Background & Definitions

ODE vs PDE

ODEs involve one or more functions of a single variable, with all derivatives **ordinary** ones w.r.t. that variable. **PDEs** allow functions of several variables and **partial** derivatives of the unknown functions w.r.t. those variables. All ODEs can be viewed as PDEs, i.e. ODE $\subseteq$ PDE

PDEs are generally more difficult to understand the solutions to than ODEs. Basically every big theorem about ODEs does not apply to PDEs. It's more than just the basic reason that there are more variables. For an ODE, we can often view the single independent variable as a time variable, so that ODEs govern a motion or flow of an object in time. The idea of ODEs governing "motion" allows us to use many mathematical results that have analogues in physics (for example empirical behavior regarding Newton's law) and allow us to understand the solutions much more precisely.

For a linear ODE, the set of solutions form a vector space with finite dimension. However, a linear PDE (like the heat equation) has a set of solutions that form a vector space with infinitely many dimensions.

Def. Order of a Diff Eq

The order of a differential equation is defined to be that of the highest order derivative it contains.

Def. Linear Diff Eq

A linear eq involves only y raised to the first power. The power or degree is not to be confused with the order.

Def. Separable Equation

A separable equation is one that can be written so that x and y appear on opposite sides of the equation.

Def. Exact Equation

A diff eq in the form $M(x,y)+N(x,y)y'=0$ is exact if $\partial_y M=\partial_x N$

In words, it is exact if we can find a function $\Psi(x,y)$ such that $\Psi_x=M, \Psi_y=N$

This way, $\Psi_x+\Psi_y y'=0 \overset{\text{chain rule}}{\Longrightarrow} \dfrac{d}{dx}\Big(\Psi\big(x,y(x)\big)\Big)=0 \implies \Psi\big(x,y(x)\big)=C$

Since $\Psi_{xy} = \Psi_{yx}$, we have $\begin{cases} \Psi_{xy} = (\Psi_x)_y = M_y \\ \Psi_{yx} = (\Psi_y)_x = N_x \end{cases} \implies M_y = N_x$

Seeing if this equality holds is a simple way to test exactness (finding Ψ can be cumbersome).

Def. Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Notes

- W must either = 0 everywhere, or $\neq 0$ everywhere (at all values of t)
- A fundamental pair is a pair of solutions that has $W \neq 0$

  If the pair is not linearly independent, then $W = 0$

Def. Abel Formula

$W = ce^{-\int p dt}$ can be calculated w/o knowing the solutions y_1, y_2

Derived from derivative of the Wronskian:

$$\begin{aligned}
W' &= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} \\
&= y_1 y_2'' - y_1'' y_2 \\
&= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1) \\
&= -p(y_1 y_2' - y_1' y_2) \\
&= -pW \\
\frac{dW}{W} &= -p \quad \leftarrow \text{W cannot be 0} \\
W &= ce^{-\int pdt} \quad \leftarrow \text{If W = 0, then c must = 0}
\end{aligned}$$

Hence, by setting c=1, we force W to be non-zero, and as a result we obtain a fundamental pair

Def. Cramer's Rule

To solve $\begin{cases} au_1' + bu_2' = e \\ cu_1' + du_2' = f \end{cases}$

$$u_1' = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{W}$$

$$u_2' = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{W}$$

For Lagrange's var. of parameter

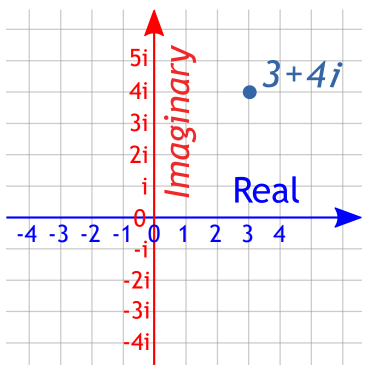
$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ RHS & y_2' \end{vmatrix}}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & RHS \end{vmatrix}}{W}$$

