KREATRYX

KNotes

ENGINEERING MATHEMATICS

Manual for K-Notes

Why K-Notes?

Towards the end of preparation, a student has lost the time to revise all the chapters from his / her class notes / standard text books. This is the reason why K-Notes is specifically intended for Quick Revision and should not be considered as comprehensive study material.

What are K-Notes?

A 40 page or less notebook for each subject which contains all concepts covered in GATE Curriculum in a concise manner to aid a student in final stages of his/her preparation. It is highly useful for both the students as well as working professionals who are preparing for GATE as it comes handy while traveling long distances.

When do I start using K-Notes?

It is highly recommended to use K-Notes in the last 2 months before GATE Exam (November end onwards).

How do I use K-Notes?

Once you finish the entire K-Notes for a particular subject, you should practice the respective Subject Test / Mixed Question Bag containing questions from all the Chapters to make best use of it.

LINEAR ALGEBRA

MATRICES

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or function) are called entries of elements of the matrix.

Example: $\begin{bmatrix} 2 & 0.4 & 8 \\ 5 & -32 & 0 \end{bmatrix}$ order = 2 x 3, 2 = no. of rows, 3 = no. of columns

Special Type of Matrices

1. Square Matrix

A m x n matrix is called as a square matrix if m = n i.e, no of rows = no. of columns

The elements a_{ij} when i = j $\left(a_{11}a_{22}.........\right)$ are called diagonal elements

Example: $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

2. Diagonal Matrix

A square matrix in which all non-diagonal elements are zero and diagonal elements may or may not be zero.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Properties

a. diag [x, y, z] + diag [p, q, r] = diag [x + p, y + q, z + r]

b. diag [x, y, z] × diag [p, q, r] = diag [xp, yq, zr]

c. $\left(\text{diag}\left[x, y, z\right]\right)^{-1} = \text{diag}\left[1/x, 1/y, 1/z\right]$

d. $\left(\text{diag}\left[x, y, z\right]\right)^{t}$ = diag [x, y, z]

e. $\left(\text{diag}\left[x, y, z\right]\right)^{n} = \text{diag}\left[x^n, y^n, z^n\right]$

f. Eigen value of diag [x, y, z] = x, y & z

g. Determinant of diag [x, y, z] = xyz

3. Scalar Matrix

A diagonal matrix in which all diagonal elements are equal.

4. Identity Matrix

A diagonal matrix whose all diagonal elements are 1. Denoted by I

Properties

a. AI = IA = A
b. $I^n = I$
c. $I^{-1} = I$
d. det(I) = 1

5. Null matrix

An m x n matrix whose all elements are zero. Denoted by O.

Properties:

a. A + O = O + A = A
b. A + (- A) = O

6. Upper Triangular Matrix

A square matrix whose lower off diagonal elements are zero.

Example: $\begin{bmatrix} 3 & 4 & 5 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{bmatrix}$

7. Lower Triangular Matrix

A square matrix whose upper off diagonal elements are zero.

Example: $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 6 & 0 \\ 5 & 7 & 9 \end{bmatrix}$

8. Idempotent Matrix

A matrix is called Idempotent if $A^2 = A$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. Involutary Matrix

A matrix is called Involutary if $A^2 = I$.

Matrix Equality

Two matrices $[A]_{m\times n}$ and $[B]_{p \times q}$ are equal if

m = p ; n = q i.e., both have same size

$a_{ij} = b_{ij}$ for all values of i & j.

Addition of Matrices

For addition to be performed, the size of both matrices should be same.

If [C] = [A] + [B]

Then $c_{ij} = a_{ij} + b_{ij}$

i.e., elements in same position in the two matrices are added.

Subtraction of Matrices

[C] = [A] – [B]

= [A] + [–B]

Difference is obtained by subtraction of all elements of B from elements of A.

Hence here also, same size matrices should be there.

Scalar Multiplication

The product of any m × n matrix A $\left[a_{jk}\right]$ and any scalar c, written as cA, is the m × n matrix cA = $\left[ca_{jk}\right]$ obtained by multiplying each entry in A by c.

Multiplication of two matrices

Let $[A]_{m \times n}$ and $[B]_{p \times q}$ be two matrices and C = AB, then for multiplication, [n = p] should hold. Then,

$$C_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

Properties

- If AB exists then BA does not necessarily exists.

  Example: $[A]_{3 \times 4}$, $[B]_{4 \times 5}$, then AB exits but BA does not exists as 5 ≠ 3

  So, matrix multiplication is not commutative.

- Matrix multiplication is not associative.

  A(BC) ≠ (AB)C .
- Matrix Multiplication is distributive with respect to matrix addition

  A(B + C) = AB +AC
- If AB = AC $\Rightarrow$ B = C (if A is non-singular)

  BA = CA $\Rightarrow$ B = C (if A is non-singular)

Transpose of a matrix

If we interchange the rows by columns of a matrix and vice versa we obtain transpose of a matrix.

eg., A = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

Conjugate of a matrix

The matrix obtained by replacing each element of matrix by its complex conjugate.

Properties

a. $\overline{\left(\overline{A}\right)} = A$

b. $\left(\overline{A+B}\right) = \overline{A} + \overline{B}$

c. $\left(\overline{KA}\right) = \overline{K}\,\overline{A}$

d. $\left(\overline{AB}\right) = \overline{A}\,\overline{B}$

Transposed conjugate of a matrix

The transpose of conjugate of a matrix is called transposed conjugate. It is represented by A^θ.

a. $\left(A^\theta\right)^\theta = A$

b. $(A+B)^\theta = A^\theta + B^\theta$

c. $(KA)^\theta = \overline{K}A^\theta$

d. $(AB)^\theta = B^\theta A^\theta$

Trace of matrix

Trace of a matrix is sum of all diagonal elements of the matrix.

