Lecture 7.

Subgroups of free groups.

Consider the free group $Fr(m)$ on the set $X = \{x_1, \dots, x_m\}$ of free generators. Let $H \leq Fr(m)$ be a subgroup. Let S be a system of representatives of cosets of H.

Def. We say that the system of representatives S is a **Schreier system** if for every $s \in S$ when s is written as a reduced product of generators, all initial segments also lie in S. That is, if $s = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k}$ is reduced then $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_j}^{\varepsilon_j} \in S$ for all $j \leq k$.

<u>Lemma</u> . For every subgroup $H \leq F_2(m)$ there exists a Schreiersystem of coset representatives.

Proof. Let $g \in F_2(m)$ and let $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ be its reduced form. Define length$(g) = k$. We also define length$(1) = 0$.

In an arbitrary coset, choose a shortest element and define the length of the coset to be the length of that shortest element. Then length$(H) = 0$

since $1 \in H$.

Choose representatives inductively as follows. Suppose that for all cosets of length less than r we have already selected representatives, such that the Schreier property is satisfied. A coset of length r contains some shortest element $x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$. Then select

$$s = \overline{x_{i_1}^{\varepsilon_1} \cdots x_{i_{r-1}}^{\varepsilon_{r-1}}}\, x_{i_r}^{\varepsilon_r}$$

as a representative for this coset.

Note that s must have length r, for if it could be shortened then the coset would not have length r.

This process yields a Schreier system of representatives. The lemma is proved.

Let S be a Schreier system of representatives of cosets of H.

<u>Lemma</u>: If $sx^{\varepsilon}\,\overline{sx^{\varepsilon}}^{-1} \neq 1$ then it is reduced.

Proof. If $sx^{\varepsilon}\,\overline{sx^{\varepsilon}}^{-1}$ is not reduced then either $s = s'x^{-\varepsilon}$ or $\overline{sx^{\varepsilon}}^{-1} = x^{-\varepsilon}w$.

If $s = s'x^{-\varepsilon}$ is a reduced form of the element s then $s' \in S$ since S is a Schreier system. Then $\overline{sx^{\varepsilon}} = \overline{s'} = s'$ and

$sx^{\varepsilon}\,\overline{sx^{\varepsilon}}^{-1} = s'\,s'^{-1} = 1$.

Now suppose that sx^{ε} is reduced. ~~and~~

~~$\overline{sx^{\varepsilon}}^{-1} = x^{-\varepsilon}w$, $\overline{sx^{\varepsilon}} = w^{-1}x^{\varepsilon}$.~~

By our construction of the Schreier system the representative $\overline{sx^{\varepsilon}}$ is also reduced. Suppose that cancellation arises at the junction

$sx^{\varepsilon}, \overline{sx^{\varepsilon}}^{-1}$. It means that the reduced form of $\overline{sx^{\varepsilon}}^{-1}$ is $x^{-\varepsilon}w$ and $\overline{sx^{\varepsilon}} = w^{-1}x^{\varepsilon}$.

Hence $sx^{\varepsilon} \sim w^{-1}x^{\varepsilon}$, $(sx^{\varepsilon})(w^{-1}x^{\varepsilon})^{-1} = sw \in H$, $s \sim w^{-1}$. Since $w^{-1}x^{\varepsilon} \in S$ it follows that $w^{-1} \in S$, hence $s = w^{-1}$! Now $sx^{\varepsilon}\,\overline{sx^{\varepsilon}}^{-1} = sx^{\varepsilon}(sx^{\varepsilon})^{-1} = 1$. This completes the proof of the lemma.

<u>Theorem</u>. The subgroup H is a free group on the set of free generators

$Y = \{ sx\,\overline{sx}^{-1} \mid s \in S, x \in X, sx\,\overline{sx}^{-1} \neq 1 \}$.

Proof. We need to show that a reduced expression in Y can not be equal to 1 in $F_2(u)$.

Consider a reduced expression

$$\dots\ r x^{\gamma}\, \overline{r x^{\gamma}}^{-1} \cdot s y^{\delta}\, \overline{s y^{\delta}}^{-1} \cdot t z^{\varepsilon}\, \overline{t z^{\varepsilon}}^{-1} \dots ,$$

where $r, s, t \in S$; $x, y, z \in X$; $\gamma, \delta, \varepsilon = \pm 1$.

Cancellations may arise on the junctions between elements from Y, Y^{-1}. Suppose that cancellation started at $\dots \overline{r x^{\gamma}}^{-1} \cdot s \dots$ and spread in both directions.