Def. Complex Numbers

are made up of a real part and an imaginary part

$z = a + ib$ where $i = \sqrt{-1}$



$r = |z| = \sqrt{a^2 + b^2}$ gives the magnitude (called modulus)

$\theta = tan^{-1}\dfrac{b}{a}$ gives the angle between the vector and the x-axis

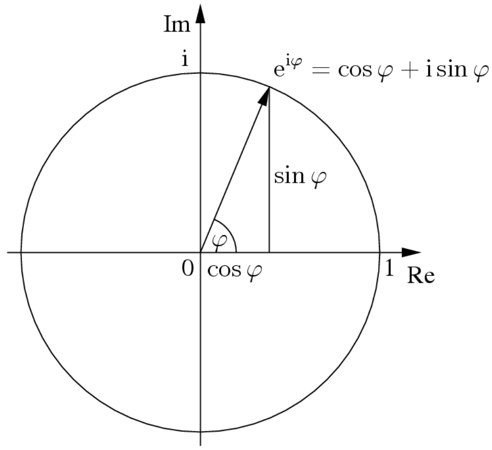
$a = r\cos\theta, b = r\sin\theta \implies z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta)$ is the polar form of a complex number

The complex conjugate of $z = a + ib$ is $\bar{z} = a - ib$, which has the same real part and opposite-signed imaginary part

- The product of a complex conjugate pair is a real number: $a^2 + b^2 > 0$

Def. Euler's Formula

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ by vector addition on the complex plane



Magnitude: $|e^{i\theta}| = \sqrt{|\cos^2\theta| + |i^2\sin^2\theta|} = 1$

Corollary: $|e^{\alpha+i\beta}| = |e^{\alpha}| \times |e^{i\beta}| = e^{\alpha}$

Property: $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$

$$\begin{aligned} Proof: \ e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \text{ using trig identities} \\ &= e^{i(\alpha+\beta)} \end{aligned}$$

Def. Bessel Equation

$x^2y'' + xy' + (x^2 - n^2)y = 0$ where n is referred to as the order

Solve using d'Alembert's if HG

$y_1 = x^{-\frac{1}{2}}\sin x$ is the solution (Bessel function) when order = 1/2

Def. Eigen Vectors & Values

Given a transformation, an eigen vector is a non-zero vector that only gets stretched/scaled by a factor (not morphed)

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \quad \text{where } \lambda \text{ is the scaling factor, called eigen value} \\ A\vec{v} &= (\lambda I)\vec{v} \quad \text{Since LHS is matrix mult, turn RHS into matrix mult as well} \\ (A - \lambda I)\vec{v} &= \vec{0} \quad A - \lambda I \text{ transforms } \vec{v} \text{ to } \vec{0} \\ \det(A - \lambda I) &= 0 \quad \text{Since the determinant is the scaling factor for a volume under a transformation, it must} = 0 \text{ here} \end{aligned}$$

To obtain the eigen value, solve the equation that arises from det = 0 (last line), called the characteristic polynomial

To obtain the eigen vectors, plug the eigen values back into the second to last line, and solve the system

Additional notes

- One eigen value can be associated with multiple eigen vectors, e.g. a transformation that scales all vectors by a factor
- Some transformations do not have real eigen values or vectors, e.g. rotations where all vectors are knocked off their span
- Eigen basis: if all basis vectors are eigen vectors, then they must form a diagonal matrix with eigen values on the diagonal.

  Since diagonal matrices are much easier to work with, we use change of basis to turn basis vectors into eigen vectors.

  Not all matrices can become diagonal, so we can't find an eigen basis for all transformations

Def. Power Series

$\sum_{n=0}^{\infty} a_n (x - x_0)^n$ where a_n are the numerical coefficients, x is a variable, and x_0 is the central point

The set of x for which the series converges is given by the convergence domain $(x_0 - r, x_0 + r)$ where r is the convergence radius. The bounds may or may not be in the domain.

