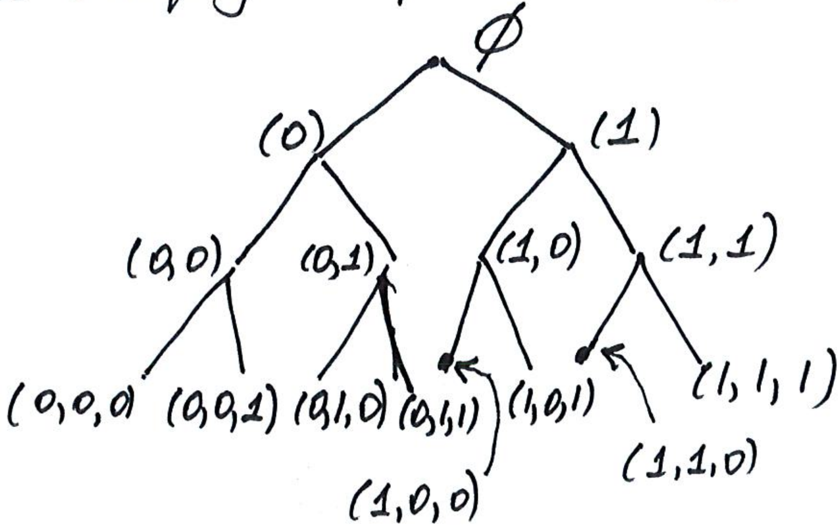
The Grigorchuk Group.

In this section we introduce the celebrated Grigorchuk group, which is another counterexample to the General Burnside Problem: it is finitely generated, torsion and infinite.

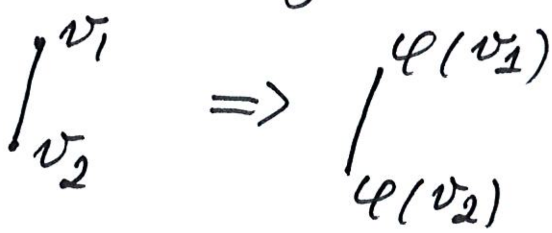
A <u>rooted binary tree</u> $T=(V,E)$ is a tree with vertex set

$V=\{$ all finite sequences in $0,1$, including the empty sequence $\emptyset \}$

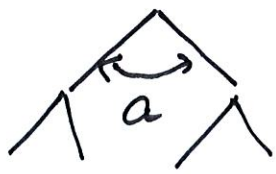


Notation: $\bar{0}=1$, $\bar{1}=0$.

<u>An automorphism φ of a tree</u> is a bijection of the set of vertices, that fixes $\emptyset$ and



<u>Example.</u>



$$a(i_1, \dots, i_n) = (\bar{i_1}, i_2, \dots, i_n)$$

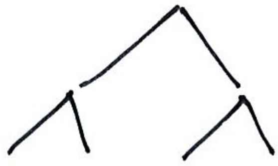
$Aut(T)$ = the group of all automorphisms of the tree T.

<u>Exercise</u>. $Aut(T)$ is uncountable.

Let $L(k) = \{$all sequences of length $k\}$ = the k-th level of T.

$$|L(k)| = 2^k.$$

$T(K)$ = the subtree of T that includes all vertices of level $\le K$.

$T(2) =$ 

Every automorphism of T keeps each $T(K)$ and each $L(K)$ invariant,

$$\varphi_K : \mathrm{Aut}(T) \to \mathrm{Aut}(T(K)),$$

automorphism $\to$ its restriction to $T(K)$,

is a homomorphism. The stabilizer

$$St(K) = \{ g \in \mathrm{Aut}(T) \mid \forall v \in L(K)\ g(v) = v \}$$

is the kernel of this homomorphism, hence a normal subgroup of finite index,

$$\bigcap_{K \ge 1} St(K) = \{1\},$$

where 1 is the identity automorphism of T.

Hence, $Aut(T)$ is a residually finite group.

