Lecture 5
Frequency Domain Representation of Discrete Time Systems

Signal & Frequency Components

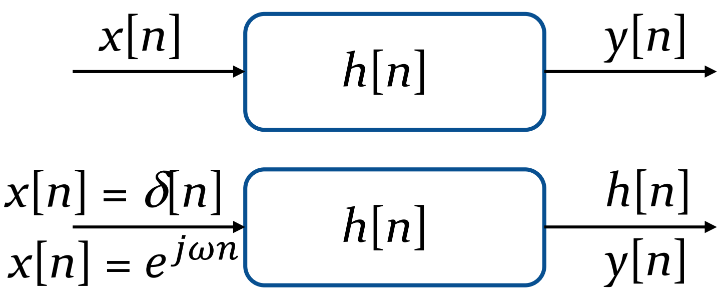
- Continuous periodic signal $\Rightarrow$ Discrete frequency components (Fourier Series)
- Continuous non-periodic signal $\Rightarrow$ Continuous frequency components (CTFT)
- Discrete-time signal $\Rightarrow$ Continuous frequency components (DTFT)

Linear Combination

- When a signal can be represented as a linear combination of complex exponentials :

$$x[n] = \sum_{k} a_k e^{j\omega_k n}$$

knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of superposition property.



$$y[n] = h[n] \circledast e^{j\omega n} = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

- Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Eigenfunction

- Then, we can write
$$y[n] = H(e^{j\omega})e^{j\omega n}$$
- Thus, for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$.
- If applying a function as an input to a system, and the output of the system is the same function multiplied by a constant, such function is an **eigenfunction** of the system.
- So, $e^{j\omega n}$, is an eigenfunction of the system.

The Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

is defined to be the **frequency response** of the LTI system with impulse response $h[n]$

- $H(e^{j\omega}) = H_{\text{re}}(e^{j\omega}) + jH_{\text{im}}(e^{j\omega})$
- $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$, where, $\theta(\omega) = \arg\{H(e^{j\omega})\}$
- $|H(e^{j\omega})|$: **magnitude response**
- $\theta(\omega)$: **phase response**

Example

- Consider the ideal delay system defined by

$$y[n] = x[n - n_d], \text{for constant integer } n_d$$

- With input $x[n] = e^{j\omega n}$, we have

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}$$

The frequency response of the ideal delay is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}$$

- An alternative method: the impulse response of the ideal delay is $h[n] = \delta[n - n_d]$. So the frequency response of the ideal delay system is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

Frequency Response in Decibels

- Gain Function:

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right|$$

the unit is in dB

- Attenuation (or loss function):

$$\mathcal{A}(\omega) = -20\log_{10}\left|H(e^{j\omega})\right|$$

is the negative of the gain function.

Symmetry of frequency Response

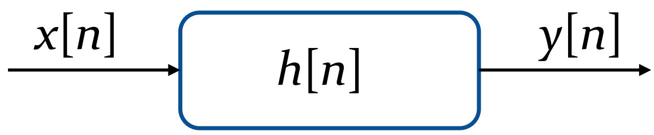
- Due to DTFT, for a real impulse response $h[n]$, $H(e^{j\omega})$ is conjugate symmetric, i.e.,
  - $H(e^{j\omega}) = H^*(e^{-j\omega})$, or
  - $|H(e^{j\omega})| = |H(e^{-j\omega})|$, and $\theta(\omega) = -\theta(-\omega)$, or
  - $H_{\mathrm{re}}(e^{j\omega})$ is even and $H_{\mathrm{im}}(e^{j\omega})$ is odd.

- For a real symmetric impulse response,
  - $H(e^{j\omega})$ is real and symmetric.

Frequency-Domain Characterization of LTI DT System

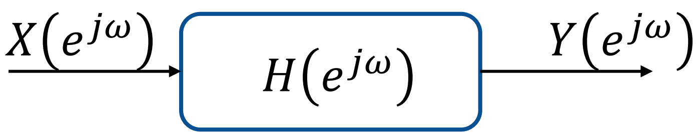
- For LTI system in time domain, we have

$$y[n] = x[n] \circledast h[n]$$



- Applying convolution property of DTFT, we have

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$



Frequency Response of FIR System

- The time-domain input-output relation of FIR system:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k], \quad N_1 < N_2$$

