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ . \\ . \\ . \\ . \\ \bar{b}_m \end{bmatrix} - \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ . \\ . \\ . \\ . \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \end{bmatrix} \tag{5.54}$$

Current solution

Column of the final Simplex tableau corresponding to the <u>surplus</u> variable associated with constraint i

Once again, these conditions establish a lower and an upper limit on Δb_i. Given these limits, the range of feasibility is easily determined.

To calculate the range of feasibility for the right-hand side value of an equality constraint, we use the column of the Simplex tableau corresponding to the artificial variable associated with that constraint. The limits on Δb_i are given by the Δb_i values satisfying (5.55).

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ . \\ . \\ . \\ . \\ \bar{b}_m \end{bmatrix} + \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ . \\ . \\ . \\ . \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \end{bmatrix} \qquad (5.55)$$

Current solution

Column of the final Simplex tableau corresponding to the <u>artificial</u> variable associated with constraint i

As long as the change in b_i is such that the value of $b_i^{'}$ stays within its range of feasibility, the same basis will remain feasible and optimal. Changes that force $b_i^{'}$ outside its range of feasibility will force us to perform additional Simplex iterations to find the new optimal basic feasible solution. More advanced linear programming texts show how this can be done without resolving the problem. In any case, the calculation of the range of feasibility for each of the $b_i^{'}$'s is valuable management information and should be included as part of the management report.

Note that the procedure for determining the range of feasibility for the right-hand side values has involved changing only *one* value at a time (i.e., only one Δb_i). Multiple changes are more difficult to analyze and in practice usually require the complete generation of a new final Simplex tableau.

5.7.3 Sensitivity Analysis – The A Matrix

A change in one of the coefficients of the A matrix can have a significant effect on the optimal solution to a linear programming problem. A complete discussion of the ramifications of making changes in the A matrix is beyond the scope of this text.

5.8 Summary LP

Postulates

(a) Objective function and constraints can be expressed as simple algebraic expressions.

(b) Objective function must be convex and the constraints must form a convex policy space.

(c) All decision variables must be positive.

Advantages and limitations

(1) Uses standard programs.

(2) User does not have to know details of method, only its philosophy and limitations.

(3) Can handle a large number of state variables.

(4) Can handle fewer decision variables than other programming techniques (for example dynamic programming) on a sequential basis, but can handle a large dimension of decision variables for one time period. (Dynamic programming must keep the dimensionality of decision variables as small as possible, preferably less than two within each unit time period.)

(5) Constraints tend to increase the computational cost.

(6) Requires convex systems (convex policy space).

(7) Requires linear relationships (nonlinear relationship can be linearized in several presently available packages).

(8) Optimum obtained is only a single set of decisions (nothing known about second or third best).

5.9 Use of Microsoft Excel Solver for solving linear programming problems

Following material is modified from the Microsoft Excel Tutorial.

Quick Tour of Microsoft Excel Solver

Month	*Q1*	*Q2*	*Q3*	*Q4*	*Total*
Seasonality	0.9	1.1	0.8	1.2	
Units Sold	3,592	4,390	3,192	4,789	15,962
Sales Revenue	$143,662	$175,587	$127,700	$191,549	$638,498
Cost of Sales	89,789	109,742	79,812	119,718	399,061
Gross Margin	53,873	65,845	47,887	71,831	239,437
Salesforce	8,000	8,000	9,000	9,000	34,000
Advertising	10,000	10,000	10,000	10,000	40,000
Corp Overhead	21,549	26,338	19,155	28,732	95,775
Total Costs	39,549	44,338	38,155	47,732	169,775
Prod. Profit	$14,324	$21,507	$9,732	$24,099	$69,662
Profit Margin	10%	12%	8%	13%	11%

Product Price	$40.00
Product Cost	$25.00

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The following examples show you how to work with the model above to solve for one value or several values to maximize or minimize another value, enter and change constraints, and save a problem model.

Row	**Contains**	**Explanation**
3	Fixed values	Seasonality factor: sales are higher in quarters 2 and 4, and lower in quarters 1 and 3.
5	=35*B3*(B11+3000)^0.5	Forecast for units sold each quarter: row 3 contains the seasonality factor; row 11 contains the cost of advertising.
6	=B5*B18	Sales revenue: forecast for units sold (row 5) times price (cell B18).
7	=B5*B19	Cost of sales: forecast for units sold (row 5) times product cost (cell B19).
8	=B6-B7	Gross margin: sales revenues (row 6) minus cost of

		sales (row 7).
10	Fixed values	Sales personnel expenses.
11	Fixed values	Advertising budget (about 6.3% of sales).
12	=0.15*B6	Corporate overhead expenses: sales revenues (row 6) times 15%.
13	=SUM(B10:B12)	Total costs: sales personnel expenses (row 10) plus advertising (row 11) plus overhead (row 12).
15	=B8-B13	Product profit: gross margin (row 8) minus total costs (row 13).
16	=B15/B6	Profit margin: profit (row 15) divided by sales revenue (row 6).
18	Fixed values	Product price.
19	Fixed values	Product cost.

This is a typical marketing model that shows sales rising from a base figure (perhaps due to the sales personnel) along with increases in advertising, but with diminishing returns. For example, the first $5,000 of advertising in Q1 yields about 1,092 incremental units sold, but the next $5,000 yields only about 775 units more.

You can use Solver to find out whether the advertising budget is too low, and whether advertising should be allocated differently over time to take advantage of the changing seasonality factor.

Solving for a Value to Maximize Another Value

One way you can use Solver is to determine the maximum value of a cell by changing another cell. The two cells must be related through the formulas on the worksheet. If they are not, changing the value in one cell will not change the value in the other cell.

For example, in the sample worksheet, you want to know how much you need to spend on advertising to generate the maximum profit for the first quarter. You are interested in maximizing profit by changing advertising expenditures.

- On the **Tools** menu, click **Solver**. In the **Set target cell** box, type **b15** or select cell B15 (first-quarter profits) on the worksheet. Select the **Max** option. In the **By changing cells** box, type **b11** or select cell B11 (first-quarter advertising) on the worksheet. Click **Solve**.

You will see messages in the status bar as the problem is set up and Solver starts working. After a moment, you'll see a message that Solver has found a solution. Solver finds that Q1 advertising of $17,093 yields the maximum profit $15,093.

- After you examine the results, select **Restore original values** and click **OK** to discard the results and return cell B11 to its former value.

Resetting the Solver Options

If you want to return the options in the **Solver Parameters** dialog box to their original settings so that you can start a new problem, you can click **Reset All**.

Solving for a Value by Changing Several Values

You can also use Solver to solve for several values at once to maximize or minimize another value. For example, you can solve for the advertising budget for each quarter that will result in the best profits for the entire year. Because the seasonality factor in row 3 enters into the calculation of unit sales in row 5 as a multiplier, it seems logical that you should spend more of your advertising budget in Q4 when the sales response is highest, and less in Q3 when the sales response is lowest. Use Solver to determine the best quarterly allocation.

- On the **Tools** menu, click **Solver**. In the **Set target cell** box, type **f15** or select cell F15 (total profits for the year) on the worksheet. Make sure the **Max** option is selected. In the **By changing cells** box, type **b11:e11** or select cells B11:E11 (the advertising budget for each of the four quarters) on the worksheet. Click **Solve**.

- After you examine the results, click **Restore original values** and click **OK** to discard the results and return all cells to their former values.

You've just asked Solver to solve a moderately complex nonlinear optimization problem; that is, to find values for the four unknowns in cells B11 through E11 that will maximize profits. (This is a nonlinear problem because of the exponentiation that occurs in the formulas in row 5). The results of this unconstrained optimization show that you can increase profits for the year to $79,706 if you spend $89,706 in advertising for the full year.

However, most realistic modeling problems have limiting factors that you will want to apply to certain values. These constraints may be applied to the target cell, the changing cells, or any other value that is related to the formulas in these cells.

Adding a Constraint

So far, the budget recovers the advertising cost and generates additional profit, but you're reaching a point of diminishing returns. Because you can never be sure that your model of sales response to advertising will be valid next year (especially at greatly increased spending levels), it doesn't seem prudent to allow unrestricted spending on advertising.

Suppose you want to maintain your original advertising budget of $40,000. Add the constraint to the problem that limits the sum of advertising during the four quarters to $40,000.

- On the **Tools** menu, click **Solver**, and then click **Add**. The **Add Constraint** dialog box appears. In the **Cell reference** box, type **f11** or select cell F11 (advertising total) on the worksheet. Cell F11 must be less than or equal to $40,000. The relationship in the **Constraint** box is **<=** (less than or equal to) by default, so you don't have to change it. In the box next to the relationship, type **40000**. Click **OK**, and then click **Solve**.