<u>Groups $Aut\,T(K)$.</u> We will start with a generalization of the concept of wreath product. Let A, B be groups. Let X be a set. Suppose that the group B acts on X: there is a mapping $B \times X \to X$, $b \times x \to bx \in X$ such that $1x = x$, $b_1(b_2 x) = (b_1 b_2)x$ for arbitrary elements $b_1, b_2 \in B$, $x \in X$. Let $Fun(X, A)$ be the set of mappings

$X \to A$ with the group structure of the Cartesian product $A^{|X|}$. Consider the semidirect product

$$A \operatorname{wr}_X B = B \cdot \operatorname{Fun}(X, A)$$

with multiplication

$$b_1 f_1 \cdot b_2 f_2 = b_1 b_2 \cdot f_1^{b_2} f_2,$$

$$f_1^{b_2}(x) = f_1(b_2 x).$$

<u>Example.</u> The group B acts on itself by left multiplications. Then

$$A \operatorname{wr}_B B = A \operatorname{wr} B.$$

A group $\operatorname{Aut} T(k)$ acts on the k-th level $L(k)$ of the tree T.

Recall that $\mathbb{Z}(2) = \mathbb{Z}/2\mathbb{Z}$ denotes the cyclic group of order 2.

Lemma. $\operatorname{Aut} T(k+1) \cong \mathbb{Z}(2) \operatorname{wr}_{L(k)} \operatorname{Aut} T(k)$.

Proof. For an automorphism $g \in \operatorname{Aut} T(k+1)$ let $\bar{g}$ denote its restriction to $T(k)$. A vertex $v = (i_1 \ldots i_{k+1})$ of level $k+1$ is mapped by the automorphism g to a vertex $(\bar{g}(i_1, \ldots, i_k), j)$, $j = 0$ or 1. Define a mapping $f_g : L_k \to \mathbb{Z}(2)$ as follows

$$f_g(v) = \begin{cases} 0 & \text{if } j = i_{k+1} \\ 1 & \text{if } j = \bar{i}_{k+1} \end{cases}$$

Exercise. Check that the mapping

$\operatorname{Aut} T(k+1) \to \mathbb{Z}(2) \operatorname{wr}_{L(k)} \operatorname{Aut} T(k)$, $g \to \bar{g} \cdot f_g$, is an isomorphism.

<u>Corollary 1.</u> $|\,Aut\ T(k)\,| = 2^{2^k - 1}$, $k \geq 1$.

<u>Proof.</u> If groups A, B and a set X are finite then

$$|\,A\ wr_X\ B\,| = |B| \cdot |A|^{|X|}$$

This implies that

$$|\,Aut\ T(k)\,| = |\,Aut\ T(k-1)\,| \cdot 2^{2^{k-1}}$$

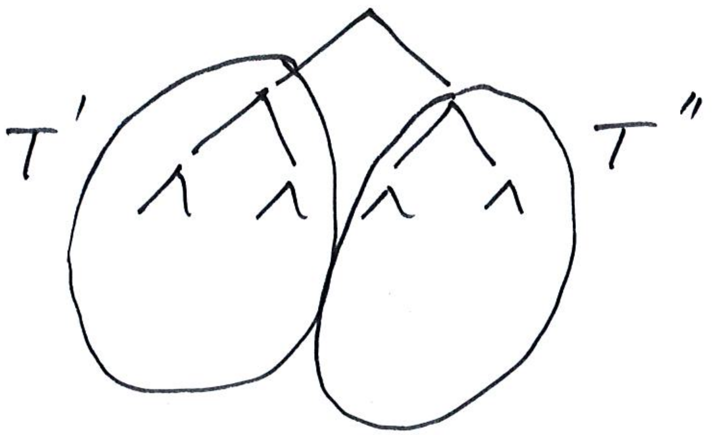
By the induction assumption on k

$$|\,Aut\ T(k)\,| = 2^{2^{k-1}-1} \cdot 2^{2^{k-1}} = 2^{2^k - 1}.$$

<u>Corollary 2.</u> The group $Aut\ T$ is a residually-2 group.

