Lecture 3
Time Domain Representation of Discrete Time Systems

Discrete Time System

$$x[n] \longrightarrow \boxed{T(\cdot)} \longrightarrow y[n]=T(x[n])$$

- The function of **discrete-time system** is to process a given sequence, called the **input sequence**, to generate another sequence, called the **output sequence**, with more desirable properties or to extract certain information about the input signal.

Examples

- **Accumulator**

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

$$= y[n-1] + x[n]$$

- The output $y[n]$ at time instant n is the sum of input sample values $x[n]$ and all past input samples.
- It accumulates all input sample values from $-\infty$ to n.

- Alternative input-output relation expression

$$y[n] = \sum_{l=-\infty}^{-1} x[l] + \sum_{l=0}^{n} x[l]$$

$$= \underbrace{y[-1]}_{\text{Initial condition}} + \underbrace{\sum_{l=0}^{n} x[l]}_{\text{A causal input sequence}}$$

Examples

- ***M*-point Moving-Average Filter**

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

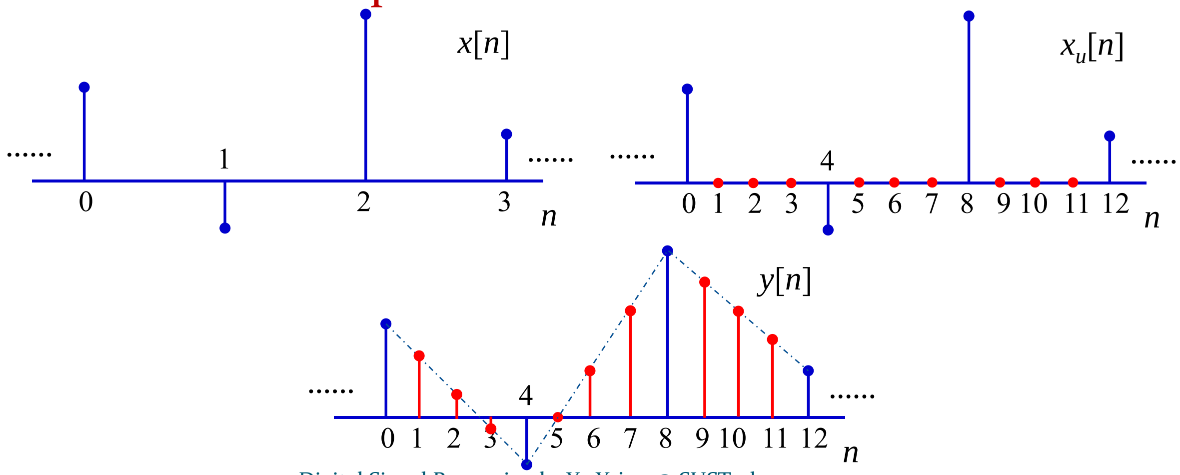
- Used in smoothing random variation in data.
- If there is no bias in measurements, an improved estimate of noise data is obtained by simply increasing M.

- Alternative expression

$$\begin{aligned} y[n] &= \frac{1}{M}\left(\sum_{l=1}^{M-1} x[n-l] + x[n] + x[n-M] - x[n-M]\right) \\ &= \frac{1}{M}\left(\sum_{l=1}^{M} x[n-l] + x[n] - x[n-M]\right) \\ &= \frac{1}{M}\left(\sum_{l=0}^{M-1} x[n-l-1] + x[n] - x[n-M]\right) \\ &= y[n-1] + \frac{1}{M}(x[n] - x[n-M]) \end{aligned}$$

Examples

- **Linear Interpolator** – employed to estimate sample values between pairs of adjacent sample values of a discrete-time sequence
- Factor-of-4 interpolation



- Factor-of-2 interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

- Factor-of-3 interpolator

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-2] + x_u[n+2]) + \frac{2}{3}(x_u[n-1] + x_u[n+1])$$

- Factor-of-2 interpolation

Example: Median Filter

- **Median**
  - The **median** of a set of $(2k+1)$ numbers is the number such that k numbers from the set have values greater than this number, and the other K numbers have values smaller.
  - Median can be determined by rank-ordering the numbers in the set by their values and then choosing the number at the middle.
  - The median of a sequence is denoted as

$$\mathrm{med}\{a_1, a_2, a_3, a_4, a_5\}$$

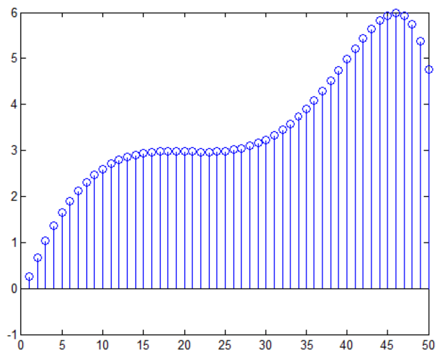
- Example: Consider the set of numbers

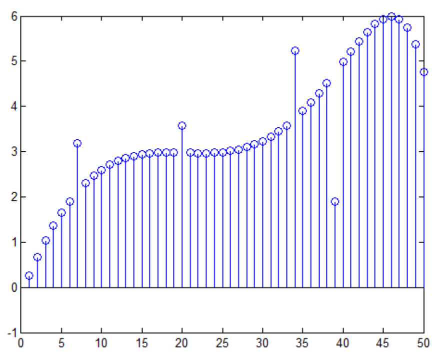
  {8, 2, 6, 12, –4}
- Rank-ordered set: {–4, 2, 6, 8, 12}
- Hence: med{8, 2, 6, 12, –4} = 6

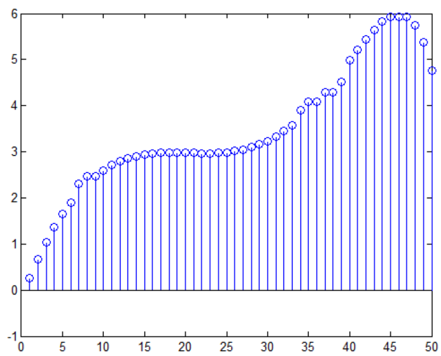
Median Filter

- Median filter is implemented by sliding a window of odd length over the input sequence $x[n]$ one sample at a time. The output $y[n]$ at the nth instant of the median filter with a window length-$(2k+1)$ is then given by
  $y[n]=\text{med}\{x[n-k], \ldots, x[n-1], x[n], x[n+1], \ldots, x[n+k]\}$.

