W1 (9.1)

HG System

Consider $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where A is a 2x2 real matrix and x is a 2-vector

- Given an initial vector $x^{(0)} := x(0)$, there must exist a unique solution – a vector function $x = x(t)$

  This function can be viewed as a parametric representation for a curve in the x_1x_2-plane, or a trajectory traversed by a moving particle whose velocity dx/dt is specified by the differential equation.

  The x_1x_2-plane itself is called the phase plane, and a representative set of trajectories is referred to as a phase portrait.

- If $x^{(0)} = \vec{0}$, then the solution is fixed at $\vec{0}$ since the derivative would be 0. This point is called the **equilibrium point**.

  If it is **stable**, then all solutions will converge onto this point. It is **unstable**, then all solutions will diverge from it.

- If $x^{(0)} \neq \vec{0}$, then $Ax = \vec{0}$ iff $\det A = 0 \implies ad - bc = 0 \implies \frac{a}{b} = \frac{c}{d}$

  Letting $h = \frac{a}{b} = \frac{c}{d}$, we have

$$\begin{aligned} \frac{dx}{dt} &= Ax = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \\ \frac{dx_1}{dt} &= ax_1 + bx_2 = b(hx_1 + x_2) \\ \frac{dx_2}{dt} &= cx_1 + dx_2 = d(hx_1 + x_2) \\ \frac{dx_2}{dt} / \frac{dx_1}{dt} &= d/b =: k \end{aligned}$$

  And integrating $\frac{dx_2}{dt} = k\frac{dx_1}{dt}$ we have $x_2 = kx_1 + C$ which means the solution is linear with slope k. Given C, the line passes through $(0, C)$

- We assume $\det A \neq 0 \iff Ax \neq 0$, so $x^{(0)} = 0$ is the only equilibrium point, called **critical point**

Real distinct roots, same sign

$$
\begin{aligned}
x &= c_1\, e^{r_1 t} \xi^{(1)} + c_2\, e^{r_2 t} \xi^{(2)} \\
\frac{dx}{dt} &= c_1\, e^{r_1 t} \underbrace{r_1 \xi^{(1)}} + c_2\, e^{r_2 t} \underbrace{r_2 \xi^{(2)}} \\
&= c_1 e^{r_1 t} \underbrace{A \xi^{(1)}} + c_2 e^{r_2 t} \underbrace{A \xi^{(2)}} \quad \textit{since} \quad A\xi^{(1)} = r_1 \xi^{(1)}, \quad A\xi^{(2)} = r_2 \xi^{(2)} \\
&= A \left[c_1 e^{r_1 t} \xi^{(1)} + c_2 e^{r_2 t} \xi^{(2)} \right] \\
&= Ax
\end{aligned}
$$

- nodal source (positive roots): as $t \to \infty, x \to \infty$ for all x, so it is asymptotically unstable (even if only 1 solution goes to infinity)
- nodal sink (negative roots): as $t \to \infty, x \to 0$, so it is asymptotically stable
- to visualize the solutions, use ξ_1, ξ_2 as new basis vectors: $x = c_1\, e^{r_1 t} \xi_1 + c_2\, e^{r_2 t} \xi_2 = y_1 \xi_1 + y_2 \xi_2$

  assume $r_1 < r_2$, then let $\alpha = \dfrac{c_2}{c_1}, \beta = \dfrac{r_2}{r_1} > 1$, so $y_2 = \alpha y_1^{\beta}$

  plotting this with arbitrary α and fixed β, we get parabolic trajectories (nodal sink below):

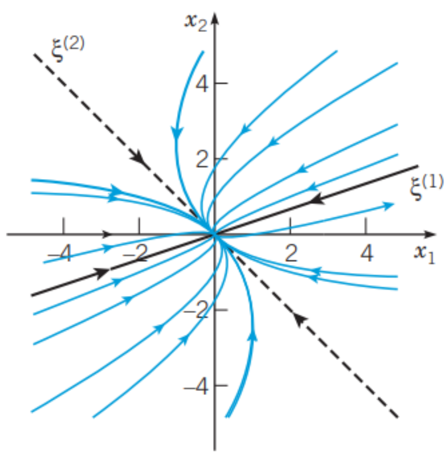


FIGURE 9.1.1 Trajectories in the phase plane when the origin is a node with $r_1 < r_2 < 0$. The solid black and dashed black curves show the fundamental solutions $\xi^{(1)} e^{r_1 t}$ and $\xi^{(2)} e^{r_2 t}$, respectively.

