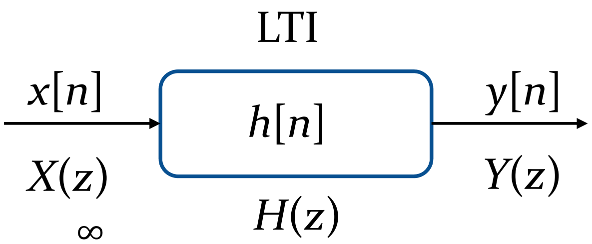
Lecture 8
LTI Discrete-Time Systems in the Transform Domain

Types of Transfer Functions



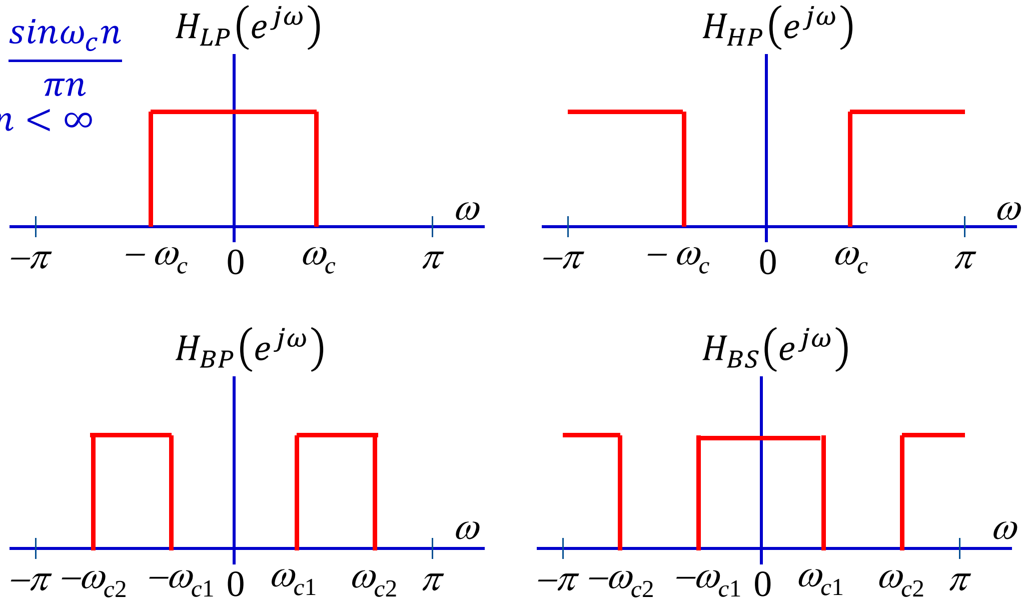
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad Y(z) = X(z)H(z)$$

- For digital transfer function with frequency selective frequency response, there are two types of classifications.
  - Based on the shape of magnitude function $\left|H\left(e^{j\omega}\right)\right|$
  - Based on the form of phase function $\theta(\omega)$

Magnitude Characteristics

- Digital Filter with Ideal Magnitude Responses:

$$h_{LP}[n] = \frac{sin\omega_c n}{\pi n}$$
$$-\infty < n < \infty$$



- The range of frequencies where the magnitude response takes the value of one is called the **passband**
- The range of frequencies where the magnitude response takes the value of zero is called the **stopband**
- The frequencies , ω_c, ω_{c1} and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a **zero phase** everywhere

Phase Characteristics

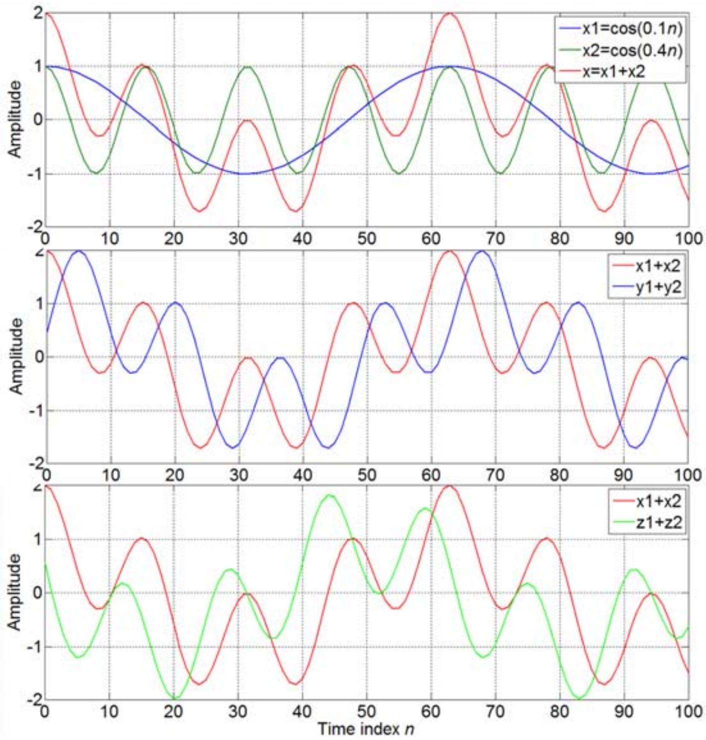
- Linear Phase Transfer Function
  - In many application, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in passband.
- The most general type of a filter with a linear phase has a frequency response given by

$$H(e^{j\omega}) = Ae^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$.

- Note: $\theta(\omega) = -\omega D$.

Phase Distortion



- $x = x_1 + x_2$

$$x_1 = \cos(0.1n)$$
$$x_2 = \cos(0.4n)$$

- $Y(e^{j\omega}) = X(e^{j\omega})H_1(e^{j\omega})$

$\theta_1(\omega) = \angle H_1(e^{j\omega}) = -5\omega$

- $y = y_1 + y_2$

$$y_1 = \cos(0.1(n-5))$$
$$y_2 = \cos(0.4(n-5))$$

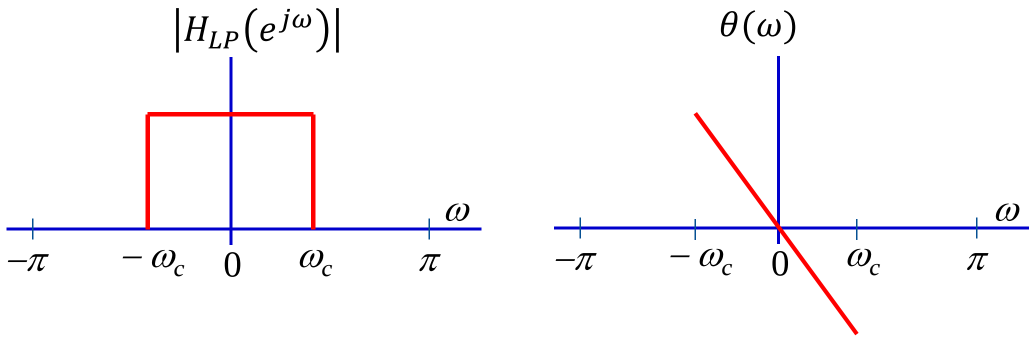
- $Z(e^{j\omega}) = X(e^{j\omega})H_2(e^{j\omega})$

$\theta_2(\omega) = \angle H_2(e^{j\omega}) = -5\text{sign}(\omega)$

- $z = z_1 + z_2$

$$z_1 = \cos(0.1n - 5)$$
$$z_2 = \cos(0.4n - 5)$$

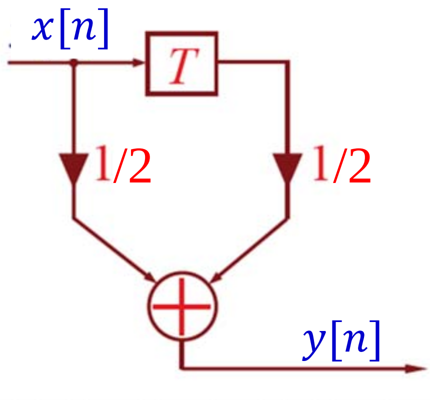
- It is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase
- The transfer function should exhibit a unity magnitude response and a linear phase response in the band of interest.



