Lecture 4

We drew important conclusions from Theorem I.3.1. Now it is time to finish the proof.

Let X be an alphabet, $X = \{x_i, i \in I\}$, the set I is equipped with an order $<$ satisfying the minimality condition.

Let R be a subset of the free algebra $F\langle X\rangle$.

We assume that

(*) for any two elements $f, g \in R$ that admit a composition this composition reduces to 0.

∓ We will show that the set Irr of irreducible words is linearly independent modulo the ideal $id(R)$.

Let us analize the process of reduction

Suppose that a word u is reducible,

$u = u' r u''$; $u', u'' \in X^*$;

$$r = \bar{r} - \sum_{v_i < \bar{r}} \alpha_i v_i \in R$$

Then $\bar{r} \to \sum \alpha_i v_i$, We replace

u by $u' (\sum_i \alpha_i v_i) u''$.

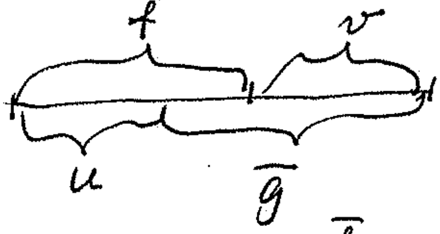
In the free algebra it is equivalent to

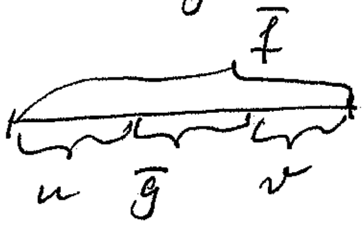
$$u = u' (\bar{r} - \sum_i \alpha_i v_i) u'' + u' (\sum_i \alpha_i v_i) u''$$

All monomials in $u' (\bar{r} - \sum_i \alpha_i v_i) u''$

are $\leq u$.

Suppose that $f, g \in R$ and their leading monomials $\bar f, \bar g$ admit a composition.

 , $w = \bar f v = u \bar g$

or  , $w = \bar f$.

For simplicity we will assume that coefficients at $\bar f, \bar g$ are $= 1$.

$$(f,g)_w = fv - u\bar g \quad \text{or} \quad (f,g)_w = f - u\bar g v.$$

In both cases the monomial w gets cancelled, so

$$(f,g)_w = \text{linear combination of}$$

monomials $< w$.

Doing reductions in these monomials we add terms $u'zu''$ (see above) and all

$u' \bar{r} u''$ are $\leq$ these monomials $< w$.

Since $(f,g)_w$ reduces to 0 (it was our assumption) we get

$$(**) \quad (f,g)_w = \sum \alpha u' r u'', \qquad \delta$$

where $\alpha \in F$; $u', u'' \in X^*$, $r \in R$ and for each summand $u' \bar{r} u'' < w$.

Now we are ready to prove that the leading monomial of a nonzero element from $id(R)$ is reducible.

Let $0 \neq f \in id(R)$. Then

$$f = \sum_i \alpha_i w_i' r_i w_i'', \qquad (1)$$

$0 \neq \alpha_i \in F$; $w_i', w_i'' \in X^*$, $r_i \in R$, ~~r_i~~. Let

$$v = \max_i (w_i' \bar{r}_i w_i'').$$

We will prove reducibility of $\bar{f}$ by lexicographical induction on pairs

$$(v, \# \text{ of times } v \text{ occurs among } w_i' \bar{r}_i w_i'')$$

If v occurs among $w_i' \bar{r}_i w_i''$'s once then $\bar{f} = v = w_i' \bar{r}_i w_i''$ for some i, so $\bar{f}$ is reducible.

If the number of times ≥ 2 then

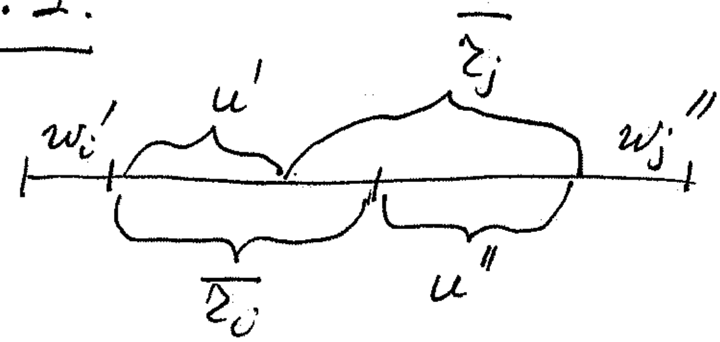
$$v = w_i' \bar{r}_i w_i'' = w_j' \bar{r}_j w_j''$$

for some $i < j$. There are two cases:

<u>Case 1.</u> The subwords $\bar{r}_i$ and $\bar{r}_j$ intersect in v

<u>Case 2.</u> They do not intersect.

