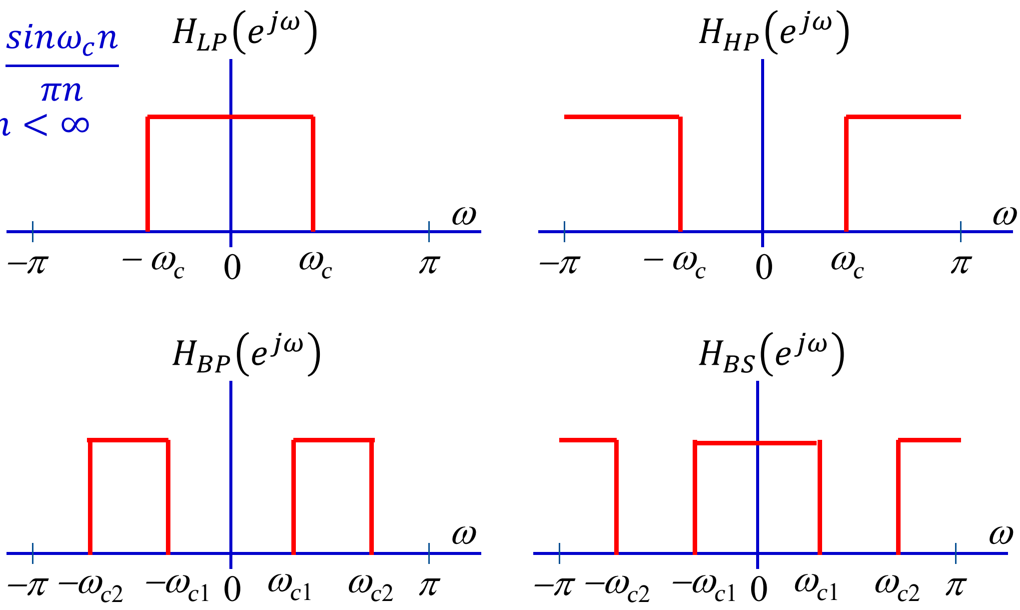
Lecture 10
Digital Filter Design

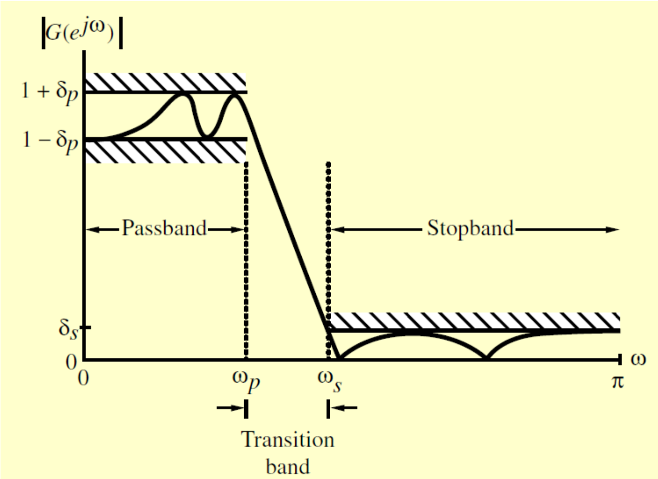
Filter Specifications

- Ideal but not practical specifications:

$$h_{LP}[n] = \frac{sin\omega_c n}{\pi n}$$
$$-\infty < n < \infty$$



Typical magnitude Specifications



- Passband edge: ω_p
- Stopband edge: ω_s
- Peak ripple value in passband: δ_p
- Peak ripple value in stopband: δ_s

- **Passband:** $\omega \le \omega_p, \quad 1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p$
- **Stopband:** $\omega_s \le \omega \le \pi, \quad |G(e^{j\omega})| \le \delta_s$
- **Transition band:** $\omega_p < \omega < \omega_s$, arbitrary response

Specifications Given as Loss function

- Loss Function

$$\mathcal{A}(\omega) = -20\log_{10}|G(e^{j\omega})|$$

- Peak passband ripple:

$$\alpha_p = -20\log_{10}(1-\delta_p), \text{in dB}$$

- Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s), \qquad \text{in dB}$$

- **Example of ripples:** the peak passband ripple α_p and the minimum stopband attenuation α_s of a digital filter are, respectively, 0. 1 dB and 35dB. Determine their corresponding peak ripple values δ_p and δ_s.
- A: $\delta_p = 1 - 10^{-\frac{\alpha_p}{20}} = 1 - 10^{-0.005} = 0.0144690$

$$\delta_s = 10^{-\frac{\alpha_s}{20}} = 10^{-1.75} = 0.01778279$$

Obtain Band Edge Frequencies

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz, along with sampling frequencies
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_{\text{sampling}}} = \frac{2\pi F_p}{F_{\text{sampling}}} = 2\pi F_p T_{\text{sampling}}$$

$$\omega_s = \frac{\Omega_s}{F_{\text{sampling}}} = \frac{2\pi F_s}{F_{\text{sampling}}} = 2\pi F_p T_{\text{sampling}}$$

- **Example** – ECG signal typically exhibits frequencies in the range from 0.01 Hz to 150 Hz. Some studies are interested to low frequency range 0.03 Hz to 0.12 Hz, and high frequency range 0.12 Hz to 0.488 Hz. If the ECG signal is sampled at 300 Hz, what are the passband edges for filters to extract the corresponding signal? How if the sampling frequency is 200 Hz?
- A: Low frequency part:

$$\omega_{p1} = \frac{0.03\times 2\pi}{300} = 0.0002\pi,\ \omega_{p2} = \frac{0.12\times 2\pi}{300} = 0.0008\pi.$$

B: High frequency part:

$$\omega_{p1} = \frac{0.12\times 2\pi}{300} = 0.0008\pi,\ \omega_{p2} = \frac{0.488\times 2\pi}{300} = 0.00325\pi.$$

Selection of Filter Type

- **Considerations:**
  - Certainly, filters must be causal and stable.
  - Complexity is proportional to the filter length. The lower order the filter, the better.
  - For FIR filter, if linear phase is required, the coefficients must symmetric.
- **Advantages in using FIR filters:**
  - Always stable
  - Can be designed with exact linear phase
- **Disadvantages in using FIR filters:**
  - Usually need a higher order than IIR

IIR Filter Design

- Most common approach to IIR filter design -

  (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications

  (2) Determine the analog lowpass filter transfer function $H(S)$

  (3) Transform $H(S)$ into the desired digital transfer function $G(z)$

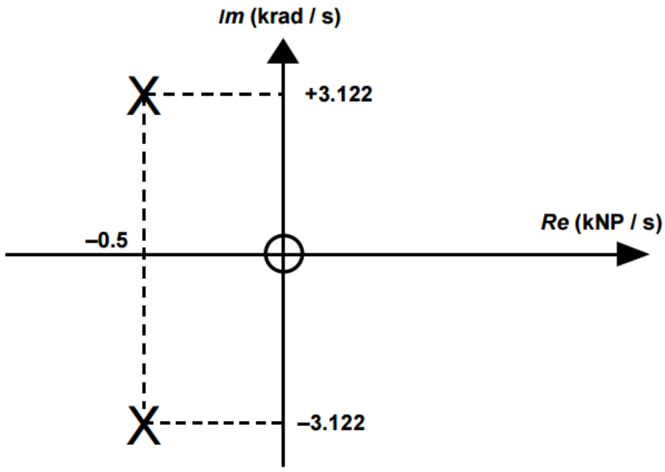
Analog Filter and *s*-plane

- The transfer function of analog filters are given by $H(s)$. The frequency response of $H(s)$ is evaluated at $s = j\Omega$.
- Example:

$$H(s) = \frac{10^3 s}{s^2 + 10^3 s + 10^7}$$

Factorizing the equation gives:

$$H(s) = \frac{10^3 s}{[s - (-0.5 + j3.122) \times 10^3][s - (-0.5 - j3.122) \times 10^3]}$$

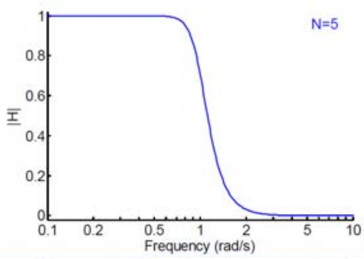


- A zero at origin and two poles
- $H(j\Omega) = \frac{10^3 j\Omega}{[j\Omega-(-0.5+j3.122)\times 10^3][j\Omega-(-0.5-j3.122)\times 10^3]}$

Analog Filters

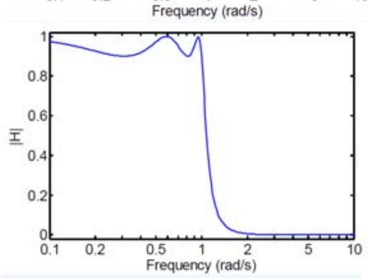
Assume cutoff frequency $\Omega_c = 1$

- Classical continuous-time filters optimize tradeoff:
  - passband ripple vs. stopband ripple vs. transition width
- Butterworth: $G_a^2(\Omega) = |H_a(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$
  - Monotonic for all Ω
  - $G_a(\Omega) = 1 - \frac{1}{2}\Omega^{2N} + \frac{3}{8}\Omega^{4N} + \cdots$
  - "Maximally flat", $2N-1$ derivatives are 0



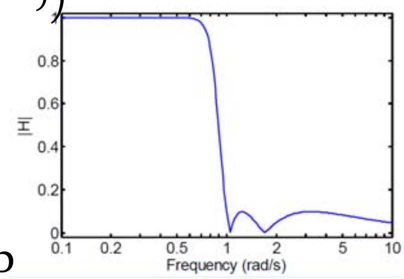
- Chebyshev: $G_a^2(\Omega) = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}$

  where polynomial $T_N(\cos x) = \cos Nx$
  - Passband equiripple + very flat at ∞

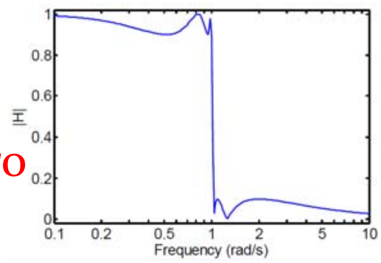


Analog Filters

- Inverse Chebyshev: $G_a^2(\Omega) = \frac{1}{1+\left(\epsilon^2 T_N^2(\Omega^{-1})\right)^{-1}}$
  - Stopband equiripple + very flat at 0