- After you examine the results, click **Restore original values** and then click **OK** to discard the results and return the cells to their former values.

The solution found by Solver allocates amounts ranging from $5,117 in Q3 to $15,263 in Q4. Total Profit has increased from $69,662 in the original budget to $71,447, without any increase in the advertising budget.

Changing a Constraint

When you use Microsoft Excel Solver, you can experiment with slightly different parameters to decide the best solution to a problem. For example, you can change a constraint to see whether the results are better or worse than before. In the sample worksheet, try changing the constraint on advertising dollars to $50,000 to see what that does to total profits.

- On the **Tools** menu, click **Solver**. The constraint, **F11<=40000**, should already be selected in the **Subject to the constraints** box. Click **Change**. In the **Constraint** box, change **40000** to **50000**. Click **OK**, and then click **Solve**. Click **Keep solver solution** and then click **OK** to keep the results that are displayed on the worksheet.

Solver finds an optimal solution that yields a total profit of $74,817. That's an improvement of $3,370 over the last figure of $71,447. In most firms, it's not too difficult to justify an incremental investment of $10,000 that yields an additional $3,370 in profit, or a 33.7% return on investment. This solution also results in profits of $4,889 less than the unconstrained result, but you spend $39,706 less to get there.

Saving a Problem Model

When you click **Save** on the **File** menu, the last selections you made in the **Solver Parameters** dialog box are attached to the worksheet and retained when you save the workbook. However, you can define more than one problem for a worksheet by saving them individually using **Save Model** in the **Solver Options** dialog box. Each problem model consists of cells and constraints that you entered in the **Solver Parameters** dialog box.

When you click **Save Model**, the **Save Model** dialog box appears with a default selection, based on the active cell, as the area for saving the model. The suggested range includes a cell for each constraint plus three additional cells. Make sure that this cell range is an empty range on the worksheet.

- On the **Tools** menu, click **Solver**, and then click **Options**. Click **Save Model**. In the **Select model area** box, type **h15:h18** or select cells H15:H18 on the worksheet. Click **OK**.

Note You can also enter a reference to a single cell in the **Select model area** box. Solver will use this reference as the upper-left corner of the range into which it will copy the problem specifications.

To load these problem specifications later, click **Load Model** on the **Solver Options** dialog box, type **h15:h18** in the **Model area** box or select cells H15:H18 on the sample worksheet, and then click **OK**. Solver displays a message asking if you want to reset the current Solver option settings with the settings for the model you are loading. Click **OK** to proceed.

Example 1: Product mix problem with diminishing profit margin.

Your company manufactures TVs, stereos and speakers, using a common parts inventory of power supplies, speaker cones, etc. Parts are in limited supply and you must determine the most profitable mix of products to build. But your profit per unit built decreases with volume because extra price incentives are needed to load the distribution channel.

C

			TV set	*Stereo*	*Speaker*
		Number to Build->	100	100	100
Part Name	*Inventory*	*No. Used*			
Chassis	450	200	1	1	0
Picture Tube	250	100	1	0	0
Speaker Cone	800	500	2	2	1
Power Supply	450	200	1	1	0
Electronics	600	400	2	1	1

Diminishing Returns Exponent: 0.9

Profits:

By Product	$4,732	$3,155	$2,208
Total	**$10,095**		

This model provides data for several products using common parts, each with a different profit margin per unit. Parts are limited, so your problem is to determine the number of each product to build from the Inventory on hand in order to maximize profits.

Problem Specifications

Target Cell	D18	Goal is to maximize profit.
Changing cells	D9:F9	Units of each product to build.
Constraints	C11:C15<=B11:B15	Number of parts used must be less than or equal to the number of parts in inventory.
	D9:F9>=0	Number to build value must be greater than or equal to 0.

The formulas for profit per product in cells D17:F17 include the factor ^H15 to show that profit per unit diminishes with volume. H15 contains 0.9, which makes the problem nonlinear. If you change H15 to 1.0 to indicate that profit per unit remains constant with volume, and then click **Solve** again, the optimal solution will change. This change also makes the problem linear.

***Example* 2: Transportation Problem.**

Minimize the costs of shipping goods from production plants to warehouses near metropolitan demand centers, while not exceeding the supply available from each plant and meeting the demand from each metropolitan area.

		Number to ship from plant x to warehouse y (at intersection):				
Plants:	*Total*	*San Fran*	*Denver*	*Chicago*	*Dallas*	*New York*
S. Carolina	5	1	1	1	1	1
Tennessee	5	1	1	1	1	1
Arizona	5	1	1	1	1	1
		---	---	---	---	---
Totals:		3	3	3	3	3
	Demands by Whse -->	180	80	200	160	220
Plants:	*Supply*	*Shipping costs from plant x to warehouse y (at intersection):*				
S. Carolina	310	10	8	6	5	4
Tennessee	260	6	5	4	3	6
Arizona	280	3	4	5	5	9
Shipping:	**$83**	$19	$17	$15	$13	$19

The problem presented in this model involves the shipment of goods from three plants to five regional warehouses. Goods can be shipped from any plant to any warehouse, but it obviously costs more to ship goods over long distances than over short distances. The problem is to determine the amounts to ship from each plant to each warehouse at minimum shipping cost in order to meet the regional demand, while not exceeding the plant supplies.

Problem Specifications

Target cell	B20	Goal is to minimize total shipping cost.
Changing cells	C8:G10	Amount to ship from each plant to each warehouse.
Constraints	B8:B10<=B16:B18	Total shipped must be less than or equal to supply at plant.
	C12:G12>=C14:G14	Totals shipped to warehouses must be greater than or equal to demand at warehouses.
	C8:G10>=0	Number to ship must be greater than or equal to 0.

You can solve this problem faster by selecting the **Assume linear model** check box in the **Solver Options** dialog box before clicking **Solve**. A problem of this type has an optimum solution at which Amounts to ship are integers, if all of the supply and demand constraints are integers.

5.10 Problems

5.1 A concrete products manufacturer makes two types of building blocks: type A and type B. For each set of 100 type A blocks the manufacturer can make a profit of \$5 whereas for each set of 100 type B blocks he can make \$8. Assume that all blocks which are produced can be sold. It takes 1 hour to make 100 type A blocks and 3 hours to make 100 type B blocks. Each day there are 12 hours available for block manufacturing. A set of 100 type A blocks requires 2 units of cement, 3 units of fine aggregate , and 4 units of coarse aggregate; a set of 100 type B blocks requires 1 unit of cement and 6 units of coarse aggregate. Each day, 18 units of cement and 24 units of coarse aggregate are available for block manufacturing. There is no restriction on the availability of fine aggregate. How many type A and type B blocks should be made during the day to maximize profits?

- *(a) Formulate the problem as a LP problem.*
- *(b) Name decision variables, objective function, and constraints.*
- *(c) Solve the problem graphically.*
- *(d) Use Excel Solver to solve the problem.*

5.2 Draw the feasible region for the following LP problem:

$5x_1 + 5x_2 = 50$

$x_2 = 6$

$x_1 = 8$

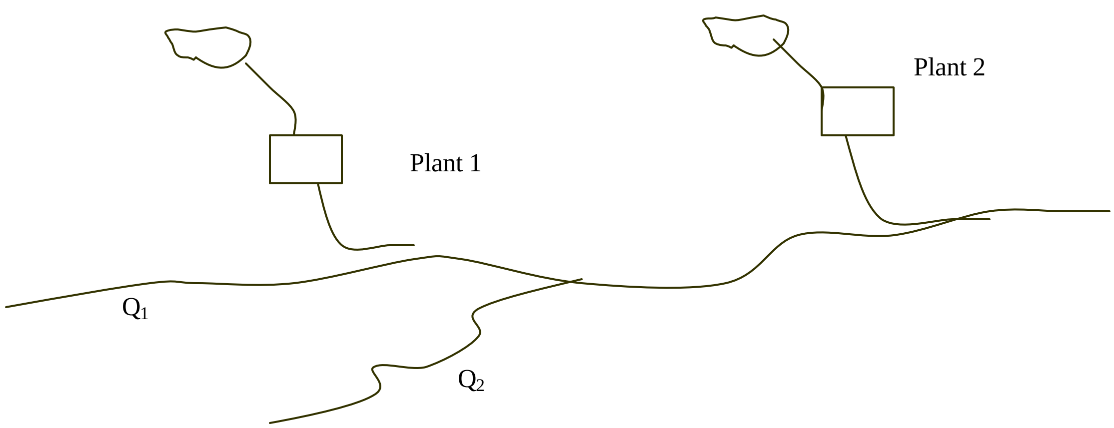
$2x_1 + 4x_2 = 28$

$x_1 + 2x_2 = 14$

- *(a) Provide the extreme points.*
- *(b) Is the feasible region convex?*
- *(c) Indicate any redundant constraints if any.*
- *(d) Introduce a new set of constraints that will make the feasible region infeasible.*
- *(e) Introduce a new constraint that can make the feasible region unbounded (assuming the objective is to maximize the objective function).*

5. 2 The Sunnyflush Company has two plants located along a stream. Plant 1 is generating 20 units of pollutants daily and plant 2 14 units. Before the wastes are discharged into the river, part of these pollutants are removed by a waste treatment facility in each plant. The costs associated with removing a unit of pollutant are \$1,000 and \$800 for plants 1 and 2. The rates of flow in the streams are Q_1 =5mgd and Q_2 =2mgd, and the flows contain no pollutants until they pass the plants. Stream standards require that the number of units of pollutants per million gallons of flow should not exceed 2. 20% of the pollutants entering the stream at plant 1 will be removed by natural processes before they reach plant 2. The company wants to determine the most economical operation of its waste treatment facilities that will allow it to satisfy the stream standards.