- Applying DTFT on both sides, we arrive at

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}X(e^{j\omega}),$$

- The frequency response of FIR system is given by

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k},$$

Frequency Response of IIR System

- The time-domain input-output relation of IIR system

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

- Applying DTFT on both sides, we arrive at

$$\sum_{m=0}^{N} b_m e^{-j\omega m} Y(e^{j\omega}) = \sum_{m=0}^{M} a_m e^{-j\omega m} X(e^{j\omega})$$

- The frequency response of IIR system is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} a_m e^{-j\omega m}}{\sum_{m=0}^{N} b_m e^{-j\omega m}}$$

Example

- Determine the frequency response of the M-point moving average filter.
- Since the input-output relation is given by:

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

- the impulse response is given by:

$$h[n] = \frac{1}{M}\sum_{l=0}^{M-1} \delta[n-l] = \begin{cases} \frac{1}{M}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Thus, the frequency response is given by

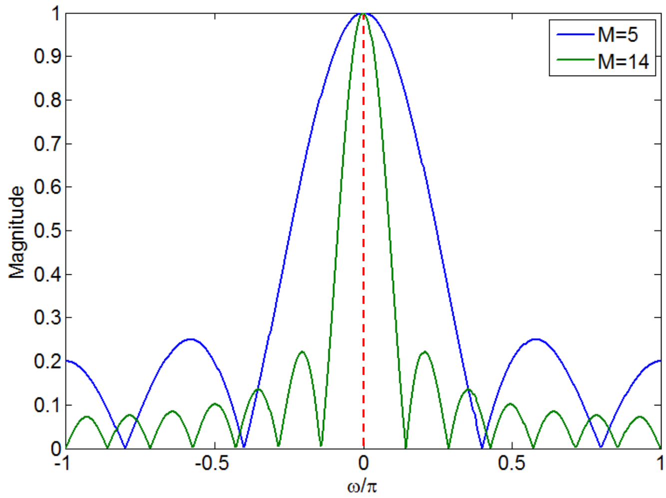
$$H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1}{M}\cdot\frac{1-e^{-jM\omega}}{1-e^{-j\omega}}$$

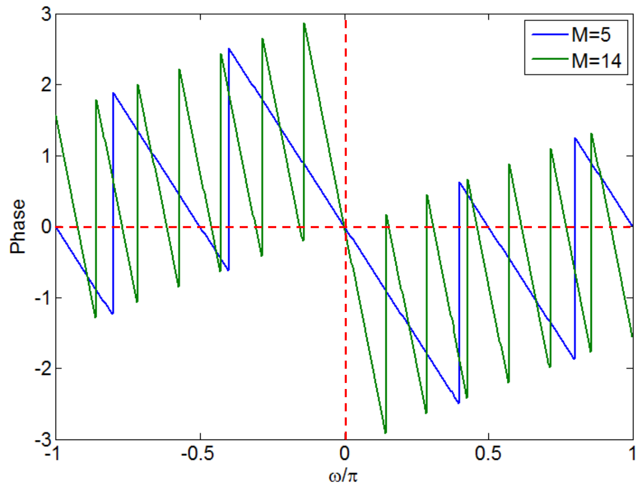
$$= \frac{1}{M}\cdot\boxed{\frac{\sin(M\omega/2)}{\sin(\omega/2)}}\boxed{e^{-j(M-1)\omega/2}}$$

$$\left|H(e^{j\omega})\right| = \frac{1}{M}\left|\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$

$$\theta(\omega) = \frac{-(M-1)\omega}{2} + \pi\sum_{k=1}^{\lfloor M/2\rfloor} \mu\left(\omega - \frac{2\pi k}{M}\right)$$

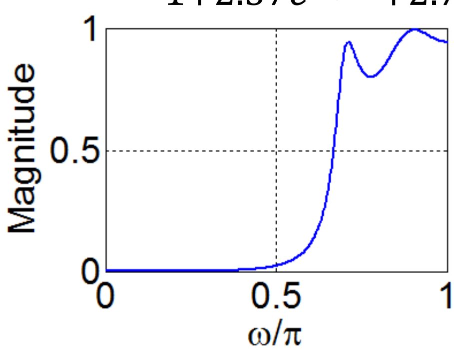
- The plots of the magnitude response and phase response of the M-point moving average filter, for M=5 and M=14

