Lecture 14.

Tensor Products.

Let R be a ring with 1. Let V_R and ${}_RW$ be a right and left R-modules respectively. As always $V \times W$ is the Cartesian product of sets V, W.

Def. Suppose that A is an abelian group. We call a mapping $\varphi : V \times W \to A$ <u>bilinear</u> if $\varphi((v_1 \pm v_2) \times w) = \varphi(v_1 \times w) \pm \varphi(v_2 \times w)$ and $\varphi(v \times (w_1 \pm w_2)) = \varphi(v \times w_1) \pm \varphi(v \times w_2)$.

Def. A bilinear mapping $\varphi : V \times W \to A$ is <u>balanced</u> if for arbitrary elements $a \in R$, $v \in V$, $w \in W$ we have

$$\varphi(va \times w) = \varphi(v \times aw).$$

<u>Objects</u>: (abelian group A, balanced bilinear mapping $\varphi : V \times W \to A$);
V and W are fixed.

A <u>morphism</u> $(A, \varphi) \to (B, \psi)$ is a homomorphism $\chi : A \to B$ of abelian groups such that the diagram

$$\begin{tikzcd}
 & A \arrow[d, "\chi"] \\
V \times W \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & B
\end{tikzcd}$$

is commutative.

<u>Def</u>. An object $(U, u : V \times W \to U)$ is <u>universal</u> if for an arbitrary object $(A, \varphi : V \times W \to A)$ there exists a unique morphism $\chi : (U, u : V \times W \to U) \to (A, \varphi : V \times W \to A)$.

As we have already done before:

(1) if $(U, u: V \times W \to U)$ is universal then $u(V \times W)$ generates the abelian group U;

(2) if a universal object exists then it is unique.

<u>Lemma</u>. For arbitrary modules V_R, ${}_RW$ a universal object exists.

Proof. Consider the set

$$X = \{ x_{v,w} \mid v \in V, w \in W \} \overset{1-1}{\longleftrightarrow} V \times W$$

Consider the free abelian group

$$\mathbb{Z}X = \bigoplus_{(v,w) \in V \times W} \mathbb{Z}\, x_{v,w}.$$

Consider the following subset S of $\mathbb{Z}X$:

$$x_{v\pm v',w} - x_{v,w} \mp x_{v',w} \text{ for all}$$

$v, v' \in V$; $w \in W$

$$x_{v,w\pm w'} - x_{v,w} \mp x_{v,w'} \text{ for all}$$

$v \in V$; $w, w' \in W$

$$x_{va,w} - x_{v,aw} \text{ for all } v \in V, w \in W,$$
$$a \in R.$$

Consider the abelian subgroup $\mathbb{Z}S$ generated by the subset S in $\mathbb{Z}X$ and the factor group $U = \mathbb{Z}X/\mathbb{Z}S$, the abelian group presented by generators X and relations $S = 0$.

The mapping $u: V \times W \to U$, $u(v \times w) = x_{v,w} + \mathbb{Z}S$ is bilinear and balanced. Let us show that it is universal. ~~If~~ Let $\varphi: V \times W \to A$ be a balanced bilinear map. ~~then φ/S~~ Consider the mapping $x_{v,w} \xrightarrow{\psi} \varphi(v \times w)$ and extend it to a homomorphism $\psi: \mathbb{Z}X \to A$. Clearly, $\psi(S) = (0)$. Hence ~~there exists~~ ψ gives rise to a homomorphism $\chi: \mathbb{Z}X/\mathbb{Z}S \to A$, which is a morphism from $(U, u: V \times W \to U)$ to $(A, \varphi: V \times W \to A)$.

We denote $U = V \otimes_R W$, $u(v \times w) = v \otimes w$.

and call the abelian group U the <u>tensor product</u> of modules V, W. An arbitrary element from $V \otimes_R W$ is a sum of simple tensors $v \otimes w$. It is easy to ~~see~~ see that $0 \otimes W = V \otimes 0 = (0)$.

<u>Example:</u> $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$

For an arbitrary simple tensor $v \otimes w$ we have $2(v \otimes w) = 2v \otimes w = 0$ and $3(v \otimes w) = v \otimes 3w = 0$. Hence $v \otimes w = 0$,

$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = (0)$.

<u>Example:</u> $V \otimes_R R \cong V$.

Indeed, let $u(v \times a) = va \in V$. Clearly, this mapping is bilinear and balanced.

Let us show that it is universal.