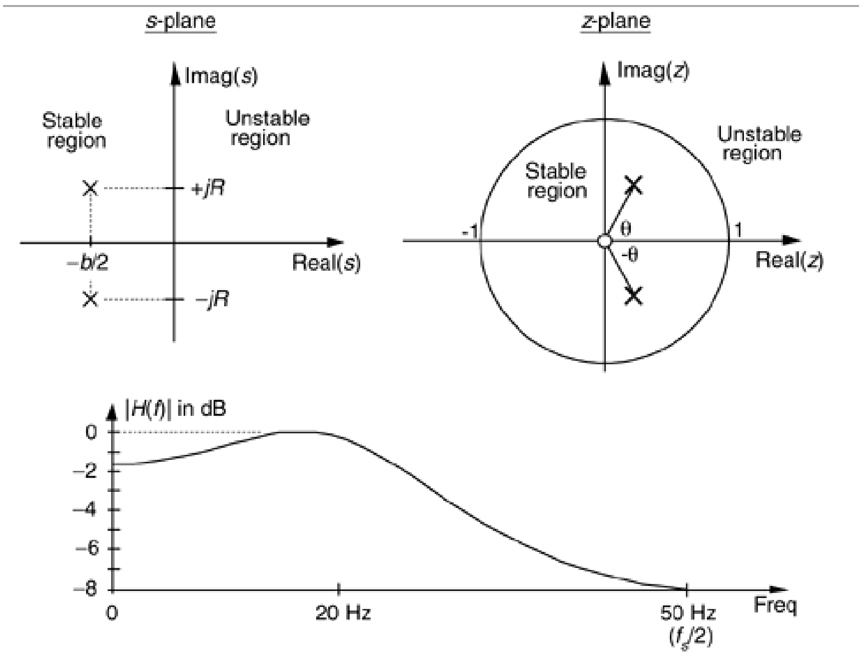
- Elliptic (No nice formula)
  - Very steep + equiripple in pass and stop bands



- There are explicit formulae for pole/zero positions.

Mapping s-plane to z-plane

- Basic idea behind the conversion of $H(s)$ into $G(z)$ is to apply a mapping from the s-domain to the z-domain so that essential properties of the analog frequency response are preserved
- Thus mapping function should be such that
  - Imaginary ($j\Omega$) axis in the s-plane be mapped onto the unit circle of the z-plane (to **preserve the frequency selective properties**)
  - Left-half of the s-plane be mapped inside the unit circle (to **ensure a stable digital transfer function**).



Bilinear Transformation

$$s = k\left(\frac{1-z^{-1}}{1+z^{-1}}\right), k > 0, \qquad \text{or } z = \frac{k+s}{k-s}$$

- Thus, relation between $G(z)$ and $H(s)$ is then given by

$$G(z) = H(s)\Big|_{s=k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

- When $s = j\Omega$, $z = \frac{k+j\Omega}{k-j\Omega} = \frac{\sqrt{k^2+\Omega^2}e^{j\tan^{-1}\frac{\Omega}{k}}}{\sqrt{k^2+\Omega^2}e^{-j\tan^{-1}\frac{\Omega}{k}}} = e^{2j\tan^{-1}\frac{\Omega}{k}}$

  $\therefore |z| = 1 \Longrightarrow z = e^{j\omega}$, and

  $$\omega = 2\tan^{-1}\frac{\Omega}{k}, \text{ or } \Omega = k\tan\left(\frac{\omega}{2}\right)$$

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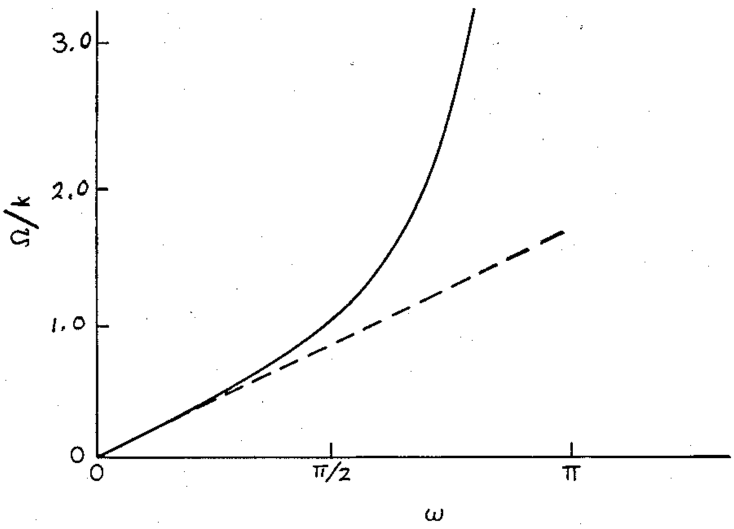


The relation between analog and digital frequency scales for the bilinear transformation.

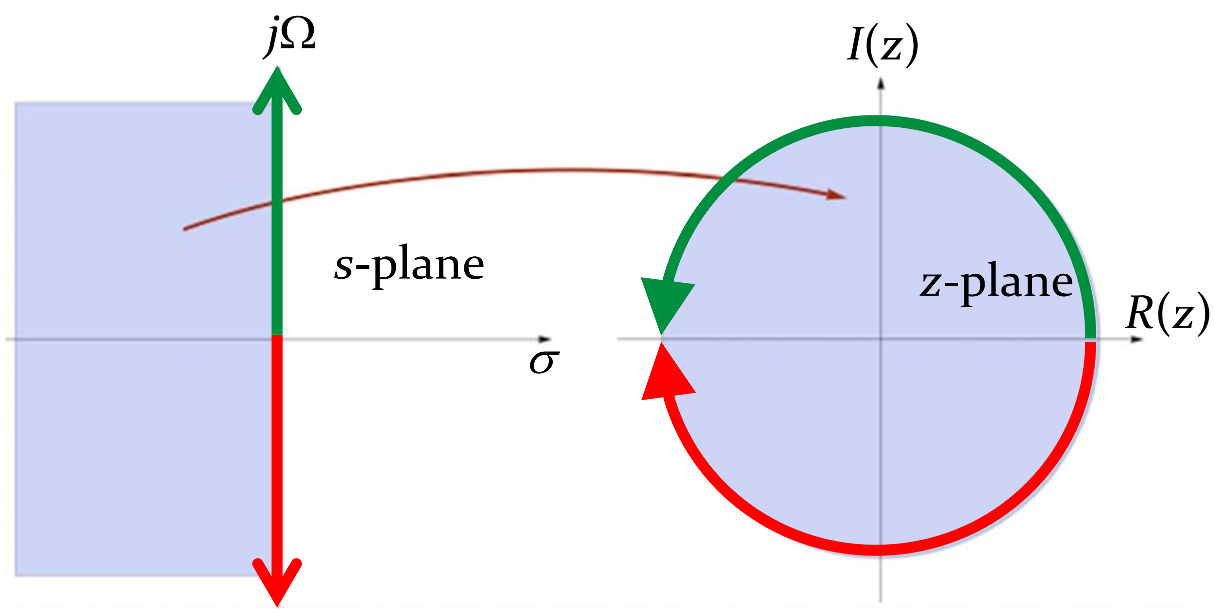
- When $\Omega = 0$, $\omega = 0$. When $\Omega \to \infty$, $\omega \to \pi$.
- Hence, the $j\Omega$-axis is mapped into the unit circle.

- For $s = \sigma_0 + j\Omega_0$, $\sigma_0 < 0$

$$z = \frac{(k+\sigma_0) + j\Omega_0}{(k-\sigma_0) - j\Omega_0} \Rightarrow |z| = \sqrt{\frac{(k+\sigma_0)^2 + {\Omega_0}^2}{(k-\sigma_0)^2 + {\Omega_0}^2}} < 1$$

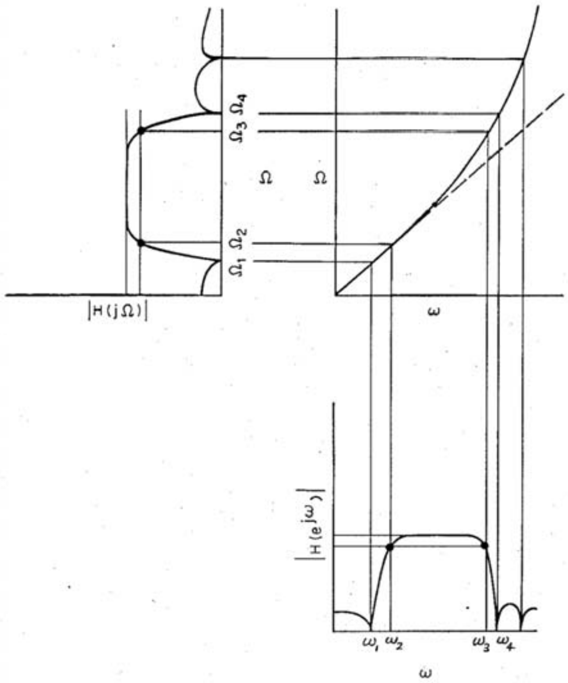
i.e., the left-half s-plane is mapped inside the unit circle.