Real distinct roots, dif signs

- saddle point:

  assuming $r_1 < 0 < r_2$, then as $t \to \infty, e^{r_1 t} \to 0, e^{r_2 t} \to \infty$

  so unless $c_2 = 0, x = c_1\, e^{r_1 t} \xi^{(1)} + c_2\, e^{r_2 t} \xi^{(2)} \to \infty$, which is unstable

- to visualize, rewrite as $y_2 = \alpha y_1^{-\beta}$

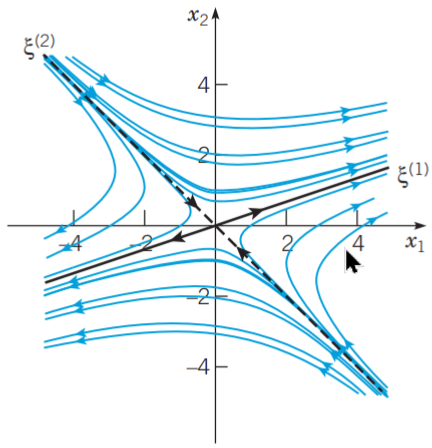


FIGURE 9.1.2 Trajectories in the phase plane when the origin is a saddle point with $r_1 > 0, r_2 < 0$. The solid black and dashed black curves show the fundamental solutions $\boldsymbol{\xi}^{(1)} e^{r_1 t}$ and $\boldsymbol{\xi}^{(2)} e^{r_2 t}$, respectively.

Duplicate roots

- proper node/star point (2 eigenvectors): $x = c_1\, e^{r_1 t}\xi_1 + c_2\, e^{r_2 t}\xi_2$

  If r < 0, then $x \to 0$ (stable); if $r > 0$, then $x \to \infty$ (unstable)