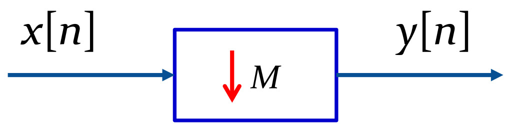
- Median Filter Example:
  - Find applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal, for example signals corrupted by impulse noise.

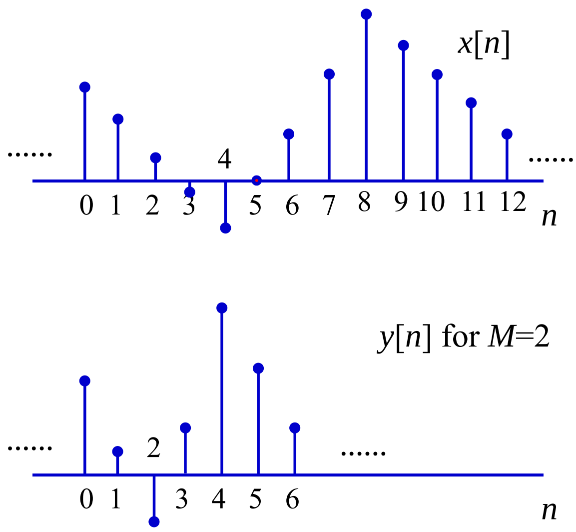






Compressor





- Compressor has an input-output relation given by

$$y[n] = x[Mn] \quad \text{for } M > 1$$

Properties of DT System

- Linearity
- Causality
- Memoryless
- Time-invariance
- BIBO-stability
- Passive and Lossless properties

Properties of DT System

- **Linearity:**

  If $y_1[n] = T\{x_1[n]\}$, and $y_2[n] = T\{x_2[n]\}$
  - Superposition:

  $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$
  - Homogeneity:

  $T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n]$

  Overall: $T\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n]$

- The above property must hold for arbitrary constant a_1 and a_2, and for all possilbe input $x_1[n]$ and $x_2[n]$.

Linearity of Accumulator

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

- Let $y_1[n] = \sum_{l=-\infty}^{n} x_1[l]$, $y_2[n] = \sum_{l=-\infty}^{n} x_2[l]$
- For an input

$$x[n] = \alpha x_1[n] + \beta x_2[n]$$

- The output is

$$y[n] = \sum_{l=-\infty}^{n} (\alpha x_1[l] + \beta x_2[l])$$

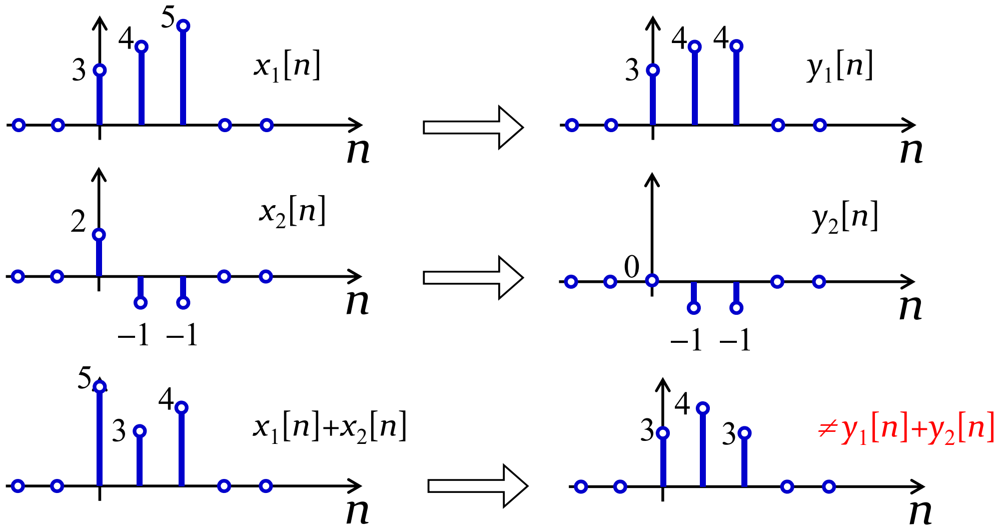
$$= \alpha \sum_{l=-\infty}^{n} x_1[l] + \beta \sum_{l=-\infty}^{n} x_2[l] = \alpha y_1[n] + \beta y_2[n]$$

- Hence, the above system is linear.

Linearity of Median Filter

- The median filter is a non-linear DT system
- $y[n] = \text{MED}\{x[n-k], \ldots, x[n+k]\}$.

Take $k = 1$



多选题

设置

Which of the following is a linear system

A $y[n] = x[n - n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[n] + 3$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\dfrac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

Properties of DT System (Cont.)

- **Causality:**
  - $y[n_0]$ depends only on $x[n]$ for $-\infty < n \le n_0$, and does not depends on input samples $n > n_0$.

- A non-causal system cannot be implemented because it uses future input signal to generate the current output signal.

- Example of causal systems:

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$
$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$$
$$y[n] = y[n-1] + x[n]$$

- Example of non-causal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$
$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+1]) + \frac{2}{3}(x_u[n-2] + x_u[n+2])$$

多选题

设置

Which of the following is a causal system

A $y[n] = x[n-n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \dots \\ 0, \qquad\qquad \text{otherwise} \end{cases}$ for $L > 1$

提交

Implementation of non-causal system

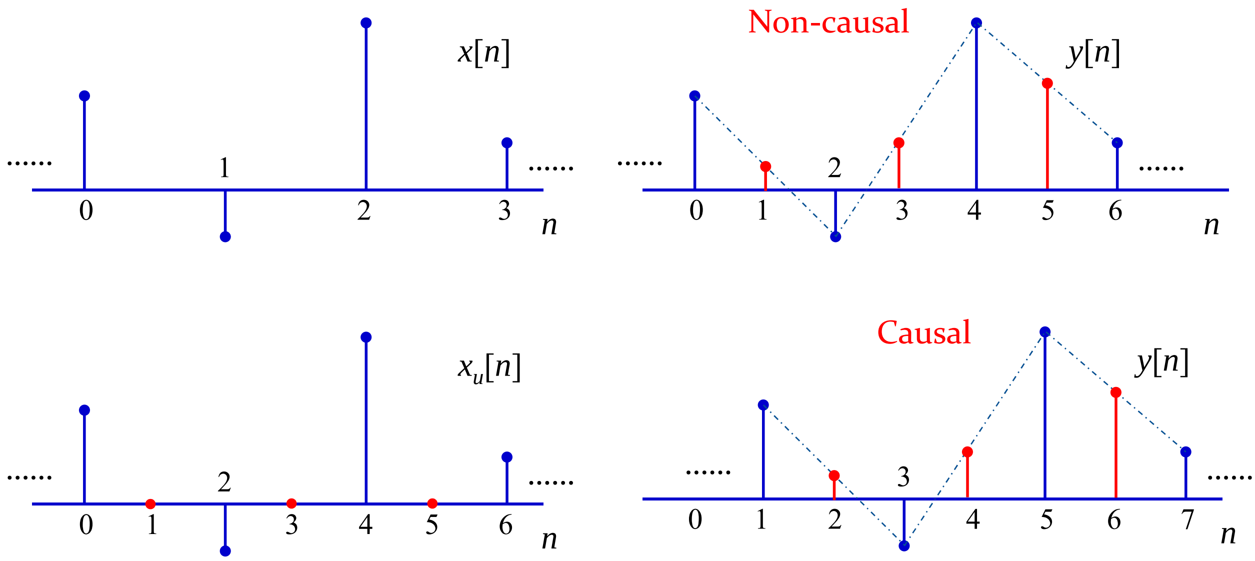
- A non-causal system may be implemented as a causal system by delaying the output by an appropriate number of samples
- For example a causal implementation of the factor-of-2 interpolator is given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