We will show that this cancellation process does not reach x^{γ}, y^{δ}.

Suppose that this is not true. There are 3 possibilities:

1) the cancellation reaches x^{γ}, y^{δ} simultaneously. Then

$\overline{r x^{\gamma}} = s, \quad x^{\gamma} y^{\delta} = 1.$

In this case $r \sim s x^{-\gamma} = s y^{\delta}$,

so $r = \overline{s y^{\delta}}$. Now

$$r \underbrace{x^{\gamma}\, \overline{r x^{\gamma}}^{-1} \cdot s y^{\delta}}_{\substack{\shortparallel \\ 1}}\, \overline{s y^{\delta}}^{-1} = r\, \overline{s y^{\delta}}^{-1} = 1.$$

2) Suppose that y^{δ} is cancelled first.

Then $\overline{r x^{\gamma}}^{-1} = w y^{-\delta} s^{-1}$ and

$\overline{r x^{\gamma}} = s y^{\delta} w^{-1}$. This is a reduced expression.

Since it lies in S it follows that

$s y^{\delta} \in S$ and therefore $s y^{\delta}\, \overline{s y^{\delta}}^{-1} = 1$.

3) x^{γ} is cancelled first. Arguing as

above we get $r x^{\gamma}\, \overline{r x^{\gamma}}^{-1} = 1$.

We proved that the cancellation process won't touch $x^\delta, y^\delta, z^\varepsilon$ etc.

This completes the proof of the theorem.

So, <u>Nielsen, Schreier</u>: every subgroup of a free group is free.

Now suppose that index of a subgroup H in $Fr(m)$ is n.

What is the cardinality of the set Y? In other words, what is the rank of the free group H?

The Cartesian product $S \times X$ contains mn elements.

For how many pairs (s, x) we have $sx\,\overline{sx}^{-1} = 1$, i.e. $sx \in S$? There are two possibilities:

(1) the generator x is cancelled. Then the

representative s has the normal form that ends with x^{-1}. The number of such pairs (s, x) is equal to the number of representatives that ends with an inverse $x^{-1}, x \in X$;

(2) the generator x is not cancelled. Then sx is the reduced form of a representative in S. The number of such pairs is equal to the number of ~~nonidentical~~ representatives in S that end with $x, x \in X$.

Hence the number of pairs (s, x) such that $sx \in S$ is equal to the number of nonidentical representatives in S, i.e. to $m-1$.

We proved the following theorem.

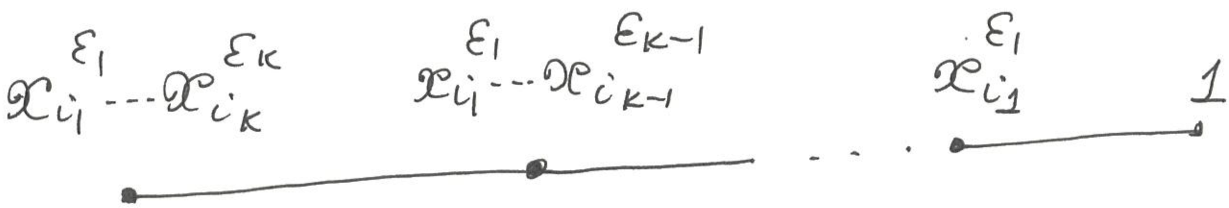
Theorem. If H is a subgroup of $F_2(m)$ of index n, then H is a free group of rank $mn-m+1$.

Corollary. If G is an m-generated group, $H \leq G$, $|G:H|=n$, then the subgroup H can be generated by $\leq mn-m+1$ elements.

Cayley Graphs.

Let G be a group generated by a set X. We construct an unoriented graph $Cay(G,X)$ with {vertices} = {elements of G}. Elements $a,b \in G$ are connected if $b = a x^{\varepsilon}$, $\varepsilon = \pm 1$, and $x \in X$.

As a first remark we notice that the Cayley graph is connected because every element is connected to 1. If $a = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ then a is connected to $x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-1}}^{\varepsilon_{k-1}}$, which is connected to $x_{i_1}^{\varepsilon_1} \cdots x_{i_{k-2}}^{\varepsilon_{k-2}}$ and so on until $x_{i_1}^{\varepsilon_1}$ is connected to 1.



Every connected graph can be viewed as a metric space with the length of each edge $= 1$.

