PERMUTATIONS & COMBINATIONS

Part I

COUNTING TECHNIQUES

PERMUTATIONS

- Permutation is the number of ways to arrange things.

  Eg : My safe code is 492.

  (order matters)

- $P(n,r) = {}^nP_r = \frac{n!}{r!}$, Where $0 \leq r \leq n$

  n → the number of things to choose from

  r → the number of things we choose

  ! → factorial.

COMBINATIONS

- Combination is the number of ways to choose things.

  Eg : My Salad is a Combination of carrot, Onion, Tomato and Lemon.

  (order doesn't matter)

- $C(n,r) = {}^nC_r = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$; Where $0 \leq r \leq n$

  n → the number of things to choose from

  r → the number of things we choose

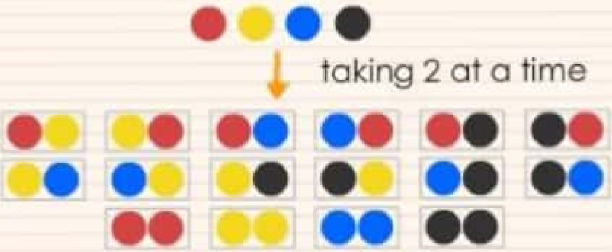
TYPES OF PERMUTATIONS & COMBINATIONS

When Repetition is Allowed.

1. Permutations with Repetition

Formula: n^r

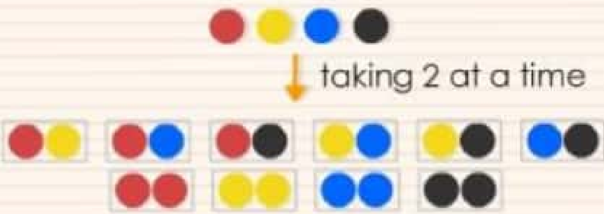
(Repetition allowed, order matters)



2. Combinations with Repetition

Formula: ${}^{(n+r-1)}C_r$

(Repetition allowed, order does not matter)

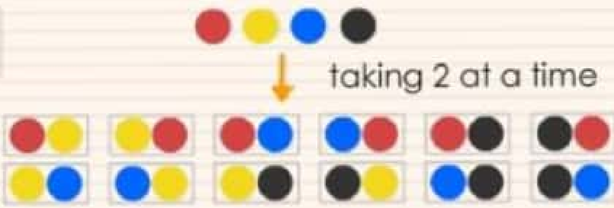


When Repetition is not Allowed.

1. Permutations without Repetition

Formula: ${}^nP_r = \frac{n!}{(n-r)!}$

(No repetition, order matters)



2. Combinations with Repetition

Formula: ${}^nC_r = \frac{n!}{r!(n-r)!}$

(No repetition, order does not matter)

PERMUTATION

Part II

If 'n' is the number of distinct things and 'r' things are chosen at a time.

1. Permutations of objects when all objects are not distinct.

Permutations = $\dfrac{n!}{P_1!\, P_2! \,.....\, P_r!}$ $P_r \rightarrow$ Number of things among 'n' are excatiy alike of r^{th} type.

2. Permutations with Repetition

Number of Permutations = n^r

3. Circular Permutations

Case 1 : When clockwise and anticlock wise arrangements are different.

Number of Permutations : $(n-1)!$

Case 2 : When clockwise and anticlock wise arrangements are not different.

Number of Permutations : $\dfrac{1}{2}(n-1)!$

4. Permutation under Restrictions

Case 1 : When 's' particular things are always to be included.

Number of Permutations : $\dfrac{(n-s)!\, r!}{(n-r)!\, (r-s)!}$

Case 2 : When a particular thing is always to be included (s = 1).

Number of Permutations : $\dfrac{(n-1)!\, r!}{(n-r)!\, (r-1)!}$

Case 3 : When 's' particular things are never be included.

Number of Permutations : $\dfrac{(n-s)!}{(n-s-r)!}$

Case 4 : When a particular thing is never included (s = 1).

Number of Permutations : $\dfrac{(n-1)!}{(n-r-1)!}$

Case 5 : When 'm' particular things always come together.

Number of Permutations : $(n-m+1)! \times m!$

Case 6 : When 'm' particular things never come together.

Number of Permutations : $n! - (n-m+1)! \times m!$

COMBINATION

Part III

If 'n' is the number of distinct things and 'r' things are chosen at a time.

1. Combinations with Repetition

Number of Combinations : ${}^{(n+r-1)}C_r$

2. Total Number of Combinations

Case 1 : Ways of selecting one or more than one things.

Number of Combinations : ${}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n - 1$

Case 2 : When 's_1' alike objects of one kind, 's_2' alike objects of 2^{nd} kind and so on ...... 's_n' alike objects of n^{th} kind.

Number of Combinations : $(s_1 + 1)(s_2 + 1) \ldots\ldots (s_n + 1) - 1$

Case 3 : When 's_1' alike objects of one kind, 's_2' alike objects of 2^{nd} kind and so on .... 's_n' alike objects of n^{th} kind and rest 'p' different objects.

Number of Combinations : $[(s_1 + 1)(s_2 + 1) \ldots\ldots (s_n + 1)]\, 2^p - 1$

3. Combinations Under Restrictions

Case 1 : When 's' particular things are always to be included.

Number of Combinations : ${}^{(n-s)}C_{(r-s)}$

Case 2 : When a particular thing is always to be included.

Number of Combinations : ${}^{(n-1)}C_{(r-1)}$

Case 3 : When 's' particular things are never included (s = 1).

Number of Combinations : ${}^{(n-s)}C_r$

Case 4 : When 'm' particular things never come together.

Number of Combinations : ${}^nC_r - {}^{(n-m)}C_{(r-m)}$

Part I

BINOMIAL THEOREM

BINOMIAL

A binomial is a polynomial with two terms. e.g

$\underbrace{5y^3}\ - \underbrace{3}$

2 terms

THE BINOMIAL THEOREM

Helps us expand binomials to any given power without direct multiplication.

General formula for $(x+y)^n$

$$(x+y)^n = {}^nC_0x^ny^0 + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}\,y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_nx^0y^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

PROPERTIES OF THE BINOMIAL EXPANSION $(x+y)^n$

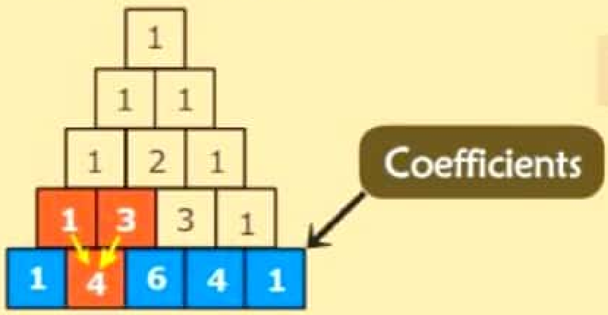
- There are $n+1$ terms.
- The first term is x^n and the final term is y^n
- The exponent of x decreases by 1 while the exponent of y increases by 1.

BINOMIAL COEFFICIENTS

Expanding $(x+y)^n$, the binomial coefficients are simply the number of ways of choosing x from a number of brackets and y from the rest and are found using pascal's triangle.

PASCAL'S TRIANGLE

Assuming $n = 4$, We have $(a+b)^4$ and pascal's triangle would look like



$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Formula for the coefficient from pascal's Triangle.

$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!}$$

It is commonly called "n choose k".

1 IMPORTANT TERMS IN THE BINOMIAL EXPANSION

Part II

IN THE EXPANSION OF $(x+y)^n$

- **GENERAL TERM :** $T_{r+1} = {}^nC_r\, x^{n-r}.\, y^r$
- **MIDDLE TERM :** $T_{(n+2)/2} = {}^nC_{n/2}.\, x^{n/2}.\, y^{n/2}$; when n is even

  $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$; when n is odd
- **NUMERICALLY GREATEST TERM :** $T_{r+1} \geq T_r \Rightarrow \frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{n-r+1}{r}\left|\frac{a}{x}\right| \geq 1 \Rightarrow r \leq \frac{(n+1)}{\left(\left|\frac{x}{a}\right|+1\right)}$
- **TERM INDEPENDENT OF x :** Term independent of x contains no x; Hence find the value of r for which the exponent of x is zero.

2 SOME RESULTS ON BINOMIAL COEFFICIENTS

- $C_0 + C_1 + C_2 + \ldots\ldots.. + C_n = 2^n$
- $C_0 + C_2 + C_4 + \ldots. = C_1 + C_3 + C_5 + \ldots. = 2^{n-1}$
- $C_0^2 + C_1^2 + \ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$
- $C_0.\, C_r + C_1.\, C_{r+1} + \ldots\ldots.. + C_{n-r}\, C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

3 SOME IMPORTANT EXPANSIONS

EXPONENTIAL SERIES

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..\infty$; where x may be any real or complex number.
- $a^x = 1 + \frac{x}{1!}\ln a + \frac{x^2}{2!}\ln^2 a + \frac{x^3}{3!}\ln^3 a + \ldots\ldots..\infty$ where $a > 0$.
- $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\infty$

LOGARITHMIC SERIES

- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty$ where $-1 < x \leq 1$
- $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\infty$ where $-1 \leq x < 1$

APPROXIMATIONS

- $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots.$

  if x be very small, then $(1 + x)^n = 1 + nx$, approximately.

FOLLOWING EXPANSION SHOULD BE REMEMBERED ($|x| < 1$)

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots.\infty$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots.\infty$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots.\infty$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.\infty$

PROBABILITY

How likely something is to happen

Part I

Probability of an event happening = $\dfrac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$

If there are 'n' equally likely events out of these 'm' are in the favour of 'A' then the probability of event A is $P(A) = \dfrac{m}{n}$

There are 'n–m' occasions when event 'A' does not happen. The event A does not happen is denoted by $\bar{A}$ and probability of event A does not happen is $P(\bar{A}) = \dfrac{n-m}{n} = 1 - \dfrac{m}{n} = 1 - P(A)$

PROBABILITY LINE

More unlikely $0 \leq P(A) \leq 1$ More likely

Impossible — Even chance — Certain

Rolling a 14
P = 0

Heads **P = 1/2**

The sun will rise
P = 1

TERMINOLOGY

	An action or operation resulting in two or more outcomes.
Sample space	A set S that consists of all possible outcomes of a random experiment.
Event	An event is defined occurrence or situation Eg : tossing a coin and the coin landing up head.
Complement of an event	The set of all out comes which are in S but not in A. The compliment event A is denoted by A , A^c , A' or not A .
Compound event	The combination of events A and B is known as Compound Event. It is denoted by AB or A and B or A ∩ B.
Mutually exclusive events	Mutually Exclusive events mean we can't get both events at the same time. It is either one or the other, but not both.
Equally likely events	When each event is as likely to occur as any other event.
Exhaustive events	A set of events is called exhaustive if all the events together consume the entire sample space.

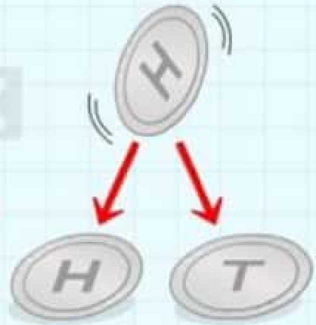
INDEPENDENT EVENTS

Independent Events are **not affected** by previous events.

This is an important concept !

Example : A coin does not know that it landed up heads before........

.... each toss of a coin is a perfect isolated thing.

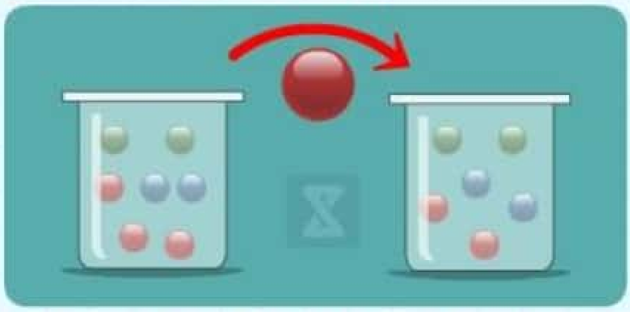


DEPENDENT EVENTS

Events that depend on what happend before

Example :

Taking colored marbles from a bag : as you take each marble, there are less marbles left in the bag, so the probabilities change.



BINOMIAL PROBABILITY DISTRIBUTION

Let probability of success of an event be p & probability of failure be q = 1 – p

The probability that the event will happen exactly 'x' times in 'n' trials is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

BAYE'S THEOREM

If an event A can occur only with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots B_n$ & if the conditional probabilities of the events.

$P(A / B_1), P(A / B_2) \ldots P(A / B_n)$ are known then,

$$P(B_1 / A) = \frac{P(B_1).\,P(A / B_1)}{\sum_{i=1}^{n} P(B_i).\,P(A / B_i)}$$

ADDITION THEOREM

1. If A & B are mutually exclusive events, then the probability of event A or B occuring is

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

2. If A & B are not mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= P(A) + P(B \cap \overline{A}) = P(B) + P(A \cap \overline{B})$$
$$= P(A \cap B) + P(A \cap B) = P(B) + P(B \cap \overline{A})$$
$$= 1 - P(\overline{A} \cap \overline{B}) = 1 - P(\overline{A \cup B})$$

3. If A & B are mutually exclusive then $\boxed{P(A \cup B) = P(A) + P(B)}$

MULTIPLICATION THEOREM

DEPENDENT EVENTS OR CONTIGENT EVENTS

How to handle Dependent Events ?

$P(B \mid A)$ is called the **conditional probability** of event B given that event A has already occured.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \Rightarrow \boxed{P(A \cap B) = P(A)\, P(B \mid A)}$$

Note: For any three events A_1, A_2, A_3

$$\boxed{P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid (A_1 \cap A_2))}$$

INDEPENDENT EVENTS

- For two independent events A & B $\boxed{P(A \cap B) = P(A) \,.\, P(B)}$
- Three events A,B & C are independent if & only if all the following conditions hold:

$$P(A \cap B) = P(A) \,.\, P(B) \,; \quad P(B \cap C) = P(B) \,.\, P(C)$$
$$P(C \cap A) = P(C) \,.\, P(A) \;\&\; P(A \cap B \cap C) = P(A).\, P(B).\, P(C)$$

i.e., they must be pairwise as well as mutually independent.

- For n events A_1, A_2, A_3 .... A_n to be independent, the number of above conditions is equal to ${}^nC_2 + {}^nC_3 +$ .... ${}^nC_n = 2^n - n - 1$

Note: Independent events are not in general mutually exclusive & vice versa.

SQUENCE AND SERIES

Part I

WHAT IS A PROGRESSION ?

A progression is a list of things (usually numbers) that are in order.

Example : 2 (1st term) 4 (2nd term) 8......... (3rd term) ← Dots Denote Infinite Progression

TYPE OF PROGRESSION

- Arithmetic Progression
- Geometric progression
- Harmonic Progression
- Arithmetico Geometric progression
- Miscellaneous Progression

Arithmetic Progression

Definition

A pattern of numbers that increases or decreases by a constant number.
E.g. 4, 7, 10, 13.............

General Progression

General form of an arithmetic progression is given as $a, a+d, a+2d..., a+(n-1)d$

Where: a – First term | d – Common difference

n^{th} term

General term of an arithmetic progression is given as $T_n = a + (n-1)d$

Sum of 'n' terms

If 'n' terms $a, a+d, a+2d..., a+(n-1)d$ are in arithmetic progression Then the sum of 'n' terms: $S_n = \frac{n}{2}[2a + (n-1)d]$

Airthmetic Mean

If $a_1, a_2, ..., a_n$ are in arithmetic progression then the Arithmetic Mean (AM) is: $A_m = \frac{a_1 + a_2 + ... + a_n}{n}$ or $= \frac{S_n}{n}$

If $A_1, A_2 .... A_n$ are 'n' arithmetic means between two numbers 'a' and 'b' then $a, A_1, A_2 ..... A_n, b$ are in AP.

Where common diffrence $d = \frac{b-a}{n+1}$ and arithmetic means are $A_i = a + i\frac{b-a}{n+1}$

Geometric Progression

Definition

The progression, where the ratio of successive terms of a progression is constant
E.g. 4, 8, 16, 32, 64, ... here the common ratio is 2.

General Progression

General form of a geometric progression is given as $a, ar, ar^2, \ldots ar^{n-1}$

Where: a – First term | r – Common ratio

n^{th} term

General term of a geometric progression is given as $T_n = a \cdot r^{(n-1)}$

Sum of 'n' terms

If 'n' terms $a, ar, ar^2, \ldots ar^{n-1}$ are in geometric progression then the sum of 'n' terms:

$$S_n = \frac{a(r^n-1)}{r-1};\ r \neq 1$$

Geometric Mean

If $a_1, a_2, \ldots, a_n$ are in geometric progression then the geometric mean (GM) is:

$$G_m = (a_1 \cdot a_2 \cdot a_3 \ldots\ldots a_n)^{1/n}$$

If $G_1, G_2 \ldots G_n$ are 'n' geometric means between two numbers 'a' and 'b' then $a, G_1, G_2 \ldots G_n, b$ are in G.P.

Where common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ and geometric means are $G_i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}$

Arithmetico Geometric Progression

Definition

The result of the multiplication of a geometric progression with the corresponding terms of an arithmetic progression

General Progression

$a, (a+d)r, (a+2d)r^2, (a+3d)r^3 \ldots$ Where:

a – First term | r – Common ratio of GP | d – Common difference of AP

n^{th} Term

General term of a arithmetico geometric progression is $T_n = [a+(n-1)d]\,r^{(n-1)}$

Sum of 'n' Terms

$a, (a+d)r, (a+2d)r^2, \ldots\ldots$ are in AGP then sum of the terms is:

$$S_n = \frac{a}{1-r} + \frac{rd(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

If $|r|<1$ and 'n' tends to infinity then sum of infinite terms is:

$$\lim_{n\to\infty} S_n = \frac{a}{(1-r)} + \frac{rd}{(1-r)^2}$$

HARMONIC PROGRESSION & MISC.

WHAT IS A PROGRESSION ?

A Progression is a list of things (usually numbers) that are in order.

Example : 2 (1st term), 4 (2nd term), 8......... (3rd term) ⟵ Dots Denote Infinite Progression

TYPE OF PROGRESSION

- Arithmetic Progression
- Geometric progression
- Harmonic Progression
- Arithmetico Geometric progression
- Miscellaneous Progression

HARMONIC PROGRESSION

- It is a sequence in which the reciprocal of the terms are in Arithmetic Progression.
- If $a, a + d, a + 2d, \ldots\ldots$ in Arithmetic Progression.

  then $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ in Harmonic Progression.
- n^{th} term of the Harmonic term is $T_n = \frac{1}{a + (n-1)d}$
- Sum of 'n' terms ⇒ No direct way but can be found with the help of A.P.
- If $a_1, a_2, \ldots\ldots a_n$ in Arithmatic Progression the Harmonic mean H_m is

  $$\frac{n}{H_m} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$
- R.M.S $\geq$ AM $\geq$ GM $\geq$ HM
- $GM^2 = AM \times HM$ ⟶ AM, GM, and HM are in Geometric Progression.

MISCELLANEONS PROGRESSION

Sequences which sometimes follow a particular pattern and sometimes not.

POWER SERIESES

- Sum of the first 'n' natural number $1 + 2 + 3 + \ldots n = \sum n = \frac{n(n+1)}{2}$
- Sum of Squares of the first 'n' natural numbers $1^2 + 2^2 + 3^2 + \ldots n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$
- Sum of cubes of the first 'n' natural numbers $1^3 + 2^3 + \ldots + n^3 = \sum n^3 = \frac{[n(n+1)]^2}{4} = (1 + 2 + \ldots n)^2$

Part I

STATISTICS

MEAN

MEAN IS THE AVERAGE

- Add up all of the values to find the total.
- Divide the total by the number of values you added together.

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$$\underbrace{2+2+3+5+5+7+8}_{\text{There are 7 Values}} = \underbrace{32}_{\text{Divide the total by 7}}$$

$32 \div 7 = 4.57$

MEDIAN

MEDIAN IS THE MIDDLE VALUE

- Put all of the values into increasing / decreasing order.
- If there are two values in the middle, find the mean of these two.

$$\text{Median} = L + \frac{n/2 - CF}{f} \times i$$

Where

- 'L' is the lower limit of the median class
- 'n' is the sample size
- 'CF' is the cumulative frequency preceding the median class
- 'f' is the frequency of the median class
- 'i' is the median class interval

MODE

MODE IS THE MOST FREQUENT VALUE

- Count how many times each value appears.
- You can have more than mode.
- The mode is the value that appears the maximum times.

2, 2, 3, 5, 5, 7, 8

THE MODES ARE 2 AND 5

RANGE

RANGE IS THE DIFFERENCE BETWEEN THE LOWEST AND HIGHEST VALUE

- Find the highest and lowest values.
- Subtract the lowest value from the highest.

2, 3, 5, 5, 7, 8

LOWEST (2) — HIGHEST (8)

$8 - 2 = 6$

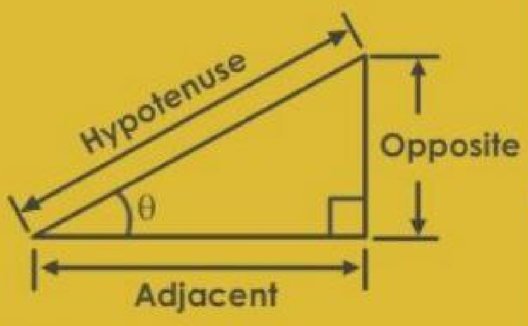
THE RANGE IS '6'

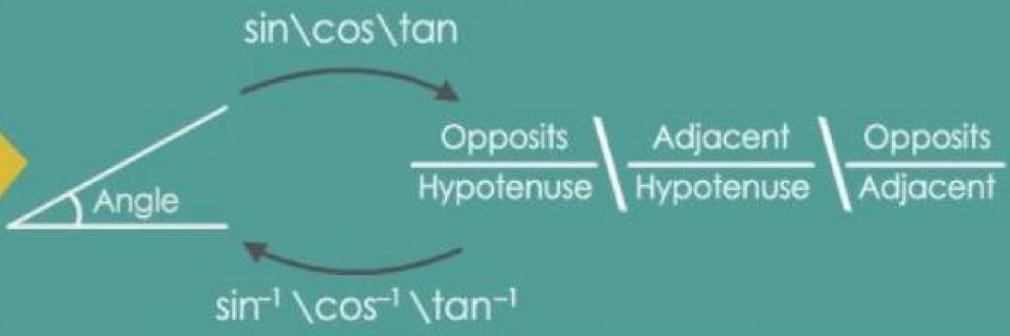
INVERSE TRIGONOMETRIC FUNCTIONS

An Inverse Function goes the other way!

when function 'f' turns the rupee into a dollor.

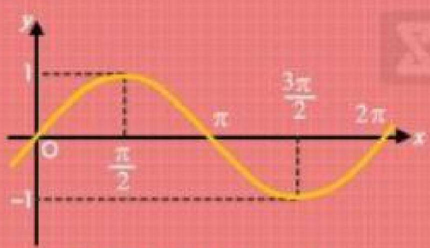
The **inverse function** 'f^{-1}' turn the dollor back to the rupee.