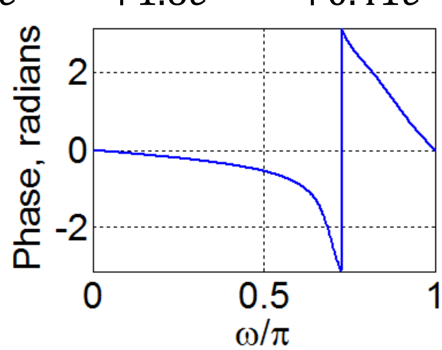




Unwrapped Phase Function

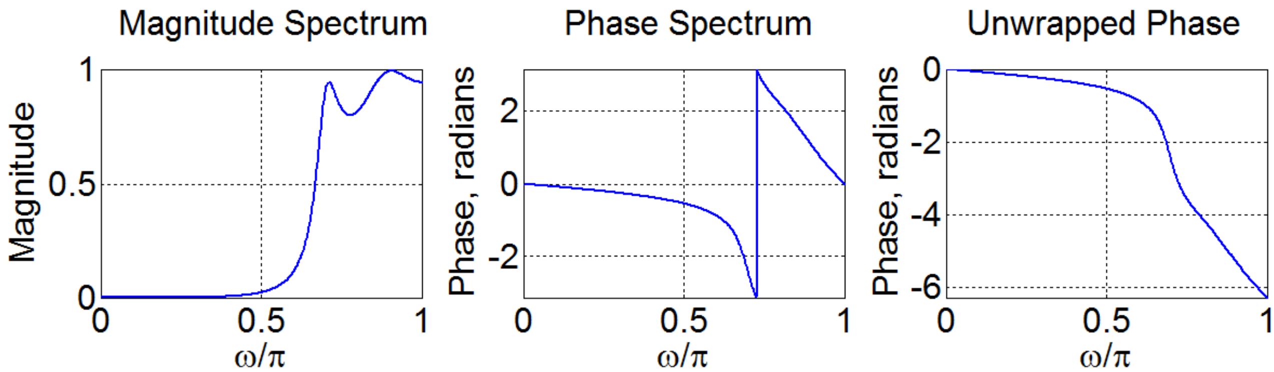
- The principle value of phase function is defined to within a range $[-\pi, \pi]$.
- The phase function of DTFT thus computed exhibits discontinuity of 2π radians in plots.
- Example: $X\left(e^{j\omega}\right) = \dfrac{0.008-0.033e^{-j\omega}+0.05e^{-j2\omega}-0.033e^{-j3\omega}+0.008e^{-j4\omega}}{1+2.37e^{-j\omega}+2.7e^{-2j\omega}+1.6e^{-j3\omega}+0.41e^{-j4\omega}}$