To find the radius, apply the ratio test:

$$\lim_{n\to\infty} \frac{\left|a_{n+1}(x - x_0)^{n+1}\right|}{\left|a_n (x - x_0)^n\right|} = |x - x_0| \underbrace{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}_{\rho}$$

The series converges absolutely at x if $|x - x_0|\rho < 1 \implies |x - x_0| < \rho^{-1} = r$

Def. Singularity

A singularity is a point at which a given mathematical object is not defined. E.g. f(x) = 1/x has a singularity at x = 0.

Def. Harmonic Series

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

W1 (2.1, 2.2)

1st Order Linear Equation

If the equation is **separable**, then separate and integrate directly to get F(x, y) = C.

If the equation is in the form $P(t)y'(t) + Q(t)y(t) = G(t)$, then

1. divide both sides by P (to get $y' + \frac{Q}{P} = \frac{G}{P}$)
2. set up equations

> **(1)** $(\mu y)' = \mu y' + \mu' y \leftarrow \textit{Leibniz}$
>
> **(2)** $\mu G = \mu y' + \mu \frac{Q}{P} y \leftarrow \textit{equation} \times \textit{integrating factor}$

3. Equate **RHS** of (1) and (2), integrate against t to get μ

   Let C = 1 if $C \cdot (*)$, C = 0 if $(*) + C$

4. Equate **LHS** of (1) and (2), plug in μ, integrate against t to find y

Homogeneous Equation

If the equation is **homogeneous** (terms depend on both y and x, not just y or just x)

1. call that term v, rearrange to get an expression for y
2. take the derivative of y w.r.t. x
3. plug in y and y', solve as usual

W2 (2.6, 2.8)

1st Order Equation (Exact Equations)

1. If given equations in the form $M(x,y) + N(x,y)y' = 0$, express it as

> $$Mdx + Ndy = 0$$

2. Check whether it is exact:

> $$\partial_y M = \partial_x N$$

If so, it is an exact equation - proceed to step 3.

If not, find an integrating factor μ to make the equation exact.

> $\partial_y(\mu M) = \partial_x(\mu N)$
>
> If μ dependent on x:
>
> $$\begin{aligned} \partial_y[\mu(x)M] &= \partial_x[\mu(x)N] \\ \mu\partial_y M &= \mu' N + \mu\partial_x N \\ \mu\partial_y M - \mu\partial_x N &= \frac{d\mu}{dx}N \\ \int \frac{d\mu}{\mu} &= \int \frac{\partial_y M - \partial_x N}{N}dx \end{aligned}$$
>
> If μ dependent on y:
>
> $$\begin{aligned} \partial_y[\mu(y)M] &= \partial_x[\mu(y)N] \\ \mu' M + \mu\partial_y M &= \mu\partial_x N \\ \frac{d\mu}{dy}M &= \mu\partial_x N - \mu\partial_y M \\ \int \frac{d\mu}{\mu} &= \int \frac{\partial_x N - \partial_y M}{M}dy \end{aligned}$$
>
> If μ dependent on xy:
>
> $$\begin{aligned} \partial_y[\mu(xy)M] &= \partial_x[\mu(xy)N] \\ \mu' xM + \mu\partial_y M &= \mu' yN + \mu\partial_x N \\ \mu' xM - \mu' yN &= \mu\partial_x N - \mu\partial_y M \\ \frac{\mu'}{\mu} &= \frac{\partial_x N - \partial_y M}{xM - yN} \end{aligned}$$

> **Note:** the resulting function in RHS must **only** depend on what μ is dependent on (either x, y, or xy).
>
> Plug in μ to obtain new equation.
>
> (Optional) Check $\partial_y(new\ M) \stackrel{?}{=} \partial_x(new\ N)$

3. Find F by integrating one of

> $M = \partial_x F \quad \text{or} \quad N = \partial_y F$
>
> **Note** The resulting constant C is a function (of the variable we are not integrating against)

4. Solve for C' by plugging F into the other

> Check that it does NOT depend on the variable it is not a function of:
>
> - x if integrated M in step 3
> - y if integrated N in step 3

