PREDICATE CALCULUS

Conversion to Normal Form

Inference Rules

$\forall x \forall y \exists z$

- **Universal elimination:**

  $\forall$x Likes(x, IceCream) with the substitution {x / Einstein} gives us **Likes(Einstein, IceCream)**

  The substitution has to be done by a ground term

- **Existential elimination: (Skolemization)**

  From **$\exists$x Likes(x, IceCream)** we may infer **Likes(Man, IceCrea)** as long as Man does not appear elsewhere in the Knowledge base

- **Existential introduction:**

  From **Likes(Monalisa , IceCream)** we can infer

  $\exists$x Likes(x, IceCream)

Basic Steps

Resolution

- Convert the set of rules and facts into clause form (conjunction of clauses/disjunction of clauses)
- Insert the negation of the goal as another clause
- Use resolution to deduce a refutation

$A(x z) \wedge P(x, y) \wedge Q(y, z)$ DNF CNF

If a refutation is obtained, then the goal can be deduced from the set of facts and rules.

Conversion to Normal Form

- A formula is said to be in clause form if it is of the form: $\forall x1\ \forall x2 \ldots\ldots \forall xn[C1 \wedge C2 \wedge \ldots\ldots \wedge Ck]$
- All first order logic formulas can be converted to clause form
- We shall demonstrate the conversion on the formula: $\forall x\ \{p(x) \rightarrow \exists z\ \{\ \neg\forall y\ [q(x,y) \rightarrow p(f(x1))] \wedge \forall y\ [q(x,y) \rightarrow p(x)]\ \}\}$

Conversion to Normal Form

- Step1:Take the existential closure and eliminate redundant quantifiers. This introduces ∃x1 and eliminates and ∃z, so:

  $\forall x \{p(x) \rightarrow \exists z \{ \neg\forall y\, [q(x,y) \rightarrow p(f(x1))] \wedge \forall y\, [q(x,y) \rightarrow p(x)] \}\}$

  $\exists x1\ \forall x \{p(x) \rightarrow \{ \neg\forall y\, [q(x,y) \rightarrow p(f(x1))] \wedge \forall y\, [q(x,y) \rightarrow p(x)] \}\}$

Conversion to Normal Form

- Step2 : Rename any variable that is quantified more than once. y has been quantified twice, so:

$\exists x1\ \forall x\ \{p(x) \rightarrow \{\ \neg\forall y\ [q(x,y) \rightarrow p(f(x1))]\ \wedge \forall \underline{y}\ [q(x,\underline{y}) \rightarrow p(x)]\ \}\}$

$\exists x1\ \forall x\ \{p(x) \rightarrow \{\ \neg\forall y\ [q(x,y) \rightarrow p(f(x1))]\ \wedge \underline{\forall z}\ [\underline{q(x,z)} \rightarrow \underline{p(x)}]\ \}\}$

Conversion to Normal Form

$A \to B$
$\neg A \vee B$

- **Step3 :** Eliminate Implication.

$$\exists x1\ \forall x\ \{p(x) \to \{\neg \forall y\ [q(x,y) \to p(f(x1))] \wedge \forall z\ [q(x,z) \to p(x)]\}\}$$

$$\exists x1\ \forall x\ \{\neg p(x) \vee \{\neg \forall y\ [\neg q(x,y) \vee p(f(x1))] \wedge \forall z\ [\neg q(x,z) \vee p(x)]\}\}$$

$\neg(A \vee B) = \neg A \wedge \neg B$

$\neg(A \wedge B) = \neg A \vee \neg B$

$\neg\forall x = \exists x$

$\neg\exists x = \forall x$

Conversion to Clausal Form

- Step4 : Move the negation all the way inwards

$$\exists x1\ \forall x\ \{\neg p(x) \vee \{\neg\forall y\ [\neg q(x,y) \vee p(f(x1))] \wedge \forall z\ [\neg q(x,z) \vee p(x)]\}\}$$

$$\exists x1\ \forall x\ \{\neg p(x) \vee \{\exists y\ [q(x,y) \wedge \neg p(f(x1))] \wedge \forall z\ [\neg q(x,z) \vee p(x)]\}\}$$

Conversion to Clausal Form

- Step5 : Push the quantifiers to the right

$$\exists x1\ \forall x\ \{\neg p(x) \vee \{\ \exists y\ [q(x,y) \wedge \neg p(f(x1))] \wedge \forall z\ [\neg q(x,z) \vee p(x)]\ \}\}$$

$$\exists x1\ \forall x\{\neg p(x) \vee \{[\exists y\ q(x,y) \wedge \neg p(f(x1))] \wedge [\forall z \neg q(x,z) \vee p(x)]\ \}\}$$

