ROB6323: Reinforcement learning and optimal control for autonomous systems I

Exercise series 2

For questions requesting a written answer, please provide a detailed explanation and typeset answers (e.g. using LaTeX[1]). Include plots where requested in the answers (or in a Jupyter notebook where relevant). For questions requesting a software implementation, please provide your code in runnable Jupyter Notebook. Include comments explaining how the functions work and how the code should be run if necessary. Code that does not run out of the box will be considered invalid.

Exercise 1 [25 points]

We would like to find a 2D point (x, y) as close as possible to the point $(1, 1)$ under the constraints that the sum $x + y$ is lower than 1, that the difference $y - x$ is lower than 1 and that the solution is inside a circle of radius 1 centered around the origin.

- Write the problem above as a minimization problem with constraints (hint: use a quadratic cost)
- Draw a geometric sketch of the problem showing the level sets of the function to minimize and the constraints
- Write the Lagrangian of the optimization problem
- Write the KKT necessary conditions for a point x^* to be optimal
- Find the minimum and the values of x and y that reach this minimum
- At the minimum, which constraints are active (if any) and what are their associated Lagrange multipliers?

Exercise 2 [25]

Consider the optimal control problem of Series 1 - Exercise 3 (control of a drone).

- Reusing the notebook of Series 1, write code to solve the same problem when the control is limited to $|5|$ for both rotors and the horizontal and vertical velocities are bounded to $2\text{m} \cdot \text{s}^{-1}$. To solve the resulting QP, use a solver (e.g. *cvxopt*) from the *qpsolvers* library[2]
- Show plots of all the states of the robot as a function of time
- Show plots of the optimal control as a function of time
- Compare the results with the results of Series 1 where no bounds were used.

[1] https://en.wikibooks.org/wiki/LaTeX, NYU provides access to Overleaf to all the community https://www.overleaf.com/edu/nyu

[2] `https://pypi.org/project/qpsolvers/`

Exercise 3 [25 points]

Implement in a Jupyter Notebook the gradient descent algorithm using a backtracking line search where $c_1 = 10^{-4}$ to minimize the following functions:

- $-\mathrm{e}^{-(x-1)^2}$, starting with $x_0 = 0$
- $(1-x)^2 + 100(y-x^2)^2$, starting with $x_0 = y_0 = 1.2$
- $x^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} -1 & 1 \end{bmatrix} x$, staring with $x_0 = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$
- $\frac{1}{2} x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$, staring with $x_0 = \begin{pmatrix} -10 \\ -10 \\ -10 \end{pmatrix}$

For each function:

- Show the convergence of the method by plotting the norm of the distance to the optimum as a function of the number of iterations
- Plot the value of α in the backtracking line search as a function of the number of iterations
- Print the optimum and explain which criteria you used to stop the algorithm

Exercise 4 [25 points]

We want to find the 0s of the following functions

- $x^2 + 3x - 1$, $x \in [-10, 10]$
- $x \sin(2x)$, $x \in [1, 10]$
- $2x^3 + 3x^2 + 2\sin(\cos(x)) - 3$, , $x \in [-1, 1]$
- $4x^4 + 5, x \in [-10, 10]$

In a Jupyter notebook, implement Newton's method to solve these problems. Evaluate the performance of the algorithm for various initial conditions.