Lecture 9
Digital Filter Structure

Block Diagram Representation

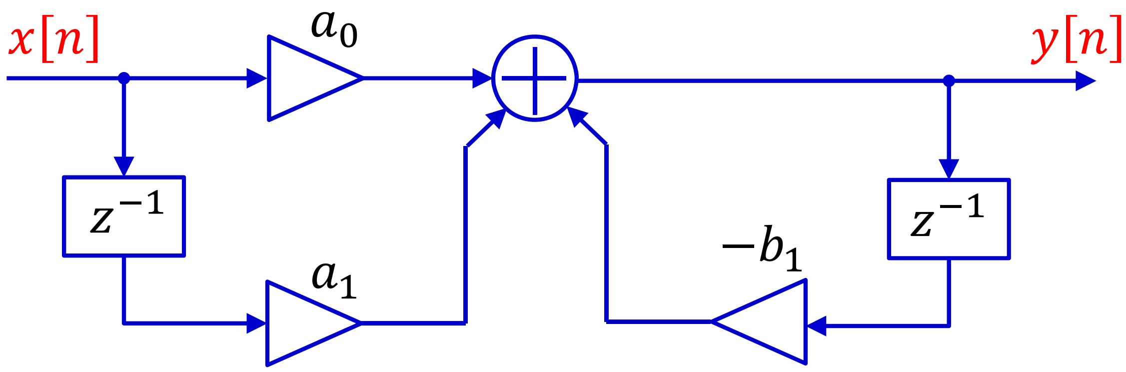
- It has advantages to represent time domain input-output relation, for example the convolution, or the difference equations, as block diagrams.

$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

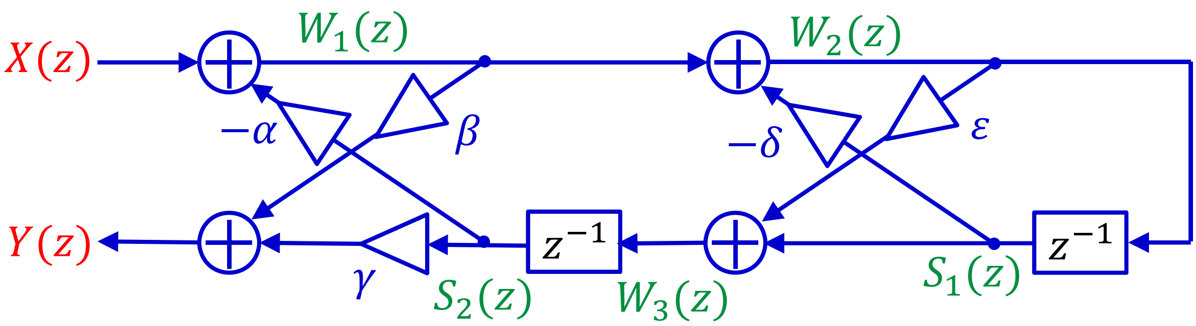
A First Order LTI Digital Filter

$$y[n] = -b_1 y[n-1] + a_0 x[n] + a_1 x[n-1]$$



$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

Analysis of Block Diagrams



$$\begin{aligned} W_1 &= X - \alpha S_2 \\ W_2 &= W_1 - \delta S_1 \\ W_3 &= \varepsilon W_2 + S_1 \\ Y &= \beta W_1 + \gamma S_2 \end{aligned}$$

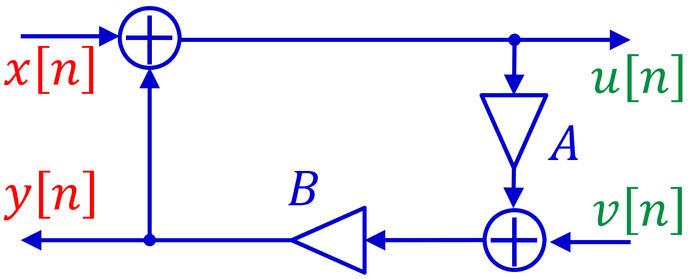
$$\begin{aligned} S_2 &= z^{-1} W_3 \\ S_1 &= z^{-1} W_2 \end{aligned}$$

$$\begin{aligned} W_1 &= X - \alpha z^{-1} W_3 \\ W_2 &= W_1 - \delta z^{-1} W_2 \\ W_3 &= \varepsilon W_2 + z^{-1} W_2 \\ Y &= \beta W_1 + \gamma z^{-1} W_3 \end{aligned}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\beta + (\beta\delta + \gamma\varepsilon) z^{-1} + \gamma z^{-2}}{1 + (\delta + \alpha\varepsilon) z^{-1} + \alpha z^{-2}}$$

Delays in Block Diagram

- A block diagram containing **delay-free loop**, i.e., a feedback loop without any delay element, is physically **IMPOSSIBLE** to achieve.