Linear-Phase FIR Transfer Functions

- It is impossible to design an IIR transfer function with an exact linear-phase
- It is always possible to design an FIR transfer function with an exact linear-phase response
- We consider real impulse response $h[n]$

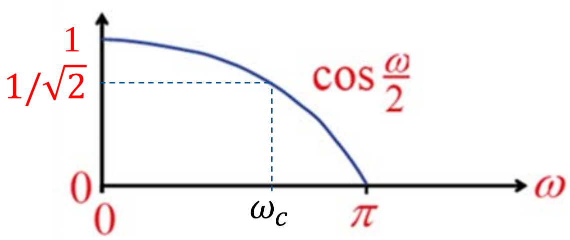
Simple Examples



- $H(z) = \frac{1+z^{-1}}{2} \leftrightarrow \{h[n]\} = \left\{\frac{1}{2}, \frac{1}{2}\right\}$

$$\uparrow$$

$$H(e^{j\omega}) = (1 + e^{-j\omega})/2$$
$$= e^{-j\omega/2}\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}$$
$$= e^{-j\omega/2}\cos(\omega/2)$$



A lowpass filter

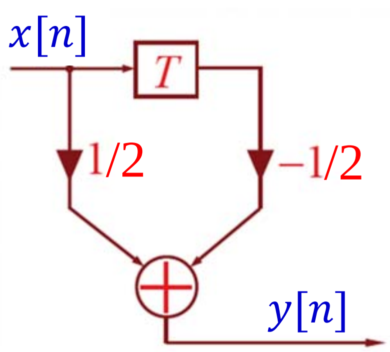
ω_c is the 3db cutoff frequency

$$\mathcal{G}(\omega_c) = 20\log_{10}\left|\frac{H(e^{j\omega_c})}{H(e^{j0})}\right| = 20\log_{10}\frac{1}{\sqrt{2}} \cong -3\text{dB}$$

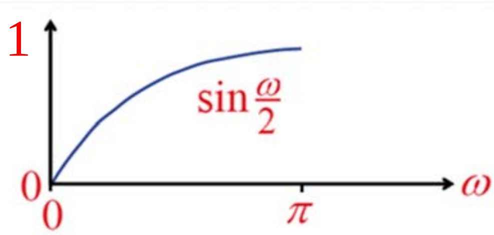
Simple Examples

- $H(z) = \frac{1-z^{-1}}{2} \leftrightarrow \{h[n]\} = \{\frac{1}{2}, -\frac{1}{2}\}$

$$\uparrow$$



$$H(e^{j\omega}) = (1 - e^{-j\omega})/2$$
$$= e^{j\pi/2}e^{-j\omega/2}\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}$$
$$= e^{j(\pi/2-\omega/2)}\sin(\omega/2)$$



A highpass filter

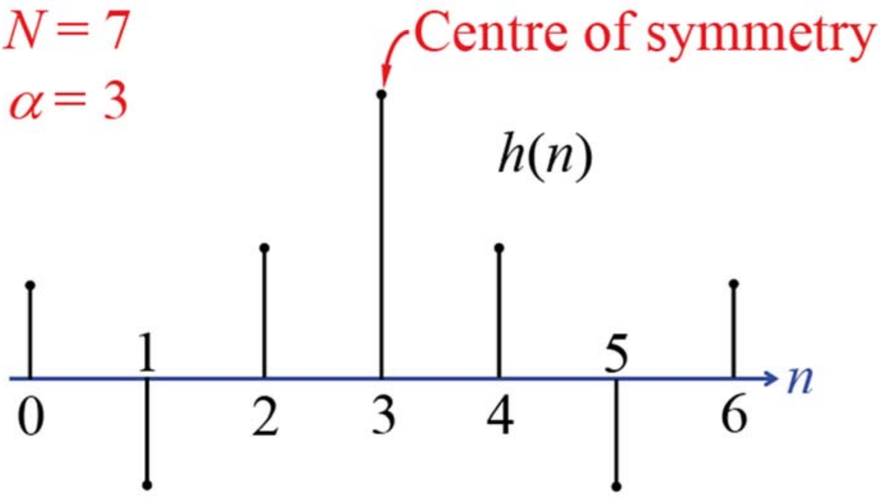
Linear Phase FIR Filter

- An FIR filter may be designed to have linear phase characteristics. The phase response, $\theta(\omega)$, of a linear phase FIR filter is $\beta - \alpha\omega$, where $\alpha = \frac{N-1}{2}$, ω is the frequency, $\beta = 0$ or $\pm 0.5\pi$ and N is the filter length.
- Its frequency response is given by $e^{-j\left(\frac{N-1}{2}\omega-\beta\right)}R(\omega)$, where $R(\omega)$ is a real function.
- The group delay is $-d\{\theta(\omega)\} = \alpha$.

- Its impulse response is either symmetrical or anti-symmetrical.
- If its impulse response is symmetrical, its phase response is $-\alpha\omega$.
- If its impulse response is anti-symmetrical, its phase response is $\pm 0.5\pi - \alpha\omega$.