non-causal expressions of a factor-of-2 interpolator:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

Numerical Example



Properties of DT System (Cont.)

- **Memoryless:**
  - $y[n_0]$ depends only on $x[n_0]$, and does not depend on input samples for $n<n_0$.
  - Example of memoryless system: $y[n] = x[n]^2$
  - Example of memory system:

$$y[n] = a_1x[n] + a_2x[n-1] + a_3x[n-2] + a_4x[n-3]$$
$$y[n] = y[n-1] + x[n]$$

多选题

设置

Which of the following is memoryless?

A $y[n] = x[n - n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \qquad \text{otherwise} \end{cases}$ for $L > 1$

提交

Properties of DT System (Cont.)

- **Time invariance:**

  If: $y[n] = T\{x[n]\}$

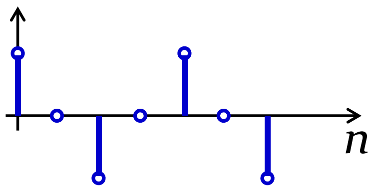
  Then: $y[n-n_0] = T\{x[n-n_0]\}$ for all integer n_0

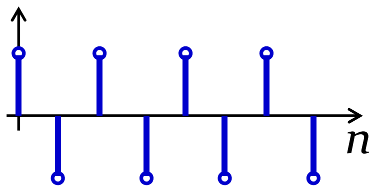
- The above relation must hold for arbitrary input and its corresponding output.
- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied.

Compressor is time-invariant or not? -- Numerical Example

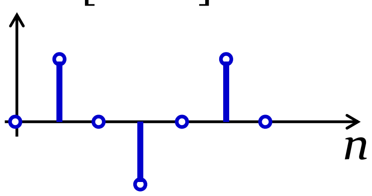
- Suppose M=2, $y[n] = x[Mn]$

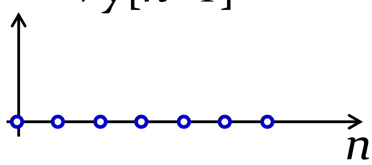
$x[n]=\cos(\pi/2\ n)$ $\Rightarrow$ $y[n]$





$x[n-1]$ $\neq y[n-1]$





- Proof:
  - If an input $\{x[n]\}$ at time n produces an output $y[n]$, the system is time invariant if a time shifted input $x[n-n_0]$ produces a time shifted output $y[n-n_0]$, i.e. if

$$x[n] \Rightarrow y[n]$$

then $x[n-n_0] \Rightarrow y'[n] = y[n-n_0]$.

  - Consider the system $y[n] = x[Mn]$, we have

$$x[n] \Rightarrow x[Mn] = y[n],$$

$$x[n-n_0] \Rightarrow x[Mn-n_0] = y'[n].$$

  - $y[n]$ shifted by n_0 is $y[n-n_0] = x[M(n-n_0)] \neq x[Mn-n_0]$ $y'[n]$.
  - Therefor, $y[n] = x[Mn]$ is not time invariant if $M \neq 1$.

多选题

设置

Which of the following is time-invariant?

A $y[n] = x[n - n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \dots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

Properties of DT System (Cont.)

- **BIBO stability:**

  If: $|x[n]| \leq B_x < \infty \qquad \forall n$

  Then: $|y[n]| \leq B_y < \infty \qquad \forall n$

- Example – The M-point moving average filter is BIBO stable:
  - For a bounded input $|x[n]| \leq B_x$, we have

$$|y[n]| = \left| \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \right| \leq \frac{1}{M} \sum_{k=0}^{M-1} |x[n-k]| \leq \frac{1}{M} M B_x$$

$$= B_x$$

多选题

设置

Which of the following is BIBO?

A $y[n] = x[n - n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

Properties of DT System (Cont.)

- **Passive and Lossless**

Passive if: $$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Lossless if the inequality is satisfied with an equal sign for every input sequence,

- **A passive Discrete-time system**

$$y[n] = \alpha x[n-N]$$

- Since $\sum_{n=-\infty}^{\infty} |y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2$

  it is a passive system if $|\alpha| \leq 1$ and is a lossless system if $|\alpha| = 1$

多选题

设置

Which of the following is passive?

A $y[n] = x[n-n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \qquad \text{otherwise} \end{cases}$ for $L > 1$

提交

Summary

	Causal	Linear	Time-Invariant	Memory-less	BIBO Stable	Passive
Time Shift $y[n] = x[n-n_d]$						
Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$						
Compressor $y[n] = x[Mn]$ for $M > 1$						
Up-sampler $y[n] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L \\ 0 & \text{otherwise} \end{cases}$ for $L > 1$						

LTI Discrete Time Systems

- **Linear Time-Invariant (LTI) System** – a system satisfying both the linearity and the time-invariant properties.

- LTI systems are mathematically easy to analyze and characterize, and consequently easy to design.

- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades.

Impulse and Step Responses

- The response of a DT system to a unit impulse sequence $\{\delta[n]\}$ is called the **unit impulse response** or simply, the **impulse response**, denoted as $\{h[n]\}$.

- The response of a DT system to a unit step sequence $\{\mu[n]\}$ is called the **unit step response** or simply, the **step response**, denoted as $\{s[n]\}$.