5. Integrate C', plug it into F, set it equal to a constant C
6. (Optional) Check that the following holds

> $$\frac{\partial F}{\partial x} = M$$
> $$\frac{\partial F}{\partial y} = N$$

Note on Existence and Uniqueness

If the equation is of the form $\frac{dy}{dt} = f(t, y)$, and we have an initial condition $y(t_0) = y_0$, then $\exists$ a unique solution $y = y(t)$. However, the solution may be defined only on a small interval of time:

> $$y(\Delta t) \approx \underbrace{y_0}_{y(0)} + \underbrace{f(0, y_0)}_{y'(0)} \Delta t$$

W3 (3.1, 3.2)

2nd Order Lin HG Eq - Constant Coefficients

If it is a linear homogeneous equation of 2nd order with <u>constant</u> coefficients: ay'' + by' + cy = 0

1. Plug in $y = e^{rt}$ and its derivative(s) to obtain the **characteristic equation**

> $ar^2 + br + c = 0$

2. Solve for r and linearly combine the solutions to get the general solution

> $y_1 = e^{r_1 t}, y_2 = e^{r_2 t} \implies y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
>
> If complex: $r = a + bi \implies y = e^{rt} = e^{(a+bi)t} = e^{-at}(c_1 \cos bx + c_2 \sin bx)$

3. If given initial conditions, differentiate y, plug in values and solve for c_1, c_2

> $y(0) = c_1 y_1(0) + c_2 y_2(0)$
>
> $y'(0) = c_1 y_1'(0) + c_2 y_2'(0)$
>
> (Can use Cramer's rule to solve)

In step 1, if $r_1 = r_2$, find the second solution by solving $W_{det} = W_{Abel}$.

2nd Order Lin HG Eq - Non-Const. Coefficients

If it is a linear homogeneous equation of 2nd order with <u>non-constant</u> coefficients: y'' + p(t)y' + q(t)y = 0

Then use the Wronskian: $W_{det} = W_{Abel}$

> Determinant method (how it's defined):

> $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$
>
> Abel Formula (can be calculated w/o knowing the solutions y_1, y_2):
>
> $$W = ce^{-\int p dt}$$
>
> Set c=1 to force W to be non-zero, so that we obtain a lin ind (fundamental) pair

Note on Existence & Uniqueness

If given equation of the form y'' + p(t)y' + q(t)y = 0 with initial conditions, then there exists a unique solution that is defined for all t (it does not escape to infinity).

If the equation is in the form a(t)y'' + b(t)y' + c(t)y = 0, then the solution could escape to infinity.

W4, 5 (3.3, 3.4)

Examples

1. Illustrates how to form a general solution using complex roots
2. Illustrates **d'Alembert's** method

   one solution must be provided

   $y_2 = vy_1$ where v is an unknown function $v(t)$

   Expand, separate the variables, plug in w = v' (**reduction of order**), solve for v
3. Illustrates Euler Equation & indicial equation
4. Combines d'Alembert's, Wronskian/Abel, integrating factor
5. Euler's where the indicial eq'n gives duplicate roots

   No need to use d'Alembert's - the solutions in this case will always be $y_1 = x^r, y_2 = x^r \log x$
6. Euler's with complex roots
7. Bessel equation (d'Alembert's)

   $x^2y'' + xy' + (x^2 - n^2)y = 0$ where n is referred to as the order

Euler-Cauchy Equation

If given an equation in the form $t^2y'' + \alpha ty' + \beta y = 0$ where $\alpha, \beta \in \mathbb{R}$:

1. Plug in $y = t^r$ and its derivatives to obtain the **indicial equation**

> $t^2(t^r)'' + \alpha t(t^r)' + \beta t^r = 0$
>
> $t^2\big(r(t^{r-1})\big)' + \alpha t(rt^{r-1}) + \beta t^r = 0$
>
> $r(r-1)\cancel{t^{r-2+2}} + \alpha r\cancel{t^{r-1+1}} + \beta \cancel{t^r} = 0$
>
> $r^2 + (\alpha - 1)r + \beta = 0$