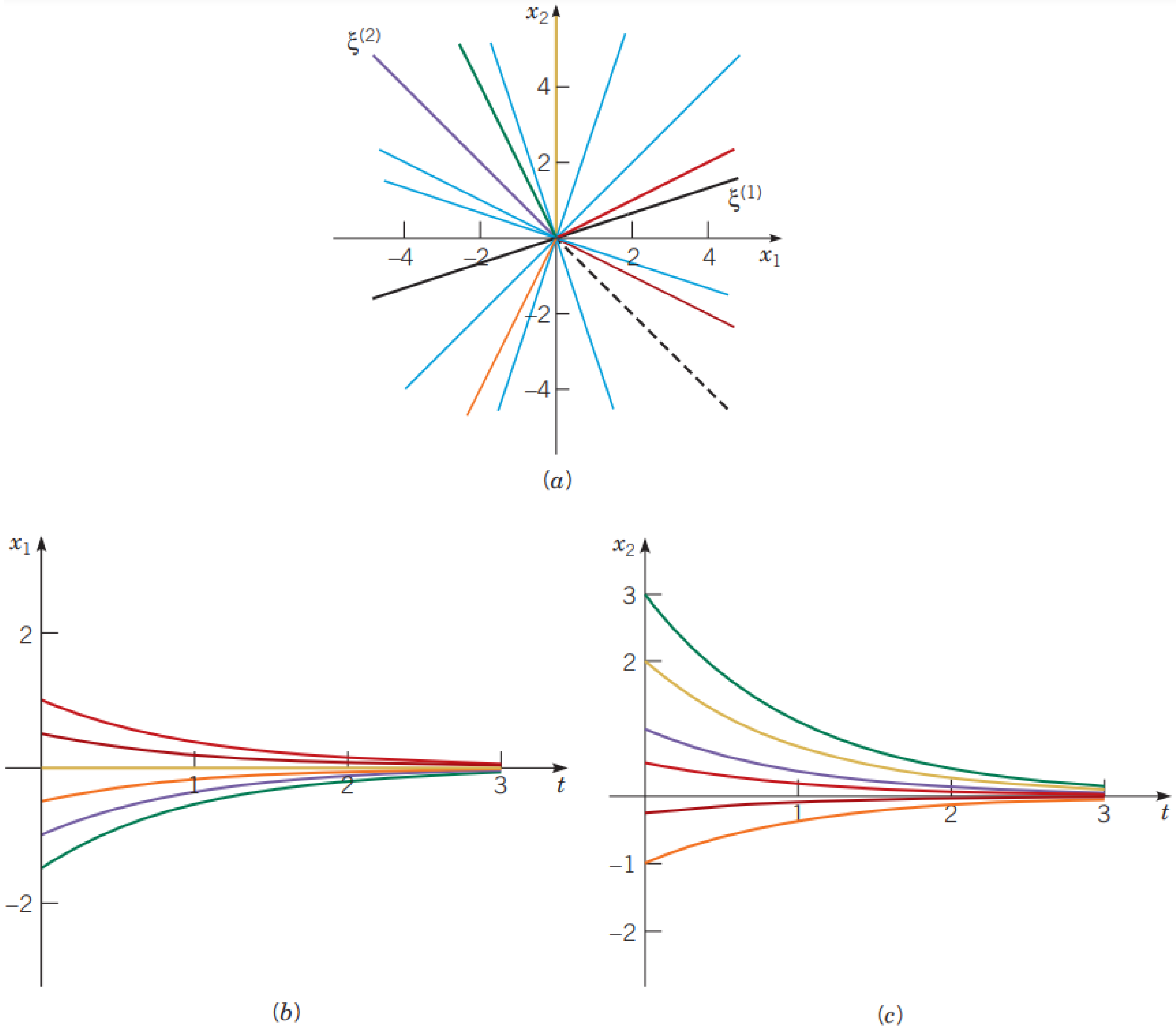


FIGURE 9.1.3 (a) Trajectories in the phase plane when the origin is a proper node with $r_1 = r_2 < 0$. (b) and (c) show the corresponding component plots for x_1 vs. t and x_2 vs. t, respectively. The solid black and dashed black curves show the fundamental solutions $\xi^{(1)} e^{r_1 t}$ and $\xi^{(2)} e^{r_2 t}$, respectively. The purple curve is for the solution that passes through $(-1, 1)$, orange through $(-1/2, -1)$, green $(-3/2, 3)$, red $(1, 1/2)$, brown $(1/2, -1/4)$, and gold $(0, 2)$.

- improper/degenerate node (1 eigenvector): $x = c_1 \xi e^{rt} + c_2(\xi t e^{rt} + \eta e^{rt})$ $\leftarrow$ Solve for η using $(A - rI)\eta = \xi$

  If r < 0, then $x \to 0$ (stable); if $r > 0$, then $x \to \infty$ (unstable)

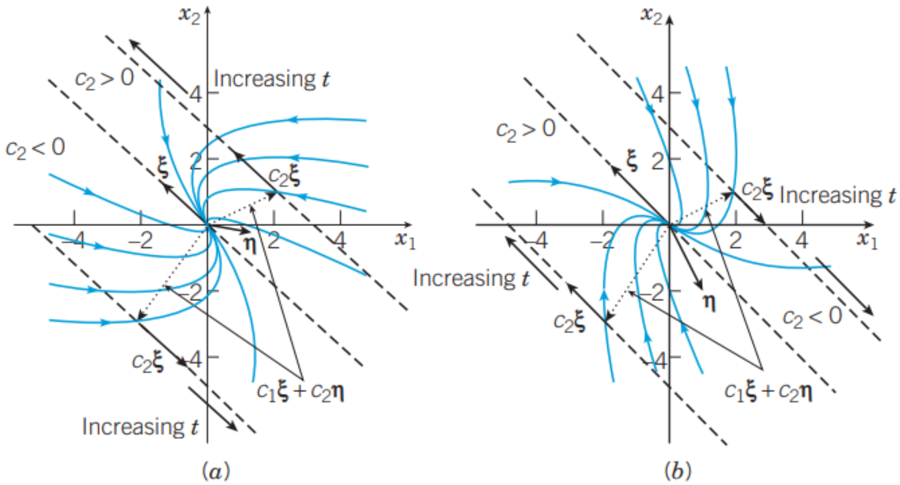


FIGURE 9.1.4 (*a*) The phase plane for an improper node with eigenvalues $r_1 = r_2 < 0$ and one independent eigenvector $\boldsymbol{\xi}$. (*b*) The phase plane for a system with the same eigenvalues $r_1 = r_2 < 0$ and eigenvector $\boldsymbol{\xi}$ but a different generalized eigenvector $\boldsymbol{\eta}$.

- To visualize, write $x = c_1 e^{rt}\xi + c_2(te^{rt}\xi + e^{rt}\eta) = \underbrace{e^{rt}(c_1 + c_2 t)}_{y_1}\xi + \underbrace{e^{rt}c_2}_{y_2}\eta = y_1\xi + y_2\eta$

  Semi-axes? Take $c_2 = 0, c_1 = 1$ and -1. So $y_2 = 0, y_1 = e^{rt}c_1$

  Tangent to which axis? Take $c_2 \neq 0$. If $|y_1| \gg |y_2|$, then the trajectory is tangent to the ξ axis

  Tangent to which semi-axis? Since sgn $y_1 =$ sgn c_2, sgn $y_2 =$ sgn c_2, we know that $y_1 > 0$ for vectors in the same half plane as η, and $y_1 < 0$ in the other half plane. Similarly for y_2. So the trajectories in the half plane $y_2 > 0$ are tangent to the positive ξ semi-axis at the origin.