Impulse Response

- Example – The impulse response of system

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

is obtained by setting $x[n] = \delta[n]$, resulting in

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

- The impulse response is thus a finite length sequence of length 4 given by

$$\{h[n]\} = \{\underset{\uparrow}{a_1}, a_2, a_3, a_4\}$$

Impulse Response

- Example – The impulse response of the discrete-time accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$

By setting $x[n]= \delta[n]$, we have

$$h(n) = \sum_{k=-\infty}^{n} \delta[k]$$

which is precisely the unit step sequence

Impulse Response

- Example – The impulse response $h[n]$ of factor-of-2 interpolator

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

is obtained by setting $x_u[n] = \delta[n]$, resulting in

$$h[n] = \delta[n-1] + \frac{1}{2}(\delta[n-2] + \delta[n])$$

- The impulse response is thus a finite length sequence of length 3 given by

$$h[n] = \{0.5, \underset{\uparrow}{1}, 0.5\}$$

Time-Domain Characterization of LTI Discrete Time Systems

- Input-output relation – A consequence of the linear and time-invariant properties is that an LTI discrete time system is completely characterized by its impulse response

- In other words, knowing the impulse response one can compute the output of the system for an arbitrary input

Compute Impulse Response

- Let $h[n]$ denote the impulse response of a LTI discrete-time system.
- We compute its output $y[n]$ for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

- As the system is linear, we can compute its outputs for each term in the input separately and add the individual outputs to determine $y[n]$.

Compute Impulse Response

- Since the system is time-invariant, we have

Input		Output
$\delta[n+2]$	$\rightarrow$	$h[n+2]$
$\delta[n-1]$	$\rightarrow$	$h[n-1]$
$\delta[n-2]$	$\rightarrow$	$h[n-2]$
$\delta[n-5]$	$\rightarrow$	$h[n-5]$

Compute Impulse Response

- Likewise, as the system is linear, we have

Input		Output
$0.5\delta[n+2]$	$\rightarrow$	$0.5h[n+2]$
$1.5\delta[n-1]$	$\rightarrow$	$1.5h[n-1]$
$\delta[n-2]$	$\rightarrow$	$h[n-2]$
$0.75\delta[n-5]$	$\rightarrow$	$0.75h[n-5]$

- Hence, because of the linear property, we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Compute Impulse Response

- Recall, an arbitrary input sequence $x[n]$ can be expressed as a linear combination of delayed and advanced unit impulse sequence in the form

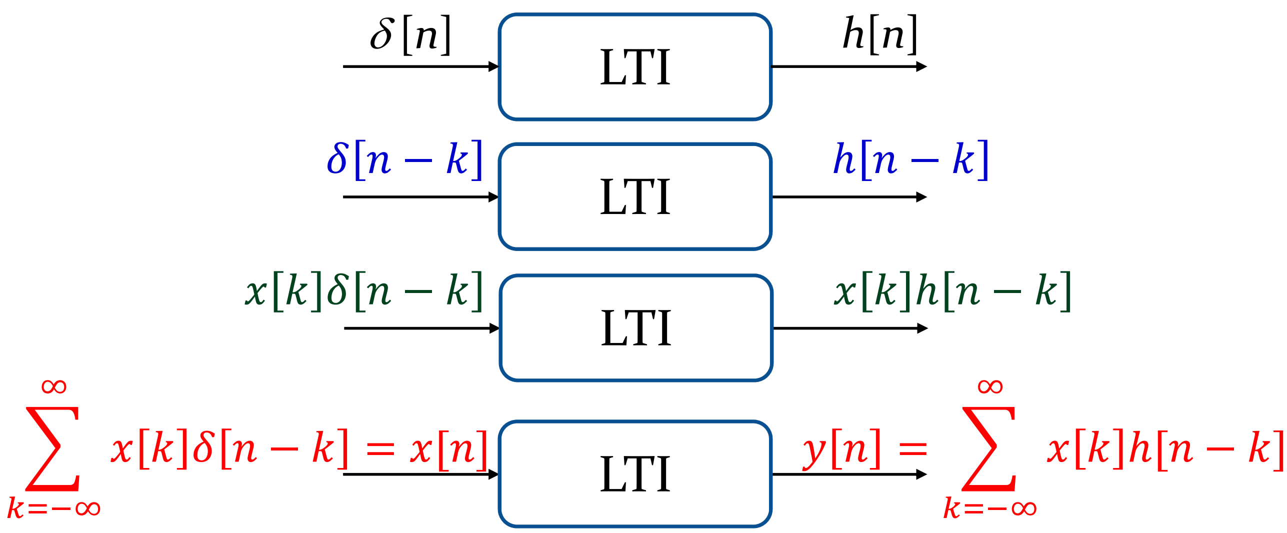
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The response of the LTI system to an input $x[k]\delta[n-k]$ will be $x[k]h[n-k]$
- Hence, the overall output is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Compute Impulse Response

- The impulse response $h[n]$ completely characterizes an LTI system. “DNA of LTI”



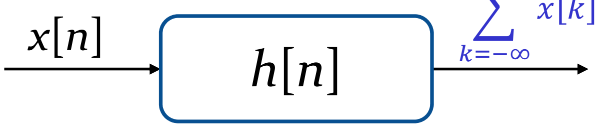
Compute Impulse Response

- Mathematically,

$$y[n] = \mathrm{LTI}\{x[n]\} = \mathrm{LTI}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathrm{LTI}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete (Linear) Convolution



$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv x[n] \circledast h[n]$$

Discrete Convolution

$$x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Sum of weighted and delayed impulse response

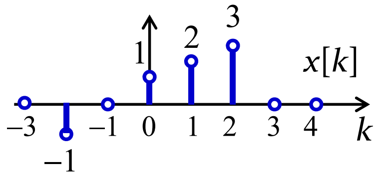
$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] \circledast x[n]$$

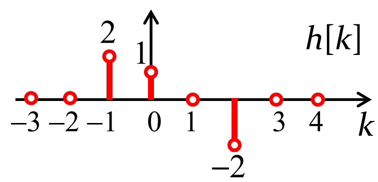
The above summation is defined to be the convolution of the sequences $x[n]$ and $h[n]$ and represented compactly as

$$y[n] = x[n] \circledast h[n]$$

An Illustrative Example

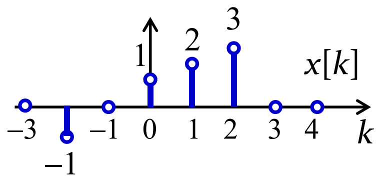
Compute the convolution of $x[n]$ and $h[n]$