1

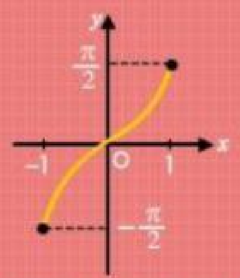
sin x



Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

$\sin^{-1}x$

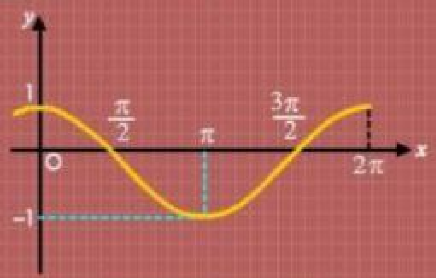


Domain: $-1 \le x \le 1$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

2

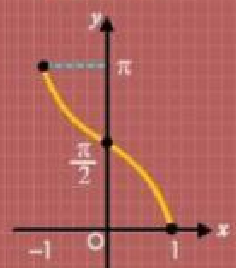
cos x



Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

$\cos^{-1}x$



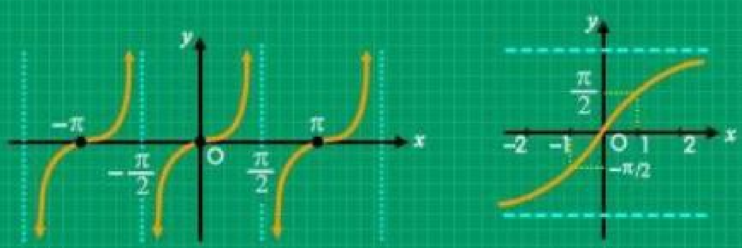
Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$

3

tan x

$\tan^{-1}x$



Domain: $x \in R - \{(2n+1)\frac{\pi}{2}, n \in I\}$

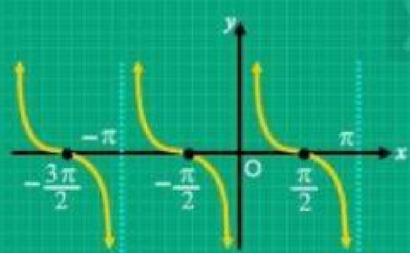
Range: $(-\infty, \infty)$

Domain: $-\infty < x < \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

4

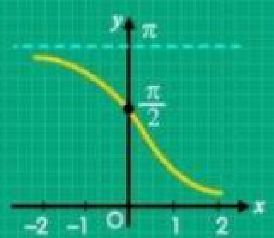
cot x



Domain: $x \in R - \{n\pi, n \in I\}$

Range: $(-\infty, \infty)$

$\cot^{-1}x$

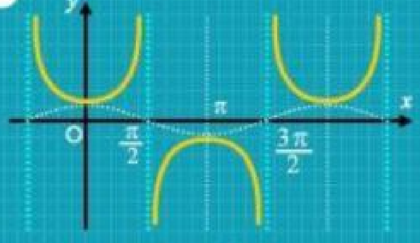


Domain: $-\infty < x < \infty$

Range: $0 < y < \pi$

5

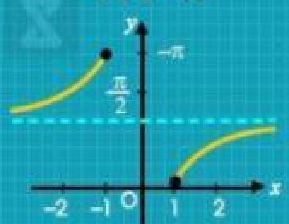
sec x



Domain: $x \ne n\pi + \frac{\pi}{2}, n \in I$

Range: $(-\infty, -1] \cup [1, \infty)$

$\sec^{-1}x$

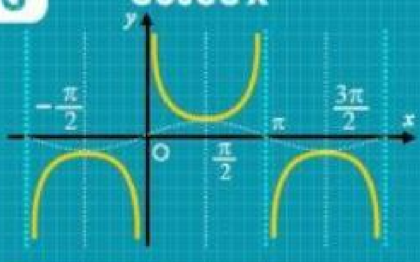


Domain: $x \le -1$ or $x \ge 1$

Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$

6

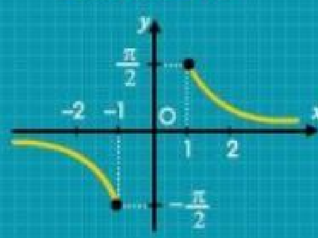
cosec x



Domain: $x \ne n\pi, n \in I$

Range: $(-\infty, -1] \cup [1, \infty)$

$\text{cosec}^{-1}x$

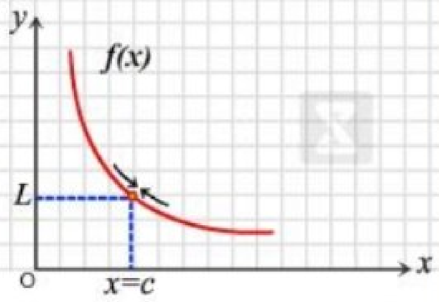


Domain: $x \le -1$ or $x \ge 1$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$

Limit

Existence of a Limit



$\lim_{x \to c} f(x) = L$ iff

$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$

Properties

If $\lim_{x \to c} f(x)$ & $\lim_{x \to c} g(x)$ exist

Scalar Multiple	$\lim_{x \to c} [b.f(x)] = b.\lim_{x \to c} f(x)$
Sum or Difference	$\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
Product	$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) / \lim_{x \to c} g(x)$
Power	$\lim_{x \to c} [f(x)^n] = [\lim_{x \to c} f(x)]^n$ for all $n \in N$

Trigonometric Functions

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} x \operatorname{cosec} x = \lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{x}{\sin^{-1} x}$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$$

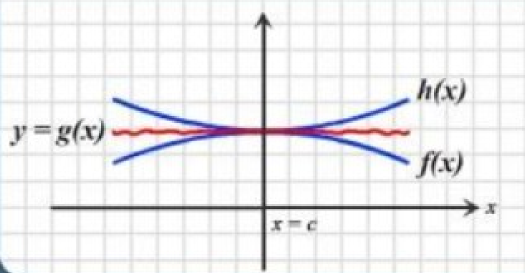
Exponential Functions

$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \; ; \; (a > 0)$	$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$
$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$	$\lim_{x \to 0} (1+x)^{1/x} = e$

Generalised Formula For 1^∞

Let $\lim_{x \to c} f(x) = 1$ & $\lim_{x \to c} \phi(x) \to \infty$ then $\lim_{x \to c} [f(x)]^{\phi(x)} = e^{\lim_{x \to c} \phi(x) \ln[f(x)]}$

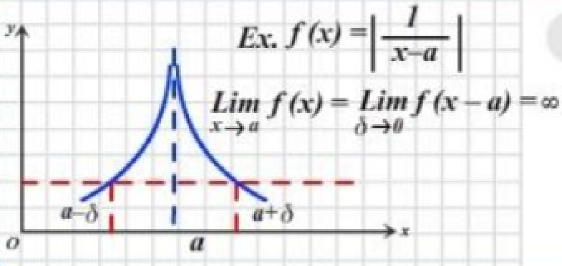
Sandwich / Squeeze Theorem



If f, g and h are three functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point $x = c$ and if

$$\lim_{x \to C} f(x) = \lim_{x \to C} h(x) = L \text{ then } \lim_{x \to C} g(x) = L$$

Limits of Infinity Theorems



If 'a' is a real number and 'r' is a positive rational number

$\lim_{x \to \infty} \frac{a}{x^r} = 0$; $\lim_{x \to -\infty} \frac{a}{x^r} = 0$

Part I

WHAT IS A MATRIX ?

Matrix is an array/arrangement of numbers

A Matrix

$$\begin{bmatrix} -1 & 3 & 0 \\ 6 & -3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} \text{Column} & \\ \begin{bmatrix} -1 & 3 & 0 \\ 6 & -3 & -1 \\ 2 & 0 & 1 \end{bmatrix} & \begin{array}{l} \text{Row} \\ \\ \\ \end{array} \end{array}$$

Order of a matrix = Number of Rows X Number of Columns

TYPE OF MATRICES

1 Row Matrix

Matrix having only one row.

$A_{1\times3} = [1\ 2\ 3]$

2 Column Matrix

Matrix having only one column.

$A_{2\times1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3 Square Matrix

Matrix having same number of rows and columns.

$A_{3\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 7 & 1 & 9 \end{bmatrix}$

4 Zero/Null Matrix

Matrix having all elements equal to zero.

$A_{3\times3} = 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5 Upper Triangular Matrix

All entries below the main diagonal are zero.

$A_{3\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$

6 Lower Triangular Matrix

All entries above the main diagonal are zero.

$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$

7 Diagonal Matrix

All entries above and below the principal diagonal are zero.

$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

8 Identity/Unit Matrix

Diagonal matrices in which all diagonal elements are unity/one.

$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

OPERATIONS ON MATRICES

Addition Matrix

Matrices must have same order.

$3 + 4 = 7$

$A+B = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$

Subtraction Matrix

Matrices must have same order.

$3 - 4 = -1$

$A-B = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$

Equality Matrix

Matrices having same order with all the corresponding elements being equal.

$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $B_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $A = B$

Transpose of a Matrix

A matrix formed by turning all the rows into columms and vice-versa. Symbol ⇨ A^T.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Part II

MATRIX MULTIPLICATION

Multiplication of Matrix with a Scalar

Each element of the Matrix is multiplied by the scalar

2 x 4 = 8

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

Multiplication of a Matrix with another Matrix

If a matrix A and another matrix B then A x B Possible if

Post factor = Pre factor

$$A_{m \times n} \qquad B_{p \times q}$$

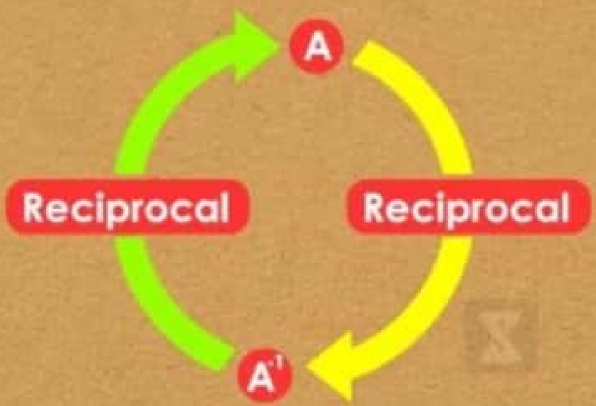
$$\begin{bmatrix} 1 & 3 & 4 \\ 7 & 5 & 6 \\ 2 & 9 & 8 \end{bmatrix}_{3 \times 3} \qquad \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}_{3 \times 1}$$

These must match.

How to Multiply a Matrix by Another Matrix ?

$$\begin{bmatrix} 1 & 3 & 4 \\ 7 & 5 & 6 \\ 2 & 9 & 8 \end{bmatrix}_{3 \times 3} \times \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 1 \times 3 + 3 \times 5 + 4 \times 2 \\ 7 \times 3 + 5 \times 5 + 6 \times 2 \\ 2 \times 3 + 9 \times 5 + 8 \times 2 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 26 \\ 58 \\ 67 \end{bmatrix}_{3 \times 1}$$

INVERSE OF A MATRIX



Reciprocal of a Matrix.

- For a matrix A inverse of this.

  i.e. $A^{-1} \neq \frac{1}{A}$ Why ?

- For a matrix A

  (matrix) × (Inverse of matrix) = I

  i.e. $A \times A^{-1} = I$ or $A^{-1} \times A = I$

  But $A \times A^{-1} \neq A^{-1} \times A$

How to find Inverse of a Matrix ?

Step - I

Check whether Matrix A is singular or non-singular i.e.

$|A| = 0 \Rightarrow$ singular

$|A| \neq 0 \Rightarrow$ Non-singular

Step - II

If Matrix A is Non-singular, then find the value of determinant and also find one adjoint matrix A.

Step - III

Follow the formula

$$A^{-1} = \frac{1}{|A|} \text{adj } A.$$

- $(A^{-1})^{-1} = A$, if A is non-singular.
- $(A^{-1})^{T} = (A^{T})^{-1}$
- If $A = \text{diag}(a_{11}, a_{22}, \ldots\ldots a_{nn})$ Then, $A^{-1} = \text{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots\ldots, \frac{1}{a_{nn}}\right)$

TYPE OF SQUARE MATRICES

Nilpotent Matrix

If $B^P = 0$ where 'P' is the least +ve integer. Then, 'B' is a Nilpotent matrix.

$$B = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

Idempotent Matrix

If $B^2 = B$. Then, 'B' is an Idempotent matrix.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Involutory Matrix

If $B^2 = I$. Then, 'B' is a Involutory matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric Matrix

If $B^T = B$. Then, 'B' is a Symmetric matrix.

$$B = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

Skew Symmetric Matrix

If $B^T = -B$ and all Principal diagonal elements are zero. Then 'B' is a skew symmetric matrix.

$$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

Unitary Matrix

If $B' (B')^T = I$ where B' is the complex conjugate of B. Then, B is a unitary matrix.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Orthogonal Matrix

A square matrix 'B' if $B^TB = I = B\ B^T$ or $B^T = B^{-1}$. Then, 'B' is an Orthogonal matrix.

Points To Remember

- In a skew-symmetric matrix all the Principal diagonal elements are zero.
- For any square matrix A, $A + A^T$ is symmetric & $A - A^T$ is skew-symmetric.
- Every square matrix can be uniquely expressed as a sum of two square matrices of which one is symmetric and the other is skew-symmetric
- $A = B + C$, where $B = \frac{1}{2}(A + A^T)$ & $C = \frac{1}{2}(A - A^T)$
- For any matrix A ⇨ $(A^T)^T = A$
- Let λ be a scalar & A be a matrix. Then $(\lambda A)^T = \lambda A^T$
- $(A_1 \pm A_2 \pm \ldots \pm A_n)^T = A_1^T \pm A_2^T \pm \ldots \pm A_n^T$ where A_j are comparable.
- $(A_1 . A_2 \ldots\ldots A_n)^T = A_n^T . A_{n-1}^T \ldots\ldots A_2^T . A_1^T$ provided the product is defined.
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $0 = [0]_{m \times n}$ is the additive identity.
- $\lambda(A + B) = \lambda A + \lambda B$

DETERMINANT

Part I

1 WHAT IS A DETERMINANT ?

Every square matrix can be associated to a number which is known as a Determinant.

If A $\longrightarrow$ square matrix

|A| or det A or $\Delta \longrightarrow$ denotes the determinant of A

$$\begin{array}{cc} & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} R_1 \rightarrow \\ \text{Rows : } R_2 \rightarrow \\ R_3 \rightarrow \end{array} & \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \end{array}$$

Columns : C_1 C_2 C_3

$\longrightarrow$ is a determinant of order '3'.

All entries (1,2,3,4,5,6,7,8,9) are called elements of the determinant.

2 SUBMATRIX

A matrix obtained by deleting some rows or columns is said to be a submatrix.

If $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$; Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}$, $\begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}$, $\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$

are all submatrices of A.

3 MINORS & COFACTORS

MINORS : are defined as the determinant of the sub matrix obtained by deleting i^{th} row and j^{th} column of the determinant (let determinant be Δ) **Denoted by M_{ij}**

COFACTOR : denoted by C_{ij} and is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

4 HOW TO FIND THE DETERMINANT ?

Matrix should be square matrix of order greater then 1, let $A = [a_{ij}]_{n \times n}$

Determinant of A is defined as sum of products of elements of any one row (or one column) with corresponding cofactors.

$$A_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\Rightarrow |A| = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$ (using 1^{st} row)

or $|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

PROPERTIES OF DETERMINANTS

Part II

P - 1 The value of a determinant remains unaltered, if the rows & columns are interchanged.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D' \Longrightarrow D \ \& \ D' \text{ are transpose of each other.}$$

P - 2 If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \boxed{D' = -D}$$

P - 3 If a determinant has any two rows (or columns) identical, then its value is zero.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then it can be verified that } \boxed{D = 0.}$$

P - 4 If all the elements of any row (or column) be multiplied by the same number then the determinant is multiplied by that number.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \boxed{D' = KD}$$

P - 5 If each element of any row (or column) can be expressed as a sum two terms then the determinant can be expressed as the sum of two determinants.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P - 6 The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_2 & b_3+nb_2 & c_3+nc_2 \end{vmatrix}. \text{ Then } \boxed{D' = D}$$

Note :- While applying this property atleast one row (or column) must remain unchanged.

Part III

DETERMINANT : CRAMER'S RULE

Simultaneous linear equations involving three unknowns x, y and z

$$a_1x + b_1y + c_1z = d_1 \ldots\ldots\ldots\ldots (i)$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots (ii)$$
$$a_3x + b_3y + c_3z = d_3 \ldots\ldots\ldots\ldots (iii)$$

The solution for the above system of linear equations

$$\boxed{x = \frac{D_1}{D} \; ; \; y = \frac{D_2}{D} \; ; \; z = \frac{D_3}{D}}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} ; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} ; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Nature of the Solutions

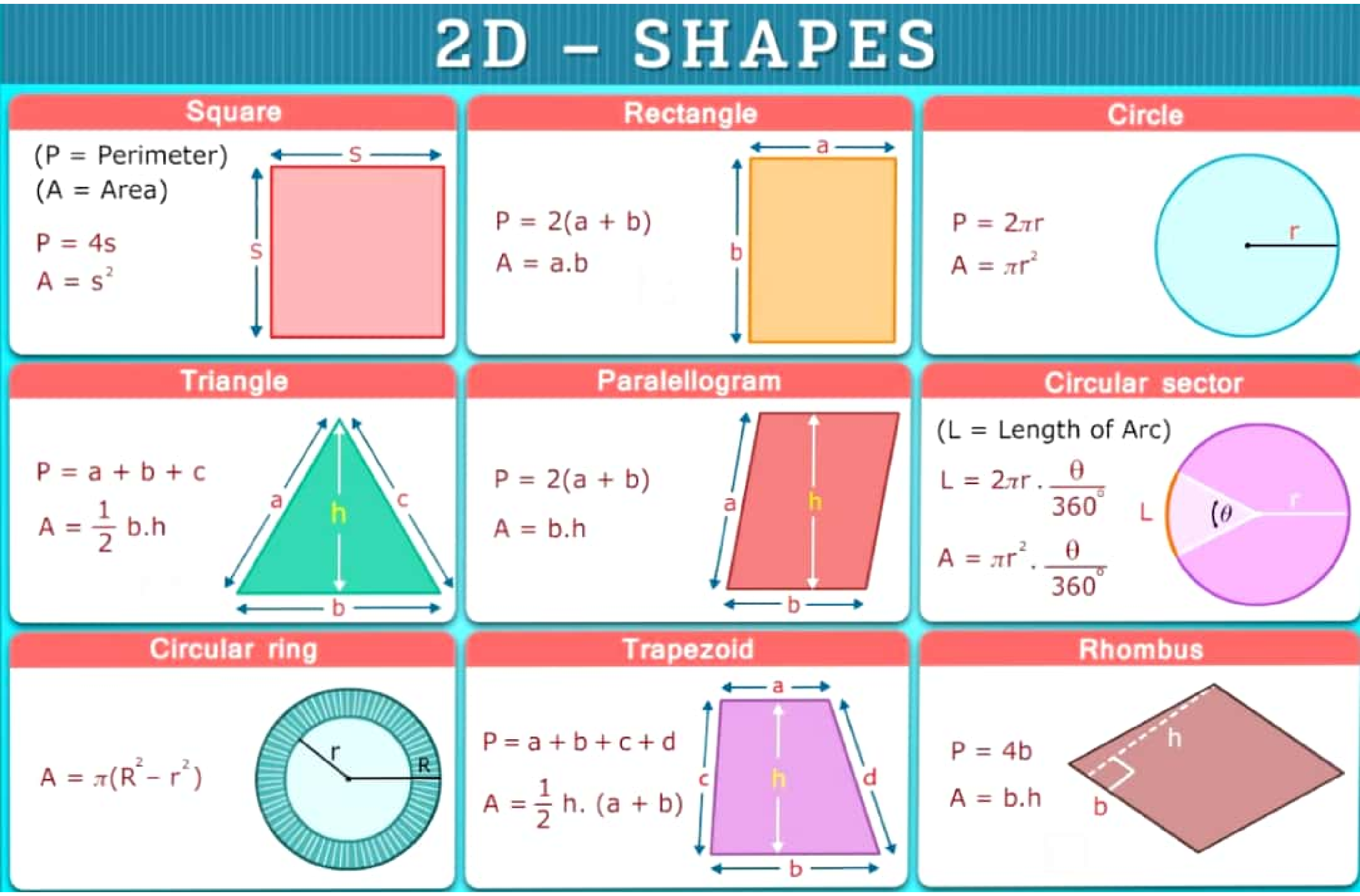
- When $D = 0$
  - At least one of D_1, D_2, D_3 is non-zero ⇒ Equations are inconsist-ent & have no solution
  - All $D_1 = D_2 = D_3 = 0$ ⇒ Equations are consistent & have infinite solution
- When $D \neq 0$
  - All least one of D_1, D_2, D_3 is non-zero ⇒ Equations are consistent & have unique solution
  - All $D_1 = D_2 = D_3 = 0$ ⇒ Equations are consistent & have a trivial solution

➡ If a given system of linear equations have only zero solution for all its variables then the given equations are said to have trivial solution.

➡ If a system of linear equations (in two variables) have definite & unique solution, then they represent intersecting lines.

➡ If a system of linear equations (in two variables) have no solution, then they represent parallel lines.

➡ If a system of linear equations (in two variables) have infinite solutions, then they represent Identical lines.


2D – SHAPES
Square
(P = Perimeter)
(A = Area)
$P = 4s$
$A = s^2$
Rectangle
$P = 2(a + b)$
$A = a.b$
Circle
$P = 2\pi r$
$A = \pi r^2$
Triangle
$P = a + b + c$
$A = \frac{1}{2}\, b.h$
Paralellogram
$P = 2(a + b)$
$A = b.h$
Circular sector
(L = Length of Arc)
$L = 2\pi r \cdot \frac{\theta}{360^\circ}$
$A = \pi r^2 \cdot \frac{\theta}{360^\circ}$
Circular ring
$A = \pi(R^2 - r^2)$
Trapezoid
$P = a + b + c + d$
$A = \frac{1}{2}\, h.\, (a + b)$
Rhombus
$P = 4b$
$A = b.h$

3D – SHAPES

SPHERE

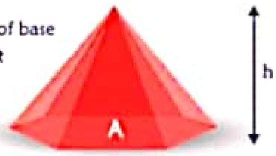
Volume $= \frac{4\pi r^3}{3}$

Surface area $= 4\pi r^2$

PYRAMID

Volume of a general pyramid $= \frac{1}{3}Ah$

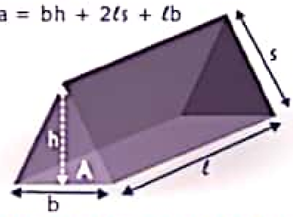
where:
A = area of base
h = height



TRIANGULAR PRISM

Volume $= A\ell$ or $\frac{1}{2}bh\ell$

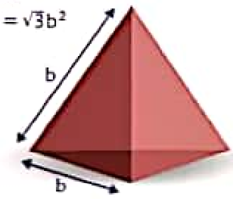
Surface area $= bh + 2\ell s + \ell b$



REGULAR TETRAHEDRON

Volume $= \frac{b^3}{6\sqrt{2}}$

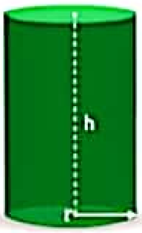
Surface area $= \sqrt{3}b^2$



RIGHT CYLINDER

Volume $= \pi r^2 h$

Surface area $= 2\pi r(r + h)$



CUBE

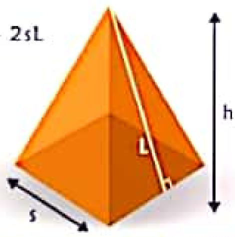
Volume $= s^3$

Surface area $= 6s^2$

SQUARE-BASED PYRAMID

Volume $= \frac{1}{3}s^2h$

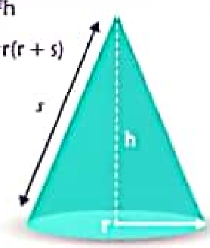
Surface area $= s^2 + 2sL$



RIGHT CIRCULAR CONE

Volume $= \frac{1}{3}\pi r^2 h$

Surface area $= \pi r(r + s)$

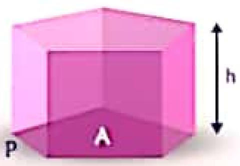


PENTAGONAL PRISM

Volume of any prism = Ah

Surface area of a closed prism $= 2A + 5(h \times p)$

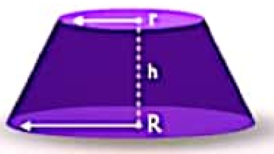
where:
A = area of base
h = height
p = perimeter of base



FRUSTUM OF A CONE

Volume $= \frac{1}{3}\pi h(r^2 + rR + R^2)$

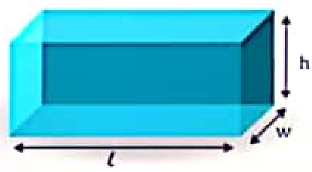
Total Surface Area $= \pi(r+R)\sqrt{(R-r)^2+h^2} + \pi(r^2+R^2)$



CUBOID

Volume $= \ell \times w \times h$

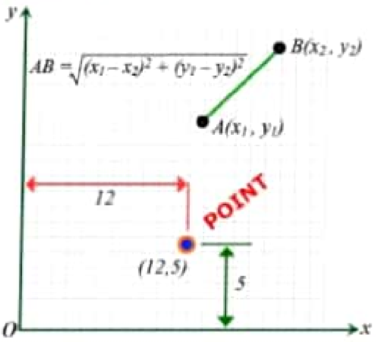
Surface area $= 2\ell h + 2\ell w + 2wh$



POINT IN 2D CARTESIAN SYSTEM

Point Definition

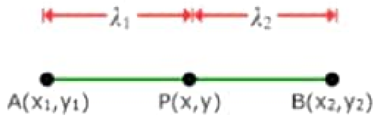
Point is an exact location. It has no size, only position.