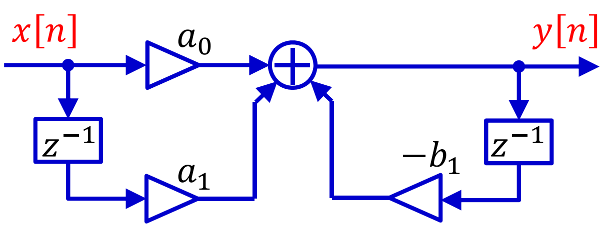
$$y[n] = B(A(x[n] + y[n]) + v[n])$$

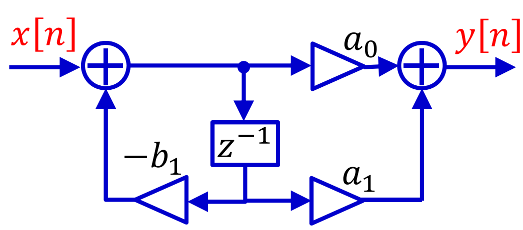
- The number of delays in a **canonic structure** is equal to the order of the transfer function (or the order of the difference equation).

Equivalent Structure

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

- Two digital filter structures are defined to be equivalent if they have the same transfer function.





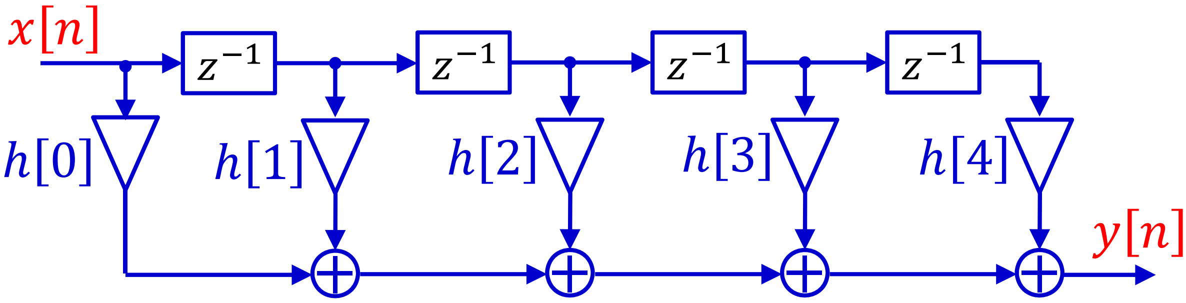
- Transpose Operation to obtain an equivalent structure
  - Reverse all paths
  - Replace branching nodes with adders, and vice versa,
  - Interchange the input and output nodes.

Basic FIR Digital Filter Structures

- Direct-Form Structures

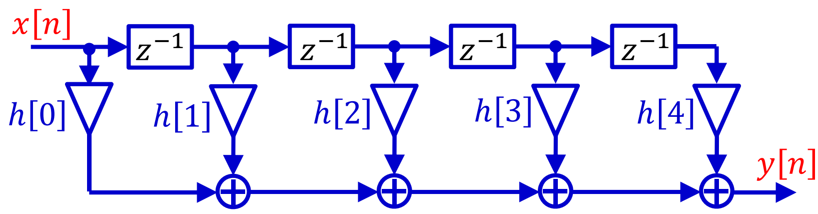
$$Y(z) = H(z)X(z) = \sum_{k=0}^{N-1} h[k]z^{-k}X(z)$$