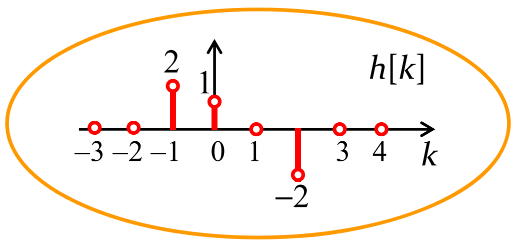


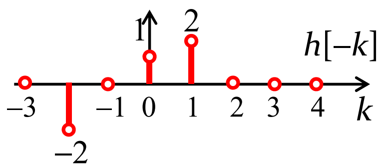


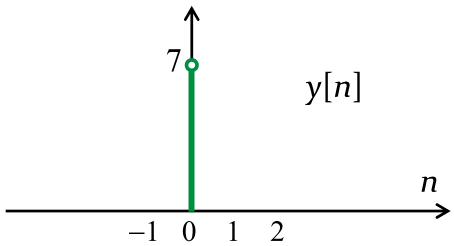
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$





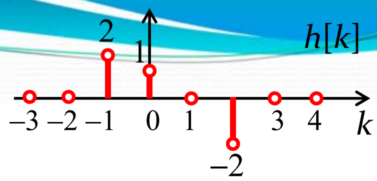


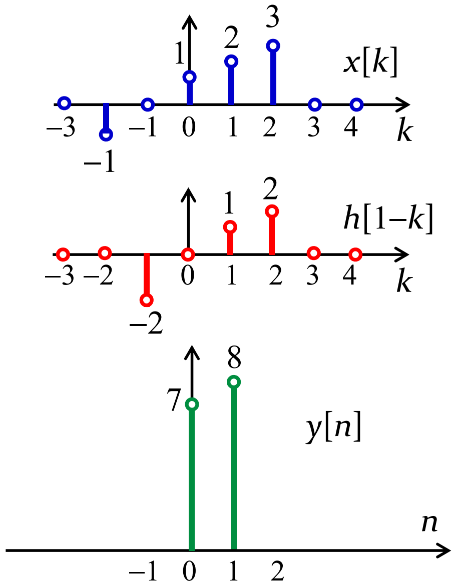


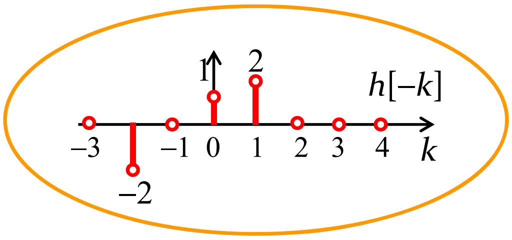
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

$$= x[-2]h[2] + x[-1]h[1] + x[0]h[0]$$
$$+ x[1]h[-1] + x[2]h[-2]$$
$$= -1 \times (-2) + 1 \times 1 + 2 \times 2 = 7$$



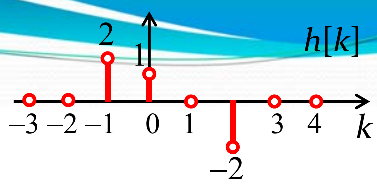


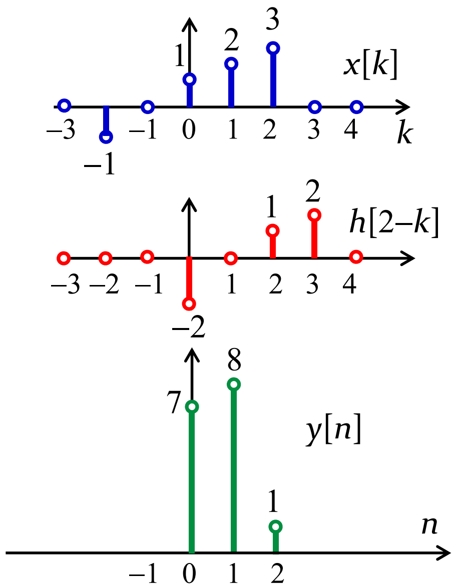


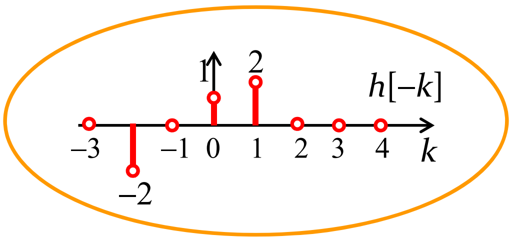
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

$$= x[-2]h[3] + x[-1]h[2] + x[0]h[1]$$
$$+x[1]h[0] + x[2]h[-1]$$
$$= 2\times1+3\times2 = 8$$



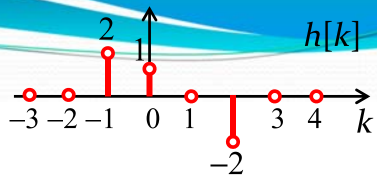


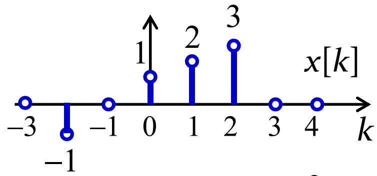


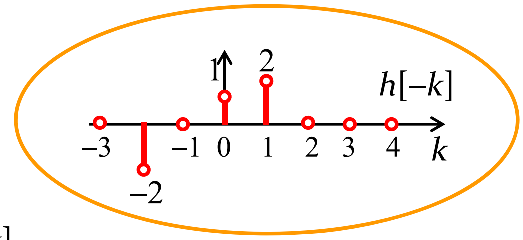
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

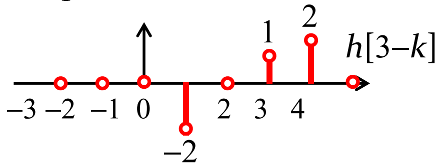
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

$$= x[-2]h[4] + x[-1]h[3] + x[0]h[2]$$
$$+x[1]h[1] + x[2]h[0]$$
$$= 1\times(-2)+3\times1=1$$

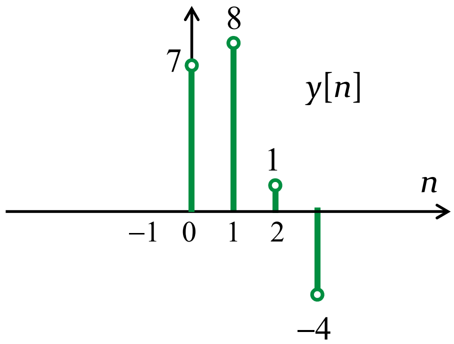






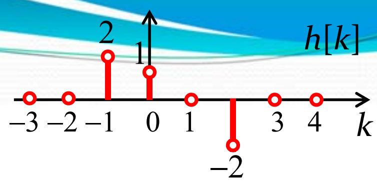


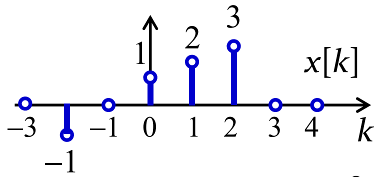
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