Section Formula

Internally

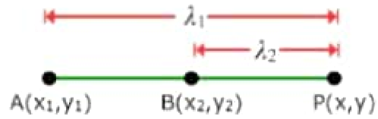
When P divides AB in ratio $\lambda_1 : \lambda_2$



$$P\left(\frac{\lambda_1 x_2 + \lambda_2 x_1}{\lambda_1 + \lambda_2}, \frac{\lambda_1 y_2 + \lambda_2 y_1}{\lambda_1 + \lambda_2}\right)$$

Externally

When P divides AB in ratio $\lambda_1 : \lambda_2$

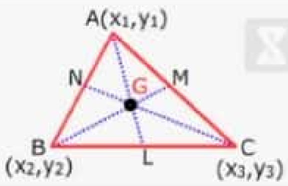


$$P\left(\frac{\lambda_1 x_2 - \lambda_2 x_1}{\lambda_1 - \lambda_2}, \frac{\lambda_1 y_2 - \lambda_2 y_1}{\lambda_1 - \lambda_2}\right)$$

Special points in a triangle with 2D co-ordinates

Centroid (G)

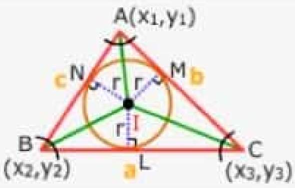
Point of intersection of medians



$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Incentre (I)

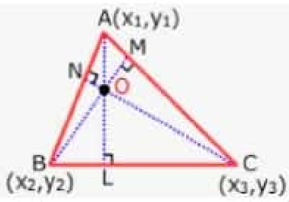
Point of intersection of angle bisectors



$$I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Orthocentre (O)

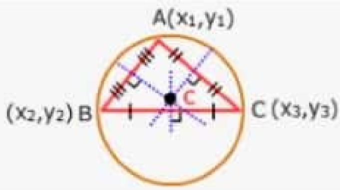
Point of intersection of Altitudes



$$O\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Circumcentre (C)

Point of intersection of perpendicular bisectors

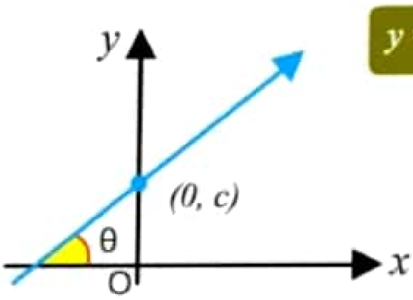


$$C\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Straight line

1 Slope - Intercept Form

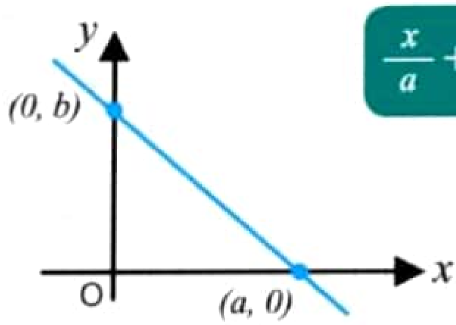
$$y = mx + c$$



where $m = \tan\theta$

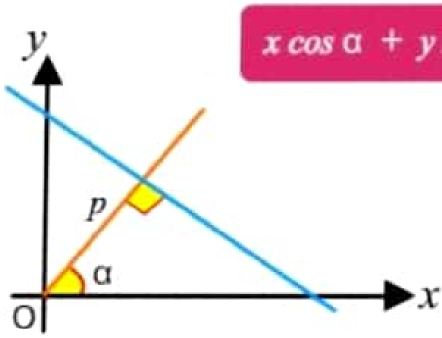
2 Double Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$



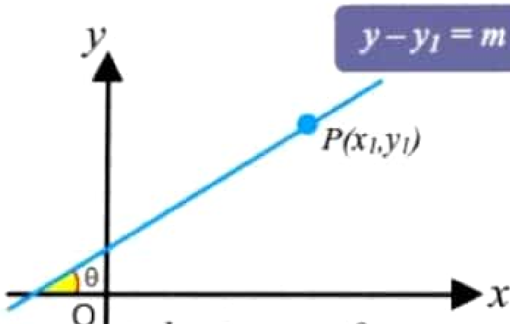
3 Normal Form

$$x\cos\alpha + y\sin\alpha = p$$



4 Slope - Point Form

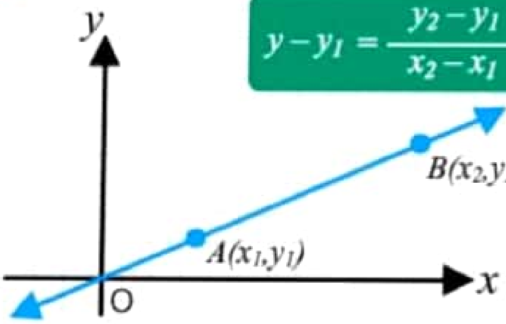
$$y - y_1 = m(x - x_1)$$



where $m = \tan\theta$

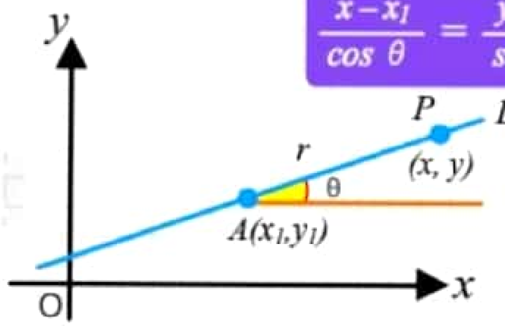
5 Two Point Form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$



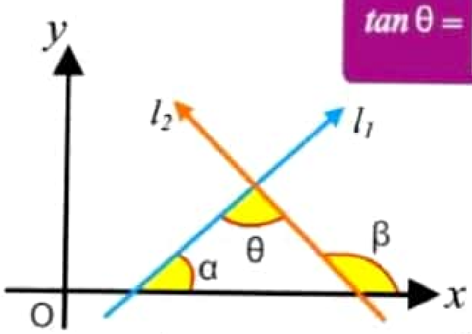
6 Parametric Form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$



7 Angle between two Straight lines

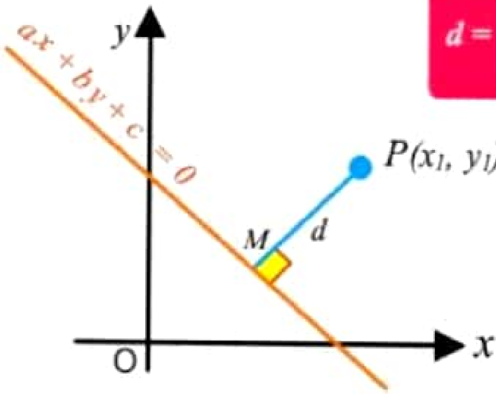
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



$m_1 = \tan\alpha$ $m_2 = \tan\beta$

8 Distance between Point & line

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$



CIRCLE PROPERTIES

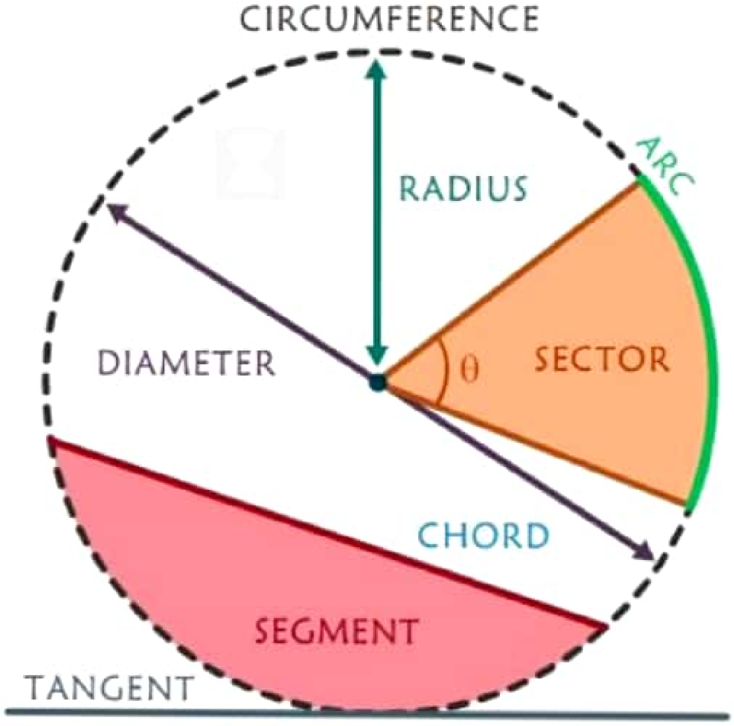
Part I

CIRCUMFERENCE
Length of the outer edge of a circle

DIAMETER
The distance from edge to edge passing through the center

CHORD
A straight line joining any two points on the circumference

SEGMENT
The area inside a circle enclosed by an arc and a chord



RADIUS
The distance from the center to the edge: half the diameter.

SECTOR
The area enclosed by an arc and two radii

ARC
A part of the circumference

TANGENT
A straight line that touches a circle at two coincident points

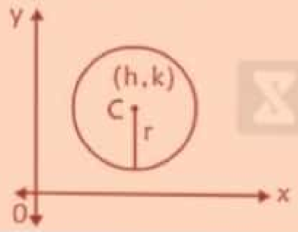
$\text{Area} = \pi r^2$	$\text{Area of Sector} = \pi r^2 \cdot \dfrac{\theta}{360^\circ}$	$\text{Circumference} = 2\pi r$	$\text{Length of Arc} = 2\pi r \cdot \dfrac{\theta}{360^\circ}$

$\theta \rightarrow$ in degrees

STANDARD EQUATION OF THE CIRCLE

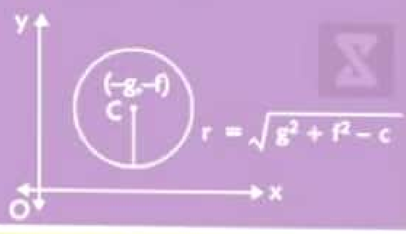
1. Central Form:

$(x-h)^2 + (y-k)^2 = r^2$



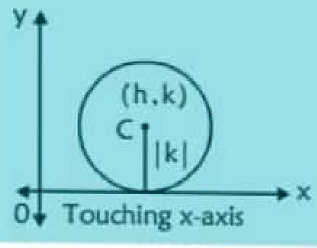
2. General equation of circle:

$x^2 + y^2 + 2gx + 2fy + c = 0$



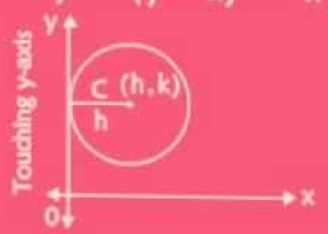
3. When circle touches x-axis

$(x-h)^2 + (y-k)^2 = k^2$



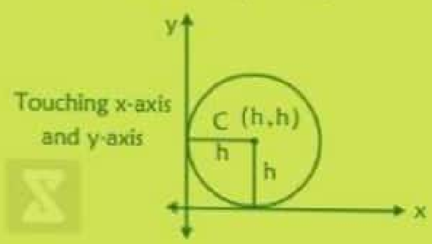
4. When circle touches y-axis

$(x-h^2) + (y-k)^2 = h^2$



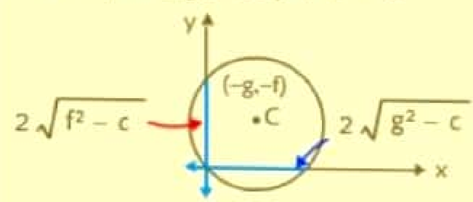
5. When circle touches both the axis

$(x-h^2) + (y-h)^2 = h^2$



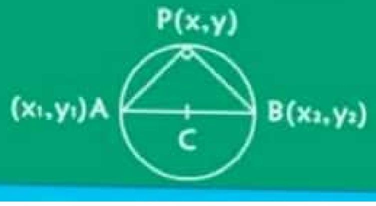
6. Intercepts cut by the circle on axes:

$x^2 + y^2 + 2gx + 2fy + c = 0$



7. Diametrical form of circle:

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$



8. The parametric forms of the circle:

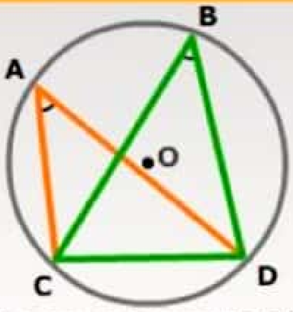
Circle	Parametric forms
$x^2 + y^2 = r^2$	$x = r\cos\theta,\ y = r\sin\theta$
$(x-h)^2 + (y-k)^2 = r^2$	$x = h + r\cos\theta,\ y = k + r\sin\theta$

where θ is the parameter $\theta \in [0, 2\pi)$

CIRCLE THEOREMS

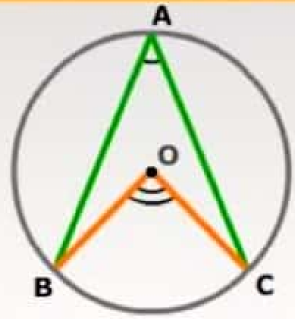
Part III

1 $\angle CAD = \angle CBD$



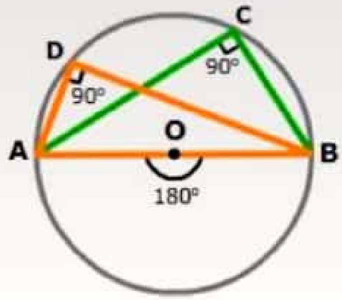
Angle in the same segment and standing on the same chord are always equal.

2 $\angle BOC = 2\angle BAC$



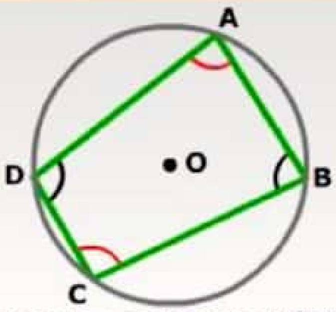
The angle at the centre of a circle is twice the angle at the circumference.

3 $\angle BCA = \angle BDA = 90^\circ$



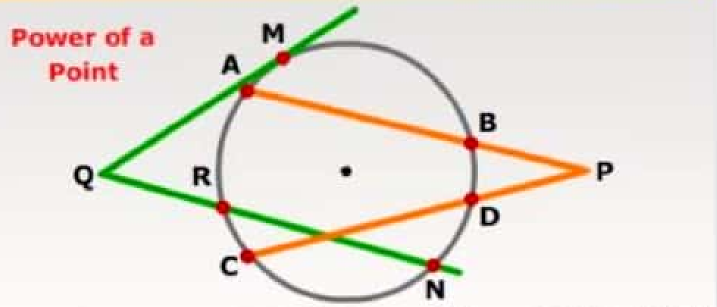
The angle in a semi-circle is always 90°.

4 $\angle B + \angle D = 180^\circ$, $\angle A + \angle C = 180^\circ$



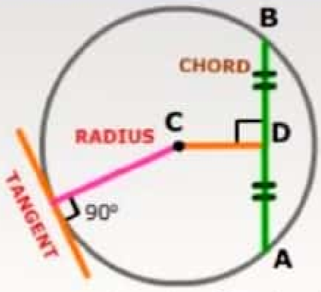
ABCD is a cyclic Quadrilateral, Diagonally opposite angles add up to 180°.

5 $QR.QN = (QM)^2$, $PA.PB = PC.PD$



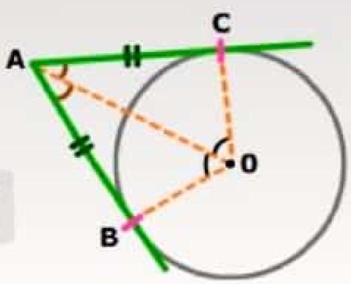
If QRN is a secant & QM is tangent then $QR.QN=(QM)^2$
If PBA & PDC are secant to circle then $PA.PB=PC.PD$

6 $CD \perp AB$, $BD = DA$, $\angle BDC = 90^\circ$



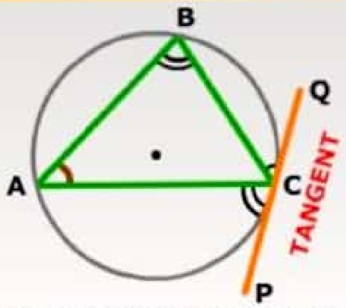
Angle between tangent and radius is always 90°.
Perpendicular bisector of any chord, pass through center.

7 $AB = AC$, $\angle AOB = \angle AOC$



Tangents from a common point (A) to a circle are always equal.

8 $\angle PCA = \angle ABC$, $\angle QCB = \angle BAC$



The angle between the tangent and the side of the triangle is equal to the interior opposite angle.

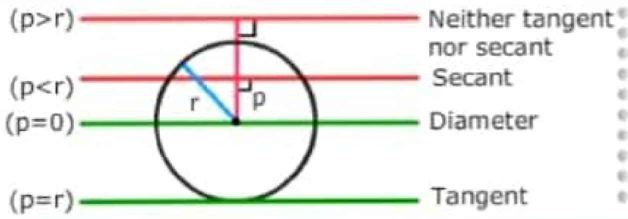
CIRCLE'S TANGENTS

Part II

Condition of Tangency :

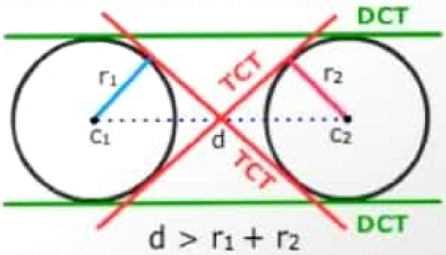
p = perpendicular distance from center to line

r = radius to circle



Common tangents of two circles

1 If two circle are separated

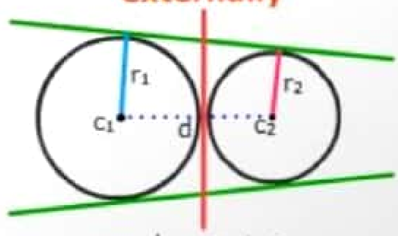


$d > r_1 + r_2$

d=distance between centers

$\Rightarrow$ 4 common tangents (2DCT,2TCT)

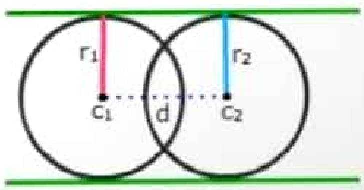
2 If two circles touch externally



$d = r_1 + r_2$

$\Rightarrow$ 3 common tangents (2DCT,1TCT)

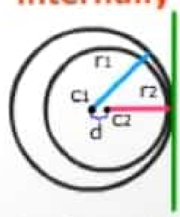
3 If two circles intersect each other



$|r_1 - r_2| < d < r_1 + r_2$

$\Rightarrow$ 2 common tangents (2DCT)

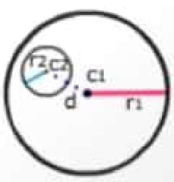
4 If two circles touch internally



$d = |r_1 - r_2|$

$\Rightarrow$ 1 common tangent (1DCT)

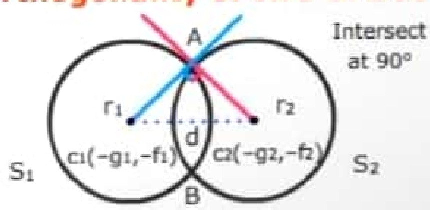
5 If one circles is completely contained in another circle



$d < |r_1 - r_2|$

$\Rightarrow$ No common tangent

6 Orthogonality of two circles



$x^2+y^2+2g_1x+2f_1y+d_1=0$

$x^2+y^2+2g_2x+2f_2y+d_2=0$

Condition : $2g_1g_2 + 2f_1f_2 = d_1+d_2$

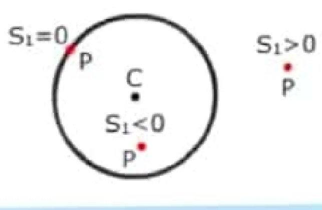
*** DCT = Direct Common Tangent, TCT = Transverse Common Tangent**

Position of a point $P(x_1,y_1)$ w.r.t. circle :

$x^2+y^2+2gx+2fy+c = 0$

deponds on

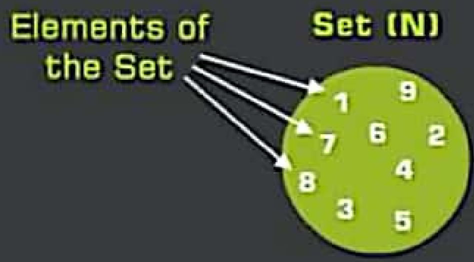
$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$



SET

Part I

A SET IS A WELL DEFINED COLLECTION OF OBJECTS.