Grigorchuk Group.

Let $\varphi \in St(1)$. Then φ maps the subtrees T' and T'' to themselves.



T' and T'' are copies of the tree T. Hence φ gives rise to automorphisms $\varphi' \in \operatorname{Aut} T'$ and $\varphi'' \in \operatorname{Aut} T''$. On the other hand φ' and φ'' define φ. We will write

$$\varphi = (\varphi', \varphi''),$$

here $\varphi', \varphi'' \in \operatorname{Aut} T$, $\varphi \in St(1)$.

Recall that the automorphism a switches the 2 branches of T,

$$a(i_1 i_2 \dots i_n) = (\bar{i}_1 \, i_2 \, i_3 \cdots i_n),$$

$$a \notin St(1).$$

Define automorphisms $b, c, d \in St(1)$ as follows:

$$b = (a, c), \quad c = (a, d), \quad d = (1, b).$$

It may look as a vicious circle, but it is not.

<u>Example.</u> $v = (1011)$

$$b(v) = (1 \; c(011)) = (1 \; 0 \; a(11)) = (1001)$$

The first index i_1 stays the same and then, depending on whether $i_1 = 0$ or 1 we know what to do with $(i_2 i_3 \cdots i_n)$.

Define the <u>Grigorchuk group</u> Γ as

$$\Gamma = \langle a, b, c, d \rangle \leq \operatorname{Aut} T.$$

<u>Theorem</u>. The group Γ is an infinite torsion group.

<u>Lemma</u>. $a^2 = b^2 = c^2 = d^2 = 1$.

<u>Proof</u>. For the element a it is obvious. For b, c, d we will use induction on n to show that all vertices of the level n are fixed by b^2, c^2, d^2. By definitio

it is true for $L(1)$. Assume that all vertices from $L(k)$ are fixed by b^2, c^2, d^2 for $k < n$.

We will prove that b^2 fixes $L(n)$. For c^2, d^2 it is done similarly.

Recall that $b = (a, c)$. Let $v = (i_1 \cdots i_n)$.

If $v = (0\ i_2\ i_3 \cdots i_n)$ then $b(v) = (0, a(i_2 \cdots i_n))$

and $b^2(v) = (0, a^2(i_2 \cdots i_n)) = (0\ i_2 \cdots i_n)$.

If $v = (1\ i_2 \cdots i_n)$ then $b(v) = (1\ c(i_2 \cdots i_n))$,

$b^2(v) = (1\ c^2(i_2 \cdots i_n))$.

But $c^2(i_2 \cdots i_n) = (i_2 \cdots i_n)$ by the induction assumption. This completes the proof of the lemma.

<u>Lemma</u>. $bc = cb = d$, $cd = dc = b$, $db = bd = c$.

<u>Proof</u>. We will prove by induction on n that these relations hold on $L(n)$.

For $n = 1$ it is obvious.

We have

$$bc = (a, c)(a, d) = (a^2, cd) = (1, cd).$$

By the induction assumption $cd = b$ on $L(n-1)$. Hence $bc = (1, b) = d$ on $L(n)$.

Other equalities are proved similarly.

<u>Corollary</u>. $\langle b, c, d \rangle \cong \mathbb{Z}(2) \oplus \mathbb{Z}(2)$, the Klein group

The elements aba, aca, ada lie in $St(1)$.

Lemma. $aba = (c, a)$, $aca = (d, a)$, $ada = (b, 1)$.

Proof. Again we consider the action on $L(n)$ and use induction on n. On $L(1)$ both sides induce the identical mapping. Suppose that the equalities hold for $L(k)$, $k < n$. We will prove that $aba = (c, a)$. Other equalities are proved similarly.