Complex roots, non-zero real part

Plug $\xi = u + iv$ into $A\xi = \underbrace{(\lambda + i\mu)}_{r}\xi$

$$A(u + iv) = (\lambda + i\mu)(u + iv)$$
$$Au + iAv = (\lambda u - \mu v) + i(\mu u + \lambda v)$$

$$Au = \lambda u - \mu v$$
$$Av = \mu u + \lambda v$$

The RHS can be written as $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

$$\frac{dx}{dt} = Ax$$
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1' = \lambda x_1 + \mu x_2$$

Use polar coordinates. First differentiate $r^2 = x_1^2 + x_2^2$

$$\begin{aligned}
\left(r^2\right)' &= 2x_1x_1' + 2x_2x_2' \\
2rr' &= 2x_1\left(\lambda x_1 + \mu x_2\right) + 2x_2\left(-\mu x_1 + \lambda x_2\right) \\
rr' &= \lambda\left(x_1^2 + x_2^2\right) = \lambda r^2 \\
r' &= \lambda r
\end{aligned}$$

Solve $\dfrac{dr}{dt} = \lambda r$ (separable ODE) to get $\boxed{r = Ce^{\lambda t}}$ where $C \geq 0$ is an arbitrary constant.

Now differentiate $\tan\theta = \dfrac{x_2}{x_1}$ to get $\theta' = -\mu$, so $\boxed{\theta = -\mu t + \theta_0}$

$$\begin{aligned}
(\tan\theta)' &= \frac{x_1x_2' - x_2x_1'}{x_1^2} = \frac{x_1\left(-\mu x_1 + \lambda x_2\right) - x_2\left(\lambda x_1 + \mu x_2\right)}{x_1^2} \\
\frac{\theta'}{\cos^2\theta} &= \frac{-\mu\left(x_1^2 + x_2^2\right)}{x_1^2} = -\mu\frac{r^2}{x_1^2} = -\mu\frac{1}{\frac{x_1^2}{r^2}} = -\mu\frac{1}{\cos^2\theta} \\
\theta' &= -\mu
\end{aligned}$$

- spiral sink (negative real part): $\lambda < 0 \implies r \to 0 \implies x \to 0$, so it is asymptotically stable
- spiral source (positive real part): $\lambda > 0 \implies r \to \infty \implies x \to \infty$, so it is unstable

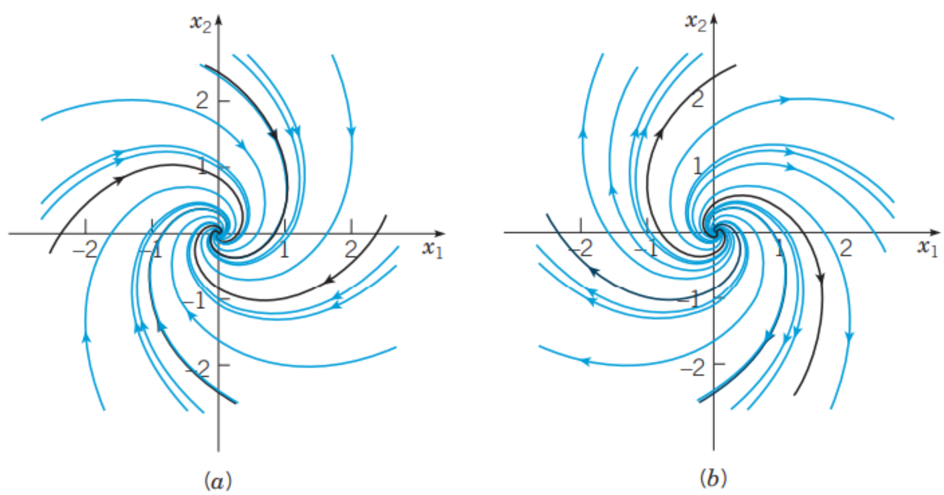


FIGURE 9.1.5 Trajectories in the phase plane for a linear system with eigenvalues $\lambda \pm i\mu$. (a) A spiral sink, $\lambda < 0$, and (b) a spiral source, $\lambda > 0$.

Complex roots, zero real part

- center: $\lambda = 0 \implies r = Ce^{\lambda t} = 0$ is a circle centered at the origin. It is stable, but not asymptotically stable.