$\in \Rightarrow$ 'IS AN ELEMENT OF'

EXAMPLE: $4 \in N$

$\notin \Rightarrow$ 'IS NOT AN ELEMENT OF'

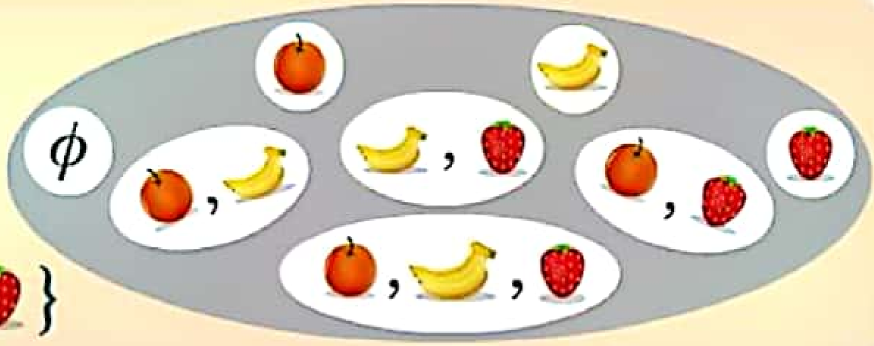
EXAMPLE: $12 \notin N$

$N = \{1,2,3,4,5,6,7,8,9\}$

1. POWER SET

POWER SET P(A)

SET A = 



2. EMPTY SET

NO ELEMENT

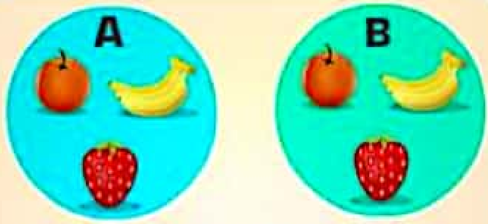
{ } or ϕ

3. FINITE SET

{1,2,3}

FINITE NUMBER OF ELEMENT

4. EQUAL SET



A = B IF $A \subseteq B$ & $B \subseteq A$

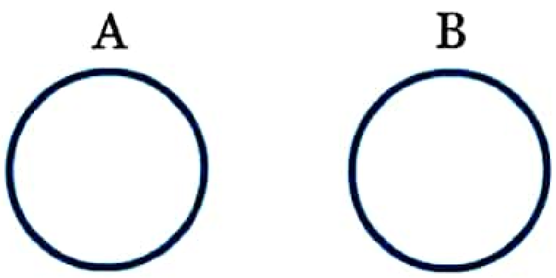
5. SUBSET

$A \subseteq B$,

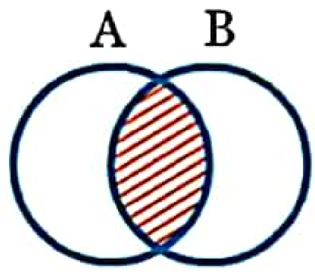
🍊 $\in A \Rightarrow$ 🍊 $\in B$

OPERATION OF SETS

Part II

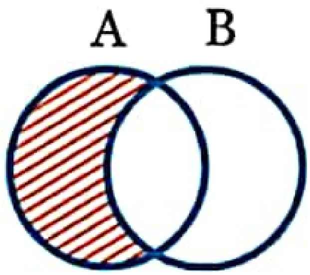


DISJOINT SET A AND B



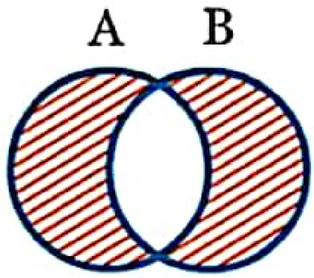
$A \cap B$

THE INTERSECTION OF A AND B



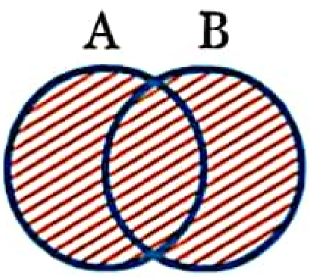
$A \setminus B$

THE RELATIVE COMPLEMENT OF B IN A



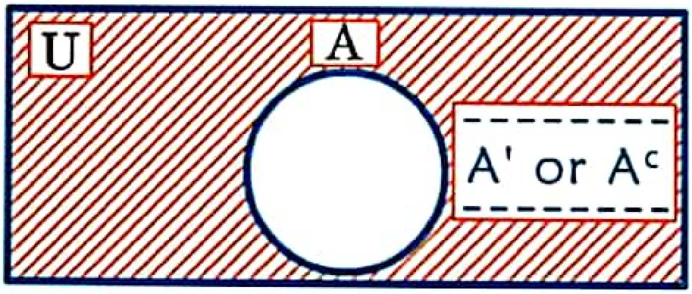
$A \Delta B$

THE SYMMETRIC DIFFERENCE OF A AND B

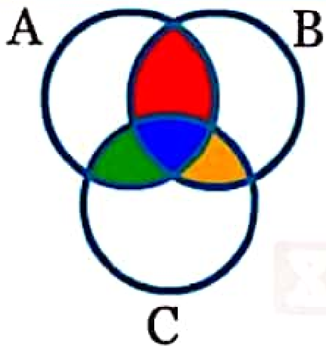


$A \cup B$

THE UNION OF A AND B



COMPLEMENT OF SET A



$A \cap B \cap C$

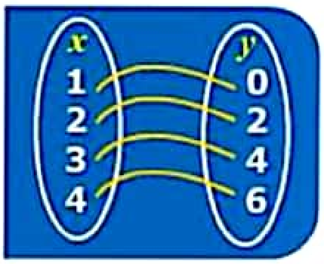
$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (C \cap A) + (A \cap B \cap C)$

FUNCTIONS

Part I

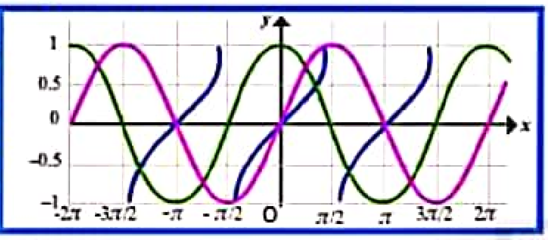
A funtion is a relationship where each input has a single output.

It is written as "$f(x)$", where 'x' is the input

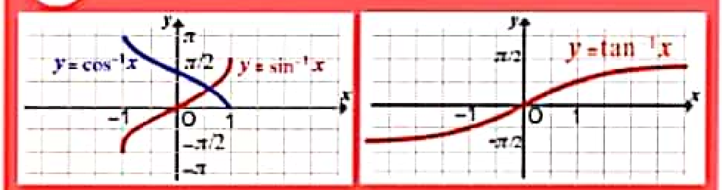


1 Trigonometric Function

$\sin x$ — $\cos x$ — $\tan x$

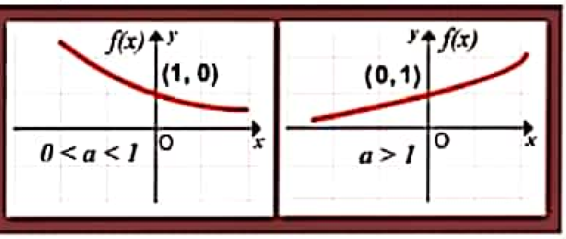


2 Inverse Trigonometric Function



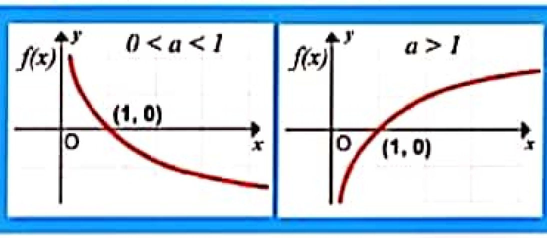
3 Exponential Function

$f(x) = a^x$, where $a > 0$, $a \neq 1$



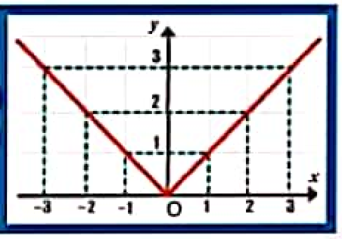
4 Logarithmic Function

$\log_a x$
$x > 0$
$a > 0$
$a \neq 1$



5 Absolute Value Function

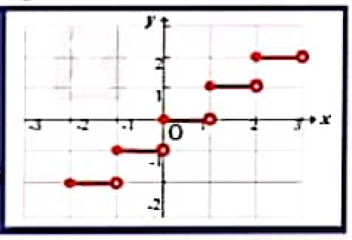
$$y = |x| = \begin{cases} x \ ; \ x \geq 0 \\ -x \ ; \ x < 0 \end{cases}$$



6 Greatest Integer Function

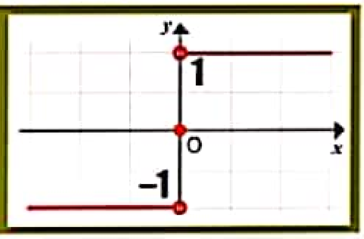
$$y = f(x) = [x]$$

$$[x] = \begin{cases} x \ ; \ x \in \mathrm{I} \\ \text{Greatest Integer ; otherwise less than } x \end{cases}$$



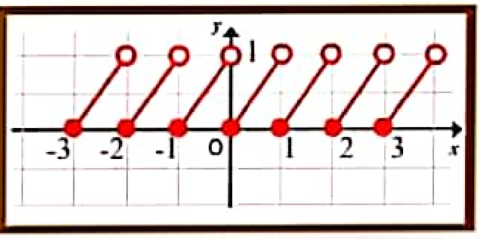
7 Signum Function

$$y = x = \begin{cases} 1 \ ; \ x > 0 \\ 0 \ ; \ x = 0 \\ -1 \ ; \ x < 0 \end{cases}$$



8 Fractional Part Function

$$y = f(x) = \{x\} = x - [x]$$



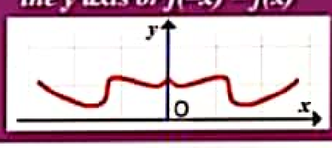
9 Algebraic Function

Constructed using $+, -, \times, \div$ & $\sqrt{\ }$

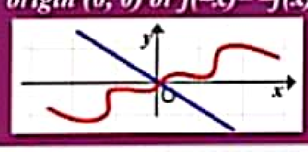
Ex. $f(x) = \sqrt{(x^4 + 5x^2)}$

10 Even - Odd Function

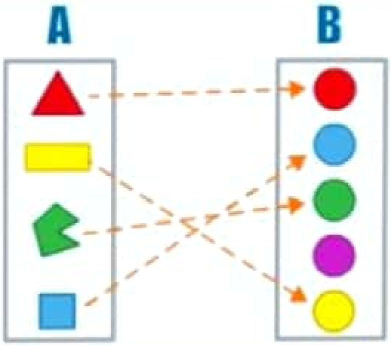
EVEN symmetrical about the y axis or $f(-x) = f(x)$



ODD symmetrical about the origin $(0, 0)$ or $f(-x) = -f(x)$

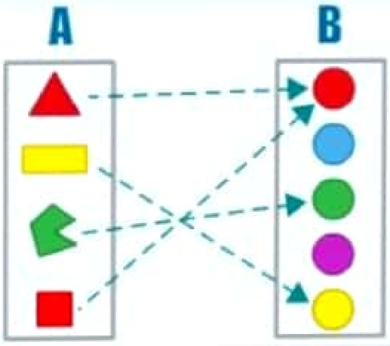


CLASSIFICATION OF FUNCTION



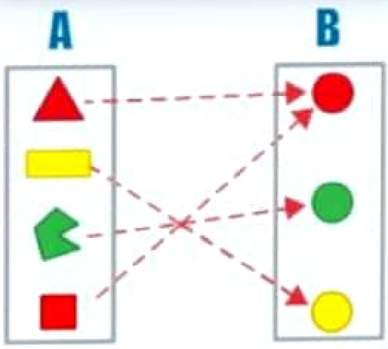
1 One-One Function

Each element of set A is connected with a different element of set B. It is also called Injective function.



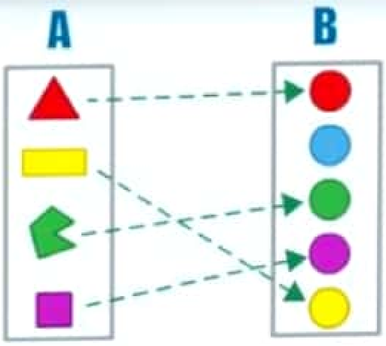
2 Many-One Function

If any two or more elements of set A are connected with a single element of set B.



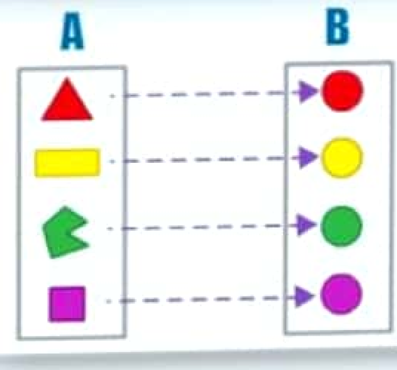
3 Onto Function

Function f from set A to set B is onto function if each element of set B is connected with elements of set A. It is also called Surjective function.



4 Into Function

Function f from set A to set B is into function if set B has at least one element which is not connected with any of the element of set A.



5 Bijective Function

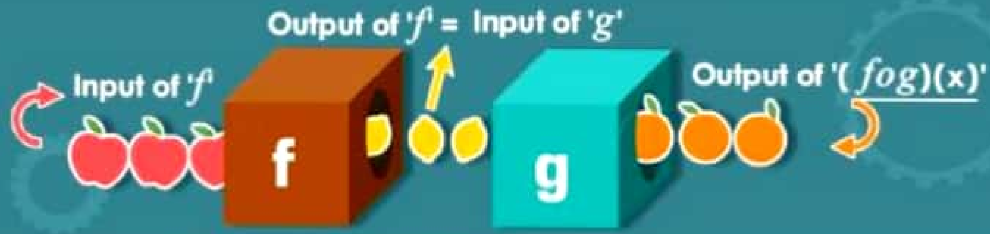
Function 'f' from set A to set B is Bijective Function if

(a) 'f' is One one function

(b) 'f' is Onto function.

SPECIAL FUNCTIONS

COMPOSITE FUNCTION



"Function Composition" means applying one function to the results of another.

PROPERTIES OF COMPOSITE FUNCTIONS

- The composite of functions is not commutative. $(gof)(x) \neq (fog)(x)$
- The composite of functions is associative. if f, g, h are three functions Then $fo(goh) = (fog)oh$
- The composite of two bijections is a bijection. if f and g are two bijections such that gof is defined, then gof is also a bijection.

INVERSE FUNCTION

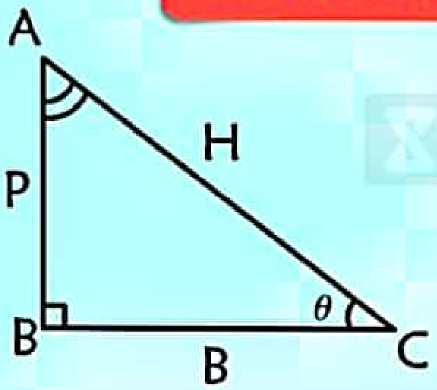
Let f : A → B be a bijective function, then there exists a unique g : B → A such that $f(x) = y \Leftrightarrow g(y) = x$, for all $x \in A$ and $y \in B$. Then 'g' is said to be inverse of 'f'.

PROPERTIES OF INVERSE FUNCTION

- The inverse of a bijection is unique.
- If $f: A \to B$ is a bijection and $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B respectively.
- The inverse of a bijection is also a bijection.
- If f & g are two bijections ; $f: A \to B$, $g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

TRIGONOMETRY RATIO

" Pandit Badri Prasad Bole Hari Hari "

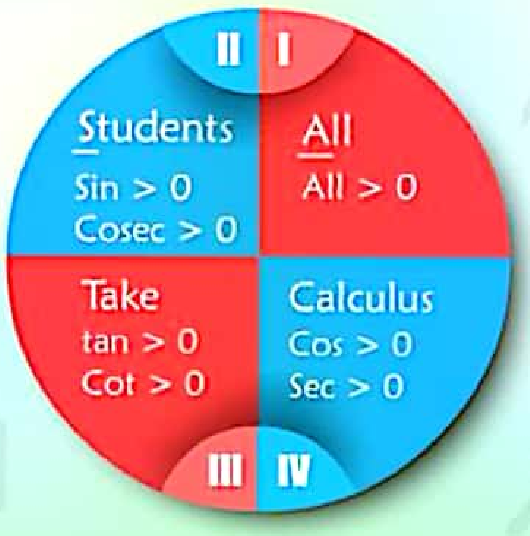


sin	cos	tan	cot	sec	cosec
P	B	P	B	H	H
H	H	B	P	B	P

Value

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D
$\cot\theta$	N.D	$\sqrt{3}$	1	$1/\sqrt{3}$	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D
$\text{cosec}\,\theta$	N.D	2	$\sqrt{2}$	$2/\sqrt{3}$	1

Quadrant



$(90^0 + \theta)$ Reduction

$\sin(90^0+\theta) = \cos\theta$

$\cos(90^0+\theta) = -\sin\theta$

$\tan(90^0+\theta) = -\cot\theta$

$\cot(90^0+\theta) = -\tan\theta$

$\sec(90^0+\theta) = -\csc\theta$

$\csc(90^0+\theta) = \sec\theta$

$(180^0 + \theta)$ Reduction

$\sin(180^0+\theta) = -\sin\theta$

$\cos(180^0+\theta) = -\cos\theta$

$\tan(180^0+\theta) = \tan\theta$

$\cot(180^0+\theta) = \cot\theta$

$\csc(180^0+\theta) = -\csc\theta$

$\sec(180^0+\theta) = -\sec\theta$

" Complementary angles are those whose sum is 90°"

$(360^0 - \theta)$ or $(2\pi - \theta)$ Reduction

$\sin(2\pi-\theta) = \sin(-\theta) = -\sin\theta$

$\cos(2\pi-\theta) = \cos(-\theta) = \cos\theta$

$\tan(2\pi-\theta) = \tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$

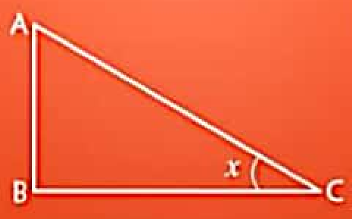
$\csc(-\theta) = -\csc\theta$

$\sec(-\theta) = \sec\theta$

TRIGONOMETRIC IDENTITIES

Part I

1 Quotient Identities



$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

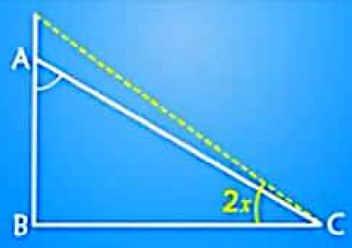
2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\operatorname{cosec}^2 x - \cot^2 x = 1$$

3 Double Angle Identities



$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

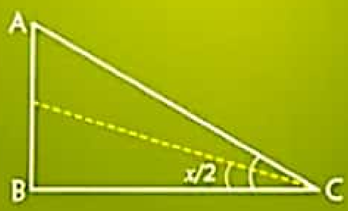
$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

4 Half Angle Identities



$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

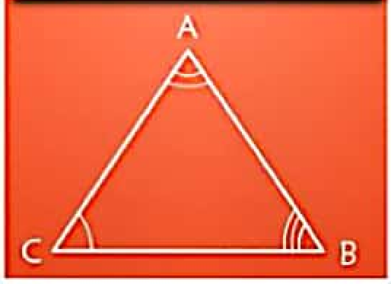
$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$$

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$$

TRIGONOMETRIC IDENTITIES

Part II

5 Angle Sum & Difference Identities



$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \ \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \, . \cot B \mp 1}{\cot B \pm \cot A}$$

6 Sum Identities

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$$

7 Product Identities

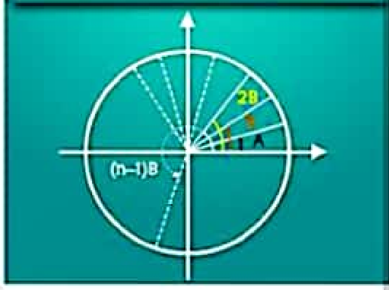
$$2\sin A \cos B = \left[\sin(A+B) + \sin(A-B)\right]$$

$$2\cos A \sin B = \left[\sin(A+B) - \sin(A-B)\right]$$

$$2\cos A \cos B = \left[\cos(A-B) + \cos(A+B)\right]$$

$$2\sin A \sin B = \left[\cos(A-B) - \cos(A+B)\right]$$

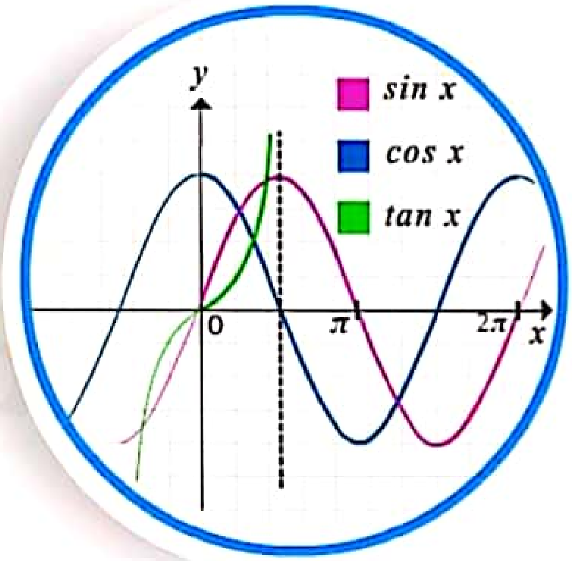
8 Summation of Trigonometric Series



$$\sin A + \sin(A+B) + \sin(A+2B) + \dots + \sin(A+(n-1)B) = \frac{\sin nB/2}{\sin B/2}.\sin\left(A + \frac{(n-1)}{2}B\right)$$

$$\cos A + \cos(A+B) + \cos(A+2B) + \dots + \cos(A+(n-1)B) = \frac{\sin nB/2}{\sin B/2}.\cos\left(A + \frac{(n-1)}{2}B\right)$$

TRIGNOMETRIC EQUATION



Principal Solution

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi]$ are called principal solutions.

Eg: $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{9\pi}{6}$ ....

But, principal solution of

$\sin x = \dfrac{1}{2}$ are $\dfrac{\pi}{6}, \dfrac{5\pi}{6} \in [0, 2\pi]$

General Solution

$\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ \alpha \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], n \in I$

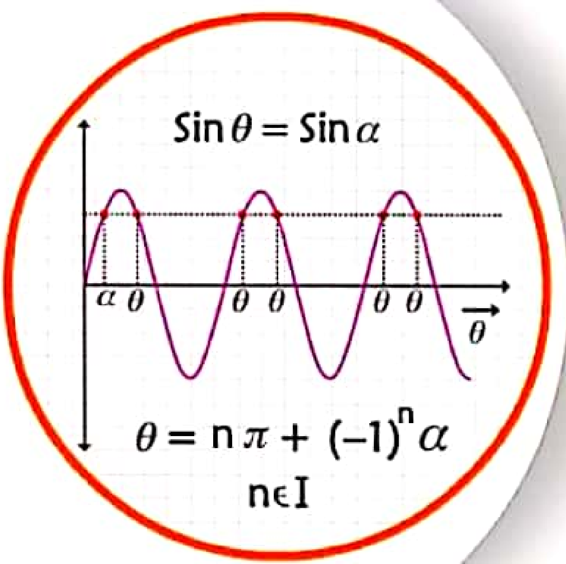
$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, \alpha \in [0, \pi], n \in I$

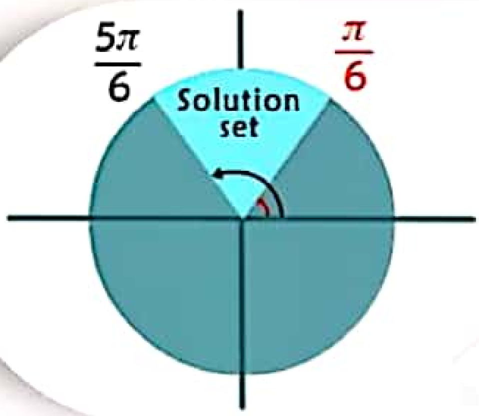
$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, \alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), n \in I$

$\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$

$\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$

$\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$; α is called one principal angle.



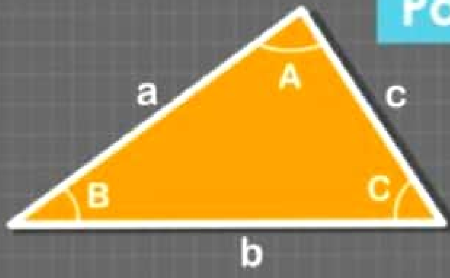


Trigonometric Inequalities

Eg: $\sin x > \dfrac{1}{2} \Rightarrow \dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$

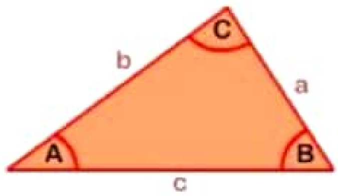
SOLUTION OF TRIANGLE

Part I



LAW OF SINES

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

PROJECTION FORMULA

$$a = b\cos C + c\cos B$$

$$b = c\cos A + a\cos C$$

$$c = a\cos B + b\cos A$$

AREA OF TRIANGLE

$$\text{Area} = \frac{1}{2}\, ab\sin C$$

$$\text{Area} = \frac{1}{2}\, bc\sin A$$

$$\text{Area} = \frac{1}{2}\, ca\sin B$$

NAIPER'S ANALOGY

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

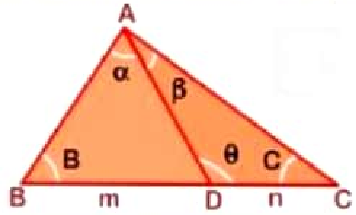
$$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

M–N THEOREM

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

$$(m+n)\cot\theta = n\cot B - m\cot C$$



TRIGONOMETRIC HALF ANGLES

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

where, $s = \dfrac{a+b+c}{2}$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

where, $s = \dfrac{a+b+c}{2}$

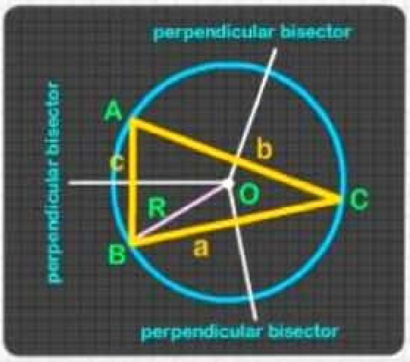
$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

where, $s = \dfrac{a+b+c}{2}$

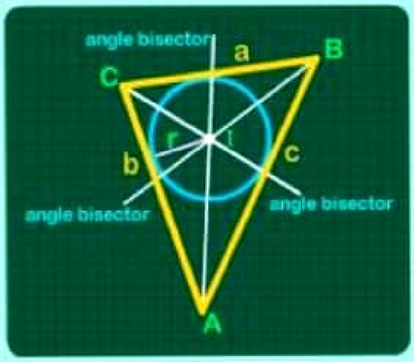
PROPERTIES OF TRIANGLES AND CIRCLES CONNECTED WITH THEM Part II



$R \rightarrow$ Circumradius of $\triangle$ ABC.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \;;\; \Delta = \frac{abc}{4R}$$

$$a\cos B\cos C + b\cos C\cos A + c\cos A\cos B = \frac{\Delta}{R}$$

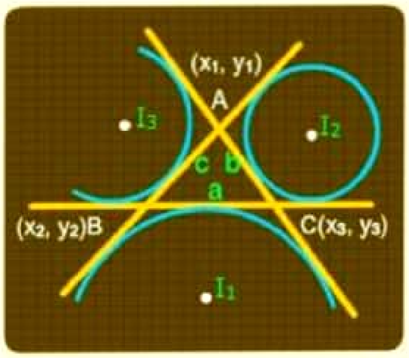


$r \rightarrow$ Inradius of $\triangle$ ABC.

