Discrete Structures

Day 13

Rule of Inference	*Tautology*	*Name*
p $p \to q \quad \therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q \quad \therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r \quad \therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p \quad \therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p $q \quad \therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r \quad \therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Q1. Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.

Rules of Inference for Quantified Statements

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$.
Eg: "All women are wise". "Lisa is wise," where Lisa is a member of the domain of all women.
Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. The element c that we select must be an arbitrary, and not a specific, element of the domain.
Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true.
Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known.

Q. Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Let $D(x)$: x is in this discrete mathematics class.

$C(x)$: x has taken a course in computer science.

Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$.

- The conclusion is $C(\text{Marla})$.
- The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Q. Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$: x is in this class.

$B(x)$: x has read the book.

$P(x)$: x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$.

The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Combining Rules of Inference for Propositions and Quantified Statements

Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**.

$\forall x(P(x) \rightarrow Q(x))$

$P(a)$, where a is a particular element in the domain

$\therefore Q(a)$

Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$\forall x(P(x) \rightarrow Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore \neg P(a)$

Q2. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Q3. What rules of inference are used in this famous argument?

"All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

Q4. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Q5. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

Q6. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

Q7. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true.

Q8**.** For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) “I am either dreaming or hallucinating.” “I am not dreaming.” “If I am hallucinating, I see elephants running down the road.”

b) “If I play hockey, then I am sore the next day.” “I use the whirlpool if I am sore.” “I did not use the whirlpool.”

c) “All insects have six legs.” “Dragonflies are insects.” “Spiders do not have six legs.” “Spiders eat dragonflies.”

d) “Every student has an Internet account.” “Homer does not have an Internet account.” “Maggie has an Internet account.”

e) “All foods that are healthy to eat do not taste good.” “Tofu is healthy to eat.” “You only eat what tastes good.” “You do not eat tofu.” “Cheeseburgers are not healthy to eat.”

f) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

a) “I am either dreaming or hallucinating.” “I am not dreaming.” “If I am hallucinating, I see elephants running down the road.”

p: I am dreaming.

q: I am hallucinating.

r: I see elephants running down the road.

Sl. No.	Steps	Reason
1	$p \vee q$	Premise
2	$\neg p$	Premise
3	$q \rightarrow r$	Premise
4	q	1,2 Disjunctive syllogism
5	r	3,6 Modus Ponens

Conclusions: "I am hallucinating“ and "I see elephants running down the road"

f) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

p: I am clever.

q: I am lucky.

r: I win the lottery.

Case 1:

I am either clever or lucky: $p \oplus q$

I am not lucky: $\neg q$

If I am lucky, then I will win the lottery:

$q \rightarrow r$

Case 2:

I am either clever or lucky: $p \vee q$

Sl.No	Steps	Reason
1	$p \oplus q$	Premise
2	$\neg[(p \rightarrow q) \wedge (q \rightarrow p)]$	$p \oplus q \equiv \neg(p \leftrightarrow q)$
3	$\neg(p \rightarrow q) \vee \neg(q \rightarrow p)$	De Morgan’s law (2)
4	$\neg(\neg p \vee q) \vee \neg(\neg q \vee p)$	$p \rightarrow q \equiv \neg p \vee q$
5	$(p \wedge \neg q) \vee (q \wedge \neg p)$	De Morgan’s law(4)
6	$(p \vee q) \wedge (p \vee \neg p) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p)$	Distributive (5)
7	$(p \vee q) \wedge (\neg q \vee \neg p)$	
8	$(p \vee q)$	Simplification (7)
9	$\neg q$	Premise
10	p	D.S. (8) and (9)