Let $v = (0\, i_2 \cdots i_n)$. Then $a(v) = (1\, i_2 \cdots i_n)$, $ba(v) = (1\; c(i_2 \cdots i_n))$,

$aba(v) = (0\, c(i_2 \cdots i_n))$. On the other hand, $(c,a): v \to (0\, c(i_2 \cdots i_n))$, so aba and (c,a) coincide on v.

Let $v = (1\, i_2 \cdots i_n)$. Then $a(v) = (0\, i_2 \cdots i_n)$, $ba(v) = (0\, \bar{i}_2\, i_3 \cdots i_n)$, $aba(v) = (1\, \bar{i}_2\, i_3 \cdots i_n)$.

On the other hand, (c,a) maps $v = (1\, i_2 \cdots i_n)$ to $(1\, a(i_2 \cdots i_n)) = (1\, \bar{i}_2 \cdots i_n)$.

Again aba and (c,a) coincide on v. This completes the proof of the lemma.

Recall that an arbitrary element $g \in St(1)$ is represented as $\varphi = (\varphi', \varphi'')$.

Let $St_\Gamma(1) = \Gamma \cap St(1)$.

Lemma . If $\varphi \in St_\Gamma(1)$ then $\varphi'' \in \Gamma$. The mapping $St_\Gamma(1) \to \Gamma$, $\varphi \longrightarrow \varphi''$, is surjective.

Proof. Clearly, the mapping $St(1) \to Aut\, T$, $\varphi \to \varphi''$, is a homomorphism. We have $b \to c$, $c \to d$, $d \to b$, $aba \to a$, $aca \to a$, $ada \to 1$. It remains notice that the group $St_\Gamma(1)$ is generated by the elements b, c, d, aba, aca, ada. Indeed, because of the Lemma and because of $a^2 = 1$ an arbitrary element $g \in \Gamma$ can be written in one of the forms:

(1) a, b, c, d

(2) $a x_1 a x_2 \dots a x_k$,

(3) $x_k a x_{k-1} a \dots x_1 a$,

(4) $x_1 a x_2 a \dots a x_{k+1}$,

(5) $a x_1 a x_2 a \dots x_k a$,

where $x_i \in \{b, c, d\}$.

Each of these elements lies in $St(1)$ if and only if the element a occurs in it even number of times. Going over the 5 cases above we see that in this case the element g lies in $\langle b, c, d, aba, aca, ada$ This ~~so~~ implies that the image of Γ lies in Γ. Since this image contains the elements , c, d, b, a , the mapping is surjective. This completes the proof of the lemma.

Now we are ready to prove that the group Γ is infinite. Indeed, the subgroup $St_\Gamma(1)$ is a proper subgroup of Γ ($a \notin St_\Gamma(1)$) and there is a surjective homomorphism $St_\Gamma(1) \to \Gamma$. This already implies that the group Γ is infinite.

Now our aim is to show that the group Γ is a 2-group, that is, for an arbitrary element $g \in \Gamma$ there exists a 2-power 2^k such that $g^{2^k} = 1$.

The group Γ is generated by the set $X = \{a, b, c, d\}$. Recall that for an element $1 \neq g \in \Gamma$ the length $l_X(g)$ of the element g is defined as a minimal number n such that g can be represented as $g = x_1 \cdots x_n$; $x_i \in X$, $\ell(1) = 0$. In the Cayley graph $Cay(\Gamma, X)$ the number $\ell_X(g)$ is the distance from g to the identity element 1.

We will use induction on $\ell_X(g)$ to show that the element g has finite order.

There are 4 elements of length 1: a, b, c, d. They have order 2 by Lemma

Lemma: For an arbitrary element g of length 2 we have $g^{16} = 1$.

<u>Proof.</u> Up to conjugation there are 3 elements of length 2 : ab, ac, ad.

We have $(ad)^2 = adad \in St_\Gamma(1)$. By lemma $ada \cdot d = (b,1)(1,b) = (b,b)$. Since $b^2 = 1$ it follows that $(ad)^4 = 1$.