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

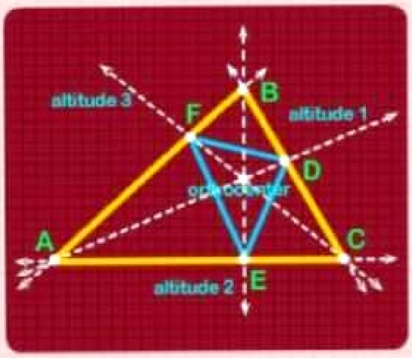
$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} \;;\; r = \frac{\Delta}{s} \;;\; s = \frac{a+b+c}{2}$$

$r_1, r_2, r_3 \rightarrow$ are radii of excircles of $\triangle$ ABC.



$$r_1 = \frac{\Delta}{s-a};\quad r_2 = \frac{\Delta}{s-b};\quad r_3 = \frac{\Delta}{s-c};\quad s = \frac{a+b+c}{2}$$

$$r_1 = s\tan\frac{A}{2};\quad r_2 = s\tan\frac{b}{2};\quad r_3 = s\tan\frac{c}{2}$$



ORTHOCENTRE AND PEDAL TRIANGLE

ORTHOCENTRE - Point of intersection of 3 altitudes.

PEDAL - Pedal triangle is formed by joining the feet of altitudes.

Area of $\boxed{\Delta DEF = 4\Delta \cos A \cos B \cos C}$

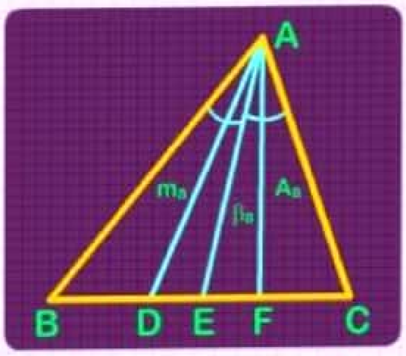
where Δ is the area of triangle ABC.

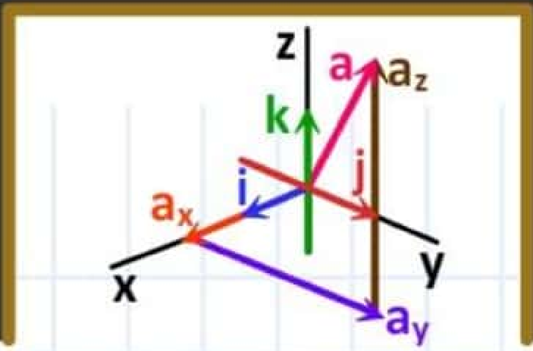
LENGTH OF ANGLE BISECTOR MEDIAN & ALTITUDE

Length of an angle bisector from the angle A = $\boxed{\beta_a = \dfrac{2bc\cos A/2}{b+c}}$

Length of median from the angle A = $\boxed{m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$

Length of altitude from the angle A = $\boxed{A_a = \dfrac{2\Delta}{a}}$ $\Delta \rightarrow$ Area of triangle ABC.





Part I

VECTOR

A quantity having both magnitude & direction

TYPES OF VECTORS

ZERO VECTOR

$\vec{a}$ is zero vector iff $|\vec{a}| = 0$

No Particular direction

UNIT VECTOR

$\vec{a}$ is unit vector iff $|\vec{a}| = 1$

Denote $\rightarrow \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

EQUAL VECTORS

$\vec{a}$ & $\vec{b}$ are equal vectors iff

$|\vec{a}| = |\vec{b}|$ & $\hat{a} = \hat{b}$

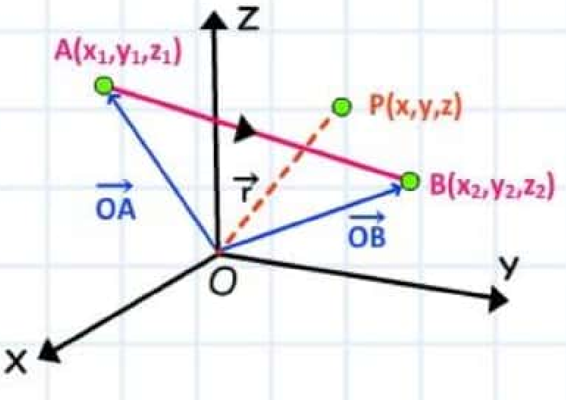
COLLINEAR VECTORS

$\vec{a}$ & $\vec{b}$ are collines vector iff $\vec{a} \parallel \vec{b}$

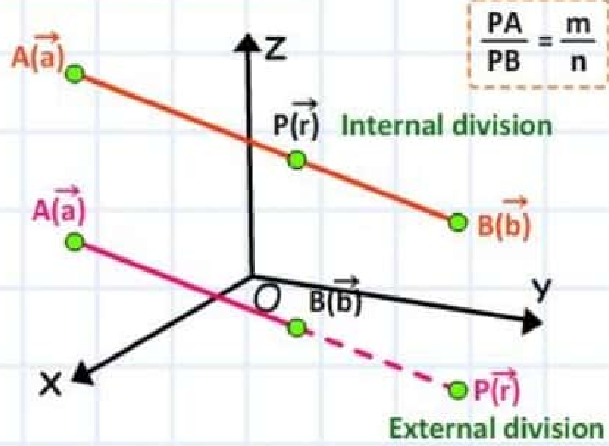
COPLANER VECTORS

vector parallel to or lying on the same plane.

POSITION VECTOR AND SECTION FORMULA



$\dfrac{PA}{PB} = \dfrac{m}{n}$



Position Vector

Postition Vector of pt. P is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Vector joining two points A & B

$\vec{AB} = \vec{OB} - \vec{OA}$

$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Internal Section

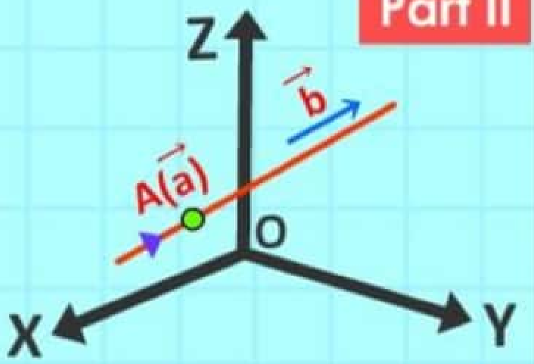
$$\vec{r} = \frac{(n\vec{a}) + (m\vec{b})}{m+n} \quad ; m + n \neq 0$$

External Section

$$\vec{r} = \frac{(m\vec{b}) - (n\vec{a})}{m-n} \quad ; m - n \neq 0$$

Part II

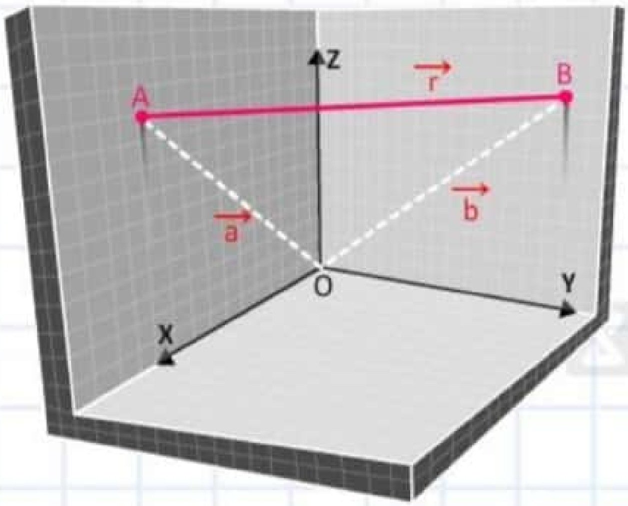
VECTOR LINES



Lines : For a Vector equation of a line one needs

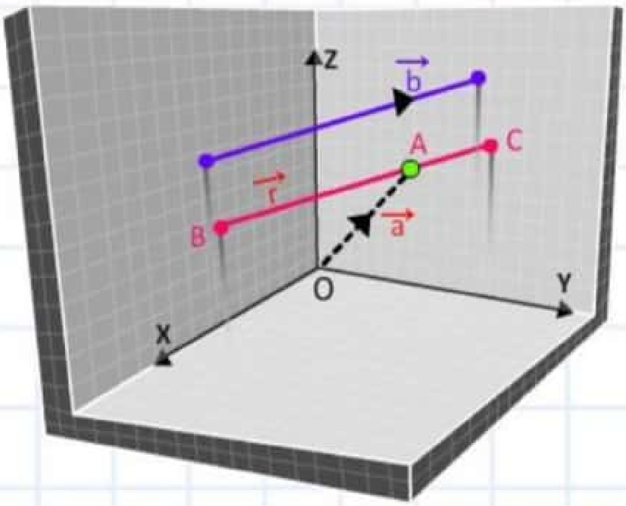
- A Point on line, let $\vec{a}$.
- A Parallel vector to the line, let $\vec{b}$.

When Two Points on Line are Known



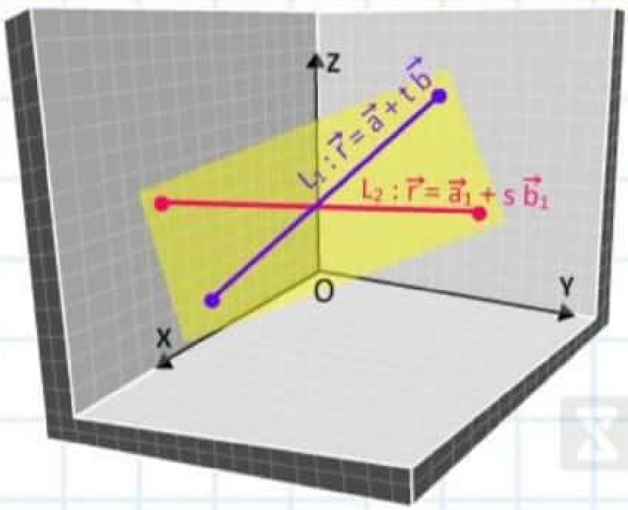
Equation of the Line AB $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

Parallel to $\vec{b}$ & Passes Through Point A



Equation of the Line BC $\vec{r} = \vec{a} + t\,\vec{b}$

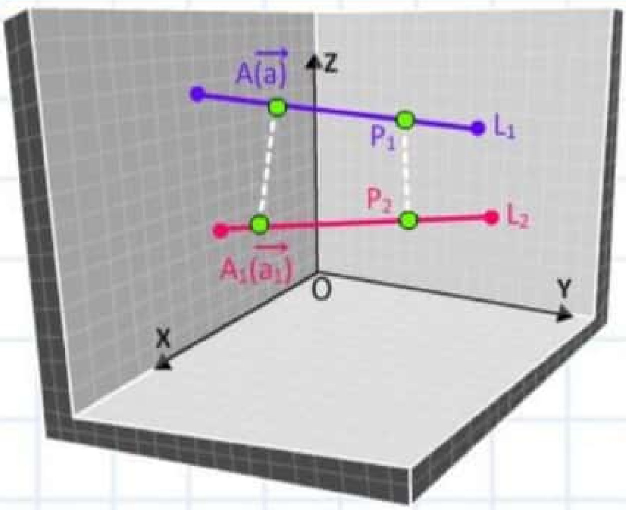
Condition for Intersection of Two Straight Lines



Condition of Intersection

- Line L_1 & L_2 will be coplaner
- Required Condition $[\vec{b}\,\vec{b}_1\,(\vec{a} - \vec{a}_1)] = 0$

Shortest distance between two Skew Lines

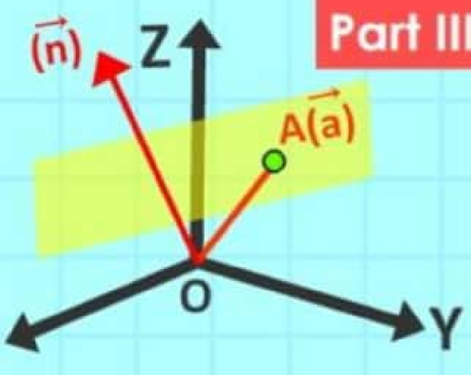


Shortest distance is measure along their common normal

$$\text{Shortest distance } P_1 P_2 = \frac{(\vec{b} \times \vec{b}_1) \cdot (\vec{a} - \vec{a}_1)}{|\vec{b} \times \vec{b}_1|}$$

Part III

PLANE OF VECTORS

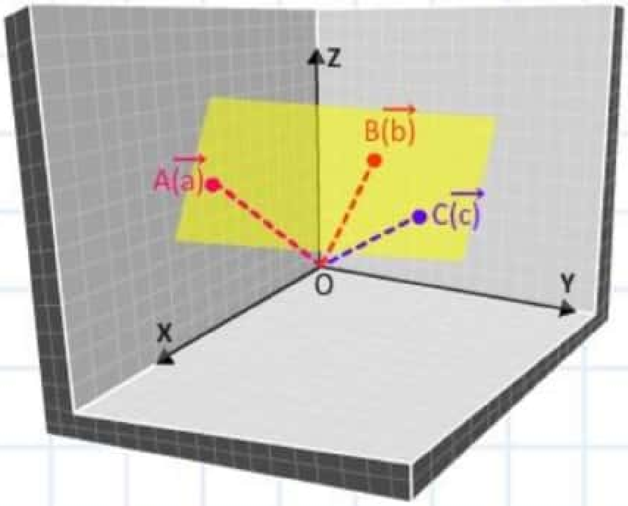


Planes : For Equation of a Plane, one needs

- A Point on Plane, let $\vec{a}$.
- A Perpendicular vector to the Plane, let $\vec{n}$.

Then Equation of Plane $(\vec{r}-\vec{a}).\vec{n}=0$

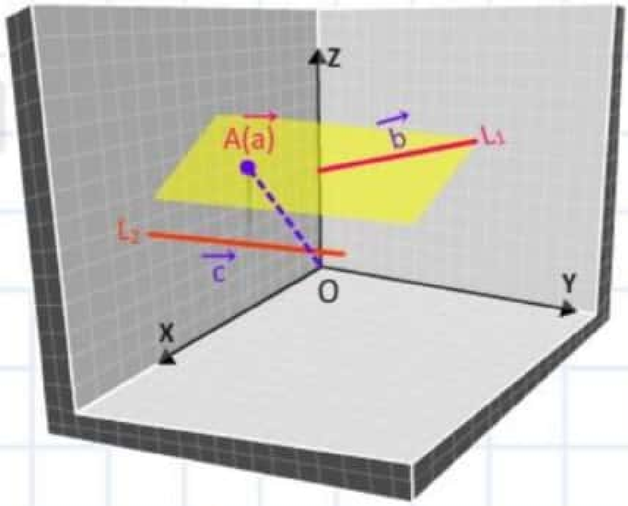
Plane Passing Through Three Points A,B,C



Equation of the Plane

$$(\vec{r}-\vec{a}).[(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})]=0$$

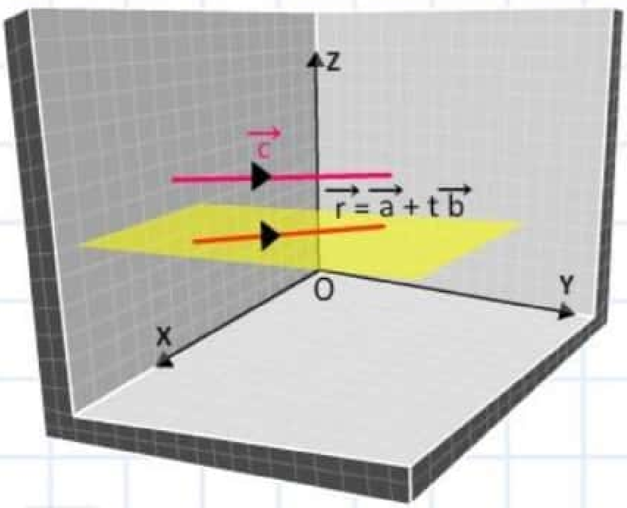
Plane Passing Through a Point and Parallel to two lines



Equation of the Plane

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

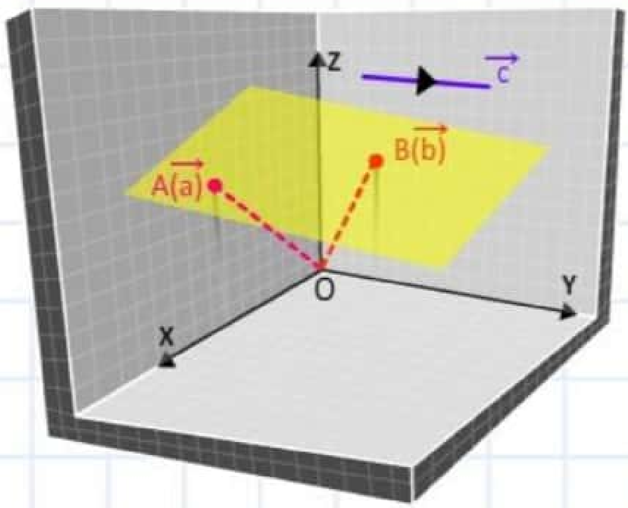
Plane Containing a Line and Parallel to another line



Equation of the Plane

$$\vec{r}.(\vec{b}\times\vec{c})=[\vec{a}\ \vec{b}\ \vec{c}]$$

Plane Passing Through Two Point and Parallel to a Lines



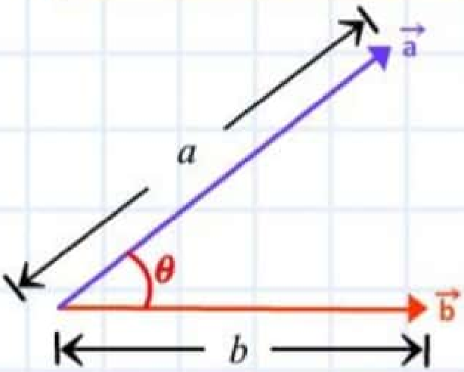
Equation of the Plane

$$\vec{r}.[(\vec{b}-\vec{a})\times\vec{c}]=[\vec{a}\,(\vec{b}-\vec{a})\,\vec{c}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

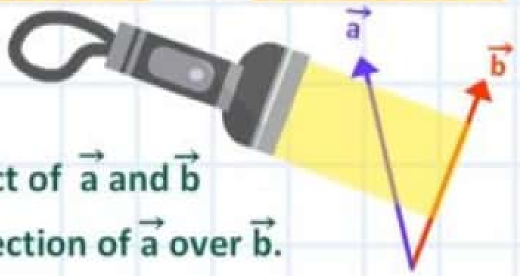
PRODUCT OF VECTORS

Part IV

Scalar or Dot Product



$\vec{a}\cdot\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$ Or $\vec{a}\cdot\vec{b} = a\,b\cos\theta$



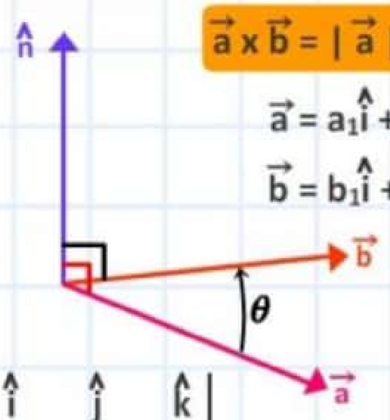
- Dot product of $\vec{a}$ and $\vec{b}$ is the projection of $\vec{a}$ over $\vec{b}$.