- *(a) Formulate the problem as LP problem.*
- *(b) Name decision variables, objective function, and constraints.*
- *(c) Solve the problem graphically.*
- *(d) Use Excel Solver to solve the problem.*



5.3 Consider the following LP problem

maximize $x_1 + x_2$

subject to

$-x_1 + x_2 = -1$

$x_1 - x_2 = -1$

$x_1 = 0$ *and* $x_2 = 0$.

(a) Find graphically the solution of the stated problem.

(b) Discuss the solution obtained in details.

(c) Modify the problem by changing the sign of both inequalities and adding one more constraint:

$x_1 + x_2 = 6$

and solve it using the Excel Solver.

5.4 The reservoir A is used for recreation (swimming, water skiing, canoeing). It is important to keep the average depth of this reservoir within prescribed limits, which vary from one month to the next. The engineers responsible for the operations of reservoir A have estimated a rapid rate of seepage and evaporation from the reservoir. Since rainfall is negligible, reservoir A must be maintained by spillage from reservoir B.

Suppose that planning horizon is 20 months. During month t, let x_t *denote the average depth of the reservoir A prior to augmenting with the water from reservoir B;* $x_1 = 25$ *for month 1. Let* y_t *be number of meters to be added to the average depth in month t (positive value for* y_t *indicates a decision to augment the reservoir A). Let* L_t *and* U_t *represent the lower and upper prescribed limits, respectively, of the average reservoir depth after augmentation in month t. Assume that* x_{t+1} *is 0.75 of the average reservoir depth in month t after augmentation.*

Suppose that the cost of augmenting the reservoir is c_t *per meter in month t. Formulate an appropriate optimization model. Do not solve the problem.*

5.5 Formulate the following linear program using the Big-M method and solve it using Excel Solver:

$$\text{Max } 4x_1 + x_2$$

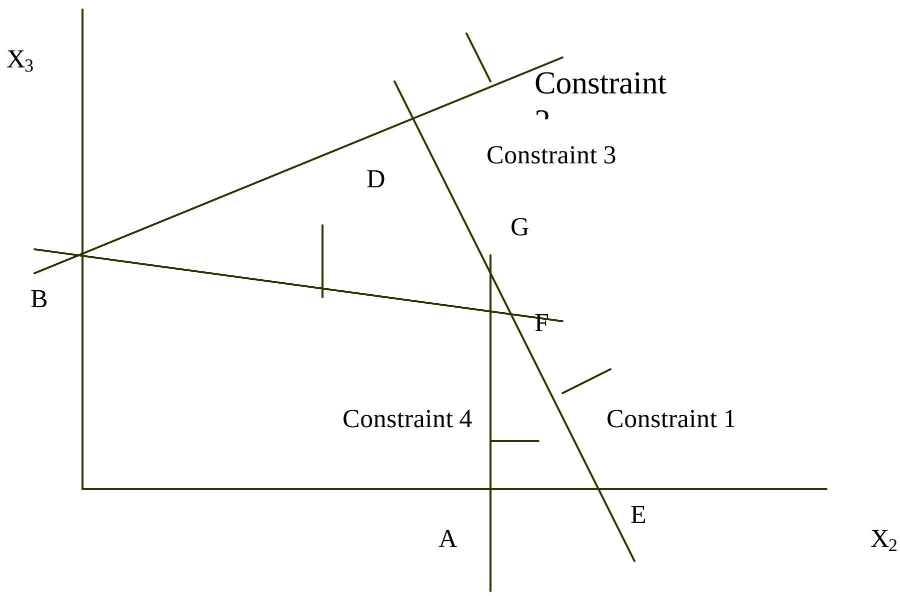
subject to

$$2x_1 + 3x_2 = 4$$

$$3x_1 + 6x_2 = 9$$

$$x_1, x_2 = 0.$$

5.6 Consider the linear programming problem described in the Figure below:



(a) How many constraints are there?

(b) Which are binding constraints? Which are redundant?

(c) How many variables are there? What are they?

(d) At point A, which variables are greater than 0?

(e) At point A, which variables are basic? Which are non-basic?

(f) At point B, which variables are greater than 0?

(g) At point B, how many basic feasible solutions are there? What are the basic variables associated with each basic feasible solution?

5.11 References

De Neufville, R., (1990), *Applied Systems Analysis: Engineering Planning and Technology Management*, McGraw Hill, New York, USA.

Hillier, F.S. and G.J. Lieberman, (1990), *Introduction to Mathematical Programming*, McGraw Hill, New York, USA.

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6. MULTI-OBJECTIVE ANALYSIS

From the previous sections we see that the single-objective programming problem consists of optimising one objective subject to a constraint set. On the other hand, a multiobjective programming problem is characterised by a p-dimensional vector of objective functions:

$$\mathbf{Z}(\mathbf{x}) = [Z_1(\mathbf{x}), Z_2(\mathbf{x}), ..., Z_p(\mathbf{x})] \tag{6.1}$$

subject to:

$\mathbf{x}$ Î $\mathbf{X}$

where **X** is a feasible region:

$\mathbf{X} = \{\mathbf{x}\textbf{:}\ \mathbf{x}$ Î $\mathbf{R^n}, g_i(\mathbf{x}) \leq 0, x_j \geq 0\ \forall\ i, j\}$ (6.2)

where	**R**	=	set of real numbers;
	$g_i(\mathbf{x})$	=	set of constraints; and
	x	=	set of decision variables.

The word "optimization" has been purposely kept out of the definition of a multiobjective programming problem since one can not, in general, optimise a priori a vector of objective functions. The first step of the multiobjective problem consists of identifying the set of nondominated solutions within the feasible region **X**. So instead of seeking a single optimal solution, a set of "noninferior" solutions is sought.

6.1 Basic concepts of multiobjective analysis

The essential difficulty with multiobjective analysis is that the meaning of the optimum is *not defined* so long as we deal with multiple objectives that are truly different. For example, suppose we are trying to determine the best design of a system of dams on a river, with the objectives of promoting "national income," reducing "deaths by flooding," and increasing "employment." Some designs will be more profitable, but less effective at reducing deaths. How can we state which is better when the objectives are so different, and measured in such different terms? How can one state with any accuracy what the relative value of a life is in terms of national income? If one resolved that question, then how would one determine the relative value of new jobs and other objectives? The answer is, with extreme difficulty. The attempts to set values on these objectives are, in fact, most controversial.

To obtain a single global optimum over all objectives requires that we either establish or impose some means of specifying the value of each of the different objectives. If all objectives can indeed be valued on a common basis, the optimization can be stated in terms of that single value. The multiobjective problem has then disappeared and the optimization proceeds relatively smoothly in terms of a single- objective.

In practice it is frequently awkward if not indefensible to give every objective a relative value. The relative worth of profits, lives lost, the environment, and other such objectives are unlikely to be established easily by anyone, or to be accepted by all concerned. One cannot hope, then, to be able to determine an acceptable optimum analytically.

The focus of multiobjective analysis in practice is to sort out the mass of clearly dominated solutions, rather than determine the single best design. The result is the identification of the small subset of feasible solutions that are worthy of further consideration. Formally, this result is known as the set of nondominated solutions.

Nondominated solutions

To understand the concept of nondominated solutions, it is necessary to look closely at the multiobjective problem (6.1) and (6.2). (Note: The nondominated solutions are sometimes referred to by other names: noninferior, Pareto optimal, efficient, etc. Throughout this text different names will be used with the same meaning.)

The essential feature of the multiobjective problem is that the feasible region of production of the solutions is much more complex than for a single-objective. In single optimization, any set of inputs, x, produces a set of results, z, that could be represented by a straight line going from bad (typically zero output) to best. In a multiobjective problem, any set of inputs, x, defines a multidimensional space of feasible solutions, as Figure 6.1 indicates. Then there is no exact equivalent of a single optimal solution.