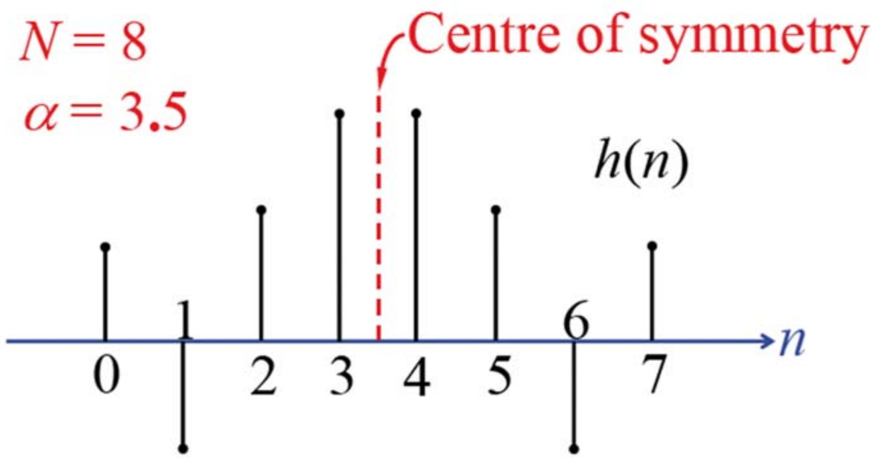
Example: Symmetrical impulse response,

N odd, where N is the length of the impulse response

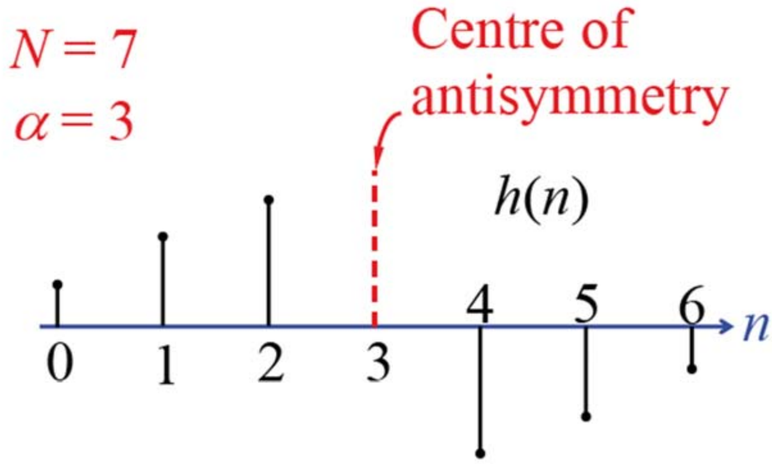


Example: Symmetrical impulse response,

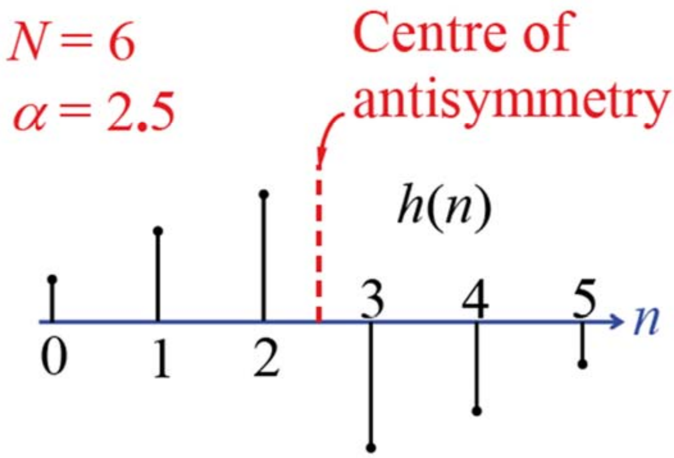
N even.



Example: Anti-symmetrical impulse response, N odd.



Example: Anti-symmetrical impulse response, N even.



Frequency response of linear phase FIR filter

- Four cases, depending on whether N is odd or even and whether the impulse response is symmetrical or anti-symmetrical.
- Type I: Symmetrical impulse response, N odd.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h[n]e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega\frac{N-1}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h[n]e^{-j\omega n}$$

$n' = N-1-n$

$$= e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{\frac{N-3}{2}} h[n]\left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right) + h\left[\frac{N-1}{2}\right]\right]$$

$$= e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{(N-3)/2} 2h[n]\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right) + h\left[\frac{N-1}{2}\right]\right]$$

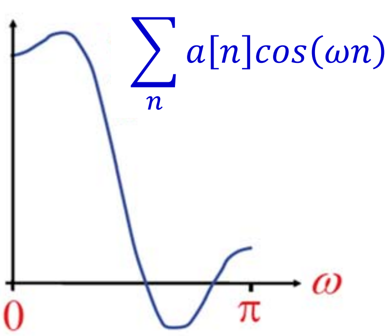
$$\xRightarrow{m=\frac{N-1}{2}-n} e^{-j\omega\frac{N-1}{2}}\left[\sum_{m=1}^{(N-1)/2} 2h\left[\frac{N-1}{2}-m\right]\cos(\omega m) + h\left[\frac{N-1}{2}\right]\right]$$

$$\therefore H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\sum_{n=0}^{(N-1)/2} a[n]\cos(\omega n)$$

$$a[0] = h\left[\frac{N-1}{2}\right]$$

$$a[n] = 2h\left[\frac{N-1}{2}-n\right],$$

$$\text{for } n = 1, 2, \dots, \frac{N-1}{2}$$

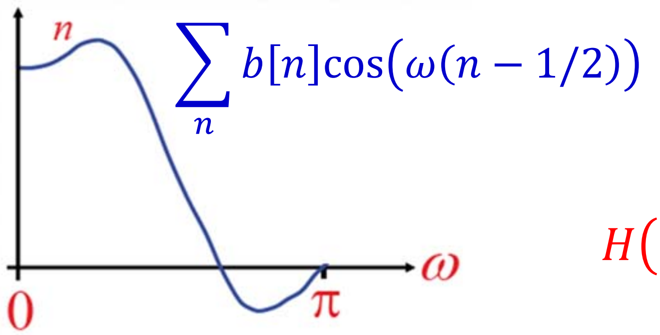


Type II:

- Symmetrical impulse response, N even

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{N-1}{2}} \sum_{n=1}^{N/2} b[n]\cos\left(\omega(n-1/2)\right)$$

$$b[n] = 2h[N/2 - n], \text{for } n = 1, 2, \dots, N/2$$



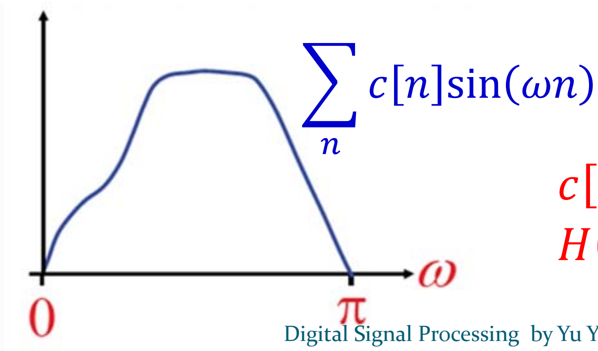
$H\left(e^{j\omega}\right) = 0 \text{ at } \omega = \pi$

Type III:

- Anti-symmetrical impulse response, N odd

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{(N-1)/2} c[n]\sin(\omega n)$$

$$c[n] = 2h\left[\frac{N-1}{2} - n\right], \text{for } n = 1, 2, \dots, \frac{N-1}{2}$$



$c[0] = 0$

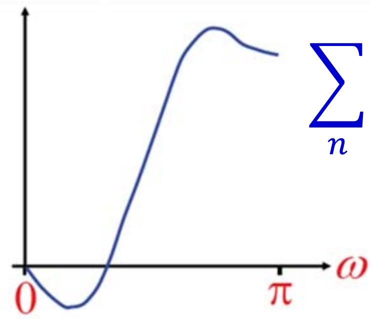
$H\left(e^{j\omega}\right) = 0 \text{ at } \omega = 0 \text{ and } \pi$

Type IV:

- Anti-symmetrical impulse response, N even

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{N/2} d[n]\sin\big(\omega(n-1/2)\big)$$

$$d[n] = 2h[N/2-n], \text{for } n = 1, 2, \ldots, N/2$$



$$\sum_n d[n]\sin\big(\omega(n-1/2)\big)$$

$H(e^{j\omega}) = 0$ at $\omega = 0$

Mirror Image Polynomial

- For an FIR filter with a symmetric impulse response, its transfer function $H(z)$ can be written as:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-1-n]z^{-n}$$

$$\xRightarrow{m=N-1-n} H(z) = \sum_{m=0}^{N-1} h[m]z^{-N+1+m} = z^{-(N-1)} \sum_{m=0}^{N-1} h[m]z^{m} = z^{-(N-1)}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **mirror-image polynomial**.