$d_X(a,b) =$ the length of a shortest path connecting a and b. The length of an element $a \in G$ is defined as

$$\ell_X(a) = d_X(a,1) = \min\{k \mid a = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}\},$$

$$d(a,b) = \ell_X(a^{-1}b).$$

We have

$$d(a,b) = d(ga, gb)$$

for any elements $a, b, g \in G$. Hence G acts on the metric space $Cay(G,X)$ by isometries.

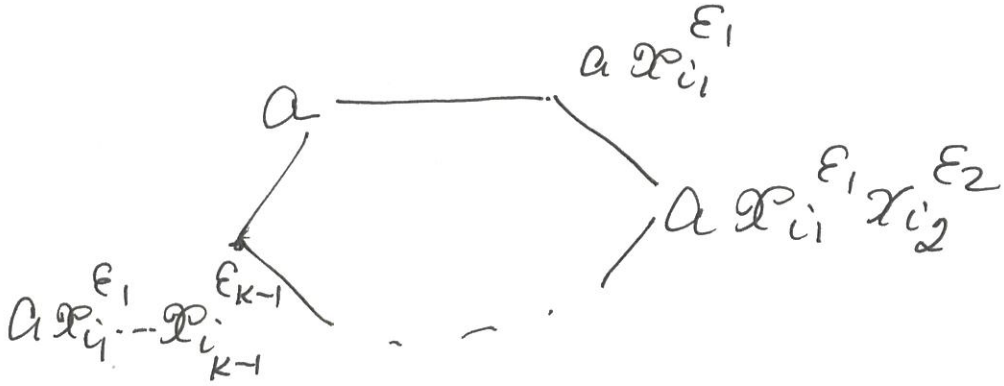
$$G \to Isom(Cay(G,X)).$$

A <u>cycle</u> in a graph is a subgraph of the type

A *tree* is a graph that does not contain any cycles.

Proposition. If $F_r(X)$ is a free group on the set of free generators X ~~them~~ then $Cay(F_r(X), X)$ is a tree.

Proof. If $Cay(F_r(X), X)$ contains a cycle then we can choose one of shortest length. It will look like the following

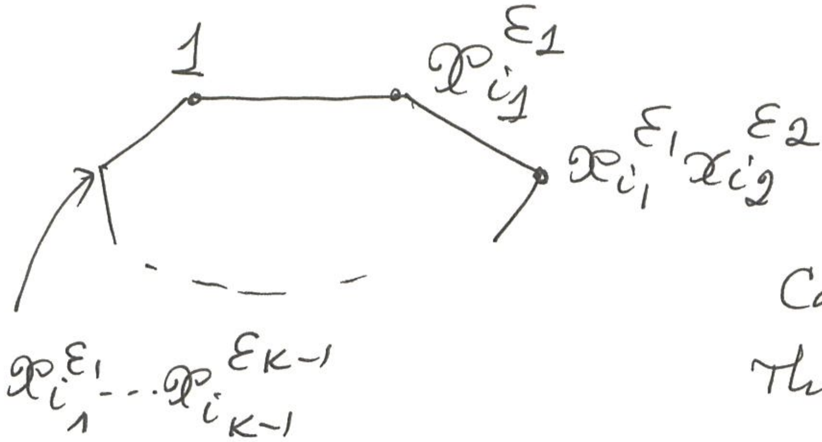


Then $a = a\, x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ and so $x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k} = 1$.
But since we have chosen a shortest cycle,

$x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ must be reduced, a contradiction to the fact that $Fr(X)$ is free on the set of free generators X.

<u>Proposition</u>. Suppose that a group G is generated by a subset X, $X \cap X^{-1} = \emptyset$ and $Cay(G, X)$ is a tree. Then G is a free group on the set of free generators X.

<u>Proof</u>. We need to check that a reduced product is not equal to 1 in G. So, suppose that $x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ is reduced and equal to 1. Take a product such that k is minimal. Notice that k can not be 2 since $X \cap X^{-1} = \emptyset$. Then we get a circle



, which contradicts $Cay(G,X)$ being a tree. This completes the proof.

We say that a group G acts on a tree T if there is a homomorphism $G \xrightarrow{\varphi} Isom(T)$. We say that G acts fixed point free if for any $1 \neq g \in G$ the isometry $\varphi(g)$ does not have fixed points.

For example, a free group $F_2(X)$ acts fixed point free on the tree $Cay(F_2(X), X)$.

Theorem (J.-P. Serre). A group which acts fixed point free on a tree is free.

Corollary. Subgroups of free groups are free.