1. Financial Data and Returns

1.3 Log returns are additive

A stock has an expected annual log-return of 8% and an annual volatility (i.e. st. deviation of annual log-return) of 20%. You want to model the daily log-return using a Normal Random Walk model. What should the mean and variance of the Normal distribution of the daily log-returns be, so that your model matches the annual parameters above? Assume that a calendar year has 252 trading days of daily returns.

The random walk model is **additive in log-returns** (r), which means that over 1 year = 252 days we have:

$$r_{1yr} = \sum_{t=1}^{252} r_t$$

Assuming i.i.d. daily log-returns with daily moments (mean return and daily volatility) $\mathbb{E}[r_t] = \mu$ and $\mathbb{V}[r_t] = \sigma^2$, we have:

$$\begin{aligned}
\mathbb{E}\left[r_{1yr}\right] &= \sum_{t=1}^{252} \mathbb{E}[r_t] \\
0.08 &= 252\mu \\
\Rightarrow \mu &= \frac{0.08}{252} = .03174603\% \\
\mathbb{V}\left[r_{1yr}\right] &= \sum_{t=1}^{252} \mathbb{V}[r_t] \qquad \text{(by indep.)} \\
0.2^2 &= 252\sigma^2 \\
\Rightarrow \sigma &= \sqrt{\frac{0.2^2}{252}} = 1.259882\%
\end{aligned}$$

2. Univariate Return Modelling

2.1 MLE of Pareto Tail Index

Prove that the MLE of the Pareto distribution's tail index is given by $\hat{\alpha} = \left(\frac{1}{n}\sum_{i=1}^{n} \ln\left(x_i/\ell\right)\right)^{-1}$, where $(x_1, \dots, x_n)$ is an i.i.d. sample and the lower bound parameter $\ell > 0$ is known.

The pdf is given by

$$f(x) = \frac{\alpha x^{-(\alpha+1)}}{\ell^{-\alpha}}, \forall x > \ell$$

The log-likelihood is

$$
\begin{aligned}
\ell(\alpha) &= \ln\left(\prod_{i=1}^{n} f\left(x_i;\alpha\right)\right) \\
&= \sum_{i=1}^{n} \ln\left(f\left(x_i;\alpha\right)\right) \\
&= \sum_{i=1}^{n} \ln\left(\frac{\alpha x_i^{-(\alpha+1)}}{x_{\min}^{-\alpha}}\right) \\
&= \sum_{i=1}^{n} \left[\ln(\alpha) - (\alpha+1)\ln\left(x_i\right) + \alpha\ln\left(x_{\min}\right)\right] = \\
&= n\left[\ln(\alpha) + \alpha\ln\left(x_{\min}\right)\right] - (\alpha+1)\sum_{i=1}^{n}\ln\left(x_i\right)
\end{aligned}
$$

Differentiating w.r.t. α and setting equal to 0, we get:

$$
\begin{aligned}
\frac{d}{d\alpha}\ell(\alpha) = \ell'(\hat{\alpha}) &= 0 \\
n\left[\frac{1}{\hat{\alpha}} + \ln\left(x_{\min}\right)\right] - \sum_{i=1}^{n}\ln\left(x_i\right) &= 0 \\
\frac{1}{\hat{\alpha}} + \ln\left(x_{\min}\right) &= \frac{1}{n}\sum_{i=1}^{n}\ln\left(x_i\right) \\
\hat{\alpha} &= \frac{1}{\left[\frac{1}{n}\sum_{i=1}^{n}\ln\left(x_i\right)\right] - \ln\left(x_{\min}\right)} \\
&= \frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{x_{\min}}\right)}
\end{aligned}
$$

2.2 Heavy tails and Stable Distributions

Heavy-tailed distributions can arise as the sum of i.i.d. RVs with infinite variance. This question will explore the properties of such limiting distributions, which are called Stable Distributions, and their emergence from Generalized Central Limit Theorem.

Note that if an RV X has a heavy-tailed distribution (i.e. one with polynomial tails), then its **moment generating function** (MGF) $m_X(t) = \mathbb{E}\left[e^{tX}\right]$ is infinite for all $t \neq 0$.

However, we can still study such distributions using their **characteristic function** (CF) $\phi_X(t) = \mathbb{E}\left[e^{itX}\right] = \mathbb{E}[\cos(tX) + i\sin(tX)], \forall t \in \mathbb{R}$, which always exists (although it can be a complex function).

> **MGF review:**
>
> It is used to compute a distribution's moments: the nth moment about 0 is the nth derivative of the MGF evaluated at 0.

MGFs provide an alternative route to analytical results (instead of working with pdfs or cdfs directly)

Moments review:

The n-th moment of a real valued continuous function f(x) of a real variable about a value c is the integral

$$\mu_n = \int_{-\infty}^{\infty} (x-c)^n f(x) \mathrm{d}x$$

For a cdf F(x), it is defined as

$$\mu_n' = \mathrm{E}\left[X^n\right] = \int_{-\infty}^{\infty} x^n \ \mathrm{d}F(x)$$

(a) Show that any heavy-tailed distribution has infinite MGF.

Hint: express the MGF as an infinite-order polynomial with coefficients given by the moments of the distribution

Recall the Taylor expansion of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The MGF of an RV X can be expressed as:

$$m_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}\left[X^k\right]$$

at all points for which the MGF exists.

Furthermore, for distributions with tail index $\alpha > 0$, we have that $\mathbb{E}\left[X^k\right] = \infty$ for all $k \geq \alpha$, so a polynomial-tail distribution with any tail index will have infinite (or, more precisely, undefined) MGF.

(b) Show that if X is symmetric around 0, then its CF only takes real values.

Hint: sin() is an odd function, i.e. $\sin(-x) = -\sin(x), \forall x \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}\left[e^{itX}\right] &= \mathbb{E}[\cos(tX) + i\sin(tX)] = \mathbb{E}[\cos(tX)] + i\cancel{\mathbb{E}[\sin(tX)]}^0 \\
\text{b/c } \mathbb{E}[\sin(tX)] &= \int_{-\infty}^{\infty} \sin(tx) f_X(x) dx \\
&= \int_{-\infty}^{0} \sin(tx) f_X(x) dx + \int_{0}^{\infty} \sin(tx) f_X(x) dx \\
&= \int_{0}^{\infty} \sin(-tx) f_X(-x) dx + \int_{0}^{\infty} \sin(tx) f_X(x) dx \\
&= -\int_{0}^{\infty} \sin(tx) f_X(x) dx + \int_{0}^{\infty} \sin(tx) f_X(x) dx = 0
\end{aligned}$$

(c) Show that the sum of independent stable RVs also follows a stable distribution.

Consider independent RVs X, Y with CFs $\phi_X(t) = e^{-c|t|^\alpha}$ and $\phi_Y(t) = e^{-d|t|^\alpha}$. Show that the CF of $Z = X + Y$ has the same form, so their distribution is stable.

Hints:

- The CF of the sum of independent RVs is the product of their CFs.
- It can be shown that all stable distributions that are symmetric around 0 have CF's of the form $\phi(t) = e^{-c|t|^\alpha}$ for some $\alpha \in (0, 2] >$ and $c > 0$.

$$\begin{aligned} \phi_Z(t) &= \mathbb{E}\left[e^{it(X+Y)}\right] = \mathbb{E}\left[e^{itX}e^{itY}\right] = \mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{itY}\right] \text{ (by independence)} \\ &= \phi_X(t)\phi_Y(t) = e^{-c|t|^\alpha}e^{-d|t|^\alpha} = e^{-\overbrace{(c+d)}^{=g}|t|^\alpha} = e^{-g|t|^\alpha} \end{aligned}$$

which has the same CF form, up to the scaling factor $g = c + d$.

(d) Show that the average of n independent t(1) RVs is also t(1).

If X follows a t-distribution with 1 degree of freedom (a.k.a. Cauchy distribution), its CF is given by $\phi_X(t) = e^{-|t|}$. Show that the average $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ of $\#n$ independent $t(1)$ RVs also follows a $t(1)$ distribution (and hence is stable).

$$\begin{aligned} \phi_{\bar{X}}(t) &= \mathbb{E}\left[e^{it\bar{X}}\right] = \mathbb{E}\left[e^{it\frac{1}{n}(X_1+\cdots+X_n)}\right] = \mathbb{E}\left[e^{i(\frac{t}{n})X_1}\right]\cdots\mathbb{E}\left[e^{i(\frac{t}{n})X_n}\right] \\ &= \left[\phi_X(t/n)\right]^n = \left(e^{-|t/n|}\right)^n = e^{-n|t/n|} = e^{-|t|} \end{aligned}$$

which is the characteristic function of the $t(1)$ distribution.

(e) Heavy tailed distributions with tail index > 2 cannot be stable.

It can be shown that all stable distributions besides the Normal have heavy tails, with tail index equal to the parameter α appearing in their CF. But not all heavy tailed distributions are stable. Explain why heavy-tailed distributions with tail index $\alpha > 2$ cannot be stable.

(Hint: You don't have to do any math, just evoke the most important theorem in statistics.)

If a distribution has a scaling factor $\alpha > 2$, it means its second order moment (variance) is finite, so we can use the CLT, which states that the sum of i.i.d. copies of any distribution with finite variance converges to a Normal, *not* the original distribution - hence unstable.

2.3 Tail index of the ratio of 2 exponential RVs

Consider the random variable $Z = X/Y$, defined as the ratio of two i.i.d. exponential random variables $X, Y \sim \text{Exp}(1)$. Show that Z follows a heavy-tailed distribution and find its tail index. (Note: it is quite common for products of RVs to have heavy tailed distributions, e.g. t-distribution.)

If $X, Y \sim \text{Exp}(1)$, then $f_X(x) = \bar{F}_X(x) = e^{-x}, x > 0$, and similarly for Y. We have

$$\begin{aligned}
\bar{F}_Z(z) &= \mathbb{P}(Z > z) = \mathbb{P}(X/Y > z) \\
&= \int_{y=0}^{\infty} \mathbb{P}(X/Y > z \mid Y = y) dF_Y(y) \\
&= \int_{y=0}^{\infty} \underbrace{\mathbb{P}(X > yz)}_{=\bar{F}_X(yz)} f_Y(y) dy \\
&= \int_{y=0}^{\infty} e^{-yz} e^{-y} dy = \int_{y=0}^{\infty} e^{-y(z+1)} dy \\
&= \left[-\frac{e^{-y(z+1)}}{z+1} \right]_{y=0}^{\infty} \\
&= \frac{1}{z+1} = (z+1)^{-1}, z > 0
\end{aligned}$$

Since $\bar{F}_Z(z) = (z+1)^{-1} \Rightarrow f_Z(z) = -\frac{d}{dz}\bar{F}_Z(z) = (z+1)^{-2} \sim z^{-(1+\alpha)}$, the distribution of $Z = X/Y$ is heavy tailed with $\alpha = 1$.

3. Multivariate Return Modelling

3.1 Copula Bounds (Frechét-Hoeffding Thm)

Consider two (marginally) Uniform $[0, 1]$ RVs U_1, U_2. Theory tells us that their joint CDF $C(u_1, u_2)$ (i.e., their 2D copula) is bounded above and below by:

$$\max\{u_1 + u_2 - 1, 0\} = \underline{C}(u_1, u_2) \le C(u_1, u_2) \le \bar{C}(u_1, u_2) = \min\{u_1, u_2\}$$

(a) Assume $U_1 = U_2$ (i.e., perfect + ve dependence), and show that their copula is $\bar{C}$.

Let $U_1 = U_2 = U$, where $U \sim \text{Uniform}(0, 1)$; the resulting 2D CDF is

$$\begin{aligned}
\mathbb{P}(U_1 \le u_1, U_2 \le u_2) &= \mathbb{P}(U \le u_1, U \le u_2) \\
&= \mathbb{P}(U \le \min\{u_1, u_2\}) \\
&= \min\{u_1, u_2\}, \quad \forall u_1, u_2 \in [0, 1] \\
&= \bar{C}(u_1, u_2)
\end{aligned}$$

(b) Assume $U_1 = 1 - U_2$ (i.e., perfect negative dependence), and show that their copula is $\underline{C}$.

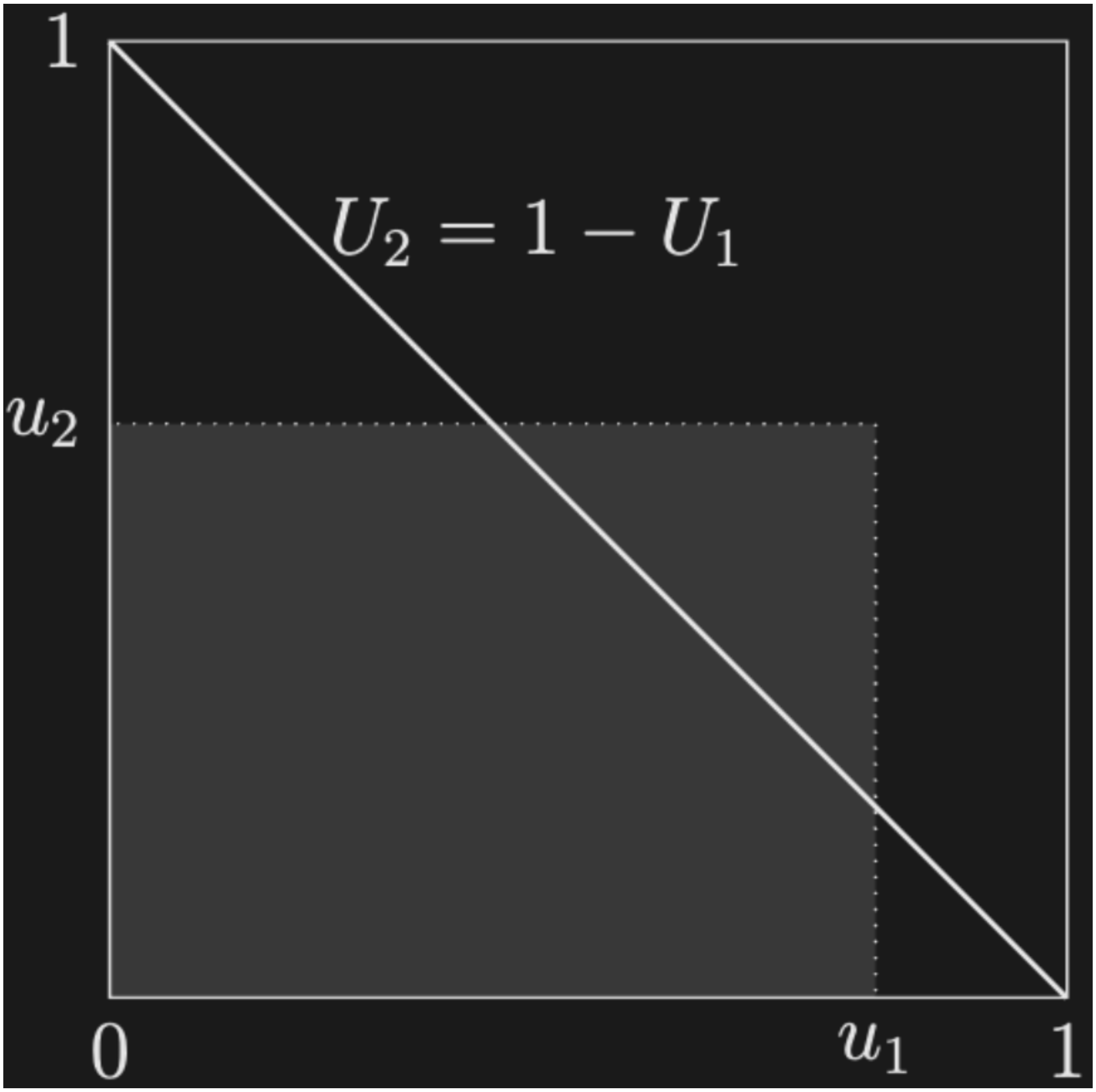
Note: In $d \ge 3$ dimensions, the upper bound corresponds to co-monotonous RVs (i.e., RVs that can be expressed as increasing functions of the same Uniform), but the lower bound is point-wise and does not correspond to any particular joint distribution.

Let $U_1 = U$ and $U_2 = 1 - U_1 = 1 - U$, where $U \sim \text{Uniform}(0,1)$; the resulting 2D CDF is

$$\begin{aligned}
\mathbb{P}\left(U_1 \leq u_1, U_2 \leq u_2\right) &= \mathbb{P}\left(U \leq u_1, 1 - U \leq u_2\right) \\
&= \mathbb{P}\left(U \leq u_1, U \geq 1 - u_2\right) \\
&= \mathbb{P}\left(1 - u_2 \leq U \leq u_1\right) \\
&= \begin{cases} 0, & 1 - u_2 \geq u_1 \Leftrightarrow u_1 + u_1 \leq 1 \\ u_1 - (1 - u_2) = u_1 + u_2 - 1, & 1 - u_2 \leq u_1 \Leftrightarrow u_1 + u_1 \geq 1 \end{cases} \\
&= \max\left\{u_1 + u_2 - 1, 0\right\}, \quad \forall u_1, u_2 \in [0,1] \\
&= \underline{C}\left(u_1, u_2\right)
\end{aligned}$$

Note that for both cases, the joint 2D distribution is **degenerate**, i.e. the RVs U_1, U_2 take values in a **lower-dimensional space**.

- In the first case, the RVs take values in the (1D) identity line of the (2D) unit square.
- In the second case they take values in the line with slope -1 and intercept 1 over the unit square, as in the plot below:



Degenerate distributions arise when some RV(s) can be **expressed as deterministic functions of the others**.

3.2 Sklar's Thm (Find Copula & CDF)

Consider an Archimedean copula with generator function $\phi(u) = \ln\left(\frac{1-\theta(1-u)}{u}\right)$, where $\theta \in [-1, 1)$

(a) Find a closed form expression for the resulting bivariate copula $C(u_1, u_2)$.

$$
\begin{aligned}
\phi(u) = \ln\left(\frac{1-\theta(1-u)}{u}\right) &= y \\
\frac{1-\theta(1-u)}{u} &= e^y \\
1-\theta(1-u) &= ue^y \\
u &= \frac{1-\theta}{e^y-\theta} \\
\implies \phi^{-1}(y) &= \frac{1-\theta}{e^y-\theta}
\end{aligned}
$$

The copula function becomes:

$$
\begin{aligned}
C\left(u_1, u_2\right) &= \phi^{-1}\left(\phi\left(u_1\right)+\phi\left(u_2\right)\right) \\
&= \frac{1-\theta}{\exp\left\{\phi\left(u_1\right)+\phi\left(u_2\right)\right\}-\theta} \\
&= \frac{1-\theta}{\exp\left\{\ln\left(\frac{1-\theta(1-u_1)}{u_1}\right)+\ln\left(\frac{1-\theta(1-u_2)}{u_2}\right)\right\}-\theta} \\
&= \frac{1-\theta}{\frac{1-\theta(1-u_1)}{u_1}\times\frac{1-\theta(1-u_2)}{u_2}-\theta} \\
&= \frac{(1-\theta)\cdot u_1\cdot u_2}{\left[\left((1-\theta)+\theta u_1\right)\cdot\left((1-\theta)+\theta u_2\right)\right]-\theta\cdot u_1\cdot u_2} \\
&= \frac{(1-\theta)\cdot u_1\cdot u_2}{\left[\left((1-\theta)^2+(1-\theta)\theta u_1+(1-\theta)\theta u_2\right)+\theta^2 u_1 u_2\right]-\theta\cdot u_1\cdot u_2} \\
&= \frac{(1-\theta)\cdot u_1\cdot u_2}{(1-\theta)\left[(1-\theta)+\theta u_1+\theta u_2\right]+\theta^2 u_1 u_2-\theta\cdot u_1\cdot u_2} \\
&= \frac{(1-\theta)\cdot u_1\cdot u_2}{(1-\theta)\left[(1-\theta)+\theta u_1+\theta u_2\right]-(1-\theta)\left(\theta u_1 u_2\right)} \\
&= \frac{u_1 u_2}{1-\theta+\theta u_1+\theta u_2-\theta u_1 u_2} \\
&= \frac{u_1 u_2}{1-\theta\left(1-u_1-u_2+u_1 u_2\right)} \\
&= \frac{u_1 u_2}{1-\theta\left(1-u_1\right)\left(1-u_2\right)}
\end{aligned}
$$

(b) Find CDF of max(uniform marginals)

Consider n random variables $U_1, \ldots, U_n$ with Uniform $[0,1]$ marginals from this copula. Find the cumulative distribution function (CDF) of their maximum.

Let $M_n = \max\{U_1, \dots, U_n\}$ so that:

$$
\begin{aligned}
F_{M_n}(m) &= P(M_n \le m) = P\left[\bigcap_{i=1}^{n} \{U_i \le m\}\right] \\
&= P(U_1 \le m, \dots, U_n \le m) \\
&= C(U_1 \le m, \dots, U_n \le m) \\
&= \phi^{-1}(\phi(m) + \dots + \phi(m)) = \phi^{-1}(n\phi(m)) \\
&= \frac{1-\theta}{\exp\left\{n \ln\left(\frac{1-\theta(1-m)}{m}\right)\right\} - \theta} \\
&= \frac{1-\theta}{\left(\frac{1-\theta(1-m)}{m}\right)^n - \theta}
\end{aligned}
$$

3.3 Independence copula = archimedean with -log generator

Show that an Archimedean copula with generator function $\phi(u) = -\log(u)$ is equal to the independence copula C_0.

We have $\phi^{-1}(y) = e^{-y}$ and

$$
\begin{aligned}
C(u_1, \dots, u_n) &= \phi^{-1}(\phi(u_1) + \dots + \phi(u_n)) \\
&= \exp(-[-\log(u_1) - \dots - \log(u_n)]) \\
&= \exp(\log(u_1) + \dots + \log(u_n)) \\
&= \exp(\log(u_1 \times \dots \times u_n)) \\
&= u_1 \times \dots \times u_n
\end{aligned}
$$

which is the independence Copula.

Show that a similar result holds for logarithms with any base.

Does the same hold when the natural logarithm is replaced by the common logarithm, i.e. $\phi(u) = -\log_{10}(u)$?

We have $-y = \log_{10} u \implies 10^{-y} = u = \phi^{-1}(y)$, so

$$
\begin{aligned}
C(u_1, \dots, u_n) &= \phi^{-1}(\phi(u_1) + \dots + \phi(u_n)) \\
&= 10^{-(-\log_{10} u_1 - \dots - \log_{10} u_n)} \\
&= 10^{(\log_{10}(u_1 \times \dots \times u_n))} \\
&= u_1 \times \dots \times u_n
\end{aligned}
$$

3.4 Convex combination of copulas is a copula

A convex combination of k joint CDFs is itself a joint CDF (finite mixture), but is a convex combination of k copula functions a copula function itself?

It suffices to show (by induction) that a convex combination of 2 copulas ($C() = wC_1() + (1-w)C_2, \quad \forall w \in [0,1]$) is also a copula.

We are given that a convex combination of two CDFs is a CDF, and this result holds for any number of dimensions. All that's left to do is to show that the marginals are Uniform $(0,1)$.

Since the initial copulas have Uniform $(0,1)$ marginals, it is straightforward to see that that mixture copula will also have Uniform $(0,1)$ marginals:

for 1D Uniform CDFs $F_1(u) = F_2(u) = u, \quad \forall u \in [0,1]$, we have

$$F(u) = wF_1(u) + (1-w)F_2(u) = wu + (1-w)u = u, \quad \forall u \in [0,1]$$

4. Portfolio Theory

4.1 Portfolio Allocation (2 risky assets)

Suppose that there are two risky assets, A and B, with expected returns equal to 2.3% and 4.5%, respectively. Suppose that the standard deviations of the returns are $\sqrt{6}\%$ and $\sqrt{11}\%$ and that the returns on the assets have a correlation of 0.17

(a) What portfolio of A and B achieves a 3% rate of expected return?

$$\begin{aligned} \mathbb{E}(wA + (1-w)B) &= w\mathbb{E}(A) + (1-w)\mathbb{E}(B) \\ 0.03 &= w \cdot 0.023 + (1-w) \cdot 0.045 \\ w &= \frac{.03 - .045}{.023 - .045} \\ &= 0.6818 \end{aligned}$$

A portfolio made up of 68% of asset A and 32% of asset B will achieve a 3% rate of expected return.

(b) What portfolios of A and B achieve a $\sqrt{5.5}\%$ standard deviation of return? Among these, which has the largest expected return?

$$
\begin{aligned}
\mathbb{V}\left(R_p\right) &= \mathbb{V}(wA+(1-w)B) \\
&= w^2\mathbb{V}(A)+(1-w)^2\mathbb{V}(B)+2\cdot w(1-w)\operatorname{Cov}(A,B) \\
&= w^2 0.06+(1-w)^2 0.11+w(1-w)2\cdot 0.17\cdot\sqrt{0.06\cdot 0.11} \\
&= w^2 0.06+\left(1-2w+w^2\right)0.11+\left(w-w^2\right)2\cdot 0.17\cdot\sqrt{0.06\cdot 0.11} \\
&= w^2(6+11-2.7621)+w(2.7621-2\cdot 11)+11 \\
0.055 &= 14.2379w^2-19.2379w+11 \\
0 &= 14.2379w^2-19.2379w+11-5.5 \\
w &= 0.4108 \text{ and } 0.9404
\end{aligned}
$$

Or use R to find roots of $5.5=6w^2+11(1-w)^2+(2)\sqrt{(6)(11)}(0.17)w(1-w)$

```
poly1 = c(-5.5, 0, 0) # coefficients of (1, w, w^2)
poly2 = c(0, 0, 6)
poly3 = 11*c(1,-2,1)
poly4 = 2*sqrt(6*11)*0.17*c(0,1,-1)
poly = poly1+poly2+poly3+poly4
roots = polyroot(poly)

R_A = 2.3
R_B = 4.5
Re(R_A * roots + R_B * (1-roots))
```

4.2 Portfolio Allocation (risk free asset + 2 risky assets)

Suppose there are two risky assets, C and D, the tangency portfolio is 65%C and 35%D, and the expected return and standard deviation of the return on the tangency portfolio are 5% and 7%, respectively. Suppose also that the risk-free rate of return is 1.5%. If you want the standard deviation of your return to be 5%, what proportions of your capital should be in the risk-free asset, asset C, and asset D?

Given: $\mu_p=0.05, \sigma_p=0.07, r_f=0.015$ Current weights: $w_C=0.65, w_D=0.35$ Want: $V(R)=0.05^2$ Find: w_f, w_C, w_D

We know that for $R=w_pR_p+(1-w_p)R_f$, $V(R)=V(w_pR_p+(1-w_p)R_f)=w_p^2V(R_p)=w_p^2\sigma_p^2$

So we have $0.05^2=w_p^2\cdot 0.07^2 \implies w_p=\frac{5}{7}, w_f=\frac{2}{7}$

The tangency portfolio composition does not change, so $w_C=0.65\cdot\frac{5}{7}, w_D=0.35\cdot\frac{5}{7}$

4.3 Stock Portfolio Weights

(a)

Suppose that stock A shares sell at $75 and stock B shares at $115. A portfolio has 300 shares of stock A and 100 of stock B. What are the weights w and $1-w$ of stocks A and B in this portfolio?

The weights are the relative proportions of the portfolio value:

$$\begin{aligned} V &= n_A \times P_A + n_B \times P_B = 300 \times 75 + 100 \times 115 = 34000 \\ w &= \frac{n_A \times P_A}{V} = \frac{22500}{34000} = 0.6617647 \\ 1-w &= \frac{n_B \times P_B}{V} = \frac{11500}{34000} = 0.3382353 \end{aligned}$$

(b)

More generally, if a portfolio has N stocks, if the price per share of the j th stock is P_j, and if the portfolio has n_j shares of stock j, then find a formula for w_j as a function of $n_1, \dots, n_N$ and $P_1, \dots, P_N$.

Similarly, we have:

$$\begin{aligned} V &= \sum_{i=1}^{N} n_i \times P_i \\ w_j &= \frac{n_j \times P_j}{V} = \frac{n_j \times P_j}{\sum_{i=1}^{N} n_i \times P_i} \end{aligned}$$

4.4 Portfolio return as a linear combination of asset returns

Suppose we have a price process $\{S(t)\}_t$. Consider $\mathcal{R}_P = w_1\mathcal{R}_1 + \dots + w_N\mathcal{R}_N$ where $\mathcal{R}_P$ is some type of return on a portfolio, and $\mathcal{R}_1, \dots, \mathcal{R}_N$ are the same type of returns on the assets in this portfolio.

Would the equation hold if $\mathcal{R}_P$ is a...

- Gross return: $R'(t) = \frac{S(t)}{S(t-1)}$
- Net return: $R(t) = R'(t) - 1 = \frac{S(t)-S(t-1)}{S(t-1)}$
- Log-return: $r(t) = \log(R'(t)) = \log(S(t)) - \log(S(t-1))$

We showed in class (Lec4 p5) that the equation *holds for net returns* (for two assets, but can apply induction)

$$R_P(t) = w_1 R_1(t) + \dots + w_N R_N(t)$$

We can add 1 to both sides and show that the equation also *holds for gross returns*:

$$\begin{aligned} R_p'(t) = R_P(t) + 1 &= w_1 R_1(t) + \dots + w_N R_N(t) + 1 \\ &= w_1 R_1(t) + \dots + w_N R_N(t) + (w_1 + \dots + w_N) \\ &= w_1 (R_1(t) + 1) + \dots + w_N (R_N(t) + 1) \\ &= w_1 R_1'(t) + \dots + w_N R_N'(t) \end{aligned}$$

However, this equation *does not hold for log-returns*

$$\begin{aligned} r_p(t) = \log(R'_P(t)) &= \log(w_1 R'_1(t) + ... + w_N R'_N(t)) \\ &= \log(w_1 R'_1(t)) \times ... \times \log(w_N R'_N(t)) \\ &= [\log(w_1) + \log(R'_1(t))] \times ... \times [\log(w_N) + \log(R'_N(t))] \\ &= [\log(w_1) + r_1(t)] \times ... \times [\log(w_N) + r_N(t)] \\ &\neq w_1 r_1(t) + ... + w_N r_N(t) \end{aligned}$$

4.5 Sampling (Can skip)

Suppose one has a sample of monthly log returns on two stocks with sample means of 0.0032 and 0.0074, sample variances of 0.017 and 0.025, and a sample covariance of 0.0059. For purposes of resampling, consider these to be the "true population values."

A bootstrap resample has sample means of 0.0047 and 0.0065, sample variances of 0.0125 and 0.023, and a sample covariance of 0.0058.

(a)

Using the resample, estimate the efficient portfolio of these two stocks that has an expected return of 0.005; that is, give the two portfolio weights.

(0.0047)w + (0.0065)(1 − w) = 0.005 so that the estimated efficient portfolio is 57.14% stock 1 and 42.86% stock 2.

(b)

What is the estimated variance of the return of the portfolio in part (a) using the resample variances and covariances?

$$(0.5714)^2(0.0125) + (0.4286)^2(0.023) + (2)(0.5714)(0.4286)(0.0058) = 0.01098$$

(c)

What are the actual expected return and variance of return for the portfolio in (a) when calculated with the true population values (e.g with using the original sample means, variances, and covariance)?

Actual expected return is (0.5714)(0.0031) + (0.4286)(0.0074) = 0.004943. Actual variance of return is

$$(0.5714)^2(0.017) + (0.4286)^2(0.025) + 2(0.5714)(0.4286)(0.0059) = 0.01303$$

5. Factor Models

5.1 One-factor model weights

Assume a market of N assets with returns following the 1-factor CAPM model

$$R_i = \beta R_M + \varepsilon_i, \quad i = 1, \ldots, N$$

where $R_M \sim \mathrm{N}\left(\mu_M, \sigma_M^2\right)$ and $\varepsilon_i \sim^{i.i.d.} \mathrm{N}\left(0, \sigma_\varepsilon^2\right), \quad \forall i$. Therefore, the model assumes all assets have the same systematic risk $\left(\beta^2 \sigma_M^2\right)$ and the the same idiosyncratic risk $\left(\sigma_\varepsilon^2\right)$.

(a) Find minimum-variance portfolio weights.

The weights are given by $w = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$ (W4, multiple asset portfolio)

First, find Σ:

$$R = \beta 1 R_M + \varepsilon \iff \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix} = \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix} R_M + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{bmatrix}$$

$$\begin{aligned}
\text{where } \mathbb{E}[\varepsilon] &= \mathbf{0}, \quad \mathbb{V}[\varepsilon] = \sigma_\varepsilon^2 I, \quad \mathbb{E}\left[R_M\right] = \mu_M, \quad \mathbb{V}\left[R_M\right] = \sigma_M^2 \\
\text{therefore } \mathbb{E}[R] &= E\left[\beta \mathbf{1} R_M + \varepsilon\right] = \beta \mathbf{1}\left[R_M\right] + \mathbb{E}[\varepsilon] = \beta \mu_M \mathbf{1} + \mathbf{0} = \beta \mu_M \mathbf{1} \\
\mathbb{V}[R] &= \mathbb{V}\left[\beta 1 R_M + \varepsilon\right] = \beta^2 1 1^T \mathbb{V}\left[R_M\right] + \mathbb{V}[\varepsilon] = \beta^2 \sigma_M^2 1 1^T + \sigma_\varepsilon^2 I \\
&= \beta^2 \sigma_M^2 \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \sigma_\varepsilon^2 \begin{bmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \\
&= \begin{bmatrix} \beta^2 \sigma_M^2 + \sigma_\varepsilon^2 & \cdots & \beta^2 \sigma_M^2 \\ \vdots & \ddots & \vdots \\ \beta^2 \sigma_M^2 & \cdots & \beta^2 \sigma_M^2 + \sigma_\varepsilon^2 \end{bmatrix} \\
&= \Sigma
\end{aligned}$$

Since Σ is symmetric, changing the order of the assets in the vector R does not change anything. Thus, the weights must be all equal, i.e. $w = \frac{1}{N} 1$.

Alternatively, you can verify that for any N-dimensional matrix of the form

$$\Sigma = a \mathbf{1}\mathbf{1}^T + bI = \begin{bmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \cdots & a \end{bmatrix} + \begin{bmatrix} b & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b \end{bmatrix}$$

with $a, b > 0$, the inverse is given by

$$\Sigma^{-1} = c \mathbf{1}\mathbf{1}^{\mathbf{T}} + dI = \begin{bmatrix} c & \ldots & c \\ \vdots & \ddots & \vdots \\ c & \cdots & c \end{bmatrix} + \begin{bmatrix} d & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d \end{bmatrix}$$

where $c = -\frac{1}{b} \frac{a}{(Na+b)}$ and $d = \frac{1}{b}$. Thus, the row-sums of the inverse matrix are all equal, which means that the minimum-portfolio weights are also equal, and hence have value $1/N$ (because they need to sum to 1).

(b) Minimum variance lower bound (due to systematic risk)

Show that the minimum variance can never be smaller than $\beta^2\sigma_M^2$, no matter how many assets we have (i.e., no matter how large N is).

For the minimum-variance portfolio (R_{mv}) we have:

$$\begin{aligned}
\mathbb{E}\left[R_{mv}\right] &= \mathbb{E}\left[w^T R\right] = w^T E[R] = \frac{1}{N}\mathbf{1}^T\left(\beta\mu_M\mathbf{1}\right) \\
&= \beta\mu_M\frac{\mathbf{1}^T\mathbf{1}}{N} = \beta\mu_M \quad (\text{ since } \mathbf{1}^T\mathbf{1} = N) \\
\mathbb{V}\left[R_{mv}\right] &= \mathbb{V}\left[w^T R\right] = w^T\mathbb{V}[R]w = \frac{1}{N^2}\mathbf{1}^T\left(\beta^2\sigma_M^2\mathbf{1}\mathbf{1}^T + \sigma_\varepsilon^2 I\right)\mathbf{1} \\
&= \frac{\beta^2\sigma_M^2}{N^2}\left(\mathbf{1}^T\mathbf{1}\right)\left(\mathbf{1}^T\mathbf{1}\right) + \frac{\sigma_\varepsilon^2}{N^2}\left(\mathbf{1}^T\mathbf{1}\right) = \beta^2\sigma_M^2 + \frac{\sigma_\varepsilon^2}{N} > \beta^2\sigma_M^2, \forall N \geq 1
\end{aligned}$$

Thus, the minimum variance portfolio will always have variance at least $\beta^2\sigma_M^2$. This problem illustrates that **you cannot "diversify away" systemic risk** (i.e., risk from common factors) the same way you can with idiosyncratic risk.

5.2 Two factor model weights

Consider the following 2-factor model with 3 assets:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \Leftrightarrow R = \beta^\top F + \varepsilon$$

where

$$\mathbb{V}[F] = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} = \sigma^2 I_2, \quad \mathbb{V}[\varepsilon] = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} = \sigma^2 I_3, \quad \operatorname{Cov}[\varepsilon, F] = 0$$

Find the minimum variance portfolio weights for this model. (Hint: You can use R to invert the matrix.)

This can be solved without computation - notice that R_1 and R_3 are uncorrelated, so splitting (or "diversifying") the portfolio between these uncorrelated assets will give the highest reduction in variance.

Verify with computation:

The model's covariance matrix is

$$
\begin{aligned}
\Sigma_R =& \mathbb{V}[R] = \mathbb{V}\left[\beta^\top F + \varepsilon\right] = \\
=& \beta^\top \mathbb{V}[F]\beta + \mathbb{V}[\varepsilon] \\
=& \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \\
=& \begin{bmatrix} \sigma^2 & \sigma^2 & 0 \\ \sigma^2 & 2\sigma^2 & \sigma^2 \\ 0 & \sigma^2 & \sigma^2 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \\
=& \begin{bmatrix} 2\sigma^2 & \sigma^2 & 0 \\ \sigma^2 & 3\sigma^2 & \sigma^2 \\ 0 & \sigma^2 & 2\sigma^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\
\Rightarrow \Sigma_R^{-1} =& \frac{1}{\sigma^2}\frac{1}{8} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{bmatrix}
\end{aligned}
$$

The minimum variance portfolio weights are:

$$
w = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} = \frac{\begin{bmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}
$$

6. Risk Management

6.1 Find VaR given CDF

Find a closed form expression for the VaR at confidence level $(1-\alpha)$ of the following continuous loss distributions.

Formula:

$$
\begin{aligned}
F_L(VaR_\alpha) = P(L \leq \mathrm{VaR}_\alpha) = 1 - \alpha \\
VaR_\alpha = F_L^{-1}(1-\alpha)
\end{aligned}
$$

(a) Pareto distribution

$F_L(x) = 1 - (x/m)^{-\beta}, x > m$, with shape parameter (tail index) $\beta > 0$.

$$
\begin{aligned}
1 - \left(\mathrm{VaR}_\alpha / m\right)^{-\beta} &= 1 - \alpha \\
\mathrm{VaR}_\alpha = m\alpha^{-1/\beta} &= \frac{m}{\alpha^{1/\beta}}
\end{aligned}
$$

(b) Gumbel distribution

$F_L(x) = \exp\left\{-\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right\}, \quad \forall x \in \mathbb{R},\, \mu \in \mathbb{R},\, \sigma > 0.$

$$\begin{aligned}
\exp\left\{-\exp\left\{-\frac{\mathrm{VaR}_\alpha - \mu}{\sigma}\right\}\right\} &= 1-\alpha \\
\exp\left\{-\frac{\mathrm{VaR}_\alpha - \mu}{\sigma}\right\} &= -\ln(1-\alpha) \\
\mathrm{VaR}_\alpha &= \mu - \sigma \ln(-\ln(1-\alpha)) \\
&= \mu + \sigma \ln\left(\frac{1}{\ln\left(\frac{1}{1-a}\right)}\right)
\end{aligned}$$

(c) Fréchet distribution

$F_L(x) = \exp\left\{-(x/\sigma)^{-\beta}\right\}, \quad \forall x > 0$, with shape parameter $\beta > 0$ and scale parameter $\sigma > 0$.

$$\begin{aligned}
\exp\left\{-\left(\frac{\mathrm{VaR}_\alpha}{\sigma}\right)^{-\beta}\right\} &= 1-\alpha \\
\left(\frac{\mathrm{VaR}_\alpha}{\sigma}\right)^{-\beta} &= -\ln(1-\alpha) \\
\mathrm{VaR}_\alpha &= \sigma[-\ln(1-\alpha)]^{-1/\beta} \\
&= \frac{\sigma}{\left[\ln\left(\frac{1}{1-a}\right)\right]^{1/\beta}}
\end{aligned}$$

6.2 CVaR definition derivation (integral → cond. exp)

For a loss RV L with continuous distribution, show that the integral definition of Conditional Value-at-Risk (CVaR), a.k.a. Expected Shortfall (ES), is equivalent to the conditional expectation:

$$\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u \, du = \mathbb{E}\left[L \mid L \geq \mathrm{VaR}_\alpha\right]$$

Hint: Recall that for absolutely continuous L with CDF $F(\cdot)$, the VaR is given by the inverse CDF: $\mathrm{VaR}_\alpha = F^{-1}(1-\alpha)$. Perform the change of variable $x = F^{-1}(1-u)$.

Derivative of an inverse:

$$\left[F^{-1}(x)\right]' = 1/F'\left(F^{-1}(x)\right)$$

$$
\begin{aligned}
\mathrm{CVaR}_\alpha = \mathrm{ES}_\alpha &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u du \\
&= \frac{1}{\alpha}\int_0^\alpha F^{-1}(1-u) du \\
&\left\{\begin{aligned} \text{change of var: } x &= F^{-1}(1-u) \\ dx &= \frac{1}{-F'\left(F^{-1}(1-u)\right)} du \\ &= \frac{1}{-F'(x)} du \\ du &= -f(x)dx \end{aligned}\right\} \\
&= -\frac{1}{\alpha}\int_{F^{-1}(1)}^{F^{-1}(1-\alpha)} x f(x) dx \\
&= -\int_\infty^{\mathrm{VaR}_\alpha} x\frac{f(x)}{\alpha} dx \\
&= \int_{\mathrm{VaR}_\alpha}^\infty x \underbrace{\frac{f(x)}{\mathbb{P}(L \geq \mathrm{VaR})}}_{=\text{cond. PDF}} dx \\
&= \int_{\mathrm{VaR}_\alpha}^\infty x f_{L|L\geq \mathrm{VaR}_\alpha}(x) dx \\
&= \mathbb{E}\left[L \mid L \geq \mathrm{VaR}_\alpha\right]
\end{aligned}
$$

6.3 Find VaR at 0.5% given VaR at 5%

Assume that the loss distribution has a polynomial tail with tail index $\alpha = 3.1$. If $\mathrm{VaR}_{5\%} = 252$, what is $\mathrm{VaR}_{0.5\%}$? (Hint: Read section 19.6)

For loss distributions with polynomial tail β, the complementary cdf is given by

$$\bar{F}(x) = P(X > x) \propto \int_x^\infty s^{-(\alpha+1)} ds \propto \left[-s^{-\alpha}\right]_{s=x}^\infty = x^{-\alpha}$$

Therefore,

$$P\left(X > \mathrm{VaR}_p\right) = p \Rightarrow \left(\mathrm{VaR}_p\right)^{-\alpha} \propto p \Rightarrow \mathrm{VaR}_p \propto \frac{1}{p^{1/\alpha}}$$

which implies that

$$
\begin{aligned}
\frac{\mathrm{VaR}_p}{\mathrm{VaR}_q} = \left(\frac{q}{p}\right)^{1/\alpha} \Rightarrow \frac{\mathrm{VaR}_{.005}}{\mathrm{VaR}_{.05}} = \left(\frac{.05}{.005}\right)^{1/\alpha} &\Rightarrow \frac{\mathrm{VaR}_{.005}}{252} = 10^{1/3.1} \\
&\Rightarrow \mathrm{VaR}_{.005} = 252 \times 10^{1/3.1}
\end{aligned}
$$

6.4 EVaR

Consider the example with the two risky zero-coupon bonds priced at \$95 per \$100 face value, where each has 4% default probability independently of the other.

(a) EVaR of one bond

Calculate the $\alpha = 5\%$ Entropic Value-at-Risk (EVaR) for one of these bonds. Note that you will need to use numeric minimization, e.g. optimize() in R, to find EVaR.

The (marginal) loss distribution of each bond $\left(L_{1/2}\right)$ is given by the PMF

$$p_L(\ell) = \mathbb{P}(L = \ell) = \begin{cases} 0.04, & \ell = 95 - 0 = 95 \\ 0.96, & \ell = 95 - 100 = -5 \end{cases}$$

with MGF

$$M_L(z) = \mathbb{E}\left[e^{zL}\right] = 0.04e^{95z} + 0.96e^{-5z}$$

The EVaR at α is given by

$$\begin{aligned} EVaR_\alpha &= \inf_{z>0} \left\{\ln\left(M_L(z)/\alpha\right)/z\right\} \\ &= \inf_{z>0} \left\{\ln\left\{\left(0.04e^{95z} + 0.96e^{-5z}\right)/0.05\right\}/z\right\} \end{aligned}$$

Running this minimization w.r.t. z in R:

```
fn = function(z){ log( (0.04*exp(95*z) + 0.96*exp(-5*z)) / 0.05 ) / z }
optimise(fn, c(0, 1))
```

Output:

```
$minimum
[1] 0.06690106
$objective
[1] 92.10402
```

So the minimum is $\text{EVaR}_{0.05}(L) = 92.10402$, occurring at $z = 0.06690106$.

(b) EVaR of portfolio; sub-additivity property

Calculate the EVaR of a portfolio of two of these bonds, and show that it is sub-additive.

The loss distribution for the sum of the two bonds $(L_1 + L_2)$ is

$$p_{L_1+L_2}L(\ell) = \mathbb{P}\left(L_1 + L_2 = \ell\right) = \begin{cases} (0.04)^2 = 0.0016, & \ell = 95 + 95 = 190 \\ 2(0.96)(0.04) = .0768, & \ell = 95 - 5 = 90 \\ (0.96)^2 = .9216, & \ell = -5 - 5 = -10 \end{cases}$$

with MGF

$$M_{L_1+L_2}(z) = \mathbb{E}\left[e^{z(L_1+L_2)}\right] = 0.0016e^{190z} + 0.0768e^{90z} + 0.09216e^{-10}$$

Running this minimization w.r.t. z in R, we get

```
$minimum
[1] 0.03841828
$objective
[1] 122.0294
```

So we have $\mathrm{EVaR}_{0.05}\left(L_1+L_2\right) = 122.0294$, occurring at $z = 0.03841828$.

Note that this EVaR is sub-additive, since

$$\begin{gathered}\mathrm{EVaR}_{0.05}\left(L_1+L_2\right) \leq \mathrm{EVaR}_{0.05}\left(L_1\right) + \mathrm{EVaR}_{0.05}\left(L_2\right) \\ 122.0294 \leq 2 \times 92.10402 = 184.208\end{gathered}$$

6.5 Heavy tail risk

Consider a loss distribution with fat upper tail and some tail index $\beta > 0$.

(a) Can you find the EVaR for such distributions?

No! EVaR requires the use of the MGF, but distributions with fat tails do not have an MGF, because moments of order $\geq \beta$ are infinite (refer to 2.2a).

(b) If the tail index is $\beta = 1$ (e.g., a Cauchy distribution), can you find the CVaR/ES? Justify your answer.

No! We know that distributions with tail index ≥ 1 have infinite mean, which means that their conditional tail distributions (which are proportional to the original ones) will also have infinite mean.

6.6 VaR Sub-additivity

Suppose the risk measure is VaR for some α. Let P_1, P_2 be two portfolios whose returns have a joint (2D) normal distribution with means μ_1, μ_2, standard deviations σ_1, σ_2, and correlation ρ. Suppose the initial investments are S_1, S_2. Show that VaR is sub-additive, i.e.,

$$\mathrm{VaR}_\alpha\left(P_1+P_2\right) \leq \mathrm{VaR}_\alpha\left(P_1\right) + \mathrm{VaR}_\alpha\left(P_2\right), \forall \alpha < 1/2$$

> Aside: This is the Markowitz framework of mean-variance analysis. Note that because VaR is sub-additive in this case, it will encourage diversification. In fact, minimizing VaR and minimizing variance will give us the same portfolio.

Since $P_i \sim N\left(\mu_i, \sigma_i^2\right)$ with initial investments S_i for $i = 1, 2$, we have that

$$\mathrm{VaR}_\alpha\left(P_i\right) = -S_i \cdot \left(\mu_i + z_\alpha \sigma_i\right) = -S_i\mu_i - S_i z_\alpha \sigma_i, \forall i = 1, 2$$

where $z_\alpha = \Phi^{-1}(\alpha)$ is the standard Normal quantile function.

Thus,

$$\mathrm{VaR}_\alpha\left(P_1\right) + \mathrm{VaR}_\alpha\left(P_2\right) = -\left(S_1\mu_1 + S_2\mu_2\right) - z_\alpha\left(S_1\sigma_1 + S_2\sigma_2\right)$$

The portfolio has weights $w_i = \frac{S_i}{S_1+S_2}, \quad i = 1, 2$

$$w_1P_1 + w_2P_2 \sim N\left(w_1\mu_1 + w_2\mu_2, w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho\right)$$

The resulting VaR is

$$\begin{aligned}\mathrm{VaR}_\alpha\left(P_1 + P_2\right) &= -\left(S_1 + S_2\right) \times \left(\left(w_1\mu_1 + w_2\mu_2\right) + z_\alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho}\right) \\ &= -\left(S_1\mu_1 + S_2\mu_2\right) - z_\alpha\sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2\rho}\end{aligned}$$

But for any $\rho \in [-1, +1]$, we have:

$$\begin{aligned}S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2\rho &\leqslant S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2 = \left(S_1\sigma_1 + S_2\sigma_2\right)^2 \\ \sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2\rho} &\leqslant S_1\sigma_1 + S_2\sigma_2 \\ -\sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2\rho} \times z_\alpha &\leqslant -\left(S_1\sigma_1 + S_2\sigma_2\right) \times z_\alpha \quad (\text{ for } \alpha < .5 \to z_\alpha < 0) \\ -\left(S_1\mu_1 + S_2\mu_2\right) z_\alpha - \sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2\rho} \times z_\alpha &\leqslant -\left(S_1\mu_1 + S_2\mu_2\right) \times z_\alpha - \left(S_1\sigma_1 + S_2\sigma_2\right) \times z_\alpha \\ \mathrm{VaR}_\alpha\left(P_1 + P_2\right) &\leqslant \mathrm{VaR}_\alpha\left(P_1\right) + \mathrm{VaR}_\alpha\left(P_2\right)\end{aligned}$$

Midterms

2016 Q1. Statistical Factor Model

Consider a market with $N = 3$ risky assets, whose returns $\mathbf{R} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$ all have the same mean, i.e. $\mathbb{E}[\mathbf{R}] = \mu\mathbf{1} = \mu$. Let the variance-covariance matrix of the returns be one of the two possibilities below:

C1: $\mathbb{V}[\mathbf{R}] = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$, and C2: $\mathbb{V}[\mathbf{R}] = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$

(a) Given variance, find 1 factor model

Represent C1 as a statistical factor model with one factor. I.e. find the values of the loadings matrix (β) and the idiosyncratic variances $\left(\left\{\sigma^2_{\epsilon,i}\right\}_{i=1,2,3}\right)$ in the representation $\mathbf{R} = \mu + \beta\mathbf{F} + \epsilon$ that has the same variance-covariance matrix (where $\mathbf{F}$ are the factors and ϵ are the idiosyncratic errors).

(Hint: The values of β are all equal, and the same holds for the $\sigma^2_{\epsilon,i}$'s.)

Letting $\beta = 1/\sqrt{2}\mathbf{1}$ and $\Sigma_\epsilon = 1/2\mathbf{I}$, we have

$$\begin{aligned}
\mathbb{V}(R) &= \beta^T \underbrace{V(F)}_{I} \beta + \underbrace{\mathbb{V}(\epsilon)}_{\Sigma_\epsilon} \\
&= \left(\frac{1}{\sqrt{2}}\right)^2 1^T I 1 + \frac{1}{2} I \\
&= \frac{1}{2} 1^T 1 + \frac{1}{2} I
\end{aligned}$$

(b) Given variance, find 2 factor model

Represent C2 as a statistical factor model with two factors. This time, fix the idiosyncratic variances to $\sigma^2_{\epsilon,1} = \sigma^2_{\epsilon,3} = 1/2, \sigma^2_{\epsilon,2} = 0$, and only find the values of the 3×2 loading matrix.

(Hint: Some of the values of β are zero.)

Notice that the 1st and 3rd assets are uncorrelated, hence they should not share the same factor. This suggests:

- the 1st asset is only exposed to the 1st factor
- the 3rd asset is only exposed to the 2nd factor
- the 2nd asset is exposed to both

The following loading matrix does this:

$$\beta = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

(c) Minimum variance portfolio

Calculate the variance of the minimum-variance portfolios for both C1 and C2. Which model results has the smallest variance?

C1. Since the betas for all assets are the same, the minimum variance portfolio has uniform weights $\mathbf{w} = 1/3\mathbf{1}$.

The resulting portfolio variance is $\beta^2\sigma^2_F + \frac{\sigma^2_\epsilon}{N} = (1/2)1 + \frac{1/2}{3} = 2/3$.

(Refer to 5.1b for the variance formula above.)

C2. The minimum variance portfolio weights $\mathbf{w} = \frac{\Sigma^{-1}1}{1^\top\Sigma^{-1}1}$.

$$\Sigma^{-1} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ -1 & 2 & -1 \\ 1/2 & -1 & 3/2 \end{bmatrix}, \text{ so } \mathbf{w} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

The variance is $\mathbf{w}^\top \Sigma \mathbf{w} = \cdots = 1/2$.

```
Sig = matrix(c(1, 1/2, 0, 1/2, 1, 1/2, 0, 1/2, 1), 3, 3)
SigInv = solve(Sig)
SigInv
# min variance ptf weights
w = rowSums(SigInv) / sum(SigInv)
w
# min variance
t(w) %*% Sig %*% w
```

2016 Q3. Continuous Mixture Model

Consider the continuous mixture $X \mid Z \sim N(Z, 1)$ where $Z \sim N(0, 1)$ (i.e. the random mean of X comes from a standard Normal)

(a) Find the (unconditional) mean of X.

$$\mathbb{E}[X] = \mathbb{E}[\overbrace{\mathbb{E}[X \mid Z]}^{=Z}] = \mathbb{E}[Z] = 0$$

(b) Find the (unconditional) variance of X.

(Hint: Use the law of total variance $\mathbb{V}[X] = \mathbb{V}[\mathbb{E}[X \mid Z]] + \mathbb{E}[\mathbb{V}[X \mid Z]]$.)

$$\mathbb{V}[X] = \mathbb{V}[\overbrace{\mathbb{E}[X \mid Z]}^{=Z}] + \mathbb{E}[\overbrace{\mathbb{V}[X \mid Z]}^{=1}] = \mathbb{V}[Z] + 1 = 1 + 1 = 2$$

(c) Does the distribution of X have fat tails? (justify your answer)

(Hint: How would you generate X random values?)

The random variable X is a Normal centred around a random Normal mean Z, which is equivalent to $X = Z + \epsilon$, where $\epsilon \sim N(0, 1)$ and $Z \perp \epsilon$ (i.e. similar to a Normal error random mean model). Thus, the distribution of X is a Normal (actually $X \sim N(0, 2)$), as a sum of independent Normals, which does not have fat tails.

(Note: you could use this distributional result to find the moments of X, but the law of total expectation/variance is more widely applicable to mixture models, even in cases when you cannot find the mixture distribution).

Now consider the continuous mixture $Y \mid Z \sim N\left(0, Z^{-2}\right)$ where $Z \sim N(0, 1)$.

(d) Find the (unconditional) mean and variance of Y.

Notice that $Z^2 \sim \chi^2(\nu = 1)$ and $Y \mid Z \sim \sqrt{\frac{1}{Z^2}} N(0, 1)$.

We know that a Normal divided by the square root of an independent χ^2 over its degrees of freedom (ν) follows a t-distribution with the same degrees of freedom.

Thus, $Y \sim t(\nu = 1)$, and therefore both its mean and variance are undefined.

(Note: You could reach the same result using the law of total expectation/variance, where the moments of $\sqrt{\frac{1}{Z^2}}$ are also undefined.)

(e) Does the distribution of Y have fat tails? (justify your answer)

Yes, the distribution of Y has fat tails with tail index $\alpha = \nu = 1$

2017 Q2. General Factor Model

Consider the general factor model $R = \beta^\top F + \varepsilon$ with n asset returns (variables) and p factors, where $\Sigma_F = I$ and Σ_ε is diagonal.

(a) Return covariance

For specific assets i, j with $R_i = \beta_i^\top F + \varepsilon_i$ and $R_j = \beta_j^\top F + \varepsilon_j$, find the condition on their beta coefficients $\left(\beta_i, \beta_j\right)$ so that their returns are uncorrelated, i.e. $\text{Cov}\left(R_i, R_j\right) = 0$.

We want

$$\begin{aligned}
\text{Cov}\left(R_i, R_j\right) &= \text{Cov}\left(\beta_i^\top F + \varepsilon_i, \beta_j^\top F + \varepsilon_j\right) \\
&= \text{Cov}\left(\beta_i^\top F, \beta_j^\top F\right) + \text{Cov}\left(\beta_i^\top F, \varepsilon_j\right) + \text{Cov}\left(\varepsilon_i, \beta_j^\top F\right) + \text{Cov}\left(\varepsilon_i, \varepsilon_j\right) \\
&= \beta_i^\top \text{Cov}(F, F)\beta_j = 0 \Rightarrow \beta_i^\top \beta_j = 0
\end{aligned}$$

(b) Given variance, find model

For $n = 3$ and $p = 1$, find the values of the loadings matrix and idiosyncratic variances that result in the covariance matrix: $\mathbb{V}(R) = \Sigma_R = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 4 \end{bmatrix}$

We want

$$\Sigma_R = \beta^\top \Sigma_F \beta + \Sigma_\varepsilon \Leftrightarrow \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \beta^\top \beta + \Sigma_\varepsilon$$

The following β and Σ_ε achieve that:

$$\beta = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \Sigma_\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

```
beta = c(1, -1, 1)
beta %*% t(beta)
```

(c) Minimum variance portfolio

For the model in part (b), find the minimum variance portfolio consisting of only 2 out of the 3 assets (i.e. the weight on one asset is 0 , and the sum of the other two weights is 1). Find the optimal weights and the minimum variance.

The pair R_1, R_2 leads to the smallest portfolio variance. The optimal weights are $w^\star = \frac{\Sigma_R^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma_R^{-1}\mathbf{1}}$, where:

$$\begin{aligned}
\Sigma_R^{-1} &= \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\
\Sigma_R^{-1}\mathbf{1} &= \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \\
w^\star &= \frac{\Sigma_R^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma_R^{-1}\mathbf{1}} = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}, \sigma_\star^2 = \frac{1}{\mathbf{1}^\top \Sigma_R^{-1}\mathbf{1}} = 5/7
\end{aligned}$$

2017 Q3. First EVT on Pareto

Let X_i be i.i.d. random variables from the Pareto distribution with CDF $F_X(x) = \mathbb{P}(X \leq x) = 1 - x^{-\alpha}$, for $x \geq 1, \alpha > 0$. Define their maximum $M_n = \max\{X_1, \ldots, X_n\}$.

This question looks at the limiting behaviour of the maximum as $n \to \infty$ (the broader area of such questions is known as Extreme Value Theory).

(a) Find CDF of maximum

Show that the CDF of the maximum is $F_{M_n}(x) = (1 - x^{-\alpha})^n$.

(Hint: Express the condition $M_n \leq x$ as a set of conditions on the X_i 's)

$$\begin{aligned}
\mathbb{P}(M_n \leq x) &= \mathbb{P}(\max\{X_1, \ldots, X_n\} \leq x) \\
&= \mathbb{P}(\{X_1 \leq x\} \cap \cdots \cap \{X_n \leq x\}) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) \\
&= [\mathbb{P}(X_i \leq x)]^n = [F_X(x)]^n = (1 - x^{-\alpha})^n
\end{aligned}$$

(b) Show convergence to Fréchet distribution

Notice that, as $n \to \infty$, the variable M_n converges in probability to infinity (i.e. as you draw more variables, their maximum increases indefinitely). But properly scaled down maxima can converge to non-trivial distributions.

Show that if you divide M_n by $n^{1/\alpha}$, the CDF of the resulting variables $M_n/n^{1/\alpha}$ converges to the Fréchet distribution's CDF: $F_{\text{Fréchet}}(x) = \exp\left\{-x^{-\alpha}\right\}, x > 0$.

(Hint: use the fact that $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}, \forall x \in \mathbb{R}$)

$$
\begin{aligned}
\mathbb{P}\left(\frac{M_n}{n^{1/\alpha}} \leq x\right) &= \mathbb{P}\left(M_n \leq x n^{1/\alpha}\right) = F_{M_n}\left(x n^{1/\alpha}\right) \\
&= \left[1-\left(x n^{1/\alpha}\right)^{-\alpha}\right]^n = \left(1-\frac{x^{-\alpha}}{n}\right)^n \\
&\to \exp\left\{-x^{-\alpha}\right\}, \text{ as } n \to \infty
\end{aligned}
$$

(c) Simulate random value from Frechet

Describe a method to simulate a random value from the Fréchet distribution, based on a Uniform $(0,1)$ random value U.

(Hint: use the inverse CDF method)

First, find inverse:

$$
\begin{aligned}
F(x) = \exp\left\{-x^{-\alpha}\right\} = y \implies -x^{-\alpha} &= \log y \\
\left(x^{-\alpha}\right)^{-1/\alpha} &= (-\log y)^{-1/\alpha} \\
x &= (-\log y)^{-1/\alpha}
\end{aligned}
$$

If $X \sim F \Rightarrow F(X) \sim \text{Unif}(0,1) \Rightarrow F^{-1}(U) \sim X$. So,

$$
\begin{aligned}
F_{\text{Fréchet}}(x) = \exp\left\{-x^{-\alpha}\right\} &\Rightarrow F_{\text{Fréchet}}^{-1}(x) = [-\log(y)]^{-1/\alpha} \\
&\Rightarrow X = [-\log(U)]^{-1/\alpha} \sim F_{\text{Fréchet}}
\end{aligned}
$$

2019 Q1. N-th power of log-normal is log-normal

Let the random variable Y follow a log-Normal distribution with drift μ and volatility σ, i.e. $Y \sim \log N\left(\mu, \sigma^2\right) \Leftrightarrow \log(Y) \sim N\left(\mu, \sigma^2\right)$.

Show that the ν th power of Y also follows a log-Normal distribution, with parameters given by $Y^\nu \sim \log N\left(\nu\mu, \nu^2\sigma^2\right), \forall \nu \in \mathbb{R}$

(Hint: take the logarithm of Y^ν)

$$
\begin{aligned}
Y \sim \log N\left(\mu, \sigma^2\right) &\Rightarrow \log(Y) \sim N\left(\mu, \sigma^2\right) \\
&\Rightarrow \log Y^\nu = \nu \log Y \sim N\left(\nu\mu, \nu^2\sigma^2\right) \\
&\Rightarrow Y^\nu \sim \log\left(\nu\mu, \nu^2\sigma^2\right)
\end{aligned}
$$

2019 Q2. Exponential distribution

Let Λ and $X \mid \Lambda$ both be exponentially distributed as:

$$\Lambda \sim \text{Exp}(1) \Rightarrow F_\Lambda(\lambda) = 1 - e^{-\lambda}, \lambda > 0$$
$$X \mid \Lambda = \lambda \sim \text{Exp}(\lambda) \Rightarrow F_{X|\Lambda=\lambda}(x) = 1 - e^{-\lambda x}, x > 0$$

(a) Find the unconditional CDF of X, i.e. $F_X(x)$.

$$\begin{aligned}
F_X(x) = P(X \leq x) &= \int_0^\infty P(X \leq x \mid \Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\
&= \int_0^\infty \left(1 - e^{-\lambda x}\right) e^{-\lambda} d\lambda \\
&= \int_0^\infty e^{-\lambda} d\lambda - \int_0^\infty e^{-\lambda(x+1)} d\lambda \\
&= 1 - \frac{1}{x+1} \int_0^\infty (x+1) e^{-(x+1)\lambda} d\lambda \\
&= 1 - \frac{1}{x+1}, x > 0
\end{aligned}$$

(b) Is the distribution of X heavy-tailed? If so, what is its tail index?

The PDF of X as

$$\begin{aligned}
f_X(x) &= \frac{d}{dx}\left(1 - \frac{1}{x+1}\right) \\
&= \frac{1}{(x+1)^2}, x > 0
\end{aligned}$$

This shows that the distribution has a polynomially decreasing tail, with tail index 1.

2019 Q4. Two factor model

Consider a market consisting of $2N$ assets whose returns follow a two-factor model, where:

- the first N assets depend only on the 1st factor
- the last N assets depend only on the 2nd factor
- all assets have the same betas and idiosyncratic risk

Specifically, the model is given by:

$$R_i = \begin{cases} \mu + \beta F_1 + \varepsilon_i, & i = 1, \ldots, N \\ \mu + \beta F_2 + \varepsilon_i, & i = N+1, \ldots, 2N \end{cases} \quad \text{where}$$
$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), \quad \varepsilon_i \sim^{iid} N\left(0, \sigma^2\right), \quad F_1, F_2 \perp \varepsilon_i$$

(a) Find covariance matrix.

Show that the return covariance matrix has the form:

$$\mathbb{V}(R) = \Sigma = \left[\begin{array}{c|c} V & \mathbf{0} \\ \hline \mathbf{0} & V \end{array}\right]$$

where $V = \beta^2 \mathbf{1}_N \mathbf{1}_N^\top + \sigma^2 I_{N\times N}$.

For any $i, j = 1, \dots, N$ and $k, l = N+1, \dots, 2N$, we have

$$\begin{aligned}
\mathbb{V}[R_i] = \mathbb{V}[R_k] &= \mathbb{V}[\mu + \beta F_1 + \varepsilon_i] = \mathbb{V}[\beta F_1] + \mathbb{V}[\varepsilon_i] = \beta^2 + \sigma^2 \\
\mathrm{Cov}\left[R_i, R_j\right] = \mathrm{Cov}[R_k, R_l] &= \mathrm{Cov}\left[\mu + \beta F_1 + \varepsilon_i, \mu + \beta F_1 + \varepsilon_j\right] \\
&= \beta^2 \mathbb{V}[F_1] + \beta\,\mathrm{Cov}[F_1, \varepsilon_i] + \beta\,\mathrm{Cov}\left[F_1, \varepsilon_j\right] + \mathrm{Cov}\left[\varepsilon_i, \varepsilon_j\right] \\
&= \beta^2 + 0 + 0 + 0 = \beta^2 \\
\mathrm{Cov}[R_i, R_k] &= \mathrm{Cov}[\mu + \beta F_1 + \varepsilon_i, \mu + \beta F_2 + \varepsilon_k] \\
&= \beta^2\,\mathrm{Cov}[F_1, F_2] + \beta\,\mathrm{Cov}[F_1, \varepsilon_k] + \beta\,\mathrm{Cov}[F_2, \varepsilon_i] + \mathrm{Cov}[\varepsilon_i, \varepsilon_k] \\
&= 0 + 0 + 0 + 0 = 0
\end{aligned}$$

Putting it all together, we get

$$\begin{aligned}
\Sigma &= \left[\begin{array}{cccc|cccc}
\beta^2+\sigma^2 & \beta^2 & \cdots & \beta^2 & 0 & 0 & 0 & 0 \\
\beta^2 & \beta^2+\sigma^2 & \cdots & \beta^2 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & 0 & 0 & 0 \\
\beta^2 & \beta^2 & \cdots & \beta^2+\sigma^2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \beta^2+\sigma^2 & \beta^2 & \cdots & \beta^2 \\
0 & 0 & 0 & 0 & \beta^2 & \beta^2+\sigma^2 & \cdots & \beta^2 \\
0 & 0 & 0 & 0 & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \beta^2 & \beta^2 & \cdots & \beta^2+\sigma^2
\end{array}\right] \\
&= \left[\begin{array}{c|c} \beta^2 \mathbf{1}_N \mathbf{1}_N^\top + \sigma^2 I_{N\times N} & \mathbf{0}_{N\times N} \\ \hline \mathbf{0}_{N\times N} & \beta^2 \mathbf{1}_N \mathbf{1}_N^\top + \sigma^2 I_{N\times N} \end{array}\right]
\end{aligned}$$

(b) Show that the minimum variance portfolio has equal weights.

The minimum variance portfolio weights are given by $w = \frac{\Sigma^{-1}\mathbf{1}_{2N}}{\mathbf{1}_{2N}^\top \Sigma^{-1} \mathbf{1}_{2N}}$, which means they are proportional to the sum of the rows of the inverese return covariance matrix Σ^{-1}. Since every row of Σ^{-1} has the same values, all the weights are equal, i.e. $w = \frac{1}{2N}\mathbf{1}_{2N}$

(c) Find the minimum variance.

The minimum variance is given by $w^\top \Sigma w = \frac{1}{(2N)^2}\mathbf{1}_{2N}^\top \Sigma 1_{2N}$, which is the sum of all the elements in Σ divided by $4N^2$. From the form of the matrix Σ, we get:

$$\begin{aligned}
\min \mathrm{var} &= \frac{1}{4N^2} \underbrace{(2N^2\beta^2 + 2N\sigma^2)}_{\text{sum of elements in } \Sigma} \\
&= \frac{1}{2}\beta^2 + \frac{1}{2N}\sigma^2
\end{aligned}$$

Note that $\beta^2/2$ is the undiversifiable risk in the model (cannot be reduced by increasing the number of assets $2N$).

(d) Repeat b & c if correlation != 0

Repeat the last two parts assuming the factors are correlated with coefficient ρ, i.e. $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$

(Hint: for any covariance matrix of the form $\Sigma = \left[\begin{array}{c|c} V & U \\ \hline U & V \end{array}\right]$, its inverse has a similar form $\Sigma^{-1} = \left[\begin{array}{c|c} Z & W \\ \hline W & Z \end{array}\right]$)

When there is correlations between the factors, we have

$$\begin{aligned}
\operatorname{Cov}[R_i, R_k] &= \operatorname{Cov}[\mu + \beta F_1 + \varepsilon_i, \mu + \beta F_2 + \varepsilon_k] \\
&= \beta^2 \operatorname{Cov}[F_1, F_2] + \beta \operatorname{Cov}[F_1, \varepsilon_k] + \beta \operatorname{Cov}[F_2, \varepsilon_i] + \operatorname{Cov}[\varepsilon_i, \varepsilon_k] \\
&= \beta^2 \rho + 0 + 0 + 0 = \beta^2 \rho
\end{aligned}$$

The covariance matrix becomes

$$\begin{aligned}
\Sigma &= \left[\begin{array}{cccc|cccc}
\beta^2+\sigma^2 & \beta^2 & \cdots & \beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 \\
\beta^2 & \beta^2+\sigma^2 & \cdots & \beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 \\
\vdots & \vdots & \ddots & \vdots & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 \\
\beta^2 & \beta^2 & \cdots & \beta^2+\sigma^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 \\
\hline
\rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \beta^2+\sigma^2 & \beta^2 & \cdots & \beta^2 \\
\rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \beta^2 & \beta^2+\sigma^2 & \cdots & \beta^2 \\
\rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \vdots & \vdots & \ddots & \vdots \\
\rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \rho\beta^2 & \beta^2 & \beta^2 & \cdots & \beta^2+\sigma^2
\end{array}\right] \\
&= \left[\begin{array}{c|c}
\beta^2 \mathbf{1}_N \mathbf{1}_N^\top + \sigma^2 \mathbf{I}_{N\times N} & \rho\beta^2 \mathbf{1}_N \mathbf{1}_N^\top \\
\hline
\rho\beta^2 \mathbf{1}_N \mathbf{1}_N^\top & \beta^2 \mathbf{1}_N \mathbf{1}_N^\top + \sigma^2 \mathbf{I}_{N\times N}
\end{array}\right]
\end{aligned}$$

Notice that the minimum-variance portfolio weights are all equal in this case as well. Therefore, the minimum variance is the sum of the covariance matrix elements divided by $4N^2$. We get:

$$\begin{aligned}
\min \operatorname{var} &= \frac{1}{4N^2}\left(2N^2\beta^2 + 2N^2\rho\beta^2 + 2N\sigma^2\right) \\
&= \frac{1}{4N^2}\left(2N^2\beta^2(1+\rho) + 2N\sigma^2\right) \\
&= \frac{1}{2}\beta^2(1+\rho) + \frac{1}{2N}\sigma^2
\end{aligned}$$

2019 Q5. Burr Distribution

Consider the Burr distribution for a positive random variable X, with cumulative distribution function (CDF) given by:

$$F_X(x) = 1 - (1 + x^c)^{-k}, \quad \forall x > 0, \text{ where } c, k > 0$$

(a) What is the tail index of the Burr distribution (in terms of the parameters c, k)

$$\begin{aligned}
f_X(s) &= \frac{d}{dx} F_X(x) = \frac{d}{dx}\left[1 - (1 + x^c)^{-k}\right] \\
&= k(1 + x^c)^{-k-1} \frac{d}{dx}(1 + x^c) \\
&= \frac{ckx^{c-1}}{(1 + x^c)^{k+1}}
\end{aligned}$$

The polynomial rate at which the PDF goes to 0 is $-ck - 1$, since:

$$\begin{aligned}
\lim_{x\to\infty} f_X(s) &= \lim_{x\to\infty} \frac{ckx^{c-1}}{(1 + x^c)^{k+1}} \\
&\sim \lim_{x\to\infty} \frac{x^{\phi-1}}{x^{ck+\phi}} \\
&\sim \lim_{x\to\infty} x^{-ck-1}
\end{aligned}$$

Hence, the rate is $-ck - 1$ and the tail index is ck.

(b) Simulate random value from Burr

Describe a method to simulate a random value from the Burr distribution, based on a Uniform $(0, 1)$ random value U.

(Hint: use the inverse CDF method)

From the inverse CDF method:

$$F_X(X) = U \sim \text{Uniform}(0, 1) \Rightarrow X = F_X(U)^{-1} \sim \text{Burr}(c, k)$$

We have:

$$\begin{aligned}
U = F_X(X) &= 1 - (1 + X^c)^{-k} \\
\Rightarrow (1 + X^c)^{-k} &= 1 - U \\
\Rightarrow 1 + X^c &= (1 - U)^{-1/k} \\
X &= \left[(1 - U)^{-1/k} - 1\right]^{1/c}
\end{aligned}$$

2020 Q2. Factor Models

Assume the standard Brownian motion $\{W_t\}$, with $W_0 = 0$ and independent Normal increments $(W_t - W_s) \sim N(0, (t-s)), \forall t > s$

(a) Brownian Motion Properties

Write down the joint distribution of $W = \begin{bmatrix} W_r & W_s & W_t \end{bmatrix}^\top$, for $0 < r < s < t$, specifying its mean vector and variance-covariance matrix.

Using the fact that $\mathrm{Cov}(W_s, W_t) = \min(s,t)$, we have

$$\begin{bmatrix} W_r \\ W_s \\ W_t \end{bmatrix} \sim N_3\left(\mathbf{0}_3, \begin{bmatrix} r & r & r \\ r & s & s \\ r & s & t \end{bmatrix}\right)$$

(b) Given variance, find model

Consider the following 3-factor model for $W = \begin{bmatrix} W_r & W_s & W_t \end{bmatrix}^\top$

$$\begin{bmatrix} W_r \\ W_s \\ W_t \end{bmatrix} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \Leftrightarrow W = \beta^\top F$$

where $F \sim N_3(\mathbf{0}, I)$ and there is no idiosyncratic errors (similar to a simple linear transformation of the factors). Find the values of the coefficients in β so that the factor model has the same covariance matrix $\mathbb{V}[W]$ as in the previous part.

(Hint: Think of each factor F_i as representing an independent increment of the Brownian motion.)

Let

$$\beta^\top = \begin{bmatrix} \sqrt{r} & 0 & 0 \\ \sqrt{r} & \sqrt{s-r} & 0 \\ \sqrt{r} & \sqrt{s-r} & \sqrt{t-s} \end{bmatrix}$$

Thus, each factor is multiplied by the same value, giving the correct Normal increment distribution (e.g. $\sqrt{s-r} \times F_2 \sim N(0, s-r)$). It is straightforward to verify that:

$$\beta^\top \beta = \begin{bmatrix} \sqrt{r} & 0 & 0 \\ \sqrt{r} & \sqrt{s-r} & 0 \\ \sqrt{r} & \sqrt{s-r} & \sqrt{t-s} \end{bmatrix} \begin{bmatrix} \sqrt{r} & \sqrt{r} & \sqrt{r} \\ 0 & \sqrt{s-r} & \sqrt{s-r} \\ 0 & 0 & \sqrt{t-s} \end{bmatrix} = \begin{bmatrix} r & r & r \\ r & s & s \\ r & s & t \end{bmatrix}$$

(c) Find conditional distribution

Find the (joint) conditional distribution of $\begin{bmatrix} W_r \\ W_t \end{bmatrix} \mid (W_s = x)$, specifying its mean vector and covariance matrix.

$$\begin{bmatrix} W_r \\ W_s \\ W_t \end{bmatrix} \sim N_3\left(\mathbf{0}_3, \begin{bmatrix} r & r & r \\ r & s & s \\ r & s & t \end{bmatrix}\right)$$

$$\Leftrightarrow \begin{bmatrix} W_r \\ W_t \\ W_s \end{bmatrix} \sim N_3\left(\mathbf{0}_3, \left[\begin{array}{cc|c} r & r & r \\ r & t & s \\ \hline r & s & s \end{array}\right]\right)$$

$$\Rightarrow \begin{bmatrix} W_r \\ W_t \end{bmatrix} \mid (W_s = x) \sim N_2(\dagger, \ddagger), \text{ where}$$

$$\dagger = \mathbf{0}_2 + \begin{bmatrix} r \\ s \end{bmatrix} \frac{1}{s}(x - 0) = \begin{bmatrix} x\frac{r}{s} \\ x \end{bmatrix}$$

$$\ddagger = \begin{bmatrix} r & r \\ r & t \end{bmatrix} - \begin{bmatrix} r \\ s \end{bmatrix} \frac{1}{s} \begin{bmatrix} r & s \end{bmatrix} = \begin{bmatrix} r\left(1 - \frac{r}{s}\right) & 0 \\ 0 & t - s \end{bmatrix}$$

2021 Q1. GPD (2nd EVT)

The Generalized Pareto Distribution (GPD) has the following CDF:

$$F_X(x) = P(X \le x) = \begin{cases} 1 - (1 + \gamma x/\sigma)^{-1/\gamma}, & \gamma \ne 0, \begin{cases} x \ge 0, & \text{if } \gamma > 0 \\ 0 \le x \le -\sigma/\gamma, & \text{if } \gamma < 0 \end{cases} \\ 1 - e^{-x/\sigma}, & \gamma = 0, x \ge 0 \end{cases}$$

where γ is the shape parameter and $\sigma > 0$ is the scale parameter (note that $\gamma = 0$ gives the Exponential distribution). The GPD arises in probability theory when considering the conditional excess CDF, defined as:

$$F_u(x) = P(X - u \le x \mid X > u) = \frac{F(u + x) - F(u)}{1 - F(u)}$$

It can be shown that for a large class of distributions, their conditional excess distribution will approach the GPD for large u.

a)

Show that the conditional excess CDF $F_u(x)$ of an Exponential distribution with rate parameter λ (i.e. $X \sim \text{Exp}(\lambda) \Rightarrow \mathbb{E}[X] = 1/\lambda$) belongs to the GPD family for any value of u.

The Exponential with mean $1/\lambda$ has CDF

$$F(x) = 1 - e^{-\lambda x}, x \ge 0$$

and conditional excess CDF

$$\begin{aligned}
F_u(x) &= \frac{F(u+x) - F(u)}{1 - F(u)} \\
&= \frac{1 - e^{-\lambda(u+x)} - (1 - e^{-\lambda u})}{1 - (1 - e^{-\lambda u})} \\
&= \frac{e^{-\lambda u} - e^{-\lambda(u+x)}}{e^{-\lambda u}} \\
&= 1 - e^{-\lambda(u+x)+\lambda u} \\
&= 1 - e^{-\lambda x}
\end{aligned}$$

Which is a GPD for $\gamma = 0$ and $\sigma = 1/\lambda$.

b)

Show the conditional excess CDF $F_u(x)$ of a GPD with $\gamma \neq 0$ is another GPD with the same shape parameter γ, but different scale parameter $\sigma' = \sigma + \gamma u$.

The conditional excess CDF becomes

$$\begin{aligned}
F_u(x) &= \frac{F(u+x) - F(u)}{1 - F(u)} \\
&= \frac{1 - (1 + \gamma(x+u)/\sigma)^{-1/\gamma} - \left[1 - (1 + \gamma u/\sigma)^{-1/\gamma}\right]}{1 - \left[1 - (1 + \gamma u/\sigma)^{-1/\gamma}\right]} \\
&= \frac{(1 + \gamma u/\sigma)^{-1/\gamma} - (1 + \gamma(x+u)/\sigma)^{-1/\gamma}}{(1 + \gamma u/\sigma)^{-1/\gamma}} \\
&= 1 - \left(\frac{1 + \gamma u/\sigma + \gamma x/\sigma}{1 + \gamma u/\sigma}\right)^{-1/\gamma} \\
&= 1 - \left(1 + \frac{\gamma x}{\sigma + \gamma u}\right)^{-1/\gamma}
\end{aligned}$$

Which is a GPD distribution with the same shape parameter γ and scale parameter $\sigma' = \sigma + \gamma u$.

c)

Consider the Pareto Type II (a.k.a. Lomax) distribution with PDF:

$$f(x) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-\alpha-1}, x \geq 0$$

Write this as a special case of the GPD, and express its parameters (γ, σ) in terms of (α, λ).

The Pareto Type II distribution has CDF

$$
\begin{aligned}
F(x) &= \int_0^x \frac{\alpha}{\lambda}\left(1+\frac{t}{\lambda}\right)^{-\alpha-1} dt \\
&= \left[-\left(1+\frac{t}{\lambda}\right)^{-\alpha}\right]_{t=0}^{x} \\
&= (1+0)^{-\alpha} - (1+x/\lambda)^{-\alpha} \\
&= 1-(1+x/\lambda)^{-\alpha}
\end{aligned}
$$

This can be expresses as a GPD with $\gamma = 1/\alpha$ and $\sigma = \lambda/\alpha$

d) Simulate random value from GPD

Show that one can generate values from the GPD with $\gamma \neq 0$ using the following transformation of a uniform random variable:

$$X = \frac{\sigma}{\gamma}\left[(1-U)^{-\gamma}-1\right], \qquad \text{where } U \sim \text{Unif}(0,1)$$

(Hint: Use the inverse CDF transform.)

For a random variable X following GPD($\gamma \neq 0, \sigma$), we have

$$
\begin{aligned}
F_X(X) = 1-(1+\gamma X/\sigma)^{-1/\gamma} &= U \sim \text{Unif}(0,1) \\
\Rightarrow 1-U &= (1+\gamma X/\sigma)^{-1/\gamma} \\
\Rightarrow (1-U)^{-\gamma} &= 1+\gamma X/\sigma \\
\Rightarrow \frac{\sigma}{\gamma}\left[(1-U)^{-\gamma}-1\right] &= X \sim \text{GPD}(\gamma,\sigma)
\end{aligned}
$$

e) Continuous mixture of exponential and gamma

Show that the GPD(γ, σ) can arise as a continuous mixture model of an Exponential distribution with a random rate parameter following a Gamma distribution, i.e.:

$$
\begin{aligned}
X \mid \Lambda &\sim \text{Exp}(\Lambda) \\
\Lambda &\sim \text{Gamma}(\alpha,\beta)
\end{aligned}
$$

where the Gamma distribution scale (α) and rate (β) parameters are such that $(\alpha = 1/\gamma, \beta = \sigma/\gamma) \Leftrightarrow (\gamma = 1/\alpha, \sigma = \beta/\alpha)$.

(Hint: Show that the complementary CDF $\bar{F}_X(x) = P(X > x)$ of the mixture model is equal to that of a GPD.)

Gamma function:

For a positive integer n:

$$\Gamma(n) = (n-1)!$$

For complex numbers with a positive real part:

$$\Gamma(\alpha) = \int_0^\infty \lambda^{\alpha-1} e^{-\lambda}\, dx$$

To find this information in RStudio, use `?GammaDist` or `?Distributions`

$$\begin{aligned}
\bar{F}_X(x) &= P(X > x) \text{ (Law of Total Prob.)} \\
&= \int_0^\infty \overbrace{P(X > x \mid \Lambda = \lambda)}^{\sim\text{Exp}(\lambda)} \quad \overbrace{f_\Lambda(\lambda)}^{\sim\text{Gamma}(\alpha,\beta)} \quad d\lambda \\
&= \int_0^\infty e^{-\lambda x} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} e^{-(\beta+x)\lambda} d\lambda \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta+x)^\alpha} \\
&= \left(\frac{\beta}{\beta+x}\right)^\alpha \\
&= 1 - 1 + \left(\frac{\beta+x}{\beta}\right)^{-\alpha} \\
&= 1 - \quad \overbrace{[1-(1+x/\beta)^{-\alpha}]}^{\text{CDF or GPD}(\gamma=1/\alpha,\sigma=\beta/\alpha)}
\end{aligned}$$

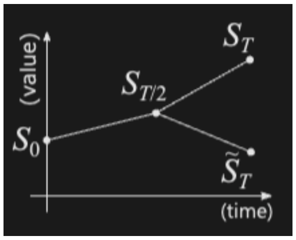
2016 Q2 Simulation

Assume you are using simulation to estimate the price of a European option. The discounted payoff is a function (f) of the asset price (S_T) at expiration (T), i.e. we want to estimate $\pi = \mathbb{E}[f(S_T)]$. Further assume the asset price follows geometric Brownian motion (GBM) under the risk-neutral measure, i.e. $dS_t = S_t(rdt + \sigma dW_t)$, where r is the risk-free interest rate, σ is the asset volatility, and $\{W_t\}$ is standard Brownian motion. Finally, assume you can only simulate standard Normal random variables (Z).

(a) Generate random values

Give the formula for generating asset values S_t (at an arbitrary time $t > 0$) in terms of a standard Normal value Z, given the current value S_0.

$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z\right\}$



Assume you generate final values of S_T in the following way: first you generate a single value half-way, i.e. $S_{T/2}$, and starting from there you generate two final values S_T and $\tilde{S}_T$ (see figure). Each value requires a different Normal $Z_{T/2}, Z_T, \tilde{Z}_T$. You estimate π by generating $2n$ final values like this, and taking their average, i.e. $\tilde{\pi} = \frac{1}{n}\sum_{i=1}^{n} \frac{f\left(S_T^{(i)}\right)+f\left(\tilde{S}_T^{(i)}\right)}{2}$. Note that the final values $S_T^{(i)}$ and $\tilde{S}_T^{(i)}$ share the same half-way value $S_{T/2}^{(i)}$.

(b) RN pricing

Show that the final asset value S_T is a function of the sum of the two Normals $Z_{T/2} + Z_T$

$$
\begin{aligned}
S_T &= S_{T/2}\left(\frac{S_T}{S_{T/2}}\right) \\
&= S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) T/2 + \sigma\sqrt{T/2}Z_{T/2}\right\} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) T/2 + \sigma\sqrt{T/2}Z_T\right\} \\
&= S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\frac{Z_{T/2} + Z_T}{\sqrt{2}}\right\}
\end{aligned}
$$

(c) Show that $\tilde{\pi}$ is an unbiased estimator for π.

The term $\frac{Z_{T/2}+Z_T}{\sqrt{2}}$ follows standard Normal distribution as the sum of two Normals with $\mathbb{E}\left[\frac{Z_{T/2}+Z_T}{\sqrt{2}}\right] = 0$ and $\mathbb{V}\left[\frac{Z_{T/2}+Z_T}{\sqrt{2}}\right] = \frac{1}{2}\left(\mathbb{V}\left[Z_{T/2}\right] + \mathbb{V}\left[Z_T\right]\right) = \frac{1}{2}(1+1) = 1$. The same holds for $\frac{Z_{T/2}+\tilde{Z}_T}{\sqrt{2}}$, thus $\mathbb{E}\left[f\left(S_T\right)\right] = \mathbb{E}\left[f\left(\tilde{S}_T\right)\right] = \pi$. We have:

$$
\begin{aligned}
\mathbb{E}[\tilde{\pi}] &= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} \frac{f\left(S_T^{(i)}\right) + f\left(\tilde{S}_T^{(i)}\right)}{2}\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{\mathbb{E}\left[f\left(S_T^{(i)}\right)\right] + \mathbb{E}\left[f\left(\tilde{S}_T^{(i)}\right)\right]}{2} \\
&= \frac{1}{n} n \frac{\pi + \pi}{2} = \pi
\end{aligned}
$$

(d) Variance reduction

Find the variance of $\tilde{\pi}$ in terms of the moments of $f(S_T)$ and $f\left(\tilde{S}_T\right)$. How does this variance compare to that of the simple estimator $\hat{\pi} = \frac{1}{2n}\sum_{i=1}^{2n} f\left(S_T^{(i)}\right)$?

The two terms $\frac{Z_{T/2}+Z_T}{\sqrt{2}}, \frac{Z_{T/2}+\tilde{Z}_T}{\sqrt{2}}$ are positively correlated since they share the same $Z_{T/2}$. As a result, S_T and $\tilde{S}_T$ are also positively correlated, and their payoff $f(S_T)$ and $f\left(\tilde{S}_T\right)$ will be positively correlated for "most" payoff functions (e.g. for monotonic payoffs like that of call or put). Thus, we have:

$$\begin{aligned}
\mathbb{V}[\tilde{\pi}] &= \frac{1}{n^2}\mathbb{V}\left[\sum_{i=1}^{n}\frac{f\left(S_T^{(i)}\right)+f\left(\tilde{S}_T^{(i)}\right)}{2}\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\frac{\mathbb{V}\left[f\left(S_T^{(i)}\right)+f\left(\tilde{S}_T^{(i)}\right)\right]}{2^2} \\
&= \frac{1}{(2n)^2}n(\mathbb{V}\left[f\left(S_T^{(i)}\right)\right] + \overbrace{\mathbb{V}\left[f\left(\tilde{S}_T^{(i)}\right)\right]}^{=\mathbb{V}\left[f\left(S_T^{(i)}\right)\right]} + 2\operatorname{Cov}\left[f\left(S_T^{(i)}\right), f\left(\tilde{S}_T^{(i)}\right)\right]) \\
&= \frac{1}{2n}\left(\mathbb{V}\left[f\left(S_T^{(i)}\right)\right] + \operatorname{Cov}\left[f\left(S_T^{(i)}\right), f\left(\tilde{S}_T^{(i)}\right)\right]\right)
\end{aligned}$$

The simple Monte Carlo estimator has variance $\frac{1}{2n}\mathbb{V}\left[f\left(S_T^{(i)}\right]\right.$, so as long as the covariance $\operatorname{Cov}\left[f\left(S_T^{(i)}\right), f\left(\tilde{S}_T^{(i)}\right)\right]$ is positive (most likely situation) $\tilde{\pi}$ will do worse than $\hat{\pi}$.

(e) Antithetic variables

Assume you use antithetic variables instead of drawing independents Normal for each of the two final values that share the same half-way value. I.e. you use $\tilde{Z}_T^{(i)} = -Z_T^{(i)}, \forall i$. Would you expect the variance of the estimator $\tilde{\pi}$ to increase or decrease? (justify your answer)

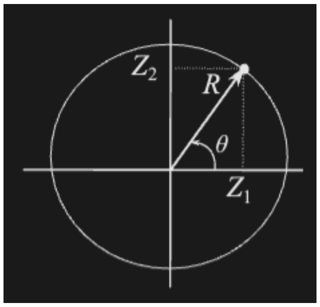
If we use antithetic variables, the correlation between $\frac{Z_{T/2}+Z_T}{\sqrt{2}}, \frac{Z_{T/2}+\tilde{Z}_T}{\sqrt{2}}$ is 0 , since:

$$\begin{aligned}
\operatorname{Cov}\left[\frac{Z_{T/2}+Z_T}{\sqrt{2}}, \frac{Z_{T/2}-Z_T}{\sqrt{2}}\right] = \frac{1}{2}&\overbrace{\operatorname{Cov}\left[Z_{T/2}, Z_{T/2}\right]}^{=1} - \overbrace{\operatorname{Cov}\left[Z_{T/2}, -Z_T\right]}^{=0} + \\
&+ \underbrace{\operatorname{Cov}\left[Z_T, Z_{T/2}\right]}_{=0} - \underbrace{\operatorname{Cov}\left[Z_{T/2}, Z_{T/2}\right]}_{=1}] = 0
\end{aligned}$$

This means that the $\tilde{\pi}$ estimator will have the same variance as the $\hat{\pi}$ estimator, i.e. it will decrease.

2016 Q4. Misc

The Rayleigh distribution has cdf $F(x) = 1 - \exp\left\{-x^2/2\right\}$. This distribution arises in the following setting: if $Z_1, Z_2 \overset{\text{iid}}{\sim} N(0,1)$ then $R = \sqrt{Z_1^2 + Z_2^2}$ follows F. In other words, the Euclidean norm of a bivariate Normal follows Rayleigh (see figure). Note that the angle (θ) of the Normal coordinate vector follows Uniform distribution.



(a)

What is the distribution of R^2 ?

$R^2 = Z_1^2 + Z_2^2 \sim \chi^2(\mathrm{df}=2)$, i.e. chi-square with 2 degrees of freedom, as the sum of the squares of two i.i.d. standard Normals.

(b)

How would you generate a random value of R given a Uniform $(0,1)$ random value U? (Hint: Use the inverse cdf method.)

Set $U = F(R) = 1 - \exp\left\{-\frac{R^2}{2}\right\} \Rightarrow \log(\overbrace{1-U}^{\sim U}) = -\frac{R^2}{2} \Rightarrow R = \sqrt{-2\log(U)}$

(c)

Assume you can only generate Uniform random numbers in $(0,1)$. Come up with a scheme to simulate two standard Normal random values based on two Uniform random values. (Hint: Simulate the angle and the length of the random vector using the Uniforms, and then take the vector coordinates.)

Consider two Uniform $(0,1)$ variables U_1, U_2. For the random angle take $\theta = 2\pi U_1$, and for the vector length use the previous part, i.e. $R = \sqrt{-2\log(U_2)}$. The two standard Normals Z_1, Z_2 are the coordinates of this vector, i.e.

$$\begin{aligned} Z_1 &= R\cos(\theta) = \sqrt{-2\log(U_2)}\cos(2\pi U_1) \\ Z_2 &= R\sin(\theta) = \sqrt{-2\log(U_2)}\sin(2\pi U_1) \end{aligned}$$

(d)

Consider the joint distribution of the maximum (M_T) and minimum (m_T) of a Brownian motion. Are these extrema independent or positively/negatively related? (justify your answer)

The minimum and maximum over the same path are positively correlated, since when the one is higher (e.g. the path does an upward excursion) the other will also tend to be higher.

2017 Q1

Consider an investment whose cumulative gain/loss, as a proportion of the initial investment, follows a standard Brownian motion W_t. I.e. if you initially invest $\$X$ at time 0 , you will have won/lost (depending on the sign) $\$X \times W_t$ by time t. Assume you have finite credit and can only sustain losses of up to twice your initial amount. This means that if your cumulative loss proportion drops at or below -2 , you will have to close your position and go "bankrupt".

(a) Folded normal

Find the probability that you go bankrupt within a year ($t=1$). Express the probability in terms of the standard Normal CDF: $\Phi(x)=\mathbb{P}(Z \leq x)$, where $Z \sim N(0,1)$

(Hint: Use the symmetry of Brownian Motion.)

Let $m_T=\min_{0 \leq t \leq T}\{W_t\}$ be the minimum of W_t by time T.

By symmetry of the standard Brownian motion, we have $-m_T \sim M_T=\max_{0 \leq t \leq T}\{W_t\}$, where $M_T \sim |W_T|$.

We have:

$$\begin{aligned}
\mathbb{P}(\text{ "bankrupt by } t=1\text{ "}) & =\mathbb{P}\left(m_1 \leq-2\right)=\mathbb{P}\left(M_1 \geq 2\right)=\mathbb{P}\left(\left|W_1\right| \geq 2\right) \\
& =2 \mathbb{P}(Z \geq 2)=2[1-\mathbb{P}(Z \leq 2)]=2[1-\Phi(2)]=2 \Phi(-2)
\end{aligned}$$

(b) Reflection principle

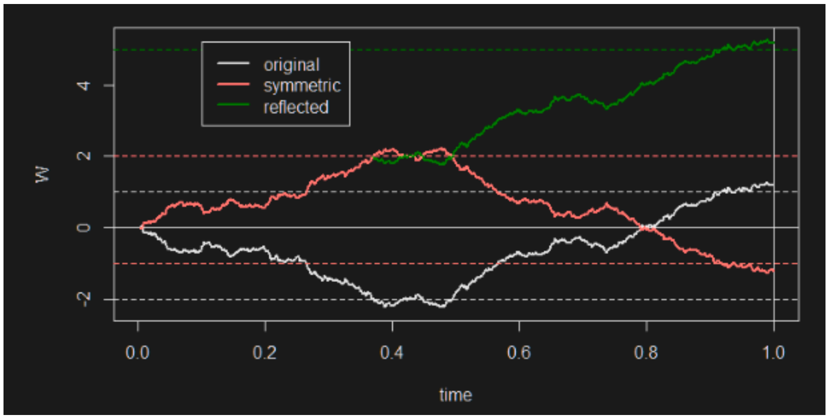
Find the probability that you go bankrupt within a year, given that the investment's cumulative gain at the end of the year is greater than 100% (i.e. $W_1 \geq 1$). (Hint: use the reflection principle)

We want $\mathbb{P}\left(m_1 \leq-2 \mid W_1 \geq 1\right)$. By symmetry, this is equal to

$$\mathbb{P}\left(M_1 \geq 2 \mid W_1 \leq-1\right)=\frac{\mathbb{P}\left(M_1 \geq 2, W_1 \leq-1\right)}{\mathbb{P}\left(W_1 \leq-1\right)}$$

By the reflection principle (see graph below), the numerator becomes

$$\begin{aligned}
\mathbb{P}\left(M_1 \geq 2 \mid W_1 \leq-1\right) & =\frac{\mathbb{P}\left(W_1 \geq 2 \times 2-(-1)\right)}{\mathbb{P}\left(W_1 \leq-1\right)} \\
& =\frac{\mathbb{P}\left(W_1 \geq 5\right)}{\Phi(-1)}=\frac{\Phi(-5)}{\Phi(-1)}
\end{aligned}$$



(c) Brownian Bridge (Rayleigh)

Find the probability that you go bankrupt within a year, given that the investment's cumulative gain at the end of the year is exactly 100% (i.e. $W_1 = 1$).

(Hint: use the result for the maximum of a Brownian bridge)

We want $\mathbb{P}\left(m_1 \leq -2 \mid W_1 = 1\right)$. By symmetry, this is equal to $\mathbb{P}\left(M_1 \geq 2 \mid W_1 = \text{-}1\right)$. The CDF of the conditional maximum of an (arithmetic) Brownian motion is $\mathbb{P}\left(M_T \leq m \mid W_T = b\right) = 1 - \exp\left\{-2\frac{m(m-b)}{\sigma^2 T}\right\}$.

Applying this result with $T = 1, m = 2, b = -1, \sigma = 1$, we get:

$$
\begin{aligned}
\mathbb{P}\left(M_1 \geq 2 \mid W_1 = -1\right) &= 1 - \mathbb{P}\left(M_1 \leq 2 \mid W_1 = -1\right) \\
&= 1 - \left(1 - \exp\left\{-2\frac{2(2-(-1))}{1}\right\}\right) = e^{-12}
\end{aligned}
$$

2017 Q4. Monte Carlo

Consider a portfolio of two options defined on the same underlying asset S. The value of these options is given by their expected discounted payoff, $v = \mathbb{E}[f(S(T))]$ & $w = \mathbb{E}[g(S(T))]$, and the value of the portfolio is $x = v + w$. Assume you cannot calculate the values explicitly, so you use Monte Carlo simulation:

$$
\hat{v} = \frac{1}{n}\sum_{i=1}^{n} f\left(S_i(T)\right), \quad \hat{w} = \frac{1}{n}\sum_{i=1}^{n} g\left(S_i'(T)\right)
$$

where $\{S_i, S_i'\}_{i=1}^{n}$ are i.i.d. random draws from the risk-neutral distribution. You can run the simulation in two ways:

1. Draw $S_i(T), S_i'(T)$ independently from their common risk-neutral distribution, or
2. Draw only once from the risk-neutral distribution, and set $S_i = S_i'$

Obviously, the second one is more efficient. In this question you will examine which method is preferable under which circumstances, for estimating the value of the portfolio $\hat{x} = \hat{v} + \hat{w}$

(a) Show that $\hat{x}$ is an unbiased estimate of x under both I. & II.

$$\begin{aligned}
\mathbb{E}[\hat{x}] &= \mathbb{E}[\hat{v}+\hat{w}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} f\left(S_i(T)\right) + \frac{1}{n}\sum_{i=1}^{n} g\left(S_i'(T)\right)\right] \\
&= \frac{1}{n}[\sum_{i=1}^{n}\overbrace{\mathbb{E}\left[f\left(S_i(T)\right)\right]}^{=v} + \sum_{i=1}^{n}\overbrace{\mathbb{E}\left[g\left(S_i'(T)\right)\right]}] = \frac{n}{n}(v+w) = v+w
\end{aligned}$$

(b) Find the variance of $\hat{x}$ under I.

$$\begin{aligned}
\mathbb{V}[\hat{x}] &= \mathbb{V}[\hat{v}+\hat{w}] = \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} f\left(S_i(T)\right) + \frac{1}{n}\sum_{i=1}^{n} g\left(S_i'(T)\right)\right] \\
&=^{(S_i \perp S_i')} \frac{1}{n^2}\left[\mathbb{V}\left[\sum_{i=1}^{n} f\left(S_i(T)\right)\right] + \mathbb{V}\left[\sum_{i=1}^{n} g\left(S_i'(T)\right)\right]\right] \\
&=^{(i.i.d.)} \frac{1}{n^2}\sum_{i=1}^{n}\left\{\mathbb{V}\left[f\left(S_i(T)\right)\right] + \mathbb{V}\left[g\left(S_i'(T)\right)\right]\right\} = \frac{1}{n}\left\{\mathbb{V}[f(S(T))] + \mathbb{V}\left[g\left(S'(T)\right)\right]\right\}
\end{aligned}$$

(c) Find the variance of $\hat{x}$ under II.

Under which condition(s) on the moments of the discounted payoff functions (f, g) is this variance smaller than the one in part (b)?

$$\begin{aligned}
\mathbb{V}[\hat{x}] =&\mathbb{V}[\hat{v}+\hat{w}] = \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} f\left(S_i(T)\right) + \frac{1}{n}\sum_{i=1}^{n} g\left(S_i'(T)\right)\right] \\
=&^{(S_i'=S_i)} \frac{1}{n^2}\left[\mathbb{V}\left[\sum_{i=1}^{n} f\left(S_i(T)\right)\right] + \mathbb{V}\left[\sum_{i=1}^{n} g\left(S_i(T)\right)\right] + \right. \\
&\left. +2\operatorname{Cov}\left(\sum_{i=1}^{n} f\left(S_i(T)\right), \sum_{i=1}^{n} g\left(S_i(T)\right)\right)\right] \\
=&^{(i.i.d.)} \frac{1}{n^2}\sum_{i=1}^{n}\left\{\mathbb{V}\left[f\left(S_i(T)\right)\right] + \mathbb{V}\left[g\left(S_i(T)\right)\right] + 2\operatorname{Cov}\left(f\left(S_i(T)\right), g\left(S_i(T)\right)\right)\right\} \\
=&\frac{1}{n}\{\mathbb{V}[f(S(T))] + \mathbb{V}[g(S(T))] + 2\operatorname{Cov}(f(S(T)), g(S(T)))\}
\end{aligned}$$

The variance is smaller under II when $\operatorname{Cov}(f(S(T)), g(S(T))) < 0$.

7. Betting Strategies

7.1 Kelly criterion for bets with arbitrary odds

Consider a sequence of i.i.d. bets with a probability of winning p and of losing $q = 1 - p$. Suppose a fraction $f \in (0, 1)$ of the capital is bet each turn. Rather than even payoff (i.e. \$1 won/lost per \$1 bet), we consider the more general scenario where $a are won or $b are lost per $1 bet.

(a) Total wealth after n bets

Find an expression for the total wealth after the nth bet (V_n) as a function of a, b, f, n, the initial wealth V_0, and the random variable W_n which counts the number of wins in n bets.

For a single bet of a fraction f of wealth V_{t-1},

$$V_t = \begin{cases} V_{t-1}(1+fa) & win \\ V_{t-1}(1-fb) & lose \end{cases}$$

Using an indicator function, the resulting wealth can be succinctly written as:

$$V_t = V_{t-1}(1+fa)^{I_t}(1-fb)^{1-I_t}$$

After a sequence of bets, we have

$$\begin{aligned} V_n &= V_0 \prod_{t=1}^{n} (1+fa)^{I_t}(1-fb)^{1-I_t} \\ &= V_0(1+fa)^{\sum_{t=1}^{n} I_t}(1-fb)^{\sum_{t=1}^{n}(1-I_t)} \\ &= V_0(1+fa)^{W_n}(1-fb)^{n-W_n} \end{aligned}$$

(b) Optimal Bet Fraction Derivation

Show that the optimal bet fraction under the Kelly criterion (max log-wealth) is given by $f^* = \frac{ap-bq}{ab}$.

The Kelly-optimal bet fraction f^* maximizes the *expected log wealth*, i.e. $f^* = \arg\max_f \mathbb{E}\left[\log\left(V_n\right)\right]$. The expected log-wealth is:

$$\begin{aligned} \mathbb{E}\left[\log\left(V_n\right)\right] &= \mathbb{E}\left[\log\left(V_0(1+fa)^{W_n}(1-fb)^{n-W_n}\right)\right] \\ &= \mathbb{E}\left[\log\left(V_0\right) + W_n \log(1+fa) + (n-W_n)\log(1-fb)\right] \\ &= \log\left(V_0\right) + \log(1+fa)\mathbb{E}\left[W_n\right] + \log(1-fb)\left(n - \mathbb{E}\left[W_n\right]\right) \\ &= \log\left(V_0\right) + \log(1+fa)np + \log(1-fb)nq \end{aligned}$$

To maximize $\mathbb{E}\left[\log\left(V_n\right)\right]$, set the derivative w.r.t. f equal to 0 and solve for f^* :

$$\begin{aligned} \frac{d}{df}\mathbb{E}\left[\log\left(V_n\right)\right] &= 0 \\ \frac{d}{df}\left[\log\left(V_0\right) + \log(1+fa)np + \log(1-fb)nq\right] &= 0 \\ \frac{\cancel{n}p}{1-f\cdot a}a &= \frac{\cancel{n}q}{1-f\cdot b}b \\ ap\left(1-f^*b\right) &= bq\left(1+f^*a\right) \\ ap - bq &= f^*ab\,\underbrace{(p+q)}_{=1} \\ f^* &= \frac{ap-bq}{ab} \end{aligned}$$

7.2 Simultaneous betting

With a sequence of gambles, we can use Kelly's criterion to maximize the expected log-return of each individual bet. For this question we consider a different set-up, where we have a set of *simultaneous* gambles. We wish to find the optimal way to distribute our wealth *among* them.

This setup occurs naturally in sports betting, where you have multiple parallel matches on a game day. Assume we can bet on two events with fractional odds given by $a_i, i = 1, \ldots, m$. This means that for each \$1 we bet, we get back $\$\,(1 + a_i)$ if we win and nothing if we lose.

(a)

Let $m = 2$ and assume the two events have the following probabilities: | | | bet 2 | | | :—: | :— | :— | :— | | | | win | lose | | bet 1 | win | p_{WW} | p_{WL} | | | lose | p_{LW} | p_{LL} | What condition on the odds and the probabilities ensure a favourable bet for each event individually?

Define the marginal probability of winning the i th bet as p_i, where in our case $p_1 = p_{WW} + p_{WL}$ and $p_2 = p_{WW} + p_{LW}$.

We want each bet to result in an expected increase in wealth.

Assuming \$1 placed on the i th bet, we have:

$$
\begin{aligned}
E(\text{i-th bet gain/loss}) = a_i p_i - (1 - p_i) > 0 \\
\Rightarrow p_i > \frac{1}{1 + a_i}, \quad i = 1, 2
\end{aligned}
$$

(b)

Assume you bet fractions $f_1, f_2 > 0$ (where $f_1 + f_2 \leq 1$) of your wealth v_0 on both events simultaneously. Find an expression for your wealth in terms of the outcomes of the bets, i.e. using indicators I_1, I_2 to indicate which of the bets are won.

Let $I_i = \begin{cases} 1, & \text{win i}^{\text{th}} \\ 0, & \text{o/w} \end{cases}$ be the indicator variable of the i th bet's outcome. We have

$$
\begin{aligned}
V_1 &= \begin{cases} V_0 \left(1 + f_1 a_1 + f_2 a_2\right), & \text{win both bets} \\ V_0 \left(1 + f_1 a_1 - f_2\right), & \text{win } 1^{\text{st}} \text{ \& lose } 2^{\text{nd}} \text{ bet} \\ V_0 \left(1 - f_1 + f_2 a_2\right), & \text{lose } 1^{\text{st}} \text{ \& win } 2^{\text{nd}} \text{ bet} \\ V_0 \left(1 - f_1 - f_2\right), & \text{lose both bets} \end{cases} \\
&= V_0 \left[1 - f_1 \left(-a_1\right)^{I_1} - f_2 \left(-a_2\right)^{I_2}\right]
\end{aligned}
$$

The main point here is that parallel bet gains/losses do not compound.

(c)

Find a closed-form expression for the expected log-wealth $G(f_1, f_2) = \mathbb{E}[\log(V_1)]$, in terms of the odds, the probabilities and the fractions.

$$\begin{aligned} G(f_1, f_2) =& \mathbb{E}[\log(V_1)] \\ &= \log V_0 + \log(1 + f_1 a_1 + f_2 a_2)\, p_{WW} + \log(1 + f_1 a_1 - f_2)\, p_{WL} \\ &\quad + \log(1 - f_1 + f_2 a_2)\, p_{LW} + \log(1 - f_1 - f_2)\, p_{LL} \end{aligned}$$

(d)

Derive 1 st order conditions for the optimal fractions f_1^*, f_2^*

$$\begin{aligned} \frac{\partial}{\partial f_1} G(f_1, f_2) = 0 \Rightarrow & \frac{a_1}{1 + f_1 a_1 + f_2 a_2} p_{WW} + \frac{a_1}{1 + f_1 a_1 - f_2} p_{WL} + \\ & - \frac{1}{1 - f_1 + f_2 a_2} p_{LW} - \frac{1}{1 - f_1 - f_2} p_{LL} = 0 \\ \frac{\partial}{\partial f_2} G(f_1, f_2) = 0 \Rightarrow & \frac{a_2}{1 + f_1 a_1 + f_2 a_2} p_{WW} - \frac{1}{1 + f_1 a_1 - f_2} p_{WL} + \\ & + \frac{a_2}{1 - f_1 + f_2 a_2} p_{LW} - \frac{1}{1 - f_1 - f_2} p_{LL} = 0 \end{aligned}$$

8. Statistical Arbitrage

8.1 Pairs Trading

Suppose you are trading the pair of stocks (P, S)

(a)

Assume the prices of the stocks have a stationary linear relationship $P_t - \lambda S_t$. Furthermore, assume that you open a trade when $P_o - \lambda S_o > 0$, and you close it when $P_c - \lambda S_c = 0$. What quantities of each stock do you have to buy/sell at the start of the trade in order to make a profit? Find the resulting profit amount.

- Since $P_0 > \lambda S_0$, open the position by selling 1 share of stock P and buying λ shares of stock S (buy low sell high), for a net initial cashflow of $P_o - \lambda S_o > 0$.
- Since $P_c - \lambda S_c = 0$, close the position by buying back 1 share of stock P and selling back λ / S_o shares of stock S, for a net final cashflow of $-P_c + \lambda S_c = 0$
- The total net profit (disregarding interest) is thus $(P_o - \lambda S_o) - (P_c - \lambda S_c) > 0$.

Notes:

- When the actual prices are cointegrated, the pairs trading strategy involves amounts proportional to the co-integrating coefficients $(1, -\lambda)$.

- Note that if $P_o - \lambda S_o < 0$ instead, we would reverse our position to make a profit, i.e. we would buy 1 share of stock P and sell λ shares of stock S.

(b)

Assume the log-prices have a stationary linear relationship $\log(P_t) - \lambda \log(S_t)$. Further assume that you open a trade when $\log(P_o) - \lambda \log(S_o) > 0$, and you close it when $\log(P_c) - \lambda \log(S_c) = 0$. Show that if you sell $1/P_o$ units of P and buy λ/S_o units of S at the start of the trade, you will make an approximate profit of $\log(P_o) - \lambda \log(S_o)$

(Hint: Use the log/net-return approximate relationship: $\log\left(\frac{P_c}{P_o}\right) \approx \frac{P_c - P_o}{P_0}$.)

> **Why and when $\frac{x-y}{y}$ is a good approximation for $\log(\frac{x}{y})$:** We can use the Taylor series expansion of $\log(x)$ about y:
>
> $$\begin{aligned} \log(x) &= \log(y) + \frac{x-y}{y} - \frac{(x-y)^2}{2y^2} + \frac{(x-y)^3}{3y^3} - \cdots \\ \log(x) - \log(y) &= \frac{x-y}{y} \underbrace{- \frac{(x-y)^2}{2y^2} + \frac{(x-y)^3}{3y^3} - \cdots}_{\to 0} \\ \log(x) - \log(y) &\approx \frac{x-y}{y} \end{aligned}$$
>
> If x is close to y and y isn't too small, the higher-order terms become negligible.

Open: $\frac{1}{P_0} P_0 - \frac{\lambda}{S_0} S_0$

Close: $-\frac{1}{P_o} P_c + \frac{\lambda}{S_o} S_c$

The total net profit (disregarding interest) is thus

$$\begin{aligned} \text{profit } &= \left(\frac{P_o}{P_o} - \lambda \frac{S_o}{S_o}\right) + \left(-\frac{P_c}{P_o} + \lambda \frac{S_c}{S_o}\right) \\ &= \frac{P_o - P_c}{P_o} + \lambda \frac{S_c - S_o}{S_o} \approx -\log\left(\frac{P_c}{P_o}\right) + \lambda \log\left(\frac{S_c}{S_o}\right) \\ &= (\log(P_o) - \lambda \log(S_o)) - \underbrace{(\log(P_c) - \lambda \log(S_c))}_{=0} \\ &= \log(P_o) - \lambda \log(S_o) > 0 \end{aligned}$$

Note that if $\log(P_o) - \lambda \log(S_o) < 0$ and $\log(P_c) - \lambda \log(S_c) = 0$, we would reverse our position, i.e. we would buy $1/P_o$ shares of stock P and sell λ/S_o shares of stock S, to make an approximate profit of $-(\log(P_o) - \lambda \log(S_o)) > 0$.

8.2 Integrated Series

Consider two stocks whose log-returns follow the CAPM/single-factor model

$$
\begin{aligned}
r_t^P &= \log(P_t/P_{t-1}) = \beta_P r_t^M + \epsilon_t \\
r_t^S &= \log(S_t/S_{t-1}) = \beta_S r_t^M + \eta_t
\end{aligned}
$$

where $\{r_t^M\}$ are the stationary returns of the "market" portfolio, and $\{\epsilon_t\}, \{\eta_t\}$ are stationary idiosyncratic errors which are independent of r_t^M. Show that for $\log(P_t), \log(S_t)$ to be cointegrated with stationary relation $\log(P_t) - \lambda \log(S_t)$, it is necessary (but not sufficient) to have $\lambda = \beta_P/\beta_S$

Notice that $\log(P_t)$ is integrated, since we have:

$$
\begin{aligned}
\log(P_t/P_{t-1}) &= \beta_P r_t^M + \epsilon_t \\
\Leftrightarrow \log(P_t) &= \log(P_{t-1}) + \overbrace{\beta_P r_t^M + \epsilon_t}^{\text{stationary}} \\
&= \log(P_0) + \sum_{i=1}^{t} \beta_P r_i^M + \sum_{i=1}^{t} \epsilon_i
\end{aligned}
$$

for some known starting value P_0, and similarly for $\log(S_t)$. So we have

$$
\begin{aligned}
\log(P_t) - \lambda \log(S_t) &= \left(\log(P_0) + \sum_{i=1}^{t} \beta_P r_i^M + \sum_{i=1}^{t} \epsilon_i \right) - \lambda \left(\log(S_0) + \sum_{i=1}^{t} \beta_S r_i^M + \sum_{i=1}^{t} \eta_i \right) \\
&= \underbrace{\log(P_0) - \lambda \log(S_0)}_{\text{const.}} + (\beta_P - \lambda \beta_S) \underbrace{\sum_{i=1}^{t} r_i^M}_{\text{random walk}} + \sum_{i=1}^{t} (\epsilon_i - \lambda \eta_i)
\end{aligned}
$$

In order for the difference $\log(P_t) - \lambda \log(S_t)$ to be stationary, we must get rid of the common random walk $\sum_{i=1}^{t} r_i^M$, which implies that

$$
\beta_P - \lambda \beta_S = 0 \Rightarrow \lambda = \beta_P/\beta_S
$$

Note that this is not sufficient for cointegration, as the last term $\sum_{i=1}^{t} (\epsilon_i - \lambda \eta_i)$ must also be stationary.

As a consequence of this model, if two cointegrated stocks have the same β (as would be reasonable to assume in the example of Exxon with Chevron), then $\lambda = 1$ and we are just using the difference.

8.3 Cointegration

Consider two stocks whose log-prices follow correlated random walks with

$$
\begin{aligned}
\log(P_t) &= \log(P_{t-1}) + \mu_P + \sigma_P W_t \\
\log(S_t) &= \log(S_{t-1}) + \mu_S + \sigma_S V_t
\end{aligned}
$$

where $\{W_t, V_t\}$ is a bivariate sequence of i.i.d. standard Normals with

$$
\begin{bmatrix} W_t \\ V_t \end{bmatrix} \sim^{iid} \mathrm{N}_2 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)
$$

Determine whether the log-prices are cointegrated or not.

Two integrated series $\{X_t, Y_t\}$ are *cointegrated* if there exists a linear combination of them that is stationary.

The individual log-prices are integrated (i.e., not stationary) since they are arithmetic Random Walks. This should be obvious from $\mathbb{V}[\log(P_t)] = \sigma_P^2 t$ is linear in t (for known P_0).

Consider any linear combination of the log-prices, $X_t = a\log(P_t) + b\log(S_t)$, for arbitrary weights a, b. We know that a linear combination of correlated arithmetic Random Walks also follows an arithmetic Random Walk with drift $a\mu_P + b\mu_S$ and volatility $\sqrt{a^2\sigma_P^2 + b^2\sigma_s^2 + 2ab\sigma_P\sigma_S\rho}$.

But since $|\rho| < 1$, the volatility of the linear combination will always be strictly positive, $\forall a, b$. To see this, note that

$$
\begin{aligned}
& (a\sigma_P - b\sigma_S)^2 \geq 0 \\
\Rightarrow\ & a^2\sigma_P^2 + b^2\sigma_s^2 - 2ab\sigma_P\sigma_S \geq 0 \\
\Rightarrow\ & a^2\sigma_P^2 + b^2\sigma_s^2 \geq 2|a||b|\sigma_P\sigma_S > 2|a||b|\sigma_P\sigma_S|\rho| \\
\Rightarrow\ & a^2\sigma_P^2 + b^2\sigma_s^2 + 2ab\sigma_P\sigma_S\rho > 0, \quad \forall a, b \in \mathbb{R}
\end{aligned}
$$

Thus, the variance of the linear combination X_t will never be constant, therefore X_t will never be stationary, therefore the log-prices cannot be cointegrated.

9. Monte Carlo Simulation

Consider the arithmetic Brownian motion $X_t = \mu t + \sigma W_t$, where $\{W_t\}$ is standard Brownian motion. Find the conditional distribution of

$$
\begin{bmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{bmatrix} \mid X_1 = x, \qquad \text{where } 0 < t_1 < \dots < t_n < 1
$$

also known as the multivariate Brownian bridge. Write the parameters of the distribution in terms of $\mu, \sigma, t_1, \dots, t_n$.

The (unconditional) joint distribution is

$$
\left[\begin{array}{c} X_{t_1} \\ \vdots \\ X_{t_n} \\ \hline X_1 \end{array}\right] \sim \mathrm{N}\left(\mu \left[\begin{array}{c} t_1 \\ \vdots \\ t_n \\ \hline 1 \end{array}\right], \quad \sigma^2 \left[\begin{array}{ccc|c} t_1 & \cdots & t_1 & t_1 \\ \vdots & \ddots & \vdots & \vdots \\ t_1 & \cdots & t_n & t_n \\ \hline t_1 & \cdots & t_n & 1 \end{array}\right] \right)
$$

Using the Normal conditional distribution formula, we get:

$$
\begin{aligned}
\begin{bmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{bmatrix} \mid (X_1 = x) &\sim \mathrm{N}\left(\mu \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} + \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} (x - \mu)\ , \ \sigma^2 \left(\begin{bmatrix} t_1 & \cdots & t_1 \\ \vdots & \ddots & \vdots \\ t_1 & \cdots & t_n \end{bmatrix} - \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix} \right) \right) \\
&\sim N\left(x \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}, \sigma^2 \begin{bmatrix} t_1(1-t_1) & t_1(1-t_2) & \cdots & t_1(1-t_n) \\ t_1(1-t_2) & t_2(1-t_2) & \cdots & t_2(1-t_n) \\ \vdots & \vdots & \ddots & \vdots \\ t_1(1-t_n) & t_2(1-t_n) & \cdots & t_n(1-t_n) \end{bmatrix} \right)
\end{aligned}
$$

10. Pricing Exotic Derivatives

Consider the Rayleigh distribution for the (conditional) maximum of an Arithmetic Brownian Bridge $(M_T \mid (X_T = b))$ with CDF:

$$\mathbb{P}\left(M_T \leq m \mid X_T = b\right) = 1 - \exp\left\{-2\frac{m(m-b)}{\sigma^2 T}\right\}, \quad \forall m \geq \max\{0, b\}.$$

Use the inverse CDF method to verify that the transformed Uniform $(0,1)$ random variate below follows this Rayleigh distribution.

$$M_T \mid (X_T = b) = \frac{b + \sqrt{b^2 - 2\sigma^2 T \log(U)}}{2}$$

For RV X with CDF $F_X(x)$ and $U \sim \text{Uniform}(0,1)$, we have $F_X(X) \sim U$ and $F_X^{-1}(U) \sim X$.

We have:

$$\begin{aligned}
F_{M_T|X_T=b}(m) &= \mathbb{P}\left(M_T \leq m \mid X_T = b\right) = 1 - \exp\left\{-2\frac{m(m-b)}{\sigma^2 T}\right\} \\
&\Rightarrow 1 - \exp\left\{-2\frac{M_T\left(M_T - b\right)}{\sigma^2 T}\right\} \sim U \\
&\Rightarrow -2\frac{M_T\left(M_T - b\right)}{\sigma^2 T} \sim \log(\overbrace{1-U}^{U'}), \quad (\text{ where } = U' \sim \text{ Uniform } (0,1)) \\
&\Rightarrow M_T^2 - M_T b + \frac{\sigma^2 T \log\left(U'\right)}{2} = 0, \quad (\text{ solve quadratice}) \\
&\Rightarrow M_T \mid X_T = b = \frac{b \pm \sqrt{b^2 - 4\sigma^2 T \log\left(U'\right)/2}}{2}, \quad (\text{ keep + ve root}) \\
&= \frac{b \pm \sqrt{b^2 - 2\sigma^2 T \log\left(U'\right)}}{2}, \quad (\geq \max\{0, b\}, \quad \forall b \in \mathbb{R})
\end{aligned}$$

11. Simulation: Variance Reduction Techniques

Antithetic variables are closely related to stratification with only 2 strata, i.e. with positive and negative values for Z. Their only difference is that for antithetic variables you have the same absolute values for positive and negative variates, whereas with stratification you can have different absolute values for positive and negative variates.

(a) Antithetic estimators with odd functions have zero variance

For what type of function $f(\cdot)$ does the use of antithetic variables completely eliminate the variability of the Monte-Carlo estimator, i.e. $\text{Var}\left[\bar{Y}_{AV}\right] = \frac{1}{n}\text{Var}\left[\frac{Y_i + \tilde{Y}_i}{2}\right] = 0$, where $Y_i = f\left(Z_i\right), \tilde{Y}_i = f\left(-Z_i\right)$ and $Z_i \sim^{iid} N(0,1)$.

The variability of the estimator will be 0 when the function $f(\cdot)$ is odd, i.e., when $f(-x) = -f(x)$. In this case the expectation we try to estimate is 0 , i.e., $E[f(Z)] = 0$, and the antithetic variable estimator does a perfect job for any sample size n, i.e.,

$$
\begin{aligned}
\bar{Y}_{AV} &= \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i + \tilde{Y}_i}{2} \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{f(Z_i) + f(-Z_i)}{2} \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{f(Z_i) - f(Z_i)}{2} = 0, \quad \forall n
\end{aligned}
$$

Best case: odd functions; worse case: even functions

(b) Condition for variance of naive mean to be zero

For the type of function you identified in part (a), show that the variance of the simple Monte Carlo estimator is not necessarily 0, i.e. $\mathrm{Var}[\bar{Y}] = \frac{1}{2n}\mathrm{Var}[Y_i] \neq 0$.

The simple MC estimator $\bar{Y} = \frac{1}{2n}\sum_{i=1}^{2n} Y_i$ has variance $\mathrm{Var}[\bar{Y}] = \frac{1}{2n}\mathrm{Var}[Y_i]$ where

$$
\mathrm{Var}[Y_i] = \mathrm{Var}[f(Z)] = E\left[f^2(Z)\right] - \overbrace{(E[f(Z)])^2}^{=0} = E\left[f^2(Z)\right] > 0
$$

unless $f(x) = 0, \forall x$

(c) Variance of stratified estimator

For the type of function you identified in part (a), show that the variance of the stratified estimator (with 2 equi-probable strata of size n each) is also not 0 and that it is smaller than that from part (b), i.e. $0 \neq \mathrm{Var}\left[\bar{Y}_{Str}\right] < \mathrm{Var}[\bar{Y}]$.

We have shown in class that the variance of the stratified estimator with $m = 2$ equiprobable strata, is given by $\mathrm{Var}\left[\bar{Y}_{Str}\right] = \frac{1}{2n}\left\{E\left[f^2(Z)\right] - \sum_{j=1}^{2}\frac{\mu_j^2}{2}\right\}$, where $\mu_j = E\left[f(Z) \mid Z \in A_j\right]$, and $\left\{A_j\right\}_{j=1,2}$ are the two strata. For our specific case, we have $A_1 = \{Z \leqslant 0\}$ and $A_2 = \{Z > 0\}$.

Since the function is odd and Z is Normal, we have

$$
\mu_1 = -\mu_2 = \mu \Rightarrow \mathrm{Var}\left[\bar{Y}_{Str}\right] = \frac{1}{2n}\left\{E\left[f^2(Z)\right] - \mu^2\right\}
$$

Comparing this to the result from part (b), it is obvious that the stratified estimator has smaller variance than the simple MC estimator, and its performance increases with μ.

Finals

2016 Q1. VaR

Assume the distribution of a portfolio's loss (L) has cdf $F_L(\ell) = \mathbb{P}(L \leq \ell) = \exp(-\exp(-\ell)), \forall \ell \in \mathbb{R}$.

(a) Generate random values

Assume you can generate Uniform $(0,1)$ random values U. Describe a method for generating random values from the above distribution.

$U \sim F_L(l) \implies U = e^{-e^{-L}} \implies L = F^{-1}(U) = -\log(-\log U)$

(b) Find VaR

Find a closed-form expression for the value at risk (VaR) of this portfolio at level $\alpha \in (0,1)$.

$$\begin{aligned} P(L > VaR_\alpha) &= \alpha \\ P(L \leq VaR_\alpha) &= 1-\alpha \\ VaR_\alpha &= F_L^{-1}(1-\alpha) \\ &= -\log(-\log(1-\alpha)) \end{aligned}$$

(c) VaR of maximum loss - iid losses

Consider n independent and identically distributed losses ($L_i, i = 1, \ldots, n$) with the above cdf $\left(F_{L_i}\right)$. Find a closed-form expression for the VaR of their maximum loss $M = \max_{1 \leq n \leq n} \{L_i\}$

$$F_M(m) = P(M \leq m) = P(\max_{1 \leq i \leq n} \{L_i\} \leq m) = \prod_{i=1}^{n} P(L_i \leq m) = (e^{-e^{-m}})^n = e^{-ne^{-m}}$$

So we have

$$\begin{aligned} P(M \leq VaR_\alpha) &= 1-\alpha \\ e^{-ne^{-VaR}} &= 1-\alpha \\ e^{-VaR} &= -\frac{1}{n}\log(1-\alpha) \\ \text{VaR} &= -\log\left(-\frac{1}{n}\log(1-\alpha)\right) \end{aligned}$$

(d) VaR of maximum loss - dependent losses

Repeat part (c) assuming the loss distributions are dependent according to the Gumbel copula with parameter $\theta \geq 1$. That is, the losses of the n portfolios have the same marginal cdf $\left(F_{L_i}\right)$, and are related through the copula function:

$$C\left(u_1, \ldots, u_n\right) = \exp\left[-\left(\sum_{i=1}^{n}\left(-\log\left(u_i\right)\right)^{\theta}\right)^{1/\theta}\right]$$

$$\begin{aligned}
P(M \leq m) &= C(F_L(m), \ldots, F_L(m)) \\
&= \exp\left[-\left(\sum_{i=1}^{n}(-\log(e^{-e^{-m}}))^{\theta}\right)^{1/\theta}\right] \\
&= \exp\left[-\left(\sum_{i=1}^{n} e^{-m\theta}\right)^{1/\theta}\right] \\
&= \exp[-(ne^{-m\theta})^{1/\theta}] \\
&= \exp[-n^{1/\theta}e^{-m}] \\
&= e^{-\sqrt[\theta]{n}e^{-m}} \\
\implies P(M \leq \text{VaR}) &= e^{-\sqrt[\theta]{n}e^{-VaR}} = 1-\alpha \\
\text{VaR} &= -\log\left(-\frac{1}{-\sqrt[\theta]{n}}\log(1-\alpha)\right)
\end{aligned}$$

2016 Q2. Control Variates

Assume you want to estimate the mean $\mathbb{E}[Y] = \mu_Y$ of $Y = f(Z)$ using simulation, where Z is some random variable. The simple Monte Carlo estimator is $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, where $Y_i = f\left(Z_i\right)$ for i.i.d. random values $Z_i, i = 1, \ldots, n$. Now consider two other variables $X_1 = g(Z), X_2 = h(Z)$ whose means $\mathbb{E}\left[X_1\right] = \mu_1, \mathbb{E}\left[X_2\right] = \mu_2$ you know. You can use both X_1, X_2 as control variates as follows:

$$\bar{Y}_{cv} = \bar{Y} - b_1\left(\bar{X}_1 - \mu_1\right) - b_2\left(\bar{X}_2 - \mu_2\right)$$

where b_1, b_2 are constants and $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} g\left(Z_i\right), \bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} h\left(Z_i\right)$. Your answers to the questions below should be in terms of the means and the 2 nd order moments of the variables:

$$\begin{aligned}
&\mathbb{V}[Y] = \sigma_Y^2 \quad \mathbb{V}\left[X_1\right] = \sigma_1^2 \quad \mathbb{V}\left[X_2\right] = \sigma_2^2 \\
&\operatorname{Cov}\left[Y, X_1\right] = \sigma_{Y,1} \quad \operatorname{Cov}\left[Y, X_2\right] = \sigma_{Y,2} \quad \operatorname{Cov}\left[X_1, X_2\right] = \sigma_{1,2}
\end{aligned}$$

(a) Show that $\bar{Y}_{cv}$ is an unbiased estimator of μ_Y.

$$E(\bar{Y}_{cv}) = E(\bar{Y}) - b_1 E(\underbrace{\bar{X}_1 - \mu_1}_{0}) - b_2 E(\underbrace{\bar{X}_2 - \mu_2}_{0}) = \mu_Y$$

(b) Express the variance of $\bar{Y}_{cv}$ in terms of the moments of the variables.

$$
\begin{aligned}
V\left[\bar{Y}_{cv}\right] &= \mathbb{E}\left[\left[\left(\bar{Y}-\mu_y\right)-b_1\left(\bar{X}_1-\mu_1\right)-b_2\left(\bar{X}_2-\mu_2\right)\right]^2\right] \\
&= [1 - b_1 - b_2]\begin{bmatrix} \sigma_y^2 & \sigma_{y_1} & \sigma_{y_2} \\ \sigma_{y_1} & \sigma_1^2 & \sigma_{12} \\ \sigma_{y_2} & \sigma_{21} & \sigma_2^2 \end{bmatrix}\begin{bmatrix} 1 \\ -b_1 \\ -b_2 \end{bmatrix} \\
&= [1 - b_1 - b_2]\begin{bmatrix} \sigma_y^2 - b_1\sigma_{y_1} - b_2\sigma_{y_2} \\ \sigma_{y_1} - b_1\sigma_1^2 - b_2\sigma_{12} \\ \sigma_{y2} - b_1\sigma_{21} - b_2\sigma_2^2 \end{bmatrix} \\
&= \sigma_y^2 - b_1\sigma_{Y_1} - b_2\sigma_{y_2} - b_1\sigma_{y_1} + b_1^2\sigma_1^2 + b_1b_2\sigma_{21} - b_2\sigma_{y2} + b_1b_2\sigma_{12} + b_2^2\sigma_2^2 \\
&= \sigma_y^2 + b_1^2\sigma_1^2 + b_2^2\sigma_2^2 - 2b_1\sigma_{y1} - 2b_2\sigma_{y2} + 2b_1b_2\sigma_{12}
\end{aligned}
$$

(c) Find optimal values of b_1, b_2 that minimize the variance in part (b).

$$
\begin{aligned}
&\frac{\partial}{\partial b_1} = 2b_1\sigma_1^2 - 2\sigma_{y_1} + 2b_2\sigma_{12} = 0 \\
&\frac{\partial}{\partial b_2} = 2b_2\sigma_2^2 - 2\sigma_{y_2} + 2b_1\sigma_{12} = 0 \\
&\Rightarrow \begin{cases} b_1 = \left(\sigma_{y_1} - b_2\sigma_{12}\right)/\sigma_1^2 \\ b_2\sigma_2^2 + \left(\sigma_{y_1} - b_2\sigma_{12}\right)/\sigma_1^2 - \sigma_{12} = \sigma_{y_2} \end{cases}
\end{aligned}
$$

Solving this, we get

$$
b_2 = \frac{\sigma_1^2\sigma_{y_2} - \sigma_{y_1}\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \Rightarrow b_1 = \frac{\sigma_2^2\sigma_{y_1} - \sigma_{y_2}\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}
$$

(d) Find variance of $\bar{Y}_{cv}$ for the optimal choice of b_1, b_2.

$$
\begin{aligned}
\sigma_y - \Sigma_{xy}^T\Sigma_{xx}^T\Sigma_{XY} &= \sigma_y^2 - \begin{bmatrix} \sigma_{y_1} & \sigma_{y_2} \end{bmatrix}\frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}\begin{bmatrix} \sigma_{y_1} \\ \sigma_{y_2} \end{bmatrix} \\
&= \sigma_y^2 - \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\left[\sigma_{y_1}\sigma_{y_2}\right]\cdot\begin{bmatrix} \sigma_2^2\sigma_{y_1} - \sigma_{12}\sigma_{y_2} \\ -\sigma_{12}\sigma_{y_1} + \sigma_1^2\sigma_{y_2} \end{bmatrix} \\
&= \sigma_y^2 - \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}[\sigma_2^2\sigma_{y_1}^2 - \sigma_{12}\sigma_{y_2}\sigma_{y_1} - \sigma_{12}\sigma_{y_1}\sigma_{y_2} + \sigma_1^2\sigma_{y_2}^2]
\end{aligned}
$$

2016 Q3. Brownian Motion

Consider a standard Brownian motion $\{W_t\}$, and its running maximum $M_t = \max_{0\leq s\leq t}\{W_s\}$

(a) Multivariate Normal Properties

Find the conditional distribution of $(W_t - W_s) \mid W_1 = 0$, where $0 \le s, t \le 1$.

$$\begin{bmatrix} w_s \\ w_t \\ w_1 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{bmatrix} \right) \Rightarrow \begin{bmatrix} w_s \\ w_t \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} s & s \\ s & t \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix} [s \quad t]}_{\begin{bmatrix} s \cdot (1-s) & s \cdot (1-t) \\ s(1-t) & t \cdot (1-t) \end{bmatrix}} \right)$$

(b) Max of BM - conditional dist

Find the conditional cdf of the maximum given that the Brownian motion is negative by time 1. That is, find $\mathbb{P}(M_1 \le m \mid W_1 \le 0)$ for $m \ge 0$, expressed in terms of the standard Normal cdf $\Phi(\cdot)$.

By reflection principle, we have

$$\begin{aligned} P(M_1 > m | W_1 \le 0) &= \frac{P(W_1 \ge 2m)}{1/2} = 2\Phi(-2m) \\ P(M_1 \le m | W_1 \le 0) &= 1 - 2\Phi(-2m) \\ &= 1 - 2(1 - \Phi(2m)) \\ &= 2\Phi(2m) - 1 \end{aligned}$$

(c) Max of BM is a folded normal

Find the expected value of M_1.

$$\begin{aligned} M_1 \sim |W_1| \implies E(M_1) &= E(|W_1|) \text{ where } W_1 \sim N(0,1) \\ &= \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \\ &= 2 \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (-e^{-z^2/2})' \, dz \\ &= \sqrt{\frac{2}{\pi}} \left[-e^{-z^2/2} \right]_0^{\infty} \\ &= \sqrt{\frac{2}{\pi}} \end{aligned}$$

2016 Q4. Two Asset Portfolio

Consider two assets with net returns over time $0 \to T$ denoted by $\mathbf{R} = \begin{bmatrix} R_1 & R_2 \end{bmatrix}^\top$, and following a bivariate distribution with 1 st and 2 nd moments given by:

$$\mathbb{E}[\mathbf{R}] = \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathbb{V}[\mathbf{R}] = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{bmatrix}$$

Assume you start with initial wealth W_0 which you allocate to the two assets according to fractions $\mathbf{f} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^\top$, respectively ($f_1, f_2$ do not have to be in $[0,1]$ and $f_1 + f_2$ does not have to equal 1). Furthermore, let the risk-free rate be r_f.

(a) Final wealth

Write an expression for the final wealth W_T at time T.

$$\begin{aligned} W_T &= W_0 \cdot \left[f_1 \cdot (1+R_1) + f_2 \cdot (1+R_2) + (1-f_1-f_2)\left(1+r_f\right)\right] \\ &= W_0 \cdot \left[1 + r_f + f_1\left(R_1 - r_f\right) + f_2\left(R_2 - r_f\right)\right] \\ &= W_0 \cdot \left[\left(1+r_f\right) + \mathbf{f}^\top \cdot \left(\mathbf{R} - r_f \mathbf{1}\right)\right] \end{aligned}$$

(b) Taylor Expansion

Use a 2nd order Taylor approximation to show that :

$$\log\left(\frac{W_T}{W_0}\right) \approx \log\left(1+r_f\right) + \frac{1}{1+r_f}\mathbf{f}^\top\left(\mathbf{R} - r_f\mathbf{1}\right) - \frac{1}{2}\frac{1}{\left(1+r_f\right)^2}\mathbf{f}^\top\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\mathbf{R} - r_f\mathbf{1}\right)^\top \mathbf{f}$$

$$\begin{aligned} \log\left(\frac{W_T}{W_0}\right) &= \log(\underbrace{1+r_f}_{x_0} + \underbrace{\mathbf{f}^\top(\mathbf{R} - r_f\mathbf{1})}_{\delta}) \\ &\approx \log(x_0) + \log'(x_0)\delta + \frac{1}{2}\log''(x_0)\delta\delta^\top \end{aligned}$$

(c)

From part (b) derive an expression for the expected (approximate) log-return, in terms of the moments μ and Σ

$$\begin{aligned} E\left(\log\left(\frac{W_T}{W_0}\right)\right) &\approx E\left(\log\left(1+r_f\right) + \frac{1}{1+r_f}\mathbf{f}^\top\left(\mathbf{R} - r_f\mathbf{1}\right) - \frac{1}{2}\frac{1}{\left(1+r_f\right)^2}\mathbf{f}^\top\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\mathbf{R} - r_f\mathbf{1}\right)^\top \mathbf{f}\right) \\ &= \log(1+r_f) + \frac{1}{1+r_f}\mathbf{f}^\top(\mu - r_f\mathbf{1}) - \frac{1}{2}\frac{1}{(1+r_f)^2}\mathbf{f}^\top[\Sigma + (\mu - r_f\mathbf{1})(\mu - r_f\mathbf{1})^\top]f \end{aligned}$$

(d) Kelly Criterion

Derive an expression for the fraction f that maximizes the expected log-return in part (c), i.e. find the Kelly-optimal fraction.

$$\frac{d}{df} = 0 \implies 0 = \frac{1}{1+r_f}(\mu - r_f\mathbf{1}) - \frac{1}{(1+r_f)^2}(\Sigma + (\mu - r_f)^2)f$$
$$f = (1+r_f)(\Sigma + (\mu - r_f)^2)^{-1}(\mu - r_f\mathbf{1})$$

2016 Q5. Integrated Series

Consider two assets with prices $\{P_1(t), P_2(t)\}_{t\geq 0}$ whose log-returns $R_i(t) = \log(P_i(t)/P_i(t-1)), i = 1, 2$ follow a 1-factor model

$$R_i(t) = \beta F(t) + \epsilon_i(t), i = 1, 2$$

where $F(t) \sim^{iid} \mathrm{WN}(0, 1)$ and $\epsilon_i(t) \sim^{\text{iid}} \mathrm{WN}\left(0, \sigma_i^2\right), i = 1, 2$ are mutually independent.

(a) Integrated process

Show that the log-price $\log\left(P_i(t)\right)$ of each asset $i = 1, 2$ is an integrated process.

A process is integrated if $\nabla X(t)$ is stationary.

$$\begin{aligned}
\nabla \log(P_i(t)) &= \log(P_i(t)) - \log(P_i(t-1)) \\
&= \log\left(\frac{P_i(t)}{P_i(t-1)}\right) = R_i(t) \\
E(R_i(t)) &= 0 \\
Var(R_i(t)) &= \beta^2 + \sigma_i^2 \\
Cov(R_i(t), R_i(s)) &= 0,\ \forall t \neq s
\end{aligned}$$

Thus it is stationary.

(b) Cointegration

Determine whether the two log-prices are cointegrated.

2 series are cointegrated if $\exists$ a **stationary** linear combo.

$$\log(P_i(t)) = R_i(t) + \log(P_i(t-1)) = \dots = \sum_{h=0}^{t} R_i(t-h) = \sum_{h=0}^{t} \beta F(t-h) + \epsilon_i(t-h)$$

$$\begin{aligned}\log(P_1(t)) + \alpha \log(P_2(t)) &= \sum_{h=0}^{t} \{\beta F(t-h) + \epsilon_1(t-h) - \alpha[\beta F(t-h) + \epsilon_2(t-h)]\} \\ &= (1-\alpha)\beta \sum_{h=0}^{t} F(t-h) + \sum_{h=0}^{t} \epsilon_1(t-h) - \alpha \sum_{h=0}^{t} \epsilon_2(t-h)\end{aligned}$$

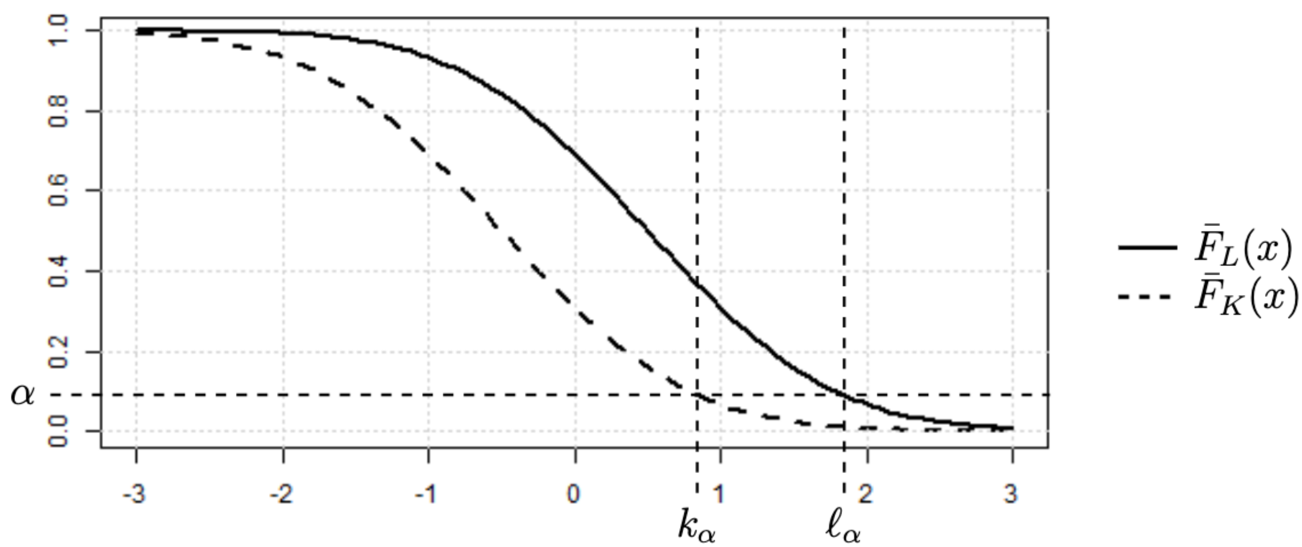
The linear combination is $I(1) \implies$ not stationary $\implies$ not co-integrated.

2017 Q1. VaR

A random variable L is said to 1 st-order stochastically dominate K, denoted $L \succeq K$, if

$$\mathbb{P}(L > x) \geq \mathbb{P}(K > x) \Leftrightarrow \bar{F}_L(x) \geq \bar{F}_K(x), \quad \forall x \in \mathbb{R}$$

where $\bar{F}_L(x) = P(L > x)$ is the complementary CDF. In what follows, assume $L \succeq K$ are continuous RVs describing the losses of two different investments. The plot below illustrates the relationship in terms of their complementary CDFs.



(a) Prove that $\text{VaR}_\alpha(L) \geq \text{VaR}_\alpha(K), \forall \alpha \in (0,1)$**.**

The VaR_α is given by the α-level quantile of the complementary CDF, i.e. the value ℓ_α such that:

$$\mathbb{P}(L > \ell_\alpha) = \alpha \Leftrightarrow \bar{F}_L(\ell_\alpha) = \alpha \Leftrightarrow \ell_\alpha = \bar{F}^{-1}(\alpha)$$

Thus, VaR is equal to the inverse complementary CDF $\text{VaR}_\alpha = \bar{F}^{-1}(\alpha)$ evaluated at α.

$$
\begin{aligned}
\bar{F}_L(x) &\geq \bar{F}_K(x), \forall x \\
\bar{F}_L^{-1}(\alpha) &\geq \bar{F}_K^{-1}(\alpha) \\
\mathrm{VaR}_\alpha(L) &\geq \mathrm{VaR}_\alpha(K), \forall \alpha \in (0,1)
\end{aligned}
$$

(This can be also be seen by the plot above: the VaR is given by the intersection of the horizontal line at α with the complementary CDF. If one complementary CDF dominates another, then its inverse also dominates the inverse of the other.)

Alternatively, you can use the stochastic dominance relationship to get $\mathbb{P}(K > \mathrm{VaR}_\alpha(L)) \leq \mathbb{P}(L > \mathrm{VaR}_\alpha(L)) = \alpha$, which implies that $\mathrm{VaR}_\alpha(K) \leq \mathrm{VaR}_\alpha(L)$ in order to have $\mathbb{P}\left(K > \mathrm{VaR}_\alpha(K)\right) = \alpha$.

(b) Prove that $\mathrm{CVaR}_\alpha(L) \geq \mathrm{CVaR}_\alpha(K), \forall \alpha \in (0,1)$.

We have $\mathrm{CVaR}_\alpha(L) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(L)du$, and similarly for K. Since

$$
\begin{aligned}
\mathrm{VaR}_\alpha(L) &\geq \mathrm{VaR}_\alpha(K) \\
\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(L)du &\geq \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(K)du \\
\mathrm{CVaR}_\alpha(L) &\geq \mathrm{CVaR}_\alpha(K)
\end{aligned}
$$

(c) Prove that if $g(\cdot)$ is a strictly increasing function, then $g(L) \succeq g(K)$.

For strictly increasing $g(\cdot)$, its inverse $g^{-1}(\cdot)$ is also strictly increasing.

$$
\begin{aligned}
\mathbb{P}\left(L > g^{-1}(x)\right) &\geq \mathbb{P}\left(K > g^{-1}(x)\right) \\
\mathbb{P}(g(L) > x) &\geq \mathbb{P}(g(K) > x), \quad \forall x \in \mathbb{R}
\end{aligned}
$$

(d) Prove that $\mathbb{E}[L] \geq \mathbb{E}[K]$ for non-negative L, K.

(Hint: for RV $X \geq 0 \Rightarrow \mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)dx$.)

For non-negative L we have $\mathbb{E}[L] = \int_0^\infty \mathbb{P}(L > x)dx$, and similarly for K.

$$
\begin{aligned}
\mathbb{P}(L \geq x) &\geq \mathbb{P}(K \geq x) \\
\int_0^\infty \mathbb{P}(L \geq x)dx &\geq \int_0^\infty \mathbb{P}(K \geq x)dx \\
\mathbb{E}[L] &\geq \mathbb{E}[K]
\end{aligned}
$$

(e) Prove that $\mathrm{EVaR}_\alpha(L) \geq \mathrm{EVaR}_\alpha(K), \forall \alpha \in (0,1)$

(Hint: Use the last two results.)

$\mathrm{EVaR}_\alpha(X) = \inf_{z>0}\left\{z^{-1}\ln\left(M_X(z)/\alpha\right)\right\}$, where $M_X(z) = \mathbb{E}\left[e^{zX}\right]$ is the moment generating function. Note that e^{zX} is a non-negative and strictly increasing (for $z > 0$) function of X.

Applying the last two results, we get that

$$
\begin{aligned}
\mathbb{E}\left[e^{zL}\right] &\geq \mathbb{E}\left[e^{zK}\right] \\
M_L(z) &\geq M_K(z) \\
z^{-1}\ln\left(M_L(z)/\alpha\right) &\geq z^{-1}\ln\left(M_K(z)/\alpha\right) \Rightarrow \\
\inf_{z>0}\left\{z^{-1}\ln\left(M_L(z)/\alpha\right)\right\} &\geq \inf_{z>0}\left\{z^{-1}\ln\left(M_K(z)/\alpha\right)\right\} \Rightarrow \\
\mathrm{EVaR}_\alpha(L) &\geq \mathrm{EVaR}_\alpha(K)
\end{aligned}
$$

2016 Q2. Factor Model

Consider $\#n$ asset returns $R = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}^\top$ with mean $\mathbb{E}[R] = \mu\mathbf{1}$.

(a) Given variance, find factor Model

Assume the return covariance matrix is $\mathbb{V}[R] = \Sigma_R = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$. Find a factor model representation $\Sigma_R = \beta^\top\beta + \Sigma_\epsilon$ using $\#(n-1)$ factors, and give the values of β and Σ_ϵ.

Each asset is correlated with its neighbours, so it must share one factor with each of its neighbours. The coefficient matrix is thus

$$
\beta_{(n-1)\times n} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
\Sigma_R = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{bmatrix} &= \beta^\top\beta + \Sigma_\epsilon \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}
\end{aligned}
$$

(b) Minimum variance portfolio

Now consider a different covariance matrix of the form:

$$\Sigma_R = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \text{ with inverse } \Sigma_R^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}.$$

Calculate the variance of the minimum variance portfolio. (Hint: $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$)

The minimum variance portfolio has variance $\left(\mathbf{1}^\top \Sigma_R^{-1} \mathbf{1}\right)^{-1}$, which is the inverse of the sums of the elements of Σ^{-1}. From the form of Σ^{-1}, the sum of the elements is

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & 1 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 1 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \\ &= n \times n + (n-1) \times (n-1) + \cdots + 2 \times 2 + 1 \times 1 \\ &= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Thus, the minimum portfolio variance is $\frac{6}{n(n+1)(2n+1)}$. Note that as $n \to \infty$, the variance goes to zero. This is not surprising since the assets are negatively correlated, so investing in more of them allows asymptotically perfect diversification of risk.

2017 Q3. Betting Strategies

Assume an asset whose returns follow an AR(1) model $R_t = \phi R_{t-1} + \varepsilon_t$, with $\phi > 0$, $\varepsilon_t \sim^{\text{iid}} \mathrm{N}\left(0, \sigma^2\right)$, and where the risk-free rate is 0 . You can think of this as a basic momentum model: whenever the last return was positive, the next one will also tend to be positive, and vice-versa.

Consider a strategy where at time t you invest a fraction $f_t = \alpha R_t$ of your wealth in the asset, where $\alpha > 0$. So, if your wealth is V_t at time t, it will become $V_{t+1} = V_t\left(1 + f_t R_{t+1}\right)$ at time $t+1$.

(a) Taylor Expansion

Use a 2nd-order Taylor expansion of the log function to approximate the expected growth rate as:

$$\mathbb{E}\left[\log\left(V_{t+1}/V_t\right)\right] \approx \alpha\mathbb{E}\left[R_t R_{t+1}\right] - \frac{1}{2}\alpha^2\mathbb{E}\left[R_t^2 R_{t+1}^2\right]$$

We have

$$\log\left(V_{t+1}/V_t\right) = \log\left(1 + f_t R_{t+1}\right) = \log\left(1 + \alpha R_t R_{t+1}\right)$$

Take Taylor expansion around 1:

$$\log\left(V_{t+1}/V_t\right) \approx \log(1) + \underbrace{\log'(1)}_{=1}\left(\alpha R_t R_{t+1}\right) + \frac{1}{2}\underbrace{\log''(1)}_{=-1}\left(\alpha R_t R_{t+1}\right)^2$$

$$\mathbb{E}\left[\log\left(V_{t+1}/V_t\right)\right] \approx \alpha\mathbb{E}\left[R_t R_{t+1}\right] - \frac{1}{2}\alpha^2\mathbb{E}\left[R_t^2 R_{t+1}^2\right]$$

(b) Cholesky Decomposition

The joint stationary distribution of a Gaussian AR(1) model is

$$\begin{bmatrix} R_t \\ R_{t+1} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}\right).$$

Express (R_t, R_{t+1}) as a linear transformation of independent standard Normals $Z, W \overset{iid}{\sim} N(0,1)$

$$\begin{bmatrix} R_t \\ R_{t+1} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a_{11}Z + a_{12}W \\ a_{21}Z + a_{22}W \end{bmatrix},$$

where the linear coefficients are given in terms of ϕ and σ. (Hint: use Cholesky decomposition)

Let $\sigma_R^2 = \frac{\sigma^2}{1-\phi^2}$. From Cholesky decomposition, we have

$$\begin{bmatrix} R_t \\ R_{t+1} \end{bmatrix} = \underbrace{\sigma_R\begin{bmatrix} 1 & 0 \\ \phi & \sqrt{1-\phi^2} \end{bmatrix}}_{\text{square root of}\frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}}\begin{bmatrix} Z \\ W \end{bmatrix} = \sigma_R\begin{bmatrix} Z \\ \phi Z + \sqrt{1-\phi^2}W \end{bmatrix}$$

(c) Compute moments

Use the previous result to calculate the moments in part 1. (Hint: if $Z \sim \mathrm{N}(0,1) \Rightarrow \mathbb{E}\left[Z^4\right] = 3$)

$$
\begin{aligned}
\mathbb{E}\left[R_t R_{t+1}\right] &= \sigma_R^2 \mathbb{E}\left[Z\left(\phi Z + \sqrt{1-\phi^2} W\right)\right] \\
&= \sigma_R^2 (\phi \underbrace{\mathbb{E}\left[Z^2\right]}_{V(Z)} + \sqrt{1-\phi^2} \underbrace{\mathbb{E}[ZW]}_{0 \cdot 0}) \\
&= \sigma_R^2 \phi \\
\mathbb{E}\left[R_t^2 R_{t+1}^2\right] &= \sigma_R^4 \mathbb{E}\left[Z^2\left(\phi Z + \sqrt{1-\phi^2} W\right)^2\right] \\
&= \sigma_R^4 \left\{\phi^2 \mathbb{E}\left[Z^4\right] + 2\phi\sqrt{1-\phi^2} \underbrace{\mathbb{E}\left[Z^3 W\right]}_{0} + \left(1-\phi^2\right) \mathbb{E}\left[Z^2 W^2\right]\right\} \\
&= \sigma_R^4 \left(3\phi^2 + 1 - \phi^2\right) = \sigma_R^4 \left(1 + 2\phi^2\right)
\end{aligned}
$$

(d) Optimal bet fraction

Find the optimal value of α that maximizes the expected growth rate.

The (approximate) expected growth rate is a quadratic in α. In order to find the maximum, just differentiate and set to 0:

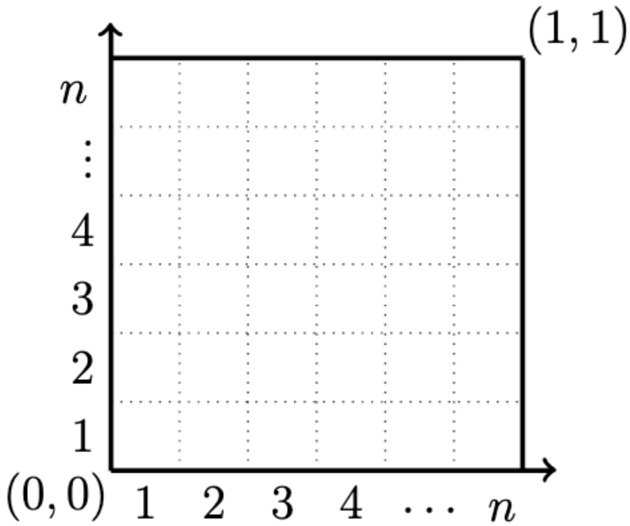
$$
\begin{aligned}
\frac{d}{d\alpha}\left\{\alpha \mathbb{E}\left[R_t R_{t+1}\right] - \frac{1}{2}\alpha^2 \mathbb{E}\left[R_t^2 R_{t+1}^2\right]\right\} &= 0 \\
\mathbb{E}\left[R_t R_{t+1}\right] - \alpha \mathbb{E}\left[R_t^2 R_{t+1}^2\right] &= 0 \\
\alpha = \frac{\mathbb{E}\left[R_t R_{t+1}\right]}{\mathbb{E}\left[R_t^2 R_{t+1}^2\right]} &= \frac{\phi}{\sigma_R^2\left(1 + 2\phi^2\right)}
\end{aligned}
$$

2017 Q4. Stratification

Imagine you want to simulate pairs of independent Exponential(1) random values using stratification over n^2 equiprobable strata. You can achieve this in two steps:

1. draw stratified Uniform $(0, 1)$ pairs
2. transform them to Exponential pairs

The plot below illustrates the $n \times n$ equiprobable Uniform strata on the unit square.



Assume you can generate independent Uniform $(0, 1)$ random numbers. Describe in pseudo-code an algorithm that draws an Exponential pair from every stratum. Explain clearly how you generate the Uniform stratified pair from the independent Uniform $(0, 1)$ numbers, and how you transform them to exponential.

(Hint: the Exponential(1) CDF is $F(x) = 1 - e^{-x}, \forall x > 0$)

Note that within each stratum (i.e. each sub-square in the graph), the points will also be uniformly distributed. Thus, we can use a for-loop to go over all squares. Then we can apply the inverse CDF transform to the stratified Uniform values, to get the Exponential values. The inverse CDF transform is given by:

$$F(X) \sim U \Rightarrow X \sim F^{-1}(U) = -\ln(1 - U)$$

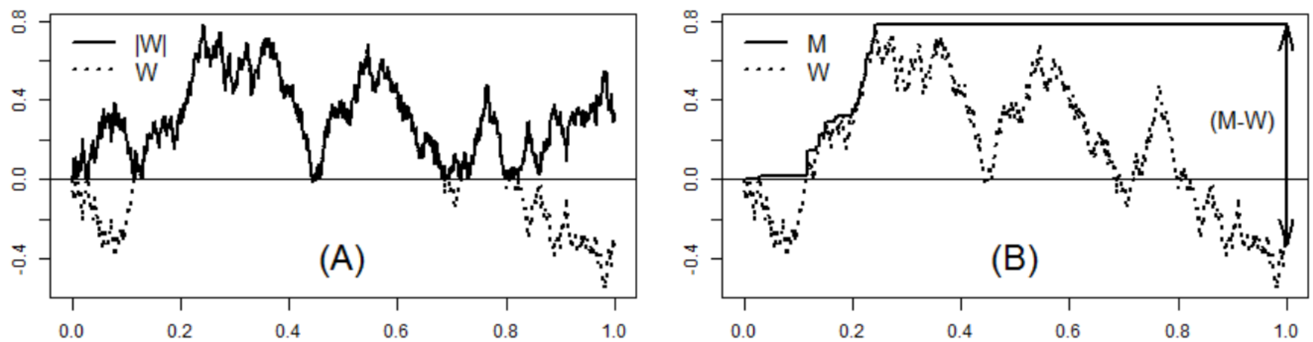
In what follows, let $i, j \in (1, \ldots, n)$ denote the stratum, U_i, U_j denote independent uniform RVs, U_i^s, U_j^s denote stratified Uniform RVs, and X_i^s, X_j^s denote stratified Exponential RVs.

$\delta = 1/n;$
for $i = 1, \ldots, n$ **do**
  for $j = 1, \ldots, n$ **do**
    generate $U_i,\ U_j \sim^{iid}$ Uniform$(0, 1)$;
    set $U_i^s = (i - 1 + U_i)\delta,\ U_j^s = (j - 1 + U_j)\delta;$
    set $X_i^s = -\ln(1 - U_i^s),\ X_j^s = -\ln(1 - U_j^s);$
  end
end

2017 Q5. Brownian Motion

The process $|W_t|$, where W_t is standard Brownian motion (BM), is called a reflected BM. You can think of it as a regular BM which is reflected at the origin, as shown on plot (A).

Now consider the distance $(M_t - W_t)$ between the BM's running maximum $M_t = \max_{0 \le s \le t} \{W_s\}$ and its current value W_t, as shown on plot (B).



For this question you will show that the distance is distributed like reflected BM, i.e. $(M_t - W_t) \sim |W_t|$, and use this fact to price a derivative.

Note: We already know that $M_t \sim |W_t|$, but it is not the same thing.

(a) Show that $\mathbb{P}(M_t - W_t \ge x, W_t \le -x) = \mathbb{P}(W_t \le -x)$ for $x > 0$

Argue that for arbitrary $x > 0$, $\mathbb{P}(M_t - W_t \ge x \mid W_t \le -x) = 1$. Use this to show that $\mathbb{P}(M_t - W_t \ge x, W_t \le -x) = \mathbb{P}(W_t \le -x)$.

Hint: $\mathbb{P}(X \ge x, Y \le y) = \int_{-\infty}^{y} \mathbb{P}(X \ge x \mid Y = u) f_Y(u) du$.

The standard BM always starts at 0 (hence its maximum is ≥ 0).

If the BM is below $-x \le 0$ at time t, then the distance between the maximum and the value of the process at time t must be $\ge x$, i.e. $\mathbb{P}(M_t - W_t \ge x \mid W_t \le -x) = 1$.

This implies that $\forall w \le -x$, we have $\mathbb{P}(M_t - W_t \ge x \mid W_t = w) = 1$, so

$$\begin{aligned}
\mathbb{P}(M_t - W_t \ge x, W_t \le -x) &= \int_{-\infty}^{-x} \overbrace{\mathbb{P}(M_t - W_t \ge x \mid W_t = w)}^{=1} f_{W_t}(w) dw \\
&= \int_{-\infty}^{-x} f_{W_t}(w) dw = \mathbb{P}(W_t \le -x)
\end{aligned}$$

(b) Show that $\mathbb{P}\left(M_t - W_t \geq x, W_t \geq -x\right) = \mathbb{P}\left(W_t \geq x\right)$**, for** $x > 0$**.**

Hint: Use the Rayleigh distribution result for the conditional CDF of the maximum $P\left(M_t - W_t \geq x \mid W_t = w\right) = P\left(M_t \geq x + w \mid W_t = w\right)$, together with the previous hint.

$$
\begin{aligned}
\mathbb{P}\left(M_t - W_t \geq x, W_t \geq -x\right) &= \int_{-x}^{\infty} \mathbb{P}\left(M_t - W_t \geq x \mid W_t = w\right) f_{W_t}(w) dw \\
&= \int_{-x}^{\infty} \mathbb{P}\left(M_t - w \geq x \mid W_t = w\right) f_{W_t}(w) dw \\
&= \int_{-x}^{\infty} \mathbb{P}\left(M_t \geq x + w \mid W_t = w\right) f_{W_t}(w) dw
\end{aligned}
$$

We know that the conditional maximum of the Brownian bridge has the Rayleigh CDF

$$
\begin{aligned}
\mathbb{P}\left(M_t \leq m \mid W_t = b\right) &= 1 - \exp\left\{-2\frac{m(m-b)}{\cancel{\sigma_W^2}{}^1 t}\right\}, \forall m \geq \max\{0, b\} \\
\Rightarrow \mathbb{P}\left(M_t \geq m \mid W_t = b\right) &= 1 - \mathbb{P}\left(M_t \leq m \mid W_t = w\right) = \exp\left\{-2\frac{m(m-b)}{t}\right\}
\end{aligned}
$$

Substituting this back to the integral:

$$
\begin{aligned}
\mathbb{P}\left(M_t - W_t \geq x, W_t \geq -x\right) &= \int_{-x}^{\infty} \overbrace{\mathbb{P}\left(M_t \geq x + w \mid W_t = w\right)}^{=\exp\left\{-2\frac{(x+w)(x+w-w)}{t}\right\}} f_{W_t}(w) dw \\
&= \int_{-x}^{\infty} e^{-2x(x+w)/t} \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t} dw \\
&= \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{w^2 + 4wx + 4x^2}{2t}\right\} dw \\
&= \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(w+2x)^2}{2t}\right\} dw \\
&\quad (\text{ change of variable: } v = w + 2x) \\
&= \int_{+x}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{v^2}{2t}\right\} dv \\
&= \mathbb{P}\left(W_t \geq x\right)
\end{aligned}
$$

(c) Show that $(M_t - W_t) \sim |W_t|$

Using the Law of Total Probability and combining the previous parts, we have

$$
\begin{aligned}
\mathbb{P}\left(M_t - W_t \geq x\right) &= \mathbb{P}\left(M_t - W_t \geq x, W_t \leq -x\right) + \mathbb{P}\left(M_t - W_t \geq x, W_t \geq -x\right) \\
&= \mathbb{P}\left(W_t \leq -x\right) + \mathbb{P}\left(W_t \geq x\right) = \mathbb{P}\left(|W_t| \geq x\right) \\
&\Rightarrow (M_t - W_t) \sim |W_t|
\end{aligned}
$$

(d) Risk Neutral Pricing

Assume an asset with price process $S_t = S_0 e^{\sigma W_t}$ under the risk-neutral measure, which implies a risk-free rate of $r = \sigma^2/2$, since

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}$$

Consider a derivative that pays at expiration T the ratio of the maximum over the starting price, i.e. payoff $= V_T = \max_{0\le t\le T}\{S_t\}/S_0$. Use risk-neutral pricing to find the price of the derivative at 0 .

(Hint: If $Z \sim N(0,1)$, then $m_{|Z|}(x) = \mathbb{E}\left[e^{x|Z|}\right] = 2e^{x^2/2}\Phi(x)$, where $\Phi(\cdot)$ is the standard Normal CDF.)

Note that the maximum stock price can be expressed as

$$\max_{0\le t\le T}\{S_t\} = \max_{0\le t\le T}\left\{S_0 e^{\sigma W_t}\right\} = S_0 \exp\{\sigma \max_{0\le t\le T}\{W_t\}\} = S_0 e^{\sigma M_T}$$

where $M_T \sim |W_T| \sim \sqrt{T}Z$ since $W_T \sim N(0,T)$.

The option price at $t=0$ is given by its discounted expected payoff under the risk neutral measure:

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}\left[V_T\right] = e^{-rT}\mathbb{E}\left[\frac{S_0 e^{\sigma M_T}}{S_0}\right] \\ &= e^{-rT}\mathbb{E}\left[e^{\sigma|W_T|}\right] = e^{-rT}\mathbb{E}\left[e^{\sigma\sqrt{T}|Z|}\right] \\ &= e^{-\sigma^2 T/2}\left(2e^{\sigma^2 T/2}\Phi(\sigma\sqrt{T})\right) = 2\Phi(\sigma\sqrt{T}) \end{aligned}$$

2019 Q1. VaR

Assume the return on an asset follows a logistic distribution with CDF

$$F_X(x) = \frac{1}{1+\exp\left(-\frac{x-\mu}{\lambda}\right)}, \quad \forall x \in \mathbb{R}, \lambda > 0,$$

where μ and λ are the location and scale parameters, respectively. Find a closed-form expression for $\mathrm{VaR}(\alpha)$ and $\mathrm{CVaR}(\alpha)$.

(Hint: $\int \log(x)dx = x\log(x) - x + c$)

For continuous distributions, the quantile function is given by the inverse CDF:

We want $\mathbb{P}(R \leq -VaR(\alpha)) = \alpha \Rightarrow$

$$\Rightarrow F(-VaR(\alpha)) = \alpha \Rightarrow VaR(\alpha) = -F^{-1}(\alpha)$$

$$\begin{cases} F(x) = \dfrac{1}{1+e^{-\frac{x-\mu}{\lambda}}} = \alpha \Rightarrow e^{-\frac{x-\mu}{\lambda}} = \dfrac{1}{\alpha} - 1 = \dfrac{1-\alpha}{\alpha} \Rightarrow \\ \Rightarrow \dfrac{x-\mu}{\lambda} = -\log\left(\dfrac{1-\alpha}{\alpha}\right) \Rightarrow x = \mu - \lambda \log\left(\dfrac{1-\alpha}{\alpha}\right) \end{cases}$$

$$\Rightarrow VaR(\alpha) = -\mu - \lambda \log\left(\frac{\alpha}{1-\alpha}\right)$$

$$= \log(u) - \log(1-u)$$

$$CVaR(\alpha) = \frac{1}{\alpha}\int_0^\alpha VaR(a) = \frac{1}{\alpha}\cdot\int_0^\alpha -\mu - \lambda \underbrace{\log\left(\frac{u}{1-u}\right)} du$$

$$= -\mu - \frac{\lambda}{\alpha}\left\{\int_0^\alpha \log(u)\,du - \int_0^\alpha \log(1-u)\,du\right. =$$

$$= -\mu - \frac{\lambda}{\alpha}\cdot\left\{(\alpha\log(\alpha) - \cancel{\alpha}) - 0 - [(\alpha-1)\log(1-\alpha) - \cancel{\alpha}] - 0\right)$$

$$= -\mu - \lambda\cdot\left(\log(\alpha) - \frac{\alpha-1}{\alpha}\log(1-\alpha)\right)$$

Page 2 of 14 (contd...)

2019 Q2. Mean variance analysis

Assume the returns on all assets in a market follow multivariate Normal distribution with mean μ and variance-covariance matrix Σ. Furthermore, assume investors pick portfolios with minimum VaR(α) for some α. Show that the only portfolios that investors would consider are the ones lying on the efficient frontier of mean-variance analysis.

(Hint: show that minimizing VaR for a Normal distribution is equivalent to minimizing the portfolio variance for a given mean return level.)

We know that the portfolio returns follow

$$R_{port} \sim N(\mu_{port}, \sigma^2_{port}), \text{ where: } \begin{cases} \mu_{port} = \underline{w}^T \underline{\mu} \\ \sigma^2_{port} = \underline{w}^T \Sigma \underline{w} \\ \text{for weights } \underline{w} \end{cases}$$

$$\Rightarrow VaR(\alpha) = -\{\mu_{port} + \sigma_{port}\Phi^{-1}(\alpha)\}$$

$\Rightarrow$ For given $(\mu_{port}, \alpha < \frac{1}{2})$, the minimum $VaR(\alpha)$ is given when $-\sigma_{port}\Phi^{-1}(\alpha)$ is minimized when

$\Rightarrow$ (if $\alpha < \frac{1}{2} \Rightarrow \Phi^{-1}(\alpha) < 0 \Rightarrow$) σ_{port} is minimized!

$\Rightarrow$ for given μ_{port}, $\min\{\sigma_{port}\}$, which is equivalent to mean-variance analysis

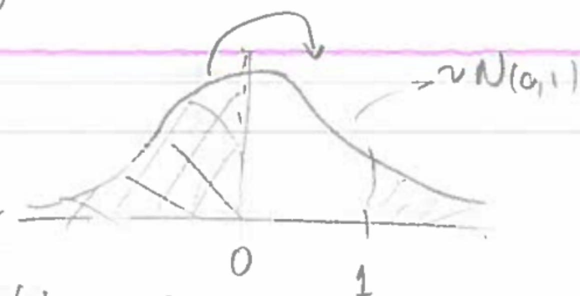
2019 Q3. Brownian Motion

Consider a standard Brownian motion $\{W_l\}$ with $W_0 = 0$.

a)

Let $M_1 = \max\{W_t : t \in [0,1]\}$ be the maximum of the process by time 1. Find the conditional expectation of M_1 given $(M_1 > 1)$, i.e $\mathbb{E}[M_1 \mid M_1 > 1]$, in terms of the standard Normal CDF $\Phi(z)$.

We know that $M_t \sim |W_t| \Rightarrow M_1 \sim |W_1|$, where $W_1 \sim N(0,1)$. So, essentially, we want to find the conditional expectation of the tail of a "flipped" std Normal (this is similar to the CVaR(α)/ES calculation for Normal)



We have: $\mathbb{E}[M_1 | M_1 > 1] = \int_1^\infty x \cdot \underbrace{f_{M_1|M_1>1}}_{\text{conditional PDF}}(x)\,dx =$

$$= \int_1^\infty x \cdot \frac{\overbrace{f_{M_1}(x)}^{\to \text{twice } N(0,1)}}{\mathbb{P}(M_1>1)}\,dx =$$

$$= \int_1^\infty x \cdot \frac{\cancel{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{\cancel{2} \cdot \Phi(-1)}\,dx = \frac{1}{\Phi(-1)} \cdot \int_1^\infty \frac{1}{\sqrt{2\pi}} \left(-e^{-x^2/2}\right)' dx =$$

$$= \frac{1}{\Phi(-1)} \cdot \left[\underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}_{= \varphi(x) \sim \text{std Normal PDF}}\right]_\infty^1 = \frac{\varphi(1)}{\Phi(-1)}$$

Page 4 of 14 (contd...)

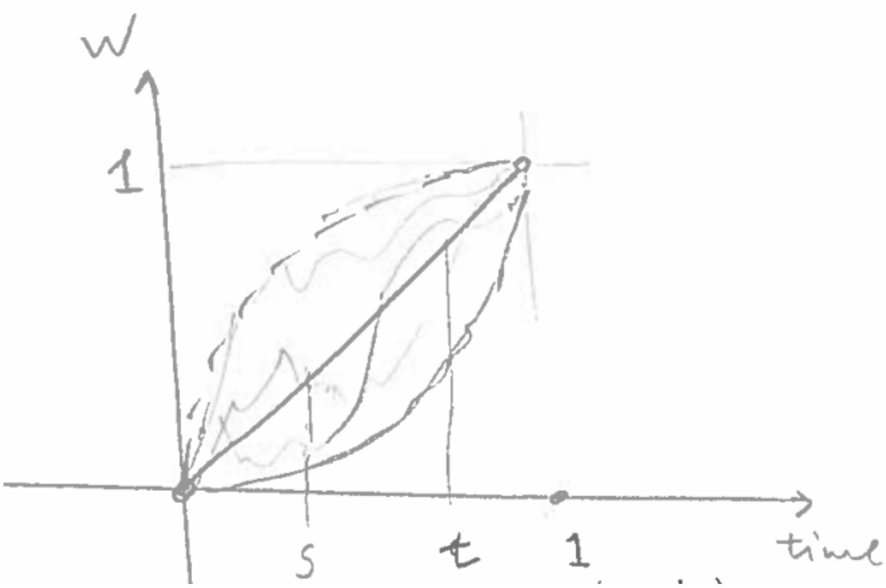
b)

Condition on the event $(W_1 = 1)$, and find the conditional (bivariate) distribution of $\begin{bmatrix} W_s \\ W_t \end{bmatrix} | (W_1 = 1)$, where $0 < s < t < 1$ $X_1 | (X_2 = x_2) \sim N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{12}^{-1}\Sigma_{21}\right)$

(b) $\begin{bmatrix} W_s \\ W_t \\ W_1 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{bmatrix} \right) \Rightarrow$

$$\Rightarrow \begin{bmatrix} W_s \\ W_t \end{bmatrix} \Big| (W_1 = 1) \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix} 1^{-1} \cdot (1-0), \begin{bmatrix} s & s \\ s & t \end{bmatrix} - \begin{bmatrix} s \\ t \end{bmatrix} 1^{-1} [s \ \ t] \right)$$

$$\Rightarrow \sim N\left(\begin{bmatrix} s \\ t \end{bmatrix}, \underbrace{\begin{bmatrix} s - s^2 & s - st \\ s - st & t - t^2 \end{bmatrix}}_{= \begin{bmatrix} s(1-s) & s(1-t) \\ s\cdot(1-t) & t\cdot(1-t) \end{bmatrix}} \right)$$



2019 Q4. VaR

Consider two assets whose revenues are independently and uniformly distributed as $R_1, R_2 \sim$ iid $Unif(-1,1)$

a)

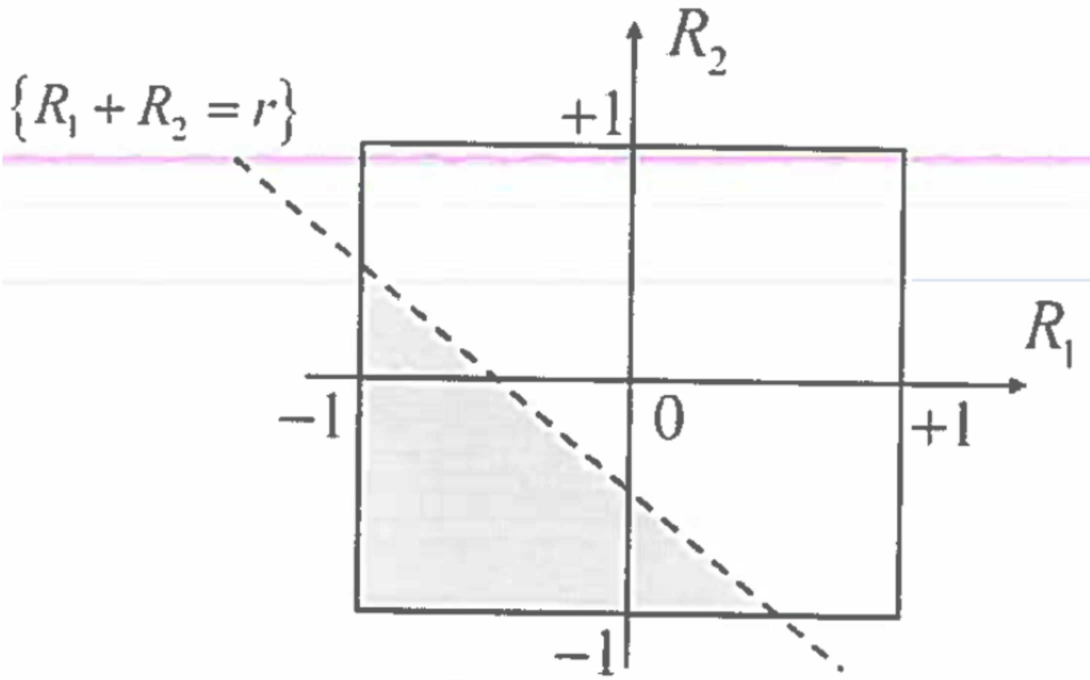
Find a closed-form expression for the individual asset value at risk, i.e. find $\mathrm{VaR}_\alpha(R_1)$ as a function of α.

b)

Show that VaR is not sub-additive in this case, i.e. show that $\mathrm{VaR}_\alpha(R_1+R_2) \leq \mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2)$ for some α (Hint: try $\alpha = 7/8$)

The CDF of R_i is $F_{R_i}(r) = \mathbb{P}(R_i \leq r) = \frac{r+1}{2}, r \in [-1,1], \forall i = 1,2$. Thus, $\mathrm{VaR}_\alpha(R_i) = = -F_{R_1}^{-1}(\alpha) = -(2\alpha - 1) = 1 - 2\alpha, \forall i = 1,2$, and $\mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2) = 2(1-2\alpha)$.

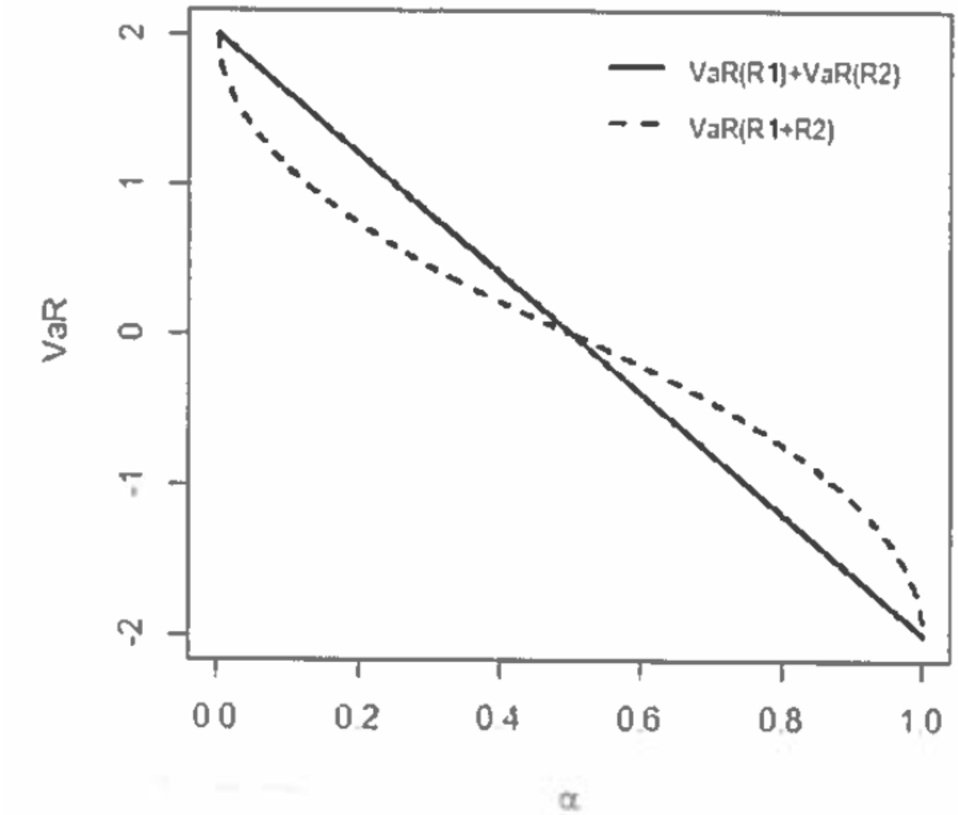
The CDF of $R_1 + R_2$ is $F_{R_1+R_2}(r) = \mathbb{P}(R_1 + R_2 \leq r), r \in [-2,2]$, where the probability is represented by the shaded region in the plot below:



$$\mathbb{P}(R_1 + R_2 \leq r) = \begin{cases} \frac{(2+r)^2}{8}, & r \leq 0 \\ 1 - \frac{(2-r)^2}{8}, & r > 0 \end{cases}$$

$$\mathrm{VaR}_\alpha(R_1 + R_2) = \begin{cases} -2(\sqrt{2\alpha} - 1), & \alpha \leq \frac{1}{2} \\ -2(1 - \sqrt{2(1-\alpha)}), & \alpha > \frac{1}{2} \end{cases}.$$

As the following plot shows, the VaR is not subadditive in this case, since $\forall\alpha > \frac{1}{2}$ we have $\mathrm{VaR}_\alpha(R_1+R_2) \nleq \mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2)$. (e.g. for $\alpha = \frac{7}{8}$ we have $\mathrm{VaR}_\alpha(R_1+R_2) = -2\left(1-\sqrt{2\left(1-\frac{7}{8}\right)}\right) = -2\left(1-\sqrt{\frac{1}{4}}\right) = -1$, but $\mathrm{VaR}_\alpha(R_1) + \mathrm{VaR}_\alpha(R_2) = 2\left(1-2\frac{7}{8}\right) = 2\left(1-\frac{7}{4}\right) = 2\left(-\frac{3}{4}\right) = -\frac{3}{2} \leq$



2019 Q5. Minimum variance portfolio

Consider two sets of assets whose return vectors R_1, R_2 are independent and have variance-covariance matrices $\mathbb{V}[R_1] = \Sigma_1, \mathbb{V}[R_2] = \Sigma_2$. Let the minimum-variance weights for a portfolio consisting only of assets in R_1 be w_1, and let the achieved minimum variance be σ_1^2. Similarly for the assets in R_2, let the minimum-variance portfolio weights be w_2 and the minimum variance be σ_2^2. Now turn attention to the minimum variance portfolio for both sets of assets $\left[\frac{R_1}{R_2}\right]$.

a)

Show that the combined minimum-variance portfolio weights are given by $w = \frac{1}{\sigma_1^2+\sigma_2^2}\left[\frac{\sigma_2^2 w_1}{\sigma_1^2 w_2}\right]$.

a. For the individual minimum variance portfolios we have

$$\left.\begin{array}{l} \mathbf{w}_i = \dfrac{\boldsymbol{\Sigma}_i^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}} \\ \sigma_i^2 = \mathbb{V}\left[\mathbf{w}_i^\mathsf{T}\mathbf{R}_i\right] = \mathbf{w}_i^\mathsf{T}\underbrace{\mathbb{V}\left[\mathbf{R}_i\right]}_{=\boldsymbol{\Sigma}_i}\mathbf{w}_i = \dfrac{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}}\cancel{\boldsymbol{\Sigma}_i}\dfrac{\cancel{\boldsymbol{\Sigma}_i^{-1}}\mathbf{1}}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}} = \dfrac{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}}{\left(\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}\right)^2} = \dfrac{1}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_i^{-1}\mathbf{1}} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathbf{w}_i = \sigma_i^2 \times \boldsymbol{\Sigma}_i^{-1}\mathbf{1}, \quad \forall i = 1,2$$

We also know by independence that the combined asset's variance-covariance matrix is $\boldsymbol{\Sigma} = \mathbb{V}\begin{bmatrix}\mathbf{R}_1 \\ \mathbf{R}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2\end{bmatrix}$. The combined minimum variance portfolio is thus given by

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{1}} = \frac{\begin{bmatrix}\boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1} \\ \mathbf{1}\end{bmatrix}}{\begin{bmatrix}\mathbf{1}^\mathsf{T} & \mathbf{1}^\mathsf{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1} \\ \mathbf{1}\end{bmatrix}} = \frac{\begin{bmatrix}\boldsymbol{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{1} \\ \mathbf{1}\end{bmatrix}}{\begin{bmatrix}\mathbf{1}^\mathsf{T} & \mathbf{1}^\mathsf{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^{-1}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1} \\ \mathbf{1}\end{bmatrix}} =$$

$$= \frac{\begin{bmatrix}\boldsymbol{\Sigma}_1^{-1}\mathbf{1} \\ \boldsymbol{\Sigma}_2^{-1}\mathbf{1}\end{bmatrix}}{\underbrace{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_1^{-1}\mathbf{1}}_{=1/\sigma_1^2} + \underbrace{\mathbf{1}^\mathsf{T}\boldsymbol{\Sigma}_2^{-1}\mathbf{1}}_{=1/\sigma_2^2}} = \frac{1}{1/\sigma_1^2 + 1/\sigma_2^2}\begin{bmatrix}\boldsymbol{\Sigma}_1^{-1}\mathbf{1} \\ \boldsymbol{\Sigma}_2^{-1}\mathbf{1}\end{bmatrix} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\begin{bmatrix}\boldsymbol{\Sigma}_1^{-1}\mathbf{1} \\ \boldsymbol{\Sigma}_2^{-1}\mathbf{1}\end{bmatrix} = \frac{1}{\sigma_1^2 + \sigma_2^2}\begin{bmatrix}\sigma_2^2\mathbf{w}_1 \\ \sigma_1^2\mathbf{w}_2\end{bmatrix}$$

b)

Show that the attained minimum variance is given by $\sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

b. The attained minimum variance is $\sigma^2 = \mathbb{V}\left[\mathbf{w}^T\mathbf{R}\right] = \mathbf{w}^T\mathbb{V}[\mathbf{R}]\mathbf{w} =$

$$= \frac{1}{\sigma_1^2+\sigma_2^2}\left[\sigma_2^2\mathbf{w}_1^T \;\vdots\; \sigma_1^2\mathbf{w}_2^T\right]\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}\begin{bmatrix}\sigma_2^2\mathbf{w}_1\\ \sigma_1^2\mathbf{w}_2\end{bmatrix}\frac{1}{\sigma_1^2+\sigma_2^2} =$$

$$= \left(\frac{1}{\sigma_1^2+\sigma_2^2}\right)^2\left(\sigma_2^2\mathbf{w}_1^T\Sigma_1\mathbf{w}_1^2\sigma_2^2 + \sigma_1^4\mathbf{w}_2^T\Sigma_2\mathbf{w}_2\sigma_1^4\right) =$$

$$= \frac{\sigma_2^4\sigma_2^2+\sigma_1^4\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)^2} = \frac{\sigma_1^2\sigma_2^2\cancel{\left(\sigma_1^2+\sigma_2^2\right)}}{\left(\sigma_1^2+\sigma_2^2\right)^2} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

2019 Q6. Variance reduction

Consider a standard Normal random variable $Z \sim N(0,1)$ and let $Y = e^Z$. Assume you want to estimate $\mathbb{E}[Y]$ using simulation, i.e. using $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{1}{n}\sum_{i=1}^n e^{Z_i}$, where $Z_i \sim$ iid $N(0,1)$

a)

Find the variance of your estimate $\mathbb{V}[\bar{Y}]$, as a function of n only. (Hint: the moment generating function of $Z \sim N(0,1)$ is $m_Z(t) = \mathbb{E}\left[e^{tZ}\right] = e^{t^2/2}$)

$$\mathbb{V}[\bar{Y}] = \frac{1}{n}\cdot\mathbb{V}[Y] = \frac{1}{n}\mathbb{V}[e^Z] = \frac{1}{n}\left\{\mathbb{E}[(e^Z)^2] - (\mathbb{E}[e^Z])^2\right\}$$

$$= \frac{1}{n}\cdot\left\{\mathbb{E}[e^{2Z}] - (\mathbb{E}[e^Z])^2\right\} = \frac{1}{n}\cdot\left\{m_Z(2) - [m_Z(1)]^2\right\}$$

$$= \frac{1}{n}\left(e^{2^2/2} - (e^{1/2})^2\right) = \frac{1}{n}\cdot e^2 - e = \frac{1}{n}\cdot e\cdot(e-1$$

b)

Assume you use $X = Z$ as a control variable for your simulation. Find the relative reduction in estimation accuracy, i.e. find the value of $\mathbb{V}\left[\bar{Y}_{\text{ctrl}}\right]/\mathbb{V}[\bar{Y}]$. (Hint: for $Z \sim N(0,1)$, we have $\mathbb{E}\left[Ze^Z\right] = \sqrt{e}$.)

$$\mathbb{V}[\bar{Y}_{ctrl}]/\mathbb{V}[Y] = 1-\rho^2 \text{, where } \rho = \frac{Cov(X,Y)}{\sqrt{\mathbb{V}(X)\cdot\mathbb{V}(Y)}}$$

$$Cov(X,Y) = \mathbb{E}\Big[\big(X-\mathbb{E}(X)\big)\cdot\big(Y-\mathbb{E}(Y)\big)\Big] =$$

$$= \mathbb{E}\Big[\big(Z-\cancelto{0}{\mathbb{E}[Z]}\big)\cdot\big(e^Z-\mathbb{E}[e^Z]\big)\Big] =$$

$$= \mathbb{E}\Big[Z\cdot\big(e^Z - m_Z(1)\big)\Big]$$

$$= \mathbb{E}[Ze^Z] - \cancelto{0}{\mathbb{E}[Z]}\cdot m_Z(1) = \sqrt{e}$$

$$\mathbb{V}(X) = \mathbb{V}(Z) = 1$$

$$\text{Thus } \rho = \frac{Cov(X,Y)}{\sqrt{\mathbb{V}(X)\,\mathbb{V}[Y]}} = \frac{\sqrt{\cancel{e}}}{\sqrt{\cancel{e}(e-1)\cdot 1}} = \frac{1}{\sqrt{e-1}}$$

$$\Rightarrow 1-\rho^2 = 1-\frac{1}{e-1} = \frac{e-2}{e-1} = .418$$

$$= 41.8\%$$

2020 Q1. Brownian Motion

Assume the prices of two assets (S_t, P_t) follow correlated geometric Brownian motions:

$$\begin{cases} d\log(S_t) = \mu_S dt + \sigma_S dW_t \\ d\log(P_t) = \mu_P dt + \sigma_P dV_t \end{cases}$$

where $S_0 = P_0 = 1$, and W_t, V_t are correlated standard Brownian motions with correlation $|\rho| < 1$

(a) GBM

Express the (marginal) distribution of S_t as a log-Normal distribution, i.e. $S_t = S_0 \times \exp\{N(\cdot,\cdot)\}$, and state the parameter values of the Normal distribution in the exponent.

GBM definition:

$$
\begin{aligned}
S_t &= S_0 e^{\log\left(\frac{S_t}{S_0}\right)} \quad \text{where } \log\left(\frac{S_t}{S_0}\right) \sim N(\mu t, \sigma^2 t) \\
&\sim S_0 \log N(\mu t, \sigma^2 t)
\end{aligned}
$$

$$
\begin{aligned}
S_t &= \underbrace{S_0}_{=1} \times \exp\left\{\log\frac{S_t}{S_0}\right\} = \exp\left\{\mu_S t + \sigma_S W_t\right\} \\
&\sim \exp\left\{N\left(\mu_S t, \sigma_S^2 t\right)\right\}
\end{aligned}
$$

(b) GBM

Show that S_t^a where $a > 0$ also follows log-Normal distribution, and find its parameter values as before.

$$
S_t^a = S_0 \times \exp\left\{a\mu_S t + a\sigma_S W_t\right\} \sim \exp\left\{N\left(a\mu_S t, a^2\sigma_S^2 t\right)\right\}
$$

(c) Conditional distribution

Show that the conditional distribution of $S_t \mid (P_t = e)$ is also log-Normal, and find its parameter values as before.

We have

$$
\begin{aligned}
P_t = P_0 \times \exp\left\{\mu_p t + \sigma_p V_t\right\} &= e \\
\mu_P t + \sigma_p V_t &= 1 \\
V_t = \frac{1 - \mu_P t}{\sigma_P} &:= c
\end{aligned}
$$

So we are looking for the conditional distribution of

$$
S_t \left|(P_t = e) = S_t\right| (V_t = c) = S_0 \times \exp\left\{\mu_S t + \sigma_S \underbrace{W_t \mid (V_t = c)}_{\star}\right\}
$$

First, we need to find the distribution of $\star$:

We are given that W_t, V_t are correlated standard BM with correlation ρ

$$
\begin{bmatrix} W_t \\ V_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, t\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \Rightarrow W_t \mid (V_t = c) \sim N\left(\rho c, t\left(1 - \rho^2\right)\right)
$$

Plugging it in, we have

$$\begin{aligned} S_t \mid (P_t = e) &= \exp\left\{\mu_S t + \sigma_S N\left(\rho c, t\left(1-\rho^2\right)\right)\right\} \\ &= \exp\left\{N\left(\mu_S t + \sigma_S \rho c, \sigma_S^2 t\left(1-\rho^2\right)\right)\right\} \\ &= \exp\left\{N\left(\mu_S t + \rho\frac{\sigma_S}{\sigma_P}\left(1-\mu_P t\right), \sigma_S^2 t\left(1-\rho^2\right)\right)\right\} \end{aligned}$$

(d)

Find the probability $\mathbb{P}\left(S_t^2 > P_t\right)$ in terms of the model parameters and the standard Normal CDF $\Phi(\cdot)$.

$$\mathbb{P}\left(S_t^2 > P_t\right) = \mathbb{P}\left(\frac{S_t^2}{P_t} > 1\right), \text{ where } \begin{cases} S_t^2 = S_0^2 e^{(\mu t + \sigma W_t)2} \\ \\ P_t = P_0 e^{\mu t + \sigma V_t} \end{cases}$$

$$\begin{aligned} \frac{S_t^2}{P_t} &= \frac{S_0^2}{P_0}\exp\left\{2\mu_S t + 2\sigma_S W_t - \mu_P t - \sigma_P V_t\right\} \\ &= \exp\left\{\underbrace{\left(2\mu_S - \mu_P\right)t}_{\mu} + 2\sigma_S W_t - \sigma_P V_t\right\} \end{aligned}$$

Find σ^2 using $V(aX+bY) = a^2V(X) + b^2V(Y) + 2abCov(X,Y)$ and $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$

$$2\sigma_S W_t - \sigma_P V_t \sim N\left(0, \left(4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P\right)t\right)$$

So we have

$$\begin{aligned} \mathbb{P}\left(S_t^2/P_t > 1\right) &= \mathbb{P}\left(N(\overbrace{\left(2\mu_S - \mu_P\right)t}^{=\mu}, \overbrace{\left(4\sigma_S^2 + \sigma_P^2 - 4\rho\sigma_S\sigma_P\right)t}^{=\sigma^2}) > 0\right) \\ &= \mathbb{P}(Z > -\mu/\sigma) = 1 - \Phi(-\mu/\sigma) \end{aligned}$$

2020 Q2. Betting Strategies

Consider a sequence of independent gambles, where the probability of winning each one is $1/2 < p < 1$. Moreover, for each gamble you either win or lose the amount you bet. Assume you start with initial wealth V_0 and at each step you bet a fixed fraction f of your wealth.

(a)

Show that if you win half and lose half of the first $2n$ bets (i.e. win any of the n bets), then the resulting wealth V_{2n} will always be less than your initial wealth V_0, for any $f > 0$.

Let the Binomial random variable $W_n \sim \text{Binomial}(n, p)$ denote the number of wins in n bets. At each bet we wager a proportion f of our wealth, so the total wealth after n gambles becomes

$$V_n = V_0(1+f)^{W_n}(1-f)^{n-W_n} = V_0\left(\frac{1+f}{1-f}\right)^{W_n}(1-f)^n$$

In this case, we have $W_{2n} = n$, therefore:

$$\begin{aligned} V_{2n} &= V_0\left(\frac{1+f}{1-f}\right)^{W_{2n}}(1-f)^{2n} = V_0\left(\frac{1+f}{1-f}\right)^{n}(1-f)^{2n} \\ &= V_0(1+f)^n(1-f)^n = V_0[(1+f)(1-f)]^n = V_0\overbrace{(1-f^2)}^{<1,\forall f>0}{}^{n} \end{aligned}$$

which is always smaller than the initial wealth.

(b)

Show that if your goal is to maximize the expected square-root of the return (i.e. $\mathbb{E}\left[\sqrt{V_n/V_0}\right]$), then the optimal fraction for this strategy is $f^* = \frac{(p/q)^2-1}{(p/q)^2+1}$, where $q = 1-p$.

(Hint: the probability generating function of a Binomial RV $X \sim \text{Binomial}(n, p)$ is given by $g_X(z) = \mathbb{E}\left[z^X\right] = [q+pz]^n$.)

The expected square root of the wealth after n bets is:

$$\begin{aligned} \mathbb{E}\left[\sqrt{X_n/X_0}\right] &= \mathbb{E}\left[\sqrt{\left(\frac{1+f}{1-f}\right)^{W_n}(1-f)^n}\right] \\ &= (1-f)^{n/2}\mathbb{E}\left[\left(\frac{1+f}{1-f}\right)^{W_n/2}\right] \\ &= (\sqrt{1-f})^n\mathbb{E}\left[\left(\sqrt{\frac{1+f}{1-f}}\right)^{W_n}\right] \\ &\overset{\text{hint}}{=} (\sqrt{1-f})^n\left[q+p\sqrt{\frac{1+f}{1-f}}\right]^n \\ &= [q\sqrt{1-f}+p\sqrt{1+f}]^n \end{aligned}$$

We maximize this by taking the derivative w.r.t. f and setting it to 0:

$$\begin{aligned}
0 &= \frac{d}{df}[q\sqrt{1-f} + p\sqrt{1+f}]^n \\
0 &= \underbrace{n[q\sqrt{1-f} + p\sqrt{1+f}]^{n-1}}_{>0}\left[-\frac{q}{2\sqrt{1-f}} + \frac{p}{2\sqrt{1+f}}\right] \\
(1+f)q^2 &= (1-f)p^2 \\
\\
f\left(p^2+q^2\right) &= p^2 - q^2 \\
f^* &= \frac{p^2-q^2}{p^2+q^2} = \frac{(p/q)^2-1}{(p/q)^2+1}
\end{aligned}$$

2020 Q3. VaR

Assume the loss of a portfolio is $L = X - \mu$, where X follows Exponential(1) distribution, i.e. $f_X(x) = e^{-x}, \forall x > 0$.

(a) Find VaR at level α

The complementary CDF of the exponential is $\bar{F}_X(x) = 1 - (1 - e^{-x}) = e^{-x}$, for all $x > 0$. Therefore:

$$\begin{aligned}
\mathbb{P}(L > VaR) &= \alpha \\
\mathbb{P}(X - \mu > VaR) &= \alpha \\
\mathbb{P}(X > VaR + \mu) = \bar{F_X}(VaR + \mu) &= \alpha \\
e^{-VaR+\mu} &= \alpha \\
-VaR + \mu &= \ln(\alpha) \\
VaR &= \mu - \ln(\alpha)
\end{aligned}$$

(b) Find conditional VaR at level

$$\begin{aligned}
CVaR &= \frac{1}{\alpha}\int_0^\alpha VaR(u)du = \frac{1}{\alpha}\int_0^\alpha \mu - \ln(u)du \\
&= \mu - \frac{1}{\alpha}\int_0^\alpha [u\ln(u) - u]'du = \mu - \frac{1}{\alpha}[u\ln(u) - u]_0^\alpha \\
&= \mu - \frac{1}{\alpha}[\alpha\ln(\alpha) - \alpha - \underbrace{\lim_{u\to 0} u\ln(u)}_{=0}] \\
&= \mu - \ln(\alpha) - 1
\end{aligned}$$

2021 Q1. VaR & Betting Strategies

Assume you start with an initial wealth of V_0 and you can invest any amount on a sequence of investment opportunities with net returns $\{R_i\}_{i\geq 1}$ which follow independent and identically distributed Uniform $(-1, 2)$ distributions. For example, if you invest \$1 in such an opportunity, your resulting wealth will be $(1 + R) \in (0, 3)$, i.e. anywhere between losing to tripling your wealth.

(a) VaR

Assume you bet your entire initial capital of $V_0 = \$1$ on the first opportunity. Calculate the Value-at-Risk (VaR) of the investment at the $\alpha = 5\%$ level.

The loss distribution is $L = -R \sim$ Uniform $(-2, 1)$, whose 95% quantile is

$$\text{VaR}_{.05} = -2 + 0.95 \times 3 = 1 - .05 \times 3 = 0.85$$

(b) CVaR

Calculate the Conditional VaR (CVaR) at the $\alpha = 5\%$ level for the investment in part (a).

For the Uniform, the (conditional) expected value is the the midpoint of the range:

$$\text{CVaR}_{.05} = \frac{0.85 + 1}{2} = 0.925$$

(c) EVaR

Calculate the Entropic VaR (EVaR) at the $\alpha = 5\%$ level for the investment in part (a). This is defined as $\text{EVaR}_\alpha = \inf_{z>0} \{z^{-1} \ln(M_L(z)/\alpha)\}$, where $M_L(z)$ is the moment generating function of the loss (L). You are given that the infimum occurs at $z \approx 18.122$.

The MGF of the uniform distribution for the loss (L) is

$$\begin{aligned} M_L(z) &= \mathbb{E}\left[e^{zL}\right] = \int_{-2}^{1} e^{z\ell}/3 d\ell \\ &= \frac{1}{3z}\left[e^{z\ell}\right]_{\ell=-2}^{1} = \frac{e^{z} - e^{-2z}}{3z} \end{aligned}$$

```
EVaR = function(z, alpha = .05, a=-2, b=1 ){
  return( 1/z * log( ( exp(b*z) - exp(a*z) ) / (z*(b-a)* alpha) )  )
}
optimize( EVaR, c(1,100) )
```

Output:

```
$minimum
[1] 18.122
$objective
[1] 0.9448
```

The EVaR_α is

$$\begin{aligned}\text{EVaR}_{0.05} &= z^{-1}\ln\left(\frac{e^z - e^{-2z}}{3z(0.05)}\right)|_{z=18.122}\\ &= \frac{1}{18.122}\ln\left(\frac{e^{18.122} - e^{-2(18.122)}}{0.15(18.122)}\right) = 0.9448181\end{aligned}$$

Note that: $\text{VaR}_{.05} \leq \text{CVaR}_{.05} \leq \text{EVaR}_{.05}$, as expected.

(d) Wealth computation

Assume your strategy is to invest a fixed fraction (f) of your current wealth in each step. Show that your final wealth after n investments is

$$V_n = V_0 \prod_{i=1}^{n} (1 + fR_i)$$

Assuming you invest a fraction f, then your wealth after the first step will be:

$$V_1 = (1-f)V_0 + fV_0(1+R_1) = V_0 - fV_0 + fV_0 + fV_0R_1 = V_0(1+fR_1)$$

Similarly for the next steps, we get

$$\begin{aligned}V_n &= V_{n-1}(1+fR_n)\\ &= V_{n-2}(1+fR_{n-1})(1+fR_n)\\ &= \vdots\\ &= V_0\prod_{i=1}^{n}(1+fR_i)\end{aligned}$$

(e) Probability of Ruin

Assume you invest your entire wealth ($f = 1$) at each step. Find the probability of going bankrupt ($\mathbb{P}(V_n = 0)$) after n steps?

If $f = 1$, you will go bankrupt if and only if any of the returns R_i is exactly equal -1 . Since the return distribution is continuous, the probability of that happening is 0.

(f) Expected wealth

Assume you invest your entire wealth ($f = 1$) at each step. Find the expected value of your wealth ($\mathbb{E}[V_n]$) after n steps.

For $f = 1$

$$
\begin{aligned}
\mathbb{E}[V_n] &= V_0 \mathbb{E}\left[\prod_{i=1}^{n}(1+R_i)\right] \\
&= \prod_{i=1}^{n} \mathbb{E}[1+R_i] \quad \text{(by independence)} \\
&= \left(\mathbb{E}[1+R_i]\right)^n \\
&= (1+0.5)^n
\end{aligned}
$$

(g) Expected log wealth

For general $f \in (0,1)$, show that the expected log-wealth after n investments is

$$
\mathbb{E}[\ln(V_n)] = \ln(V_0) + \frac{n}{3}\left[\left(2+\frac{1}{f}\right)\ln(1+2f) - \left(\frac{1}{f}-1\right)\ln(1-f) - 1\right]
$$

Hint: integral of $\ln x$ is $x \ln x - x$

$$
\int \ln(1+fx)dx = (x+1/f)\ln(1+fx) - x + c
$$

$$
\begin{aligned}
\mathbb{E}[\ln(V_n)] &= \mathbb{E}\left[\ln\left(V_0 \prod_{i=1}^{n}(1+fR_i)\right)\right] \\
&= \mathbb{E}\left[\ln(V_0) + \sum_{i=1}^{n}\ln(1+fR_i)\right] \\
&= \ln(V_0) + n\mathbb{E}[\ln(1+fR)] \\
&= \ln(V_0) + n\int_{-1}^{2}\frac{1}{3}\ln(1+fx)dx \\
&= \ln(V_0) + \frac{n}{3}\left[\left(x+\frac{1}{f}\right)\ln(1+fx) - x\right]_{x=-1}^{2} \\
&= \ln(V_0) + \frac{n}{3}\left[\left(2+\frac{1}{f}\right)\ln(1+f2) - 2 - \left(-1+\frac{1}{f}\right)\ln(1-f) + 1\right] \\
&= \ln(V_0) + \frac{n}{3}\left[\left(2+\frac{1}{f}\right)\ln(1+2f) - \left(\frac{1}{f}-1\right)\ln(1-f) - 1\right]
\end{aligned}
$$

2021 Q2. Brownian Motion & Simulation

Consider the standard Brownian motion $\{W_t\}_{t\geq 0}$ and its running maximum $M_t = \max_{0\leq s\leq t}\{W_s\}$. For this question you will prove that the conditional distribution of the maximum given the final point of the Brownian motion (i.e. $M_1 \mid W_1$) follows a Rayleigh distribution, something we used in class.

(a) Joint PDF

Show that the joint PDF of M_1, W_1 is

$$f_{W_1,M_1}(x,y) = \frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}, \forall y > 0, x < y$$

(Hint: Differentiate the joint CDF $F_{W_1,M_1}(x,y)$ w.r.t. x and y, using the fact that $\mathbb{P}(W_1 \leq x, M_1 \leq y) = \mathbb{P}(W_1 \leq x) - \mathbb{P}(W_1 \leq x, M_1 > y)$, where the latter probability is given by the reflection principle.)

Let $\Phi()/\phi()$ be the standard normal CDF/PDF. The joint CDF of W_1, M_1 is given by:

$$\begin{aligned}
F_{W_1,M_1}(x,u) &= \mathbb{P}(W_1 \leq x, M_1 \leq y) \\
&= \mathbb{P}(W_1 \leq x) - \mathbb{P}(W_1 \leq x, M_1 > y) \\
&= \Phi(x) - \mathbb{P}(W_1 \geq 2y - x) \\
&= \Phi(x) - \Phi(-(2y-x)), \quad \forall y > 0, x < y
\end{aligned}$$

The joint PDF is

$$\begin{aligned}
f_{W_1,M_1}(x,u) &= \frac{d^2}{dxdy} F_{W_1,M_1}(x,u) \\
&= \frac{d^2}{dxdy}\left[\Phi(x) - \Phi(-(2y-x))\right] \\
&= \frac{d}{dx}[-\phi(-(2y-x))(-2)] \\
&= \frac{d}{dx}[2\phi(2y-x)] \\
&= \frac{d}{dx}\left[\frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}\right] \\
&= \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\} \frac{d}{dx}\left(-\frac{(2y-x)^2}{2}\right) \\
&= \frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}, \quad \forall y > 0, x < y
\end{aligned}$$

(b) Conditional PDF

Show that the conditional PDF of $M_1 \mid (W_1 = x)$ is given by

$$f_{M_1|W_1}(y \mid x) = 2(2y-x)e^{-2y(y-x)}, \forall y > 0, x < y$$

$$
\begin{aligned}
f_{M_1|W_1}(y \mid x) &= \frac{f_{W_1,M_1}(x,y)}{f_{W_1}(x)} \\
&= \frac{\frac{2(2y-x)}{\sqrt{2\pi}} \exp\left\{-\frac{(2y-x)^2}{2}\right\}}{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}} \\
&= 2(2y-x)\exp\left\{-\frac{(2y-x)^2 - x^2}{2}\right\} \\
&= 2(2y-x)\exp\left\{-\frac{4y^2 - 4yx + x^2 - x^2}{2}\right\} \\
&= 2(2y-x)\exp\{-2y(y-x)\}, \quad \forall y > (0 \vee x)
\end{aligned}
$$

(c) Conditional CDF

Show that the conditional CDF of $M_1 \mid (W_1 = x)$ is given by

$$
F_{M_1|W_1}(y \mid x) = 1 - e^{-2y(y-x)}, \forall x \in \mathbb{R}, y > \min\{0, x\}
$$

$$
\begin{aligned}
F_{M_1|W_1}(y \mid x) &= \int_{0 \vee x}^{y} 2(2u-x)e^{-2u(u-x)} du \\
&= \int_{0 \vee x}^{y} \left(-e^{-2u(u-x)}\right)' du \\
&= \left[-e^{-2u(u-x)}\right]_{u=(0\vee x)}^{y} \\
&= -e^{-2y(y-x)} + e^0, \quad \text{because } (0 \vee x)(x - (0 \vee x)) = 0 \\
&= 1 - e^{-2y(y-x)}, \quad y > (0 \vee x)
\end{aligned}
$$

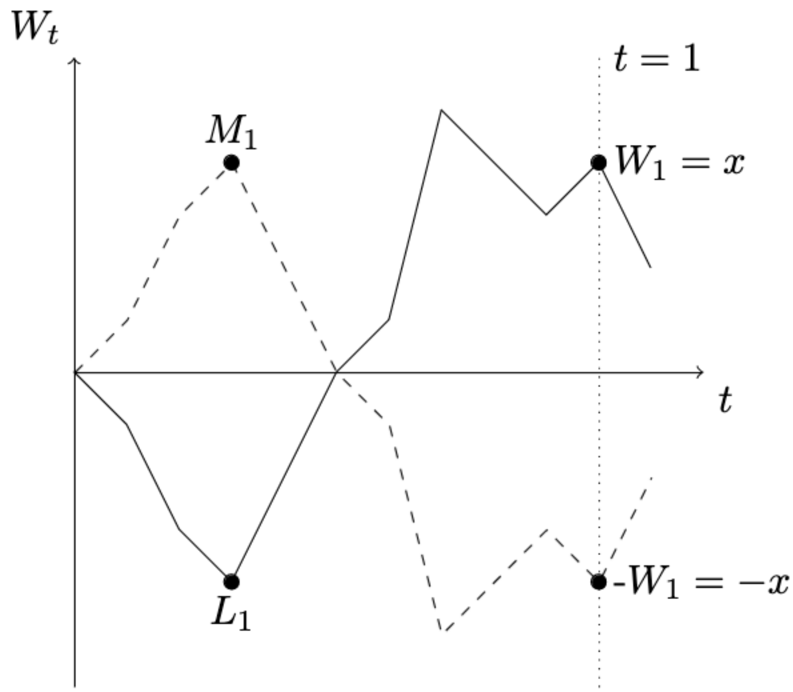
(d) Generate random values

Let $L_t = \min_{0 \le s \le t} \{W_t\}$ be the running minimum of the standard Brownian motion. Find a formula that simulates the conditional minimum $L_1 \mid (W_1 = x)$ of the standard Brownian motion given $W_1 = x \in \mathbb{R}$, based on a single Uniform $(0, 1)$ random number U.

(Hint: use the inverse conditional CDF of $M_1 \mid (W_1 = x)$ together with the symmetry of standard Brownian motion.)

From the symmetry of standard Brownian motion, we have that the minimum conditional on $W_1 = x$ has the same distribution as minus the maximum conditional on $W_1 = -x$, i.e. $L_1|(W_1 = x) \sim -M_1|(W_1 = -x)$

The following plot illustrates this:



We can simulate the conditional maximum M_1 given $W_1 = x$ using the inverse CDF method. Letting $M = M_1 \mid (W_1 = x)$, we have

$$\begin{aligned}
F_{M_1|(W_1=x)}(M) &\sim U \\
1 - e^{-2M(M-x)} &= U \\
-2M(M-x) &= \ln(1-U) \\
2M^2 - 2xM + \ln(1-U) &= 0 \\
\Rightarrow M &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(2)\ln(1-U)}}{2(2)} \\
M_1 \mid (W_1 = x) &= \frac{x + \sqrt{x^2 - 2\ln(1-U)}}{2} \qquad \text{(rejecting the negative root)}
\end{aligned}$$

Putting everything together, we get the minimum as:

$$\begin{aligned}
L_1 \,|(W_1 = x) = -M_1|\,(W_1 = -x) &= -\frac{-x + \sqrt{x^2 - 2\ln(1-U)}}{2} \\
&= \frac{x - \sqrt{x^2 - 2\ln(1-U)}}{2}
\end{aligned}$$