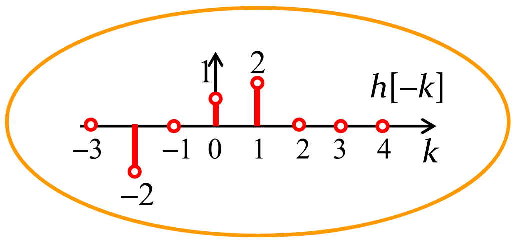


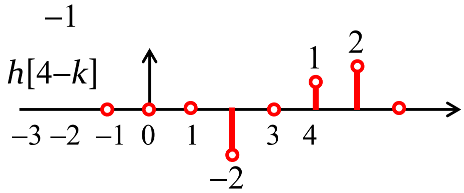
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$

$$= x[-2]h[5] + x[-1]h[4] + x[0]h[3]$$
$$+x[1]h[2] + x[2]h[1]$$
$$= 2\times(-2) = -4$$

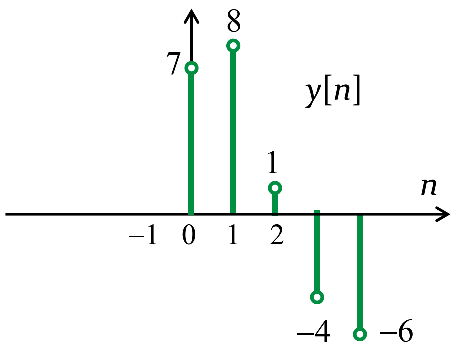






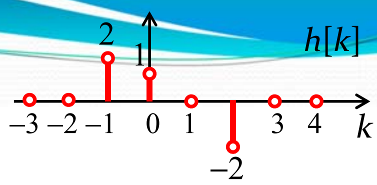


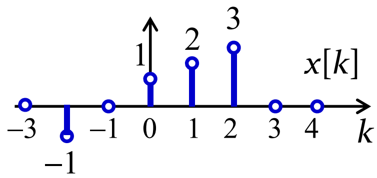
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

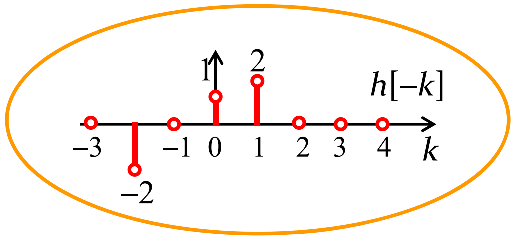


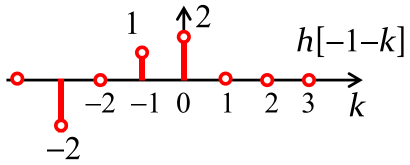
$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

$$= x[-2]h[6] + x[-1]h[5] + x[0]h[4]$$
$$+x[1]h[3] + x[2]h[2]$$
$$= 3\times(-2) = -6$$

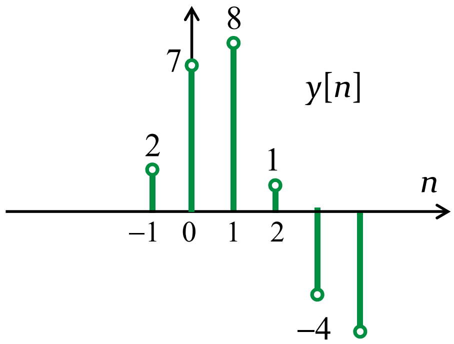






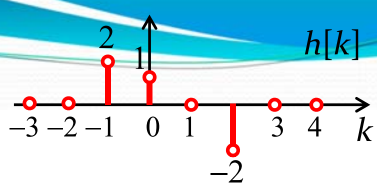


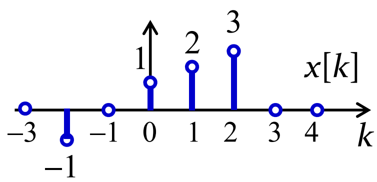
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

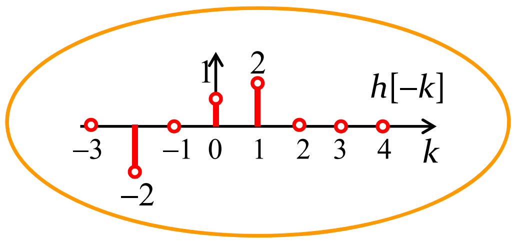


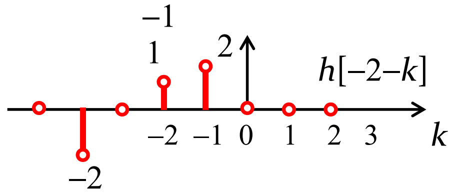
$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]h[-1-k]$$

$$= x[-2]h[1] + x[-1]h[0] + x[0]h[-1]$$
$$+x[1]h[-2] + x[2]h[-3]$$
$$= 1 \times 2 = 2$$

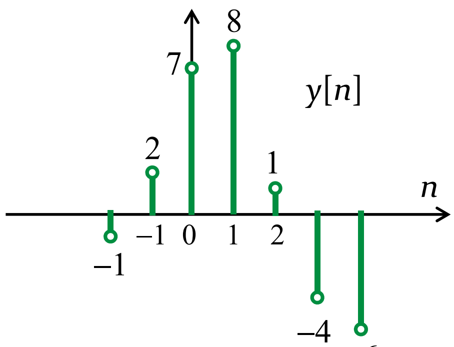






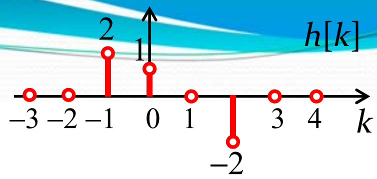


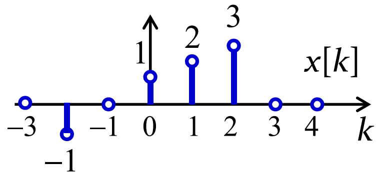
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