Classification of real Matrix

a. Symmetric Matrix : $(A)^T = A$
b. Skew symmetric matrix : $(A)^T = -A$
c. Orthogonal Matrix : $\left(A^T = A^{-1};\ \ AA^T = I\right)$

Note:

a. If A & B are symmetric, then (A + B) & (A – B) are also symmetric
b. For any matrix AA^T is always symmetric.
c. For any matrix, $\left(\frac{A + A^T}{2}\right)$ is symmetric & $\left(\frac{A - A^T}{2}\right)$ is skew symmetric.
d. For orthogonal matrices, $|A| = \pm 1$

Classification of complex Matrices

a. Hermitian matrix : $\left(A^\theta = A\right)$
b. Skew – Hermitian matrix : $A^\theta = -A$
c. Unitary Matrix : $\left(A^\theta = A^{-1}; AA^\theta = 1\right)$

Determinants

Determinants are only defined for square matrices.
For a 2 × 2 matrix

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Minors & co-factor

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of element a_{21} : $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Co-factor of an element $a_{ij} = (-1)^{i+j} M_{ij}$

To design cofactor matrix, we replace each element by its co-factor

$\Rightarrow$ Determinant

Suppose, we need to calculate a 3 × 3 determinant

$$\Delta = \sum_{j=1}^{3} a_{1j}\text{cof}\left(a_{1j}\right) = \sum_{j=1}^{3} a_{2j}\text{cof}\left(a_{2j}\right) = \sum_{j=1}^{3} a_{3j}\text{cof}\left(a_{3j}\right)$$

We can calculate determinant along any row of the matrix.

Properties

- Value of determinant is invariant under row & column interchange i.e., $\left|A^T\right| = |A|$
- If any row or column is completely zero, then $|A| = 0$
- If two rows or columns are interchanged, then value of determinant is multiplied by -1.
- If one row or column of a matrix is multiplied by 'k', then determinant also becomes k times.
- If A is a matrix of order n × n , then
  $$|KA| = K^n |A|$$
- Value of determinant is invariant under row or column transformation
- $|AB| = |A| * |B|$
- $\left|A^n\right| = |A|^n$
- $\left|A^{-1}\right| = \dfrac{1}{|A|}$

Adjoint of a Square Matrix

Adj(A) = $\left[\text{cof}(A)\right]^T$

Inverse of a matrix

Inverse of a matrix only exists for square matrices

$$\left(A^{-1}\right)=\frac{Adj(A)}{|A|}$$

Properties

a. $AA^{-1}=A^{-1}A=I$

b. $(AB)^{-1}=B^{-1}A^{-1}$

c. $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

d. $\left(A^{T}\right)^{-1}=\left(A^{-1}\right)^{T}$

e. The inverse of a 2 × 2 matrix should be remembered

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

I. Divide by determinant.

II. Interchange diagonal element.

III. Take negative of off-diagonal element.

Rank of a Matrix

a. Rank is defined for all matrices, not necessarily a square matrix.

b. If A is a matrix of order m × n,
then Rank (A) ≤ min (m, n)

c. A number r is said to be rank of matrix A, if and only if

- There is at least one square sub-matrix of A of order 'r' whose determinant is non-zero.
- If there is a sub-matrix of order (r + 1), then determinant of such sub-matrix should be 0.

Linearly Independent and Dependent

Let X_1 and X_2 be the non-zero vectors

- If $X_1=kX_2$ or $X_2=kX_1$ then X_1,X_2 are said to be L.D. vectors.
- If $X_1 \neq kX_2$ or $X_2 \neq kX_1$ then X_1,X_2 are said to be L.I. vectors.

Note

Let X_1,X_2 .... X_n be n vectors of matrix A

- if rank(A)=no of vectors then vector X_1,X_2.......... X_n are L.I.
- if rank(A)<no of vectors then vector X_1,X_2.......... X_n are L.D.

System of Linear Equations

There are two type of linear equations

- **Homogenous equation**

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0$$

This is a system of 'm' homogenous equations in 'n' variables

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & -- & -- & a_{2n} \\ -- & & & & \\ -- & & & & \\ a_{m1} & a_{m2} & -- & -- & a_{mn} \end{bmatrix}_{m \times n} ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix}_{n \times 1} ; \; 0 = \begin{bmatrix} 0 \\ 0 \\ - \\ - \\ 0 \end{bmatrix}_{m \times 1}$$

This system can be represented as

$$AX = 0$$

Important Facts

- **Inconsistent system**

  Not possible for homogenous system as the trivial solution

  $\left[x_1, x_2 ....., x_n\right]^T = [0,0,......,0]^T$ always exists .

- **Consistent unique solution**

  If rank of A = r and r = n $\Rightarrow$ $|A| \neq 0$, so A is non-singular. Thus trivial solution exists.

- **Consistent infinite solution**
  If r < n, no of independent equation < (no. of variables) so, value of (n – r) variables can be assumed to compute rest of r variables.

- **Non-Homogenous Equation**
  $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
  $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

  $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_n$

This is a system of 'm' non-homogenous equation for n variables.