Case 1.1.

$v =$ 

Then $w_i'' = u'' w_j''$, $w_j' = w_i' u'$.

We have

$$\alpha_i w_i' z_i w_i'' + \alpha_j w_j' z_j w_j'' = \tag{2}$$
$$= (\alpha_i + \alpha_j) w_i' z_i w_i'' + \alpha_j (w_j' z_j w_j'' - w_i' z_i w_i'').$$

Furthermore,

$$w_j' z_j w_j'' - w_i' z_i w_i'' = \boxed{\cancel{w_j z_j w_i}}'.$$

$$w_i' u' z_j w_j'' - w_i' z_i u'' w_j'' =$$
$$= w_i' (u' z_j - z_i u'') w_j''$$

Composition of z_i and z_j.

the fact that $u'r_j - r_i u''$ reduces to zero means that

$$u'r_j - r_i u'' = \sum \alpha\, v' r v'',$$

where $\alpha \in F$; $v', v'' \in X^*$, $r \in R$, and for each summand $v'\bar{r}v'' \leq$ maximal monomial in $u'r_j - r_i u''$ that is less than $u'\bar{r}_j = \bar{r}_i u''$.

Hence

$$w_j' r_j w_j'' - w_i' r_i w_i'' = \sum \beta\, t' r t'',$$

$\beta \in F$; $t', t'' \in X^*$; $r \in R$, and for each summand $t'\bar{r}t'' < v$.

Replacing $\alpha_i w_i' r_i w_i'' + \alpha_j w_j' r_j w_j''$ in the sum (1) by the right hand side of (2), we reduce the number of occurrences of v among leading monomials of the summonds.

If $\alpha_i + \alpha_j = 0$ and v does not occur in other summands then we reduce v.
In any case the assertion follows from the induction assumption.

<u>Case 1.2.</u>

$$v = w_i' \overbrace{u' \underbrace{\bar{r}_j}_{} u''}^{\bar{r}_i} w_i''$$

~~The~~ We have $w_j' = w_i' u'$, $w_j'' = u'' w_i''$,

$$\alpha_i w_i' r_i w_i'' + \alpha_j w_j' r_j w_j'' = (\alpha_i + \alpha_j) w_i' r_i w_i''$$

$$+ \alpha_j (w_i' r_i w_i'' - w_j' r_j w_j''),$$

$$w_i' r_i w_i'' - w_j' r_j w_j'' = w_i' (r_i - u' r_j u'') w_i'',$$

the expression $r_i - u' r_j u''$ is a composition of r_i and r_j, which allows us to argue literally as in the case 1.1.

<u>Case 2.</u> $v = |w_i' \underbrace{\quad}_{\overline{r_i}} u \underbrace{\quad}_{\overline{r_j}} w_j''|$

we have $w_j' = w_i' \overline{r_i}\, u$, $w_i'' = u \overline{r_j}\, w_j''$.

Hence

$$w_i' r_i w_i'' - w_j' r_j w_j'' = w_i' r_i u \overline{r_j}\, w_j''$$

$$- w_i' \overline{r_i}\, u\, r_j w_j'' = w_i' (r_i u \overline{r_j} - \overline{r_i} u r_j) w_j''.$$

Let $r_i = \overline{r_i} - \{r_i\}$, where $\{r_i\}$ is a linear combination of monomials $< \overline{r_i}$.

Similarly $r_j = \overline{r_j} - \{r_j\}$.

Then

$$r_i u \overline{r_j} - \overline{r_i}\, u\, r_j = r_i u (r_j + \{r_j\})$$

$$- (r_i + \{r_i\}) u r_j = r_i u \{r_j\} - \{r_i\} u r_j.$$

Since every monomial in $\{r_i\}$ is $< \bar{r}_i$ and every monomial in $\{r_j\}$ is $< \bar{r}_j$ we get

$$w_j' r_j w_j'' - w_i' r_i w_i'' = \sum \beta\, t' r t'',$$

$$t' \bar{r} t'' < v,$$

which allows us to complete the proof of the theorem as we did above.

