Discrete Structures

Day 14

Q1**.** For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) “I am either dreaming or hallucinating.” “I am not dreaming.” “If I am hallucinating, I see elephants running down the road.”

b) “If I play hockey, then I am sore the next day.” “I use the whirlpool if I am sore.” “I did not use the whirlpool.”

c) “All insects have six legs.” “Dragonflies are insects.” “Spiders do not have six legs.” “Spiders eat dragonflies.”

d) “Every student has an Internet account.” “Homer does not have an Internet account.” “Maggie has an Internet account.”

e) “All foods that are healthy to eat do not taste good.” “Tofu is healthy to eat.” “You only eat what tastes good.” “You do not eat tofu.” “Cheeseburgers are not healthy to eat.”

f) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

g) “If I take the day off, it either rains or snows.” “I took Tuesday off or I took Thursday off.” “It was sunny on Tuesday.” “It did not snow on Thursday.”

a) “I am either dreaming or hallucinating.” “I am not dreaming.” “If I am hallucinating, I see elephants running down the road.”

p: I am dreaming.

q: I am hallucinating.

r: I see elephants running down the road.

Sl. No.	Steps	Reason
1	$p \vee q$	Premise
2	$\neg p$	Premise
3	$q \rightarrow r$	Premise
4	q	1,2 Disjunctive syllogism
5	r	3,6 Modus Ponens

Conclusions: "I am hallucinating“ and "I see elephants running down the road"

f) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

p: I am clever.

q: I am lucky.

r: I win the lottery.

Case 1:

I am either clever or lucky: $p \oplus q$

I am not lucky: $\neg q$

If I am lucky, then I will win the lottery:

$q \rightarrow r$

Case 2:

I am either clever or lucky: $p \vee q$

Sl.No	Steps	Reason
1	$p \oplus q$	Premise
2	$\neg[(p\rightarrow q)\wedge (q \rightarrow p)]$	$p \oplus q \equiv \neg(p \leftrightarrow q)$
3	$\neg(p\rightarrow q)\vee \neg(q \rightarrow p)$	De Morgan’s law (2)
4	$\neg(\neg p \vee q) \vee \neg(\neg q \vee p)$	$p \rightarrow q \equiv \neg p \vee q$
5	$(p \wedge \neg q) \vee (q \wedge \neg p)$	De Morgan’s law(4)
6	$(p\vee q) \wedge(p\vee \neg p) \wedge(\neg q \vee q) \wedge(\neg q \vee \neg p)$	Distributive (5)
7	$(p\vee q) \wedge(\neg q \vee \neg p)$	
8	$(p\vee q)$	Simplification (7)
9	$\neg q$	Premise
10	p	D.S. (8) and (9)

g) “If I take the day off, it either rains or snows.” “I took Tuesday off or I took Thursday off.” “It was sunny on Tuesday.” “It did not snow on Thursday.”

$O(x)$ = “I took x off”

$R(x)$ = “it rains on x”

$S(x)$ = “it snows on x”

“If I take the day off, it either rains or snows.”

$\forall x(O(x) \rightarrow (R(x) \vee S(x)))$

“I took Tuesday off or I took Thursday off.”

$O(Tue) \vee O(Thu)$

“It was sunny on Tuesday.”

$\neg S(Tue) \wedge \neg R(Tue)$

“It did not snow on Thursday.”

$\neg S(Thu)$

Sl.No.	Steps	Reason
1	$\forall x(O(x) \rightarrow (R(x) \vee S(x)))$	Premise
2	$O(Tue) \rightarrow (R(Tue) \vee S(Tue))$	U.I (1)
3	$\neg S(Tue) \wedge \neg R(Tue)$	Premise
4	$\neg(R(Tue) \vee S(Tue))$	De Morgan’s Law (3)
5	$\neg O(Tue)$	M.T. (2), (4)
6	$O(Tue) \vee O(Thu)$	Premise
7	$O(Thu)$	D.S. (5) (6)
8	$O(Thu) \rightarrow (R(Thu) \vee S(Thu))$	U.I (1)
9	$R(Thu) \vee S(Thu)$	M.P. (7) (8)
10	$\neg S(Thu)$	Premise
11	$R(Thu)$	D.S. (9) (10)

Q2. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.

b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.

P(x): x is enrolled in the university.

Q(x): x has lived in a dormitory.

