1. Forwards and Futures

Basics

Forward Contract vs Futures Contract

Forwards $\supset$ Futures

- Futures are traded on exchanges, forwards are over the counter (OTC)
- Futures have a clearing house, forwards are between 2 parties
- Futures markets are regulated, forward markets are self-regulated
- Futures are marked to market, forwards are marked to model

Margins

All futures holders (long and short) need to post margins, which are deposits that allow the exchange to reduce counterparty risk

- T-bills and stocks can be deposited in lieu of cash
  - T-bills - 90% of face value
  - Stocks - 50% of face value

2 types of margins

- Initial margin (IM) - initial deposit
- Maintenance margin (MM) - min balance required by the exchange
  - = $\mu + 3\sigma$ per contract, where μ is the avg change in the underlying's value
- IM = (1.1 to 1.2)MM

Margin call

- A contract holder receives a margin call when their account balance $\leq$ MM
- At which point they must top up to IM. The amount added is called the variation margin

Margins are adjusted frequently to reflect volatility and prices

(Extra) short sellers own a margin account which allows them to borrow money and invest (no voting right). The lender will still receive dividends (they won't even know that the broker has handed their stock over to the short seller). This dividend comes from the short seller, who pays a fee to the broker.

Forward Contracts on Assets without Cash Income

Cost of Carry Model

Carrying cost is the cost of buying at S today and carrying it (e.g. interest on tied up funds, storage cost, etc.)

The model measures the relationship between forward price F and spot price S.

Deriving the model:

Suppose we have 2 ways to buy gold, one has storage cost u, one does not:

	Time 0 (today)	Time T (delivery date)
Spot (storage)	Se^{uT}	S_T
	$\parallel$	$\parallel$
Forward	Fe^{-rT}	$F\ (\rightarrow S_T)$

Since at time T, both choices are equal, they must be equal at time t.
Rearranging $Se^{uT} = Fe^{-rT}$, and adding other exponents that apply to other assets gives us $\boxed{F = Se^{(r+u-q-y)T}}$,
where
r = risk free rate
u = storage and insurance
q = yield or return / year
y = convenience yield (benefit of holding)
T = time to maturity
(added to exponent if it's a carrying cost, and subtracted if it's a benefit)

In dollar terms, $\boxed{\text{F} = \text{S} + \text{carry cost - yield}}$

Cash and Carry Trade

If $F > Se^{(r+u-q-y)t}$, then sell futures, borrow cash to buy spot, carry to time T

For x units of the underlying (assume 1 contract = 1 unit), the cash flows are as follows:

Time 0 (today)	Time T (delivery date)
- sell futures at F (+0) - borrow cash (+$xSe^{(u-q)T}$) - buy asset at S (-$xSe^{(u-q)T}$)	- $xe^{(u-q)T}$ grows to x units (+xS_T) - sell asset at F (+$xF - xS_T$) - pay back loan & int (-$xSe^{(r+u-q-y)T}$)
Total cash flow = 0	Total cash flow = $xF - xSe^{(r+u-q-y)T}$

Reverse Cash and Carry Trade

If $F < Se^{(r+u-q-y)t}$, then buy futures, short spot and invest proceeds, return asset at time T

For x units of the underlying (assume 1 contract = 1 unit), the cash flows are as follows:

Time 0 (today)	Time T (delivery date)
- buy futures at $F = S_T$ (-0) - short asset at S (+$xSe^{(u-q)T}$) - invest proceeds (-$xSe^{(u-q)T}$)	- buy asset at F (-$xS_T + xS_T$) - cover short i.e. return asset (-xS_T) - get principal + int (+$xSe^{(r+u-q-y)T}$)
Total cash flow = 0	Total cash flow = $-xF + xSe^{(r+u-q-y)T}$

Note The above arbitrage arguments require that there is no transaction costs, bid ask spread, or difficulty with shorting.

Forward Contract Valuation

A forward contract will have a non-zero value f (= zero at time of transaction) since spot price S fluctuates.

Note

F: fixed price written in contract;

f: price after contract initiation; reset to 0 everyday

Derivation

Suppose for a long position with delivery price K, portfolio A contains 1 unit of the underlying (S), and portfolio B contains 1 forward (f) and cash ($Ke^{-rT} = S_0$)

- At maturity, portfolio B will have $K(= F = S_0e^{rT})$, which is used to purchase 1 unit of the underlying
  $\implies V_A = V_B$
- Since they are equal at maturity, they must be equal now (at time 0)
  $\implies f + Ke^{-rT} = S \implies \boxed{f = S - Ke^{-rT} = S - S_0}$

Forward Contracts on Assets with Cash Income

Cost of Carry Model

If the underlying pays dividends/coupons, then denoting PV(income) as I, we have $\boxed{F = (S - I)e^{rT}}$

Cash and Carry trade

If $F > (S - I)e^{rT}$,

- Today: short forward (+0), short T-bill/borrow S dollars (+S), buy asset(-S)
  - net CF = 0
- Maturity: sell asset/get forward proceeds (+F), pay back loan with interest (-Se^{rT}), get income from asset (Ie^{rT})
  - net CF = $F - (S - I)e^{rT}$

Reverse Cash and Carry trade

If $F < (S - I)e^{rT}$,

- Today: long forward (-0), short asset (+S), long T-bill/invest S dollars (-S)
  - net CF = 0
- Maturity: buy asset/cover short (-F), account for income rendered to broker ($-Ie^{rT}$), get int on investment (Se^{rT})
  - net CF = $(S - I)e^{rT} - F$

Forward Contract Valuation

$\boxed{f = S - I - Ke^{-rT}}$

Arbitrage If $f < S - I - Ke^{-rT} \iff f + I + Ke^{-rT} < S$,

- Today: short asset (+S), long forward (-f), long T-bill/invest money ($-I - Ke^{-rT}$)
  - net CF = $S - f - I - Ke^{-rT}$
  - note: in cash and carry arb, shorting/longing forward has a CF of 0, since f = 0 initially.
- Maturity: buy stock and cover short (-K), return with div ($-Ie^{rT}$), collect int on T=bill ()$Ie^{rT} + K$)
  - net CF = 0
  - CF from time 0 = $e^{rT}(S - f - I - Ke^{-rT})$

Stock Index Futures

Income stream of dividends is almost continuous, so we have $F = Se^{(r-q)T}$ and $f = Se^{-qT} - Ke^{-rT}$

Most indices are market portfolios (more or less), i.e. $\beta = 1$

Hedging with Index Futures

Depending on portfolio management style, beta may not = 1

Recall CAPM: $r_i = r_f + \beta_i(r_m - r_f) \implies \beta_p =$ portfolio beta = (portfolio return - risk free rate) / (market risk premium)

If $\beta_p = 1.2$, then the portfolio will move 1.2% when the market moves 1%

So if $\beta_p > 1$, need to hold more contracts to reduce risk/hedge, and vice versa

The number of index futures to short = $\boxed{\frac{V_p}{V_I}\beta_p}$ where V_I = value of an index futures contract = index value * multiplier

For a hedged portfolio, E(return) = risk free rate, so beta = 0

Using index futures, we can alter portfolio beta to any level (not necessarily 0)

- To **reduce** β_p to β_p^*, the number of index futures to **short** = $\boxed{\frac{V_p}{V_I}(\beta_p - \beta_p^*)}$

- To **increase** β_p to β_p^*, the number of index futures to **long** = $\boxed{\frac{V_p}{V_I}(\beta_p^* - \beta_p)}$

Forward Contracts on Currencies

$\boxed{F = Se^{(r-r_f)T}}$ Obtained by setting the yield/return $q =$ risk free rate, and convience yield = 0

$\boxed{f = Se^{-r_fT} - Ke^{-rT}}$

Derivation

	Time 0 (today)	Time T (delivery date)
Forward	$f + Ke^{-rT}$	$f + K \overset{buy}{\rightarrow} F = Se^{(r-r_f)T}$
	$\parallel$	$\parallel$
Spot	Se^{-r_fT}	$Se^{(r-r_f)T}$

Since both portfolios have the same value at time T, they must have the save value at time 0.

Detailed transactions for porfolio A:

- Today: long futures ($-f$), long T-bill ($-Ke^{-rT}$)
  - net CF = $-f - Ke^{-rT}$
- Maturity: risk free asset has grown ($+K$), honor futures ($f - f - K$), obtain asset ($+F = Se^{(r-r_f)T}$)
  - net CF = $F = Se^{(r-r_f)T}$

Predicting Future Spot Price

$\boxed{F = E(S_T)e^{(r-\mu)T}}$ where μ = expected return

This follows from the cost of carry model: if $E(S_T) = Se^{\mu T}$, then $F = Se^{rT} \implies S = Fe^{-rT}$

Derivation

- Today: long futures (-0), buy risk free asset ($-Fe^{-rT}$)
  - net CF = $-Fe^{-rT}$
- Maturity: risk free asset has grown ($+F$), use it to honor futures ($-F$), obtain asset ($+S_T$)

- net CF = $S_T = F = E(S_T)e^{(r-\mu)T}$

- Thus, Expected PV = CF_0 + (discounted CF_T) $\implies 0 = -Fe^{-rT} + E(S_T)e^{(-\mu)T} \implies Fe^{-rT} = E(S_T)e^{(-\mu)T}$

Implication (of formula above)

$r < \mu \iff F < E(S_T) \iff$ positive systematic risk

$r > \mu \iff F > E(S_T) \iff$ negative systematic risk

$r = 0 \iff F = E(S_T) \iff$ zero systematic risk

Suppose $F < E(S_T)$. Since F converges to S_T as the contract matures, investors who long futures expect to make a profit.

Forward vs Spot - Terminology

F>S <— Contango

F<S <— Backwardation

If comparing F to E(S_T), then add the word "normal" in front

Note: it is possible for a market to be contango and normal backwardation at the same time (i.e. E(S_T) > F > S)

Interest Rate Futures

Forward Rate Agreement (FRA)

A FRA is a forward contract on an interest rate for a specified future period, settled in cash.

The agreed forward rate $R_K = R_F$ = 0 at initiation, so the contract is worth 0 initially. As rates fluctuate, R_F will have a non-zero value.

If R_2 = interest rate for the next 2 years (2 year rate), R_1 = interest rate for the next year (1 year rate), and $T_2 - T_1 = x$
then the x-year forward rate, beginning at T_1, can be found by converting the **continuous compound rate** $\boxed{R = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}}$ to **annual compound rate** $\rightarrow$ $\boxed{R_F = e^R - 1}$

Contract Valuation

Suppose the contract is for $T_2 - T_1$, principal = L, agreed rate = R_K

	T_1	T_2
Contract	$-L$	$L + L \cdot R_K(T_2 - T_1)$
Forward	$-L$	$L + L \cdot R_F(T_2 - T_1)$

- For the lender, PV(contract) is

  $$V = \Big(L\big[1 + R_K(T_2 - T_1)\big] - L\big[1 + R_F(T_2 - T_1)\big]\Big)e^{-R_2 T_2} \implies \boxed{V = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}}$$

- For the borrower, PV(contract) is

  $$\boxed{V = L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2}}$$

Day Count Conventions

Int earned in period T = $\boxed{\text{(Int earned in reference period)}\textcolor{red}{\backslash\text{cp}}\frac{\#\text{ days in T}}{\#\text{days in ref period}}}$

- Treasury bonds: actual/actual
- Treasury bills: actual / 360
- Corporate/municip bonds: 30/360

Eurodollar/Interest Rate Futures (Short Term)

Eurodollar: U.S. dollar deposits made outside of U.S.

Eurodollar futures: futures contracts on the interest rate

CME Quotes:

- Quoted in dollars: if rate = 2.75%, then the quote would be 97.25

- **Front month**: 1/4 basis points = 0.000025 p.a. = 0.00000625 per quarter

  Since contract size = $1,000,000, minimum price tick = $6.25/contract

- **Other months**: 1/2 bp —> $12.5/contract

- **Quarterly compounding** is used by CME:

Let Q = quote, then price per contract = $10000\underbrace{\left(100 - \underbrace{0.25(100-Q)}_{\text{int over 3 month period}}\right)}_{\text{price of a 3 month "T-bill"}}$

- **Speculating**: if 3-month LIBOR is predicted to go down, then long Eurodollar futures (since quotes will go up)

  Hedging: if a loan is needed in 9 months, then short Eurodollar futures to hedge

Treasury Bond Futures (Long Term)

Structure

- Typically: T-note if maturity < 10 years, T-bond otherwise
- The standard bond is used as a benchmark for quoting purposes
  - face val = \$100k, coupon = 6%, maturity = 30 years
- Prices of T-bond futures are quoted the same way as T-bonds
  - are in the format XX-YY
  - $\boxed{\text{Quoted price} = \frac{\text{face value}}{100}(XX + \frac{YY}{32})}$
    - a.k.a. clean price (price assuming the current time to be coupon pmt time)

      Background: bond prices drop right after each coupon payment. This price drop has nothing to do with the interest rate movements. So, we remove the accrued interests from the dirty price to obtain the clean price. This way we are able to better track the impact of interest rate changes on bond prices.
  - $\boxed{\text{Cash price} = (\text{quoted price}) + (\text{int accrued since last coupon})}$
    - a.k.a. dirty price
- For the short position:
  - Since the quoted futures price is for the standard bond, we need to adjust the price via a conversion factor: $\boxed{\text{Cash received} = (\text{quoted future price})(\text{conversion factor}) + (\text{int accrued since last coupon})}$
  - There is a timing option, since the bond can be delivered any day of the maturity month

- There is cheapest to delivery option (a.k.a. quality option), since any T-bond with maturity > 15 years without call in the first 15 years can be delivered
    - To determine which bond is cheapest to deliver, calculate the contract value for each bond using the conversion factor, and compare that to its quote to see which one has the largest savings.

- Conversion factor: a ratio between market price(delivery bond) & market price(standard bond)

1. Calculate T - first round down to the nearest 3 months.

   If a multiple of half years, assume next coupon is in 6 months;

   o/w, assume next coupon is in 3 months (need to account for accrued interest in step 2)

2. Calculate bond price using coupon = 6% and T (in half years) from step 1: $\boxed{P_{bond} = \sum_{t=1}^{T} \frac{100(\frac{r}{2})}{1.03^t} + \frac{100}{1.03^T}}$

   If assumed next coupon is in 3 months in step 1, then add it to P_{bond}, and discount P_{bond} back for 3 months, and subtract the interest accrued in 3 months (half a coupon) i.e. $\boxed{\frac{P_{bond} + 100(\frac{r}{2})}{1.03^{1/2}} - 100(\frac{r}{4})}$

3. Divide by 100 to get conversion factor.

Optimal Hedge Ratio

Generally, # of futures to hedge = # of underlying assets.

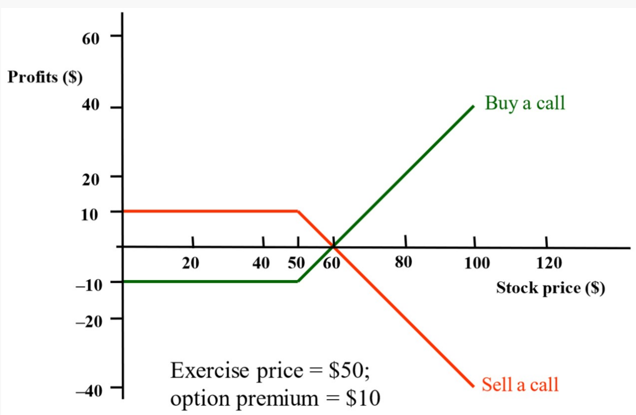
E.g. By entering into futures contracts to sell 200 tons of soybeans at $600/ton, we effectively lock into a fixed price of $600, in case the price crashes.

If for some reason (e.g. hedging horizon does not match futures maturity, or the futures' underlying / asset used to hedge is different from the hedged asset), then there exists an optimal hedge ratio (= units of futures to hold / units of underlying)

$$h = \frac{cov(\Delta S, \Delta F)}{var(\Delta F)} = \frac{\sigma_s \sigma_F \rho}{\sigma_F^2} = \frac{\sigma_s \rho}{\sigma_F}$$

2. Options and Trading Strategies

Basics

Call (right to buy in the future at X, if $X < S_T$)	Put (right to sell in the future at X, if $X > S_T$)
Long: strictly bullish **Short**: neutral/bearish	**Long:** strictly bearish **Short:** neutral/bullish
Long: unlimited profit, limited loss **Short**: unlimited loss, limited profit	**Long**: limited P&L **Short**: limited P&L
Long: payoff = spot - exercise **Short**: payoff = -(spot - exercise)	**Long**: payoff = exercise - spot **Short**: payoff = -(exercise - spot)
	

Notation

S for spot price (stock price), X for exercise price, T for time to maturity

C for call value, P for put value — uppercase if American, lowercase if European

Factors Affecting Option Prices

Factor	Call Price	Put Price
Higher S	increase	decrease
Higher X	decrease	increase

Factor	Call Price	Put Price
Longer maturity	increase if American may decrease if European (e.g. unable to exercise early to earn div)	increase if American may decrease if European (e.g. unable to exercise early to earn int)
Higher r	increase	decrease
Higher volatility	increase	increase

Margins

- If maturity < 9 months, options cannot be bought on margin (i.e. can't use borrowed money to buy)
- If maturity > 9 months, up to 25% margin is allowed (i.e. up to 25% of the investment can be sourced from a loan)
- Apply to short positions only:

  For a naked call, margin = $\boxed{\max(mc_1, mc_2)}$
  where mc_1 = option proceeds + 20% of the underlying share price - amt by which the option is OTM
  mc_2 = option proceeds + 10% of the underlying share price

  For a naked put, margin = $\boxed{\max(mp_1, mp_2)}$
  where mc_1 = option proceeds + 20% of the underlying share price - amt by which the option is OTM
  mc_2 = option proceeds + 10% of the exercise price

- For complex strategies, margin can vary (e.g. 0 for covered call writing)

Bounds for Option Prices

Bounds for Call Price

$\boxed{\max(0, S - Xe^{-rT}) \leq c \leq C \leq S}$

Proof WTS $S - Xe^{-rt} \leq c$

Portfolio	Value Now	Value at Maturity

Portfolio	**Value Now**	**Value at Maturity**
Call + T-bill with face value X, maturity T	$c + Xe^{-rT}$	if $X \leq S_T$, then $X - X + S_T$ (T-bill matures, pay X, get stock) if $X > S_T$, then X (T-bill matures, do nothing)
Stock	S	S_T either way

If exercise price < spot price, both portfolios have the same value. Otherwise, the first portfolio has a higher value.

Since $V_A \geq V_B$ at maturity, $V_A \geq V_B$ must also hold at present $\implies c + Xe^{-rT} \geq S \implies c \geq S - Xe^{-rT}$

Arbitrage If $c < S - Xe^{-rT}$, then short a stock, and buy a call + T-bill (obtain this by rearranging the inequality)

$V_0 = S - c - Xe^{-rT}$

$$V_T = \text{V(stock)} + \text{V(call and T-bill)} = \begin{cases} -S_T + (X - X + S_T) = 0 & X \leq S_T \\ -S_T + X & X > S_T \end{cases} \implies V_T \geq 0$$

Bounds for Put Price

American $\boxed{\max(0, X - S) \leq P \leq X}$

European $\boxed{\max(0, Xe^{-rT} - S) \leq p \leq Xe^{-rT}}$

Proof

WTS $p \leq Xe^{-rT}$

The put value is maxed if the stock price is zero at maturity.
In this case, the put will be worth $X - \underbrace{S_T}_{0} = X$ at maturity, which is Xe^{-rT} today.

WTS $Xe^{-rT} - S \leq p$

Portfolio	**Value Now**	**Value at Maturity**
Put + Stock	$p + S$	if $X \leq S_T$, then S_T (do nothing, keep stock) if $X > S_T$, then $X - S_T + S_T$ (exercise and sell stock)
T-bill with face value X, maturity T	Xe^{-rT}	X either way

If exercise price > spot price, both portfolios have the same value. Otherwise, the first portfolio has a higher value.

Since $V_A \geq V_B$ at maturity, $V_A \geq V_B$ must also hold at present $\implies p + S \geq Xe^{-rT} \implies p \geq Xe^{-rT} - S$

Arbitrage If $p < Xe^{-rT} - S$, then buy put + stock, and short T-bill (i.e. borrow PV of X)

$V_0 = p + S - Xe^{-rT}$

$$V_T = \text{V(T-bill)} + \text{V(put and stock)} = \begin{cases} -X + S_T & X \leq S_T \\ -X + (X - S_T + S_T) = 0 & X > S_T \end{cases}$$

Bounds for European Calls with Dividends

Let D = PV(div before maturity), then $\boxed{c \geq \max(0,\ S - Xe^{-rT} - D)}$

Proof

Portfolio	Value Now	Value at Maturity
Call + T-bill with face value $De^{rT} + X$	$c + D + Xe^{-rT}$	if $X > S_T$, then $De^{rT} + X$ (T-bill matures, do nothing) if $X \leq S_T$, then $De^{rT} + X - X + S_T$ (T-bill matures, pay X, get stock)
Stock	S	$S_T + De^{rT}$ either way

If exercise price < spot price, both portfolios have the same value. Otherwise, the first portfolio has a higher value.

Since $V_A \geq V_B$ at maturity, $V_A \geq V_B$ must also hold at present $\implies c + D + Xe^{-rT} \geq S \implies c \geq S - D - Xe^{-rT}$

Arbitrage If $c < S - Xe^{-rT} - D$, then short a stock, and buy a call + T-bill (face val = $De^{rT} + X$)

$V_0 = S - (c + Xe^{-rT} + D) > 0 \quad \leftarrow \text{the dif between LHS \& RHS of bounds eqn}$

$$V_T = \text{V(stock)} + \text{V(call and T-bill)} = \begin{cases} -(S_T + De^{rT}) + (De^{rT} + X - X + S_T) = 0 & X \leq S_T \\ -(S_T + De^{rT}) + (De^{rT} + X) = X - S_T & X > S_T \end{cases}$$

OR equivalently: short a stock, and buy a call + T-bill (face val = $S - c$)

$V_0 = S - (c + (S - c)) = 0$

$$V_T = \text{V(stock)} + \text{V(call and T-bill)} = \begin{cases} -(S_T + De^{rT}) + ((S - c)e^{rT} - X + S_T) > 0 & \leftarrow \text{the dif with int} & X \leq S_T \\ -(S_T + De^{rT}) + (S - c)e^{rT} > 0 & \leftarrow = X - S_T + \text{the dif with int} & X > S_T \end{cases}$$

Bounds for European Puts with Dividends

Let D = PV(div before maturity), then $\boxed{p \geq \max(0, Xe^{-rT} - S + D)}$

Proof

Portfolio	Value Now	Value at Maturity
Put + Stock	$p + S$	if $X \leq S_T$, then $S_T + De^{rT}$ (do nothing, keep stock and div) if $X > S_T$, then $X - S_T + S_T + De^{rT}$ (exercise and sell stock, keep div)
T-bill with face value $De^{rT} + X$	$D + Xe^{-rT}$	$De^{rT} + X$ either way

If exercise price > spot price, both portfolios have the same value. Otherwise, the first portfolio has a higher value.

Since $V_A \geq V_B$ at maturity, $V_A \geq V_B$ must also hold at present $\implies p + S \geq D + Xe^{-rT} \implies p \geq D + Xe^{-rT} - S$

Arbitrage If $p < Xe^{-rT} - S + D$, then short T-bill (face val = $De^{rT} + X$), and buy put + stock

$$V_0 = D + Xe^{-rT} - (p + S) > 0 \quad \leftarrow \text{the dif between LHS \& RHS of bounds eqn}$$
$$V_T = \text{V(T-bill)} + \text{V(put and stock)} = \begin{cases} -(De^{rT} + X) + (S_T + De^{rT}) = S_T - X > 0 & X \leq S_T \\ -(De^{rT} + X) + (X - S_T + S_T + De^{rT}) = 0 & X > S_T \end{cases}$$

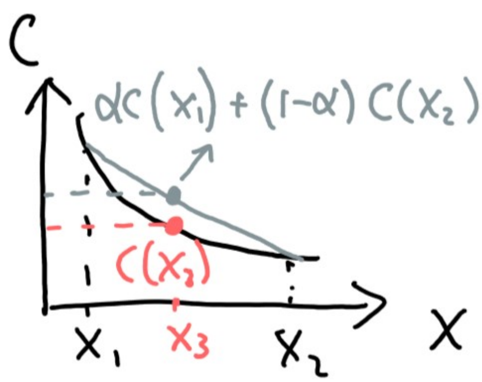
OR equivalently: short T-bill (face val = $(p + S)e^{rT}$), and buy put + stock

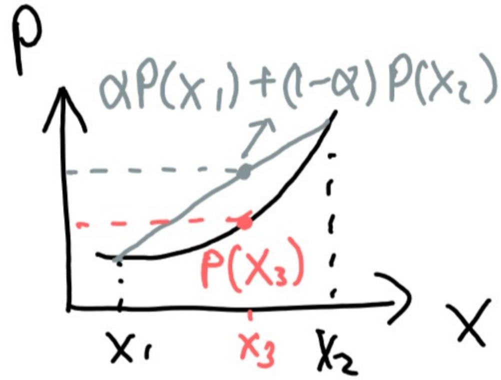
$$V_0 = p + S - (p + S) = 0$$
$$V_T = \text{V(T-bill)} + \text{V(put and stock)} = \begin{cases} -(p + S)e^{rT} + (S_T + De^{rT}) > 0 & \leftarrow = S_T - X + \text{the dif with int} & X \leq S_T \\ -(p + S)e^{rT} + (X - S_T + S_T + De^{rT}) > 0 & \leftarrow \text{the dif with int} & X > S_T \end{cases}$$

Convexity (Option on Index vs Many Individual Options)

Suppose that exercise price X_3 is a convex combination of X_1, X_2 i.e. $X_3 = \alpha X_1 + (1 - \alpha) X_2$ where $0 \leq \alpha \leq 1$

Then $\boxed{C(X_3) \leq \alpha C(X_1) + (1 - \alpha) C(X_2)}$ $\boxed{P(X_3) \leq \alpha P(X_1) + (1 - \alpha) P(X_2)}$ where C, c and P, p are interchangeable





Implication Having an option on a portfolio made of 2 stocks (pink) is worse off than having 2 options, one on each stock (grey). Higher risk/volatility = more chances to win.

Arbitrage If $C(X_3) > \alpha C(X_1) + (1-\alpha)C(X_2)$, short C_3, long α units of C_1 and $(1-\alpha)$ units of C_2

$$V_0 = C_3 - \alpha C_1 - (1-\alpha)C_2$$

$$V_T = V(c_3) + V(c_1) + V(c_2) = \begin{cases} 0+0+0=0 & S_T \leq X_1 \\ 0+\alpha(S_T - X_1) + 0 = \alpha(S_T - X_1) & X_1 < S_T \leq X_3 \\ -(S_T - X_3) + \alpha(S_T - X_1) + 0 = (1-\alpha)(X_2 - S_T) & X_3 < S_T \leq X_2 \\ -(S_T - X_3) + \alpha(S_T - X_1) + (1-\alpha)(S_T - X_2) = 0 & X_2 \leq S_T \end{cases}$$

Relative Pricing (Options with different X)

For European options, the difference between V(options with different X) $\leq$ PV(difference of X):

$$\boxed{c(X_1) - c(X_2) \leq (X_2 - X_1)e^{-rT}}$$
$$\boxed{p(X_2) - p(X_1) \leq (X_2 - X_1)e^{-rT}}$$

where $X_2 > X_1$

Arbitrage (call) If $c(X_1) - c(X_2) > (X_2 - X_1)e^{-rT} \iff c(X_1) + X_1 e^{-rT} > c(X_2) + X_2 e^{-rT}$, then short c_1 and T-bill with a face value of X_1, and long c_2 and T-bill with a face value of X_2

$$V_0 = c_1 + X_1 e^{-rT} - c_2 - X_2 e^{-rT}$$

$$V_T = V_1 + V_2 = \begin{cases} -X_1 + X_2 > 0 & S_T \leq X_1 \\ -S_T + X_2 > 0 & X_1 < S_1 \leq X_2 \\ -S_T + S_T = 0 & X_2 \leq S_T \end{cases} \implies V_T \geq 0$$

Arbitrage (put) If $p(X_2) - p(X_1) > (X_2 - X_1)e^{-rT} \iff p(X_2) + X_1 e^{-rT} > p(X_1) + X_2 e^{-rT}$, then short p_2 and T-bill with a face value of X_1, and long p_1 and T-bill with a face value of X_2

$$V_0 = p_2 + X_1 e^{-rT} - p_1 - X_2 e^{-rT}$$

$$V_T = V_1 + V_2 = \begin{cases} -S_T + S_T = 0 & S_T \leq X_1 \\ -X_1 + S_T > 0 & X_1 < S_1 \leq X_2 \\ -X_1 + X_2 > 0 & X_2 \leq S_T \end{cases} \implies V_T \geq 0$$

Put-Call Parity

For European options **without dividends** $\boxed{c + Xe^{-rT} = p + S}$

Proof

Portfolio	Value Now	Value at Maturity
Call + T-bill with face value X	$c + Xe^{-rT}$	if $X \leq S_T$, then $X - X + S_T$ (T-bill matures, pay X, get stock) if $X > S_T$, then X (T-bill matures, do nothing)
Put + Stock	$p + S$	if $X \leq S_T$, then S_T (do nothing, keep stock) if $X > S_T$, then $X - S_T + S_T$ (exercise and sell stock)

At maturity, the two portfolios have the same return.

Note Rearrange to get that $c - p = S - Xe^{-rT} =$ the minimum value of a call

For European options **with dividends** $\boxed{c + D + Xe^{-rT} = p + S}$

Proof

Portfolio	Value Now	Value at Maturity
Call + T-bill with face value $De^{rT} + X$	$c + D + Xe^{-rT}$	if $X > S_T$, then $De^{rT} + X$ (T-bill matures, do nothing) if $X \leq S_T$, then $De^{rT} + X - X + S_T$ (T-bill matures, pay X, get stock)
Put + Stock	$p + S$	if $X \leq S_T$, then $S_T + De^{rT}$ (do nothing, keep stock and div) if $X > S_T$, then $X - S_T + S_T + De^{rT}$ (exercise, sell stock, keep div)

At maturity, the two portfolios have the same return.

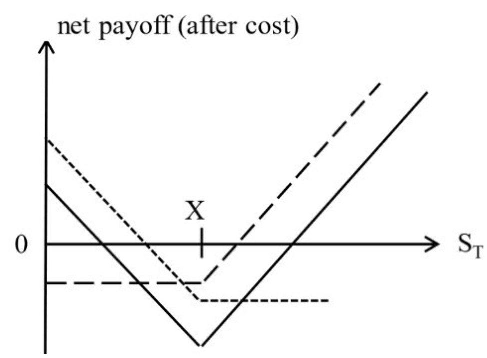
Early Exercise of American Options

	Call	Put

	Call	Put
No Dividends	Don't exercise early since - $S - X < S - Xe^{-rT}$ - the option acts as a shield against large losses - there is interest to be earned on the exercise price	May be exercised early (so $P \geq p$) since - $P \geq X - S$ from the bounds we derived - consider the case where a stock is worth 0, ex. now, get X and int; if ex. at time T, get X only
With Dividends	May be exercised early since price drop (D) on ex-div date (t_i) may > int on X i.e. if $D_i \geq X(1 - e^{-r(t_{i+1}-t_i)})$ then exercise	May be exercised early but unlikely since div payments would be lost

Trading Strategies

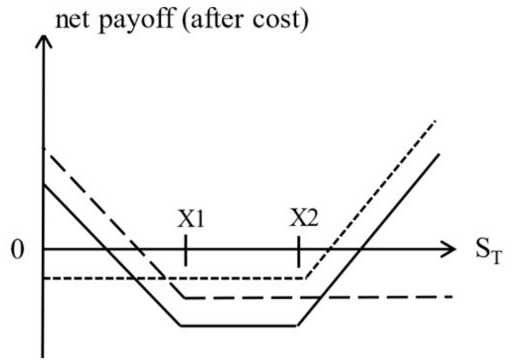
Straddle: 1C + 1P, same X



Use if market is expected to move a lot either way (sell if market is expected to be stable)

	payoff from call	payoff from put	total payoff	net profit
$S_T \leq X$	0	$X - S_T$	$X - S_T$	$X - S_T - c - p$
$S_T \geq X$	$S_T - X$	0	$S_T - X$	$S_T - X - c - p$

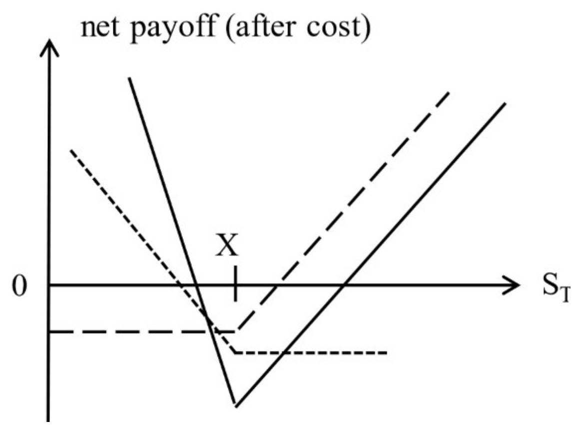
Strangle: 1C + 1P (low X)



Similar to a straddle, but less expensive since the call has a higher X and the put has a lower X (so less attractive).

	payoff from call	payoff from put	total payoff	net profit
$S_T \le X_1$	0	$X_1 - S_T$	$X_1 - S_T$	$X_1 - S_T - c - p$
$X_1 \le S_T \le X_2$	0	0	0	$-c-p$
$S_T \ge X_2$	$S_T - X_2$	0	$S_T - X_2$	$S_T - X_2 - c - p$

Strip: 1C + 2P, same X

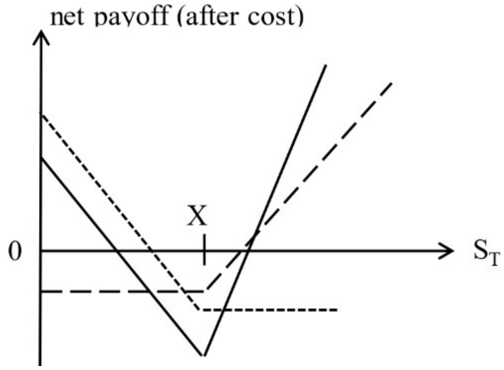


Use if market is expected to move either way but down is more likely

Note Can take a more aggressive position by choosing options with very high exercise prices. This guarantees profit even if the stock goes down very little.

	payoff from call	payoff from put	total payoff	net profit
$S_T \le X$	0	$2(X - S_T)$	$2(X - S_T)$	$2(X - S_T) - c - 2p$
$S_T \ge X$	$S_T - X$	0	$S_T - X$	$S_T - X - c - 2p$

Strap: 2C + 1P, same X

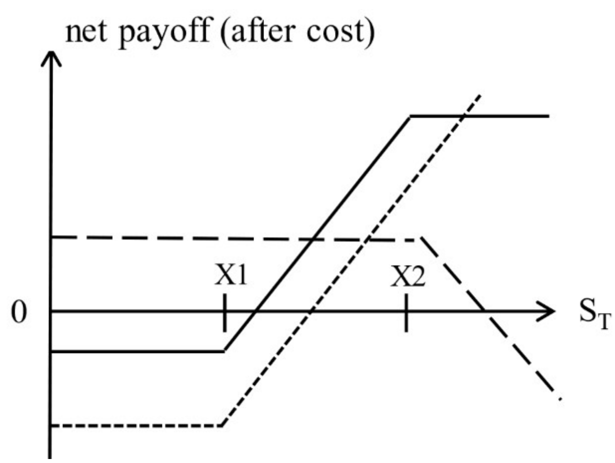


Use if market is expected to move either way but up is more likely

Note Can take a more aggressive position by choosing options with very low exercise prices. This guarantees profit even if the stock goes up very little.

	payoff from call	payoff from put	total payoff	net profit
$S_T \le X$	0	$X - S_T$	$X - S_T$	$X - S_T - 2c - p$
$S_T \ge X$	$2(S_T - X)$	0	$2(S_T - X)$	$2(S_T - X) - 2c - p$

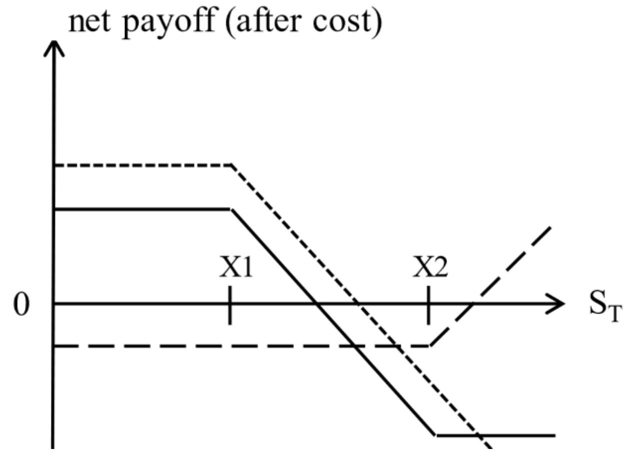
Bull spread: 1C - 1C (high X)



Use if market is expected to go up; less expensive than buying a single call; limits P&L

	payoff from call	payoff from put	total payoff	net profit
$S_T \le X_1$	0	0	0	$-c_1 + c_2$
$X_1 \le S_T \le X_2$	$S_T - X_1$	0	$S_T - X_1$	$S_T - X_1 - c_1 + c_2$
$S_T \ge X_2$	$S_T - X_1$	$-(S_T - X_2)$	$X_2 - X_1$	$X_2 - X_1 - c_1 + c_2$

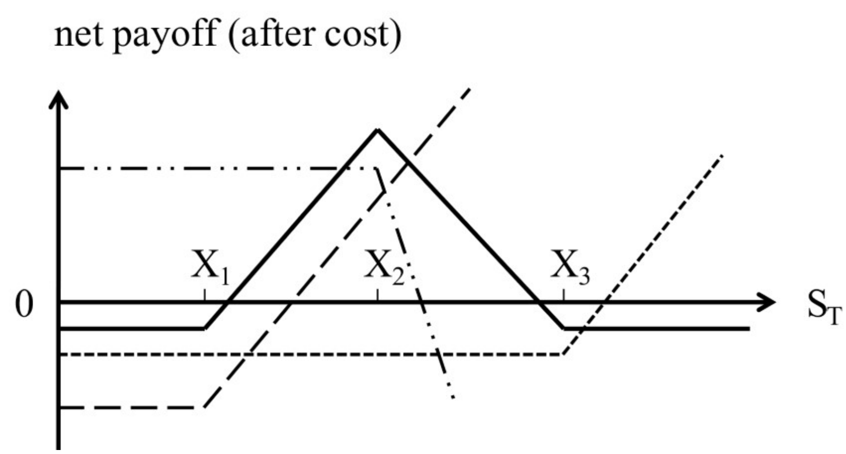
Bear spread: 1C - 1C (low X)



Use if market is expected to go down; less expensive than buying a single call; limits P&L

	payoff from call	payoff from put	total payoff	net profit
$S_T \leq X_1$	0	0	0	$c_1 - c_2$
$X_1 \leq S_T \leq X_2$	$-(S_T - X_1)$	0	$-(S_T - X_1)$	$-(S_T - X_1) + c_1 - c_2$
$S_T \geq X_2$	$-(S_T - X_1)$	$S_T - X_2$	$-(X_2 - X_1)$	$-(X_2 - X_1) + c_1 - c_2$

Butterfly: 1C (low X) + 1C (high X) - 2C (avg X)



Use if market is expected to be stable; different from selling straddle since loss is limited

Using Calls	payoff C_1	payoff $-2(C_2)$	payoff C_3	net profit
$S_T \leq X_1$	0	0	0	$-c_1 + 2c_2 - c_3$
$X_1 \leq S_T \leq X_2$	$S_T - X_1$	0	0	$-c_1 + 2c_2 - c_3 + S_T - X_1$
$X_2 \leq S_T \leq X_3$	$S_T - X_1$	$-2(S_T - X_2)$	0	$-c_1 + 2c_2 - c_3 + X_3 - S_T$
$S_T \geq X_3$	$S_T - X_1$	$-2(S_T - X_2)$	$S_T - X_3$	$-c_1 + 2c_2 - c_3$

Using Puts	payoff P_1	payoff $-2(P_2)$	payoff P_3	net profit
$S_T \leq X_1$	$X_1 - S_T$	$-2(X_2 - S_T)$	$X_3 - S_T$	$-p_1 + 2p_2 - p_3$
$X_1 \leq S_T \leq X_2$	0	$-2(X_2 - S_T)$	$X_3 - S_T$	$-p_1 + 2p_2 - p_3 + S_T - X_1$
$X_2 \leq S_T \leq X_3$	0	0	$X_3 - S_T$	$-p_1 + 2p_2 - p_3 + X_3 - S_T$
$S_T \geq X_3$	0	0	0	$-p_1 + 2p_2 - p_3$

Stock Replicate

C - P = long stock

P - C = short stock

Other spreads to know: box spread, calendar spread, diagonal spread, condor spread

Exotic & Innovative Options

Range Forward Contract

- A **zero cost** forward contract constructed using options; allows holder to buy (or sell) an asset at 2 possible prices

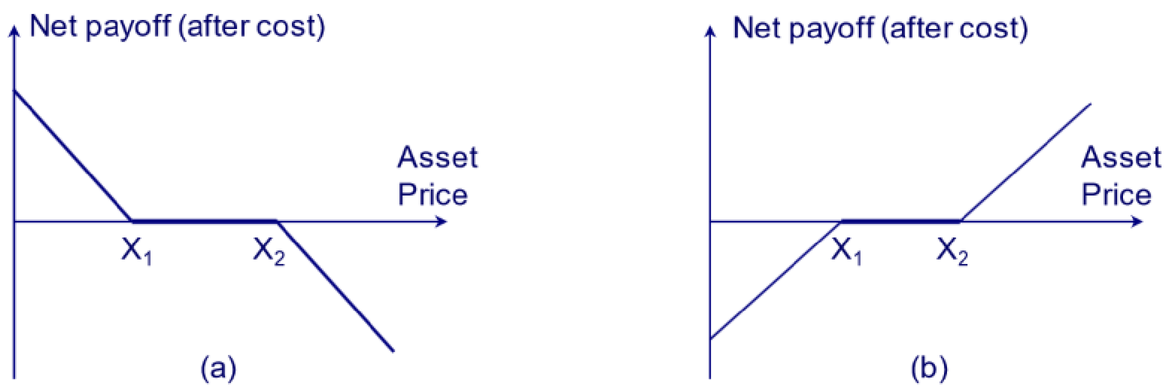


Figure 1 Payoffs from (a) short and (b) long range-forward contract

- Short range forward: **P(X_2) - C(X_1)** s.t. initial cost = 0

  if $S < X_1$, sell at X_1; if $S > X_2$, sell at X_2

- Long range forward: **C(X_2) - P(X_1)** s.t. initial cost = 0

  if $S < X_1$, buy at X_1; if $S > X_2$, buy at X_2

- It becomes a regular forward at $X_1 = X_2 = Se^{rT}$, (derive using put-call parity) since the call and the put would have the same value

Chooser Option

- Holder can choose whether to receive a call or a put at T_1
- Buy if direction is unknown, so it's similar to a straddle but holding cost is lower
- If both underlying options are European with the same X, use put-call parity:

  Let T_2 = maturity of the options, then at T_1 its value is
  $\max(c,p) = \max(c, c + Xe^{-r(T_2-T_1)-S_1}) = c + \max(0, Xe^{-r(T_2-T_1)-S_1})$

  Thus it is equivalent to a combo of $\begin{cases} C & (X, T_2) \\ P & (Xe^{-r(T_2-T_1)}, T_1) \end{cases}$

Barrier Option

- Option whose existence depends on whether the underlying's price reaches a barrier
- **Knock out** options:

  up and out – if price above barrier, option is knocked out

  down and out – if below barrier, option is knocked out

- **Knock in** options:

  up and in – if price above barrier, option comes into existence

  down and in – if price below barrier, option comes into existence

- The following are equivalent to a regular option:
  - A **combo** of the 2 (combine in and out, NOT up and down)
  - Up and out with barrier = ∞

    Down and out with barrier = 0
  - Up and in with barrier = current asset price

    Down and in with barrier = current asset price

Lookback Option

- Option with X = max or min stock price; normally for commodities
- Payoff at maturity = $\begin{cases} \max(0, S_T - S_{T_{\min}}) & \text{if call} \\ \max(0, S_{T_{\max}} - S_T) & \text{if put} \end{cases}$

Shout Option

- European option that gives the holder 1 chance to "shout" to lock in the exercise price

Asian Option

- Option whose payoff depends on the avg price of the underlying
- **Average price** option: payoff = $\begin{cases} \max(0, S_{\text{avg}} - X) & \text{if call} \\ \max(0, X - S_{\text{avg}}) & \text{if put} \end{cases}$

  Cheaper because volatility is lower

- **Average strike** option: payoff = $\begin{cases} \max(0, S_T - S_{\text{avg}}) & \text{if call} \\ \max(0, S_{\text{avg}} - S_T) & \text{if put} \end{cases}$

Exchange Option

- Option to give up an asset (V) in exchange for another (U)
- Payoff = $\max(0, U_T - V_T)$
- $E(X) = U \cdot N(d_1) - V \cdot N(d_2)$

  $d_1 = \dfrac{\ln(U/V) + \Phi^2 T/2}{\Phi\sqrt{T}},\ d_2 = d_1 - \Phi\sqrt{T} \qquad \Phi = \sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$

- If U = V, then it's worthless if $\phi = 0$ or $T = 0$; want to have high negative correlation.
- When U is high or V is low, E(X) = U - V since it is a certainty that the option will be exercised.

Weather Derivatives

- According to CME, 1/3 of the world economy is impacted by weather
- There are products on temperature, namely cumulative **HDD** (Heating Degree Days) or **CDD** (Cooling Degree Days)

  Daily HDD = max(0, 65°F - daily avg temp)

Daily CDD = max(0, daily avg temp - 65°F)

- Options: calculate normally using S_T = actual HDD or CDD
- Can hedge downside by shorting/buying futures

  If revenue drops by x per unit decrease in HDD, then short futures and choose tick size = x

  change in rev = $-$(avg HDD - actual HDD) x

  futures payoff = (avg HDD - actual HDD) x

  So (change in revenue) = - (futures payoff)

3. Binomial Trees

One Period Binomial Tree

1. Form riskless portfolio, i.e. $V_u = V_d$

$$\begin{aligned} V_u &= V_d \\ Call: S_u - wC_u &= S_d - wC_d \quad \leftarrow \text{covered call (buying a stock to protect call)} \\ Put: S_u + wP_u &= S_d + wP_d \quad \leftarrow \text{protective put (buying a put to protect stock)} \end{aligned}$$

2. Find w, plug it back in to find V

3. Solve for C or P

$$\begin{aligned} Call: S - wC_u &= Ve^{-rT} \\ Put: S + wP_u &= Ve^{-rT} \end{aligned}$$

4. Black Scholes

Modelling Stock Price

The **% change in stock price** over a short time interval is **normal**: $\frac{\Delta S}{S} \sim N\left(\mu \Delta t, \sigma\sqrt{\Delta t}\right)$

The **stock price** is **lognormal**, so its log is normally distributed: $\boxed{\ln S_T \sim N\left(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}\right)}$

Proof $\quad S_T = Se^{RT} \implies \ln S_T = \ln S + RT$. Since R is a normal r.v., $\ln S_T$ is also normal.

$\boxed{E(S_T) = S_0 e^{\mu T}, Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)}$

To estimate volatility using historical data:

1. Collect stock prices for period with length = maturity of the option (if maturity < 6 months, use 6 months of data minimum). Exclude ex-dividend days.
2. Calculate log returns (n+1 stock prices -> n return values): $u_i = \ln(S_i / S_{i-1})$
3. Calculate standard deviation: $s = \sqrt{\frac{\sum_{i=1}^{n}(u_1 - \bar{u})^2}{n-1}}$
4. Annualize to get $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$
5. The standard error is $\frac{\hat{\sigma}}{\sqrt{2n}}$

90% CI: $\mu \pm 1.6449\sigma$

95% CI: $\mu \pm 1.9600\sigma$

99% CI: $\mu \pm 2.5758\sigma$

Black Scholes Model

Assumptions

- log normal stock price
- no transaction costs or income within maturity
- continuous security trading
- investors can lend and borrow at a fixed risk free rate
- no arbitrage

$$\boxed{\begin{cases} c = SN(d_1) - Xe^{-rT}N(d_2) \\ p = Xe^{-rT}N(-d_2) - SN(-d_1) \end{cases}}$$ where $d_1 = \dfrac{\ln \dfrac{S}{X} + (r + \dfrac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

Options on Stocks with Dividends

European options replace S with adjusted stock price = $S - PV(D)$

American calls with 2 div approx value = max(c_1, c_2) where

c_1 = European call value (case where option to exercise early is not used)

c_2 = European call value with maturity just before the 2nd dividend (case where $D_2 > X\left(1 - e^{-r(T-t_2)}\right)$ so exercise at t_2)

If there is a large dividend at t_1, then also calculate:

c_3 = European call value with maturity just before the 1st dividend (case where $D_1 > X\left(1 - e^{-r(T-t_1)}\right)$ so exercise at t_1)

Options on Stock Indexes

Replace S with Se^{-qT}

Options on Currencies

Replace S with Se^{-r_fT}

Options on Futures

Replace S with Se^{-rT}

Call futures: right to enter into a long futures contract at a certain price X

Put futures: right to enter into a short futures contract at a certain price X

Portfolio Insurance

Covered Position

- write a call (want $X > S_T$), buy a share, and hold until maturity

- if $X < S_T$ at maturity, then the option will be exercised (sell the share and receive X + premium)

  if $X > S_T$, then the option won't be exercised (receive premium only), stuck with shares that have gone down in value

Stop-loss Strategy

- buy a share as soon as $S > X$, sell as soon as $S < X$
- covered when exercise is imminent, naked when exercise is not possible
- can be costly - frequent buy high $(X + \epsilon)$ sell low$(X - \epsilon)$

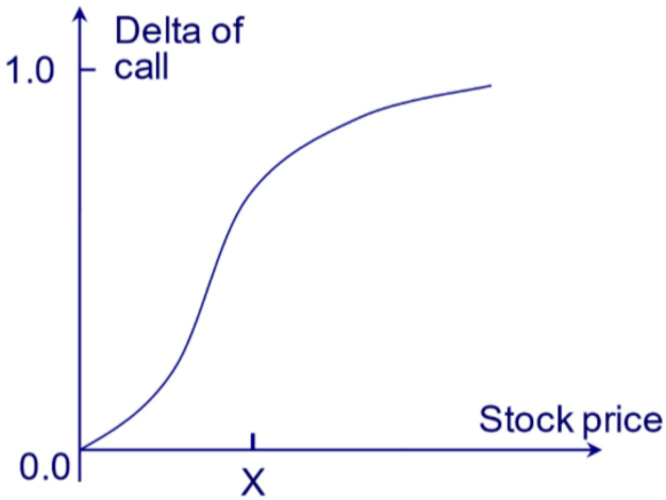
Insurance with Puts

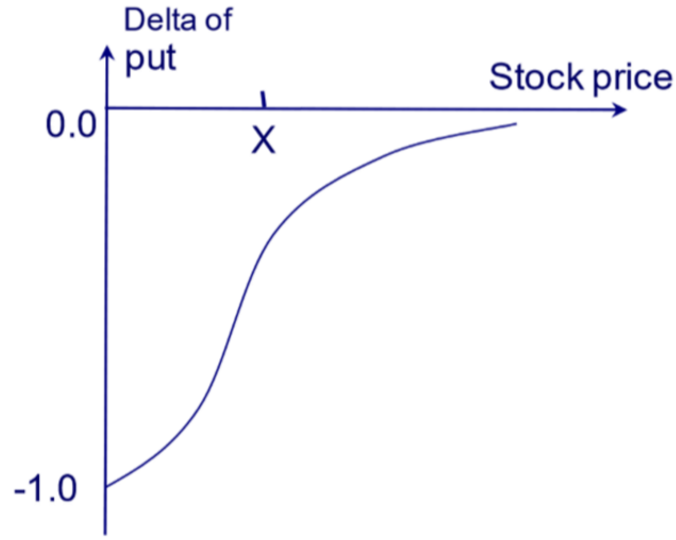
- insure a portfolio with puts

> **E.g.** $V = \$100M$, can't lose more than 10% $\implies$ floor = $90M
>
> - Insure the portfolio with puts on an index with level at $4000 = S$ with $X = 90\%(4000) = 3600$
>
>   Given r = 0.04, q = 0.025, σ = 0.25, so using Black Scholes, p = 186.23
>
>   Thus, total cost = (# contracts) (cost per contract) = $\frac{100M}{4000(1000)}(100(186.23)) = 4.7M$
>
> - Suppose there is a 20% drop, then $S_T = 80\%(4000) = 3200$, and $V = 80\%(100) = 80M$
>
>   Exercise puts: total payoff = $\frac{100M}{4000(1000)}(100(3600 - 3200)) = 10M$
>
>   V' = -4.7M + 80M + 10M = 85.3M

Delta Hedging

$$\Delta = \frac{\partial\,(\text{option value})}{\partial S}$$





For a call with no dividend, $\Delta = N(d_1) > 0$

For a call with dividends, $\Delta = e^{-qT}N(d_1) > 0$

For a put with no dividend, $\Delta = N(d_1) - 1 < 0$

For a put with dividends, $\Delta = e^{-qT}(N(d_1) - 1) < 0$

Insurance with Synthetic Puts

Let $\Theta = -\dfrac{\partial p}{\partial S} = -\Delta = \boxed{e^{-qT}(1 - N(d_1))}$. Then forming a portfolio $V = p + \Theta S \implies \dfrac{\partial V}{\partial S} = \underbrace{\dfrac{\partial p}{\partial S}}_{-\Theta} + \Theta = 0$

$-\Theta S$, i.e. shorting a fraction of S is like buying synthetic puts (since it's derivative = $-\Theta$, which is equal to $\dfrac{\partial P}{\partial S}$). It moves the same amount in the same direction as p.

Thus, to insure a portfolio, sell a portion of it and invest in T-bills.

Notes

- can also form a riskless portfolio with a call and $-\Delta$ shares: $V = c - \Delta S \implies \dfrac{\partial V}{\partial S} = \underbrace{\dfrac{\partial C}{\partial S}}_{\Delta} - \Delta = 0$
- requires continuous adjustments, thus there exists a trade-off between accuracy and transaction costs
- total hedging cost $\approx$ option premium

Index Futures

If the portfolio is worth K_1 times the index, and each futures contract is on K_2 times the index,

then the # of contracts to short is $\boxed{\frac{\Theta}{e^{(r-q)T^*}} \cdot \frac{K_1}{K_2}} = \boxed{e^{-q(T^*-T)} e^{-rT^*}(1 - N(d_1)) \cdot \frac{K_1}{K_2}}$

Derivation want $\Theta S = xF \implies x = \Theta \frac{S}{F} = \left(e^{-qT}(1 - N(d_1))\right) \frac{S}{Se^{(r-q)T^*}} = e^{-q(T^*-T)} e^{-rT^*}(1 - N(d_1))$

Note This is what caused the crash of 1987.

5. Applications

Equity and Debt as Options

Assume a firm only has debt and equity, all debt mature at the same time and have no coupon payments

Let F = face value of debt, D = market value of debt, E = market value of equity

V = market value of firm's assets = D + E

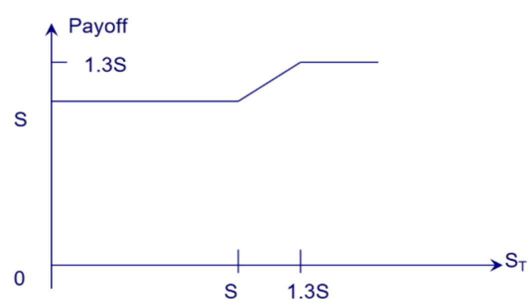
- $E_T = \begin{cases} V_T - F & V_T \geq F \\ 0 & V_T < F \end{cases}$ so $\quad E_T = \max(0, V_T - F) \quad \leftarrow \text{Euro call on V}$
- $D_T = \min(F, V_T) = F + \min(0, V_T - F) = F - \max(0, F - V_T) \quad \leftarrow \text{(T-bill with par = F) - (put with X = F)}$

$\implies D_T = Fe^{-rT} - p = V - E \quad \leftarrow \text{put call parity}$

Embedded Options in Investment Products

GIC: guaranteed investment certificate

- Index linked GIC: rate of return tied to a stock index (= S)
- Let $R = \dfrac{S_T}{S}$ denote the total return over the product's life
- Payoff = $\begin{cases} S & S_T < S \\ S_T & S \leq S_T < 1.3S \\ 1.3S & 1.3S \leq S_T \end{cases}$



- It is a combination of a (T-bill with face = S) and a bull spread (C with X = S) – (C with X = 1.3S)

Payoff at mat.	$S_T < S$	$S \leq S_T \leq 1.3S$	$S_T \geq 1.3S$

Payoff at mat.	$S_T < S$	$S \leq S_T \leq 1.3S$	$S_T \geq 1.3S$
T-bill	S	S	S
C with X=S		$S_t - S$	$S_T - S$
-C with X = 1.3S		0	$-(S_T - 1.3S)$
Total	S	S_T	$1.3S$