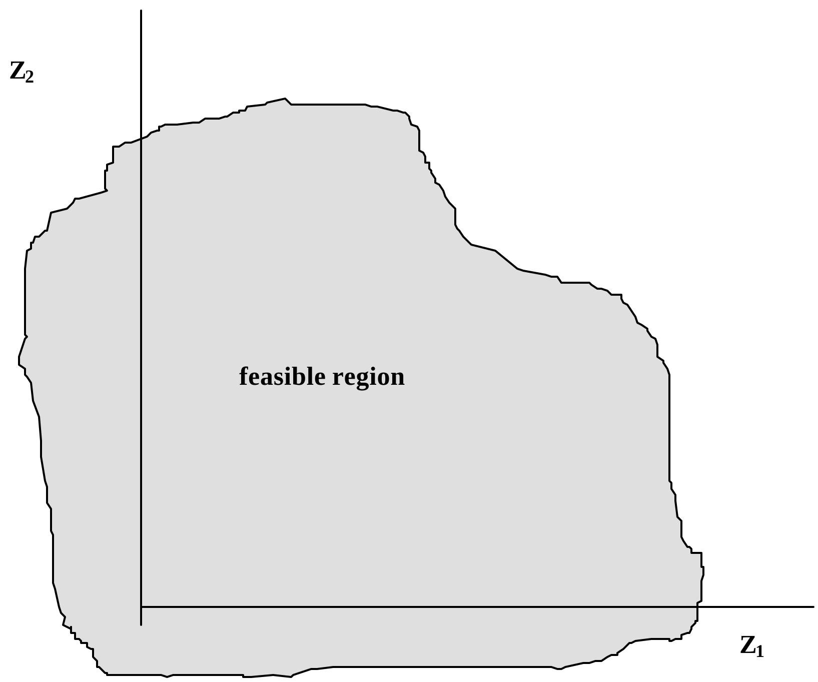

Figure 6.1 Feasible region of multiobjective problem

The nondominated solutions are the conceptual equivalents, in multiobjective problems, of a single optimal solution in a single-objective problem. The main characteristic of the nondominated set of solutions is that for *each solution outside the set, there is a nondominated solution for which all objective functions are unchanged or improved and at least one which is strictly improved*.

The preferred design for any problem should be one of the nondominated solutions. So long as all objectives worth taking into account have been considered, no design that is not among the nondominated solutions is worthwhile: it is dominated by some designs that are preferable on all accounts. This is the reason multiobjective analysis focuses on the determination of the nondominated solutions.

It is often useful to group the noninferior solutions into major categories. The purpose of this exercise is to facilitate discussions about which solution to select. Indeed, to the extent that it is not possible to specify acceptable relative values for the objectives, and thus impossible to define the best design analytically, it is necessary for the choice of the design to rest on judgement. As individuals find it difficult to consider a large number of possibilities, it is helpful to focus attention on major categories.

If we introduce the levels of acceptability for each of the objectives the nondominated solutions are best divided into two types of categories: the major alternatives and the compromises. A *major alternative* group of dominated solutions represents the best performance on some major objective. As Figure 6.2 indicates, the major alternatives represent polar extremes. A *compromise group* lies somewhere in between the major alternatives.

The remainder of the feasible region of solutions is likewise usefully categorised into dominated and excluded solutions. *Dominated* solutions are those that are inferior in all essential aspects to the other solutions. They can thus be set aside from further consideration. *Excluded* solutions are those that perform so badly on one or more dimensions that they lie beneath the threshold of acceptability. Thus, they may be dropped from further consideration.

The concepts of nondominated solutions and of major categories are often highly useful in a practical sense. They organise the feasible designs into a small number of manageable ideas and draw attention to the choices that must be made. These ideas can be applied even when the feasible region is not defined analytically.

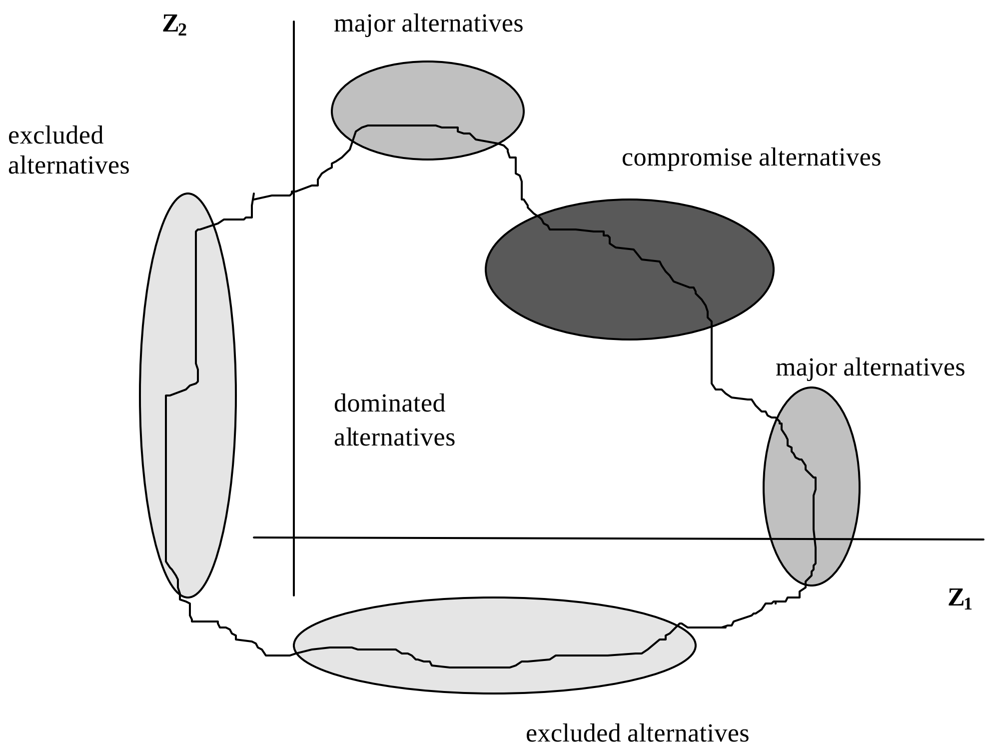

Figure 6.2 Classification of feasible multiobjective alternative solutions

Given a set of feasible solutions $\mathbf{X}$, the set of nondominated solutions is denoted as $\mathbf{S}$ and defined as follows:

$$\mathbf{S} = \{\mathbf{x}: \mathbf{x} \in \mathbf{X}, \text{there exists no other } \mathbf{x}' \in \mathbf{X}$$
$$\text{such that } z_q(\mathbf{x}') > z_q(\mathbf{x}) \text{ for some } q \in \{1, 2, ..., p\}$$
$$\text{and } z_k(\mathbf{x}') \leq z_k(\mathbf{x}) \text{ for all } k \neq q\} \tag{6.3}$$

It is obvious from the definition of $\mathbf{S}$ that as one moves from one nondominated solution to another nondominated solution and one objective function improves, then one or more of the other objective functions must decrease in value.

Classification of multiobjective techniques

Multiobjective techniques are classified into three groups: (a) methods for generating the nondominated set; (b) methods with prior articulation of preferences; and (c) methods with progressive articulation of preferences.

(a) Methods for generating the nondominated set. A generating method does just one thing - it considers a vector of objective functions and uses this vector to identify and generate the subset of nondominated solutions in the initial feasible region. In doing so, these methods deal strictly with the physical realities of the problem (i.e. the set of constraints) and make no attempt to consider the preferences of a decision maker (DM). The desired outcome, then, is the identification of the set of nondominated solutions to help the DM gain insight into the physical realities of the problem at hand.

There are several methods available to generate the set of nondominated solutions, and four of these methods are widely known. These methods are:

- Weighting method;
- ε-constraint method;
- Phillip's linear multiobjective method; and
- Zeleny's linear multiobjective method.

The first two methods transform the multiobjective problem into a single-objective programming format and then, by parametric variation of the parameters used to effect the transformation, the set of nondominated solutions can be generated. The weighting and constraint methods can be used to obtain nondominated solutions when the objective functions and/or constraints are nonlinear.

The last two methods generate the nondominated set for linear models only. However, these two approaches do not require the transformation of the problem into a single-objective programming format. These methods operate directly on the vector of objectives to obtain the nondominated solutions.

Further insight into the weighting method will be presented in one of the following sections of this document.

(b) Methods with prior articulation of preferences. Methods in this class are further divided into continuous and discrete methods.

<u>*Continuous.*</u> Once the set of nondominated solutions for a multiobjective problem have been identified using any of the methods mentioned in (a), the decision maker (DM) is able to select one of those nondominated solutions as his final choice. Then, this solution is one that meets the physical constraints of the problem and happens to satisfy the value structure of the DM as well. However, more likely is the situation where the DM is unwilling or unable to "settle down" for one of those solutions made available to him and would like to voice his preferences regarding the various objective functions in the search for that choice solution. Various methods are available where the DM is asked to articulate his worth or preference structure, and these preferences are then built into the formulation of the mathematical model for the multiobjective problem.

To assist the DM in articulating his/her preferences, a series of questions may be put to him/her where he/she is asked to consider specific tradeoffs among several objectives and elicit a preference for a particular allocation for each objective. In the process, use is made of basic elements of utility theory and probability.

The net effect of articulating the preferences of the DM prior to solving the multiobjective problem is to reduce the set of nondominated solutions to a much smaller set of solutions, facilitating the task of selecting a final choice. Depending on the method used, this smaller set may contain several solutions, one solution, or none at all. Preferences, then, provide an ordering of solutions stronger than that provided by the concept of nondominated solutions.

The most known methods in the group of continuous methods with prior articulation of preferences are:

- Goal programming;
- Utility function assessment; and

- The surrogate worth tradeoff method.

Discrete. There are many decision situations in which the DM must choose among a finite number of alternatives which are evaluated on a common set of noncommensurable multiple objectives or criteria. Problems of this sort occur in many practical situations, for example, which one of five candidates should be hired, which of ten suppliers of an important component should be selected, which of eight systems should be chosen and implemented, etc.

In problems of the above type, the solution process can be described as follows. First, a statement of the general goals relating to the situation is made. Second, the alternatives must be identified or developed. Third, the common set of relevant criteria for evaluation purposes must be specified. Fourth, the levels of the criteria for each alternative must be determined. Finally, a choice based on a formal or informal evaluation procedure is made.

The structure of the discrete problem can be represented in a payoff matrix as shown in Table 6.1. The rating of the ith objective on the jth alternative ($i = 1, 2, ..., m$ and $j = 1, 2, ..., n$) is represented by r_{ij}.

Clearly the problem of choice in a problem such as that depicted by Table 6.1 is sufficiently complex as to require some type of formal assistance. Determining the worth of the alternatives that vary on many dimensions presents formidable cognitive difficulties. People faced with such complex decisions react by reducing the task complexity by using various heuristics. Unfortunately, it has been observed that decision makers who rely on heuristic decision rules systematically violate the expected utility principle. Moreover, decision makers tend to ignore many relevant variables in order to simplify their problem to a scale consistent with the limitations of the human intellect. While such simplification facilitates the actual decision making, it can clearly result in suboptimal behaviour.

In effect, as the decision-making task increases, researchers have observed systematic discrepancies between rational theory and actual behaviour. Evidence exists that even experts have great difficulty in intuitively combining information in appropriate ways. In fact, these studies and many others indicate that global judgements (i.e., combinations of attributes) are not nearly as accurate as analytical combinations. Due to severe limitations of the intuitive decision-making process, it is evident that analytical methods are needed to help determine the worth of multiattributed alternatives.

Table 6.1 Payoff matrix

Alternatives		1	2	...	n
Criteria	1	r_{11}	r_{12}	...	r_{1n}
	2	r_{21}	r_{22}	...	r_{2n}
	...				
	...				
	m	r_{m1}	r_{m2}	...	r_{mn}

Ideally, the alternative which maximises the utility of the DM should be chosen. Therefore, the obvious first step in the application of any discrete multiattributed method is the elimination of all dominated alternatives. Occasionally, for discrete problems, this dominance analysis will yield only one nondominated alternative, in which case the problem is solved; no further analysis is needed.

The methods available in this group range from the very simple to the very complex. Some of the methods are:

- Exclusionary screening;
- Conjunctive ranking;
- Weighted average;
- ELECTRE I and II;
- Indifference tradeoff method;

- Direct-rating method; etc.

(c) Methods of progressive articulation of preferences. The characteristic of the methods in this group is the following general algorithmic approach. First, a nondominated solution is identified. Second, the decision maker (DM) is solicited for tradeoff information concerning this solution, and the problem is modified accordingly. These two steps are repeated until the DM indicates acceptability of a current achievement level, provided one exists.

The methods typically require greater DM involvement in the solution process. This has the advantage of allowing the DM to gain a greater understanding and feel for the structure of the problem. On the other hand, the required interaction has the disadvantage of being time-consuming. The DM may not feel that the investment of the time required provides any better decision making than ad hoc approaches. That is, the DM may perceive the costs to be greater than any benefits. In fact, Cohon and Marks (1975) point out that decision makers have less confidence in the interactive algorithms and find them more difficult to use and understand than the trial and error methods. Certainly, one can not ignore these behavioural difficulties. What they seem to indicate is that more research is needed on how analysts can successfully interact with decision makers to implement improved but complex decision aids.

At any rate, knowledge of some of the advantages and disadvantages should help the management scientists or system analysts in choosing the appropriate decision aid. Certainly, the use of any of the methods as aids to decision making will depend on the analyst's assessment of the personality and tastes of the DM. The methods of progressive articulation of preferences vary in the degree of sophistication and the degree of required interaction. Some of them are:

- Step method (stem);
- Method of Geoffrion;
- Compromise programming;
- SEMOPS method; and others.

6.2 Multiobjective analysis – application examples

River Basin Planning

Alternative strategies for the Santa Cruz River Basin, as proposed by the Corps of Engineers, are examined by Gershon et al. (1980) with the methods ELECTRE I and II. Combinations of flood control actions (e.g. levee construction, channelization, dams, reservoirs, and flood plain management) and water supply actions (e.g. wastewater reclamation, new groundwater development, the Central Arizona project, and conservation measures) are combined to represent 25 alternative systems.

An adaptation of the constraint method is demonstrated by Cohon et al. (1979) in the development of the Lehigh River Basin. Maximisation of economic efficiency and the minimisation of reservoir capacity (as a surrogate for environmental quality) are the two objectives considered.

Conjunctive Water Uses

Simultaneous utilisation of surface and groundwater sources is often a desirable management alternative, particularly in urban water supply. In a case study of Western Skane, Sweden, Hashimoto (1980) considers local groundwater and two pipeline systems to supply five municipalities. A two-level hierarchy is structured to aid in the decision process and the STEM method is used to obtain tradeoffs among five objectives pertaining to lake water levels, downstream releases, and operating costs.

Multiattribute utility functions are used by Keeney and Wood (1977) to evaluate and rank five water systems along the Tisza River Basin in Hungary.

Northeastern United States Water Supply (NEWS) Study (Corps of Engineers, July 1977) is one of the better known studies conducted in the U.S.A. The region extends about 1,000 miles from northern Maine to southern Virginia, averaging 200 miles in width from the

Atlantic Coast. It is significant because of the scale of the problem, the application of state-of-the-art planning techniques, and the intense institutional analysis performed.

Reservoir Operation

In developing a plan of operation of a reservoir, primary consideration may be given to reducing the damaging peak flood stages at principal downstream flood centres. Other objectives may involve recreation and water quality enhancement.

Tauxe et al. (1979) have applied multiobjective dynamic programming (MODP) to the operation of the Shasta Reservoir in California. The three objectives considered are (1) maximisation of cumulative dump energy generated above the level of firm energy; (2) minimisation of the cumulative evaporation or loss of the resource; and (3) maximisation of the firm energy.

Cooperative n-person game theory is used by Fronza et al. (1977) to establish yearly contract volumes in the operation of the Lake Maggiore reservoir in northern Italy. A situation is described in which no "water sellers" or supervisors exist and the solution is shown to result from a trade-off between two users.

Simonovic and Burn (1989) used multiobjective compromise programming for short-term operation of a single multipurpose reservoir. An improved methodology for short-term reservoir operation is derived which considers the operating horizon as a decision variable which can change in real time. The optimal value of the operating horizon is selected based on the trade-off between a more reliable inflow forecast for shorter horizons and better reservoir operations associated with the use of longer operating horizons.

The methodology, based on combined use of simulation and multiobjective analysis, has been developed by Simonovic (1991). It has been used to modify the existing operating rules of the Shellmouth Reservoir in Manitoba, Canada. Flood control and the water supply objectives are confronted with the use of reservoir water for dilution of the heated effluent from a thermal generating plant and the improvement of water quality in the river.

Flood Plain Management

Both structural and nonstructural measures can reduce flood damages. Structural means include levees and dams to physically prevent floods from reaching an area. Nonstructural means include (1) restrictions on the use of flood-prone areas and reduction of the runoff produced by storms; and (2) the purchase of flood plain land that is already developed and its conversion to flood-compatible uses to protect life and property.

Novoa and Halff (1977) evaluate eight alternative flooding remedies ranging from mono-action to stream channelization to complete development of portions of the City of Dallas, Texas. The method of weighted averages is used to evaluate and rank the eight alternative plans; evaluation criteria reflect: (1) relative flood protection; (2) relative neighbourhood improvement; (3) number of relocated families; (4) project cost; (5) maintenance cost; and (6) legal considerations.