Example: $H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6}$

Amplitude response: $\breve{H}_1(\omega) = 6 - 6\cos(\omega) + 4\cos(2\omega) - 2\cos(3\omega)$

Phase response: $\theta_1(\omega) = -3\omega$

- If $H_2(z) = -H_1(z)$, then $\breve{H}_2(\omega) = \breve{H}_1(\omega)$, and $\theta_2(\omega) = -3\omega + \pi$

Antimirror-Image Polynomial

- Similarly, for an FIR filter with an anti-symmetric impulse response, its transfer function $H(z)$ can be written as:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} -h[N-1-n]z^{-n} = -z^{-(N-1)}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called an **antimirror-image polynomial**.

Example: $H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6}$

Amplitude response: $\breve{H}_3(\omega) = 6\sin(\omega) - 4\sin(2\omega) + 2\sin(3\omega)$

Phase response: $\theta_3(\omega) = -3\omega + \pi/2$

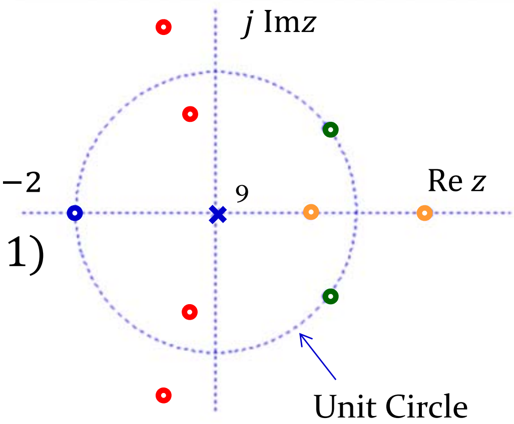
- If $H_4(z) = -H_3(z)$, then $\breve{H}_4(\omega) = \breve{H}_3(\omega)$, and $\theta_4(\omega) = -3\omega - \pi/2$

Mirror Image Symmetry of Zeros

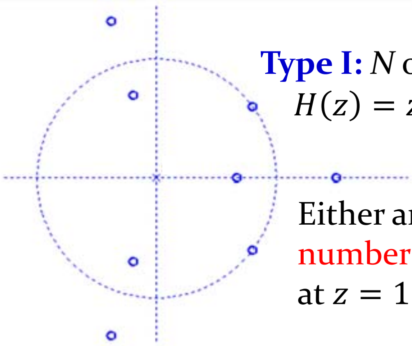
- The zeros of linear phase FIR filter with real coefficients exhibit **mirror image symmetry** with respect to unit circle.
- $H(z) = z^{-(N-1)}H(z^{-1})$ or $H(z) = -z^{-(N-1)}H(z^{-1})$
- Zeros are in forms of:
  - Real zeros
    - $z = r$ and $z = \frac{1}{r}$, $r \neq \pm 1$

      $\Rightarrow (z-r)(z-r^{-1}) \Rightarrow 1 + az^{-1} + z^{-2}$
    - $z = -1$ or $z = 1 \Rightarrow (z+1)$ or $(z-1)$
  - Complex zeros
    - $z = re^{\pm j\varphi}$ and $z = \frac{1}{r}e^{\pm j\varphi}$, $r \neq \pm 1$

      $\Rightarrow 1 + az^{-1} + cz^{-2} + az^{-3} + z^{-4}$
    - $z = e^{\pm j\varphi} \Rightarrow 1 + az^{-1} + z^{-2}$



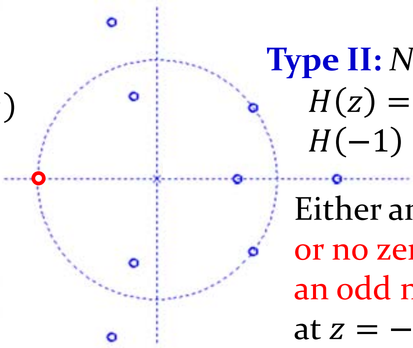
Zero-Locations of FIR Filters



Type I: N odd

$H(z) = z^{-(N-1)} H(z^{-1})$

Either an even number or no zeros at $z = 1$ and $z = -1$

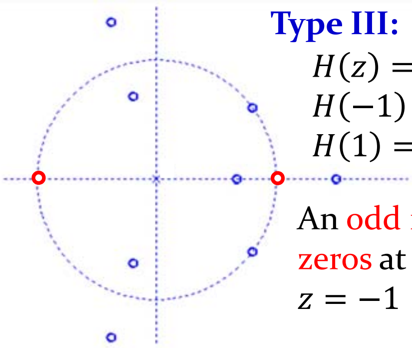


Type II: N even

$H(z) = z^{-(N-1)} H(z^{-1})$

$H(-1) = -H(-1)$

Either an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$



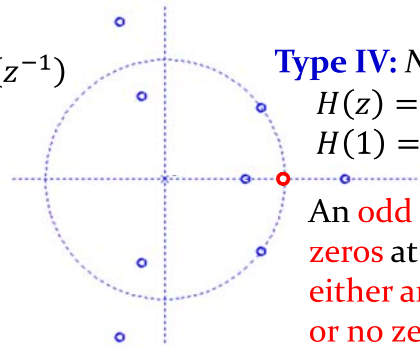
Type III: N odd

$H(z) = -z^{-(N-1)} H(z^{-1})$

$H(-1) = -H(-1)$

$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and $z = -1$



Type IV: N even

$H(z) = -z^{-(N-1)} H(z^{-1})$

$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$

Zero-Phase FIR Filters

- The impulse response $h[n]$ is symmetric around $h[0]$, i.e., $h[n] = h[-n]$.
- Thus, the frequency response $H(e^{j\omega})$ is real, but not necessary positive (unlike $|H(e^{j\omega})|$)
- For example, the type I FIR filter:

$$H(e^{j\omega}) = \sum_{n=-\frac{N-1}{2}}^{-1} h[n]e^{-j\omega n} + h[0] + \sum_{n=1}^{\frac{N-1}{2}} h[n]e^{-j\omega n}$$

$$= h[0] + 2\sum_{n=1}^{\frac{N-1}{2}} h[n]\cos(\omega n)$$

- Zero-phase FIR filter is not causal, but the causality may be recovered by delaying the impulse response