- Mapping is highly nonlinear
  - Complete negative imaginary axis in the s-plane from $\Omega = 0$ to $\Omega = -\infty$ is mapped into the lower half of the unit circle in the z-plane from $z = 1$ to $z = -1$
  - Complete positive imaginary axis in the s-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the z-plane from $z = 1$ to $z = -1$



Frequency Warping

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- • Effect of warping



Design Steps

- **Step 1:** Develop the specifications of $H(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$
- **Step 2**: Design $H(s)$
- **Step 3:** Determine $G(z)$ by applying bilinear transformation to $H(s)$

Example: (About Step 3)

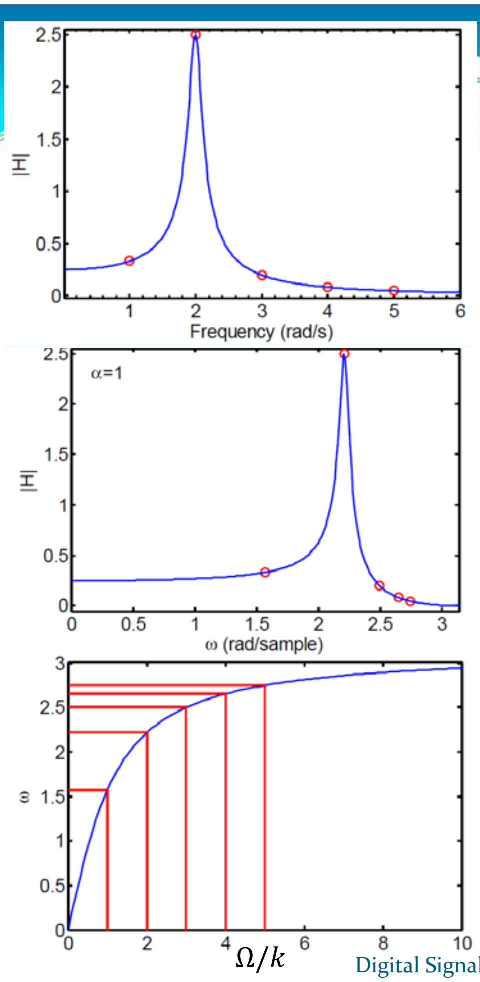
- Transform $H(s) = \frac{1}{s^2+0.2s+4}$ to digital filter
- A: Choose $k = 1$. Substitute $s = k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ to $H(s)$

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 4}$$

Extra zeros at z=−1

$$= \frac{(1+z^{-1})^2}{(1-z^{-1})^2 + 0.2(1-z^{-1}) + 4(1+z^{-1})^2}$$

$$= 0.19\frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$



- Frequency response is identical (both magnitude and phase) but with a distorted frequency axis
- Frequency mapping:

$$\omega = 2\tan^{-1}\frac{\Omega}{k}$$

  - $\Omega = [k, 2k, 3k, 4k, 5k]$
  $\rightarrow \omega = [1.6, 2.2, 2.5, 2.65, 2.75]$

- Choices of k
  - Set $k = \frac{\Omega_0}{\tan\frac{\omega_0}{2}}$ to map $\Omega_0 \rightarrow \omega_0$
  - Set $k = 2f_s = \frac{2}{T}$ to map low frequencies to themselves.

Example: Butterworth Lowpass

- Transform $H_a(s) = \frac{1}{s+1}$ into a lowpass digital filter transfer function with 32 kHz sampling frequency and 4kHz 3dB frequency. (General form: $H_a(s) = \frac{\Omega_c}{s+\Omega_c}$)
- A: $H_a(j\Omega) = \frac{1}{j\Omega+1}$,

  $\therefore$ the 3dB frequency of the analog filter is $\Omega = 1$

  It is given that the 3dB frequency of digital filter is at

$$\omega = 2\pi \frac{4k\text{Hz}}{32k\text{Hz}} = \frac{\pi}{4}$$

- Use the bilinear transformation

$$s \rightarrow k\left(\frac{1-z^{-1}}{1+z^{-1}}\right),$$

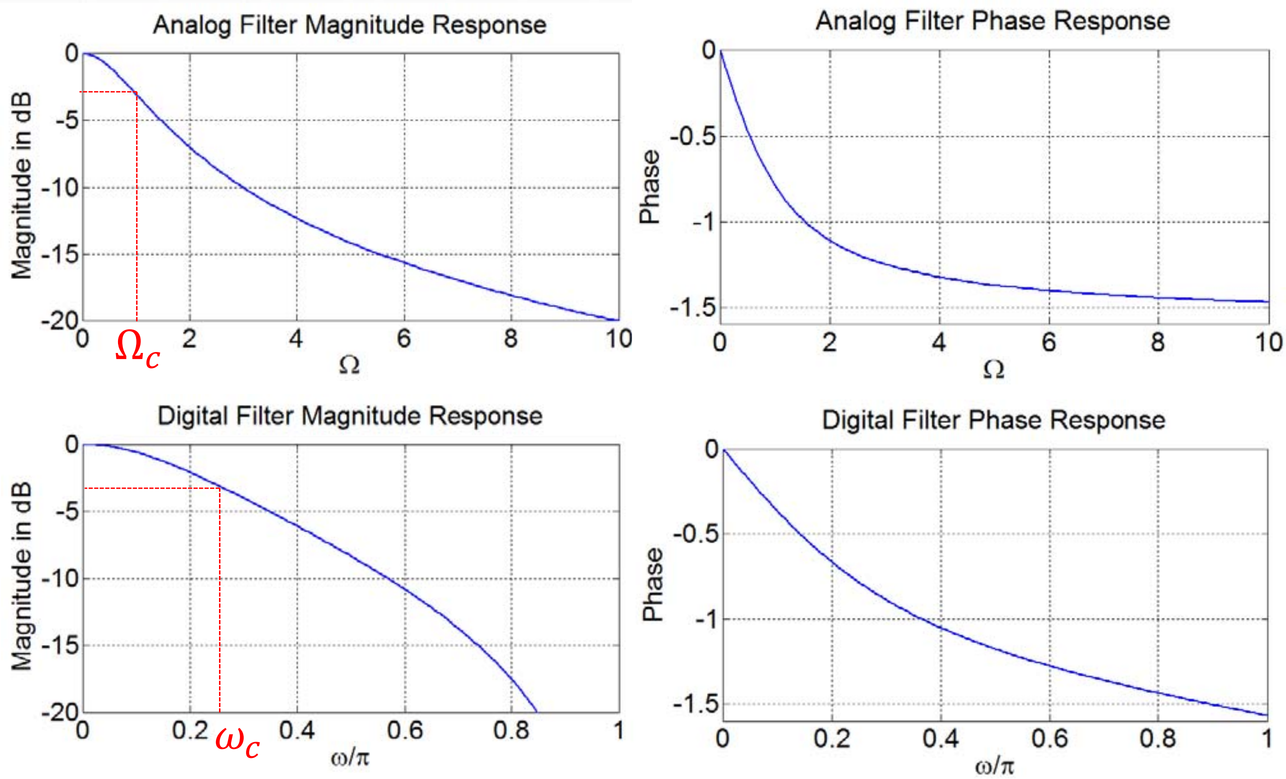
  where k is given by

$$\Omega = k\tan\left(\frac{\omega}{2}\right)$$

- Hence, we have $1 = k\tan\left(\frac{\pi}{8}\right)$

  $\therefore k = 2.414$ and

$$H(z) = \frac{1}{1 + 2.414\dfrac{1-z^{-1}}{1+z^{-1}}}$$



Determine Analog Filter Specifications (Step 1)

- **Objective:**
  - from $\{\omega_p, \omega_s, \delta_p, \delta_s\}$ to determine $\{\Omega_p, \Omega_s, \delta_p, \delta_s\}$
- **Solution 1:**
  - We can arbitrary choose Ω_p, for example to be Ω_0, because by choosing $k = \frac{\Omega_0}{\tan\frac{\omega_p}{2}}$ we can map ω_p to any chosen Ω_0.
  - Using the chosen k to determine Ω_s, i.e.,

$$\Omega_s = \frac{\Omega_0}{\tan\frac{\omega_p}{2}} \tan\left(\frac{\omega_s}{2}\right)$$

- **Solution 2:**
  - We can arbitrary choose k, for example choose $k = 1$.
  - Using the chosen k to determine Ω_p and Ω_s, i.e.,

$$\Omega_p = k \tan\left(\frac{\omega_p}{2}\right)$$
$$\Omega_s = k \tan\left(\frac{\omega_s}{2}\right)$$

Example (About Step 1)

- A lowpass digital filter is supposed to pass the frequency components with frequencies lower than 4kHz, and block the frequency components with frequencies higher than 8kHz, assuming that the sampling frequency is 32 kHz. Determine the frequency bandedges for an analog prototype filter when bilinear transform is used for the digital filter design. (Using solution 1)
- A: $\omega_p = \frac{4k}{32k} 2\pi = 0.25\pi$, $\omega_s = \frac{8k}{32k} 2\pi = 0.5\pi$

  We arbitrary choose $\Omega_p = 1$ radian/second $\Longrightarrow k = 2.414$

  Thus, $\Omega_s = 2.414 \tan\left(\frac{0.5\pi}{2}\right) = 2.414$ radian/second

Spectral Transformation

- **Lowpass to lowpass transformation:**
- We can transform z-plane to change the cutoff frequency by substituting

$$z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha \hat{z}^{-1}} \Leftrightarrow \hat{z}^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

  - where z^{-1} and $\hat{z}^{-1}$ denote the unit delay in the proto-type lowpass digital filter and the transformed filter.
- Frequency mapping
  - If $z = e^{j\omega}$, $\hat{z} = \frac{1+\alpha e^{-j\omega}}{e^{-j\omega}+\alpha} \Longrightarrow |\hat{z}| = 1$. Hence, the unit circle is preserved.

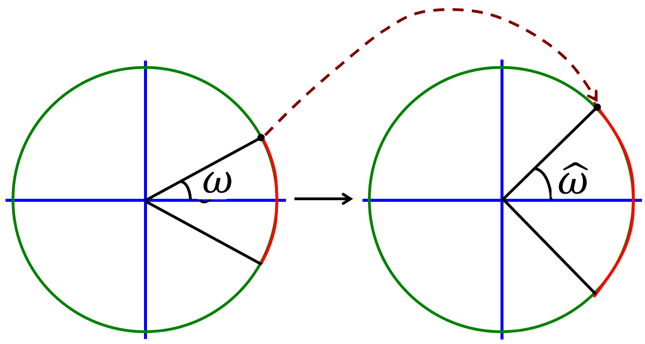
- On unit circle

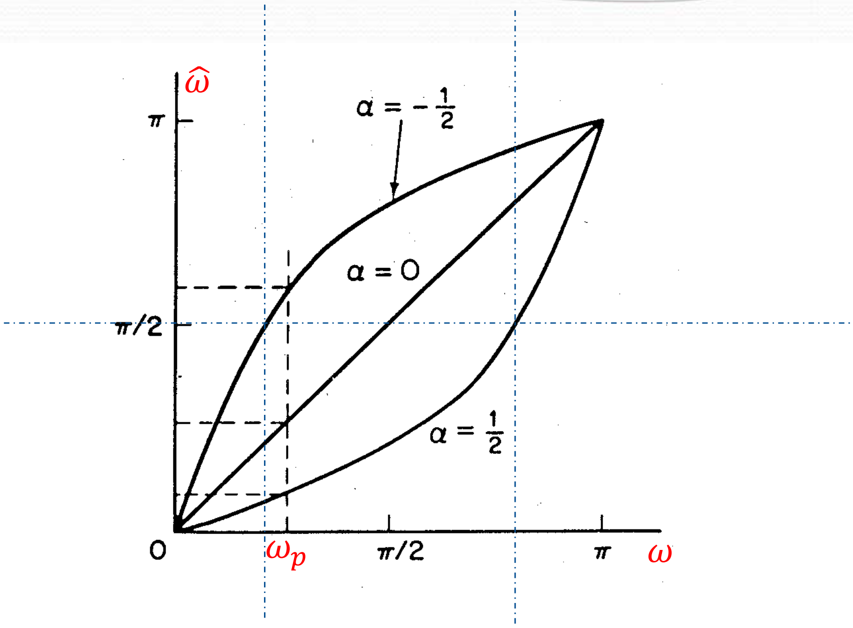
$$e^{-j\omega} = \frac{e^{-j\widehat{\omega}} - \alpha}{1 - \alpha e^{-j\widehat{\omega}}}$$

from which we arrive at

$$\tan\left(\frac{\omega}{2}\right) = \left(\frac{1+\alpha}{1-\alpha}\right)\tan\left(\frac{\widehat{\omega}}{2}\right)$$

$$\alpha = \frac{\sin\left(\frac{\omega - \widehat{\omega}}{2}\right)}{\sin\left(\frac{\omega + \widehat{\omega}}{2}\right)}$$