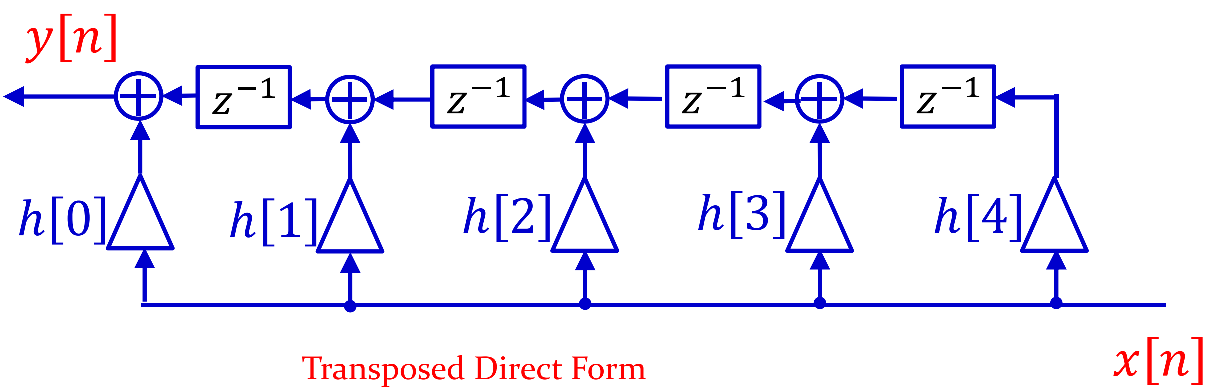
$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$



- Transposed Direct Form Structures

Direct Form





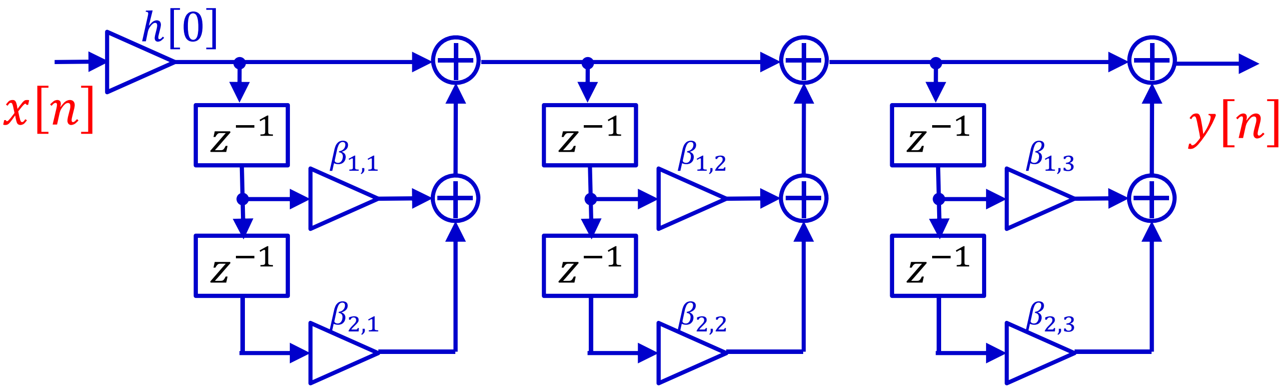
Transposed Direct Form

Cascade-Form Structures

$$H(z) = h[0]\prod_{k=1}^{K}\left(1 + \beta_{1,k}z^{-1} + \beta_{2,k}z^{-2}\right)$$