Simple IIR Filters

- General Form

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{m=1}^{N} b_m z^{-m}}$$

Lowpass & Highpass IIR Filter

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1, \qquad \alpha, K \text{ are real}$$

- Its squared-magnitude function is given by

$$\left|H\left(e^{j\omega}\right)\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

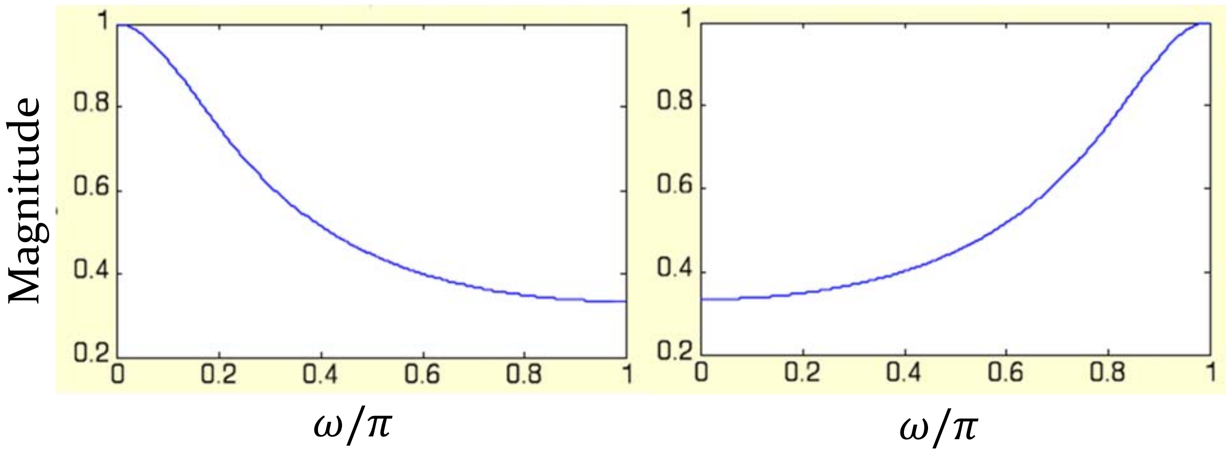
- When $\alpha > 0$, $\left|H\left(e^{j\omega}\right)\right|^2{}_{\max} = \frac{K^2}{(1+\alpha^2)-2\alpha}$, at $\omega = 0$,

$$\left|H\left(e^{j\omega}\right)\right|^2{}_{\min} = \frac{K^2}{(1+\alpha^2)+2\alpha}, \text{ at } \omega = \pi.$$

- When $\alpha < 0$, $\left|H(e^{j\omega})\right|^2_{\max} = \frac{K^2}{(1+\alpha^2)+2\alpha}$, at $\omega = \pi$,

$$\left|H(e^{j\omega})\right|^2_{\min} = \frac{K^2}{(1+\alpha^2)-2\alpha}, \text{ at } \omega = 0.$$

$\alpha > 0$ $\qquad K = \pm(1 - |\alpha|)$ $\qquad \alpha < 0$



Improved Lowpass IIR Filters

$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- A factor $1 + z^{-1}$ added to the numerator to force the magnitude function to have a zero at $\omega = \pi$

$$\left|H_{LP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

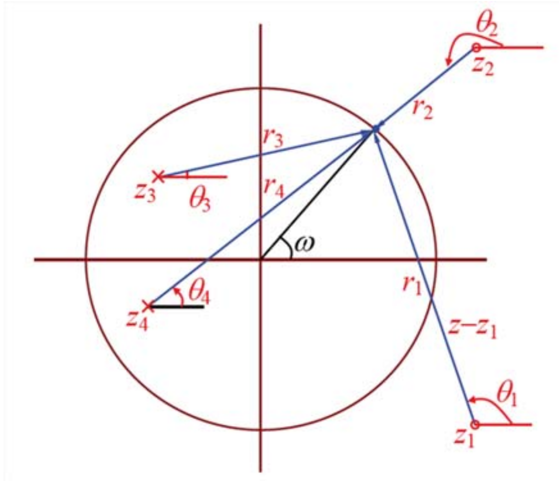
$$\left|H_{LP}(e^{j0})\right| = 1, \text{and } \left|H_{LP}(e^{j\pi})\right| = 0$$

Zeros, Poles and Geometric Interpolation of Frequency Response

$$H(z) = \frac{(z - z_1)(z - z_2)}{(z - z_3)(z - z_4)}$$

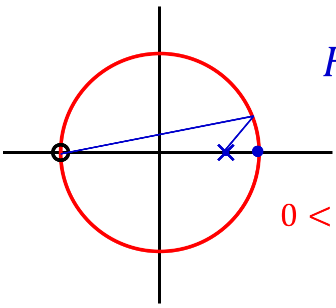
$$H(e^{j\omega}) = \frac{(e^{j\omega} - z_1)(e^{j\omega} - z_2)}{(e^{j\omega} - z_3)(e^{j\omega} - z_4)}$$

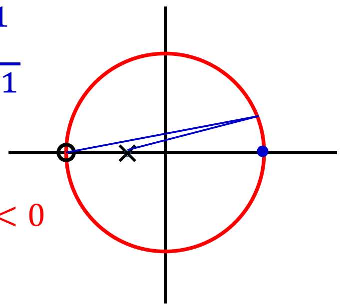
$$= \frac{r_1 r_2}{r_3 r_4} \underline{/\theta_1 + \theta_2 - \theta_3 - \theta_4}$$

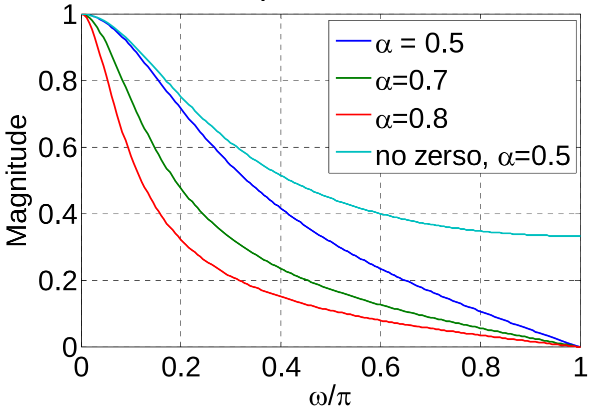


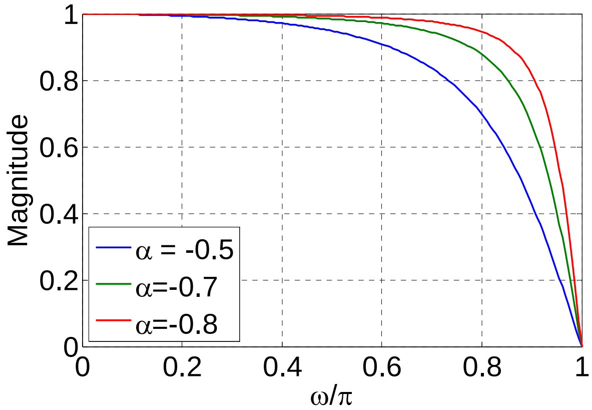
Pole, Zero, and Response

$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}$$









Improved Highpass IIR Filters

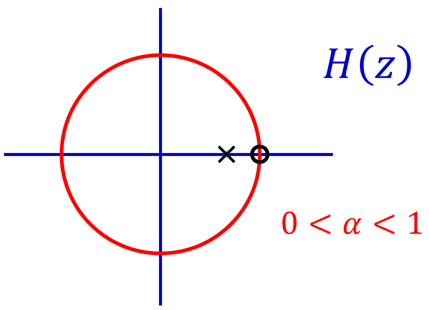
$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- A factor $1 - z^{-1}$ added to the numerator to force the magnitude function to have a zero at $\omega = 0$