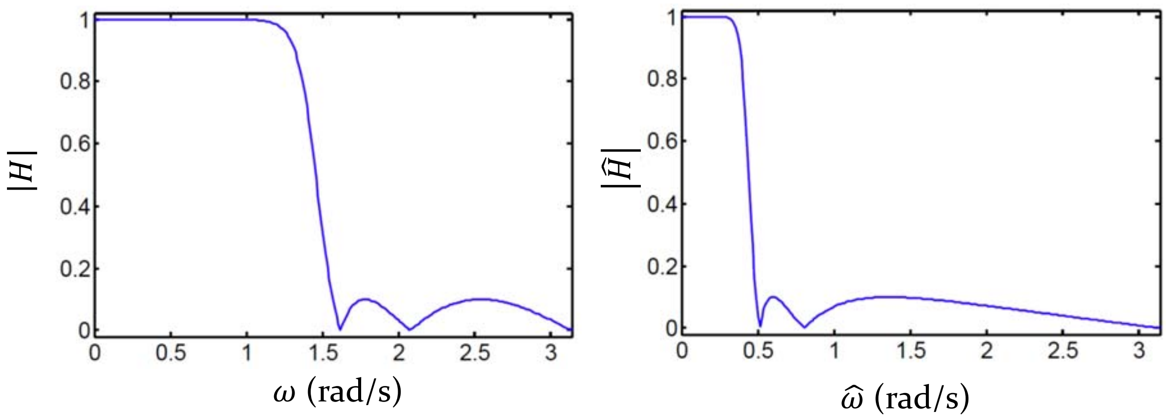




Warping of the frequency scale in
Low-pass – low-pass transformation.

Example

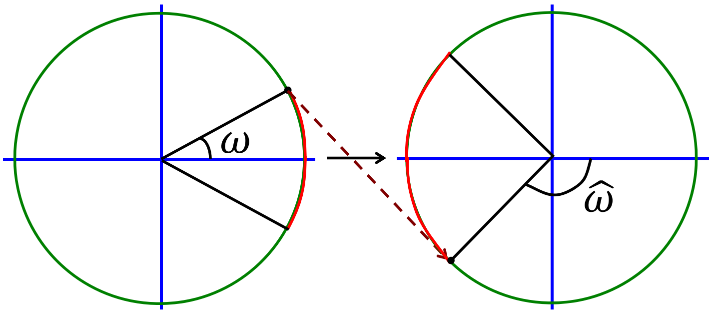
- A 5-th order inverse Chebyshev:
- $\omega_0 = \frac{\pi}{2} = 1.57 \xrightarrow{\alpha=0.6} \widehat{\omega}_0 = 0.49$



- **Lowpass to Highpass Transformation**

$$z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\hat{z}^{-1}} \quad \Longrightarrow \quad \alpha = -\frac{\cos\left(\frac{\omega - \widehat{\omega}}{2}\right)}{\cos\left(\frac{\omega + \widehat{\omega}}{2}\right)}$$

$$\operatorname{ctan}\left(\frac{\omega}{2}\right) = \left(\frac{-1 + \alpha}{1 + \alpha}\right)\tan\left(\frac{\widehat{\omega}}{2}\right)$$



FIR Digital Filter Design

- A causal FIR transfer function $H(z)$ of length N is a polynomial in z^{-1} of degree $N-1$:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$$

- The corresponding frequency response is given by

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n}$$

- $h[n] = \pm h[N-1-n]$ is enforced to ensure a linear phase design.

Basic Approaches

- Windowed Fourier series approach
- Frequency sampling approach
- Computer based digital filter design method

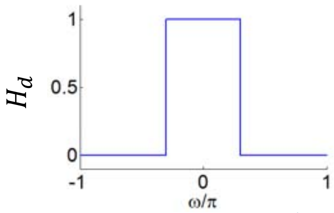
Window Method

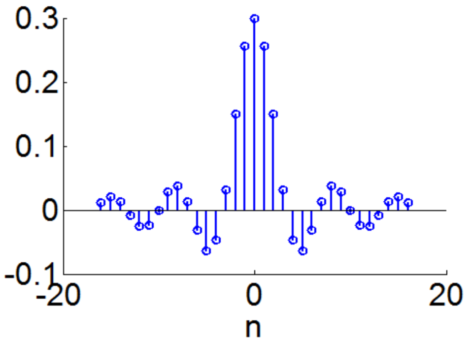
- Inverse DTFT
  - For any BIBO stable filter, $H(e^{j\omega})$ is the DTFT of $h[n]$

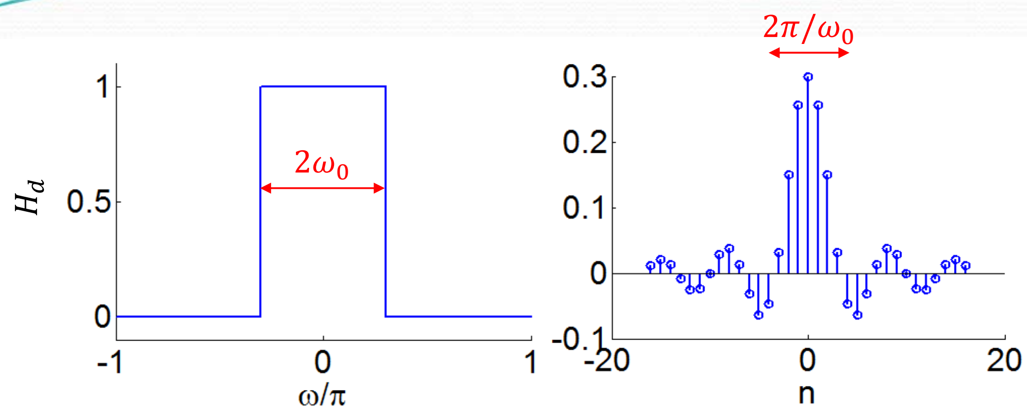
$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \Leftrightarrow h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}\, d\omega$$

  - If we know $H\left(e^{j\omega}\right)$ exactly, the IDTFT gives the ideal $h[n]$
- Example: ideal lowpass filter

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \Leftrightarrow h_d[n] = \frac{\sin \omega_0 n}{\pi n}$$







- **Note:** Width in ω is $2\omega_0$, and width in n is $\frac{2\pi}{\omega_0}$
  - Product is 4π always
- Sadly $h_d[n]$ is infinite and non-causal.
- **Solution:** Multiply $h_d[n]$ by a window

Rectangular Window

- Truncate to $\pm M$ to make finite; $h[n]$ is now of length $2M+1$.
- Mean square error (MSE) Optimality:
  - Define MSE in frequency domain

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right)\right|^2 d\omega$$

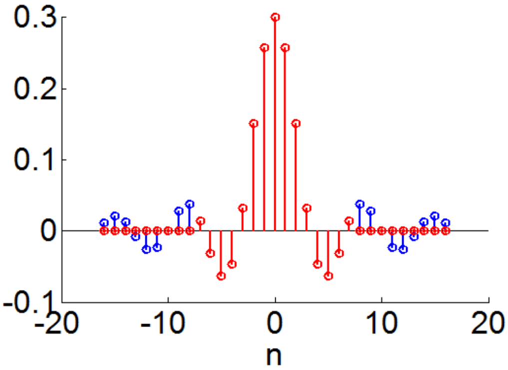
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right) - \sum_{n=-M}^{M} h[n]e^{-j\omega n}\right|^2 d\omega$$

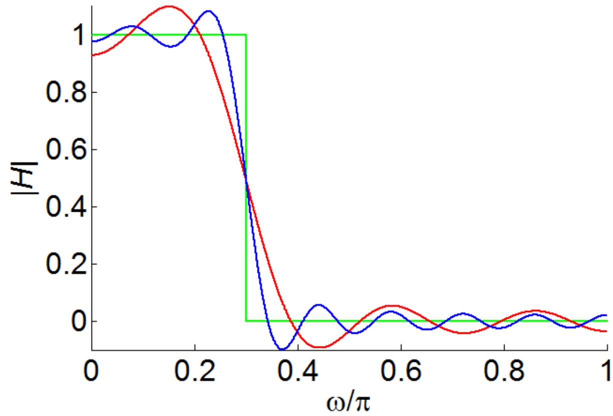
  - Minimum E is when $h[n] = h_d[n]$ for $-M \le n \le M$

- Proof: From Parseval:

$$E = \sum_{n=-M}^{M} |h_d[n] - h[n]|^2 + \sum_{|n|>M} |h_d[n]|^2$$

- However, 9% overshoot at a discontinuity even for large M.