Next, consider the element ac. We have $(ac)^2 = aca \cdot c = (d,a)(a,d) = (da, ad)$.

Both elements da and ad have orders 4. Hence $(ac)^8 = 1$.

Finally, consider the element ab. We have $(ab)^2 = aba \cdot b = \cancel{(ca, ac)} (c,a)(a,c) = (ca, ac)$, which implies $(ab)^{16} = 1$. This completes the proof of the lemma.

Now suppose that $l_X(g) = n \geq 3$ and all words of length $< n$ have finite orders.

Again, up to a conjugation we can assume that $g = ax_1a \cdots ax_k$; $x_i \in \{b,c,d\}$.

Indeed, if $g = ag'a$ then $a^{-1}ga = g'a^2 = g'$; if $g = xg'y$, where $x, y \in \{b,c,d\}$, then $x^{-1}gx = g'yx$, but the element yx is one of $1, b, c, d$; if $g = x_1ax_2 \cdots x_ka$, $x_i \in \{b,c,d\}$ then $aga^{-1} = ax_1a \cdots ax_k$.

Hence, $n = 2k$, $k \geq 2$.

Case 1. $g \in St_\Gamma(1)$, which means that k is even.

Then $g = (ax_1a)x_2(ax_3a) \cdots (ax_{k-1}a)x_k$
$= (u_1, v_1)(x_2', x_2'')(u_2, v_2) \cdots (u_{k-1}, v_{k-1})(x_k', x_k'')$,

where u_i, v_i have length ≤ 1 and x_j', x_j'' have length ≤ 1. Hence $g = (u, v)$, where both the elements u, v have length $\leq K$. By the induction assumption the elements u, v have finite orders, hence the element g has a finite order as well.

We remark that if the element $d = (1, b)$ occurs among x_i's, $i = 2, 4, 6, \ldots$ $\ldots K$, then $l_X(u) < K$. If the element d occurs among x_j's, $j = 1, 3, 5, \ldots, K-1$ then $l_X(v) < K$, since $ada = (b, 1)$.

<u>Case 2.</u> $g \notin st_\Gamma(1)$, the number K is odd. Then $g^2 = ax_1ax_2 \ldots ax_K ax_1 a \ldots ax_K$, $l_X(g^2) \leq 4K$.

Arguing as above, we get $g^2 = (u, v)$,

$l_x(u) \leq 2k$, $l_x(v) \leq 2k$.

So far we have not won anything.

We have

$\{$the x_i's at even positions in $g^2\}$ =

= $\{$the x_j's at odd positions in $g^2\}$ =

$\{x_1, \ldots, x_k\}$.

Indeed, let $g = a x_1 a x_2 a x_3$,

$g^2 = a x_1 a x_2 a x_3 a x_1 a x_2 a x_3 =$

$= (a x_1 a)\, x_2\, (a x_3 a)\, x_1\, (a x_2 a)\, x_3$

odd: $\{x_1, x_2, x_3\}$

Suppose that d occurs among $x_1, \dots, x_k$.

Then $\ell_x(u), \ell_x(v) \le 2k-1$

and we are done by the induction assumptio

~~Suppose that d occurs among $x_1, \dots, x_k$. Then~~

We proved that elements $a x_1 a x_2 \cdots a x_k$,

$n = 2k$; $x_1, \dots, x_k \in \{b, c, d\}$, $d \in \{x_1, \dots, x_k\}$ have

finite orders.

Now suppose that c occurs among

$x_1, \dots, x_k$.

We have $c = (a, d)$, $aca = (d, a)$. Hence both

u, v include the element d. By what we

proved above the elements u, v have finite

orders. Hence the element g has a finite order

The only remaining case:

$$g = abab\cdots ab = (ab)^k.$$

By Lemma the element ab has order

16. This completes the proof of Theorem