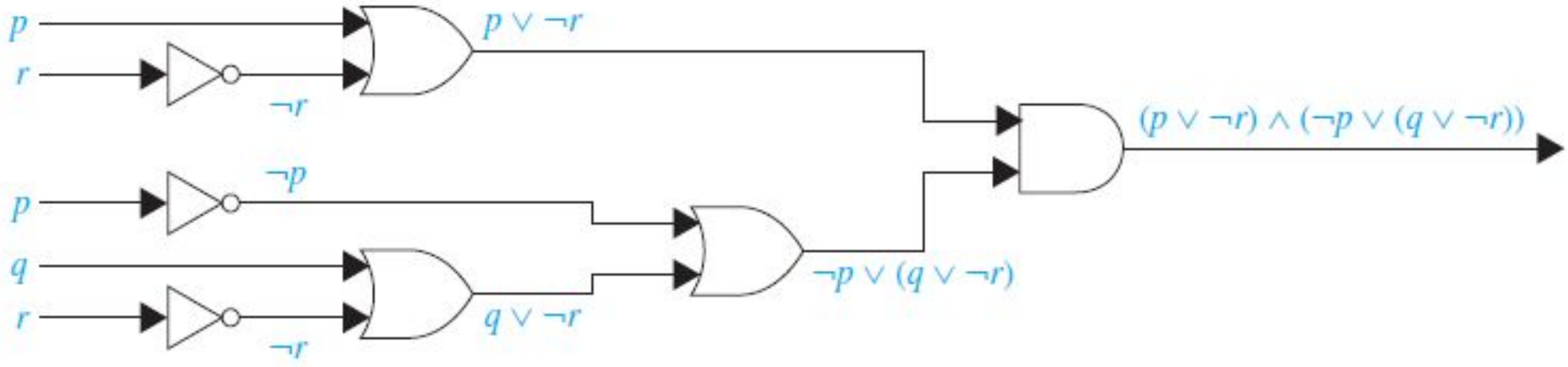
Discrete Structures

Day 8

Logic Circuits

Q. Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p, q, and r.



Propositional Equivalences

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.

A compound proposition that is always false is called a *contradiction*.

A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

- **Logical Equivalences:** Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. The symbol ≡ or ⇔is used to denote logical equivalence.
- The compound propositions *p* and *q* are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

- **De Morgan laws**:

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Q. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Q. Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Logical Equivalences.

Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$	Negation laws

Logical Equivalences Involving Conditional Statements.

- $p \rightarrow q \equiv \neg p \vee q$
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- $p \vee q \equiv \neg p \rightarrow q$
- $p \wedge q \equiv \neg (p \rightarrow \neg q)$
- $\neg (p \rightarrow q) \equiv p \wedge \neg q$
- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
- $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
- $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
- $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements.

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Q. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ since $p \rightarrow q \equiv \neg p \vee q$

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law

$\equiv p \wedge \neg q$ by the double negation law

Q. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

- Q. Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.

$\neg(p \lor (\neg p \land q)) \quad \equiv \neg p \land \neg(\neg p \land q)$ by the second De Morgan law

$\equiv \neg p \land [\neg(\neg p) \lor \neg q]$ by the first De Morgan law

$\equiv \neg p \land (p \lor \neg q)$ by the double negation law

$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$ by the second distributive law

$\equiv \mathbf{F} \lor (\neg p \land \neg q)$ because $\neg p \land p \equiv \mathbf{F}$

$\equiv (\neg p \land \neg q) \lor \mathbf{F}$ by the commutative law for disjunction

$\equiv \neg p \land \neg q$ by the identity law for **F**

- Consequently $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Q. "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

r: You can ride the roller coaster.

u: you are under 4 feet tall.

o: you are older than 16 years old.

$$(u \land \neg o) \rightarrow \neg r$$

- You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old

p → q ≡ q unless ¬p ≡If p, q

≡¬r if u unless o

≡$\neg o$ →(¬r if u)

≡$\neg o$ →(u → ¬r)

≡ $\neg o \rightarrow (\neg u \lor \neg r)$

≡ $\neg(\neg o) \lor (\neg u \lor \neg r)$

≡ $o \lor (\neg u \lor \neg r)$

≡$(o \lor \neg u) \lor \neg r$

≡ ¬(¬o ∧u) ∨ $\neg r$

≡¬(u ∧ ¬o) ∨ $\neg r$

≡(u ∧ ¬o)→ $\neg r$

If a then b unless c ≡ $(a \land \neg c) \rightarrow b$

Q. Show that $(p \land q) \to (p \lor q)$ is a tautology.

$(p \land q) \to (p \lor q) \quad \equiv \neg(p \land q) \lor (p \lor q)$

$\equiv (\neg p \lor \neg q) \lor (p \lor q)$ by the first De Morgan law

$\equiv (\neg p \lor p) \lor (\neg q \lor q)$ by the associative and commutative laws for disjunction

$\equiv \mathbf{T} \lor \mathbf{T}$

$\equiv \mathbf{T}$ by the domination law

Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

Q. Determine whether each of the compound propositions

i. $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$

ii. $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$, is satisfiable.

i, When p=T; q=T; r=T; $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ will be T, hence it satisfiable

Q. Show that each of these conditional statements is a tautology

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

e) $\neg(p \rightarrow q) \rightarrow p$

f) $\neg(p \rightarrow q) \rightarrow \neg q$

a) $(p \wedge q) \rightarrow p \equiv \neg(p \wedge q) \vee p \equiv \neg p \vee \neg q \vee p \equiv \neg p \vee p \vee \neg q \equiv T \vee \neg q \equiv T$

f) $\neg(p \rightarrow q) \rightarrow \neg q \equiv \neg(\neg p \vee q) \rightarrow \neg q \equiv (p \wedge \neg q) \rightarrow \neg q$

$\equiv \neg(p \wedge \neg q) \vee \neg q \equiv \neg p \vee q \vee \neg q \equiv \neg p \vee T \equiv T$

Q. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.

Q. Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.

Prove

i. $p \vee (p \wedge q) \equiv p$

ii. $p \wedge (p \vee q) \equiv p$

doubt !