Lecture 8. Groups presented by generators and relations.

Let G be an m-generated group, $G = \langle a_1, \ldots, a_m \rangle$. Let $Fr(m)$ be the free group on the set of free generators $X = \{x_1, \ldots, x_m\}$. Then the mapping $x_i \to a_i$, $1 \leq i \leq m$, extends to a surjective homomorphism $\varphi: Fr(m) \to G$. Let $H = \ker \varphi$. Then

$$G \cong Fr(m) / H.$$

Let $R \subseteq H$ be a subset that generates H as a normal subgroup. It means that H is the smallest normal subgroup of $Fr(m)$ that contains R. Let

$$R^{Fr(m)} = \{ r^g = g^{-1} r g \mid r \in R, g \in Fr(m) \}$$

be the set of all conjugates of elements of R. Then H is the subgroup of $Fr(m)$ generated by the set $R^{Fr(m)}$.

We say that the group G is presented by the set of generators X and the set of relations R,

$$G = \langle X \mid R = 1 \rangle.$$

Dehn's Algorithmic Problem: assume $|X| < \infty$, $|R| < \infty$. Does there exist an algorithm that decides if two elements $v, w \in Fr(m)$ are equal modulo H? Equivalently, if $v(a_1, \ldots, a_m) \overset{?}{=} w(a_1, \ldots, a_m)$

1959 P. S. Novikov: a finitely presented group for which such an algorithm does not exist.

If $h \in H$ then h can be presented as

$$h = (r_1^{g_1})^{\pm 1} \cdots (r_s^{g_s})^{\pm 1},$$

where $r_1, \dots, r_s \in R$ (not necessarily distinct), $g_1, \dots, g_s \in F_r(m)$.

Such a presentation may be not unique.

$$\|h\| = \min s$$

Let $B(n)$ be the ball of radius n in $Cay(F_r(m), X)$ with the center at 1.

What is $B(n)$? If $w = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ is a reduced form then $\mathrm{dist}(w, 1) = k$, so

$$B(n) = \{ x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}, \text{ reduced forms}, k \le n \}$$

Dehn functions.

$$D_X(n) = \max \{ \|h\| \mid h \in H \cap B(n) \}, \quad n \geq 1.$$

As in the case of semigroups if

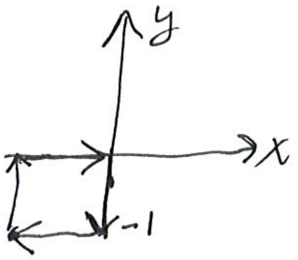
$$\langle X \mid R_1 = 1 \rangle \cong \langle Y \mid R_2 = 1 \rangle,$$

X, Y, R_1, R_2 are finite sets, then $D_X(n)$ is asymptotically equivalent to $D_Y(n)$.

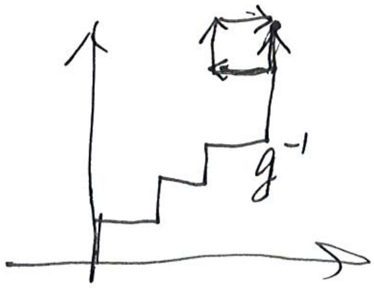
Dehn function $\longleftrightarrow$ nondeterministic time complexity.

Example. $\langle x, y \mid y^{-1} x^{-1} y x = 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ free abelian group of rank 2.

Consider the coordinate system and move $\rightarrow$ for x, $\leftarrow$ for x^{-1}, $\uparrow$ for y, $\downarrow$ for y^{-1}. Then $y^{-1}x^{-1}yx$ stands for the contour of a unit square.

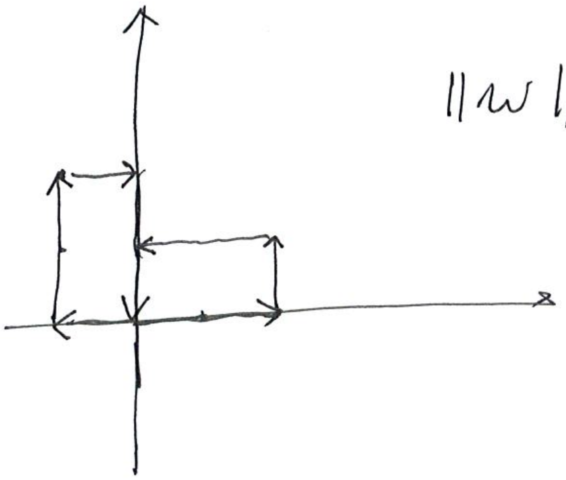


What is $(y^{-1}x^{-1}yx)^{g}$?



Unit square, but moved away from the origin.

Consider $w = x^{-1}y^{2}xy^{-2}x^{2}yx^{-2}y^{-1}$.



$\|w\|$ = Area bounded by the curve $w = 4$.

Now the problem looks as: the length of a curve is $\leq n$. What is the maximal area bounded by this curve? We did not fill in all details.

Isoperimetric problem: $\sim n^2$.

Free Product of Algebras.

Let F be a field and let A, B be associative F-algebras with 1. We always assume that a homomorphism maps $1 \to 1$.

Objects: triples (an associative F-algebra C with 1, homomorphisms $\varphi: A \to C$, $\psi: B \to C$).