Water Quality Management

The public's increasing concern with water pollution demands that the attention be directed to the analysis of water quality determinants, both physical and chemical. These determinants can include dissolved oxygen, coliform count, temperature, and concentrations of various pollutants.

An excellent compilation of papers in water supply and water quality has been published by the Institute for Water Resources, U.S. Army Corps of Engineers (IWR report 81-R04, April 1981). It discusses water quality in the context of a variety of activities such as interbasin water transfer, urban water supply, coal strip-mining, municipal reuse of wastewater, agricultural and industrial uses. A second excellent discussion of water quality in the context of regional planning is given by Goodman (1982). This particular book is strongly recommended as it presents a comprehensive view of water resources planning with a multitude of case studies.

A regional planning framework that integrates land and water quality management models is that developed by Das and Haimes (1979). The Surrogate Worth Tradeoff (SWT) method is applied here to investigate tradeoffs among four planning objectives: (1) sheet erosion control; (2) phosphorous loading; (3) biological oxygen demand; and (4) nonpoint source pollution cost control.

In a hypothetical example based on the institutional arrangements of the Yodo River basin of Japan, Nakayama et al. (1980) discuss a water quality control problem with three objectives: (1) BOD concentration at the inflow point into Osake Bay; (2) treatment cost in the upper reach (Katsura and Uji River basins); and (3) treatment cost in the lower reach (Yodo River basin).

Brill et al. (1976) have analysed an effluent charge program for the Delaware River and proposed metrics for a distribution objective that allows the explicit consideration of efficiency-equity tradeoffs.

Water and Related Land Resources

Development of water resources frequently affects or relates to the development of land resources and vice versa. Activities such as coal strip-mining, land reclamation, watershed management, and land use planning relate closely to the hydrology of the area and the availability of water resources (e.g. surface and groundwater).

Goicoechea et al. (1982) have applied mathematical programming to the problem of reclaiming lands disturbed by coal strip-mining activities in the Black Mesa region of northern Arizona. This study suggests a reclamation program to enable the land to sustain agricultural, livestock grazing, fish pond-harvesting, and recreational uses. The PROTRADE method is used in the process to examine levels of attainment and the trade-off for five objectives: (1) livestock production; (2) water runoff augmentation; (3) farming of selected crops; (4) sediment control; and (5) fish yield. This method, as noted earlier, allows the use of random variables to represent parameters in the objective functions; to

generate objective tradeoffs the decision maker is able to consider both the level of attainment and the probability of attainment for each objective function.
The same authors have investigated the implementation of mechanical and chemical vegetation treatments to promote water runoff in the San Pedro River basin, Tucson, Arizona. The TRADE method was used to examine tradeoffs among five objectives: (1) runoff augmentation; (2) sediment control; (3) animal wildlife balance; (4) recreation; and (5) commercial benefits.

Gershon et al. (1981) have investigated the impact of alternative river basin development plans using the ELECTRE I technique for screening and the ELECTRE II technique for the final ranking. They illustrate the approach by means of the case study of the Tucson basin, Arizona. A systems analysis of the problem leads to defining an array of 25 alternative systems versus 13 criteria, only 5 of which are quantified. In the same study sensitivity of the results to changes in weights and scales is investigated.

6.3 The weighting method

The weighting method belongs to the group of techniques for generating a nondominated set. It is based on the idea of assigning weights to the various objective functions, combining these into a single-objective function, and parametrically varying the weights to generate the nondominated set.

Mathematically, the weighting method can be stated as follows:

$$\max z(\mathbf{x}) = w_1 z_1(\mathbf{x}) + w_2 z_2(\mathbf{x}) + \ldots + w_p z_p(\mathbf{x}) \tag{6.4}$$

subject to:

$$\mathbf{x} \in \mathbf{X}$$

which can be thought of as an operational form of the formulation

$$\text{max-dominate } z(\mathbf{x}) = [z_1(\mathbf{x}), z_2(x) \ldots, z_p(\mathbf{x})]$$

subject to:

$$\mathbf{x} \in \mathbf{X}$$

In other words, a multiobjective problem has been transformed into a singleobjective optimization problem for which the solution methods exist. The coefficient w_i operating on the ith objective function, $z_i(x)$, is called a weight and can be interpreted as "the relative weight or worth" of that objective when compared to the other objectives. If the weights of the various objectives are interpreted as representing the relative preferences of some DM, then the solution to (6.4) is equivalent to the best-compromise solution, that is, the optimal solution relative to a particular preference structure. Moreover, the optimal solution to (6.4) is a nondominated solution, provided all the weights are positive. The reasoning behind the nonnegativity requirement is as follows. Allowing negative weights would be equivalent to transforming the maximisation problem into a minimisation one, for which a different set of nondominated solutions will exist. The trivial case where all the weights are zero will simply identify every $\mathbf{x} \in \mathbf{X}$ as an optimal solution, and will not distinguish between the dominated and the nondominated solutions.

Conceptually, the generation of the nondominated set using the method of weights appears simple. However, in practice the generation procedure is quite demanding. Several weight sets can generate the same nondominated point. Furthermore, moving from one set of weights to another set of weights may result in skipping a nondominated extreme point. Subsequent linear combinations of the observed adjacent extreme points would, in many cases, yield a set of points that are only "close" to the nondominated border. In other words, in practice, it is quite possible to miss using weights that would lead to an extreme point. Consequently, the most that should be expected from the weighting method is an approximation of the nondominated set.

The sufficiency of the approximation obviously relates to the proportion of the total number of extreme points that are identified. For example, assume each weight is varied systematically between zero and some upper limit using a predetermined step size. It seems reasonable to believe that the choice of a large increment will result in more skipped extreme points than the choice of a small increment. However, the smaller the increment the greater the computational requirements. There exists a tradeoff between the accuracy of the specification of the nondominated set and the costs of the computational requirements. Judgement by the DM and the analyst must be exercised to determine the desired balance.

Class exercise:

A State water agency is responsible for the operation of a multipurpose reservoir used for (a) municipal water supply; (b) groundwater recharge; and (c) the control of water quality in the river downstream from the dam. Allocating the water to the first two purposes is, unfortunately, in conflict with the third purpose. The agency would like to minimise the negative effect on the water quality in the river and, at the same time, maximise the benefits from the municipal water supply and groundwater recharge.

Thus, there are two objectives: minimise the increase in river pollution and maximise profits. Tradeoffs between these two objectives are sought to assist the water agency in the decision-making process. The available data is listed in Table 6.2

The following assumptions are made:

- *(i)* *One time period is involved; t = 0, 1.*
- *(ii)* *Allocation is limited to two restrictions: (a) pump capacity is 8 hr per period; and (b) labour capacity is 4 man-hours per period.*

Table 6.2 Available data for an illustrative example

	Water Supply	*Groundwater Recharge*
Number of units of water delivered	x_1	x_2
Number of units of water required	*1.0*	*5.0*
Pump time required (hr)	*0.5*	*0.25*
Labour time required (man-hour)	*0.2*	*0.2*
Direct water costs ($)	*0.25*	*0.75*
Direct labour costs ($)	*2.75*	*1.25*
Sales price of water per unit ($)	*4.0*	*5.0*

(iii) Total amount of water in the reservoir available for allocation is 72 units.

(iv) The pollution in the river increases by three units per unit of water used for water supply and two units per unit of water used for groundwater recharge.