2. Solve for r and linearly combine the solutions to get the general solution

> If $r_1 \neq r_2$, then $y_1 = t^{r_1}$, $y_2 = t^{r_2} \implies y = c_1t^{r_1} + c_2t^{r_2}$
>
> If $r_1 = r_2$, then $y_1 = t^r, y_2 = t^r \log t \implies y = c_1t^r + c_2t^r \log t$ (alternatively, use d'Alembert's to obtain this)
>
> If $r = a \pm ib$, then use $t = e^{\log t}$
>
> $\implies y = t^r = (e^{\log t})^r = e^{\log t(a+ib)} = e^{a\log t}e^{ib\log t} = e^{\log t^a}\big(\cos(b\log t) + i\sin(b\log t)\big)$
>
> $\implies y_1 = t^a\cos(b\log t), y_2 = t^a\sin(b\log t) \implies y = t^a(\cos(b\log t) + \sin(b\log t))$

D'Alembert's Method

If 1 solution has been provided, find the other solution using d'Alembert's:

$y = vy_1$

$y' = v'y_1 + vy_1'$

$y'' = v''y_1 + 2v'y_1' + vy_1''$

1. Write out the above

2. Plug them into the DE, take out terms with v, and the equation should look like

> $P(v''y_1 + 2v'y_1') + Q(v'y_1'')$ where P is the coeff. in $Py'' + Qy' + Ry = 0$

3. Plug in y_1 and simplify.

4. Solve by subbing $w = v'$, so $v'' = \dfrac{dw}{dt}$ $\leftarrow$ order reduction

5. Find v by integrating $w = v'$. Plug v into $y_2 = vy_1$.

W6, 7 (3.5, 3.6)

Examples

1. variation of params
2. ^ on Bessel
3. var of params
4. ^ on Euler
5. system of 2 eqs with real duplicate roots (finding generalized eigen vector)
6. system of 2 eqs with complex roots
7. Non HG system
8. example 5 but non HG
9. Non HG system
10. system of 3 eqs
11. system of 3 eqs with complex roots

Non-homogeneous Equation (Lagrange)

If the equation is non-zero on the RHS, it is not homogeneous.

1. Solve the homogeneous equation and obtain fundamental solutions y_1, y_2

2. Solve $y = u_1(t)y_1 + u_2(t)y_2$ using **Lagrange's Variation of Parameters**

> 2 unknowns, so need 2 equations:
>
> (1) orthogonality: $u_1'y_1 + u_2'y_2 = 0$
>
> (2) plug $y = u_1y_1 + u_2y_2$ (and its derivatives) into the non-HG eq
>
> $y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

> $$\begin{aligned} y'' &= (u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2')' \\ &= (u_1y_1' + u_2y_2')' \leftarrow \text{by orthogonality eq} \\ &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \end{aligned}$$
>
> factor out $u_1(y_1'' + y_1)$ and $u_2(y_2'' + y_2)$, which = 0 ($y_k'' + y_k = 0$ since y_k is a solution of the HG eq)
>
> should be left with $u_1'y_1' + u_2'y_2' = RHS$
>
> Left with 2 unknowns u_1', u_2' and 2 equations (**Lagrange system**)
>
> Compute the Wronskian to check that $\exists$ independent u_2, u_2

3. Solve for u_1', u_2' and integrate to get u_1, u_2

4. The general solution is $y = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2$

   Be sure to check that the newly found y_1, y_2 are independent of the first set of fundamental solutions

Systems of Lin Eq with Constant Coefficients

If the DE is in the form $x' - Ax = 0$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then letting r be eigen value(s) and ξ be the corresp. eigen vector(s), we have that

$A\xi = r\xi \implies (A - rI)\xi = 0 \implies \det(A - rI) = 0 \implies \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = 0 \implies (a-r)(d-r) - bc = 0$

Solving the characteristic equation results in 3 possible cases:

Case 1: real, distinct r_1, r_2

> The general solution is $x = c_1\, e^{r_1t}\xi^{(1)} + c_2\, e^{r_2t}\xi^{(2)}$ where $\xi^{(1)}, \xi^{(2)}$ are eigenvectors

$$\begin{aligned} Proof: \frac{dx}{dt} &= c_1\, e^{r_1t} \underbrace{r_1\xi^{(1)}} + c_2\, e^{r_2t} \underbrace{r_2\xi^{(2)}} \\ &= c_1e^{r_1t} \underbrace{A\xi^{(1)}} + c_2e^{r_2t} \underbrace{A\xi^{(2)}} \quad since \quad A\xi^{(1)} = r_1\xi^{(1)}, \quad A\xi^{(2)} = r_2\xi^{(2)} \\ &= A\left[c_1e^{r_1t}\xi^{(1)} + c_2e^{r_2t}\xi^{(2)}\right] \\ &= Ax \end{aligned}$$

Case 2: real, duplicate r_1, r_2

If there are 2 independent eigen vectors associated with the duplicate r,

then $x = c_1 e^{rt}\xi^{(1)} + c_2 e^{rt}\xi^{(2)} = e^{rt}\left(c_1\xi^{(1)} + c_2\xi^{(2)}\right)$

If there is 1 eigen vector, then add an additional solution:

$$\begin{aligned} x &= te^{rt}\xi + e^{rt}\eta \quad \text{so the general solution becomes } x = c_1 e^{rt}\xi + c_2[te^{rt}\xi + e^{rt}\eta] \\ \frac{dx}{dt} &= e^{rt}\xi + tre^{rt}\xi + re^{rt}\eta \quad (1) \\ Ax &= te^{rt}A\xi + e^{rt}A\eta = te^{rt}r\xi + e^{rt}A\eta \quad (2) \end{aligned}$$

Equalize (1) & (2):

$$\begin{aligned} e^{rt}\xi + tre^{rt}\xi + re^{rt}\eta &= te^{rt}r\xi + e^{rt}A\eta \\ \xi + \cancel{tr\xi} + r\eta &= \cancel{tr\xi} + A\eta \\ \xi + r\eta &= A\eta \\ (A - r)\eta &= \xi \end{aligned}$$

This is the **generalized eigen value equation**.

Case 3: complex r_1, r_2

1. Find the eigenvector ξ associated with r_1 (r_2 is redundant)
2. Expand out $e^{rt}\xi$ using Euler's, and separate the real and imaginary parts
3. $x = c_1[\text{real part}] + c_2[\text{imaginary part}]$

Non HG systems of Lin Eq

If the DE is in the form $x' - Ax = RHS$ where RHS is non-zero, then:

1. Solve the HG system

2. Use Lagrange's variation of parameter to solve for the particular solution x_0

2 equations:

(1) Take the general solution x, replace c_1, c_2 with u_1, u_2, replace x with x_0

(2) Take the general solution x, replace c_1, c_2 with u_1', u_2', replace x with RHS

> From (2), obtain a system of equations, and solve for u_1', u_2'
>
> Integrate to get u_1, u_2, and plug them into equation 1 to get the particular solution

3. General solution to Non-HG system = (gen soln to HG system) + (particular soln)

W8-12 (Ch5)

For the equation $P(x)y'' + Q(x)y' + R(x)y = 0$, a point x_0 is **ordinary** if $P(x_0) \neq 0$, and **singular** if $P(x_0) = 0$

At ordinary points, we want to find solutions of the form

$$y = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \leftarrow \text{power series}$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

At singular points, we want to find solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \quad \leftarrow \text{Euler solutions} \times \text{power series}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n (x - x_0)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x - x_0)^{n+r-2}$$

A singular point is **regular** if $\lim_{x \to x_0} (x - x_0) \dfrac{Q(x)}{P(x)}$ and $\lim_{x \to x_0} (x - x_0)^2 \dfrac{R(x)}{P(x)}$ are both finite. O/w it is irregular.