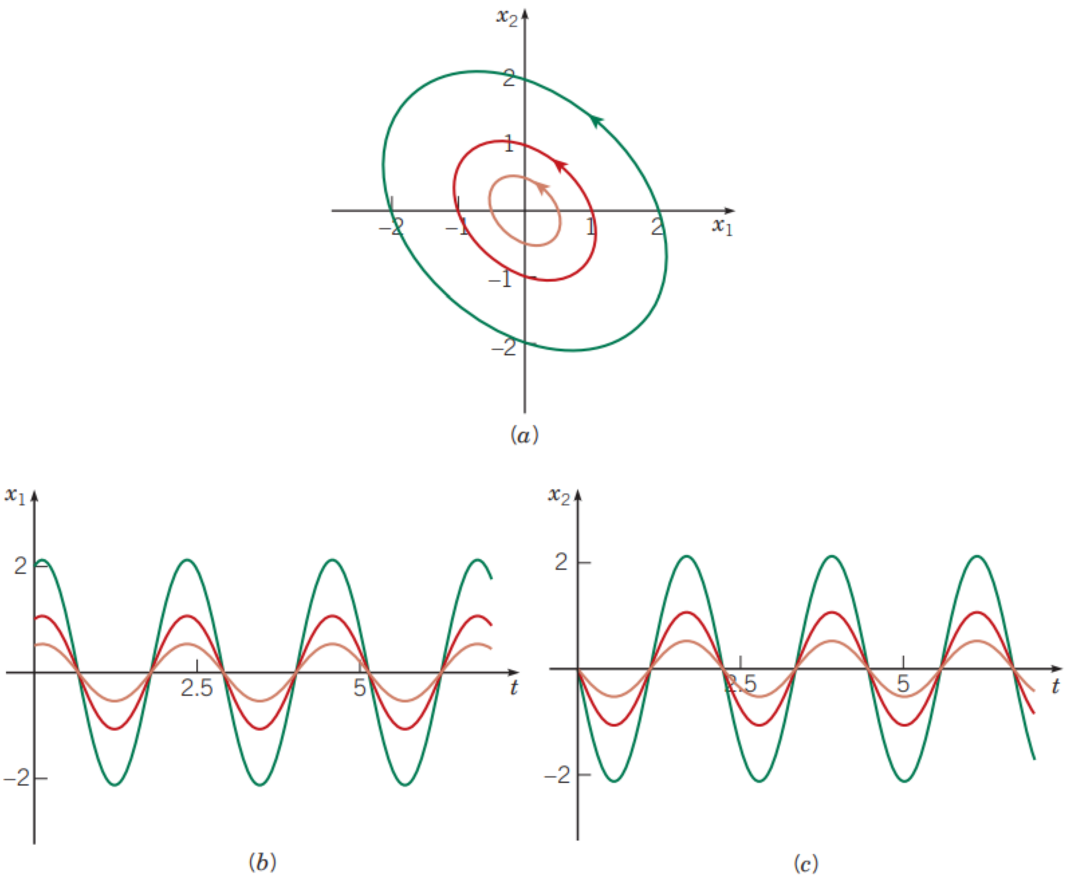


FIGURE 9.1.6 (a) Trajectories in the phase plane when the linear system has eigenvalues $\pm i\mu$. (b) and (c) show the component plots for x_1 vs. t and x_2 vs. t, respectively. The orange curves pass through $(1/2, 0)$, red through $(1, 0)$, and the green through $(2, 0)$.

NHG System

If given $\dfrac{dx}{dt} = Ax - b$, then let $x = x^{(0)} + u$ where u is a new unknown vector.

Sub it in: $\dfrac{dx}{dt} = A(x^{(0)} + u) - b = \dfrac{du}{dt}$

Kill the NHG term by solving $Ax^{(0)} = b$ so that the above becomes $Au = \dfrac{du}{dt}$

The solution $x^{(0)}$ is the equilibrium point.

Plotting Phase Portraits

1. Plot the eigen vectors as secondary axes at th

   Their directions are determined by the sign of r. If positive, point away from origin.