Let A be an abelian group, $\varphi : V \times R \to A$ is a bilinear balanced map. Then for arbitrary elements $v \in V, a \in A$, we have $\varphi(v \times a) = \varphi(v \times a \cdot 1) = \varphi(va \times 1)$.

Let $\chi : V \to A$, $\chi(v) = \varphi(v \times 1) \in A$.

We have

$$\begin{tikzcd} & va \arrow[d, "\chi"] \\ v \times a \arrow[ur] \arrow[r, "\varphi"'] & \varphi(v \times a) = \varphi(va \times 1) \end{tikzcd}$$

the diagram is commutative. We proved that $(V, u : v \times a \to va)$ is universal.

Example. Let $V = \bigoplus_i V_i$, $W = \bigoplus_j W_j$ be direct sums of modules. Then

$$V \otimes_R W = \bigoplus_{i,j} (V_i \otimes_R W_j).$$

Let it be your <u>exercise</u>: if

$u_{ij} : V_i \times W_j \to V_i \otimes_R W_j$ are universal

balanced bilinear maps, then

$$u\left(\sum_i v_i \times \sum_j w_j\right) = \bigoplus_{i,j} u_{ij}(v_i \times w_j) \in \bigoplus_{i,j} (V_i \otimes_R W_j)$$

is a universal bilinear balanced map.

In particular, let $R = F$ be a field and let V, W be vector spaces over F. Let $\{e_i\}_i$, $\{f_j\}$ be bases of the spaces V, W respectively. Then

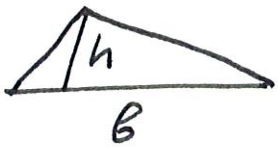
$V = \bigoplus_i F e_i$, $W = \bigoplus_j F f_j$ and therefore

$$V \otimes_F W = \bigoplus_{i,j} (F e_i \otimes F f_j) = \bigoplus_{i,j} F (e_i \otimes f_j).$$

We showed that $\{e_i \otimes f_j\}_{i,j}$ is a basis of $V \otimes_F W$.

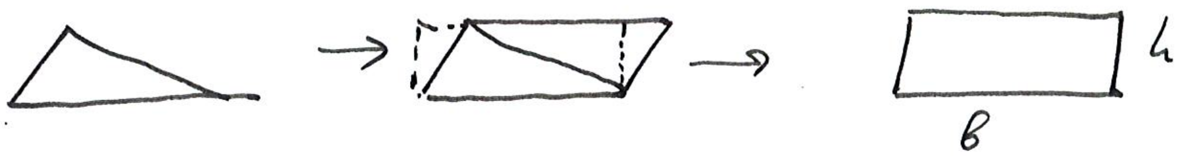
Among other things it implies that if $v_1, \dots, v_k \in V$ are linearly independent elements and $w_1, \dots, w_k \in W$ then $v_1 \otimes w_1 + \dots + v_k \otimes w_k = 0$ implies $w_1 = \dots = w_k = 0$. In other words linearly independent elements in V are "linearly independent in $V \otimes_F W$ over W

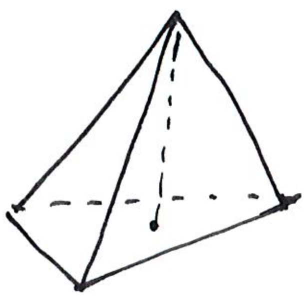
Application: Hilbert's Problem.

The area of a triangle 

is $\frac{1}{2} bh$.

"Greek" proof: double the triangle and cut it in pieces





Volume of a Piramid = $\frac{1}{3} S \cdot h$, S = area of the base.

It is proved by integration.

Hilbert: is there a "Greek" proof?

Can we triple a piramide and cut that object into pieces, putting them back together to assemble a box?

Max Dehn: not always.

To each polyhedron P we assign a tensor from $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}\pi$: for each edge of the polyhedron consider the tensor (length of the edge) $\otimes$ (the angle the two faces meet at that edge $+\mathbb{Q}\pi$),

$$D(P) = \sum_{\text{edge}} \text{length} \otimes (\text{angle} + \mathbb{Q}\pi) =$$

the Dehn invariant of the polyhedron.

<u>Example</u>. The Dehn invariant of a box is a sum of $l \otimes (\pi/2 + \mathbb{Q}\pi)$, but $\pi/2 + \mathbb{Q}\pi = 0$ in $\mathbb{R}/\mathbb{Q}\pi$, so the Dehn invariant $= 0$.