Conversion to Normal Form

- Step6 : Eliminate Existential Quantifiers (Solemnization)
  - Pick out the left most $\exists yB(y)$ and replace it with $B(f(x_{i1},x_{i2},x_{i3}\ldots.. x_{in}))$ where:

1. $x_{i1},x_{i2},x_{i3}\ldots.. x_{in}$ are all the free variables of $\exists yB(y)$ that are universally quantified to the left of $\exists yB(y)$ and,

  - f is any n-ary function constant which does not occur already

Conversion to Normal Form

- Skolemization

$\exists x1\ \forall x\{\neg p(x) \vee \{[\exists y\ q(x,y) \wedge \neg p(f(x1))] \wedge [\forall z \neg q(x,z) \vee p(x)]\}\}$

$\forall x\ \{\neg p(x) \vee \{[q(x,g(x)) \wedge \neg p((f(a))] \wedge [\forall z \neg q(x,z) \vee p(x)]\}\}$

Conversion to Normal Form

- Step7 : Move all universal quantifiers to the left

$$\underline{\forall x}\ \{\neg p(x) \lor \{[q(x,g(x)) \land \neg p((f(a))] \land [\underline{\forall z}\neg q(x,z) \lor p(x)]\}\}$$

$$\forall x\ \forall z\ \{\neg p(x) \lor \{[q(x,g(x)) \land \neg p((f(a))] \land [\neg q(x,z) \lor p(x)]\}\}$$

Conversion to Normal Form

- Step8 : Distribute ∧ over ∨

$$\forall x\ \forall z\ \{\neg p(x) \vee \{[q(x,g(x)) \wedge \neg p((f(a))] \wedge [\neg q(x,z) \vee p(x)]\}\}$$

T_1

$$\forall x\ \forall z\ \{[\neg p(x) \vee q(x,g(x))] \wedge$$

T_2

$$[\neg p(x) \vee \neg p((f(a))] \wedge$$

T_3

$$[\neg p(x) \vee \neg q(x,z) \vee p(x)]\}$$

CNF

Conversion to Normal Form

- Step9 : Simplify (Optional)

$$\forall x\ \forall z\ \{[\neg p(x) \vee q(x,g(x))] \wedge [\neg p(x) \vee \neg p((f(a))] \wedge [\neg p(x) \vee \neg q(x,z) \vee p(x)]\ \}$$

$$\forall x\ \forall z\ \{[\neg p(x) \vee q(x,g(x))] \wedge [\neg p(x) \vee \neg p((f(a))] \wedge [\neg q(x,z)]\ \}$$

Summarize

- **Step1**:Take the existential closure and eliminate redundant quantifiers and introduce existential closure to unhandled variables.
- **Step2:** Rename any variable that is quantified more than once.
- **Step3:** Eliminate Implication.
- **Step4:** Move the negation all the way inwards.
- **Step5:** Push the quantifiers to the right.
- **Step6:** Eliminate Existential Quantifiers (Solemnization)
- **Step7:** Move all universal quantifiers to the left.
- **Step8:** Distribute $\wedge$ over $\vee$
- **Step9:** Simplify (Optional)

Exercise

- $\forall x\ \forall y\ \forall z\ Gaul(x) \wedge Potion(y) \wedge Hostile(z) \wedge Sells(x,y,z) \rightarrow Criminal(x)$
- $\forall x\ \exists y\ married(x,y) \wedge member(x) \rightarrow member(y)$
- $\forall x\ \forall y\ \exists z\ \forall y\ \exists a\ father(x,z) \wedge father(z,y) \rightarrow grandfather(x,y)$
- $\neg\forall x\ \exists z\ bird(x) \rightarrow fly(x)$

Step 1:eliminate redundant introduce existential to unhandled variables

- $\forall x \forall y \forall z$ Gaul(x) $\wedge$ Potion(y) $\wedge$ Hostile(z) $\wedge$ Sells(x,y,z) $\rightarrow$ Criminal(x)
- $\forall x\ \exists y$ married(x,y) $\wedge$ member(x) $\rightarrow$ member(y)
- $\forall x\ \exists z$ father(x,z) $\wedge$ $\forall y$father(z,y) $\rightarrow$ grandfather(x,y)
- $\neg \forall x$ bird(x) $\rightarrow$ fly(x)