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & -- & a_{1n} \\ a_{21} & a_{22} & -- & -- & a_{2n} \\ -- & & & & \\ -- & & & & \\ a_{m1} & a_{m2} & -- & -- & a_{mn} \end{bmatrix}_{m \times n} ; \; X = \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix} ; \; B = \begin{bmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{bmatrix}_{m \times 1}$$

$$\text{Augmented matrix} = [A \mid B] = \begin{bmatrix} a_{11} & a_{12} & -- & a_n & b_1 \\ a_{21} & a_{22} & -- & -- & b_2 \\ -- & & & & \\ -- & & & & \\ a_{m1} & a_{m2} & -- & a_{mn} & b_m \end{bmatrix}$$

Conditions

- **Inconsistency**
  If r(A) ≠ r(A | B), system is inconsistent

- **Consistent unique solution**
  If r(A) = r(A | B) = n, we have consistent unique solution.

- **Consistent Infinite solution**
  If r(A) = r (A | B) = r & r < n, we have infinite solution

The solution of system of equations can be obtained by using Gauss elimination Method. (Not required for GATE)

Note

- Let $A_{n \times n}$ and rank(A)=r, then the no of L.I. solutions of Ax = 0 is "n-r"

Eigen values & Eigen Vectors

If A is n × n square matrix, then the equation

$$Ax = \lambda x$$

is called Eigen value problem.

Where λ is called as Eigen value of A.

x is called as Eigen vector of A.

Characteristic polynomial $\Rightarrow |A - \lambda I| = \begin{bmatrix} a_{11} - \lambda & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} - \lambda & -- & a_{2n} \\ -- & & & \\ a_{m1} & a_{m2} & -- & a_{mn} - \lambda \end{bmatrix}$

Characteristic equation $\Rightarrow |A - \lambda I| = 0$

The roots of characteristic equation are called as characteristic roots or the Eigen values.

To find the Eigen vector, we need to solve

$[A - \lambda I][x] = 0$

This is a system of homogenous linear equation.

We substitute each value of λ one by one & calculate Eigen vector corresponding to each Eigen value.

Important Facts

a. If x is an eigenvector of A corresponding to λ, the KX is also an Eigenvector where K is a constant.
b. If a n × n matrix has 'n' distinct Eigen values, we have 'n' linearly independent Eigen vectors.
c. Eigen Value of Hermitian/Symmetric matrix are real.
d. Eigen value of Skew - Hermitian / Skew – Symmetric matrix are purely imaginary or zero.
e. Eigen Value of unitary or orthogonal matrix are such that | λ | = 1.
f. If $\lambda_1, \lambda_2 ......, \lambda_n$ are Eigen value of A, $k\lambda_1, k\lambda_2 ......, k\lambda_n$ are Eigen values of kA.
g. Eigen Value of A^{-1} are reciprocal of Eigen value of A.

h. If $\lambda_1, \lambda_2 \ldots, \lambda_n$ are Eigen values of A, $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \ldots\ldots\ldots, \frac{|A|}{\lambda_n}$ are Eigen values of Adj(A).

i. Sum of Eigen values = Trace (A)

j. Product of Eigen values = |A|

k. In triangular or diagonal matrix, Eigen values are diagonal elements.

Cayley - Hamiltonian Theorem

Every matrix satisfies its own Characteristic equation.

e.g., If characteristic equation is

$$C_1\lambda^n + C_2\lambda^{n-1} \ldots\ldots + C_n = 0$$

Then

$$C_1A^n + C_2A^{n-1} + \ldots\ldots + C_nI = O$$

Where I is identity matrix

O is null matrix

CALCULUS

Important Series Expansion

a. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

b. $(1+x)^{-1} = 1 + x + x^2 + \ldots\ldots\ldots\ldots$

c. $a^x = 1 + x \log a + \frac{x^2}{2!}(x\log a)^2 + \frac{x^3}{3!}(x\log a)^3 + \ldots\ldots\ldots\ldots\ldots\ldots$

d. $\sin x = x - x^3/3! + x^5/5! \ldots\ldots\ldots\ldots\ldots\ldots$

e. $\cos x = 1 - x^2/2! + x^4/4! \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

f. $\tan x = x + x^3/3! + \frac{2}{15}x^5 + \ldots\ldots\ldots$

g. $\log(1+x) = x - x^2/2 + x^3/3 + \ldots\ldots\ldots\ldots, \; |x| < 1$

Important Limits

a. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

b. $\lim_{x \to 0} \frac{\tan x}{x} = 1$

c. $\lim_{x \to 0} (1+nx)^{1/x} = e^n$

d. $\lim_{x \to 0} \cos x = 1$

e. $\lim_{x \to 0} (1+x)^{1/x} = e$

f. $\lim_{x \to \infty} \left(1 + 1/x\right)^x = e$

L – Hospitals Rule

If f (x) and g(x) are to function such that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

If f′(x) and g′(x) are also zero as $x \to a$, then we can take successive derivatives till this condition is violated.

For continuity, $\lim_{x \to a} f(x) = f(a)$

For differentiability, $\lim_{h \to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \right]$ exists and is equal to $f'(x_0)$

If a function is differentiable at some point then it is continuous at that point but converse may not be true.

Mean Value Theorems

- **Rolle's Theorem**

  If there is a function f(x) such that f(x) is continuous in closed interval a ≤ x ≤ b and f′(x) is existing at every point in open interval a < x < b and f(a) = f(b).
  Then, there exists a point 'c' such that f′(c) = 0 and a < c < b.