2. Draw trajectories.

   - if real, distinct roots:

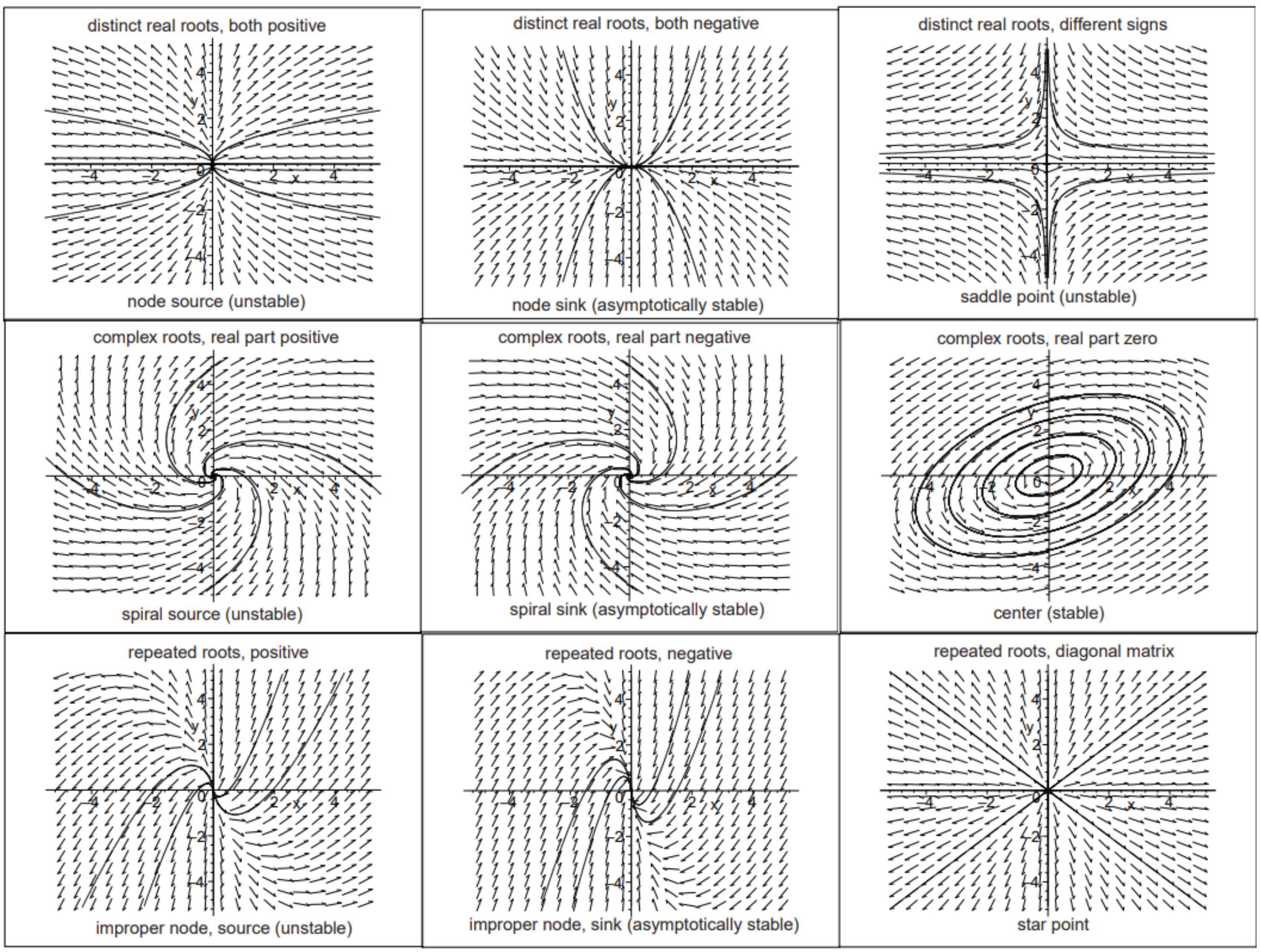
- dif signs: saddle; just follow the directions on the secondary axes (4 curves, i.e).
- same signs: node; see which axis has a bigger eigen value $|r|$ (means faster convergence), make the trajectory parallel to that axis and curve it towards the other axis until it's tangent, then make a U-turn (trace a parabola)

- if duplicate roots:

  - 2 eigen vectors: proper node/star point; just draw straight lines through the origin.
  - 1 eigen vector: improper node; arrow test: go to (1, 0) and plot $A\begin{bmatrix}1\\0\end{bmatrix}$, if unclear go to (0, 1) and plot $A\begin{bmatrix}0\\1\end{bmatrix}$ and connect/follow the arrow(s)

- if complex roots:

  - non-zero real part: spiral; arrow test
  - purely imaginary: circle; arrow test



W2 (9.2-9.4)

Autonomous Systems

We have 2 diff eq'ns of the form $\dfrac{dx}{dt} = F(x, y), \dfrac{dy}{dt} = G(x, y)$

There exists a unique solution $x = x(t), y = y(t)$ satisfying the initial conditions $x(t_0) = x_0, y(t_0) = y_0$

In vector form, $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ and there exists a solution $\mathbf{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ satisfying the initial cond. $\mathbf{x}(t_0) = \mathbf{x}^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

Since F, G do not depend on t, the system is said to be **autonomous**.