Step2: Rename any variable that is quantified more than once

- $\forall x\ \forall y\ \forall z\ (Gaul(x) \wedge Potion(y) \wedge Hostile(z) \wedge Sells(x,y,z)) \rightarrow Criminal(x)$
- $\forall x\ \exists y\ (married(x,y) \wedge member(x)) \rightarrow member(y)$
- $\forall x\ \forall y\ \exists z\ (father(x,z) \wedge father(z,y)) \rightarrow grandfather(x,y)$
- $\neg\forall x\ bird(x) \rightarrow fly(x)$

$P \rightarrow Q$

$\neg P \vee Q$

Step3: Eliminate Implication

- $\forall x\ \forall y\ \forall z\ \{\neg\{Gaul(x) \wedge Potion(y) \wedge Hostile(z) \wedge Sells(x,y,z)\} \vee Criminal(x)\}$
- $\forall x\ \exists y\ \{\neg\ \{married(x,y) \wedge member(x)\} \vee member(y)\}$
- $\forall x\ \forall y\ \exists z\ \{\neg\ \{father(x,z) \wedge father(z,y)\} \vee grandfather(x,y)\}$
- $\neg\forall x\ \{\neg\ bird(x) \vee fly(x)\}$

Step4: Move the negation all the way inwards

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \vee \neg Potion(y) \vee \neg Hostile(z) \vee \neg Sells(x,y,z) \vee Criminal(x)\}$
- $\forall x\ \exists y\ \{\neg married(x,y) \vee \neg member(x) \vee member(y)\}$
- $\forall x\ \forall y\ \exists z\ \{\neg father(x,z) \vee \neg father(z,y) \vee grandfather(x,y)\}$
- $\exists x\ \{bird(x) \vee \neg fly(x)\}$

Step5: Push the quantifiers to the right

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \lor \neg Potion(y) \lor \neg Hostile(z) \lor \neg Sells(x,y,z) \lor Criminal(x)\}$
- $\forall x\ \exists y\ \{\neg married(x,y) \lor \neg member(x) \lor member(y)\}$

  $f(x)$
- $\forall x\ \forall y\ \exists z\ \{\neg father(x,z) \lor \neg father(z,y) \lor grandfather(x,y)\}$

  $f(x,y)$
- $\exists x\ \{bird(x) \lor \neg fly(x)\}$

Step6: Eliminate Existential Quantifiers

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \vee \neg Potion(y) \vee \neg Hostile(z) \vee \neg Sells(x,y,z) \vee Criminal(x)\}$
- $\forall x\ \{\neg married(x,f(x)) \vee \neg member(x) \vee member(y)\}$
- $\forall x\ \forall y\ \{\neg father(x,(g(x)) \vee \neg father((g(x),y) \vee grandfather(x,y)\}$
- $bird(M) \vee \neg fly(M)$

Step7: Move all universal quantifiers to the left

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \vee \neg Potion(y) \vee \neg Hostile(z) \vee \neg Sells(x,y,z) \vee Criminal(x)\}$
- $\forall x\ \{\neg married(x,f(x)) \vee \neg member(x) \vee member(y)\}$
- $\forall x\ \forall y\ \ \{\neg father(x,(g(x)) \vee \neg father((g(x),y) \vee grandfather(x,y)\}$
- $bird(M) \vee \neg fly(M)$

Step8: Distribute $\wedge$ over $\vee$

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \vee \neg Potion(y) \vee \neg Hostile(z) \vee \neg Sells(x,y,z) \vee Criminal(x)\}$
- $\forall x\ \{\neg married(x,f(x)) \vee \neg member(x) \vee member(y)\}$
- $\forall x\ \forall y\ \{\neg father(x,(g(x)) \vee \neg father((g(x),y) \vee grandfather(x,y)\}$
- $bird(M) \vee \neg fly(M)$

Step9: Simplify (Optional)

DNF

- $\forall x\ \forall y\ \forall z\ \{\neg Gaul(x) \vee \neg Potion(y) \vee \neg Hostile(z) \vee \neg Sells(x,y,z) \vee Criminal(x)\}$
- $\forall x\ \{\neg married(x,f(x)) \vee \neg member(x) \vee member(y)\}$
- $\forall x\ \forall y\ \ \{\neg father(x,(g(x)) \vee \neg father((g(x),y) \vee grandfather(x,y)\}$
- $bird(M) \vee \neg fly(M)$