- **Lagrange's Mean value Theorem**

  If there is a function f(x) such that, f(x) is continuous in closed interval a ≤ x ≤ b; and f(x) is differentiable in open interval (a, b) i.e., a < x < b,
  Then there exists a point 'c', such that

$$f'(c) = \frac{f(b) - f(a)}{(b - a)}$$

<u>Differentiation</u>

Properties: (f + g)′ = f′ + g′ ; (f – g)′ = f′ – g′ ; (f g)′ = f′ g + f g′

Important derivatives

a. $x^n \to n\, x^{n-1}$
b. $\ell nx \to 1/x$
c. $\log_a x \to (\log_a e)\left(1/x\right)$
d. $e^x \to e^x$

e. $a^x \to a^x \log_e a$
f. sin x → cos x
g. cos x → -sin x
h. tan x → $\sec^2 x$
i. sec x → sec x tan x
j. cosec x → - cosec x cot x
k. cot x → - cosec^2 x
l. sin h x → cos h x
m. cos h x → sin h x
n. $\sin^{-1} x \to \dfrac{1}{\sqrt{1 - x^2}}$
o. $\cos^{-1} x \to \dfrac{-1}{\sqrt{1 - x^2}}$
p. $\tan^{-1} x \to \dfrac{1}{1 + x^2}$
q. $\text{cosec}^{-1} x \to \dfrac{-1}{x\sqrt{x^2 - 1}}$
r. $\sec^{-1} x \to \dfrac{1}{x\sqrt{x^2 - 1}}$
s. $\cot^{-1} x \to \dfrac{-1}{1 + x^2}$

Increasing & Decreasing Functions

- $f'(x) \geq 0 \ \forall x \in (a, b)$, then f is increasing in [a, b]
- $f'(x) > 0 \ \forall x \in (a, b)$, then f is strictly increasing in [a, b]
- $f'(x) \leq 0 \ \forall x \in (a, b)$, then f is decreasing in [a, b]
- $f'(x) < 0 \ \forall x \in (a, b)$, then f is strictly decreasing in [a, b]

Maxima & Minima

Local maxima or minima

There is a maximum of f(x) at x = a if f′(a) = 0 and f″(a) is negative.

There is a minimum of f (x) at x = a, if f′(a) = 0 and f″ (a) is positive.

To calculate maximum or minima, we find the point 'a' such that f′(a) = 0 and then decide if it is maximum or minima by judging the sign of f″(a).

Global maxima & minima

We first find local maxima & minima & then calculate the value of 'f' at boundary points of interval given eg. [a, b], we find f(a) & f(b) & compare it with the values of local maxima & minima. The absolute maxima & minima can be decided then.

Taylor & Maclaurin series

- Taylor series

  $$f(a + h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \ldots\ldots\ldots\ldots$$

- Maclaurin

  $$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \ldots\ldots\ldots\ldots$$

Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constant, it is said to be a partial derivative.

Homogenous Function

$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots\ldots\ldots + a_ny^n$ is a homogenous function of x & y, of degree 'n'

$$= x^n\left[a_0 + a_1\left(y/x\right) + a_2\left(y/x\right)^2 + \ldots\ldots\ldots\ldots + a_n\left(y/x\right)^n\right]$$

Euler's Theorem

If u is a homogenous function of x & y of degree n, then

$$\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\right]$$

Maxima & minima of multi-variable function

$$\text{let } r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a \\ y=b}} \quad ; \quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a \\ y=b}} \quad ; \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a \\ y=b}}$$

- **Maxima**
  $rt > s^2 \quad ; \quad r < 0$
- **Minima**
  $rt > s^2 \quad ; \quad r > 0$
- **Saddle point**
  $rt < s^2$

Integration

Indefinite integrals are just opposite of derivatives and hence important derivatives must always be remembered.

Properties of definite integral

a. $\int_a^b f(x)dx = \int_a^b f(t)dt$

b. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

c. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

d. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

e. $\frac{d}{dt}\int_{\phi(t)}^{\psi(t)} f(x)dx = f(\psi(t))\psi'(t) - f(\phi(t))\phi'(t)$

Vectors

- Addition of vector
  $\vec{a}+\vec{b}$ of two vector $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$
  $\Rightarrow \vec{a} + \vec{b} = [a_1+b_1, a_2+b_2, a_3+b_3]$
- Scalar Multiplication
  $c\vec{a} = [ca_1,\ ca_2,\ ca_3]$

- Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

- Dot Product

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \gamma$, where 'γ' is angle between $\vec{a}$ & $\vec{b}$.