where $K = (N-1)/2$ if N is odd,

and $K = N/2$ if N is even, with $\beta_{2,K} = 0$

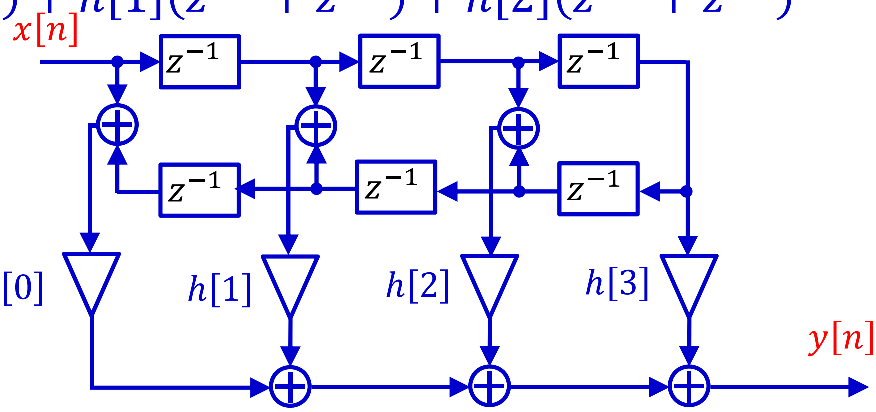


Linear-Phase FIR Filter Structure

$$h[n] = h[N-1-n], \text{or } h[n] = -h[N-1-n]$$

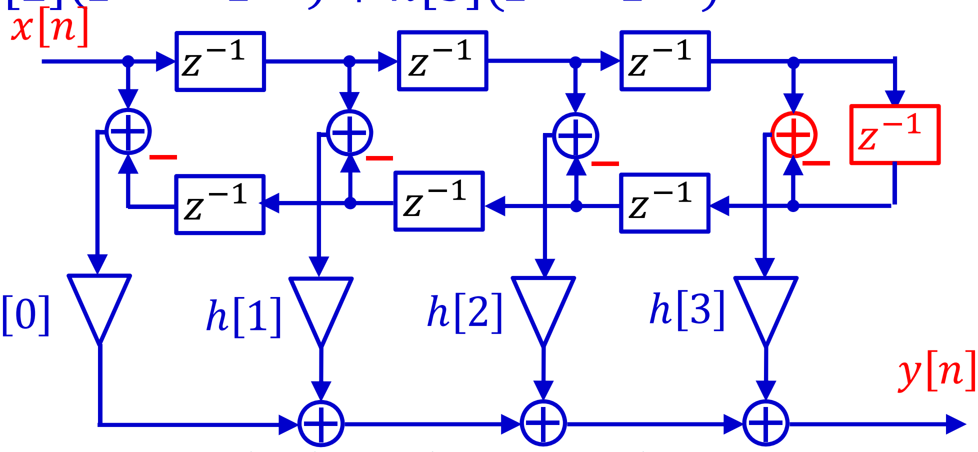
- For example, a length-7 Type I filter

$H(z)$
$= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[2]z^{-4} + h[1]z^{-5}$
$+ h[0]z^{-6}$
$= h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5}) + h[2](z^{-2} + z^{-4})$
$+ h[3]z^{-3}$



- a length-8 Type IV filter

$$H(z)$$
$$= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} - h[3]z^{-4}$$
$$- h[2]z^{-5} - h[1]z^{-6} - h[0]z^{-7}$$
$$= h[0](1 - z^{-7}) + h[1](z^{-1} - z^{-6})$$
$$+ h[2](z^{-2} - z^{-5}) + h[3](z^{-3} - z^{-4})$$



Basic IIR Digital Filter Structure

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

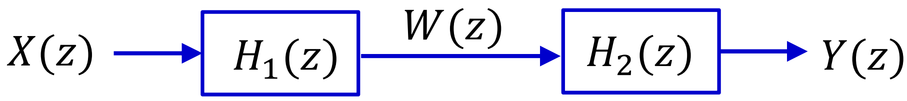
$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{m=1}^{N} b_m z^{-m}} = H_1(z)H_2(z)$$