Based on the preceding information the objective functions and constraints of the problem can be formulated. The contribution margin (selling price/unit less variable cost/unit) of each allocation can be calculated:

Municipal water supply

$4.00 -	*$0.25 -*	*$2.75*	*= $1.00 per unit of water delivered*
Sales price	*Direct water cost*	*Direct labour*	

Groundwater recharge

\$5.00 -	\$0.75 -	\$1.25	= \$3.00 per unit of water delivered
Sales price	Direct water cost	Direct labour	

and the objective function for profit becomes:

$$z_1(\mathbf{x}) = x_1 + 3x_2$$

The objective function for pollution is:

$$z'_2(\mathbf{x}) = 3x_1 + 2x_2$$

This function can be modified to $z_2 = -3x_1 - 2x_2$ *so that the maximisation criterion is appropriate for both of the objective functions.*

Finally, the technical constraints due to pump capacity, labour capacity, and water availability are:

$0.5x_1 + 0.25x_2$ £ 8 (pump capacity)

$0.2x_1 + 0.2x_2$ £ 4 (labour capacity)

$x_1 + 5x_2$ £ 72 (water)

Now, using the operational form of the weighting method, the problem to solve is:

$$\max z(x) = w_1 z_1(\mathbf{x}) + w_2 z_2(\mathbf{x})$$

$$= w_1(x_1 + 3x_2) + w_2(-3x_1 - 2x_2) \qquad (6.5)$$

subject to:

$$g_1(\mathbf{x}) = 0.5x_1 + 0.25x_2 - 8 \text{ £ } 0 \qquad (6.6)$$

$$g_2(\mathbf{x}) = 0.2x_1 + 0.2x_2 - 4 \text{ £ } 0 \tag{6.7}$$

$$g_3(\mathbf{x}) = x_1 + 5x_2 - 72 \text{ £ } 0 \tag{6.8}$$

$$g_4(\mathbf{x}) = -x_1 \text{ £ } 0 \tag{6.9}$$

$$g_5(\mathbf{x}) = -x_2 \text{ £ } 0 \tag{6.10}$$

It can be decided arbitrarily to fix $w_1 = 1$ *and increase* w_2 *at increments of one until all the nondominated extreme points have been identified. For this example, the pairs of values selected for* (w_1, w_2) *are (1, 0), (1, 1), (1, 2), (1, 3), (0, 1), as shown in Table 6.3 For the pair of weights (1, 0), the objective function to maximise is:*

$$\begin{aligned} z(\mathbf{x}) &= 1(x_1 + 3x_2) + 0(-3x_1 - 2x_2) \\ &= x_1 + 3x_2 \end{aligned}$$

subject to the stated constraints. The solution can be obtained graphically by "moving" the line $z(\mathbf{x}) = x_1 + 3x_2$ *out towards the boundary of the feasible region until it just touches the extreme point* $\mathbf{x}^* = (7, 13)$, *yielding* $z(\mathbf{x}^*) = 46$, $z_1(\mathbf{x}^*) = 46$, *and* $z_2(\mathbf{x}^*) = -47$ *(Figure 6.3). This solution, however, is not unique to the pair of weights (1, 0) as can be observed from Table 6.3.*

After a graphical presentation of the solutions in the objective space (Figure 6.4), it is possible to identify nondominated points by visual inspection. Since all of the nondominated extreme points are obviously identified, an exact representation of the nondominated set is achieved.

However, for problems with a larger number of variables and constraints, one would probably have to settle for an approximate representation of the nondominated set.

Once the nondominated set is specified, the DM can use the information to help him/her select a preferred solution. The tradeoffs are now readily apparent. For example, a decision to move from the production vector of (7, 13) to the production vector of $(0, \frac{72}{5}\ 24)$

results in a decrease of 2.8 units of profit with the resulting benefit of a reduction of 18.2 units

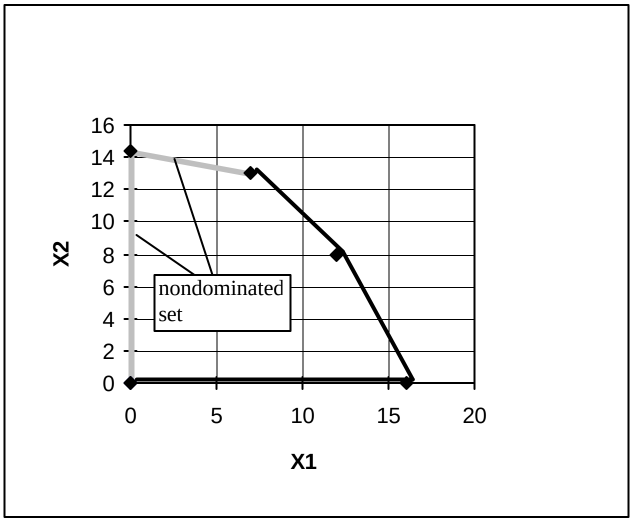

Figure 6.3 Feasible region and the nondominated set in decision space

Table 6.3 Pairs of weights and associated nondominated solutions

Weights (w_1, w_2)	*Nondominated Extreme Point* $\mathbf{x}^* = (x_1, x_2)$	$z_1(\mathbf{x}^*)$	$z_2(\mathbf{x}^*)$	$z(\mathbf{x})$
(1, 0)	*(7, 13)*	*46*	*- 47*	*46*
(1, 1)	$(0, \frac{72}{5}$ *25)*	$\frac{216}{5}$ *26*	$-\frac{144}{5}$ *27*	$\frac{72}{5}$ *28*
(1, 2)	*(0, 0)*	*0.0*	*0.0*	*0.0*
(1, 3)	*(0, 0)*	*0.0*	*0.0*	*0.0*
(0, 1)	*(0, 0)*	*0.0*	*0.0*	*0.0*

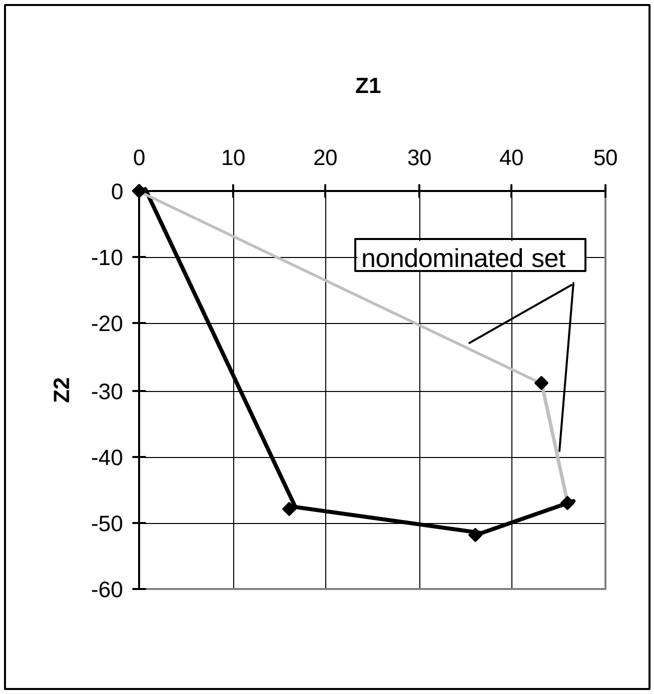

Figure 6.4 Feasible region and the nondominated set in objective space

of pollution. This may be perceived as too great a sacrifice and feasible production vectors in-between these two extremes with the corresponding tradeoffs could be examined.

6.4 Problems

6.1 A new Civil Engineering graduate has job offers from five different companies and is faced with the problem of selecting a company within a week. The individual lists in the form of a payoff matrix all of the important information:

CRITERIA	COMPANY 1	2	3	4	5
9month salary ($)	*34,000.*	*33,000.*	*36,000.*	*37,500.*	*32,000.*
Average work load (hr/season)	*9*	*7.5*	*10.5*	*12*	*6*
*Geographic location**	*2*	*3*	*5*	*2*	*4*
Expected summer salary	*4,800.*	*4,600.*	*2,600.*	*5,500.*	*6,000.*
Continuing Education	*4*	*4*	*3*	*1*	*5*
Additional benefits	*3*	*2*	*4*	*5*	*1*

* *Larger numbers represent more desirable outcomes.*

Acting as the new graduate apply the weighting method and explain your decision.

6.2 A rolling mill generates two distinct types of liquid wastes. One is pickling waste and the other, process water. These wastes can either be discharged directly into a river, subject to an effluent tax, or else may be treated in the existing plant which removes 90% of the pollutants. The pickling wastes require pretreatment before being passed through the treatment plant.

It is required to determine the level of production that maximizes expected profits which in this case is income minus waste disposal costs. The wastes costs are due to both waste treatment costs at the factory and effluent tax charges. The steel mill already has pretreatment and treatment facilities with known operating costs.

The nearby municipality controls the effluent tax on untreated wastes entering the city sewers. The higher the effluent tax the more wastes the plant will treat itself. However, this could cause a decline in plant production that would directly affect the economy of the community. On the other hand, the lower the effluent tax, the less wastes the company will *treat and production will be higher. This means the community will be subsidizing treating the plant's waste at the city sewage plant.*

Examine the problem using linear programming and solvi it using Excel Solver in order that profits for the steel plant may be optimized according to the city's effluent charges. Apply the Weighting Method for w=1,2 to assist the steel company in making profit projections that reflect the city's current effluent charges and a contemplated doubling of those charges.

The following data is available:

income on steel manufactured = $25/ton

pickle wastes generated = .1 thousand gallons/ton of steel

process water generated = 1 thousand gallons/ton of steel

cost of pretreating pickle waste = $.2/thousand gallons

cost of treating process water and pretreated pickle waste = $.1/thousand gallons

pretreatment plant capacity = 2,000,000 gallons/day

treatment plant capacity = 50,000,000 gallons/day

treatment plant efficiency = .9 (both wastes)

effluent tax on untreated pickle waste = $.15/thousand gallons

effluent tax on untreated process water = $.05/thousand gallons.