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Properties

a. $|a \cdot b| \le |a|\,|b|$ (Schwarz inequality)

b. $|a + b| \le |a| + |b|$ (Triangle inequality)

c. $|a + b|^2 + |a - b|^2 = 2\left(|a|^2 + |b|^2\right)$ (Parallelogram Equality)

- Vector cross product

$\vec{v} = \vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin \gamma$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{where } \vec{a} = [a_1, a_2, a_3] \; ; \; \vec{b} = [b_1, b_2, b_3]$$

- **Properties:**

a. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

b. $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

- Scalar Triple Product

$(a, b, c) = \vec{a} \cdot (\vec{b} \times \vec{c})$

- Vector Triple product

$\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

- Gradient of scalar field

Gradient of scalar function f (x, y, z)

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

- Directional derivative

Derivative of a scalar function in the direction of $\vec{b}$

$$D_b f = \frac{\vec{b}}{|\vec{b}|} \cdot \text{grad } f$$

- Divergence of vector field

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad ; \text{where } \vec{v} = [v_1, v_2, v_3]$$

- Curl of vector field

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \vec{v} = [v_1, v_2, v_3]$$

Some identities

a. Div grad f = $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

b. Curl grad f = $\nabla \times \nabla f = 0$

c. Div curl f = $\nabla . (\nabla \times f) = 0$

d. Curl curl f = grad div f – $\nabla^2 \times f$

Line Integral

$$\int_C F(r).dr = \int_a^b \left\{ F(r(t)). \frac{dr}{dt} \right\} dt$$

Hence curve C is parameterized in terms of t ; i.e. when ‘t’ goes from a to b, curve C is traced.

$$\int_C F(r).dr = \int_a^b (F_1 x' + F_2 y' + F_3 z') dt$$

$$\vec{F} = [F_1, F_2, F_3]$$

Green's Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\ dy = \oint_C \left(F_1 dx + F_2 dy \right)$$

This theorem is applied in a plane & not in space.

Gauss Divergence Theorem

$$\iiint_T \text{div } F.\ dv = \iint_S \vec{F}\ .\ \hat{n}\ d\ A$$

Where $\hat{n}$ is outer unit normal vector of s .

T is volume enclosed by s.

Stoke's Theorem

$$\iint_S \left(\text{curl } F \right).\ \hat{n}\ d\ A = \oint_C F.\ r'\left(s \right) ds$$

Where $\hat{n}$ is unit normal vector of S

C is the curve which enclosed a plane surface S.

DIFFERENTIAL EQUATIONS

- The order of a deferential equation is the order of highest derivative appearing in it.
- The degree of a differential equation is the degree of the highest derivative occurring in it, after the differential equation is expressed in a form free from radicals & fractions.

For equations of first order & first degree

- **Variable Separation method**

  Collect all function of x & dx on one side.
  Collect all function of y & dy on other side.
  like f(x) dx = g(y) dy
  solution: $\int f(x)dx = \int g(y)\ dy + c$

- **Exact differential equation**

  An equation of the form

  $$M(x, y)\ dx + N(x, y)\ dy = 0$$

  For equation to be exact.

  $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$; then only this method can be applied.

  The solution is

  $$a = \int M\,dx + \int (\text{terms of N not containing x})dy$$

- **Integrating factors**

  An equation of the form

  $$P(x, y)\ dx + Q(x, y)\ dy = 0$$

  This can be reduced to exact form by multiplying both sides by IF.

  If $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x, then

  $R(x) = = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

  Integrating Factor

  IF = $\exp\left(\int R(x)dx\right)$

  Otherwise, if $\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ is a function of y

$$S(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Integrating factor, IF = exp $\left(\int S(y)dy\right)$

- **Linear Differential Equations**

  An equation is linear if it can be written as:

  $$y' + P(x)y = r(x)$$

  If r(x) = 0 ; equation is homogenous

  else r(x) $\neq$ 0 ; equation is non-homogeneous

  $y(x) = ce^{-\int p(x)dx}$ is the solution for homogenous form

  for non-homogenous form, h = $\int P(x)dx$

  $$y(x) = e^{-h}\left[\int e^h r dx + c\right]$$

- **Bernoulli's equation**

  The equation $\frac{dy}{dx} + Py = Qy^n$

  Where P & Q are function of x

  Divide both sides of the equation by y^n & put $y^{(1-n)} = z$

  $$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

  This is a linear equation & can be solved easily.

- **Clairaut's equation**

  An equation of the form y = Px + f (P), is known as Clairaut's equation where P = $\left(\frac{dy}{dx}\right)$

  The solution of this equation is

  y = cx + f (c) where c = constant

Linear Differential Equation of Higher Order

Constant coefficient differential equation

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots\dots + k_n y = X$$

Where X is a function of x only

a. If $y_1, y_2, \ldots\ldots\ldots, y_n$ are n independent solution, then

$c_1y_1 + c_2y_2 + \ldots\ldots\ldots + c_ny_n = x$ is complete solution

where $c_1, c_2, \ldots\ldots\ldots, c_n$ are arbitrary constants.

b. The procedure of finding solution of n^{th} order differential equation involves computing complementary function (C. F) and particular Integral (P. I).

c. Complementary function is solution of

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + k_ny = 0$$

d. Particular integral is particular solution of

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + k_ny = x$$

e. y = CF + PI is complete solution

Finding complementary function

- Method of differential operator

Replace $\frac{d}{dx}$ by $D \rightarrow \frac{dy}{dx} = Dy$

Similarly

$$\frac{d^n}{dx^n} \text{ by } D^n \rightarrow \frac{d^ny}{dx^n} = D^ny$$

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + k_ny = 0 \text{ becomes}$$

$$\left(D^n + k_1D^{n-1} + \ldots\ldots\ldots + k_n\right)y = 0$$

Let $m_1, m_2, \ldots\ldots\ldots\ldots, m_n$ be roots of

$$D^n + k_1D^{n-1}\ldots\ldots\ldots\ldots\ldots + K_n = 0 \quad \ldots\ldots\ldots\ldots(i)$$