An autonomous system has an associated direction field that is indep of time.

Note Solutions that satisfy the same initial condition must lie on the same trajectory.

Stability at Critical Points

At a critical point, $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x}' = \mathbf{0}$

Critical points are important because they correspond to equilibrium solutions (where x(t) and y(t) are constant). These solutions determine the behavior of trajectories throughout the phase plane.

Stable: given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(0) - x^{(0)}\| < \delta$, then

$$\|x(t) - x^{(0)}\| < \epsilon \, \forall \, t$$

Asymptotically stable: given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(0) - x^{(0)}\| < \delta$, then as $t \to \infty$

$$\|x(t) - x^{(0)}\| \to 0$$

Note Asymptotic stability $\implies$ stability. The converse is false.

For the linear system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, ❓ dif in the TB for stable case (P408 thm)

- **stable** if $r_1, r_2 \leq 0$ are real, or if $r_1 = \lambda + i\mu, \mu \neq 0, \lambda \leq 0$
- **asymp. stable** if $r_1, r_2 < 0$ are real, or if $r_1 = \lambda + i\mu, \mu \neq 0, \lambda < 0$

Oscillating Pendulum

The following models a pendulum's angular momentum about the origin.

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

$\gamma = \dfrac{c}{mL}$ is the damping force (m is the mass, L is the length of the pendulum), $\omega^2 = \dfrac{g}{L}$

We can convert it to an autonomous system of 2 first-order equations by letting $x = \theta, y = \dfrac{d\theta}{dt}$, so

$$\begin{aligned} \frac{dx}{dt} &= \frac{d\theta}{dt} = y \\ \frac{dy}{dt} &= \frac{d^2\theta}{dt^2} = -\gamma\frac{d\theta}{dt} - \omega^2\sin\theta = -\gamma y - \omega^2 \sin x \end{aligned}$$

Linearization

Given $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}$, find the critical point:

$$Ax^{(0)} + b = 0, \quad Ax^{(0)} = -b, \quad x^{(0)} = -A^{-1}b$$

Variable change:

$$x = x^{(0)} + u$$

So we have

$$\begin{aligned} \frac{du}{dt} = \frac{dx}{dt} &= Ax + b \\ &= A\left(x^{(0)} + u\right) + b \\ &= Ax^{(0)} + Au + b \\ &= -b + Au + b \\ &= Au \end{aligned}$$

Local Linearization

Let $\mathbf{x}^{(0)} = \left(x^{(0)}, y^{(0)}\right)$ be a critical point, i.e.

$$\begin{aligned} F\left(x^{(0)}, y^{(0)}\right) &= 0 \\ G\left(x^{(0)}, y^{(0)}\right) &= 0 \end{aligned}$$

Use Taylor expansion, discard the error terms and write in vector form:

$$f(x,y)=\begin{bmatrix}F(x,y)\\G(x,y)\end{bmatrix}\simeq J\begin{bmatrix}x-x_0\\y-y_0\end{bmatrix}=J\left(\mathbf{x}-\mathbf{x}^{(0)}\right)$$

where J is the Jacobian at $x^{(0)}$:

$$J=\begin{bmatrix}F_x\left(x_0,y_0\right) & F_y\left(x_0,y_0\right)\\G_x\left(x_0,y_0\right) & G_y\left(x_0,y_0\right)\end{bmatrix}$$

Variable change:

$$\mathbf{x}-\mathbf{x}^{(0)}=u \iff \begin{cases}x-x_0=u_1\\y-y_0=u_2\end{cases}$$

Then we have

$$\frac{du}{dt}=\frac{dx}{dt}=f(x,y)\simeq J\left(x-x^{(0)}\right)=Ju$$

Note This method only works in the vicinity of the critical point (hence "local" linearization).

Summary Table

TABLE 9.3.1	Stability and Instability Properties of Linear and Locally Linear Systems			
	Linear System		Locally Linear System	
Eigenvalues	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	*PN or IN*	Unstable	*N or SpP*	Unstable
$r_1 = r_2 < 0$	*PN or IN*	Asymptotically stable	*N or SpP*	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	*C*	Stable	*C or SpP*	Indeterminate

Key: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Competing Species

The following yields 4 critical points

$$\frac{dx}{dt} = x\left(\epsilon_1 - \sigma_1 x - \alpha_1 y\right)$$
$$\frac{dy}{dt} = y\left(\epsilon_2 - \sigma_2 x - \alpha_2 y\right)$$

Predator Prey System

The following yields 2 critical points (saddle and center)

$$\frac{dx}{dt} = x(a - \alpha y)$$
$$\frac{dy}{dt} = y(-c + \gamma x)$$

It is integrable:

$$\frac{(-c + \gamma x)dx}{x} = \frac{(a - \alpha y)dy}{y}$$

Plotting Phase Portraits

If given a system, plot the behavior at each critical point first. Then combine to form the big picture.