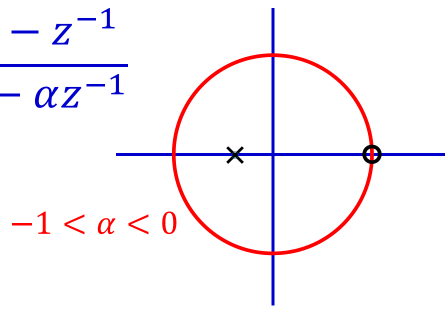
$$\left|H_{LP}\left(e^{j\omega}\right)\right|^2 = \frac{(1+\alpha)^2(1-\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

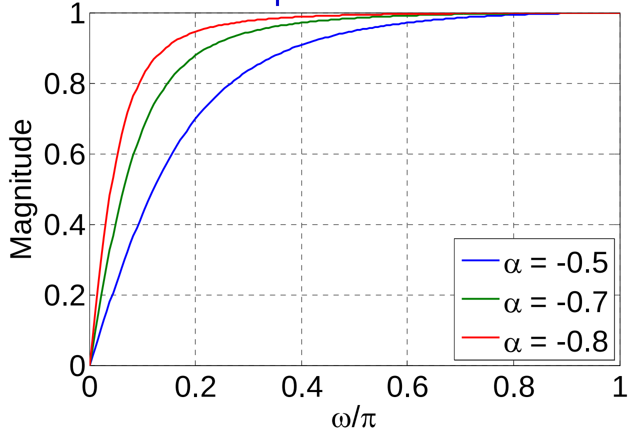
$$\left|H_{LP}\left(e^{j0}\right)\right| = 0, \text{and} \left|H_{LP}\left(e^{j\pi}\right)\right| = 1$$

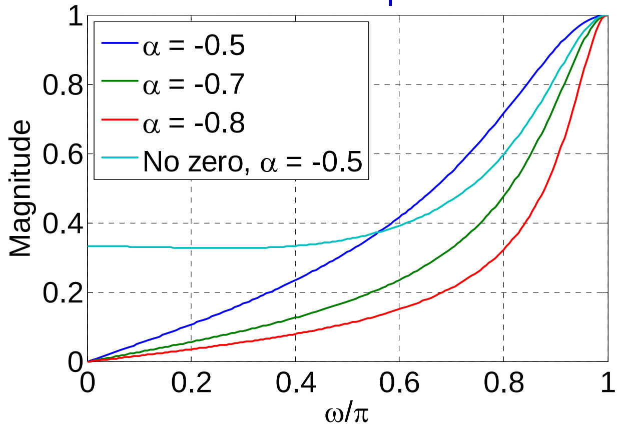
Pole, Zero, and Response



$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}$$

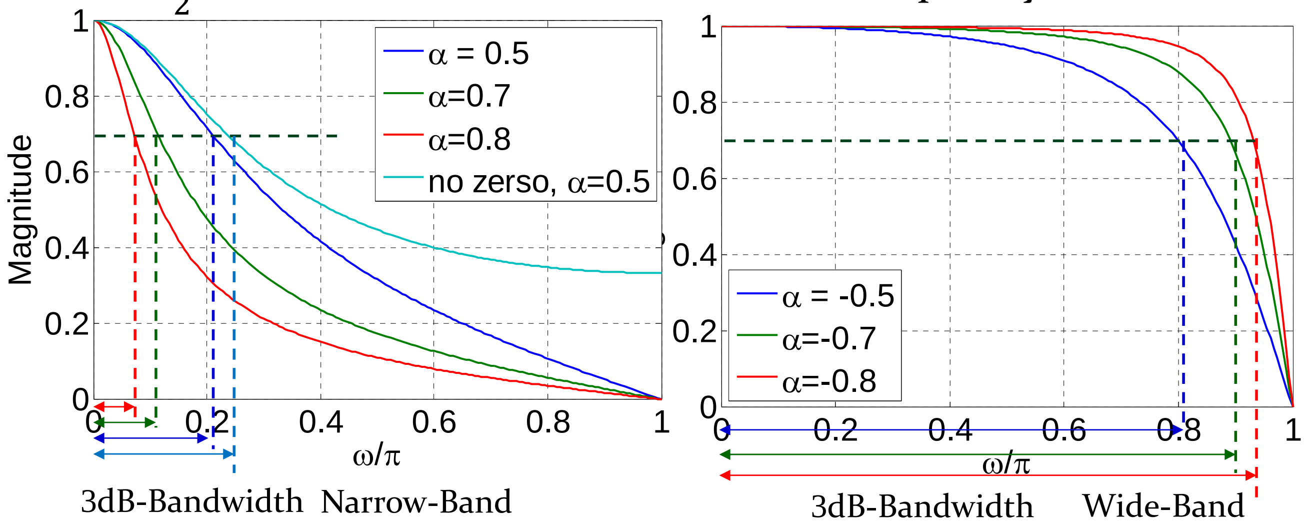






3-dB Cutoff Frequency

- The frequency where the magnitude is reduced to $\frac{1}{\sqrt{2}}$ of the ideal passband gain, or the squared magnitude is reduced to $\frac{1}{2}$ of the ideal one, is the 3-dB cutoff frequency.