Case I: All roots are real & distinct

$$(D-m_1)(D-m_2)............(D-m_n)=0 \quad \text{is equivalent to (i)}$$

$$y = c_1e^{m_1x} + c_2e^{m_2x} + ... + c_ne^{m_nx}$$

is solution of differential equation

Case II: If two roots are real & equal

i.e., $m_1 = m_2 = m$

$$y = (c_1 + c_2x)e^{mx} + c_3e^{m_3x} + ... + c_ne^{m_nx}$$

Case III: If two roots are complex conjugate

$m_1 = \alpha + j\beta \;;\; m_2 = \alpha - j\beta$

$$y = e^{\alpha x}\left[c_1'\cos\beta x + c_2'\sin\beta x\right] + ... + c_ne^{m_nx}$$

Finding particular integral

Suppose differential equation is

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + ... + k_ny = X$$

Particular Integral

$$PI = y_1\int\frac{W_1(x)}{W(x)}\times dx + y_2\int\frac{W_2(x)}{W(x)}\times dx + ... + y_n\int\frac{W_n(x)}{W(x)}\times dx$$

Where $y_1, y_2, ...y_n$ are solutions of Homogenous from of differential equations.

$$W(x) = \begin{vmatrix} y_1 & y_2 & - & y_n \\ y_1' & y_2' & - & y_n' \\ - & & & \\ y_1^{(n)} & y_2^{(n)} & - & y_n^{(n)} \end{vmatrix} \qquad W_i(x) = \begin{vmatrix} y_1 & y_2 & - & y_{i-1} & 0 & - & y_n \\ y_1' & y_2' & - & y_{i-1}' & 0 & & y_n' \\ - & & & & 0 & & \\ y_1^{(n)} & y_2^{(n)} & - & y_{i-1}^{(n)} & 1 & & y_n^{(n)} \end{vmatrix}$$

$W_i(x)$ is obtained from W(x) by replacing i^{th} column by all zeroes & last 1.

Euler-Cauchy Equation

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + k_n y = 0$$

is called as Euler-Cauchy theorem

Substitute y = x^m

The equation becomes

$$\left[m(m-1)........(m-n) + k_1 m(m-1).........(m-n+1) + ... + k_n\right] x^m = 0$$

The roots of equation are

Case I: All roots are real & distinct

$$y = c_1 x^{m_1} + c_2 x^{m_2} + ... + c_n x^{m_n}$$

Case II: Two roots are real & equal

$$m_1 = m_2 = m$$

$$y = \left(c_1 + c_2 \ell n x\right) x^m + c_3 x^{m_3} + ... + c_n x^{m_n}$$

Case III: Two roots are complex conjugate of each other

$$m_1 = \mu + j\vartheta \ ; \quad m_2 = \mu - j\vartheta$$

$$y = x^{\mu}\left[A\cos\left(\vartheta \ell nx\right) + B\sin\left(\vartheta \ell nx\right) + \right] + c_3 x^{m_3} + ... + c_n x^{m_n}$$

COMPLEX FUNCTIONS

- **Exponential function of complex variable**

$$f(z) = e^z = e^{(x+iy)}$$

$$f(z) = e^x e^{iy} = e^x(\cos y + i \sin y) = u + iv$$

- **Logarithmic function of complex variable**

If $e^w = z$; then w is logarithmic function of z

log z = w + 2inπ

This logarithm of complex number has infinite numbers of values.

The general value of logarithm is denoted by Log z & the principal value is log z & is found from general value by taking n = 0.

- **Analytic function**

A function f(z) which is single valued and possesses a unique derivative with respect to z at all points of region R is called as an analytic function.

If u & v are real, single valued functions of x & y s. t. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous throughout a region R, then Cauchy – Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$ are necessary & sufficient condition for f(z) = u + iv to be analytic in R.

- **Line integral of a complex function**

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

where C is a smooth curve represented by z = z(t), where a ≤ t ≤ b.

- **Cauchy's Theorem**

If f(z) is an analytic function and f'(z) is continuous at each point within and on a closed curve C. then

$$\int_C f(z)\,dz = 0$$

- **Cauchy's Integral formula**

  If f(z) is analytic within & on a closed curve C, & a is any point within C.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Singularities of an Analytic Function

- **Isolated singularity**

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad ; \quad a_n = \frac{1}{2\pi i}\int \frac{f(t)}{(t-a)^{n+1}}\,dt$$

$z = z_0$ is an isolated singularity if there is no singularity of f(z) in the neighborhood of z $= z_0$.

- **Removable singularity**

  If all the negative power of (z – a) are zero in the expansion of f(z),

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

  The singularity at z = a can be removed by defined f(z) at z = a such that f(z) is analytic at z = a.

- **Poles**

  If all negative powers of (z – a) after n^{th} are missing, then z = a is a pole of order 'n'.

- **Essential singularity**

  If the number of negative power of (z – a) is infinite, the z = a is essential singularity & cannot be removed.

RESIDUES

If z = a is an isolated singularity of f(z)

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots\ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots\ldots\ldots$$

Then residue of f(z) at z = a is a_{-1}

Residue Theorem

$$\int_c f(z)\,dz = 2\pi i \times \text{(sum of residues at the singular points within c)}$$

If f(z) has a pole of order 'n' at z=a

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$$

Evaluation Real Integrals

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2};\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Assume z= $e^{i\theta}$

$$\cos\theta = \frac{\left(z + \frac{1}{z}\right)}{2};\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$I = \oint_c f(z)\frac{dz}{iz} = 2\pi i\left(\sum_{k=1}^{n} \text{Res}(f(z_k))\right)$$

Residue should only be calculated at poles in upper half plane.