Separatrix:

Let P be a point in the xy-plane with the property that a trajectory passing through P ultimately approaches the critical point as t → ∞. Then this trajectory is said to be attracted by the critical point.

The set of all such points P is called the **basin of attraction** or the **region of asymptotic stability** of the critical point.

A trajectory that bounds a basin of attraction is called a **separatrix** because it separates trajectories that approach a particular critical point from trajectories that do not.

W3, 4 (Ch6)

Comparison Test for Convergence

If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges.

On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges.

Integral Transform

A relation of the form $F(s) = \displaystyle\int_a^b K(s,t)f(t)dt$ where f is transformed into F by $K(s,t)$, the kernel of the transformation. F is said to be the transform of f.

Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt \quad \text{exists if}$$

- f is piecewise continuous on $0 \leq t \leq A \; \forall \; A > 0$
- $\exists \; K, a, M \in \mathbb{R}$ such that $|f(t)| \leq Ke^{at}$ when $t \geq M$

To use it to solve a diff eq'n:

1. Transform an initial value problem for an unknown function f in the t-domain into a simpler problem for F in the s-domain

2. Solve it to find F.
3. Invert the transform - recover the desired function f from its transform F.

Inverse Transform

1. multiply numerator by denominator

Initial Value Problem

$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0) \quad \text{exists for } s > a \text{ if}$

- f is continuous, and f' is piecewise continuous on any interval $0 \leq t \leq A$
- $\exists\, K, a, M$ s.t. $|f(t)| \leq Ke^{at}\ \forall\, t \geq M$

Extending to the second derivative:

$$\begin{aligned}\mathcal{L}\left\{f''(t)\right\} &= s\mathcal{L}\left\{f'(t)\right\} - f'(0) \\ &= s(s\mathcal{L}\left\{f(t)\right\} - f(0)) - f'(0) \\ &= s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)\end{aligned}$$

Provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the n^{th} derivative $f^{(n)}$ can be derived by n successive applications of this theorem. The result is given in the following corollary.

Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, \left|f^{(n-1)}(t)\right| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists for $s > a$ and is given by

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

General Second-Order Linear Eq'n

$$\begin{aligned} ay'' + by' + cy &= f(t) \\ a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) &= F(s)\end{aligned}$$

Heaviside (Unit Step) Function

$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$

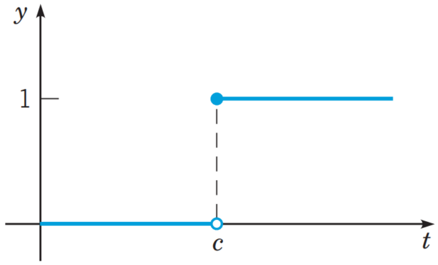


FIGURE 6.3.1 Graph of $y = u_c(t)$.

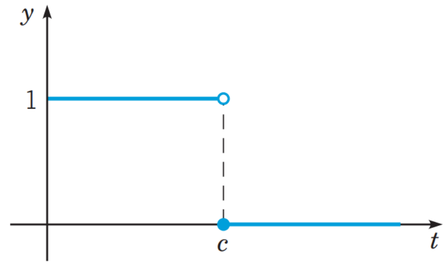


FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

The Laplace transform of u_c for $c \geq 0$ is easily determined:

$$\begin{aligned}\mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt \\ &= \frac{e^{-cs}}{s}, \quad s > 0.\end{aligned}$$

Since $u_0(t) = 1$ for all $t \geq 0$, we have that

$$\mathcal{L}\{u_0(t)\} = \frac{e^0}{s} = \frac{1}{s} = \mathcal{L}\{1\}$$

For a given function f defined for $t \geq 0$, we will often want to consider the related function g defined by

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction.

Making use of the unit step function, we can write $g(t)$ in the form

$$g(t) = u_c(t) f(t-c)$$

Translation Thm

The translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}

If the Laplace transform of $f(t)$, $F(s) = \mathcal{L}\{f(t)\}$, exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s), \quad s > a$$

Conversely, if $f(t)$ is the inverse Laplace transform of $F(s)$, $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\left\{e^{-cs} F(s)\right\}$$

Multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction

If the Laplace transform of $f(t), F(s) = \mathcal{L}\{f(t)\}$, exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$