The operation is shown in Figure 6.5. Also the city has imposed restrictions on how much untreated waste can be put into the sewer. No more than 500,000 gal/day of pickle water or 10,000,000 gal/day of process water may be discharged without treatment.

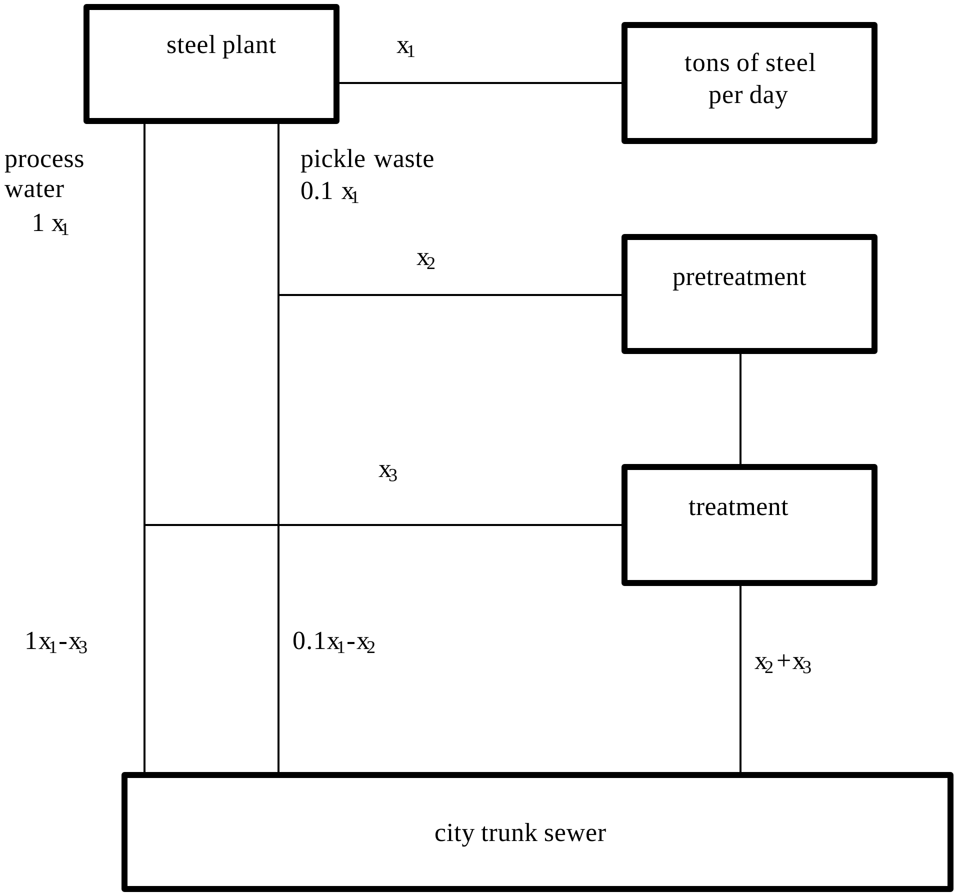

Figure 6.5 Layout of the waste facilities (after Hipel et al, 1976)

6.3 NIS consultatnts has evaluated proposed sites for a multipurpose water facility along two objectives: economic benefits (EB) and environmental quality improvement (EQUI). The optimal performance of each site is given by the following (EB, EQUI) pairs:
A(20,135) B(75,135) C(90,100) D(35,1050) E(82,250) F(60,-50) G(60,550) H(75,500) I(70,620) J(10,500) K(40,350) L(30,800) N(55,250) O(40,-80) P(30,500) Q(30,900) R(60,950) S(80,-150) T(45,550) U(25,1080) V(70,800) W(63,450).
Solve graphically.

(a) List excluded, dominated and nondominated alternative sites.

(b) Identify nondominated solutions by maximizing EB subject to EQUI ³ b where b=200,400,600 and 800.

(c) Identify nondominated solution by the weighting method assigning relative weight to (EB, EQUI) of (20,8).

6.4 Consider the multiobjective problem:

$$\max 5x_1 - 2x_2 \quad \text{and} \quad -x_1 + 4x_2$$

subject to

$$-x_1 + x_2 <= 3$$
$$x_1 + x_2 <= 8$$
$$x_1 <= 6$$
$$x_2 <= 4$$
$$x_i >= 0$$

(i) Graph the nondominated solutions in decision space.

(ii) Graph the nondominated combinations of the objectives in objective space.

(iii) Reformulate the problem to illustrate the weighting method for defining all nondominated solutions of part (i) and illustrate this method in decision and objective space.

6.5 Evaluate a plan for disposing of wastewater from a treatment plant. For the given situation you want to use the best combination of two methods: a filetr for tertiary treatment and irrigation with the effluent. The net benefits are: irrigation \$5 per m^3/s; tertiary treatment -\$2 per m^3/s. Another objective is to maximize the disolved oxygen (DO) in the river bordering the treatment plant and irrigated area. The DO will decrease by 1 mg/l for each m^3/s of effluent irrigated and increase by 4 mg/l for eaxh m^3/s given tertiary treatment.

The amount of effluent filtered must be less than 3 m^3/s plus the amount of effluent used for irrigation (measured in m^3/s). The total system (tertiary with irrigation) can not exceed 8 m^3/s; moreover, physical constraints on the capacity of the available land limit the amount *for irrigation to 6 m^3/s and the amount of effluent for tertiary treatment to 4 m^3/s. Consider both the economic and the water quality objectives, conduct a multiobjective analysis using the weighting method. What is your recommendation? Why?*

6.6 Evaluate the three alternatives presented below using the multiobjective analyses of your choice. In your evaluation use three objectives: economic (NED); environmental enhancement (EQ); and regional development of a depressed region (RD). Three alternatives may be used in various combinations: flood control; hydropower and water quality control (mostly low-flow augmentation measure in m^3/sec).

The flood control alternative, a levee that can not exceed 2m in height, will yield \$1000/m in annual benefits for both the NED and RD objectives and destroy 20 environmental units per meter of levee height.

The hydropower alternative cannot exceed 2MW of power and will yield \$1000/MW for the NED account but only \$500/MW for the RD. It will add 10 environmental units per MW.

The water qulaity alternative will yield \$500 per m^3/sec of flow NED. It cannot exceed 2 m^3/sec and will yield \$1000 per m^3/sec for RD. It will yield 10 environmental units per m^3/sec.

The sum of each meter of levee height plus each MW of power plus 1.25 m^3/sec of flow augmentation must not exceed 5.0 and the sum of levee heignt plus 1 cannot be less than 1 m^3/sec of flow augmentation.

Your goal is to maximize NED, RD and EQ.

6.7 A reservoir is planned for irrigation and low-flow augmentation for water quality control. A storage volume of $6*10^6$ m^3 will be available for these two conflicting uses each year. The maximum irrigation demand (capacity) is $4*10^6$ m^3. Let x_1 be the allocation of *water to irrigation and x_2 the allocation for downstream flow augmentation. Assume that there are two objectives, expressed as:*

$$z_1 = 4x_1 - x_2 \quad \text{and} \quad z_2 = -2x_1 + 6x_2$$

(i) Write the multiobjective planning model using the weighting approach.

(ii) Define the nondominated solutions. This requires a plot of feasible combinations of x_1 and x_2.

(iii) Assuming that various values are assigned to the weight w_1 where w_2 is constant and equal to 1, verify the following solutions to the weighting model.

w_1	x_1	x_2	z_1	z_2
>6	4	0	16	-8
6	4	0 to 2	16 to 14	-8 to 4
<6 to >1.6	4	2	14	4
1.6	4 to 0	2 to 6	14 to -6	4 to 36
<1.6	0	6	-6	36

6.8 Suppose that two polluters exist, A and B, who can provide additional treatment, x_a and x_b, at a cost $c_a(x_a)$ and $c_b(x_b)$, respectively. Lwt w_a and w_b be the waste produced at sites A and B, and $w_a(1-x_a)$ and $w_b(1-x_b)$ be the resulting waste discharges at sites A and B. These discharges must be no greater than the effluent standards E_a^{max} and E_b^{max}. The resulting pollution concentration $a_{aj}(w_a(1-x_a))+a_{bj}(w_b(1-x_b))+q_j$ at varous sites j must not exceed the stream standards S_j^{max}. Assume that the total cost and cost inequity [i.e. $c_a x_a + c_b x_b$ and $| c_a x_a + c_b x_b |$] are management objectives to be minimized.

(i) Discuss how you would model this multiobjective problem using the weighting method.

(ii) Discuss how you would use the model to identify the nondominated solutions.

(iii) Effluent standards at sites A and B and ambient stream standards at sites j could be replaced by other planning objectives (e.g., the minimization of waste discharged into the stream). What would these objectives be, and how could they be included in the multiobjective model?

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