Residue is calculated for the function: $\left(\frac{f(z)}{iz}\right)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{Res } f(z)$$

Where residue is calculated at poles in upper half plane & poles of f(z) are found by substituting z in place of x in f(x).

PROBABILITY AND STATISTICS

Types of events

- **Complementary events**

  $\{E^c\} = \{s\} - \{E\}$

  The complement of an event E is set of all outcomes not in E.

- **Mutually Exclusive Events**

  Two events E & F are mutually exclusive iff P(E ∩ F) = 0.

- **Collectively exhaustive events**

  Two events E & F are collectively exhaustive iff (E U F) = S
  Where S is sample space.

- **Independent events**

  If E & F are two independent events

  $P(E \cap F) = P(E) * P(F)$

De Morgan's Law

- $\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$
- $\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$

Axioms of Probability

$E_1, E_2, \ldots\ldots\ldots, E_n$ are possible events & S is the sample space.

a. $0 \le P(E) \le 1$

b. $P(S) = 1$

c. $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$ for mutually exclusive events

Some important rules of probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)^* P(B|A) = P(B) * P(A|B)$

$P(A|B)$ is conditional probability of A given B.

If A & B are independent events

$P(A \cap B) = P(A) * P(B)$

$P(A | B) = P(A)$

$P(B | A) = P(B)$

Total Probability Theorem

$P(A \cap B) = P(A \cap E) + P(B \cap E)$

$= P(A) * P(E|A) + P(B) * P(E|B)$

Baye's Theorem

$P(A|E) = \quad P(A \cap E) + P(B \cap E)$

$= P(A)^* P(E | A) + P(B) * P(E | B)$

Statistics

- Arithmetic Mean of Raw Data

$$\bar{x} = \frac{\sum x}{n}$$

$\bar{x}$ = arithmetic mean; x = value of observation ; n = number of observations

- Arithmetic Mean of grouped data

$$\bar{x} = \frac{\sum(fx)}{\sum f} \quad ; \quad f = \text{frequency of each observation}$$

- Median of Raw data

Arrange all the observations in ascending order

$$x_1 < x_2 < \dots\dots\dots < x_n$$

If n is odd, median = $\frac{(n+1)}{2}$ th value

If n is even, Median = $\dfrac{\left(n/2\right)^{th} \text{ value} + \left(n/2+1\right)^{th} \text{ value}}{2}$

- Mode of Raw data
  Most frequently occurring observation in the data.

- Standard Deviation of Raw Data

$$\sigma = \sqrt{\frac{n\sum x_i^2 - \left(\sum x_i\right)^2}{n^2}}$$

n = number of observations

variance = σ^2

- Standard deviation of grouped data

$$\sigma = \sqrt{\frac{N\sum f_i^2 x_i^2 - \left(\sum f_i x_i\right)^2}{N}}$$

fi = frequency of each observation

N = number of observations.

variance = σ^2

- Coefficient of variation = CV = $\frac{\sigma}{\mu}$

- **Properties of discrete distributions**
  a. $\sum P(x) = 1$
  b. $E(X) = \sum x\, P(x)$
  c. $V(x) = E(x^2) - (E(x))^2$

- **Properties of continuous distributions**
  - $\int_{-\infty}^{\infty} f(x)\,dx = 1$
  - $F(x) = \int_{-\infty}^{x} f(x)\,dx$ = cumulative distribution
  - $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$ = expected value of x
  - $V(x) = E(x^2) - [E(x)]^2$ = variance of x

- **Properties Expectation & Variance**

$$E(ax + b) = a\,E(x) + b$$

$$V(ax + b) = a^2\,V(x)$$

$$E(ax_1 + bx_2) = aE(x_1) + bE(x_2)$$

$$V(ax_1 + bx_2) = a^2V(x_1) + b^2V(x_2)$$

$$\text{cov}(x, y) = E(xy) - E(x)\,E(y)$$

Binomial Distribution

no of trials = n

Probability of success = P

Probability of failure = (1 – P)

$$P(X = x) = {}^nC_x P^x (1-P)^{n-x}$$

Mean = E(X) = nP

Variance = V[x] = nP(1 – P)

Poisson Distribution

A random variable x, having possible values 0,1, 2, 3,......., is poisson variable if

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Mean = E(x) = λ

Variance = V(x) = λ

Continuous Distributions

Uniform Distribution

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Mean = E(x) = $\dfrac{b+a}{2}$

Variance = V(x) = $\dfrac{(b-a)^2}{12}$

Exponential Distribution

$$f(x) = \begin{cases} e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Mean = E(x) = $1/\lambda$

Variance = V(x) = $1/\lambda^2$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e \times p\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), -\infty < x < \infty$$

Means = E(x) = μ

Variance = v(x) = σ^2

Coefficient of correlation

$$\rho = \frac{\operatorname{cov}(x,y)}{\sqrt{\operatorname{var}(x)\operatorname{var}(y)}}$$

x & y are linearly related, if $\rho = \pm 1$

x & y are un-correlated if $\rho = 0$

Regression lines

- $(x-\bar{x}) = b_{xy}(y-\bar{y})$
- $(y-\bar{y}) = b_{yx}(x-\bar{x})$

Where $\bar{x}$ & $\bar{y}$ are mean values of x & y respectively

$$b_{xy} = \frac{\operatorname{cov}(x,y)}{\operatorname{var}(y)} \quad ; \quad b_{yx} = \frac{\operatorname{cov}(x,y)}{\operatorname{var}(x)}$$

$$\rho = \sqrt{b_{xy} b_{yx}}$$

NUMERICAL – METHODS

Numerical solution of algebraic equations

- **Descartes Rule of sign:**

  An equation f(x) = 0 cannot have more positive roots then the number of sign changes in f(x) & cannot have more negative roots then the number of sign changes in f(-x).