Compute the 3-dB cutoff frequency

$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- Given α, compute ω_c. Squared magnitude function

$$\left|H_{LP}\left(e^{j\omega}\right)\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)} = \frac{1}{2}$$

which, when solved, yields

$$\cos\omega_c = \frac{2\alpha}{1+\alpha^2} \Rightarrow \omega_c = \cos^{-1}\frac{2\alpha}{1+\alpha^2}$$

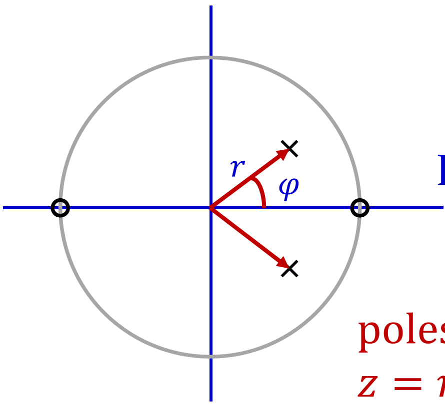
- Given ω_c, determine α. A solution resulting in a stable transfer function is

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

Bandpass IIR Digital Filter

- A 2nd-order general form

$$H_{BP}(z) = \frac{K(1 - z^{-2})}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}$$



$$r = \sqrt{\alpha}$$

$$\varphi = \cos^{-1}\left(\frac{\beta(1+\alpha)}{2}\sqrt{\alpha}\right)$$

For stability, we have $r < 1 \ \rightarrow\ |\alpha| < 1$

poles:
$z = re^{\pm j\varphi}$

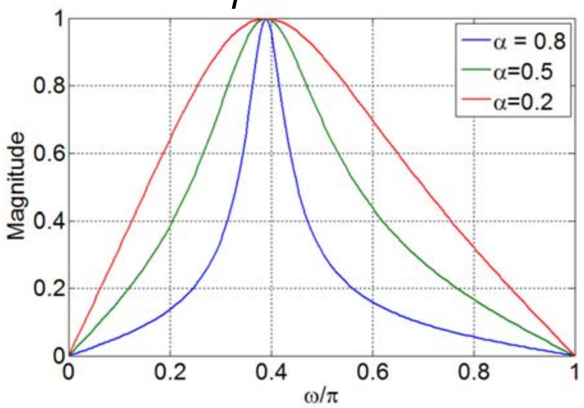
- The squared magnitude function:

$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{4K^2\sin^2\omega}{(1+\alpha)^2(\beta - \cos\omega)^2 + (1-\alpha)^2\sin^2\omega}$$

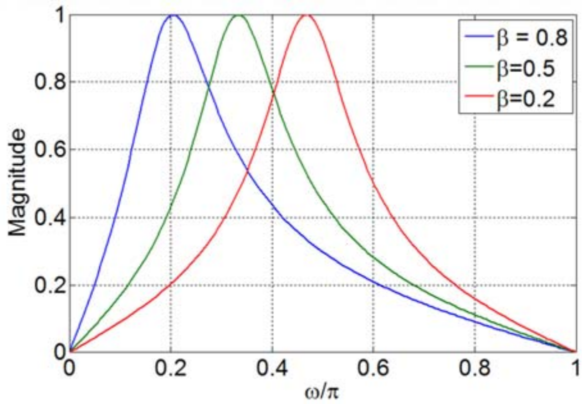
- $\left|H_{BP}(e^{j\omega})\right|^2 = 0$ at $\omega = 0$ and $\omega = \pi$
- $\left|H_{BP}(e^{j\omega})\right|^2 = \frac{2K}{1-\alpha}$, the maximum, at $\omega_0 = \cos^{-1}\beta$
  - ω_0 is called the **center frequency** of the bandpass filter
  - Choose $K = \frac{1-\alpha}{2}$ to make the maximum magnitude to be 1.
- The frequencies, ω_{c1} and ω_{c2}, where $\left|H_{BP}(e^{j\omega})\right|^2 = \frac{1}{2}$ are the 3-dB cutoff frequencies.

- **3-dB Bandwidth** $B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$
- **Quality factor** $Q = \frac{\omega_0}{B_w}$

$\beta = 0.34$



$\alpha = 0.6$



$\omega_0 = \cos^{-1}\beta$ $\qquad B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$

Example

- Design a second-order bandpass digital filter with center frequency at 0.4π and a 3dB bandwidth of 0.1π.
- A: $\beta = \cos\omega_0 = \cos(0.4\pi) = 0.309016994$

$$\frac{2\alpha}{1+\alpha^2} = \cos B_w = \cos(0.1\pi) = 0.951056516$$

$\Longrightarrow \alpha = 1.37638192$ (not stable) or $\alpha = 0.726542528$

So, the transfer function of the second-order bandpass filter is:

$$H_{BP}(e^{j\omega}) = \frac{0.136728736(1-z^{-2})}{1-0.53353098z^{-1}+0.726542528z^{-2}}$$

Allpass Filter

- **Allpass Transfer Function**: an IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

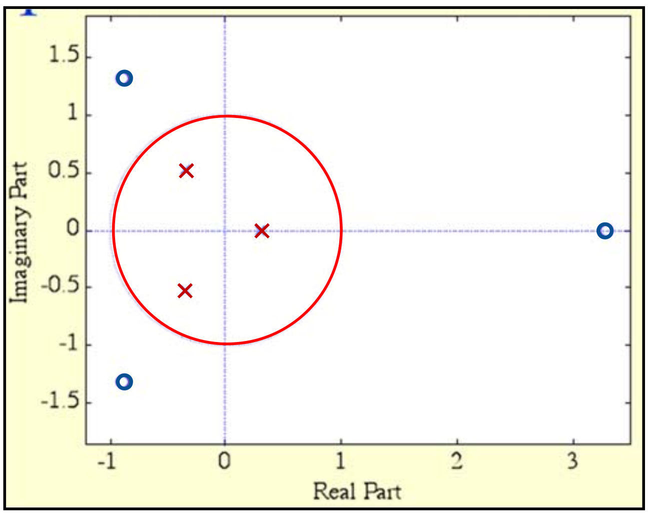
$$\left|A\left(e^{j\omega}\right)\right|^2 = 1, \qquad \text{for all } \omega$$

- An M-th order causal **real-coefficient** allpass transfer function is of form

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}}$$
$$= \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Implying that the poles and zeros exhibits mirror-image symmetry in z-plane.
- Example:

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 04z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

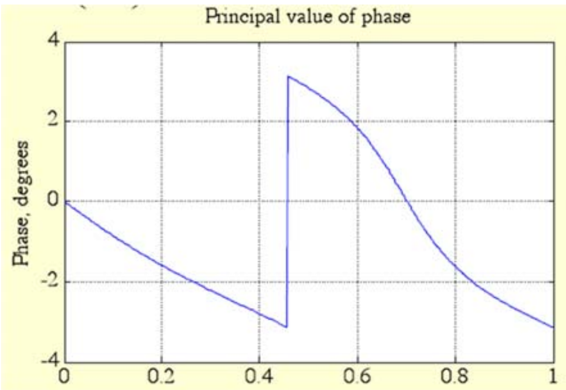


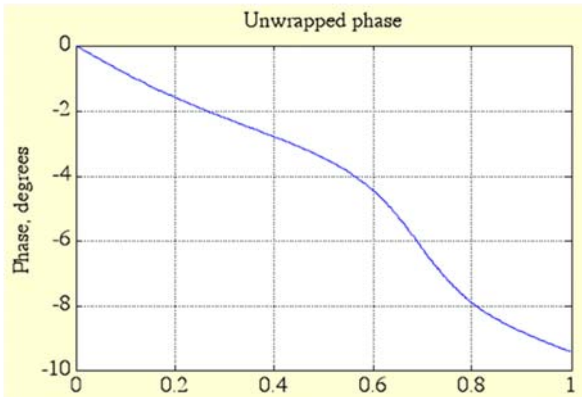
- The poles of a causal stable transfer function must lie inside the unit circle in z-plane
- Pairs of conjugated poles and zeros for real coefficient allpass filter, unless real pole or zeros.