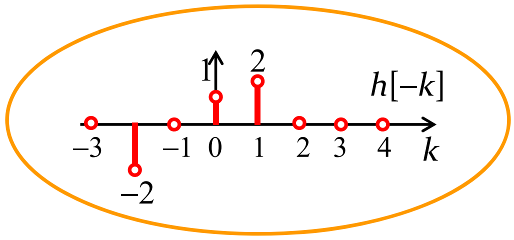


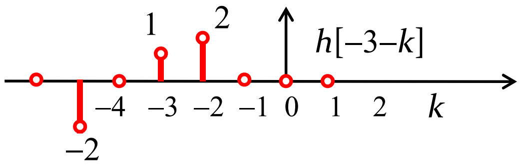
$$y[-2] = \sum_{k=-\infty}^{\infty} x[k]h[-2-k]$$

$$= x[-2]h[0] + x[-1]h[-1] + x[0]h[-2]$$
$$+x[1]h[-3] + x[2]h[-4]$$
$$= -1 \times 1 = -1$$

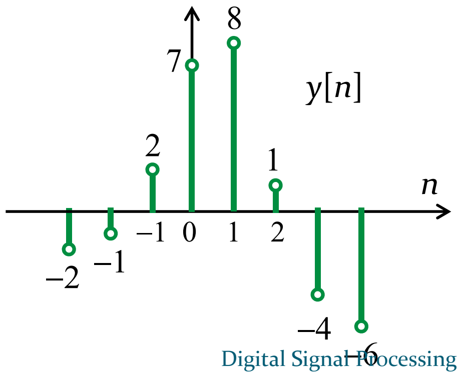








$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



$$y[-3] = \sum_{k=-\infty}^{\infty} x[k]h[-3-k]$$

$$= x[-2]h[-1] + x[-1]h[-2] + x[0]h[-3] + x[1]h[-4] + x[2]h[-5]$$

$$= -1 \times 2 = -2$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Computation of Convolution

- Fold $h[k]$ with respect to the origin to obtain $h[-k]$.
- Form the product of the corresponding samples of sequences $h[-k]$ and $x[k]$, and sum all product terms to obtain $y[0]$.
- Shift $h[-k]$ to the right by n samples (more specifically, shift to right by $|n|$ samples if $n>0$ and shift to left by $|n|$ samples if $n<0$).
- Form the product of the corresponding samples of sequences $h[n-k]$ and $x[k]$, and sum all product terms to obtain $y[n]$.
- Overall, fold, shift, product, and sum

- <u>Note:</u> The sum of indices of each sample product inside the convolution sum is equal to the index of the sample being generated by the convolution operation
  - For example, the computation of $y[-3]$ in the previous example involves the products, $x[-2]h[-1]$, $x[-1]h[-2]$, $x[0]h[-3]$, $x[1]h[-4]$, and $x[2]h[-5]$
  - The sum of indices in each of these products is equal to -3
- In general, if the lengths of the two sequences being convolved are M and N, then the sequence generated by the convolution is of length $M+N-1$

Example:

- If

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

i.e., $y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

- How about

$$x[n] \longrightarrow \boxed{h[n-m]} \longrightarrow y_1[n]$$

$$y_1[n] = x[n] \circledast h[n-m] = \sum_{k=-\infty}^{\infty} x[k]h[n-m-k]$$

$$= y[n-m]$$

Properties of Convolution

- Communitive: $x[n] \circledast h[n] = h[n] \circledast x[n]$
- Associative: $x[n] \circledast (h[n] \circledast g[n]) = (x[n] \circledast h[n]) \circledast g[n]$
- Linear:

$$x[n] \circledast (\alpha h[n] + \beta g[n]) = \alpha x[n] \circledast h[n] + \beta x[n] \circledast g[n]$$

- Sequence shifting is equivalent to convolve with a shifted impulse

$$x[n-d] = x[n] \circledast \delta[n-d]$$

Computation of LTI System Output

- In practice, if either the input or the impulse response, or both of them, are finite length, the convolution can be used to compute the output sample, as it involves a finite sum of products.
- If both the input sequence and the impulse response sequence are of infinite length, convolution can **NOT** be used to compute the output.
- For system characterized by an infinite impulse response sequence, an alternate time-domain description involving a finite sum of products have to be considered.

BIBO Stability of LTI Systems

- Recall, BIBO stability condition:
  - iff $\{y[n]\}$ remains bounded for any bounded input sequence $\{x[n]\}$

- An LTI system is BIBO stable iff $h[n]$ is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

BIBO Stability of LTI Systems

- Proof: "if" **(Sufficient Condition)**

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]|\, \underbrace{|x[n-k]|}_{\leq B_x}$$

$$\leq B_x \underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}_{< \infty} \leq B_y$$

BIBO Stability of LTI Systems

- Proof: "only if" **(Necessary Condition)**

suppose $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, show that for that $h[k]$, there always exists a bounded $x[n]$ that gives unbounded $y[n]$.

- Let:

Assume $h[n]$ is real sequence

$$x[n] = \frac{h[-n]}{|h[-n]|} = \text{sign}\{h[-n]\}$$

$$\text{sign}\{h[-n]\} \rightarrow \boxed{h[n]} \rightarrow$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} \frac{h[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

BIBO Stability of LTI Systems

- Example – Consider an LTI discrete-time system with an impulse response $h[n] = \alpha^n \mu[n]$
- For this system:

$$S = \sum_{n=-\infty}^{\infty} |\alpha^n|\mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}, \text{if } |\alpha| < 1$$

- Therefor $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable.
- If $|\alpha| = 1$, the system is not BIBO stable.

Causality of LTI Systems

- An LTI system is causal iff

$$h[k] = 0 \qquad \text{for } k < 0$$

- Proof: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$= \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k]$$

Sufficient: Since $h[k] = 0$ for $k < 0$, the first term is 0. So, $y[n_0]$ is independent of $x[n_0+1]$, $x[n_0+2]$, … .
Necessary: The system is causal, i.e., $y[n_0]$ independent of $x[n_0+1]$, $x[n_0+2]$, … , implying that the first term is equal to 0. Since $x[n]$ may not be equal to 0, $h(k)$ must be equal to 0 for $k < 0$