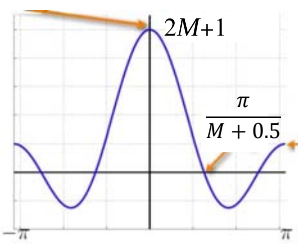


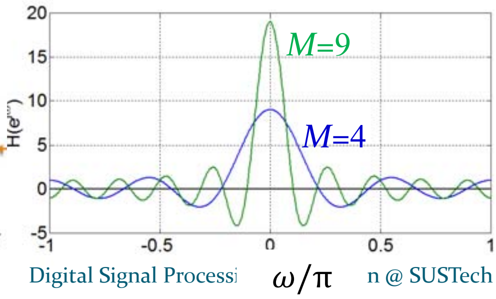


- Normal to delay by M to make causal. Multiply $H(e^{j\omega})$ by $e^{-jM\omega}$

Gibbs Phenomenon

- Truncation
  - $\Leftrightarrow$ Multiply $h_d[n]$ by a rectangular window $w_R[n] = \sum_{k=-M}^{M} \delta[n-k]$ (in time domain)
  - $\Leftrightarrow$ Convolution $H_{2M+1}(e^{j\omega}) = \frac{1}{2\pi} H_d(e^{j\omega}) \circledast W_R(e^{j\omega})$ (in frequency domain)
  - $W_R(e^{j\omega}) = \frac{\sin\frac{(2M+1)\omega}{2}}{\sin\frac{\omega}{2}}$, Width of mainlobe is $\frac{4\pi}{2M+1}$

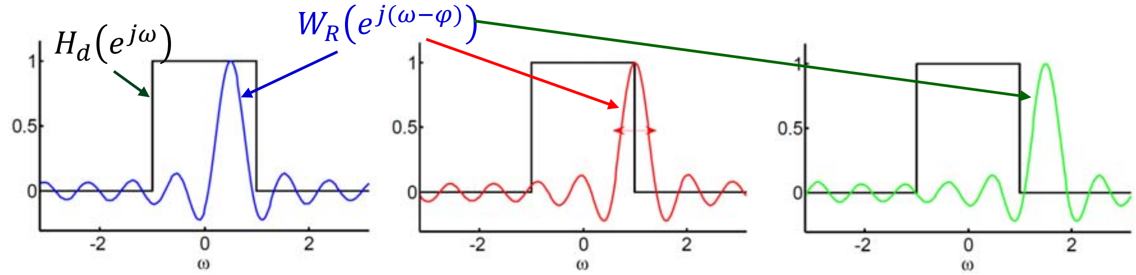


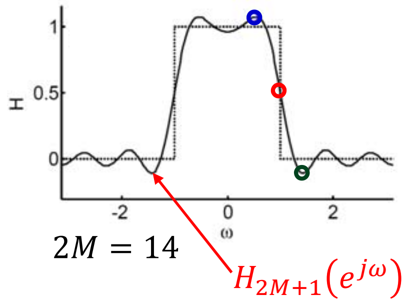


When M increase, the width of the main lobe decreases, but the height increases.

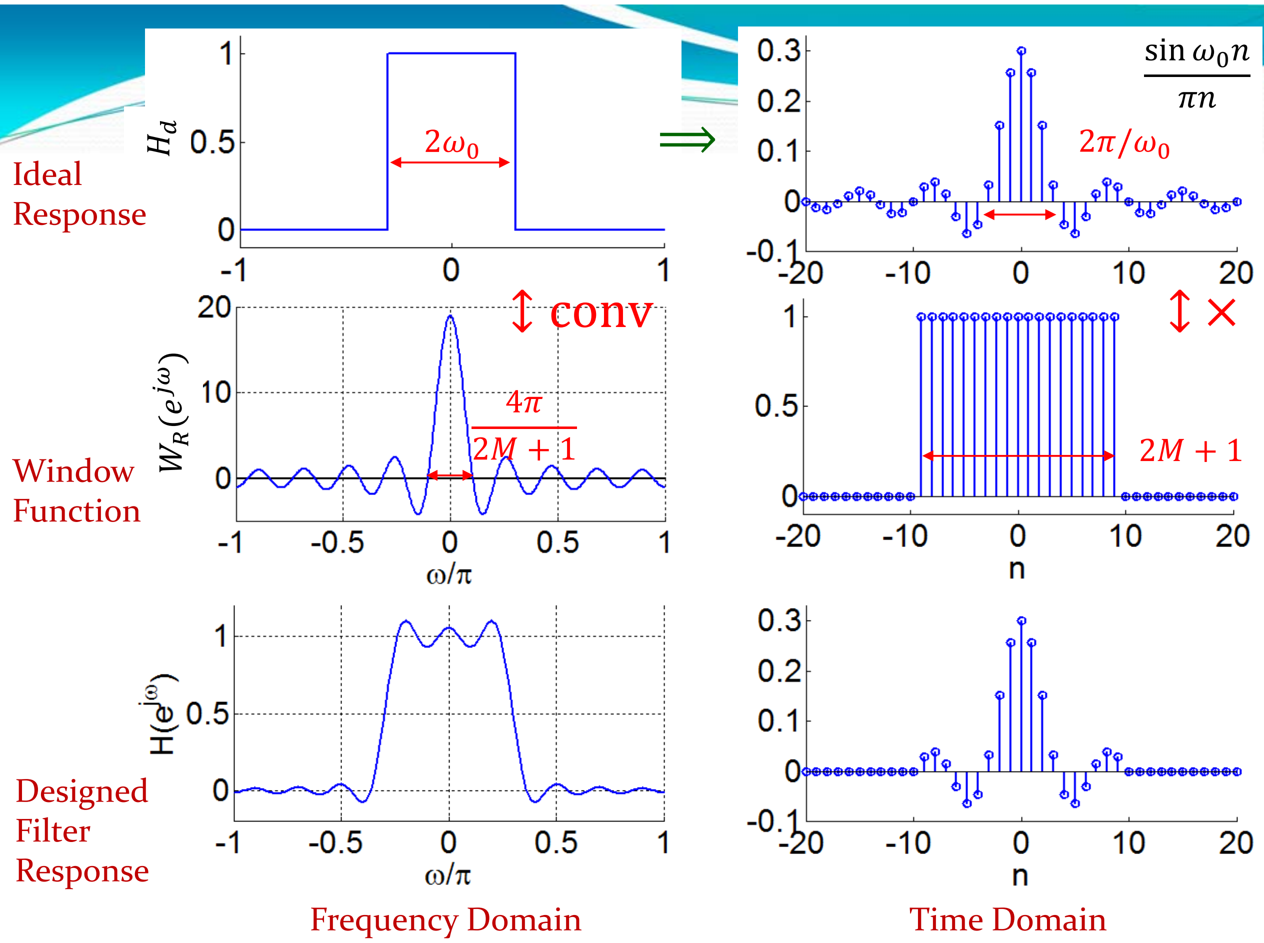
- Effects: convolve ideal frequency response with periodic sinc function.

$$H_{2M+1}\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d\left(e^{j\varphi}\right)W_R\left(e^{j(\omega-\varphi)}\right)d\varphi$$





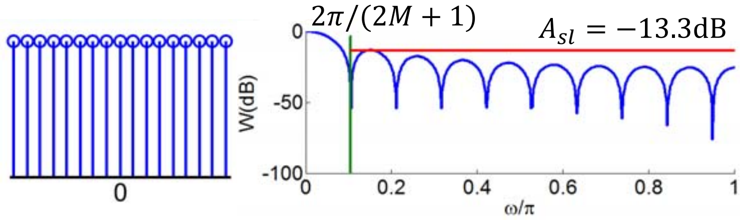
- The convolution is the area under the curve of the product of the two waveforms.
- When M increases, ripples in $H_{2M+1}\left(e^{j\omega}\right)$ around the cutoff frequency occur closely but with no decrease in amplitude.



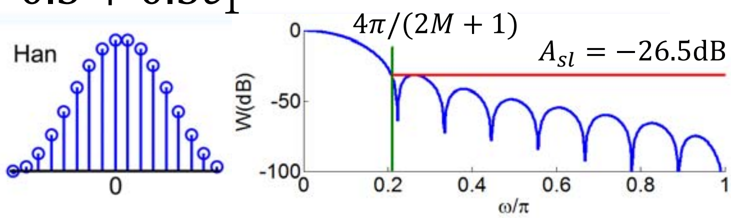
Fixed Windows

- Consider length $N = 2M + 1$ windows, for $-M \le n \le M$

- Rectangle: $w_R[n] \equiv 1$
  Don't Use

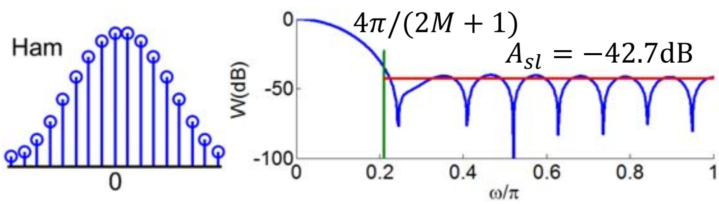


- Hanning: $w_{Hann}[n] = 0.5 + 0.5c_1$
  $c_k = \cos\frac{2k\pi n}{N}$
  Rapid sidelobe decay

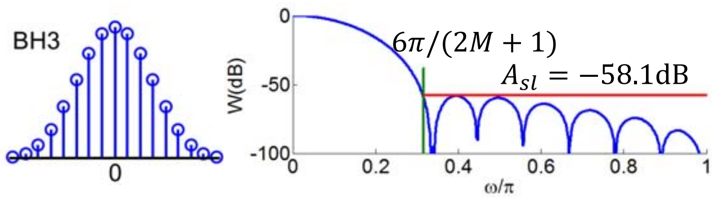


Fixed Windows (cont'd)