- It can be shown that

$$\left|A_M(e^{j\omega})\right|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}} = \frac{z^{-M}D_M(z^{-1})}{D_M(z)} \cdot \frac{z^M D_M(z)}{D_M(z^{-1})} = 1$$

- Q: what's the use of allpass filters?
- A: Its phase plays the role



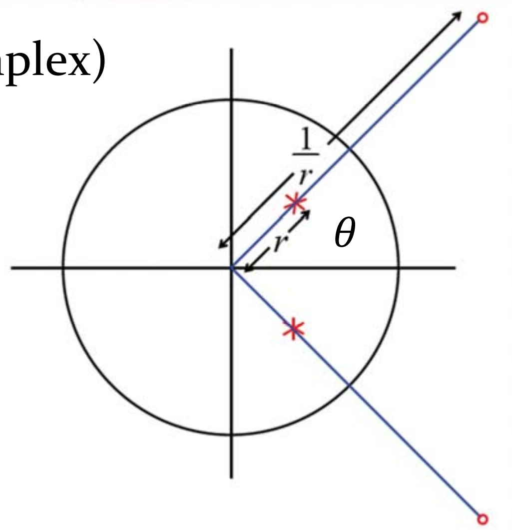


- Factorized form of real (or complex) coefficient allpass filter

$$A_M(z) = \pm \prod_{i=1}^{M} \left(\frac{-\lambda_i^* + z^{-1}}{1 - \lambda_i z^{-1}} \right)$$

Pole: $z_i = \lambda_i = r_i e^{j\theta_i}$

Zero: $z = \frac{1}{\lambda_i^*} = \frac{1}{r_i} e^{j\theta_i}$



- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles suited inside the unit circle.

A First-Order Allpass Filter

- A first-order complex coefficient allpass transfer function

$$A(z) = \frac{-\lambda^* + z^{-1}}{1 - \lambda z^{-1}}, |\lambda| < 1, \lambda = re^{j\varphi}$$

- Its frequency response is given by

$$A(e^{j\omega}) = \frac{-\lambda^* + e^{-j\omega}}{1 - \lambda e^{-j\omega}} = e^{-j\omega} \frac{1 - re^{j(\omega-\varphi)}}{1 - re^{-j(\omega-\varphi)}}$$

So the phase function is

$$\theta(\omega) = -\omega - 2\tan^{-1}\frac{r\sin(\omega-\varphi)}{1 - r\cos(\omega-\varphi)}$$

$$\frac{d\theta(\omega)}{d\omega} = \frac{-(1-r^2)}{(1 - r\cos(\omega-\varphi))^2 + r^2\sin^2(\omega-\varphi)} < 0$$

The phase of the first-order allpass filter decreases monotonically.

Minimum Phase System

- **Definition:** A causal system with all zeros located inside the unit circles in z-plane is a minimum phase system, denoted as $H_{\min}(z)$.
- **Property:** Any real coefficient causal system can be represented as

$$H(z) = H_{\min}(z)A_m(z)$$

where, $H_{\min}(z)$ has the same magnitude response as $H(z)$, and $A_m(z)$ is an allpass system.

Minimum Phase System Property

- **Proof:** Assume that $H(z)$ has only one zero $z = \frac{1}{a^*}$, $|a| < 1$, located outside the unit circle.
  Thus, $H(z)$ can be represented as
  $$H(z) = H_1(z)(z^{-1} - a^*).$$
- According to definition, $H_1(z)$ is a minimum system.
- And $H(z)$ can be equivalent to
$$H(z) = H_1(z)(z^{-1} - a^*)\frac{1 - az^{-1}}{1 - az^{-1}}$$
$$= H_1(z)(1 - az^{-1})\frac{z^{-1} - a^*}{1 - az^{-1}}$$

- i.e., $H(z) = H_{\min}(z)A_m(z)$, where
$$H_{\min}(z) = H_1(z)(1 - az^{-1})$$

and
$$A_m(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- Why it is called a minimum-phase system?
  - **The minimum phase-lag property**: the phase delay of the minimum phase system is always less than the phase delays of the other systems with the same magnitude response.

Example

- Q: Given the transfer function of a real coefficient causal stable LTI system

$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}}, |a| < 1 \text{ and } |b| < 1$$

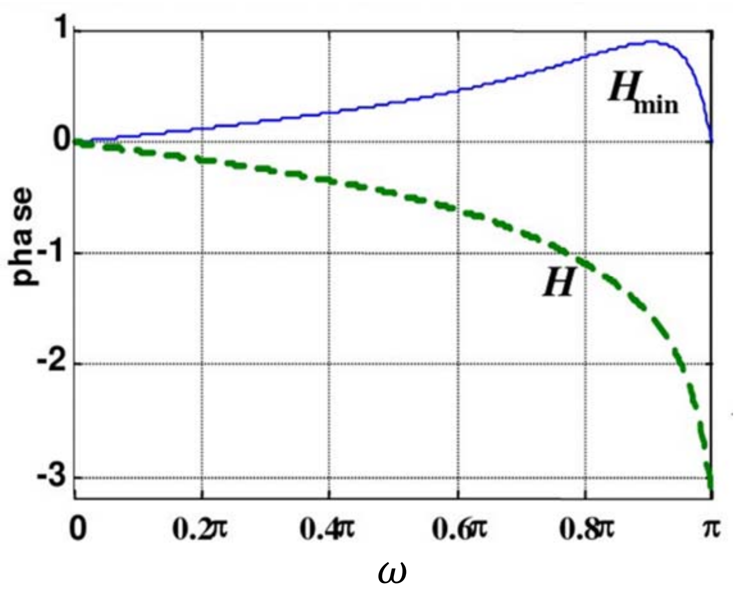
Find the minimum phase system having the same magnitude response as that of $H(z)$

- A: Since the zero of $H(z)$ is $-\frac{1}{b}$ and $|b| < 1$, it is not a minimum phase system.

$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}} \frac{1 + bz^{-1}}{1 + bz^{-1}} = \frac{1 + bz^{-1}}{1 + az^{-1}} \frac{b + z^{-1}}{1 + bz^{-1}}$$

- So the minimum phase system is

$$H_{\min}(z) = \frac{1 + bz^{-1}}{1 + az^{-1}}$$



The phase response of $H(e^{j\omega})$ and $H_{\min}(e^{j\omega})$ when $a = 0.9$, and $b = 0.4$

Maximum Phase System

- A causal system with all zeros located outside the unit circles in z-plane is a maximum phase system, denoted as $H_{\max}(z)$.
- Example: The transfer function of a real coefficient causal stable LTI system is give by

$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}}, |a| < 1 \text{ and } |b| < 1$$

- Obviously, this is a maximum phase system.

Inverse System

- Inverse system has $h_1[n] \circledast h_2[n] = \delta[n]$
- Then, in z-Domain

$$H_1(z)H_2(z) = 1$$

- If we have $H_1(z)$, then

$$H_2(z) = \frac{1}{H_1(z)}$$