Discrete Structures

Day 17

Sets

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces.

For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$,

Or $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

N = {1, 2, 3, . . .}, the set of **natural numbers**

Z = {. . . ,−2,−1, 0, 1, 2, . . .}, the set of **integers**

Z+ = {1, 2, 3, . . .}, the set of **positive integers**

Q = {p/q | p ∈ **Z**, q ∈ **Z**, and q = 0}, the set of **rational numbers**

R, the set of **real numbers**

R+, the set of **positive real numbers**

C, the set of **complex numbers**.

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.We write $A = B$ if A and B are equal sets.

Eg: The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅. The empty set can also be denoted by { }.

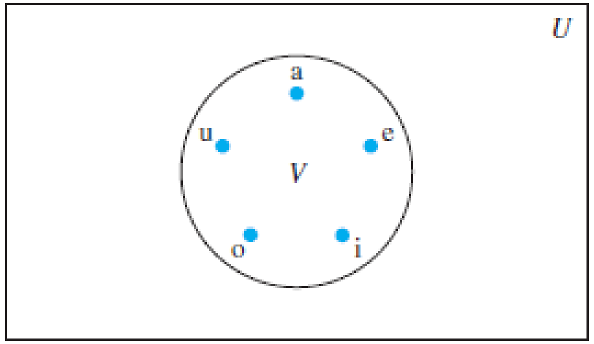
A set with one element is called a **singleton set.** A common error is to confuse the empty {∅} has one more element than ∅.

Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician JohnVenn.

In Venn diagrams the **universal set** *U,* which contains all the objects under consideration, is represented by a rectangle. Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set.

Eg: Draw a Venn diagram that represents *V,* the set of vowels in the English alphabet.



Subsets: The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$.

Every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

When we wish to emphasize that a set A is a subset of a set B but that $A = B$, we write $A \subset B$ and say that A is a **proper subset** of B. A is a **proper subset** of B denoted by $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$ is true.

Showing Two Sets are Equal : To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

That is, $A = B$ if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$, which is what it means for the A and B to be equal.

The Size of a Set: Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Eg:

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Because the null set has no elements, it follows that $|\emptyset| = 0$.

Power Sets: Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

Q. What is the power set of the set $\{0, 1, 2\}$?

The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

Q. What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

The empty set has exactly one subset, namely, itself. Consequently,

$P(\emptyset) = \{\emptyset\}$.

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

If a set has n elements, then its power set has 2^n elements.

Cartesian Products

Let A and B be sets. The *Cartesian product of A and B*, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$.

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B.

For example,

$R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from

the set $\{a, b, c\}$ to

the set $\{0, 1, 2, 3\}$.

A relation from a set A to itself is called a relation on A.

Q. What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Set Operations

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.

$$A \cup B = \{x \mid x \in A \lor x \in B\}.$$

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

Two sets are called *disjoint* if their intersection is the empty set.

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

$$A - B = \{x \mid x \in A \land x \notin B\}.$$

Q. The difference of {1, 3, 5} and {1, 2, 3} is the set ________

Let U be the universal set. The *complement* of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is

$U - A$.

An element belongs to $\bar{A}$ if and only if $x \notin A$. This tells us that

$$\bar{A} = \{x \in U \mid x \notin A\}.$$

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\bar{\bar{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan's laws
$\cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup A = U$ $A \cap A = \emptyset$	Complement laws

Q. Use set builder notation and logical equivalences to establish $\overline{A \cap B} = \bar{A} \cup \bar{B}$

We can prove this identity with the following steps.

$\overline{A \cap B}$ = $\{x \mid x \notin A \cap B\}$ by definition of complement

= $\{x \mid \neg(x \in (A \cap B))\}$ by definition of does not belong symbol

= $\{x \mid \neg(x \in A \wedge x \in B)\}$ by definition of intersection

= $\{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ by the first De Morgan law for logical equivalences

= $\{x \mid x \notin A \vee x \notin B\}$ by definition of does not belong symbol

= $\{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ by definition of complement

= $\{x \mid x \in \bar{A} \cup \bar{B}\}$ by definition of union

= $\bar{A} \cup \bar{B}$ by meaning of set builder notation

Q. Let *A*, *B*, and *C* be sets. Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Relations and Their Properties

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not{R}\, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

Relations on a Set

Relations from a set A to itself. A *relation on a set A* is a relation from A to A.

Q. Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Properties of Relations

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Q. Consider the following relations on {1, 2, 3, 4}:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R6 = \{(3, 4)\}$.

Which of these relations are reflexive?

Q. Is the "divides" relation on the set of whole number reflexive?

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

The set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b(((a, b) \in R \land (b, a) \in R) \rightarrow (a = b))$.

The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them. A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b), where $a \neq b$. A relation is not antisymmetric if we can find a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation.

Q. Consider the following relations on {1, 2, 3, 4}:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R6 = \{(3, 4)\}$.

Which of these relations are *symmetric* and which are *antisymmetric*?

R2 and R3 is symmetric and R4, R5 and R6 are antisymmetric

Q. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Q. Consider the following relations on {1, 2, 3, 4}:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R6 = \{(3, 4)\}$.

Which of these relations are transitive?

Q. Is the “divides” relation on the set of positive integers transitive?

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

- $R1 \cup R2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
- $R1 \cap R2 = \{(1, 1)\}$,
- $R1 - R2 = \{(2, 2), (3, 3)\}$,
- $R2 - R1 = \{(1, 2), (1, 3), (1, 4)\}$.

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Q. What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.