Unwrapped Phase Function

- The process to remove the 2π discontinuity is called **unwrapping the phase.**

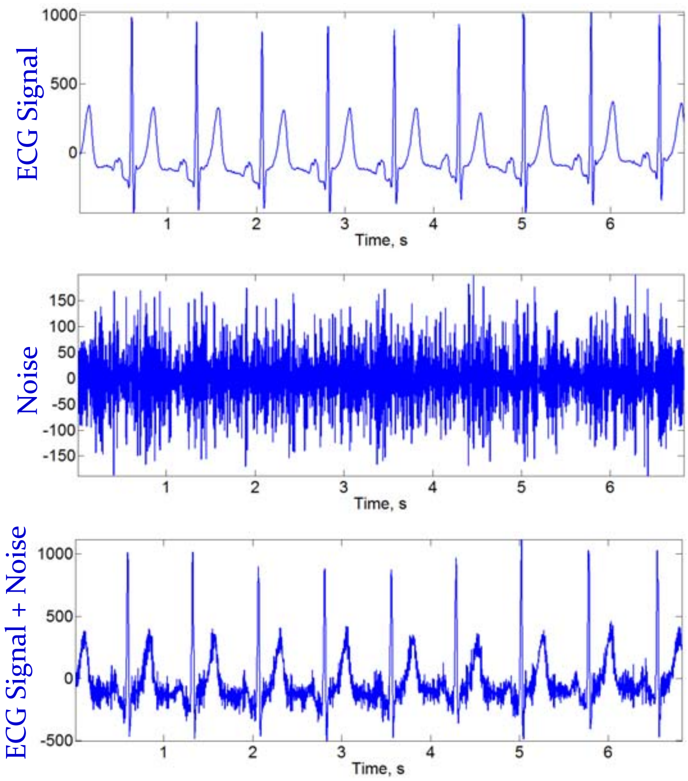


The Concept of Filtering

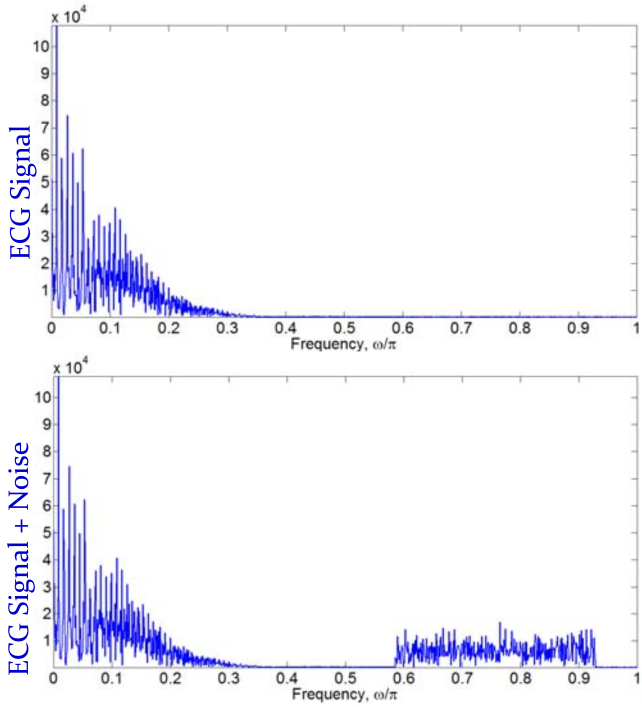
- One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components.

- Such systems are called digital filters and are one of main devices in digital signal processing

- A time domain signal with noise



- Their frequency spectrum



The Concept of Filtering

- Any discrete-time signal may be expressed as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Any frequency component $e^{j\omega n}$ may be scaled by a frequency response $H(e^{j\omega})$ at frequency ω, such that the frequency component is passed without distortion, or attenuated.
- For example, if we have an ideal LTI system with magnitude response given by

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

A Simple Example

- We apply an input $x[n]$ to the system, where
$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n,$$
$$0 < \omega_1 < \omega_c < \omega_2 < \pi$$
- Because of linearity, the output of the system is
$$y[n] \\ = A\left|H\left(e^{j\omega_1}\right)\right|\cos(\omega_1 n + \theta(\omega_1)) \\ + B\left|H\left(e^{j\omega_2}\right)\right|\cos(\omega_2 n + \theta(\omega_2))$$
- As $\left|H\left(e^{j\omega_1}\right)\right| = 1$, and $\left|H\left(e^{j\omega_2}\right)\right| = 0$, the output reduces to $y[n] = A\cos(\omega_1 + \theta(\omega_1))$
- The LTI system acts like a lowpass filter.

Design Example

- Design a very simple digital filter.
- **Requirement:** An input, consisting of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample, is to be filtered to keep the high-frequency component, but block the low-frequency component.
- For simplicity, we assume a filter of length 3 with an impulse response: $h[0]=h[2]=\alpha$, and $h[1]=\beta$.

- The input-output relation in time-domain would be:

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

- **Design objective:** Choose suitable values of α and β, such that the output contains only a sinusoidal sequence with an angular frequency 0.4 rad/sample.
- Now the frequency response of the filter is given by,

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$
$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$
$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega} = (2\alpha\cos\omega + \beta)e^{-j\omega}$$

Magnitude response

- To block the low-frequency component, let

$$H(e^{j0.1}) = (2\alpha\cos(0.1) + \beta) = 0$$

- To pass the high-frequency component, let

$$H(e^{j0.4}) = (2\alpha\cos(0.4) + \beta) = 1$$

- Result in:

$$\alpha = -6.76185, \qquad \beta = 13.456335$$

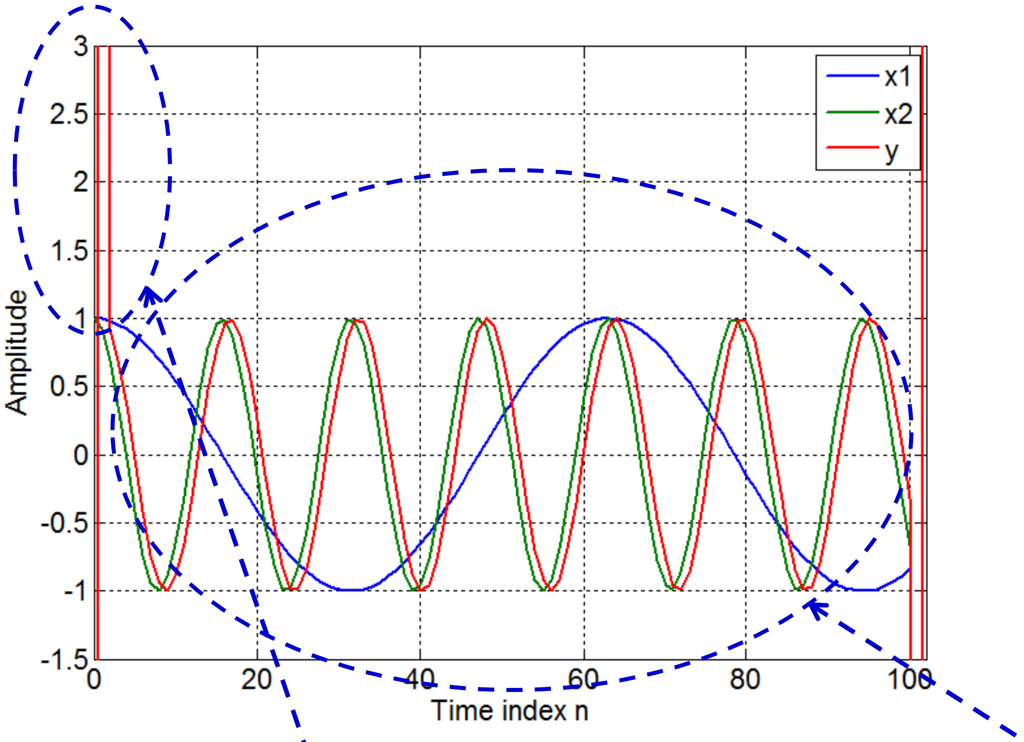
i.e., $h[n] = \{-6.76185, 13.456335, -6.76185\}$, for $n = 0, 1, 2$

- So the designed filter has the input-output relation in time-domain given by

$y[n] = -6.76185(x[n] + x[n-2]) + 13.456335x[n-1]$

and the input is $x[n] = (\cos(0.1n) + \cos(0.4n))\mu[n]$

- Input and output sequences in time-domain



Transient

Steady state output

Phase Delay and Group Delay

- Re-examine a system with an input of pure sinusoidal signal

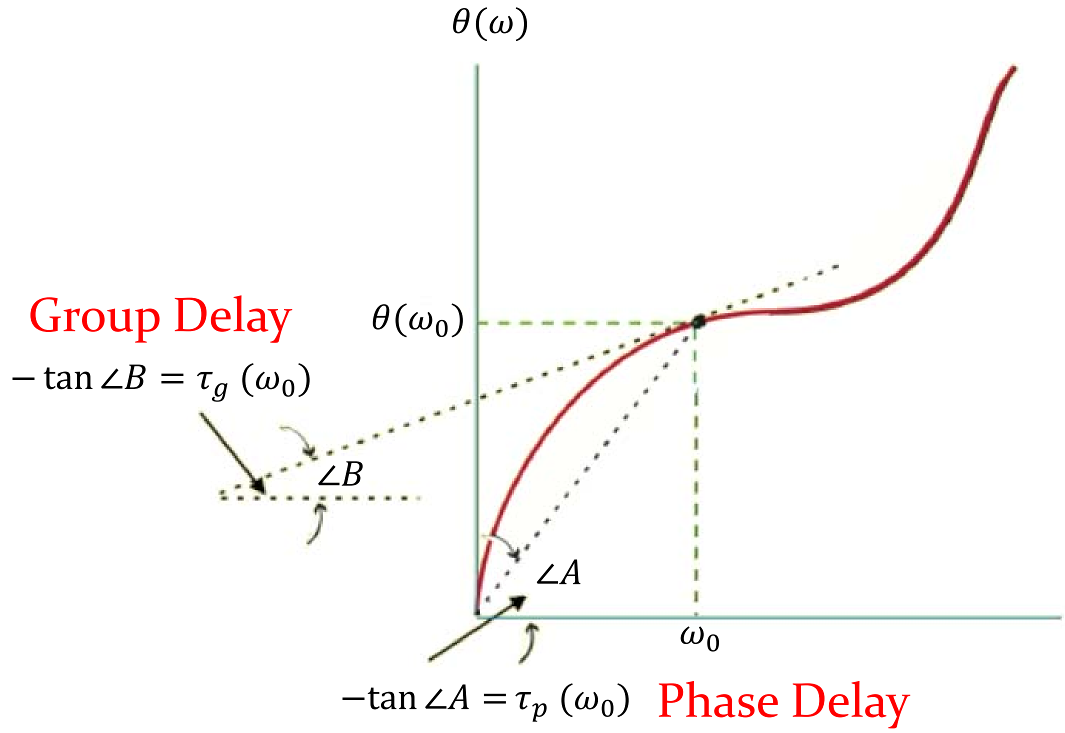
$$
\begin{aligned}
y[n] &= h[n] \circledast A\cos(\omega_0 n + \varphi) \\
&= A\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}\right)\cos(\omega_0 n + \varphi) \\
&= A\left|H\left(e^{j\omega_0}\right)\right|\cos(\omega_0 n + \theta(\omega_0) + \varphi) \\
&= A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0\left[n + \boxed{\frac{\theta(\omega_0)}{\omega_0}}\right] + \varphi\right)
\end{aligned}
$$