Vector or Cross Product

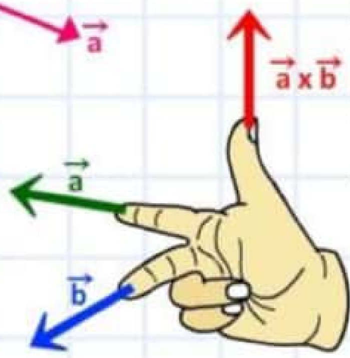
$\vec{a}\times\vec{b} = |\vec{a}|\ |\vec{b}|\ \sin\theta\,\hat{n}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

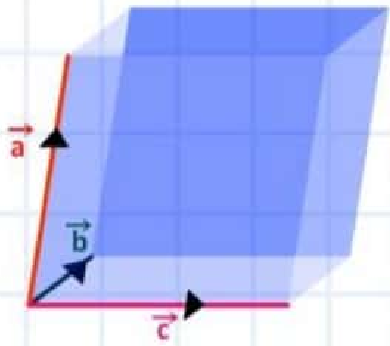


$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



- Direction of resultant vector can be found using Right Hand Rule

Scalar Triple Product



⇐ Scalar triple product represents the volume of a parallelepiped

SCALAR TRIPLE PRODUCT

$\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$ $\qquad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

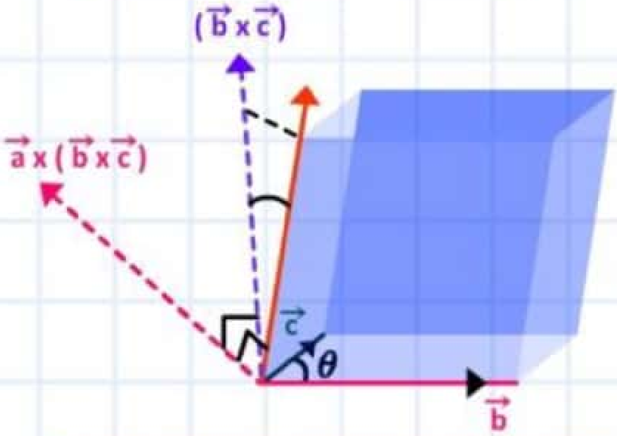
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ $\qquad \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\vec{a}\vec{b}\vec{c}]$$

NOTE

(i) $\vec{a}, \vec{b}$ & $\vec{c}$ are Coplanar iff $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Vector Triple Product



$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

NOTE

(i) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot\vec{b}\times(\vec{c}\times\vec{d})$

$= \vec{a}\cdot\{(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}\}$

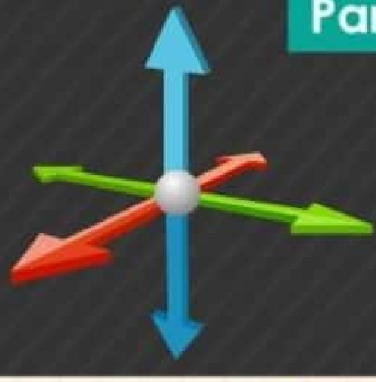
$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

(ii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

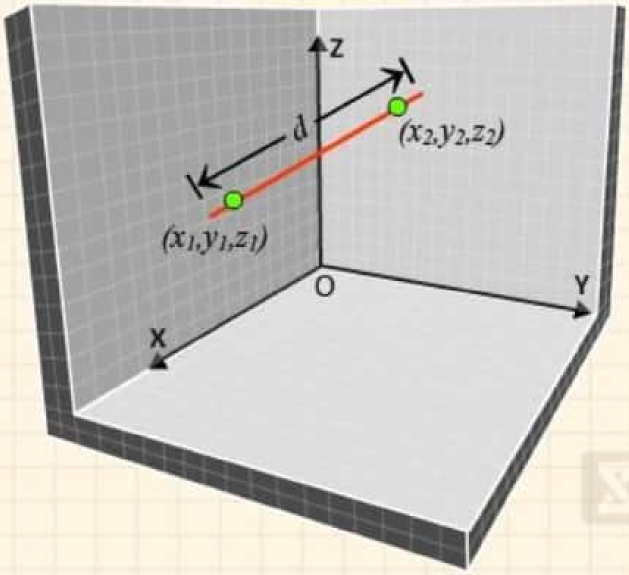
Or $= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

Part I

3D COORDINATE GEOMETRY

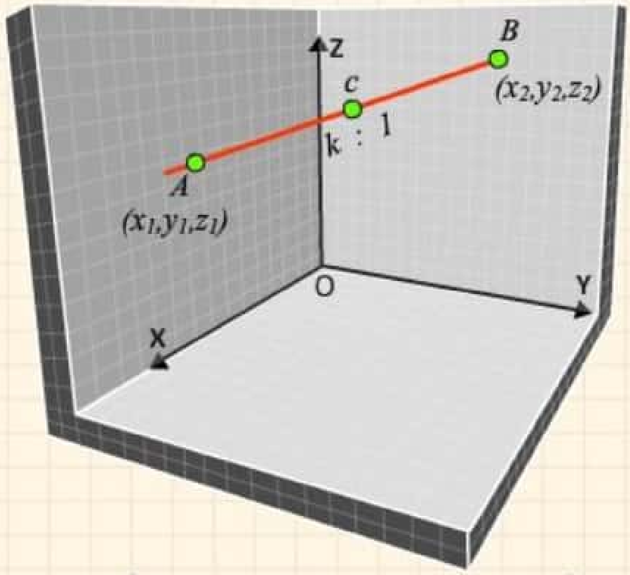


Distance between two points



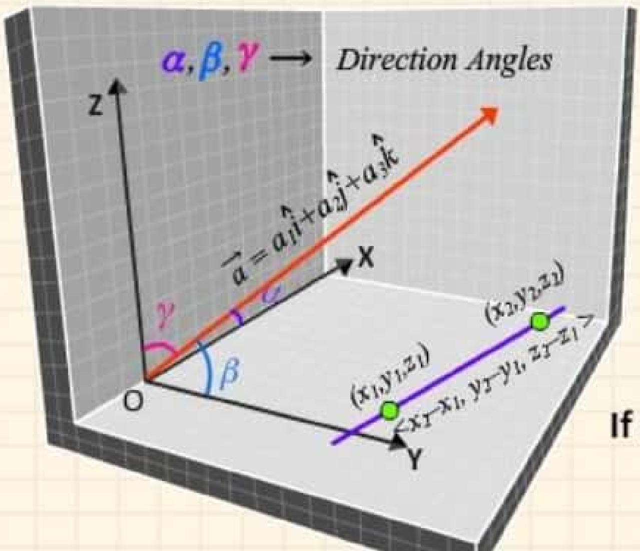
$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Section Formula



$$C\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$$

Direction Cosines & Ratios



Direction Cosines

$$\cos\alpha = \frac{\vec{a_1}}{|\vec{a}|}, \quad \cos\beta = \frac{\vec{a_2}}{|\vec{a}|}, \quad \cos\gamma = \frac{\vec{a_3}}{|\vec{a}|}$$

Note : $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos\alpha \equiv \ell$; $\cos\beta \equiv m$; $\cos\gamma \equiv n$

So $\ell^2 + m^2 + n^2 = 1$

Direction Ratios

If a,b and c are direction ratios then $a \propto \ell$; $b \propto m$; $c \propto n$

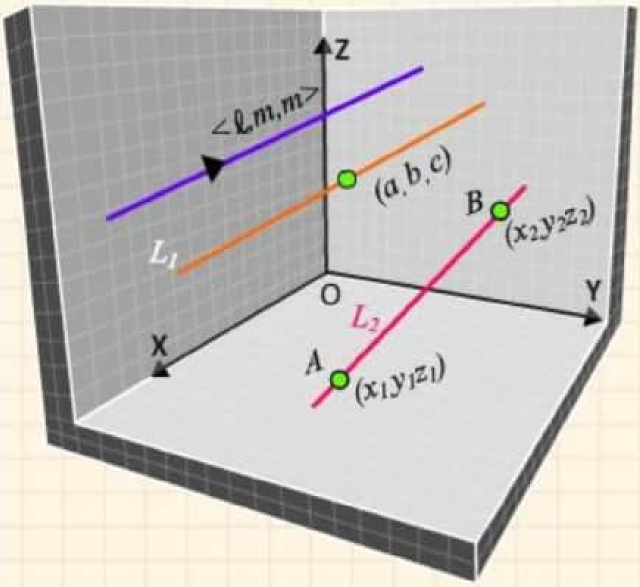
$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \lambda \text{ (say)}$$

$$\therefore (a^2+b^2+C^2)\ \lambda^2 = \ell_2 + m_2 + n_2 = 1$$

Therefore, $\ell = \pm \frac{a}{\sqrt{(a^2+b^2+c^2)}}$; $m = \pm \frac{b}{\sqrt{(a^2+b^2+c^2)}}$; $n = \pm \frac{c}{\sqrt{(a^2+b^2+c^2)}}$

Part II

THREE DIMENSIONAL LINES



Line passing through point (a, b, c) parallel to line having direction cosines ℓ,m,n is

$$L_1 : \frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n}$$

Equation of a line passing through two points A & B

$$L_2 : \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Angle between two lines

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

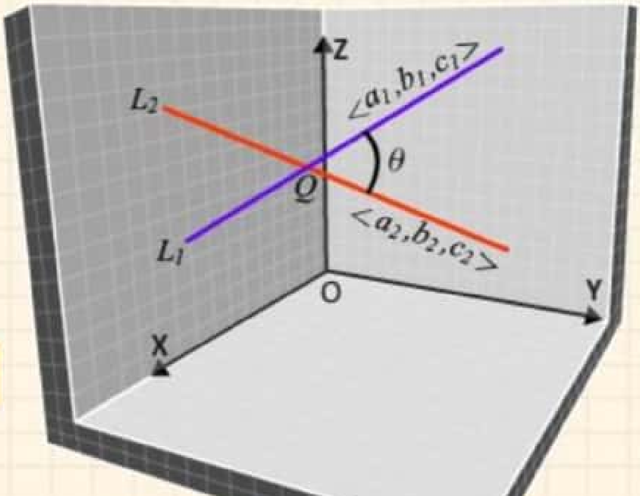
$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

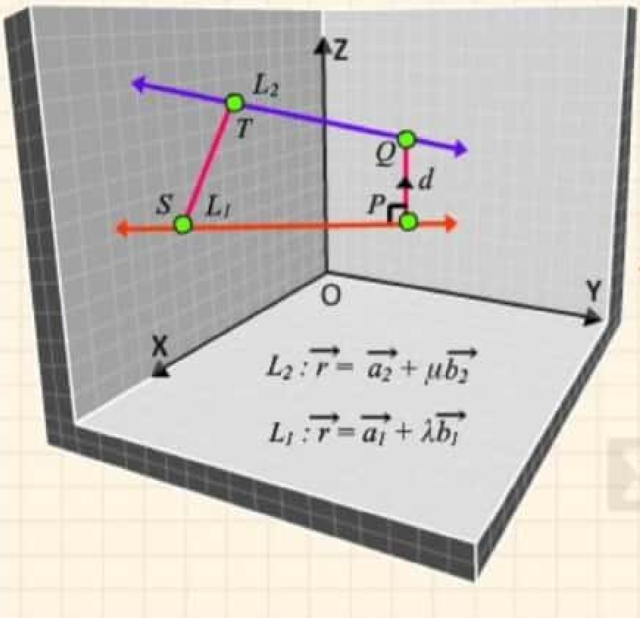
If L_1 and L_2 are Perpendicular ($\theta = 90°$)

$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ Or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

If L_1 and L_2 are Parallel

$\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ Or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$





Distance between two skew lines

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2})\ (\vec{a_2}-\vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

Cartesian form

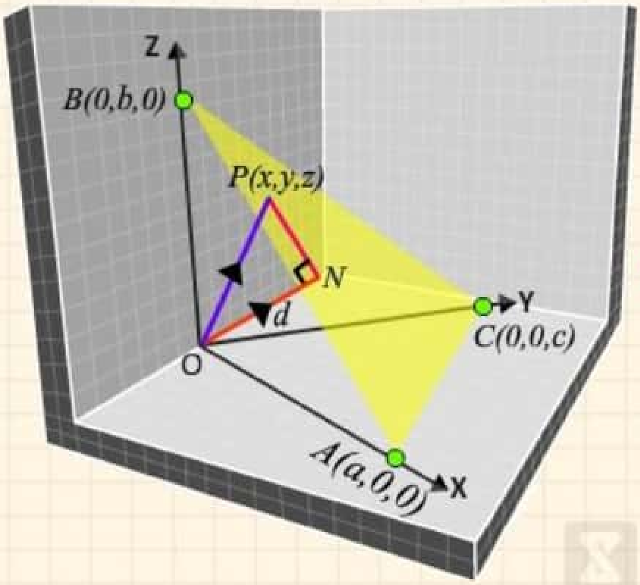
$$\text{Line } L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$\text{Line } L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

$$d = \left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}\right|$$

Part III

THREE DIMENSIONAL PLANES



Equation of a plane in Normal form

Equation : $\vec{r}\cdot\hat{n} = d$

unit normal vector along $\overrightarrow{ON}$

Perpendicular distance of plane from 'O'

Cartesian form

Equation : $\ell x + my + nz = d$

Here ℓ,m,n are the direction cosines of $\hat{n}$

Intercept form of the equation of a plane ABC

Equation : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

a, b and c are the direction ratios.

Equation of a plane perpendicular to a given vector and passing through a given point

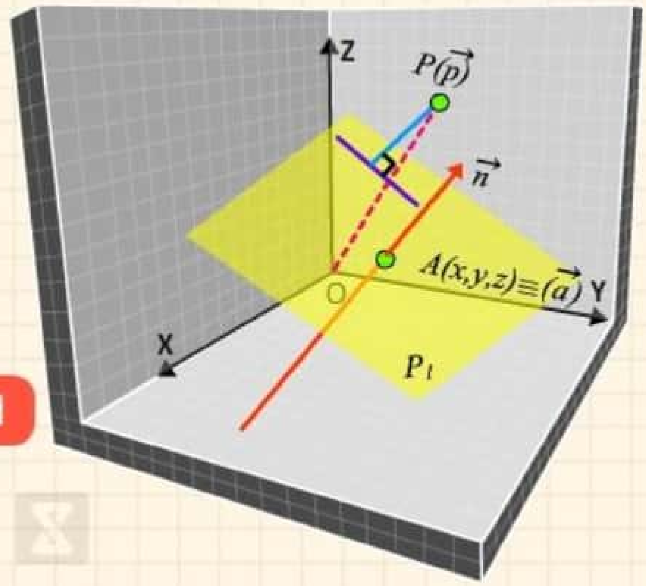
Equation : Cartesian Form

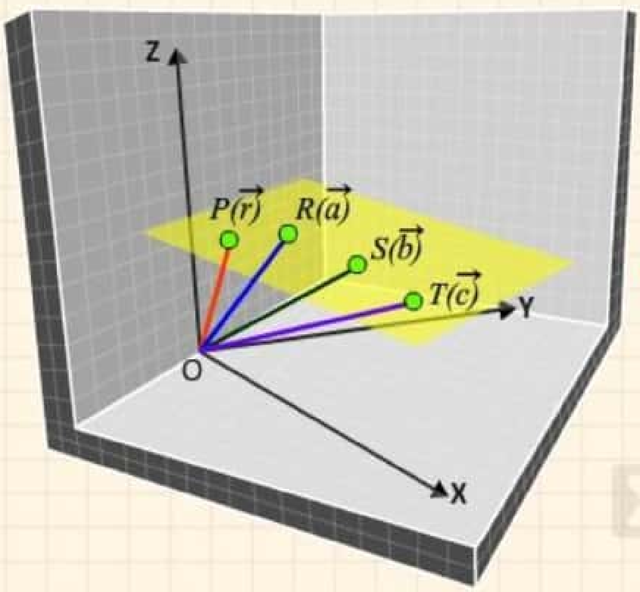
$(\vec{r}-\vec{a})\cdot\vec{n} = 0$ if $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

Plane : $a(x-x_1)+b(y-y_1)+c(z-z_1) = 0$

The Distance of a Point P From Plane $P_1 : \vec{r}\cdot\hat{n} = d$

Perpendicular distance $= \left|\frac{\vec{p}\cdot\hat{n} - d}{|\hat{n}|}\right|$





Equation of a plane passing through three non - collinear points

Equation : $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = 0$

Cartesian Form

If R $=(x_1,y_1,z_1)$, S $=(x_2,y_2,z_2)$ & T $=(x_3,y_3,z_3)$

Equation : $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

Part IV

Plane P passing through the Intersection of two given planes P_1 & P_2

$P_1 : \vec{r}.\hat{n}_1 = d_1$ Or $P_2 : \vec{r}.\hat{n}_2 = d_2$

Equation of Plane P

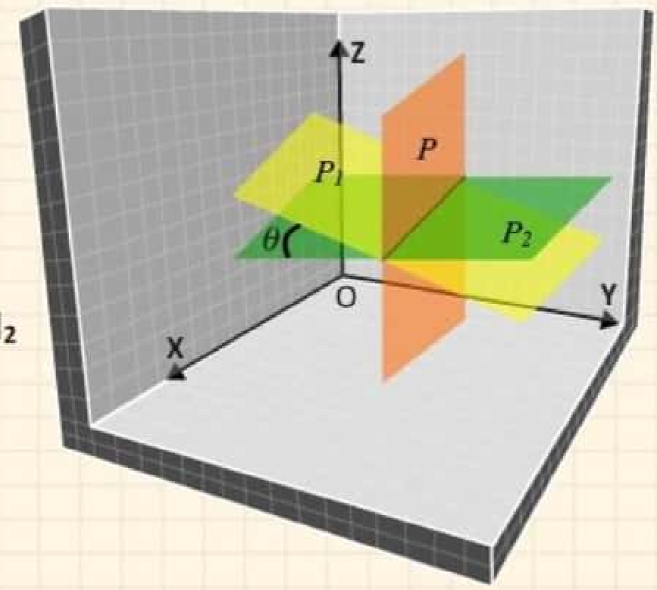
$$\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2 \; ; \quad \lambda : \text{Constant}$$

Cartesian form

$P_1 : a_1x+b_1y+c_1z = d_1$ Or $P_2 : a_2x+b_2y+c_2z = d_2$

Equation of plane P :

$$(a_1x+b_1y+c_1z-d_1) + \lambda(a_2x+b_2y+c_2z-d_2) = 0$$



If Angle between two planes P_1 & P_2 is θ

$$\cos\theta = \left|\frac{\vec{n_1}\,.\,\vec{n_2}}{|\vec{n_1}|.|\vec{n_2}|}\right|$$

Cartesian form

If $P_1 : a_1x+b_1y+c_1z+d_1 = 0$, $P_2 : a_2x+b_2y+c_2z+d_2 = 0$

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\;\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

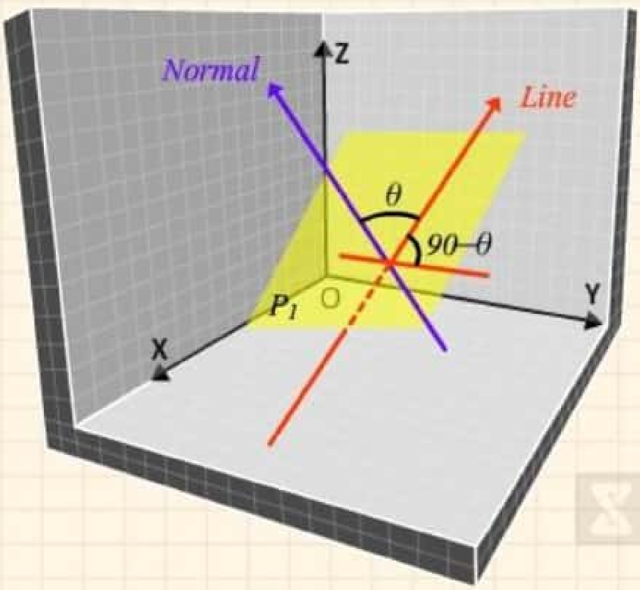
- If $P_1 \perp P_2 \Rightarrow \theta = 90^\circ$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- If $P_1 || P_2 \Rightarrow \theta = 0^\circ$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

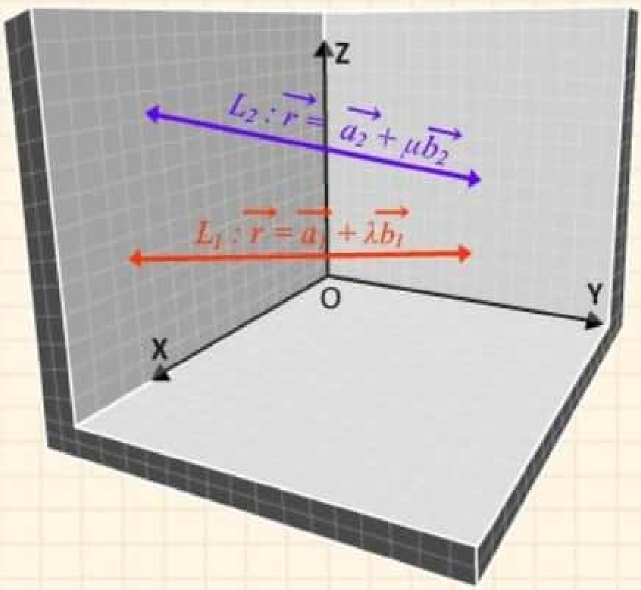
Angle Between a Line and a Plane



Line : $\vec{r} = \vec{a} + \lambda\vec{b}$ Plane : $\vec{r}.\vec{n} = d$

Therefore $$\cos\theta = \left|\frac{\vec{b}\,.\,\vec{n}}{|\vec{b}|.|\vec{n}|}\right|$$

Coplanarity of Two Lines



If L_1 & L_2 are coplaner, then

$$(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = 0$$

COMPLEX NUMBER

Part I

Complex Numbers

A number $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$; x = Real part or Re(z); y = Imaginary part or Im(z)

Magnitude

$|z| = \sqrt{x^2 + y^2}$

$|z| = |\bar{z}|$

Argument

amp (z) = arg (z) = $\theta = \tan^{-1}\frac{y}{x}$

General Argument : $2n\pi + \theta$, $n \in \mathbb{N}$

Principal Argument : $-\pi < \theta \le \pi$

Least Positive Argument : $0 < \theta \le 2\pi$

Complex Conjugate

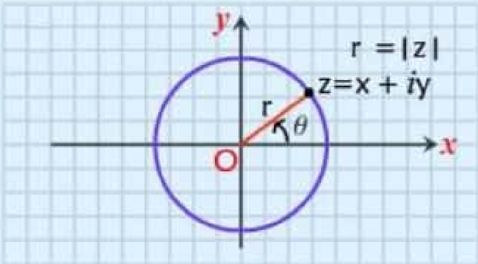
If $z = x + iy$

then the conjugate of 'z' is

$\boxed{\bar{z} = x - iy}$

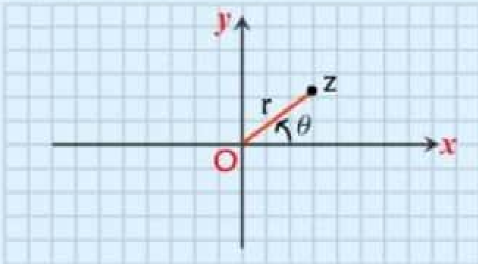
Representation

Polar Representation



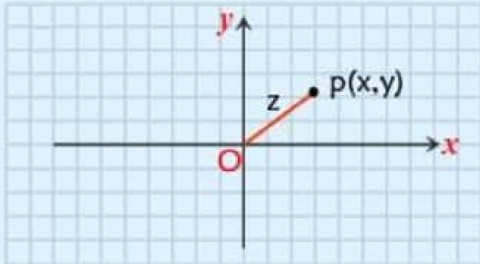
$x = r\cos\theta$, $y = r\sin\theta$

Exponential Form



$z = re^{i\theta}$ (where $e^{i\theta} = \cos\theta + i\sin\theta$)

Vector Representation



$z = x + iy$ may be considered as a position vector of point P.

Properties of argument of a Complex Number

If z, z_1 and z_2 are complex numbers, then

1. arg(any real positive number) = 0
2. arg(any real negative number) = π
3. $\arg(z - \bar{z}) = \pm\frac{\pi}{2}$
4. $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$
5. $\arg(z_1.\bar{z}_2) = \arg(z_1) - \arg(z_2)$
6. $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$
7. $\arg(\bar{z}) = -\arg(z) = \arg(1/z)$
8. $\arg(-z) = \arg(z) \pm \pi$
9. $\arg(z^n) = n\arg(z)$
10. $\arg(z) + \arg(\bar{z}) = 0$
11. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$
12. $|z_1 - z_2| = |z_1 + z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$
13. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$
14. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
15. $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$
16. $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$
17. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$
18. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ or $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2)$

COMPLEX NUMBER

Part II

Properties of Complex Conjugate

If $z = a + ib \Rightarrow \bar{z} = a - ib$

- $\overline{(\bar{z})} = z$
- $z + \bar{z} = 2a = 2\,\mathrm{Re}(z) =$ purely real
- $z - \bar{z} = 2ib = 2i\,\mathrm{Im}(z) =$ purely imaginary
- $z\bar{z} = a^2 + b^2 = |z|^2 = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$
- $z + \bar{z} = 0$ or $z = -\bar{z} \Rightarrow z = 0$ or z is purely imaginary
- $z = \bar{z} \Rightarrow z$ is purely real

Properties of Modulus

- $z\bar{z} = |z|^2$
- $z^{-1} = \dfrac{\bar{z}}{|z|^2}$
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(z_1\bar{z}_2)$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$

Square roots of a Complex Number

The square root of $z = a + ib$ is $\sqrt{a+ib} = \pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b > 0$ and $\pm\left[\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b < 0$

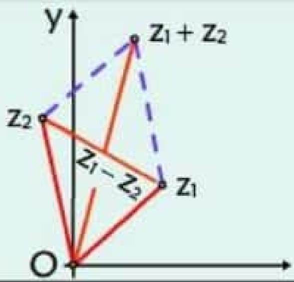
Inequalities

Triangle Inequalities

(1) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ (2) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

Parallelogram Identity

(1) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$

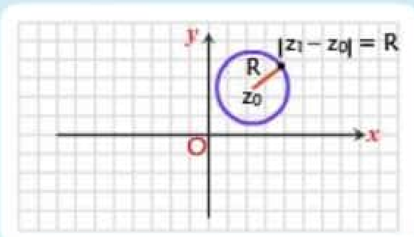


Points to Remember

- If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$
- If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$
- If z_1, z_2, z_3 are affixes of the Points A, B and C in the Argand plane, then

  (a) $\angle BAC = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$ (b) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$

- The equation of a circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$



- If a, b are positive real numbers then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$

Integral powers of iota

$i = \sqrt{-1}$ so $i^2 = -1$; $i^3 = -i$ and $i^4 = 1$ $i^{4n+3} = -i$; i^{4n} or $i^{4n+4} = 1$

Hence $i^{4n+1} = i$; $i^{4n+2} = -1$

COMPLEX THEOREM

Statement

(i) if $n \in Z$ (the set of integers), then $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(ii) if $n \in Q$ (the set of rational number), then $\cos(n\theta) + i\sin(n\theta)$ one of the values of $(\cos\theta + i\sin\theta)^n$.

Roots of Unity

Let $z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z.

Then by De Moivre's theorem $r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$.