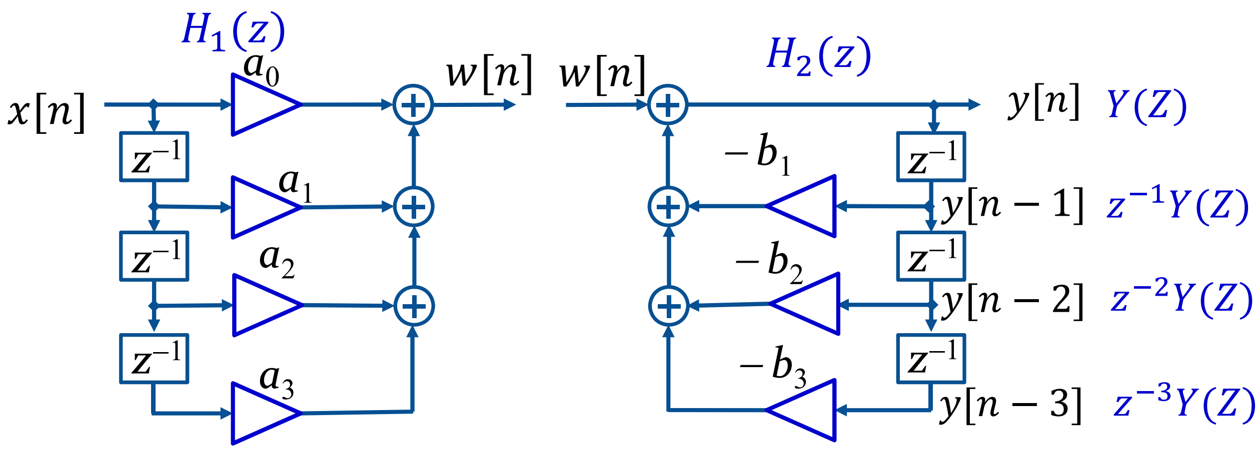
For example, when $M = N = 3$, we have

$$H_1(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$

$$H_2(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}$$

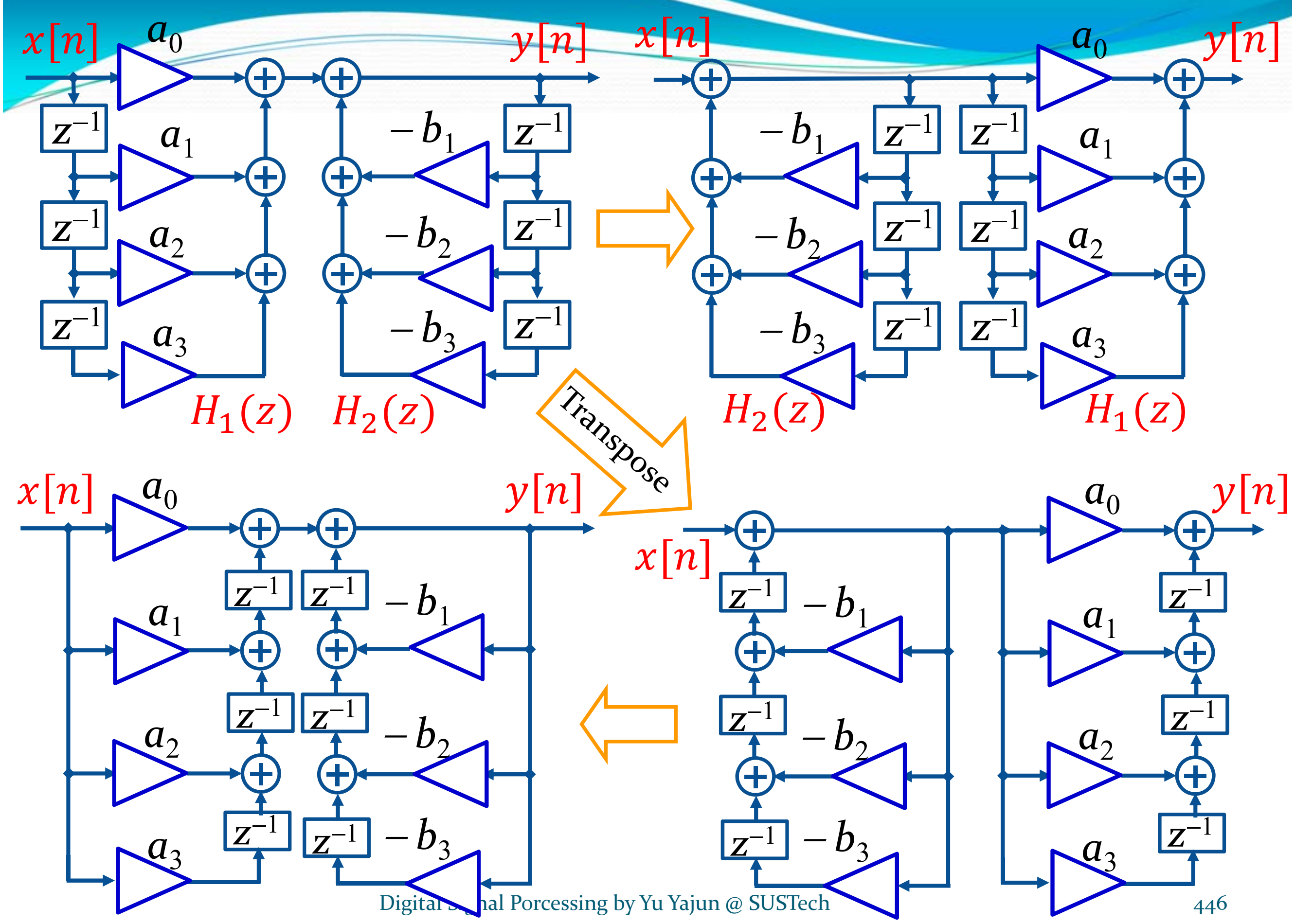
- Direct-Form Structure

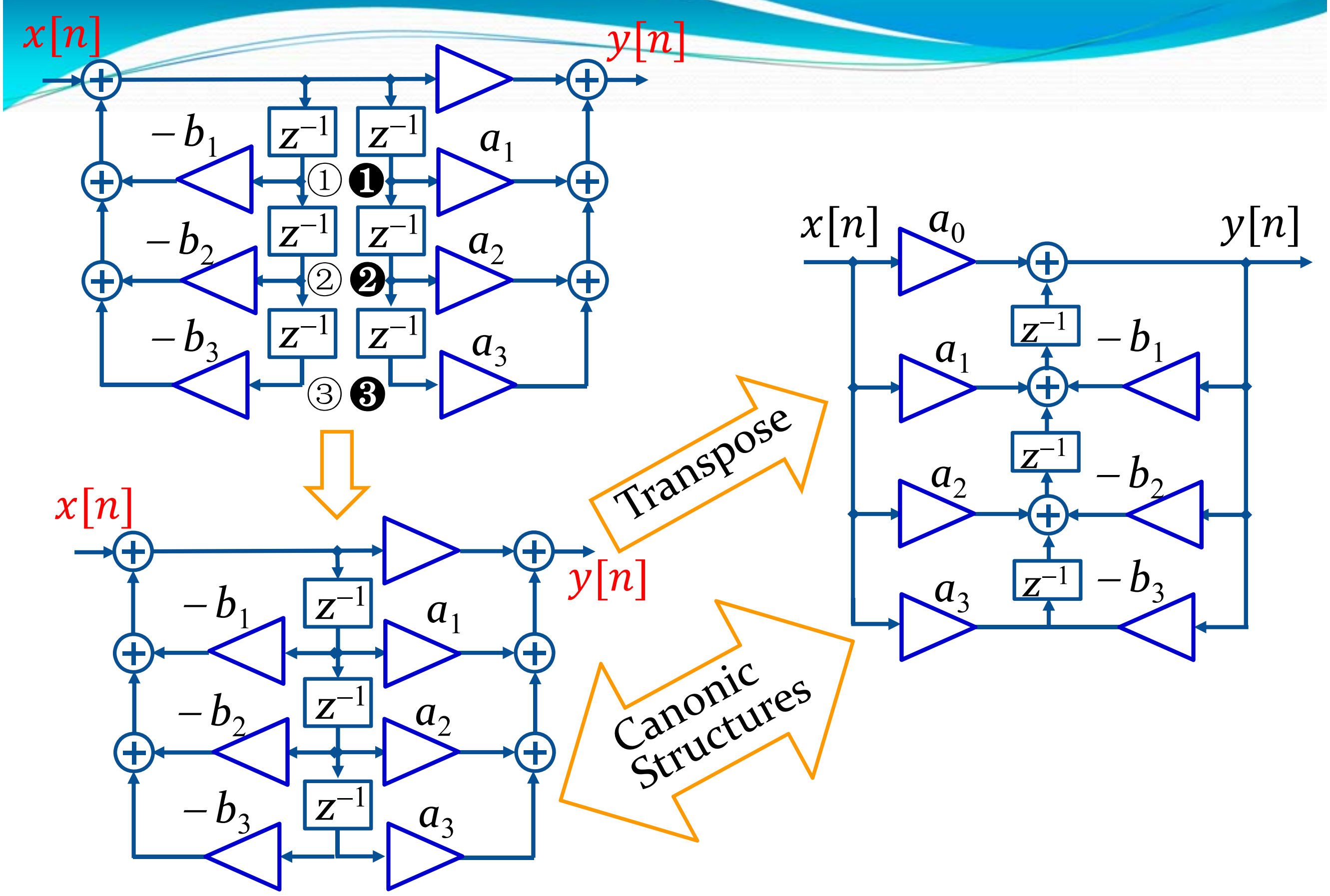




$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}$$

$$Y(z)(1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}) = W(z)$$

