Discrete Structures

Day 15

Q. Show that at least four of any 22 days must fall on the same day of the week.

Let p be the proposition "At least four of 22 chosen days fall on the same day of the week."

Suppose that $\neg p$ is true.

This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$.

Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week.

Rewriting a direct proof of $p \to q$ into proof by contradiction

In such proofs, we first assume that p and $\neg q$ are true, and showing that q must be also be true. This implies that $\neg q$ and q are both true, a contradiction.

Rewriting a proof by contraposition of a conditional statement as a proof by contradiction.

In a proof of $p \to q$ by contraposition, we assume that $\neg q$ is true. We then show that $\neg p$ must also be true. To rewrite a proof by contraposition of $p \to q$ as a proof by contradiction, we suppose that both p and $\neg q$ are true. Then, we use the steps from the proof of $\neg q \to \neg p$ to show that $\neg p$ is true. This leads to the contradiction $p \wedge \neg p$, completing the proof.

Q. Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Let p be "$3n + 2$ is odd" and q be "n is odd."

To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n + 2$ is odd and that n is not odd. Because n is not odd, we know that it is even. Because n is even, there is an integer k such that $n = 2k$. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

Because $3n + 2$ is $2t$, where $t = 3k + 1$, $3n + 2$ is even. Note that the statement "$3n + 2$ is even" is equivalent to the statement $\neg p$, because an integer is even if and only if it is not odd. Because both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd.

Mistakes in Proofs

What is wrong with this famous supposed "proof" that 1 = 2?

We use these steps, where *a* and *b* are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

- Many incorrect arguments are based on a fallacy called **begging the question**. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. This fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning**.

Eg. Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

- Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k. Let $n = 2l$ for some integer l. This shows that n is even.

This argument is incorrect. The statement "let $n = 2l$ for some integer l" occurs in the proof. This is circular reasoning because this statement is equivalent to the statement being proved, namely, "n is even."

PROOFS OF EQUIVALENCE

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p).$

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p1, p2, p3, \ldots, pn$ are equivalent. This can be written as

- $p1 \leftrightarrow p2 \leftrightarrow \cdots \leftrightarrow pn,$

One way to prove these mutually equivalent is to use the tautology

$p1 \leftrightarrow p2 \leftrightarrow \cdots \leftrightarrow pn \equiv (p1 \rightarrow p2) \wedge (p2 \rightarrow p3) \wedge \cdots \wedge (pn \rightarrow p1).$

When we prove that a group of statements are equivalent, we can establish any chain of conditional statements we choose as long as it is possible to work through the chain to go from any one of these statements to any other statement. For example, we can show that $p1$, $p2$, and $p3$ are equivalent by showing that $p1 \rightarrow p3$, $p3 \rightarrow p2$, and $p2 \rightarrow p1$.

Q. Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Let p: n is odd

q: n^2 is odd

$(p \leftrightarrow q) \equiv (p \rightarrow q) \land (q \rightarrow p)$.

We need to show, *$(p \rightarrow q)$ and $(q \rightarrow p)$ are T.*

Trying using direct proof for *$(p \rightarrow q)$*

Assume p is T.

$n=2k+1$, where k is an integer.

Squaring both sides

$n^2 = 4k^2+4k+1$

$n^2 = 2(2k^2+2k)+1$

$n^2 = 2(k')+1$, where $k'=2k^2+2k$

n^2 is odd, i.e. q is T.

By direct proof, *$(p \rightarrow q)$ is T*

Trying using proof by C.P. for *$(q \rightarrow p)$*

Assume $\neg p$ is T.

n is even, $n=2k$, where k is an integer

Squaring both sides

$n^2 = 4k^2$

$n^2 = 2(2k^2)$

n^2 is even, i.e. $\neg$ q is T.

By proof by C.P. *$(q \rightarrow p)$ is T*

Q1. Show that these statements about the integer n are equivalent:

$p1$: n is even.

$p2$: $n - 1$ is odd.

$p3$: n^2 is even.

Show that these three statements are equivalent by showing that the conditional statements $p1 \rightarrow p2$, $p2 \rightarrow p3$, and $p3 \rightarrow p1$ are true.

Q. Use a direct proof to show that the sum of two odd integers is even.

p: Let m and n be two odd integers.

q: Sum of two odd integers is even.

Let p is T

$m=2k+1$, where k is an integer.

$n=2l+1$, where l is an integer.

Adding m and n,

$$m + n = (2k+1) + (2l+1)$$

$$= 2(k+l+1)$$

$$=2k', \text{ where } k'= k+l+1$$

i.e. $m + n$ is even, q is T

Q. Use a direct proof to show that every odd integer is the difference of two squares.

P(x): x is an odd integer.

Q(x): x is the difference of two squares.

Assume P(x) is T.

$$x = 2k+1$$
$$= k^2 + 2k + 1 - k^2$$
$$= (k+1)^2 - k^2$$

Q. Prove that if *n* is a perfect square, then *n* + 2 is not a perfect square.

p: *n* is a perfect square.

q: *n* + 2 is not a perfect square

Assume $n=b^2$

Using proof by contradiction.

Assume p and $\neg$ q is T,

If n is a perfect square, there exist an integer x such that,

$$n=x^2$$

If n+2 is a perfect square, there exist an integer y such that,

$$n+2=y^2$$

$x^2+2=y^2$

$2=y^2-x^2$

$2=(y+x)(y-x)$

The only possible integers are 1 and 2

$y+x=2$ and $y-x=1$ or $y+x=1$ and $y-x=2$

Adding we get, $2y=3$ or $y=3/2$, which contradicts that y is an integer. The assuming that n+2 is perfect square is incorrect.

Hence, *n* + 2 is not a perfect square

Q. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

To prove: $x+y=z$, where x is irrational, y is rational and z is irrational.

Using proof by contradiction

Assume, sum of an irrational number and a rational number is rational.

$y=a/b$ and $z=c/d$, where b and $d \neq 0$

$x+a/b=c/d$

$x=c/d-a/b$

$x=(cb-ad)/bd$

We can say x is rational, which contradicts x is irrational.

So, our assumption that sum of an irrational number and a rational number is rational is incorrect.

Hence, sum of an irrational number and a rational number is irrational.

Q. Prove that if m and n are integers and mn is even, then m is even or n is even.

p: mn is even.

q: m is even or n is even.

Using proof by contradiction

Assume p and $\neg$ q is T,

$\neg$ q : m is odd and n is odd.

$m=2k+1$ and $n=2l+1$

$$
\begin{aligned}
mn &= (2k+1)(2l+1) \\
&= 4kl+2k+2l+1 \\
&= 2(2kl+k+l)+1 \\
&= 2k'+1, \text{ where } k'=(2kl+k+l)
\end{aligned}
$$

i.e. mn is odd which contradicts p.

Hence, using proof by contradiction, if m and n are integers and mn is even, then m is even or n is even.

Q2. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

Q3. Show that at least ten of any 64 days chosen must fall on the same day of the week.