Cube Roots of unity

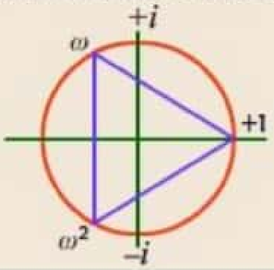
$z = (1)^{1/3}$

Roots : $1, \omega, \omega^2$, where $\omega = e^{i\frac{2\pi}{3}}$

Properties of Cube Roots of Unity

- $1 + \omega^r + \omega^{2r} = 0 \qquad r \neq 3n$
- $\omega = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

  $\omega^2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.



n^{th} Roots of unity

$z = (1)^{1/n}$

Roots : $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

$\alpha_r = e^{i\frac{2\pi r}{n}} = \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n}$

Properties of n^{th} Roots of Unity

- They are in G.P. with common ratio $e^{i\frac{2\pi}{n}}$
- $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$ if $p \neq kn$
- $1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p = n$ if $p = kn$
- $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$
- $(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd
- $(1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots \alpha_{n-1}) = (-1)^{n-1}$
- $(\omega - \alpha_1)(\omega - \alpha_2) \ldots (\omega - \alpha_{n-1}) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+\omega & \text{if } n = 3k+2 \end{cases}$

Point to Remember

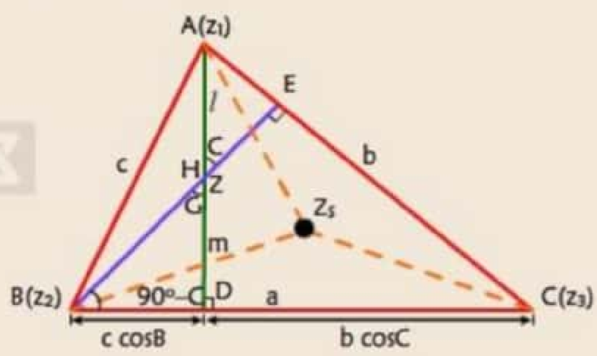
Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane

(a) Centroid 'G' $= \dfrac{z_1 + z_2 + z_3}{3}$

(b) Incentre 'I' $= \dfrac{a z_1 + b z_2 + c z_3}{a + b + c}$

(c) Orthocentre 'Z_H' $= \dfrac{Z_1 \tan A + Z_2 \tan B + Z_3 \tan C}{\tan A + \tan B + \tan C}$

(D) Circumcentre 'Z_S' $= \dfrac{Z_1(\sin 2A) + Z_2(\sin 2B) + Z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$



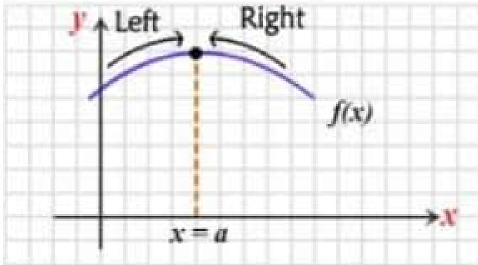
CONTINUITY

Part I

Mathematical Definition of Point Continuity

A function f(x) is said to be continuous at x = a iff,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a) = \text{finite quantity}$$



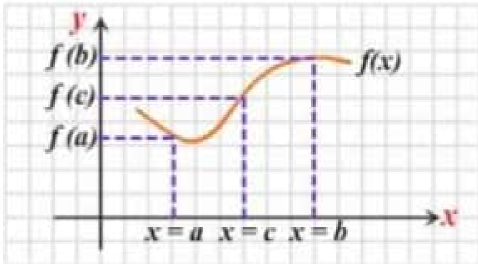
One sided Continuity

- continuous at x=a from left if $\lim_{x\to a^-} f(x) = f(a)$
- continuous at x=a from right if $\lim_{x\to a^+} f(x) = f(a)$

Continuity In An Interval

In an Open Interval (a,b) if

If it is continuous at each and every point $c \in (a,b)$.



In a Closed Interval [a,b] if

(a) f(x) is continuous in (a,b).

(b) f(x) is right continuous at x=a. i.e. $\lim_{x\to a^+} f(x) = f(a) =$ a finite quantity

(c) f(x) is left continuous at x=b. i.e. $\lim_{x\to b^-} f(x) = f(b) =$ a finite quantity

Theorems of Continuity

Theorem - 1 If 'f' & 'g' are continuous at $x = a$, then $f \pm g$, $f.g$ will also be continuous at $x = a$. And $\frac{f}{g}$ will also be continuous provided $g(a) \neq 0$

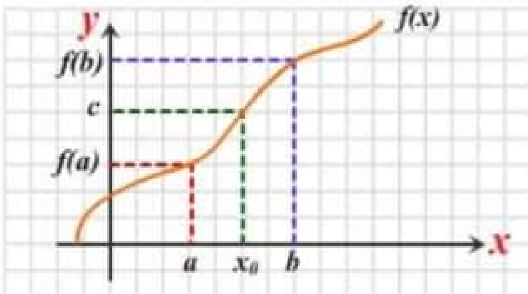
Theorem - 2 If 'f' is continuous at $x = a$ & 'g' is discontinuous at $x = a$ then $f \pm g$, must be discontinuous at $x = a$. However nothing definite can be said about f.g or f/g.

Theorem - 3 If f(x) & g(x) are discontinuous at $x = a$ then nothing definite can be said about $f \pm g$, f.g or f/g

Theorem - 4

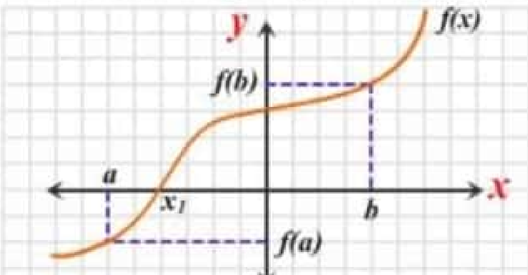
Intermediate value theorem

If 'f' is continuous on [a, b] & $f(a) \neq f(b)$ then for any value $c \in (f(a), f(b))$ there exists at least one number $x_0 \in (a, b)$ such that $f(x_0) = c$

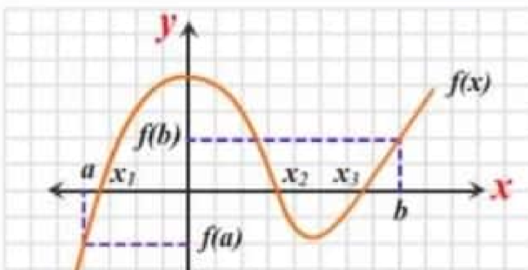


Alternatively

f(x) is continuous in [a,b] and f(a) & (b) have opposite signs then the equation f(x) = 0 has at least one root in (a,b).



one root



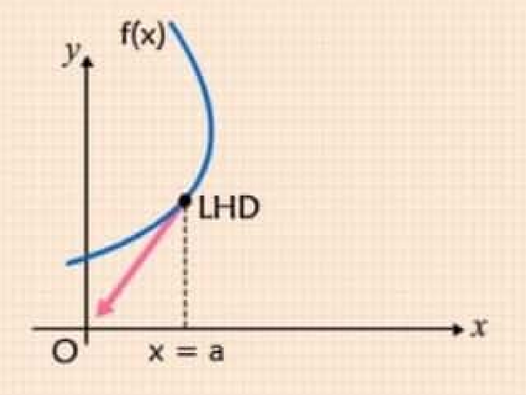
More than one root

Differentiability

Part I

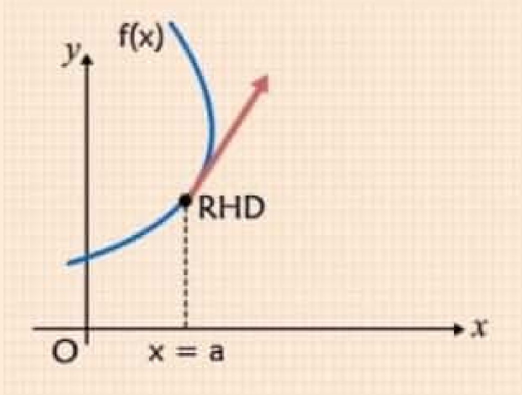
Differentiability of f(x) at x = a geometrically means that a unique tangent with finite slope can be drawn at x = a.

Left Hand Derivative (LHD)



$$f'(a^-) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} \; ; (h > 0)$$

Right Hand Derivative (RHD)



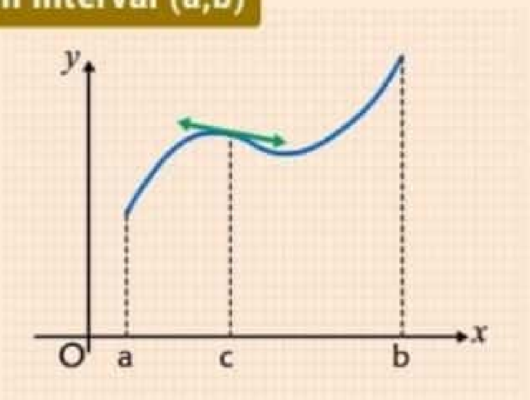
$$f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \; ; (h > 0)$$

Differentiable if $f'(a^-) = f'(a^+)$

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

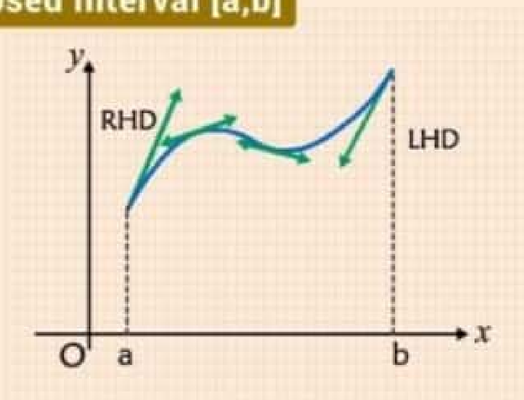
Differentiability Over an Interval

Open interval (a,b)



A function f(x) is said to be differentiable over (a, b) if it is differentiable at each and every point c ∈ (a, b).

Closed interval [a,b]



A function f(x) is said to be differentiable over [a, b] if:

- It is differentiable in (a, b).
- It is right differentiable at x = a.
- It is left differentiable at x = b.

DERIVATIVE

Part I

Derivative by First Principle

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

$\longrightarrow$ Instantaneous rate of change of y.w.r.t.x.

Fundamental Rules for Differentiation

1	PRODUCT RULE	$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
2	QUOTIENT RULE	$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x).\frac{d}{dx}\{f(x)\} - f(x).\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
3	CHAIN RULE	if $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Derivative of Standard Functions

For function $y = f(x)$, the derivative of the function is $\frac{dy}{dx}$

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert>1$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert>1$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, x \in R$
$\frac{d}{dx}(\sec x) = \sec x.\tan x$	$\frac{d}{dx}(x^n) = n.x^{n-1}; x \in R, n \in R, x > 0$
$\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x.\cot x$	$\frac{d}{dx}(a^x) = a^x.\ln a; a > 0, a \neq 1$
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$	$\frac{d}{dx}(e^{ax}) = ae^{ax}$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1<x<1$	$\frac{d}{dx}(\log_a \lvert x\rvert) = \frac{1}{x}\log_a e$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1<x<1$	$\frac{d}{dx}(\ln \lvert x\rvert) = \frac{1}{x}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, x \in R$	$\frac{d}{dx}(\text{constant}) = 0$

METHOD OF DIFFERENTIATION Part II

L' Hopital Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

f(x) & g(x) are differentiable at x = a

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \text{...till the indeterminate form vanishes.}$$

Guillaume de L' Hopital

Logarithmic Differentiation

$$\text{If } y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x) \ln [f(x)]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}\left\{g(x).\ln[f(x)]\right\} \qquad \Rightarrow \frac{dy}{dx} = [f(x)]^{g(x)}.\left\{\frac{d}{dx}[g(x) \ln f(x)]\right\}$$

Parametric Differentiation

$$\text{If } x = f(t) \ \& \ y = g(t) \text{ then } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Differentiation of Inverse Function

y = f(x) and x = g(y) are inverse function of each other $\frac{dx}{dy} = \frac{1}{dy/dx}$ or $g'(y) = \frac{1}{f'(x)}$

Derivative of a Determinant

$$\text{If } F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

where f,g,h,l,m,n,u,v,w are differentiable function of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

INDEFINITE INTEGRAL

Part I

Indefinite Integral is the opposite of derivative

If $\frac{dy}{dx} = f(x)$ ➡ $y = \int f(x)\, dx$ — with respect to 'x'

Integral Symbol — Integrand: Function we want to integrate

METHODS OF INTEGRATION

1 SUBSTITUTION

$I = \int f(g(x))\, g'(x)\, dx$ ➡ Let $g(x) = u$ ➡ then $\frac{dg(x)}{dx} = g'(x) = \frac{du}{dx}$

$I = \int f\,(g(x))\; g'(x)\; dx$

$I = \int f\,(u)\; du$

Then we can integrate f(u), and finish by putting g(x) back as u.

2 BY PARTS (PRODUCT RULE)

$I = \int f(x).g(x)dx$ ➡ $I = f(x)\int g(x)dx - \int f'(x).\left(\int g(x)dx\right)dx$

A helpful rule of thumb is ILATE. Choose f(x) based on which of these comes first:

I L A T E

- Inverse Trigonometric Functions
- Logarithmic Functions
- Algebraic Functions
- Trigonometric Functions
- Exponential Functions

3 MISCELLANEOUS

Euler's substitutions for integration $I = \int R(x, \sqrt{ax^2+bx+c})\, dx$

Substitutions

1. $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}x$; $a > 0$
2. $\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = t\,(x-x_1) = t(x-x_2)$,
3. $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$; $c > 0$.

Part II

4 PARTIAL (FRACTION)

Expressing complicated algebraic fractions into 'Partial Fractions'.

Partial Fractions with 'Repeated Linear Factors' in the denominator.

Denominator containing	Expression	Form of Partial Fractions
Linear factor	$\dfrac{f(x)}{(x+a)(x+b)}$	$\dfrac{A}{x+a}+\dfrac{B}{x+b}$
Repeated linear factors	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{x+a}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
Quadratic term (which cannot be factored)	$\dfrac{f(x)}{(ax^2+bx+c)(gx+h)}$	$\dfrac{Ax+B}{ax^2+bx+c}+\dfrac{C}{gx+h}$

BASIC INTEGRALS & TRICKS

ALGABRAIC

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$n \neq -1,\ n \in R$$

$$\int \frac{dx}{ax+b} = \frac{\ln(ax+b)}{a} + C$$

$$\int a^{px+q}\,dx = \frac{a^{px+q}}{p.\ln a}\ ;\ a>0$$

TRIGONOMETRIC

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

MISCELLANEOUS

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$$

$$\int \frac{dx}{(x-a)\sqrt{(x-a)(b-x)}}$$

Put : $x = a\cos^2\theta + b\sin^2\theta$

$$\int \frac{dx}{\sqrt{(x-a)(x-b)}}$$

Put : $x = a\sec^2\theta - b\tan^2\theta$

$$\int \frac{dx}{(ax+b)\sqrt{px+q}}$$

Put $px+q=t^2$

DEFINITE INTEGRALS

Part I

A Definite Integral represents the exact area under the curve between points 'a' and 'b'.

$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of f(x) between the limits a and b. where $\frac{d}{dx}(F(x)) = f(x)$

DERIVATIVE OF ANTIDERIVATIVE (LEIBNITZ'S RULE)

If h(x) & g(x) are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].\ g'(x)$$

DEFINITE INTEGRAL AS LIMIT OF A SUM

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+\overline{n-1}h)]$$

$$= \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh) \text{ where } h = \frac{b-a}{n}$$

WALLI'S FORMULA & REDUCTION FORMULA

$$\int_0^{\pi/2} \sin^n x.\cos^m x dx = \frac{[(n-1)(n-3)...1 \text{ or } 2]\,[(m-1)(m-3)...1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2]}K$$

Where $K = \frac{\pi}{2}$ if both 'm' and 'n' are even ($m, n \in N$); otherwise $K = 1$

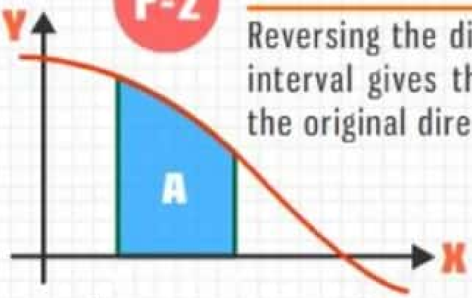
Part II

PROPERTIES OF DEFINITE INTEGRAL

P-1 **CHANGE OF VARIABLE:** $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$

P-2 **REVERSING THE INTERVAL**

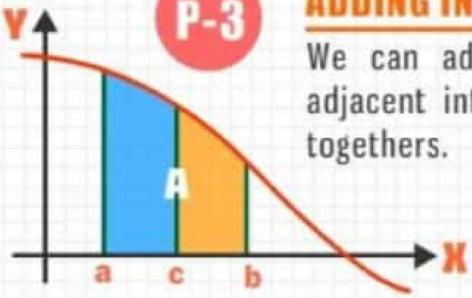
Reversing the direction of the interval gives the negative of the original direction.



$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

P-3 **ADDING INTERVAL**

We can add two adjacent intervals togethers.



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

P-4

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

P-5

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx, \text{ in particular } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

P-6

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx\ ; & \text{if } f(2a-x) = f(x) \\ 0\ ; & \text{if } f(2a-x) = -f(x) \end{cases}$$

P-7

$$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$$

where 'T' is the period of the function i.e. $f(x+T) = f(x)$

P-8

$$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx;$$ where $f(x)$ is periodic with period T & $n \in I$

Part I

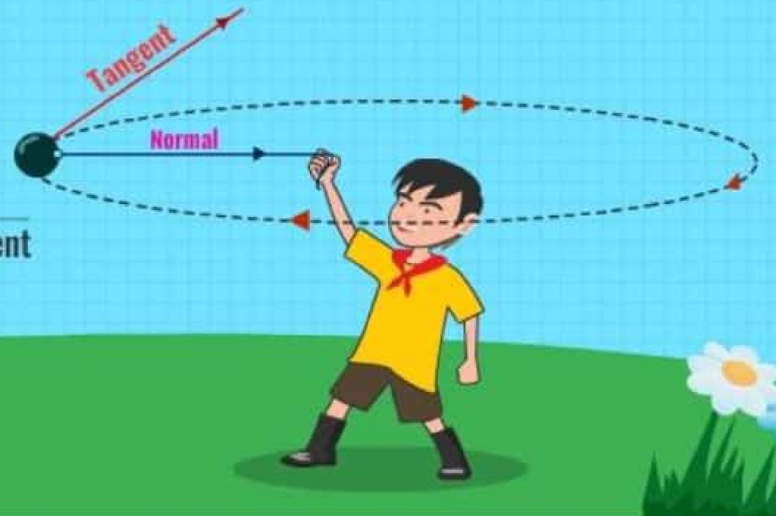
TANGENT & NORMAL

TANGENT

Tangent is a limiting case of a secant

NORMAL

A line that is perpendicular to a tangent line at the point of tangency.



CALCULATING TANGENT LINE & NORMAL LINE TO A CURVE

EQUATION OF TANGENT & ITS LENGTH

Equation: $y - y_1 = m_T\,(x - x_1)$

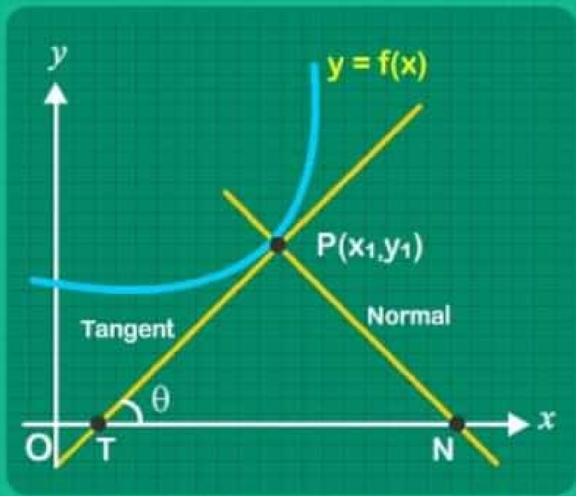
Length: $PT = \left| \dfrac{y_1\sqrt{1 + (m_T)^2}}{(m_T)} \right|$

$$m_T = \left(\frac{dy}{dx}\right)_{P(x_1, y_1)} = \tan\theta$$

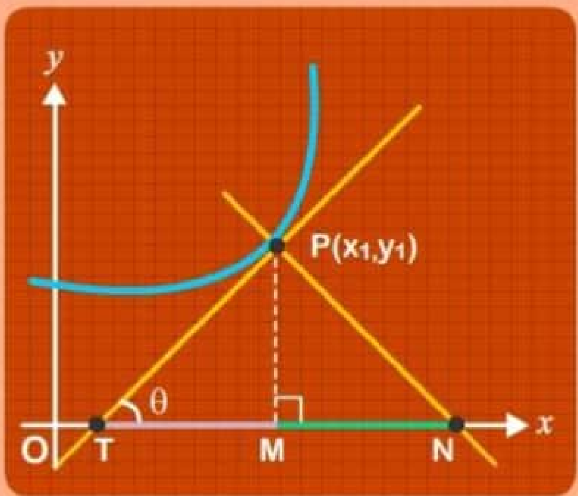
EQUATION OF NORMAL & ITS LENGTH

Equation: $y - y_1 = \dfrac{-1}{m_T}\,(x - x_1)$

Length: $PN = \left| y_1\sqrt{1 + (m_T)^2} \right|$



SUBTANGENT & SUBNORMAL



TM is the Subtangent and length of $TM = \left| \dfrac{y_1}{m_T} \right|$

MN is the Subnormal and length of $MN = \left| y_1\, m_T \right|$

ANGLE BETWEEN TWO INTERSECTING CURVES

Part II

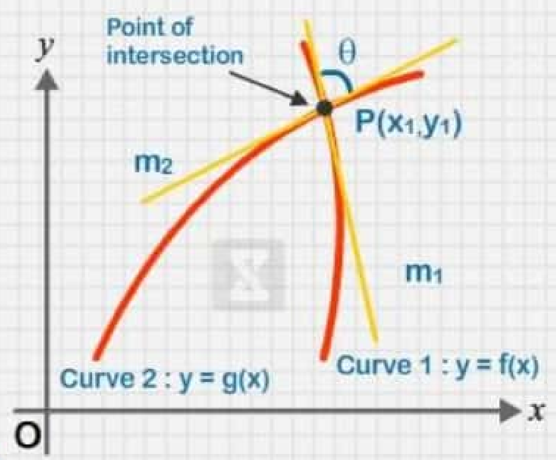
$m_1 \rightarrow$ slope to the curve 1 at point 'P'

$m_2 \rightarrow$ slope to the curve 2 at point 'P'

$$m_1 = \left.\frac{d\,f(x)}{dx}\right|_{(x_1,y_1)} ; \quad m_2 = \left.\frac{d\,g(x)}{dx}\right|_{(x_1,y_1)}$$

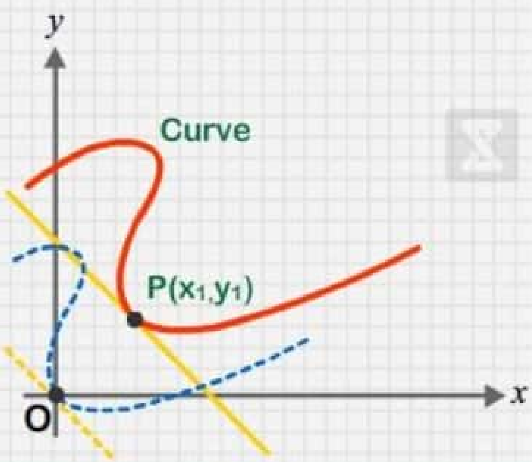
$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

If $\theta = \frac{\pi}{2}$ then the curves are called Orthogonal Curves.