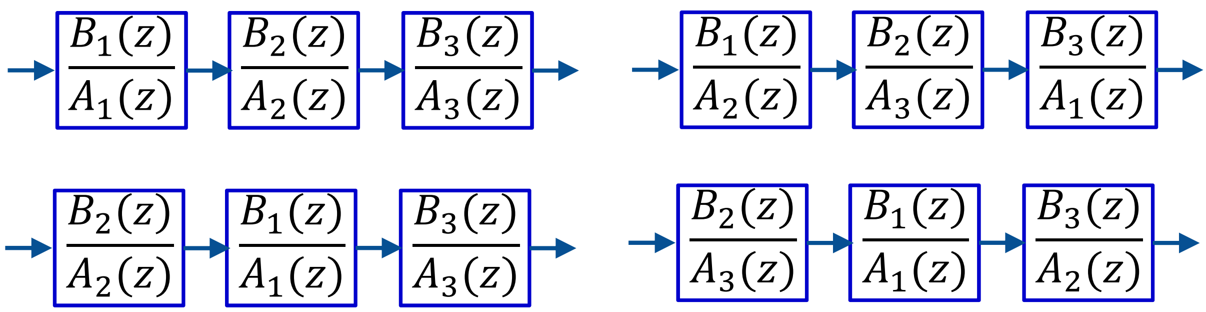




Cascade Realization

$$H(z) = \frac{\prod_{k=1}^{M_1}(1 - f_k z^{-1}) \prod_{k=1}^{M_2}(1 - g_k z^{-1})(1 - g_k{}^* z^{-1})}{\prod_{k=1}^{N_1}(1 - c_k z^{-1}) \prod_{k=1}^{N_2}(1 - d_k z^{-1})(1 - d_k{}^* z^{-1})}$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{B_1(Z)B_2(Z)B_3(Z)}{A_1(Z)A_2(Z)A_3(Z)}$$

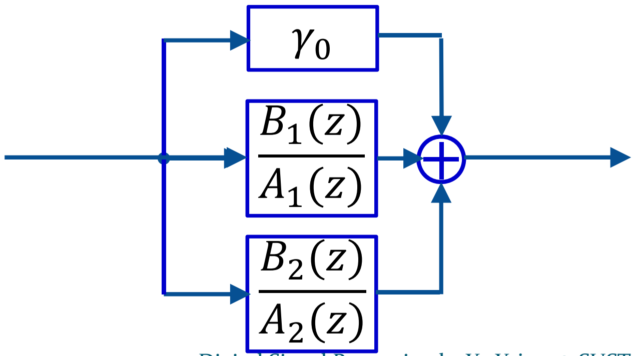


Parallel Realization