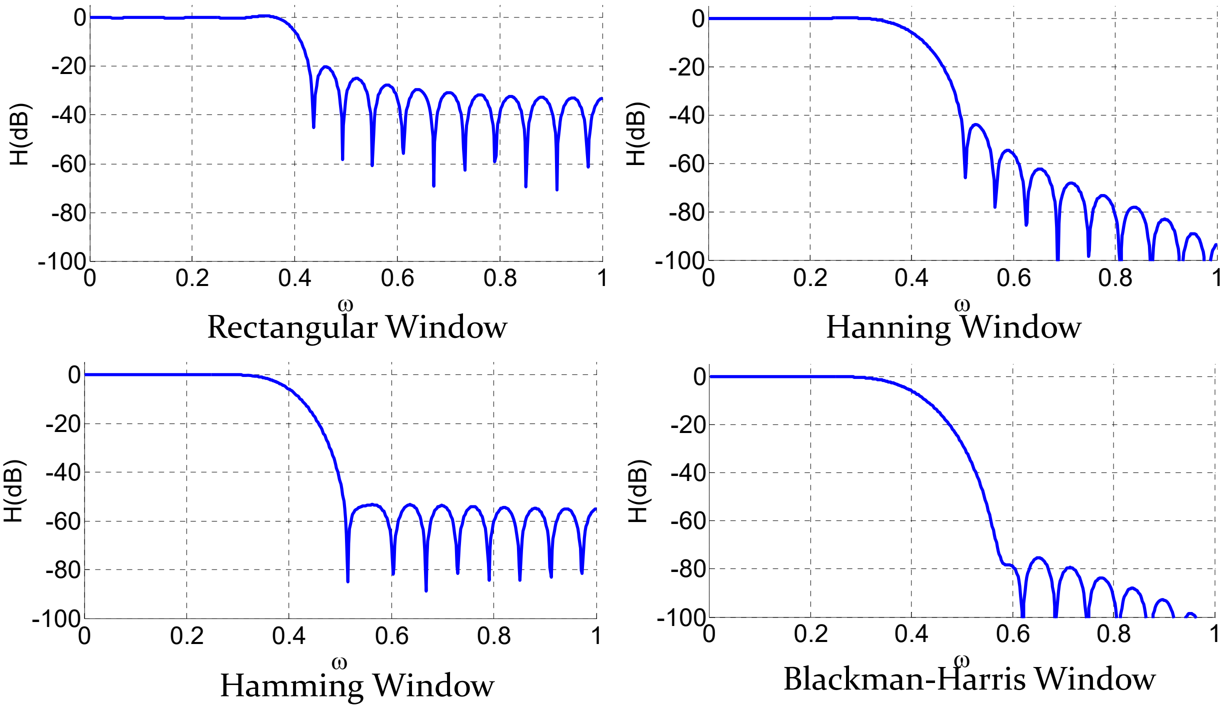
- Hamming: $w_{Hamm}[n] = 0.54 + 0.46c_1$
  Best peak sidelobe



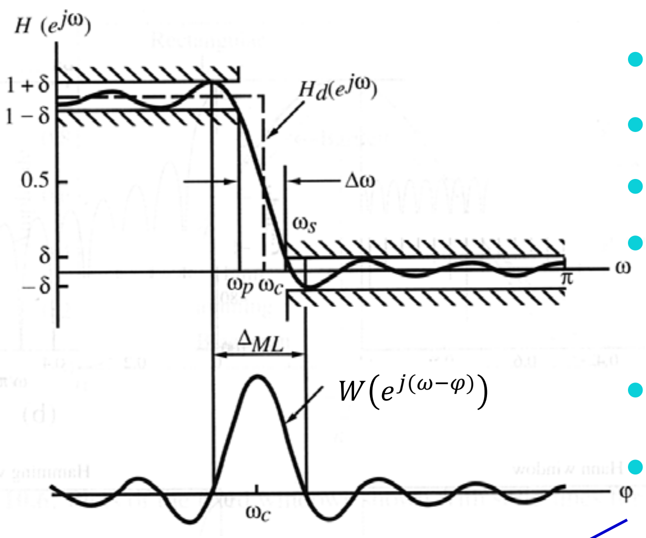
- Blackman-Harris 3 terms:
  $w_{BH}[n] = 0.42 + 0.5c_1 + 0.08c_2$
  Best peak sidelobe



Lowpass Filters designed using



Frequency Domain Relation



- $H\left(e^{j(\omega_c+\omega)}\right)+H\left(e^{j(\omega_c-\omega)}\right) \cong 1$
- $H\left(e^{j\omega_c}\right) \cong 0.5$
- $\delta_p = \delta_s = \delta$
- Distance between the maximum passband deviation and the minimum value $\cong \Delta_{ML}$
- $\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$
- A narrow transition band $\xrightarrow{\text{require}}$ small main lobe width
- Small $\delta \xrightarrow{\text{require}}$ small area under sidelobes

Contradictory requirements

Properties of Fixed Windows

Type of Window	Window function		Resultant Filter	
	Main Lobe Width Δ_{ML}	Relative Side-lobe Level A_{sl}	Minimum Stop-band Attenuation δ	Transition Bandwidth $\Delta\omega$
Rectangular	$\frac{4\pi}{2M+1}$	13.3dB	20.9dB	$\frac{0.92\pi}{M}$
Hanning	$\frac{8\pi}{2M+1}$	31.5dB	43.9dB	$\frac{3.11\pi}{M}$
Hamming	$\frac{8\pi}{2M+1}$	42.7dB	54.5dB	$\frac{3.32\pi}{M}$
Blackman-Harris	$\frac{12\pi}{2M+1}$	58.1dB	75.3dB	$\frac{5.56\pi}{M}$

- δ is independent from M, or ω_c, and is essentially constant.
- $\Delta\omega = \frac{c}{M}$

Design Example

- Using Window method to design a lowpass filter with passband edge $\omega_p = 0.3\pi$, stopband edge $\omega_s = 0.5\pi$, minimum stopband attenuation $\alpha_s = 40\text{dB}$.
- A: $\alpha_s = 40\text{dB}$. Hanning, Hamming and Blackman-Harris window meet the requirement.

$$\omega_c = \frac{\omega_p+\omega_s}{2} = 0.4\pi, \text{ and } \Delta\omega = \omega_s - \omega_p = 0.2\pi$$

Hanning has the minimum length:

$M = \left\lceil \frac{3.11\pi}{0.2\pi} \right\rceil = 16.$ Window length $N = 2M + 1 = 33$

So, $w_{Hann}[n] = 0.5 + 0.5\cos\frac{2\pi n}{33} \Longrightarrow h[n] = h_d[n]w_{Hann}[n]$

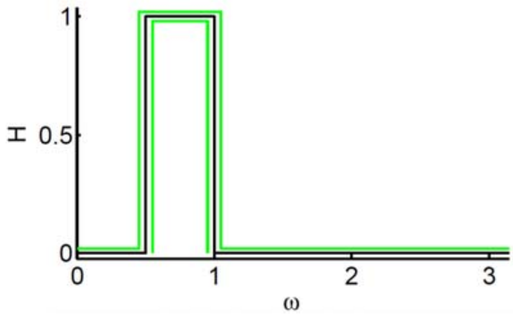
Adjustable Window

- Kaiser: $w_K[n] = \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)}$, where $I_0(\mu) = 1 + \sum_{r=1}^{\infty}\left[\dfrac{\left(\frac{\mu}{2}\right)^r}{r!}\right]^2$
  - β control minimum attenuation $\alpha_s = -20\log_{10}\delta_s$ in the stopband
  - Good compromise: width vs. sidelobe vs. decay
  - Estimation of β and filter length N from α_s and $\Delta\omega$:

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)} + 1$$

Design Example

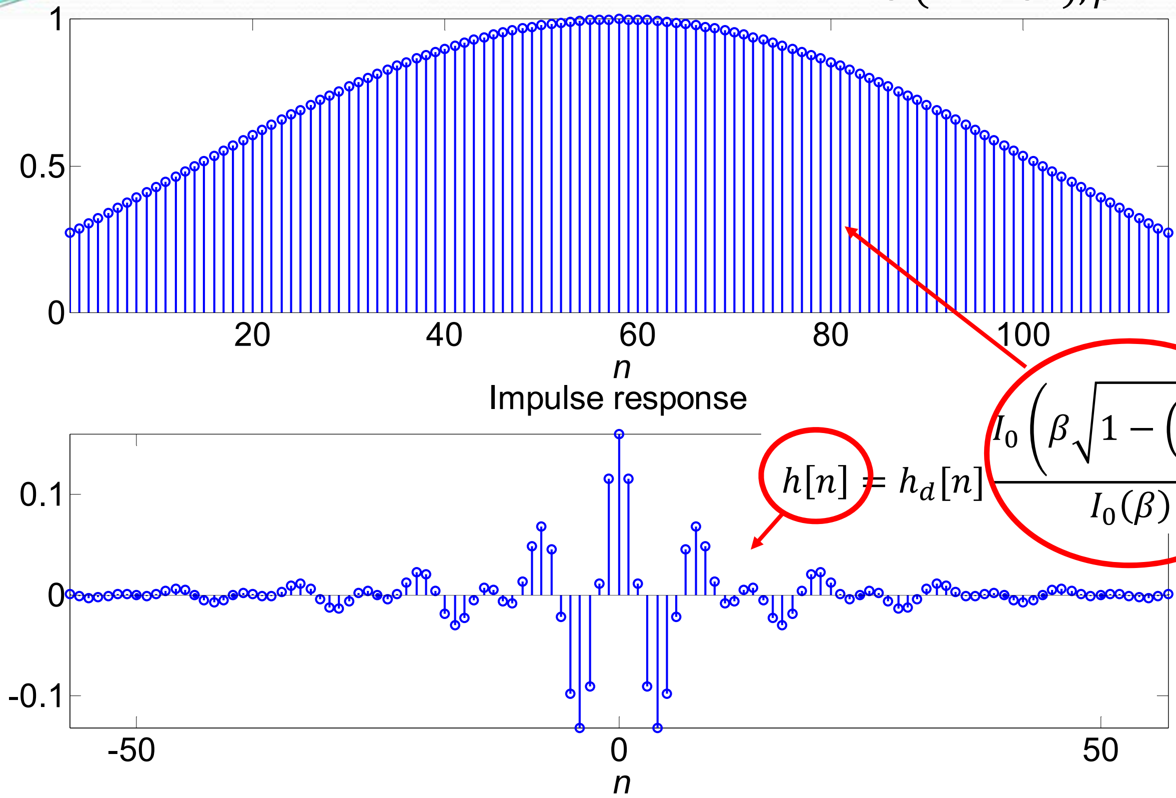


- Bandpass: $\omega_{c1} = 0.5, \omega_{c2} = 1,$
  $\Delta\omega = 0.1, \delta_p = \delta_s = 0.02$
- A: $\alpha_s = -20\log_{10}\delta_s = 34\text{dB}$
  $N = \frac{\alpha_s - 8}{2.285(\Delta\omega)} + 1 = 114 \cong 115$
- Ideal impulse response: Difference of two lowpass filters