POINTS TO REMEMBER

1



Equation of Tangent at point $P(x_1,y_1)$ to any second degree general curve

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

REPLACE:

$x^2 \rightarrow xx_1$; $y^2 \rightarrow yy_1$; $2x \rightarrow x + x_1$; $2y \rightarrow y + y_1$

$2xy \rightarrow xy_1 + x_1y$; $c \rightarrow c$

If curve passes through the 'O' then the equation of the tangent at 'O' may be directly written by comparing the lowest degree terms equal to 0.

$$2gx + 2fy = 0 \text{ or } gx + fy = 0$$

2

- If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0 \Rightarrow$ Tangent is parallel to x-axis (Horizontal Tangent).
- If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} \rightarrow \infty$ or $\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0 \Rightarrow$ Tangent is parallel to y-axis (Vertical Tangent).
- If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \pm 1 \Rightarrow$ Tangent at $P(x_1,y_1)$ is Equally inclined to the coordinate axis.

3 The shortest distance between two non-intersecting curves is always along to the common normal of the curves.

MONOTONOCITY

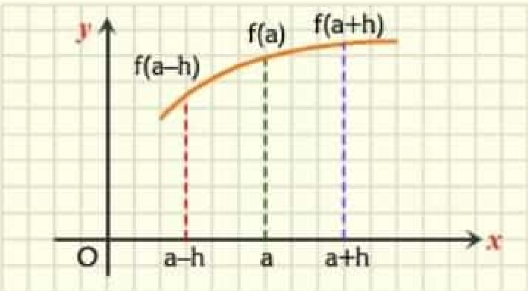
Part I

Definition

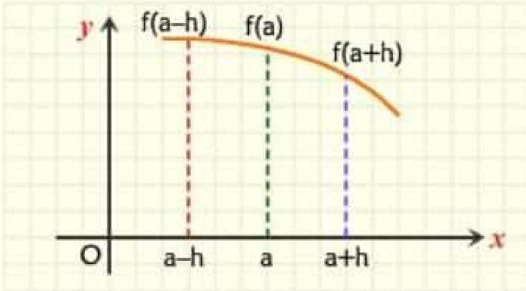
Functions are said to be monotonic if they are either increasing or decreasing in their entire domain, otherwise functions are called non–monotonic function.

Monotonocity of a Function at a Point

A function is said to be monotonically increasing at $x = a$ If $f(a + h) > f(a)$ & $f(a - h) < f(a)$ for small (+ve)h



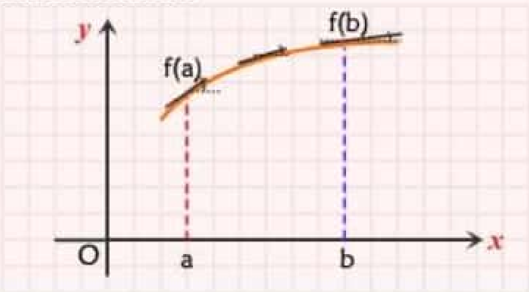
A function is said to be monotonically decreasing at $x = a$ If $f(a + h) < f(a)$ & $f(a - h) > f(a)$ for small (+ve)h



Monotonicity of a Function in an Interval (First Derivative Test)

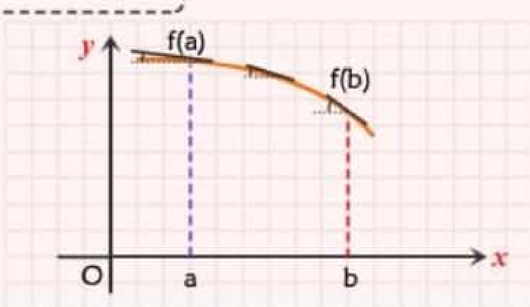
Function f(x) is said to be increasing in an interval (a, b) if

$\boxed{\frac{dy}{dx} > 0 \text{ or } f'(x) > 0}$

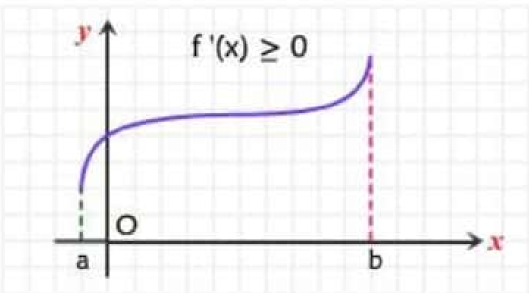


Function f(x) is said to be decreasing is an interval (a, b) if

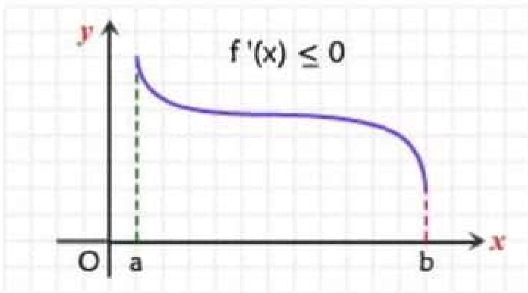
$\boxed{\frac{dy}{dx} < 0 \text{ or } f'(x) < 0}$



Non-Decreasing Function

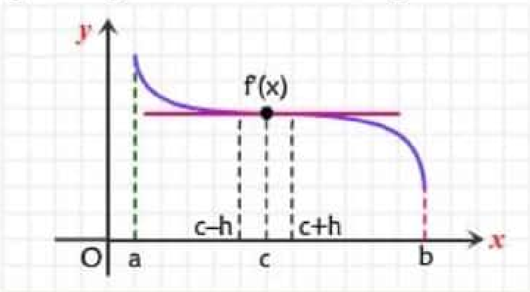


Non-Increasing Function



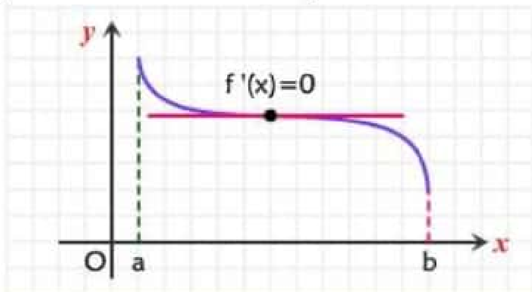
Point of Inflection

Point in the domain of f(x) where f '(x) does not changes its sign as x increases through c.



Critical Point

Point in the domain of f(x) where f '(x) is equal to zero or f '(x) fails to exist. Due to any reason.

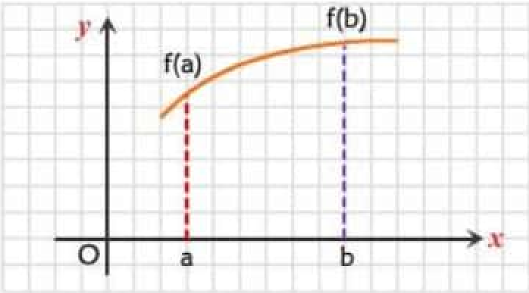


MONOTONOCITY

Part II

Greatest & Lowest Value of a Function

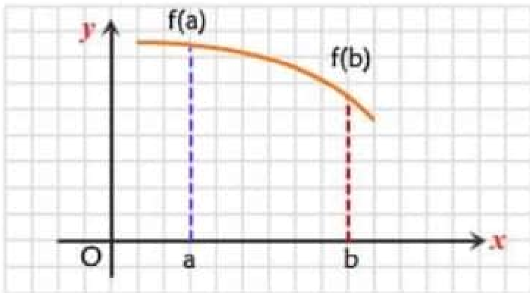
If a continuous function $y = f(x)$ is strictly increasing in $[a, b]$ then



Lowest value $= f(a)$

Greatest value $= f(b)$

If a continuous function $y = f(x)$ is strictly decreasing in $[a, b]$ then



Lowest value $= f(b)$

Greatest value $= f(a)$

Rolle's Theorem

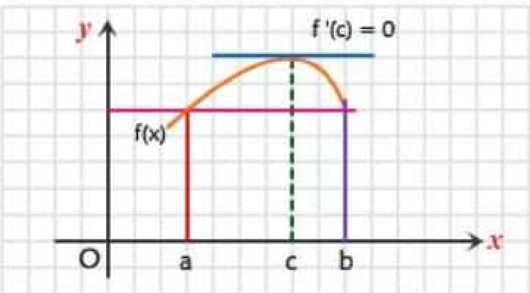
Let $f(x)$ be a function of x satisfying following conditions:

(1) $f(x)$ is continuous in $[a, b]$

(2) $f(x)$ is differentiable in (a, b)

(3) $f(a) = f(b)$

Then there exists atleast one point $x = c$, which belongs to (a, b) such that $\boxed{f'(c) = 0}$



Lagrange's Mean Value Theorem

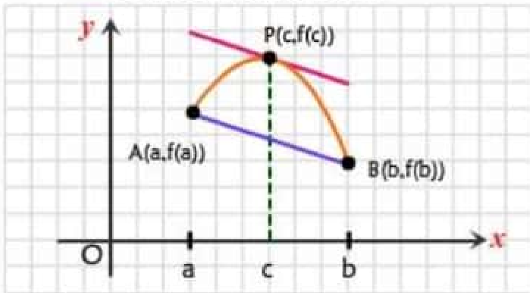
Let $f(x)$ be a function at x satisfying the following:

(1) $f(x)$ is continuous in $[a, b]$

(2) $f(x)$ is differentiable in (a, b)

(3) There exist atleast one $c \in (a, b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

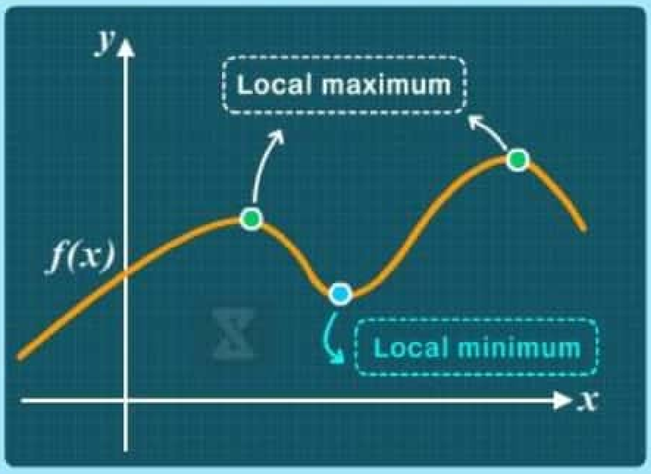


Points to Remember

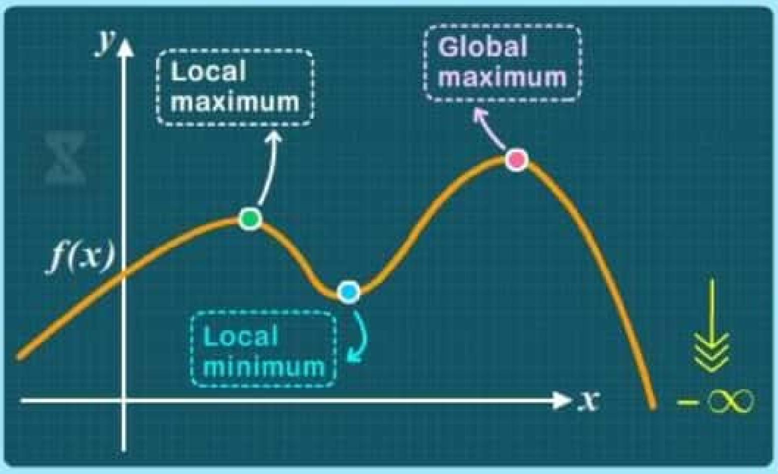
- If a function is invertible, it has to be either increasing or decreasing.
- If f is an increasing function then its negative i.e. $h = -f$ is a decreasing function.
- Reciprocal of an increasing function is a decreasing function.
- If f is an increasing function and g is also an increasing function their sum $h = f + g$ is an increasing function.
- If f and g both are increasing function then $h = f \times g$ is also an increasing function.
- If a function f is increasing (I) and takes negative values and another function g is decreasing (D) and takes positive values, then their product is an increasing function.
- Monotonocity of the difference of two function can be predicted as if I – I = can't say, I – D = increasing, D – I = decreasing, D – D = can't say.

MAXIMA AND MINIMA

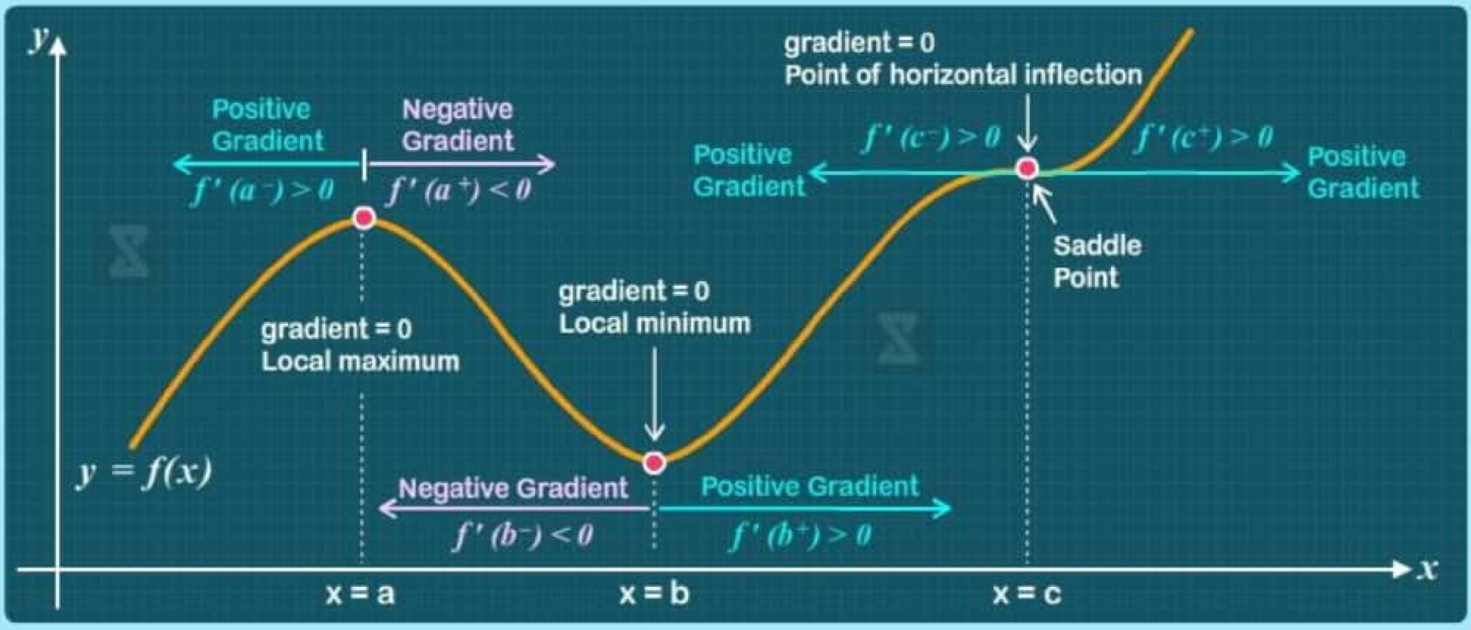
LOCAL MAXIMUM & MINIMUM



ABSOLUTE MAXIMUM & MINIMUM



FINDING MAXIMUM & MINIMUM



FIRST DERIVATIVE TEST

LOCAL MAXIMUM	$f'(a) = 0$	$f'(a^-) > 0$	$f'(a^+) < 0$
LOCAL MINIMUM	$f'(b) = 0$	$f'(b^-) < 0$	$f'(b^+) > 0$
SADDLE POINT	$f'(c) = 0$	$f'(c^-) > 0$	$f'(c^+) > 0$

- In general at saddle point (let x = c) $f'(c^+)$ and $f'(c^-)$ both are either positive or negative.

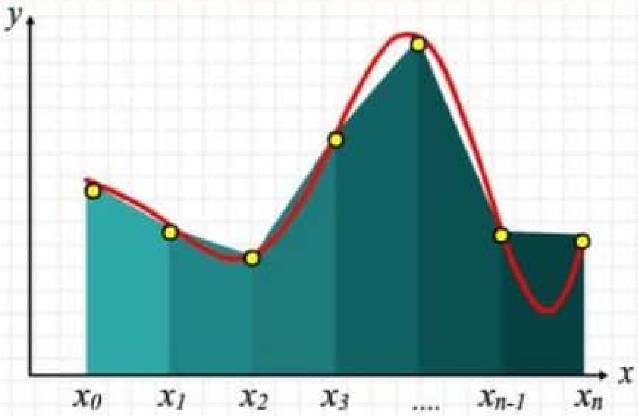
SECOND DERIVATIVE TEST

LOCAL MAXIMUM	$f'(a) = 0$	$f''(a) < 0$
LOCAL MINIMUM	$f'(b) = 0$	$f''(b) > 0$
SADDLE POINT	$f'(c) = 0$	$f''(c) = 0$

- In general at saddle point (let x = c) $f'(c) = f''(c) = \ldots\ldots = f^n(c) = 0$.

AREA UNDER THE CURVE

Trapezoidal Riemann Approximation



$$\text{Area} = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x \quad ; \Delta x = \frac{b-a}{n}$$

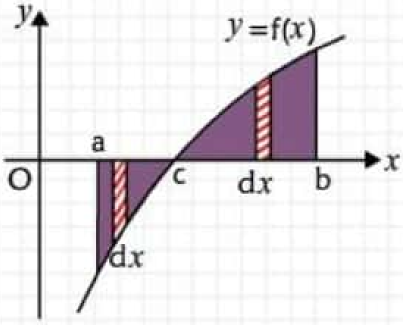
Riemann Sum:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x$$

$$= \int_a^b f(x)dx$$

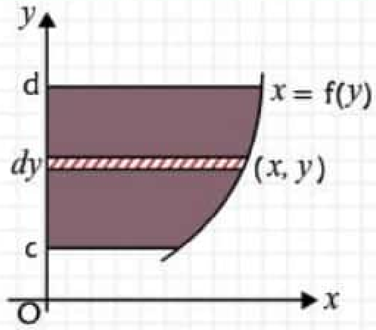
dx – infinitely small

Area By Vertical Strips



$$A = \left|\int_a^c f(x)dx\right| + \int_c^b f(x)dx$$

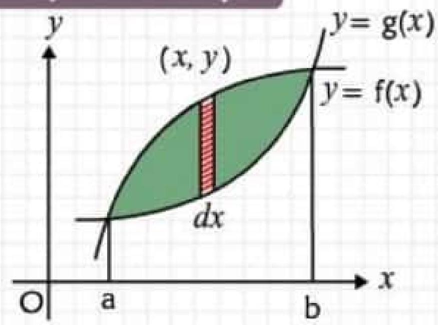
Area By Horizontal Strips



$$A = \int_c^d f(y)dy = \int_c^d x\,dy$$

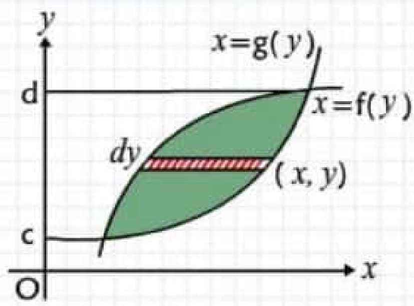
Area Enclosed Between Two Curves

Case – I : By vertical strips



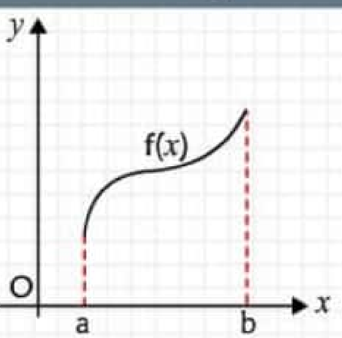
$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

Case – II : By horizontal strips



$$A = \int_c^d f(y)dy - \int_c^d g(y)dy = \int_c^d [f(y) - g(y)]dy$$

Average Value of a Function



$y = f(x)$; $a \le x \le b$

$$y_{avg} = \frac{1}{b-a}\int_c^d f(x)dx$$

Useful Results

- Whole area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab $(\text{units})^2$.
- Area enclosed between the parabola $y^2 = 4ax$ & $x^2 = 4by$ is $\frac{16\,ab}{3}$ $(\text{units})^2$.
- Area enclosed between the parabola $y^2 = 4ax$ & $y = mx$ is $\frac{8\,a^2}{3m^3}$ $(\text{units})^2$.

Part I

DIFFERENTIAL EQUATION

Differential equation is an equation that involves independent and dependent values and the derivatives of the dependent variables.

TYPES OF DIFFERENTIAL EQUATION

ORDINARY DIFFERENTIAL EQUATION

If the differential coefficients have reference to a single independent variable only.

Eg : $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$

PARTIAL DIFFERENTIAL EQUATION

If there are two or more independent variables.

Eg : $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

ORDER AND DEGREE OF DIFFERENTIAL EQUATION

Order

Degree

$$f(x,y)\left[\frac{d^{m}y}{dx^{m}}\right]^{p} + \phi(x,y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{q} + \ldots\ldots = 0$$

Order : is the highest derivative.

Degree : is the exponent of the highest derivative.

SOLVING DIFFERENTIAL EQUATION

1 ELEMENTARY TYPE OF 1ST ORDER AND 1ST DEGREE

SEPERATION OF VARIABLES METHOD :

Step 1 - Move all the y terms (including dy) to one side and all the x terms (inculding dx) to the other side.

Step 2 - Intergrate one side with respect to 'y' and the other side with respect to 'x'.

Step 3 - Simplify it.

SEPERATION OF VARIABLES TYPES :

Type-1 : Equation of the form $f(x)dx = -g(y)dy \Longrightarrow \int f(x)dx + \int g(y)dy = C$

Type-2 : Equation of the form $\frac{dy}{dx} = f(ax+by+c);\ b \neq 0 \Longrightarrow$ put $ax+by+c = t$ & reduce to Type 1

Type-3 : Equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ {if $a_2+b_1=0$} Cross multiply and note the perfect differential $d(xy)$.

Type-4 : Transformation to polar coordinates

(a) $x = r\cos\theta;\ y = r\sin\theta;\ x^2+y^2=r^2;\ \frac{y}{x} = \tan\theta;\ xdx+ydy = rdr;\ xdy - ydx = r^2d\theta$

(b) $x = r\sec\theta;\ y = r\tan\theta;\ x^2-y^2=r^2;\ \frac{y}{x} = \sin\theta;\ xdx - ydy = rdr;\ xdy - ydx = r^2\sec\theta\, d\theta$

Part II

2 HOMOGENEOUS DIFFERENTIAL EQUATION

T-(1) An equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ (where f(x, y) and g(x, y) are homogeneous functions of x and y and of same degree) To solve put $y = tx$ or $x = ty$

T-(2) Equation reducible to homogeneous differential $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ When $a_1b_2 - a_2b_1 \neq 0$ or $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. Substitution : $x = u + h$ and $y = v + k$ then reduce differential equation to Homogeneous Differential Equation T-1

3 LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be linear if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n(x)\, y = \phi(x)$$

where $a_0(x), a_1(x), a_2(x) \ldots\ldots, a_n(x)$ are called coefficient of Differential Equation.

Linear differential equation of first order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve

Calculate : Integrating factor (I.F.) $= e^{\int P(x)dx}$

Solution : $y\,(\text{I.F.}) = \int Q(x)(\text{I.F.})dx + C$

Bernoulli's Equation

Equation reducible to linear differential equation

Equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$

To solve divide by y^n and substitute $y^{1-n} = t$

$$\frac{dt}{dx} + (1-n)P(x)t = Q(x)(1-n)$$

Now solve as 1st order Linear Differential Equation

SOME IMPORTANT EXACT DIFFERENTIALS

- $xdy + ydx = d(xy)$
- $\frac{xdy - ydx}{xy} = d\left(\ln\left(\frac{y}{x}\right)\right)$
- $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
- $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$
- $\frac{ydx - xdy}{xy} = d\left(\ln\frac{x}{y}\right)$
- $d\left(\frac{e^y}{x}\right) = \left(\frac{xe^ydy - e^ydx}{x^2}\right)$
- $\frac{xdy - ydx}{y^2} = d\left(-\frac{x}{y}\right)$
- $\frac{xdy + ydx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$
- $d\left(\frac{e^x}{y}\right) = \left(\frac{ye^xdy - e^xdx}{y^2}\right)$
- $\frac{xdy + ydx}{xy} = \frac{d(xy)}{xy} = d(\ln xy)$
- $\frac{2(xdx + ydy)}{x^2 + y^2} = [\ln(x^2 + y^2)]$
- $\frac{dx + dy}{x + y} = d(\ln(x + y))$