$$H(z) = \gamma_0 + \sum_k \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

For example: $H(z) = \gamma_0 + \frac{B_1(z)}{A_1(z)} + \frac{B_2(z)}{A_2(z)}$

- Can be obtained by partial-fraction expansion

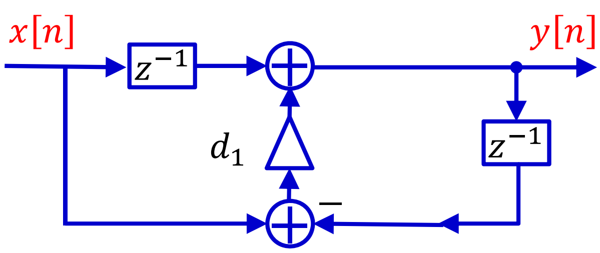


Allpass Filter Structure

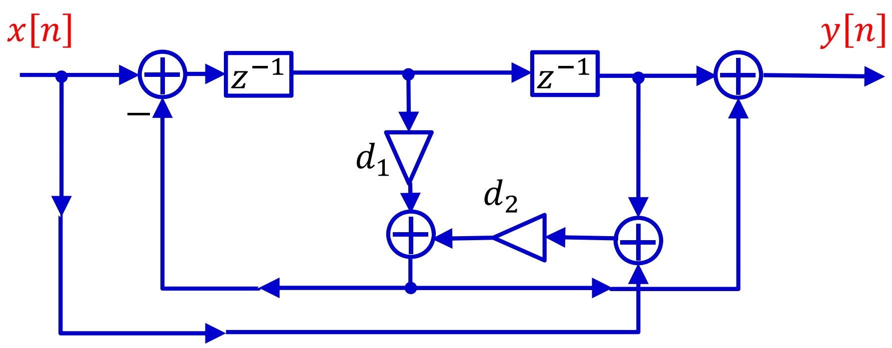
- Transfer function of **real** coefficient allpass filter

$$A_M(z) = \frac{d_M + d_{M-1} z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