Groebner - Shirshov basis for semigroups

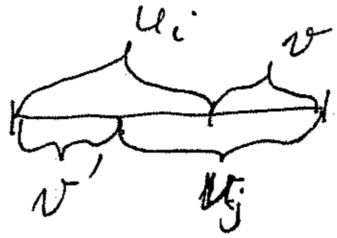
Let S be a semigroup. The semigroup algebra FS consists of formal finite linear combinations

$$FS = \{ \sum_i \alpha_i s_i \mid \alpha_i \in F,\ s_i \in S \}.$$

If
$$S = \langle X \mid u_i = v_i,\ i \in I \rangle$$

then $F\langle X \mid u_i - v_i = 0,\ i \in I \rangle$ is a presentation for the semigroup algebra FS.

Suppose that I is equipped with an order $<$ satisfying the minimality condition. Suppose that $u_i > v_i$, for every $i \in I$. Suppose that u_i and u_j admit a composition

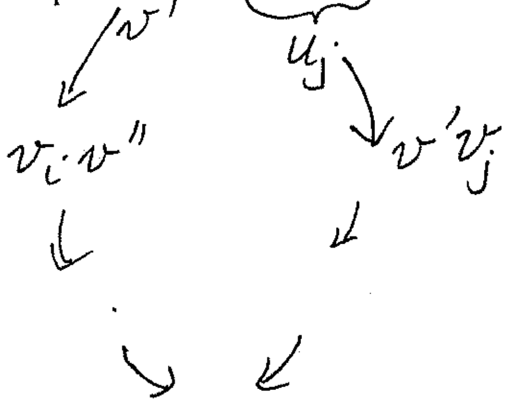


$$(u_i - v_i,\ u_j - v_j)_w = (u_i - v_i)v'' - v'(u_j - v_j)$$

$$= -v_i v'' + v' v_j,$$

again a difference of two words.

The condition (*) that compositions reduce to 0 means the following



<u>In other words:</u>

(**) if u_i, u_j intersect in a word w and we do two different reductions $u_i \to v_i$ and $u_j \to v_j$, then $v_i v''$ and $v' v_j$ can be reduced to the same word.

This is the necessary and sufficient condition for irreducible words to be all different. So,

irreducible words = normal forms.

<u>Every word can be reduced to a unique normal form.</u>

<u>Algorithmic Problems (Max Dehn, 1906)</u>:

Given a finitely presented group (semigroup, algebra) A,

(1) (<u>Word Problem</u>) and given two elements in the free group (semigroup, algebra) are they equal in A ?

(2) (<u>Isomorphism Problem</u>). Given two finitely presented groups (semigroups, algebras) are they isomorphic?

• There exists a finitely presented semigroup (and, hence, an algebra) in which the word problem is undecidable, <u>E. Post</u>, <u>A. Turing</u>, (40s),

• There exists a finitely presented group in which a word problem is undecidable, <u>P. S. Novikov</u>, 1959.

However, if a system of defining relations is closed with respect to compositions, then the Word Problem is decidable.

We will describle two important cases when it helps.

Graded Algebras, homogenous relations.

An algebra A is called <u>graded</u> if it is a direct sum of subspaces indexed by nonnegative integers

$$A = A_0 + A_1 + A_2 + \cdots$$

and $\quad A_i \cdot A_j \subseteq A_{i+j}$.

The polynomial algebra and the free algebra are graded

$$F\langle X\rangle = F\cdot 1 + \sum_{n\geq 1} F\langle X\rangle_n ,$$

where $F\langle X\rangle_n$ is the span of all words $x_{i_1}\cdots x_{i_n}$ of length n.

An element $f \in F\langle X\rangle$ is called <u>homogeneous</u> if it lies in one of $F\langle X\rangle_n$'s, i.e. it is a linear combination of words of the same length.

<u>Proposition I.4.2.</u> Graded algebras have decidable word problem.

<u>Proof.</u> Let $A = \langle X \mid R = 0\rangle$, $|R| < \infty$, all defining relations in R are homogeneous. If the set $R = R_1$ is closed with respect to compositions then we have a basis of A. If not then add all reduced nonzero compositions to R_1 and get R_2. And so on.

$$R = R_1 \subseteq R_2 \subseteq \cdots$$

We notice that the composition $(f,g)_w$ of two homogeneous elements f, g is again homogeneous and

$$\deg (f,g)_w > \max(\deg f, \deg g).$$

Hence all elements in $R_{n+1} \setminus R_n$ have degrees $> n$.

Let $h \in F\langle X\rangle_n$. If it reduces to zero via R_n then $h = 0$ in A. If it does not then $h \neq 0$ in A.

~~Commutative Algebras.~~

~~In the class of commutative algebras the algorithmic problems are decidable.~~