$\boxed{-\tau_p(\omega_0)}$

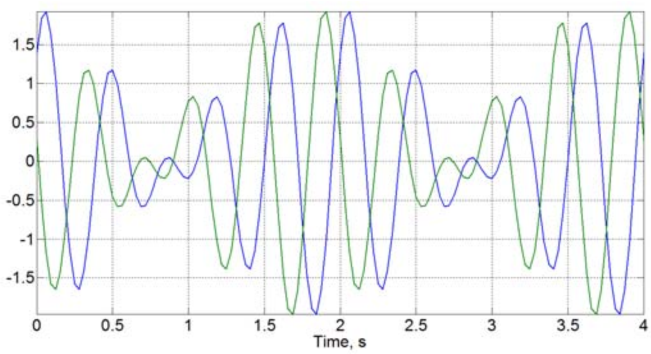
- Define **phase delay** and **group delay**, respectively, as

$$\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}, \qquad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}.$$

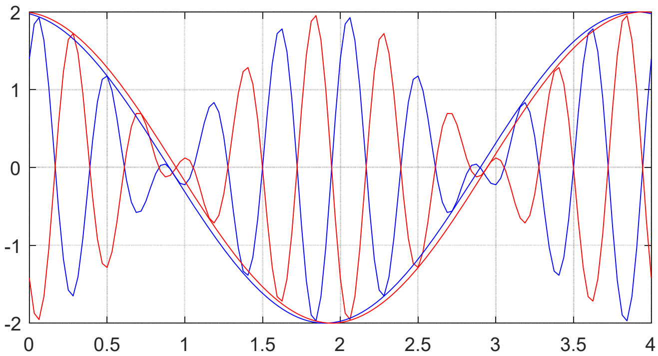
A Graphic Comparison



*Physical Meanings



Phase Delay: Delay of Samples



Group Delay: Delay of Envelopes

- $T = \frac{1}{32}, \omega_1 = 4\pi, \omega_2 = 5\pi$
- Blue Signal:

$\sin(\omega_1 n + 0.2\pi) + \sin(\omega_2 n + 0.3\pi)$

- Green Signal:

$\sin(\omega_1 n + 0.2\pi + 5\omega_1 T)$
$\quad + \sin(\omega_2 n + 0.3\pi + 5\omega_2 T)$

$\theta(\omega_1) = 5\omega_1 T,\ \theta(\omega_2) = 5\omega_2 T,$

$\tau_p(\omega_1) = \tau_p(\omega_2) = -5T$

$\tau_g(\omega_1) = \tau_g(\omega_2) = -5T$

- Red Signal:

$\sin(4\pi n + 0.2\pi + 5\omega_1 T + 0.4\pi)$
$\quad + \sin(5\pi n + 0.3\pi + 5\omega_2 T + 0.2\pi)$

$\theta(\omega_1) = 5\omega_1 T + 0.4\pi,\ \theta(\omega_2) = 5\omega_2 T + 0.2\pi\,,$

$\tau_p(\omega_1) = -5T - 0.1$

$\tau_p(\omega_2) = -5T - 0.04$

$$\tau_g(\omega_1) = \tau_g(\omega_2) = -\frac{\Delta\theta(\omega)}{\Delta\omega} - \frac{\theta(\omega_2) - \theta(\omega_1)}{\omega_2 - \omega_1}$$
$$= -(5T - 0.2) = 1.4T$$

- $H\left(e^{j\omega}\right) = (2\alpha\cos\omega + \beta)e^{-j\omega}$,
- $\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = 1, \qquad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = 1$