Discrete Structures

Day 9

Q. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

To show: $\neg(p \leftrightarrow q) \equiv \neg p \leftrightarrow q$

L.H.S

$$
\begin{aligned}
\neg(p \leftrightarrow q) &\equiv \neg[(p \rightarrow q) \land (q \rightarrow p)] \\
&\equiv \neg[(\neg p \lor q) \land (\neg q \lor p)] \\
&\equiv \neg[(\neg p \land \neg q) \lor (\neg p \land p) \lor (q \land \neg q) \lor (q \land p)] \\
&\equiv \neg[(\neg p \land \neg q) \lor F \lor F \lor (q \land p)] \\
&\equiv \neg(\neg p \land \neg q) \land T \land T \land \neg(q \land p) \\
&\equiv (p \lor q) \land (\neg q \lor \neg p) \\
&\equiv (\neg p \rightarrow q) \land (q \rightarrow \neg p) \\
&\equiv \neg p \leftrightarrow q
\end{aligned}
$$

Q. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

Q. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

To show: $p \to q \equiv \neg q \to \neg p$

L.H.S.

$$
\begin{aligned}
p \to q &\equiv \neg p \lor q \\
&\equiv q \lor \neg p \\
&\equiv \neg(\neg q) \lor \neg p \\
&\equiv \neg q \to \neg p
\end{aligned}
$$

very imp

Q. Show using truth table, $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

Q. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent without using truth table.

Hint: $A \oplus B = A\bar{B} + \bar{A}B$

Q. Determine whether $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology or not.

नही किया

Q. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.

When p, q are T and r are all F.

$(p \rightarrow q) \rightarrow r$ *is F*

$p \rightarrow (q \rightarrow r)$ *is T*

So, $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.

Predicates and Quantifiers

Predicates:

Statements involving variables, "$x > 3$", "$x = y + 3$"

These statements are neither true nor false when the values of the variables are not specified.

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part—the predicate, "is greater than 3"—refers to a property that the subject of the statement can have.

- We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable.
- The statement $P(x)$ is also said to be the value of the propositional function P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Q. Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

T F

Q. Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

F T

Q. Let $A(x)$ denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?

F T T

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.
- We will focus on two types of quantification here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

The *universal quantification* of *P(x)* is the statement

"*P(x)* for all values of *x* in the domain."

The notation $\forall xP(x)$ denotes the universal quantification of *P(x)*. Here $\forall$ is called the **universal quantifier.** We read $\forall xP(x)$ as "for all *x* *P(x)*" or "for every *x* *P(x)*."

An element for which *P(x)* is false is called a **counterexample** of $\forall xP(x)$.

Eg: Let *P(x)* be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall xP(x)$, where the domain consists of all real numbers?

- Because *P(x)* is true for all real numbers *x*, the quantification $\forall xP(x)$ is true.

Eg: Let *Q(x)* be the statement "$x < 2$." What is the truth value of the quantification $\forall xQ(x)$, where the domain consists of all real numbers?

- *Q(x)* is not true for every real number *x*, because, for instance, *Q(3)* is false. That is,

$x = 3$ is a counterexample for the statement $\forall xQ(x)$. Thus $\forall xQ(x)$ is false.

- The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists xP(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

- Besides the phrase "there exists" we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."

- Eg: Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists xP(x)$, where the domain consists of all real numbers?

- Because "$x > 3$" is sometimes true—for instance, when $x = 4$ the existential quantification of $P(x)$, which is $\exists xP(x)$, is true.

Statement	When True?	When False?
$\forall xP(x)$	*P(x)* is true for every *x*.	There is an *x* for which *P(x)* is false
$\exists xP(x)$	There is an *x* for which *P(x)* is true.	*P(x)* is false for every *x*.

THE UNIQUENESS QUANTIFIER

The **uniqueness quantifier**, denoted by $\exists!$ or $\exists 1$.

The notation $\exists !xP(x)$ [or $\exists 1xP(x)$] states "There exists a unique *x* such that *P(x)* is true."

Other phrases for uniqueness quantification include "there is exactly one" and "there is one and only one.")

Quantifiers with Restricted Domains

- An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition a variable must satisfy is included after the quantifier.
- Eg: What do the statements $\forall x < 0\ (x^2 > 0)$, $\forall y \neq 0\ (y^3 \neq 0)$, and $\exists z > 0\ (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?
- The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.
- The statement $\forall y \neq 0\ (y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.
- Finally, the statement $\exists z > 0\ (z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.

In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x (P(x) \lor Q(x))$.

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Negating Quantified Expressions

- "Every student in your class has taken a course in calculus."

  $\forall x P(x)$,

- Where,
- $P(x)$ is the statement "x has taken a course in calculus"

  and the domain consists of the students in your class.

- The negation of this statement is "It is not the case that every student in your class has taken a course in calculus."
- This is equivalent to "There is a student in your class who has not taken a course in calculus."

  $\exists x \neg P(x)$.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**.

Negation	*Equivalent Statement*
$\neg\exists x P(x)$	$\forall x \neg P(x)$
$\neg\forall x P(x)$	$\exists x \neg P(x)$

$\sim\exists x\, P(x) \equiv \forall x \sim P(x)$

Q. What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

$\sim\forall x\, Q(x)$

Let *H(x)* denote "*x* is honest."

"There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians.

The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$.

"Every politician is dishonest."

In English, the statement "All politicians are not honest" is ambiguous, this statement often means "Not all politicians are honest."

- "All Americans eat cheeseburgers"

Let $C(x)$ denote "x eats cheeseburgers."

"All Americans eat cheeseburgers": $\forall x C(x)$, where the domain consists of all Americans.

$\neg\forall x C(x)$ is given by $\exists x \neg C(x)$.

"Some American does not eat cheeseburgers"

"There is an American who does not eat cheeseburgers."

Q. What are the negation of the statements $\forall x(x^2 > x)$.

$\neg\forall x(x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$.

This can be rewritten as $\exists x(x^2 \le x)$.

$\exists x \sim(x^2 > x)$

$\exists x(x^2 \le x)$

Q. What are the negation of the statements $\exists x(x^2 = 2)$

$\forall x(x^2 \neq 2)$

What are the negation of the statements $\exists x(x^2 = 2)$

$\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$.

Q. Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x)\wedge\neg Q(x))$ are logically equivalent.

$$\begin{aligned}\neg\forall x(P(x) \rightarrow Q(x)) &\equiv \exists x(\neg(P(x) \rightarrow Q(x)))\\ &\equiv \exists x(\neg(\neg P(x) \vee Q(x)))\\ &\equiv \exists x(P(x) \wedge \neg Q(x))\end{aligned}$$

Translating from English into Logical Expressions

Q. Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

$\forall x P(x)$

Rewrite the statement by introduce a variable *x* so that we can clearly identify the appropriate quantifiers to use.

"For every student *x* in this class, *x* has studied calculus."

C(x): *x* has studied calculus.

$\forall x C(x)$, where the domain for *x* consists of the students in the class

Q. Express the statement "For every person *x*, if person *x* is a student in this class then *x* has studied calculus." using predicates and quantifiers

$\forall x(S(x) \to C(x))$

S(x): person *x* is in this class

C(x): *x* has studied calculus.

$\forall x(S(x) \to C(x))$, where the domain consist of all people.

The above statement cannot be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus