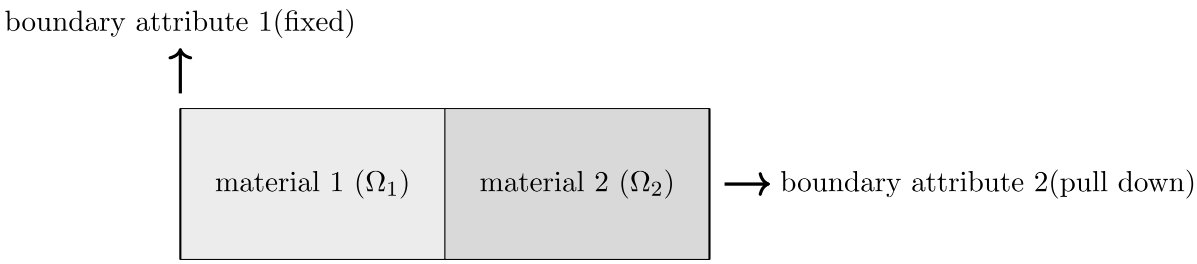
Two-Material Linear Elastic Cantilever



Governing Equations

We consider a linear elasticity problem on a domain $\Omega = \Omega_1 \cup \Omega_2$ consisting of two subdomains with different material properties.

Strong Form

Find displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that:

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{0} \qquad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda\,(\nabla \cdot \mathbf{u})\,\mathbf{I} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}}\right) \qquad \text{in each } \Omega_k. \tag{2}$$

The Lamé parameters are related to Young's modulus E and Poisson's ratio ν by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

Boundary Conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{f} \quad \text{on } \Gamma_N, \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_N).$$

Weak Formulation

Define the space of admissible test functions as

$$\mathbf{V} = \left\{ \mathbf{v} \in [H^1(\Omega)]^3 \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

The weak form reads: find $\mathbf{u} \in \mathbf{V}$ such that

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \nabla\mathbf{v}\, d\Omega = \int_{\Gamma_N} \mathbf{f} \cdot \mathbf{v}\, d\Gamma, \quad \forall \mathbf{v} \in \mathbf{V}.$$