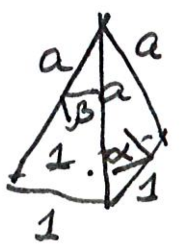
If we cut a polyhedron P into two polyhedra P_1 and P_2 then

$$D(P) = D(P_1) + D(P_2).$$

So, if we triple a piramid and the reassemble it into a box, the resulting box should have Dehn invariant 0, meaning that the original piramid had Dehn invariant 0.

Consider a piramid whose base is an equilateral triangle with sides 1. The Dehn invariant is

$$3\otimes(\alpha+\mathbb{Q}\pi) + 3a\otimes(\beta+\mathbb{Q}\pi)$$



The length a takes values in $(\frac{1}{\sqrt{3}}, \infty)$, $\alpha = \alpha(a)$ is an increasing function taking values in $(0, \pi/2)$.

It implies that there exists an irrational $a \in (\frac{1}{\sqrt{3}}, \infty)$ such that $\frac{\alpha(a)}{\pi}$ is also irrational. Indeed, the set $S = (0, \pi/2) \setminus \{\alpha(q) \,/\, q \in (\frac{1}{\sqrt{3}}, \infty)$, q is rational$\}$ is uncountable. The set $\{\alpha \in S \,/\, \frac{\alpha}{\pi} \in \mathbb{Q}\}$ is countable. Hence the set $S \setminus \{\alpha \in S \,/\, \frac{\alpha}{\pi} \in \mathbb{Q}\}$ is not empty. This proves the claim.

If the number a is irrational then 1 and a are linearly independent over $\mathbb{Q}$. By the properties of tensor product $3\otimes(\alpha+\mathbb{Q}\pi) + 3a\otimes(\beta+\mathbb{Q}\pi)=0$ implies $\alpha+\mathbb{Q}\pi=0$, which contradicts α/π being irrational.

Tensor products of bimodules and algebras.

Bimodules. Let R and S be rings. An abelian group V is called a (R,S)-bimodule if

1) ${}_R V$ is a left R-module
2) V is a right S-module
3) for arbitrary elements

$a \in R$, $b \in S$, $v \in V$ we have $(av)b = a(vb)$.

Now let R, S, T be rings. Let V be an (R,S)-bimodule and let W be an (S,T)-bimodule. Then the tensor product $V \otimes_S W$ is an (R,T)-bimodule: for elements $a \in R$, $t \in T$, $v \in V$, $w \in W$ we define

$$a(v \otimes w) = av \otimes w, \quad (v \otimes w)t = v \otimes wt.$$

Then we extend it to $\sum_i v_i \otimes w_i$ by linearity. The problem, however, is that an element from $V \otimes_S W$ may be represented as a sum of simple tensors in more than one way.

We need to show that $\sum_i v_i \otimes w_i = 0$ in $V \otimes_S W$ implies $\sum_i a v_i \otimes w_i t = 0$.

The mapping $\varphi : V \times W \to V \otimes_S W$, $\varphi(v \times w) = a v \otimes w t$ is bilinear and balanced. Hence there exists a homomorphism $\chi : V \otimes_S W \to V \otimes_S W$ of abelian groups such that the diagram

$$\begin{tikzcd} & v \otimes w \arrow[d, "\chi"] \\ v \times w \arrow[ur] \arrow[r, "\varphi"'] & a v \otimes w t \end{tikzcd}$$

is commutative.

If $\sum_i v_i \otimes w_i = 0$ then $\chi(\sum_i v_i \otimes w_i) = \sum_i a v_i \otimes w_i t = 0$, which proves the claim.

Algebras. Now suppose that $R = F$ is a field and A, B are associative F-algebras. We claim that the tensor product $A \otimes_F B$ is an F-algebra with respect to the operation

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right) = \sum_{i,j} a_i c_j \otimes b_i d_j$$

Again the problem is that elements may be represented as sums of simple tensors not uniquely. We need to prove that

$\sum_i a_i \otimes b_i = 0$ implies $\sum_i a_i c \otimes b_i d = 0$

and $\sum_j c_j \otimes d_j = 0$ implies $\sum_j a c_j \otimes b d_j = 0$.

we will prove only the first implication

The mapping $\varphi(a\times b) = ac\otimes bd$ is balanced and bilinear.

<u>Remark.</u> In the case of vector spaces the "balanced" part is automatic.

Hence there exists a homomorphism of abelian groups $\chi : A\otimes_F B \to A\otimes_F B$, $\chi(a\otimes b) = \varphi(a\times b) = ac\otimes bd$. If $\sum_i a_i\otimes b_i = 0$ then $\chi(\sum_i a_i\otimes b_i) = \sum_i a_i c\otimes b_i d = 0$.