- Objective: efficient structure using N multipliers to implement N-th order allpass filter, for example:
- First order: $A_M(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$



- Second order: $A_M(z) = \frac{d_2+d_1z^{-1}+z^{-2}}{1+d_1z^{-1}+d_2z^{-2}}$



- Allpass filters with this structure have a magnitude gain of 1 even with coefficient errors

*Allpass with Lattice Structure

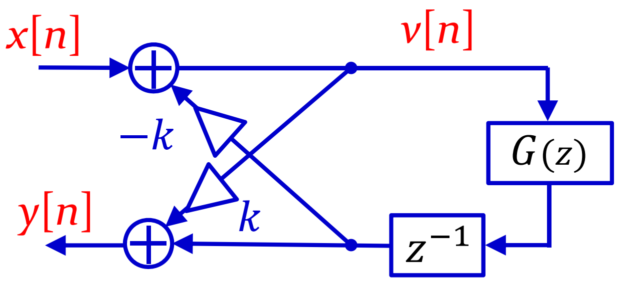
- Lattice Stage:
- Suppose $G(z)$ is allpass: $G(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$

$$V(z) = X(z) - kV(z)G(z)z^{-1}$$

$$V(z) = \frac{1}{1+kG(z)z^{-1}}X(z)$$

$$Y(z) = kV(z) + V(z)G(z)z^{-1}$$

$$= \frac{k+G(z)z^{-1}}{1+kG(z)z^{-1}}X(z)$$



$$\frac{Y(z)}{X(z)} = \frac{k+G(z)z^{-1}}{1+kG(z)z^{-1}} = \frac{kA(z)+z^{-N-1}A(z^{-1})}{A(z)+kz^{-N-1}A(z^{-1})} \triangleq \frac{z^{-(N+1)}D(z^{-1})}{D(z)}$$

- Obtaining $\{d[n]\}$ from $\{a[n]\}$:

$$d[n] = \begin{cases} 1, & n = 0 \\ a[n] + ka[N-n+1], & 1 \le n \le N \\ k, & n = N+1 \end{cases}$$

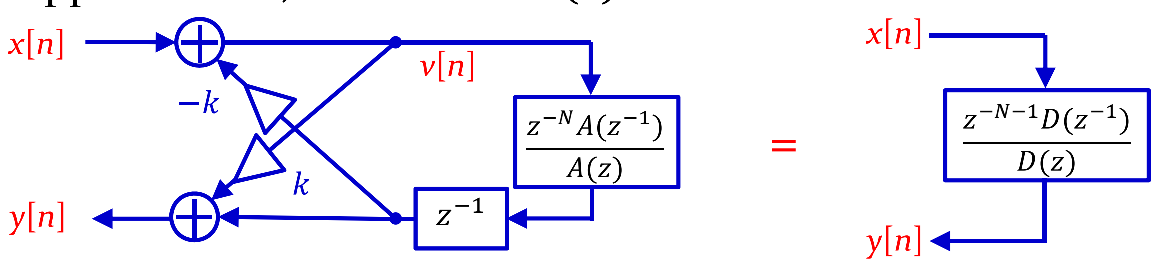
- Obtaining $\{a[n]\}$ from $\{d[n]\}$:

$$k = d[N+1], \qquad a[n] = \frac{d[n] - kd[N+1-n]}{1-k^2}$$

- If $G(z)$ is stable then $\frac{Y(z)}{X(z)}$ is stable if and only if $|k| < 1$

Example $A(z) \leftrightarrow D(z)$

- Suppose $N = 3, k = 0.5$ and $A(z) = 1 + 4z^{-1} - 6z^{-2} + 10z^{-3}$



- $A(z) \to D(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$A(z)$	1	4	-6	10	
$z^{-4}A(z^{-1})$		10	-6	4	1
$D(z)$	1	9	-9	12	0.5

$$D(z) = A(z) + kz^{-4}A(z^{-1})$$

$D(z) \to A(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$D(z)$	1	9	-9	12	0.5
$k = d[N+1]$					0.5
$z^{-4}D(z^{-1})$	0.5	12	-9	9	1
$A(z)$	1	4	-6	10	

$$A(z) = \frac{D(z) - kz^{-4}D(z^{-1})}{1-k^2}$$