- **Bisection Method**

  If a function f(x) is continuous between a & b and f(a) & f(b) are of opposite sign, then there exists at least one roots of f(x) between a & b.

  Since root lies between a & b, we assume root $x_0 = \frac{(a+b)}{2}$

  If $f(x_0) = 0$; x_0 is the root

  Else, if $f(x_0)$ has same sign as $f(a)$, then roots lies between x_0 & b and we assume

  $x_1 = \frac{x_0 + b}{2}$, and follow same procedure otherwise if $f(x_0)$ has same sign as $f(b)$, then

  root lies between a & x_0 & we assume $x_1 = \frac{a + x_0}{2}$ & follow same procedure.

  We keep on doing it, till $|f(x_n)| < \in$, i.e., $f(x_n)$ is close to zero.

  No. of step required to achieve an accuracy $\in$

$$n \geq \frac{\log_e\left(\frac{|b-a|}{\in}\right)}{\log_e 2}$$

- **Regula-Falsi Method**

  This method is similar to bisection method, as we assume two value x_0 & x_1 such that

$$f(x_0)f(x_1) < 0\ .$$

$$x_2 = \frac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$$

  If $f(x_2)$=0 then x_2 is the root , stop the process.

If $f(x_2)>0$ then

$$x_3 = \frac{f(x_2).x_0 - f(x_0).x_2}{f(x_2) - f(x_0)}$$

If $f(x_2)<0$ then

$$x_3 = \frac{f(x_1).x_2 - f(x_2).x_1}{f(x_1) - f(x_2)}$$

Continue above process till required root not found

- **Secant Method**

In secant method, we remove the condition that $f(x_0)f(x_1) < 0$ and it doesn't provide the guarantee for existence of the root in the given interval , So it is called un reliable method .

$$x_2 = \frac{f(x_1).x_0 - f(x_0).x_1}{f(x_1) - f(x_0)}$$

and to compute x_3 replace every variable by its variable in x_2

$$x_3 = \frac{f(x_2).x_1 - f(x_1).x_2}{f(x_2) - f(x_1)}$$

Continue above process till required root not found

- **Newton-Raphson Method**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note : Since N.R. iteration method is quadratic convergence so to apply this formula $f''(x)$ must exist.

Order of convergence

- Bisection = Linear
- Regula false = Linear
- Secant = superlinear
- Newton Raphson = quadratic

- **Numerical Integration**

Trapezoidal Rule

$\int_a^b f(x)\,dx$, can be calculated as

Divide interval (a, b) into n sub-intervals such that width of each interval

$$h = \frac{(b-a)}{n}$$

we have (n + 1) points at edges of each intervals

$$(x_0, x_1, x_2, \ldots\ldots\ldots, x_n)$$

$$y_0 = f(x_0);\ y_1 = f(x_1), \ldots\ldots\ldots\ldots\ldots\ldots, y_n = f(x_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \ldots\ldots\ldots + y_{n-1}\right) + y_n\right]$$

Simpson's $\frac{1}{3}$ rd Rule

Here the number of intervals should be even

$$h = \left(\frac{b-a}{n}\right)$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + y_5 + \ldots\ldots\ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots\ldots\ldots\ldots\ldots + y_{n-2}\right) + y_n\right]$$

Simpson's $\frac{3}{8}$ th Rule

Here the number of intervals should be even

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_{5\ldots\ldots\ldots}y_{n-1}) + 2(y_3 + y_6 + y_9 \ldots\ldots\ldots\ldots y_{n-3}) + y_n\right]$$

Truncation error

Trapezoidal Rule: $|T_\in|_{bound} = \frac{(b-a)}{12} h^2 \max|f''(\in)|$ and order of error =2

Simpson's $1/3$ Rule: $|T_\in|_{bound} = \frac{(b-a)}{180} h^4 \max\left|f^{(iv)}(\in)\right|$ and order of error =4

Simpson's $3/8$ th Rule: $|T_\in|_{bound} = \frac{3(b-a)}{n80} h^4 \max\left|f^{(iv)}(\in)\right|$ and order of error =5

where $x_0 \le \varepsilon \le x_n$

Note : If truncation error occurs at n^{th} order derivative then it gives exact result while integrating the polynomial up to degree (n-1).

Numerical solution of Differential equation

Euler's Method

$$\frac{dy}{dx} = f(x,\ y)$$

To solve differential equation by numerical method, we define a step size h

We can calculate value of y at $(x_0 + h,\ x_0 + 2h, ..., x_0 + nh)$ & not any intermediate points.

$$y_{i+1} = y_i + hf(x_i,\ y_i)$$

$$y_i = y(x_i)\ ;\quad y_{i+1} = y(x_{i+1})\quad ;\ X_{i+1} = X_i + h$$

Modified Euler's Method (Heun's method)

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h,\ y_0 + h)\right]$$

Runge – Kutta Method

$$y_1 = y_0 + k$$

$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h,\ y_0 + k_3\right)$$

Similar method for other iterations

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