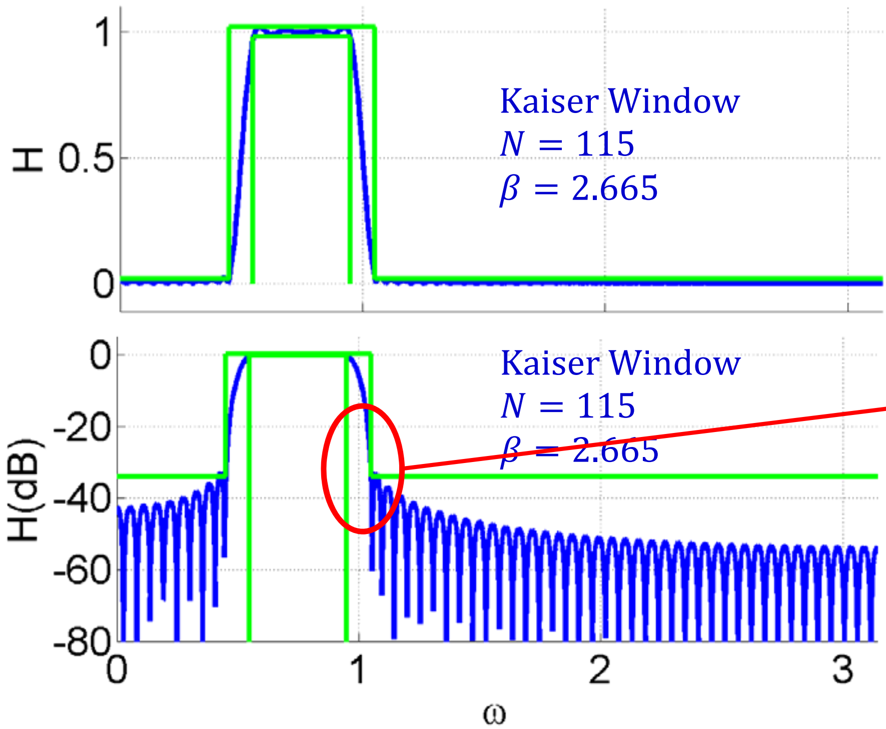
$$h_d[n] = \frac{\sin\omega_{c2}n}{\pi n} - \frac{\sin\omega_{c1}n}{\pi n}$$

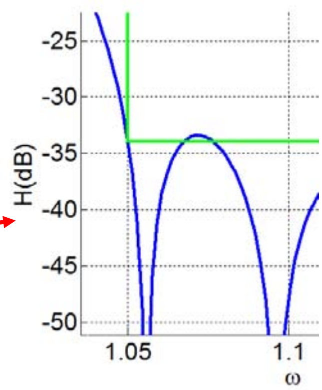
- $\beta = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) = 2.655$

Kaiser Window $N = 115\ (M = 57), \beta = 2.665$



$$h[n] = h_d[n]\frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)}$$

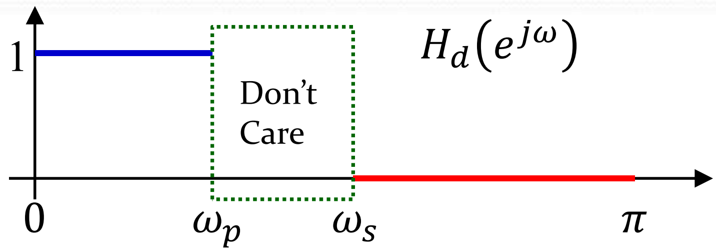




Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
- Optimal design
  - Design a filter $h[n]$ with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria, or satisfies specs.

Optimality



- **Least Squares:**

$$\text{Minimize} \int_{\omega \in \text{care}} \left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

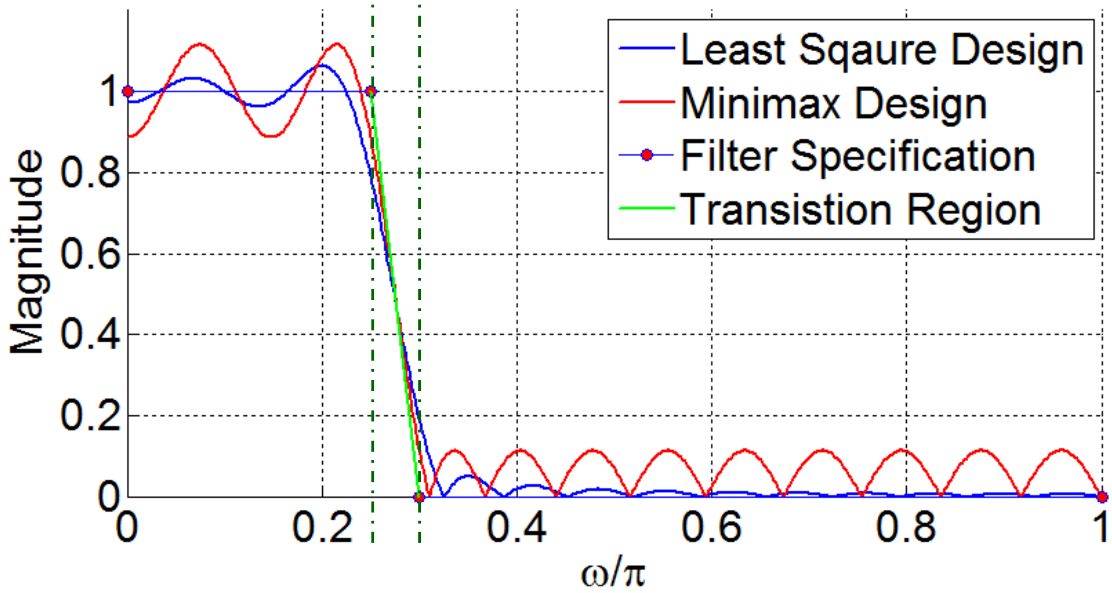
- A variation - **Weighted Least Squares:**

$$\text{Minimize} \int_{\omega \in \text{care}} W(\omega)\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

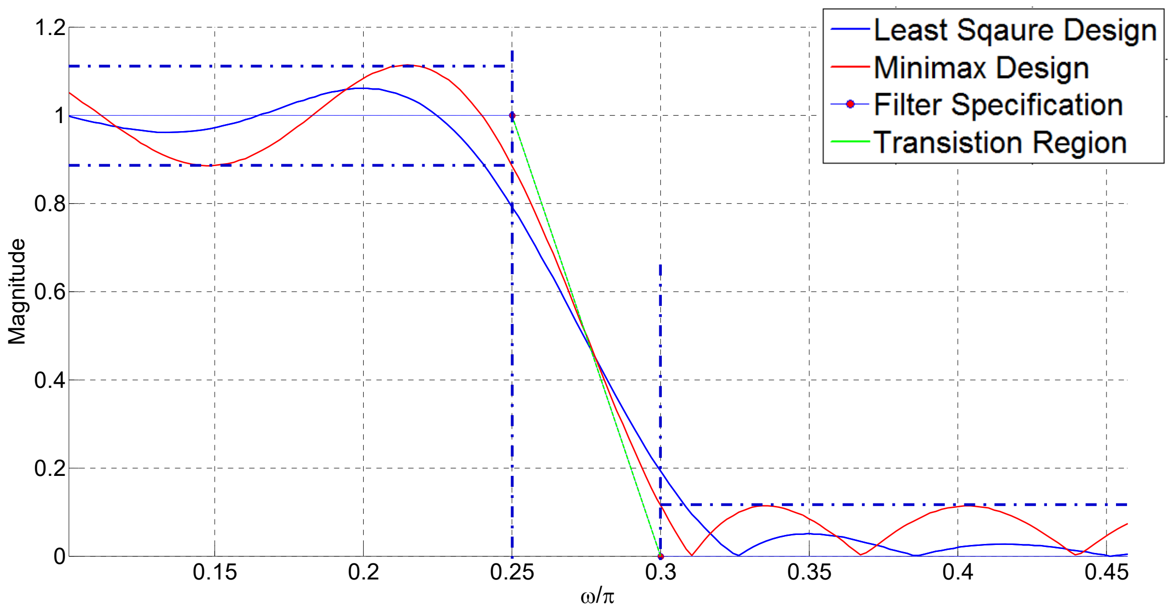
- **Chebychev Design (min-max)**

$$\text{Minimize}_{\omega \in \text{care}} \ \max\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|$$

Least Square vs. Minimax



Zoom-in View



Design Through Optimization

- **Idea:** Sample/discretize the frequency response
$$H(e^{j\omega}) \Longrightarrow H(e^{j\omega_k})$$
  - Sample points are fixed:
$$\omega_k = k\frac{\pi}{K}, \qquad -\pi < \omega_1 < \omega_2 \ldots < \omega_K \leq \pi$$
  - K has to be >> N, where N is the filter length. (Rule of thumb $K \geq 8N$)
  - Yields a (good) approximation of the original problem

Least Squares

- Target: Design a length $N = 2M + 1$ filter **Type I** filter
  - First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$.
  - Then, shift to make causal

$$h[n] = \tilde{h}[n - M]$$
$$H(e^{j\omega}) = e^{-jM\omega}\tilde{H}(e^{j\omega})$$

Least Squares Cont.

- **Matrix Formulation**

$$\tilde{\boldsymbol{h}} = \left[\tilde{h}[-M], \tilde{h}[-(M-1)], \ldots \tilde{h}[0], \ldots, \tilde{h}[M-1], \tilde{h}[M]\right]^T$$

$$\boldsymbol{b} = \left[\tilde{H}_d\left(e^{j\omega_1}\right), \tilde{H}_d\left(e^{j\omega_2}\right), \ldots, \tilde{H}_d\left(e^{j\omega_K}\right)\right]^T$$

$$\mathbf{A} = \begin{bmatrix} e^{-j\omega_1(-M)} & e^{-j\omega_1(-M+1)} & \cdots & e^{-j\omega_1 M} \\ e^{-j\omega_2(-M)} & e^{-j\omega_2(-M+1)} & \cdots & e^{-j\omega_2 M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\omega_K(-M)} & e^{-j\omega_K(-M+1)} & \cdots & e^{-j\omega_K M} \end{bmatrix}_{K\times(2M+1)}$$

Objective: $\min_{\tilde{\boldsymbol{h}}} \left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2$

Least Squares Cont.