Examples

- **Causal LTI system**
  - Accumulator: $y[n] = \sum_{l=-\infty}^{n} x[l]$

    $h[n] = \mu[n]$, a causal impulse response sequence

- **Non causal LTI system**
  - Factor-of-2 interpolator

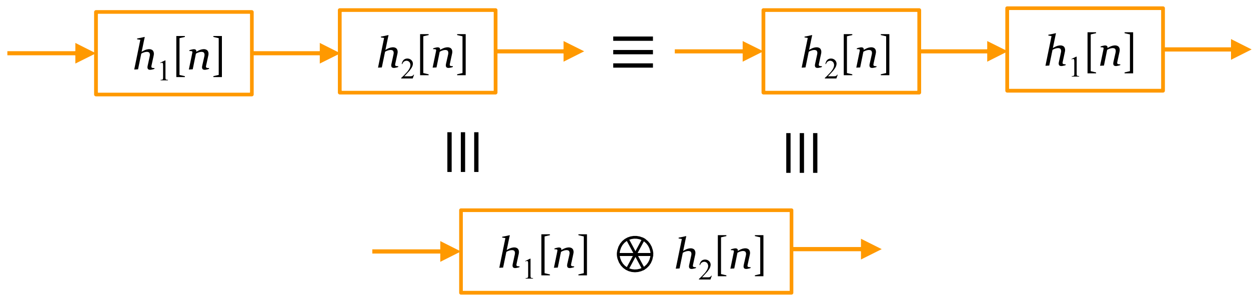
$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$h[n] = \{0.5, \underset{\uparrow}{1}, 0.5\}$$

a non causal impulse response sequence

Simple Interconnection Schemes

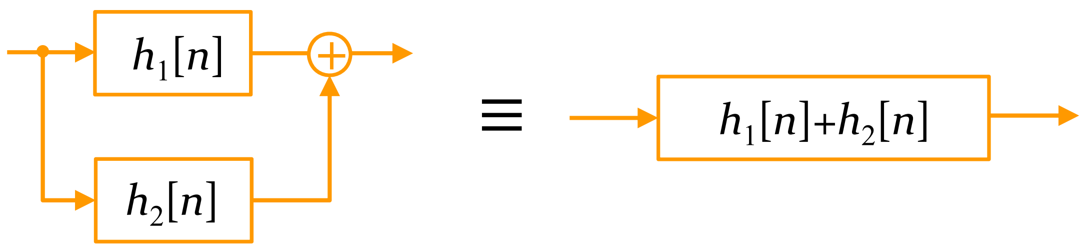
- Cascade Connection



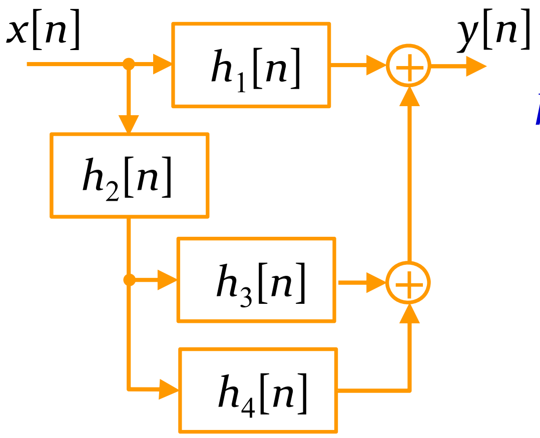
- If $h_1[n] \circledast h_2[n] = \delta[n]$

  system $h_1[n]$ is said to be the **inverse** of system $h_2[n]$, and vice versa.

- Parallel Connection



Analysis of Cascade and Parallel connections



$$y[n] = x[n] \circledast h[n]$$

$$h[n]=h_1[n] + h_2[n] \circledast (h_3[n]+h_4[n])$$

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

主观题

设置

$$h[n]=h_1[n] + h_2[n] \circledast (h_3[n]+h_4[n])$$

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

What is the equivalent $h[n]$? Use $d[n]$ to represent $\delta[n]$ if the later is not able to be keyed in.

作答

General Difference Equation

- An important subclass of LTI system is characterized by a linear constant-coefficient difference equation of the form:

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

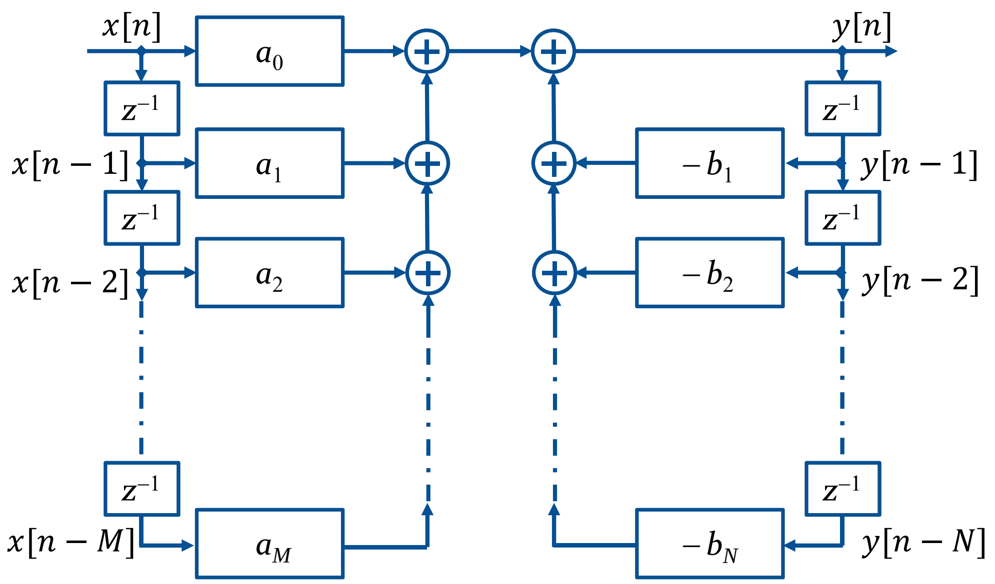
- $y[n]$ can be computed recursively

$$b_0 y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

Signal-flow graph

- When b_0 is normalized to $b_0 = 1$

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$



Classification of LTI System

- Based on Impulse Response Length
  - If $h[n]$ is of **finite** length, then it is known as a **finite impulse response (FIR)** discrete time system.

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

  - Examples: Moving averaged filter, factor-2-interpolator

$$y[n] = \sum_{k=0}^{4} \frac{1}{5} x[n-k]$$

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

- If $h[n]$ is of **infinite** length, then it is know as a **infinite impulse response (IIR)** discrete-time system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- If causal (system causal, input sequence causal),

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

- The class of IIR system we are concerned with is the causal system characterized by the linear constant coefficient difference equation.
- Example: accumulator

$$y[n] = \sum_{l=-\infty}^{n} x[l] = y[n-1] + x[n]$$

- Based on the Output Calculation Process
  - **Non-recursive** discrete time system: Computation of output samples involves only the present and past input samples. Example: FIR system.
  - **Recursive** discrete time system: Computation of output samples involves NOT ONLY the present and past input samples, but also the **past output** samples. Example: IIR system implemented using difference equation.
  - It's possible to implement an FIR system using a recursive scheme, e.g., $y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$

- Based on the Coefficient Values
  - **Real coefficient** filters.
  - **Complex coefficient** filters.