General steps

1. Plug in the series and shift the indices s.t. all $(x - x_0)$ have equal exponents.
2. Pull out the coefficients to obtain the indicial relation.
3. Set $n = 0$ and solve to get the roots. Set $n \geq 1$ to get the recurrence relation and determine a_n.

Repeated root

Consider the Euler characteristic equation $r^2+(\alpha-1)r+\beta=0$

Applying Vieta's formula (think factoring), we have $\alpha-1=-(r_1+r_2), \beta=r_1r_2$

Since $r_1=r_2=r$, $\alpha=-2r+1, \beta=r^2$

We know that $y=x^r$ solves the equation $x^2y''+\alpha(r)xy'+\beta(r)y=0$

Write it as $x^2y_{xx}+\alpha(r)xy_x+\beta(r)y=0$ so it's clear what variable y is a derivative of

Differentiate w.r.t. r $\implies x^2y_{xxr}+\alpha xy_{xr}+\beta y_r+\underbrace{\alpha_r xy_x+\beta_r y}_{*}=0$

(*) = $(-2r+1)_r x(x^r)_x+(r^2)x^r=-2(x)(rx^{r-1})+2rx^r=0$

By the mixed derivative rule, $y_{xxr}=(y_r)_{xx}$ and $y_{xr}=(y_r)_x$

So $x^2y_{xxr}+\alpha xy_{xr}+\beta y_r+\underbrace{\alpha_r xy_x+\beta_r y}_{0}=x^2(y_r)''+\alpha(y_r)'+\beta(r)y_r=0$

Hence, $y_1=x^r$, and $y_2=y_r=\partial_r x^r=\partial_r e^{r\log x}=e^{r\log x}\log x=x^r\log x$

Frobenius Algorithm

If given an equation in the form $x^2y''+x\underbrace{(xp(x))}y'+\underbrace{(x^2q(x))}y=0$, then $p_0=\lim_{x\to 0}xp(x)$ and $q_0=\lim_{x\to 0}x^2q(x)$

If given an equation in the form $P(x)y''+Q(x)y'+R(x)y=0$, then $p_0=\lim_{x\to 0}x\frac{Q(x)}{P(x)}$ and $q_0=\lim_{x\to 0}x^2\frac{R(x)}{P(x)}$

The indicial equation is $r(r-1)+p_0r+q_0=0$

3 cases:

- If $r_1\neq r_2$ and their difference is not an integer, then $y_1=\sum_{n=0}^{\infty}a_nx^{n+r_1}, y_2=\sum_{n=0}^{\infty}a_nx^{n+r_2}$
- If $r_1-r_2=N$ where N is a non-zero integer (for convenience, enumerate r_1, r_2 such that N is positive), then the recurrence equation for r_2 will have 0 in the denominator of the Nth coefficient a_N

  $y_1=\sum_{n=0}^{\infty}a_nx^{n+r_1}$ as usual

  $y_2=a(\log x)y_1+x^{r_2}\left(1+\sum_{n=1}^{\infty}c_nx^n\right)$ where $a=\lim_{r\to r_2}(r-r_2)a_N$ and $c_n=[(r-r_2)a_n]'|_{r=r_2}$

- If $r_1 = r_2$, then $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y_2 = (\log x) y_1 + x^r \sum_{n=1}^{\infty} a_n'(r) x^n$

Bessel Equation

If given an equation of the form $x^2 y'' + xy' + (x^2 \nu^2) y = 0$ where ν is the order, then apply Frobenius.

$x^2 y'' + p(x) xy' + q(x) y = 0 \implies p(x) = 1, q(x) = x^2 - \nu^2$

$r(r-1) + r - \nu^2 = 0 \implies p_0 = 1, q_0 = -\nu^2$ since $r(r-1) + p_0 r + q_0 = 0$

$r^2 - \nu^2 = 0 \implies r_1 = \nu, r_2 = -\nu$

Order 0

$x^2 y'' + xy' + x^2 y = 0$ has roots $r_1 = r_2 = 0$