Objective: $\min_{\tilde{\boldsymbol{h}}} \left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2$

- Solution: $\dfrac{d\left\|\mathbf{A}\tilde{\boldsymbol{h}}-\boldsymbol{b}\right\|^2}{d\tilde{\boldsymbol{h}}} = 0$

$$\left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2 = \left(\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right)^*\left(\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right)$$
$$= \tilde{\boldsymbol{h}}^*\mathbf{A}^*\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}^*\mathbf{A}\tilde{\boldsymbol{h}} - \tilde{\boldsymbol{h}}^*\mathbf{A}^*\boldsymbol{b} + \boldsymbol{b}^*\boldsymbol{b}$$

$$\frac{d\left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2}{d\tilde{\boldsymbol{h}}} = 2\mathbf{A}^*\mathbf{A}\tilde{\boldsymbol{h}} - 2\mathbf{A}^*\boldsymbol{b} = \mathbf{0}$$

$$\tilde{\boldsymbol{h}} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\boldsymbol{b}$$

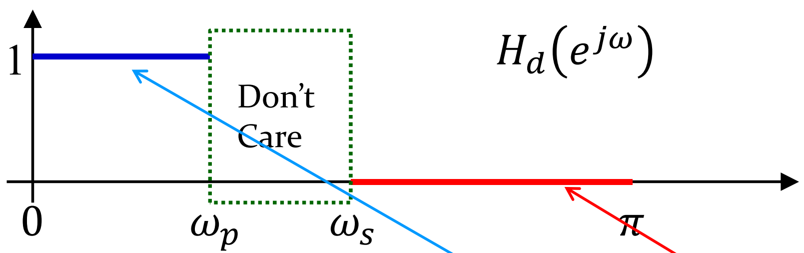
- $\dfrac{d\boldsymbol{x}^T\boldsymbol{a}}{d\boldsymbol{x}} = \dfrac{d\boldsymbol{a}^T\boldsymbol{x}}{d\boldsymbol{x}} = \boldsymbol{a}$
- $\dfrac{d\boldsymbol{x}^T\mathbf{A}\boldsymbol{x}}{d\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x}$
  - If $\mathbf{A}$ is symmetric $\mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} = 2\mathbf{A}\boldsymbol{x}$

- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}\left(e^{j\omega}\right)$ is real, $\tilde{h}[n]$ should have symmetry!

Example: Design of Linear Phase LP Filter

- Suppose
  - $\tilde{H}(e^{j\omega})$ is real and symmetric
  - Length N is odd
- Then,
  - $\tilde{h}[n]$ is real, and symmetric around $\tilde{h}[0]$
- So
  - $\tilde{H}(e^{j\omega}) = \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{j\omega} + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-1]e^{j2\omega} + \cdots$
    $= \tilde{h}[0] + 2\tilde{h}[1]\cos\omega + 2\tilde{h}[2]\cos 2\omega + \cdots$

Example: Cont.



- Given $N = 2M+1$, ω_p, and ω_s, find the best LS filter:

$$\mathbf{A} = \begin{bmatrix} 1 & 2\cos\omega_1 & \cdots & 2\cos M\omega_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos\omega_p & \cdots & 2\cos M\omega_p \\ 1 & 2\cos\omega_s & \cdots & 2\cos M\omega_s \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos\omega_K & \cdots & 2\cos M\omega_K \end{bmatrix}_{K\times(M+1)}$$

$$\boldsymbol{b} = [1\ 1\ \ldots 1\ 0\ 0\ \ldots 0]^T_{1\times K}$$

$$\tilde{\boldsymbol{h}}_+ = \left[\tilde{h}[0], \ldots, \tilde{h}[M]\right]^T_{1\times(M+1)} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\boldsymbol{b}$$

*Extension: Weighted Least Squares

- LS has no preference for passband or stopband
- Use weighting of LS to change ratio

- Solve the discrete version of

$$\text{Minimize } \int_{-\pi}^{\pi} W(\omega)\left|H\left(e^{j\omega}\right) - H_d\left(e^{j\omega}\right)\right|^2 d\omega$$

  - where $W(\omega) = \frac{1}{\delta_p}$ in passband, $W(\omega) = \frac{1}{\delta_s}$ in stopband, and $W(\omega) = 0$ in transition band.
  - Equivalently, you may set $W(\omega) = 1$ in passband, $W(\omega) = \frac{\delta_p}{\delta_s}$ in stopband, and $W(\omega) = 0$ in transition band.

Weighted Least Squares

- Written in matrix form:

$$\textbf{Objective: } \min_{\tilde{\boldsymbol{h}}} \left\|\boldsymbol{w} \cdot (\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b})\right\|^2$$

where, ‘ . ’ is inner product, and

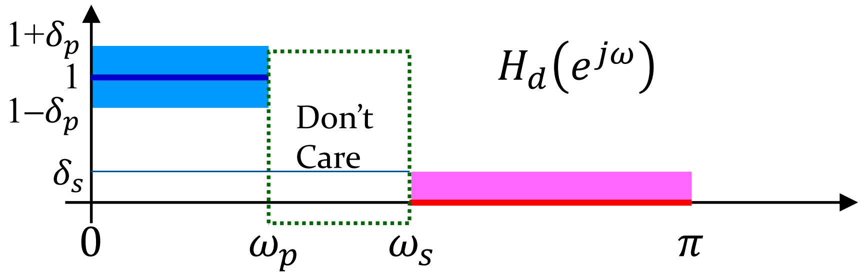
$$\boldsymbol{w} = \left[1\ 1\ \dots 1\ \frac{\delta_p}{\delta_s} \frac{\delta_p}{\delta_s}\ \dots\ \frac{\delta_p}{\delta_s}\right]^T_{1\times K}$$

- It can be written as $\min_{\tilde{\boldsymbol{h}}} \left(\mathbf{A}\tilde{\boldsymbol{h}}_+ - \boldsymbol{b}\right)^* \mathbf{W}^2 \left(\mathbf{A}\tilde{\boldsymbol{h}}_+ - \boldsymbol{b}\right)$, where

$$\mathbf{W} = \mathrm{diag}(w) = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \frac{\delta_p}{\delta_s} & & \\ & & & & \ddots & \\ 0 & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}_{K\times K}$$

Minimax Optimal Filters

- **Objective:** Minimize$_{\omega \in \text{care}}$ $\max\left|H\left(e^{j\omega}\right) - H_d\left(e^{j\omega}\right)\right|$
  - Parks-McClellan algorithm - equi-ripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Can also use convex optimization
- **Specifications:**



- Filter specifications are given in terms of boundaries

- More specifically, minimize
  - maximum passband ripple

$$1 - \delta_p \leq \left|H\left(e^{j\omega}\right)\right| \leq 1 + \delta_p, \qquad 0 \leq \omega \leq \omega_p$$

  - and, maximum stopband ripple

$$\left|H\left(e^{j\omega}\right)\right| \leq \delta_s, \qquad \omega_s \leq \omega \leq \pi$$

Estimation of the Filter Length

- Given $\omega_p, \omega_s, \delta_p, \delta_s$, estimate the filter length
- Kaiser's formula:

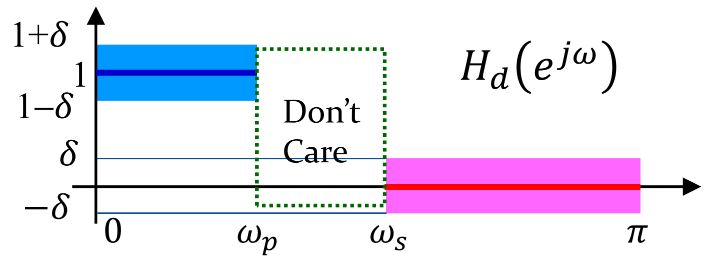
$$N \cong \frac{-20\log_{10}\left(\sqrt{\delta_p\delta_s}\right) - 13}{\frac{14.6\left(\omega_s - \omega_p\right)}{2\pi}} + 1$$

- Bellanger's formula:

$$N \cong -\frac{2\log_{10}\left(10\delta_p\delta_s\right)}{3\left(\omega_s - \omega_p\right)/2\pi}$$

Formulation of the Problem

- When $\widetilde{H}\left(e^{j\omega}\right)$ is real and symmetric
- Given $N = 2M + 1$, ω_p, ω_s, find $\widetilde{\boldsymbol{h}}_+$ to



Minimize: δ

Subject to: $1 - \delta \leq \widetilde{H}\left(e^{j\omega_k}\right) \leq 1 + \delta, 0 \leq \omega \leq \omega_p$

$$-\delta \leq \widetilde{H}\left(e^{j\omega_k}\right) \leq \delta,\ \omega_s \leq \omega \leq \pi$$

- Both the objective function and the constrains are the linear functions of variables δ and $\widetilde{\boldsymbol{h}}_+$
- A well studied class of problem

Minimax Design via Linear Programming

- Linear Programming: ($\boldsymbol{x}$ is the vector to be optimized)

  Minimize: $\boldsymbol{c}^T\boldsymbol{x}$

  Subject to: $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$

- Minimax Design Linear Phase FIR Filter

$$\boldsymbol{x} = \begin{bmatrix} \tilde{h}[0] \\ \vdots \\ \tilde{h}[M] \\ \delta \end{bmatrix}_{M+2} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{M+2}$$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2\cos\omega_1 & \cdots & 2\cos M\omega_1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2\cos\omega_p & \cdots & 2\cos M\omega_p & -1 \\ -1 & -2\cos\omega_1 & \cdots & -2\cos M\omega_1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2\cos\omega_p & \cdots & -2\cos M\omega_p & -1 \\ 1 & 2\cos\omega_s & \cdots & 2\cos M\omega_s & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2\cos\omega_K & \cdots & 2\cos M\omega_K & -1 \\ -1 & -2\cos\omega_s & \cdots & -2\cos M\omega_s & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2\cos\omega_K & \cdots & -2\cos M\omega_K & -1 \end{bmatrix}_{2K\times(M+2)} \times \begin{bmatrix} \tilde{h}[0] \\ \vdots \\ \tilde{h}[M] \\ \delta \end{bmatrix}_{M+2} = \boldsymbol{b} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2K}$$

The End