Premises:

$$\forall x\big(P(x) \rightarrow Q(x)\big).$$

$$\neg Q(Mia)$$

Conclusion:

$$\neg P(Mia)$$

Sl.No	Step	Reason
1	$\forall x\big(P(x) \rightarrow Q(x)\big)$	Premise
2	$P(Mia) \rightarrow Q(Mia)$	U.I.
3	$\neg Q(Mia)$	Premise
4	$\neg P(Mia)$	M.T.

The argument is correct using **Universal modus tollens**

Introduction to Proofs

- A proof is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems. Using these ingredients and rules of inference, the final step of the proof establishes the truth of the statement being proved.

Some Terminology:

- A **theorem** is a statement that can be shown to be true. Theorems can also be referred to as **facts** or **results**.)
- A less important theorem that is helpful in the proof of other results is called a **lemma.**
- A **corollary** is a theorem that can be established directly from a theorem that has been proved.
- A **conjecture** is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.

Methods of Proving Theorems

- **Direct Proofs:** A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

Q. Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Note that this theorem states $\forall n(P(n) \rightarrow Q(n))$, where

$P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd."

To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd.

By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd.

We can square both sides of the equation $n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer.

Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Q. Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.

Assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares.

There are integers s and t such that $m = s^2$ and $n = t^2$.

Substituting s^2 for m and t^2 for n into mn.

$mn = s^2t^2$.

Hence, $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication.

We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Proof by Contraposition

- Direct proofs often reach dead ends. Unlike direct proofs, proofs that do not start with the premises and end with the conclusion, are called **indirect proofs**.
- **Proof by Contraposition** is a type of indirect proof. Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$.
- This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.
- In a proof by contraposition of $p \to q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

Q. Prove that if $3n + 2$ is odd, then n is odd where n is an integer.

If we try using direct proof. Assume that $3n + 2$ is an odd integer.

$3n + 2 = 2k + 1$ for some integer k.

$3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd.

Trying using proof by contraposition.

Assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even.

$$n = 2k \text{ for some integer } k.$$

$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

This tells us that $3n + 2$ is even and therefore not odd.

This is the negation of the premise of the theorem.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd."

Q. Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Trying using proof by contraposition.

Assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false.

$$a > \sqrt{n} \text{ and } b > \sqrt{n}.$$

Multiply these inequalities together

$$ab > \sqrt{n} \cdot \sqrt{n}$$

$$ab > n.$$

This shows that $ab \neq n$, which contradicts the statement $n = ab$.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded.

VACUOUS PROOFS

We can quickly prove that a conditional statement $p \to q$ is true when we know that p is false, because $p \to q$ must be true when p is false. Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \to q$.

Q. Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

$P(0)$ is "If $0 > 1$, then $0^2 > 0$." The hypothesis $0 > 1$ is false. Using a vacuous proof, $P(0)$ is automatically true.

Trivial proof

We can also quickly prove a conditional statement $p \to q$ if we know that the conclusion q is true. By showing that q is true, it follows that $p \to q$ must also be true. A proof of $p \to q$ that uses the fact that q is true is called a **trivial proof**.

Q. Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a0 \geq b0$" is true.

Hence, this conditional statement, which is $P(0)$, is true. This is an example of a trivial proof

Q3. Prove that the sum of two rational numbers is rational.

Q 4. Prove that if n is an integer and n^2 is odd, then n is odd.

P

Proofs by Contradiction

- Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true. How can we find a contradiction q that might help us prove that p is true in this way?
- Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called **proofs by contradiction**. Because a proof by contradiction does not prove a result directly, it is another type of indirect proof.

Q. Prove that √2 is irrational by giving a proof by contradiction.

Let p be the proposition "√2 is irrational." To start a proof by contradiction, we suppose that $\neg p$ is true. Note that $\neg p$ is the statement "It is not the case that √2 is irrational," which says that √ 2 is rational. We will show that assuming that $\neg p$ is true leads to a contradiction.

If √2 is rational, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors.

Squaring we get,

$2 = a^2/b^2$.

Hence, $2b^2 = a^2$.

By the definition of an even integer it follows that a^2 is even. We next use the fact that if a^2 is even, a must also be even, $a = 2c$ for some integer c. Thus,

$2b^2 = 4c^2$.

Dividing both sides of this equation by 2 gives

$b^2 = 2c^2$.

By the definition of even, this means that b^2 is even. Again using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well.

Both a and b are even, that is, 2 divides both a and b. Because our assumption of $\neg p$ leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, $\neg p$ must be false. That is, the statement p, "√2 is irrational," is true. We have proved that √2 is irrational.