Morphism $(C_1, \varphi_1: A \to C, \psi_1: B \to C) \to (C_2, \varphi_2: A \to C, \psi_2: B \to C)$

is a homomorphism $\chi : C_1 \to C_2$ of associative algebras such that the diagrams

$$\begin{tikzcd} C_1 \arrow[r, "\chi"] & C_2 \\ A \arrow[u, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\chi"] & C_2 \\ B \arrow[u, "\psi_1"] \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

commute.

<u>Universal Object</u>: such a triple $(U, u_1 : A \to U, u_2 : B \to U)$ that for any other ~~triple~~ object $(C, \varphi : A \to C, \psi : B \to C)$ there exists a unique morphism $\chi : U \to C$.

<u>Lemma</u>. If $(U, u_1 : A \to U, u_2 : B \to U)$ is universal then U is generated by $u_1(A), u_2(B)$ as an F-algebra.

Proof. Suppose that the subalgebra $U' < U$ generated by $u_1(A)$, $u_2(B)$ in U is smaller than U. There exists a morphism $\chi : U \to U'$ that can be viewed as a morphism from (U, u_1, u_2) to itself. Then there are two different morphisms from (U, u_1, u_2) to itself: id and χ, a contradiction.

Let

$$A = F\langle X \mid R_1(X) = 0 \rangle, \quad B = F\langle Y \mid R_2(Y) = 0 \rangle$$

be presentations by generators and relations.

Consider the free F-algebra $F\langle X \dot{\cup} Y \rangle$ on the set of free generators $X \dot{\cup} Y$.

Let $id_{F\langle X \dot{\cup} Y \rangle}(R_1(X), R_2(Y))$ be the ideal generated by $R_1(X) \cup R_2(Y)$ in $F\langle X \dot{\cup} Y \rangle$.

Clearly,

$$\exists\; id_{F\langle X \rangle}(R_1(X)),\ id_{F\langle Y \rangle}(R_2(Y)) \subseteq id_{F\langle X \dot{\cup} Y \rangle}(R_1(X), R_2(Y))$$

Hence we can define natural homomorphism

$$u_1 : F\langle X \rangle / id_{F\langle X \rangle}(R_1(X)) \to F\langle X \dot{\cup} Y \rangle / id_{F\langle X \dot{\cup} Y \rangle}(R_1(X), R_2(Y)) = U$$

$$u_2 : F\langle Y \rangle / id_{F\langle Y \rangle}(R_2(Y)) \to F\langle X \dot{\cup} Y \rangle / id_{F\langle X \dot{\cup} Y \rangle}(R_1(X), R_2(Y)) = U$$

<u>Lemma</u>. The triple $(U, u_1 : A \to U, u_2 : B \to U)$ is universal.

<u>Proof</u>. Let C be an F-algebra and let $\varphi : A \to C$, $\psi : B \to C$ be homomorphisms.

Consider the mapping $x_i \to \varphi(x_i + id_{F\langle X\rangle}(R_1(X))) \in$ $y_j \to \psi(y_j + id_{F\langle Y\rangle}(R_2(Y))) \in C$ and extend it to a homomorphism $\tilde{\chi} : F\langle X \dot{\cup} Y\rangle \to C$.

Then $\tilde{\chi}(R_1(X)) = (0)$ and similarly $\tilde{\chi}(R_2(Y)) = ($ so $\tilde{\chi}(id_{F\langle X\dot{\cup}Y\rangle}(R_1(X), R_2(Y))) = (0)$. Hence the homomorphism $\tilde{\chi}$ gives rise to the homomorphism $\chi : U = F\langle X\dot{\cup}Y\rangle / id_{F\langle X\dot{\cup}Y\rangle}(R_1(X), R_2(Y)) \to C$

It is straightforward that the corresponding diagrams are commutative. This completes the proof of the lemma.

<u>Lemma</u>. A universal triple is unique up to isomorphism.

<u>Proof</u>. Let $(U_1, \varphi_1 : A \to U_1, \psi_1 : B \to U_1)$ and

$(U_2, \varphi_2 : A \to U_2, \psi_2 : B \to U_2)$ be universal triples. Then there exist morphisms $\chi : U_1 \to U_2$ and $\chi' : U_2 \to U_1$. For an arbitrary element $a \in A$ we have

$$\begin{tikzcd} & \varphi_1(a) \arrow[d, shift right, "\chi_1"'] \\ a \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & \varphi_2(a) \arrow[u, shift right, "\chi_2"'] \end{tikzcd},$$

hence $\chi_2 \chi_1 (\varphi_1(a)) = \varphi_1(a)$, $\chi_2 \chi_1 \big|_{\varphi_1(A)} = id_{\varphi_1(A)}$ and similarly $\chi_2 \chi_1 \big|_{\psi_1(B)} = id_{\psi_1(B)}$. Since $\varphi_1(A)$, $\psi_1(B)$ generate U_1 it follows that $\chi_2 \chi_1 = id_{U_1}$. Similarly $\chi_1 \chi_2 = id_{U_2}$. Hence both χ_1 and χ_2 are isomorphisms. This completes the proof of the lemma.

<u>Lemma</u>. The homomorphisms u_1 and u_2 are injective.

<u>Proof</u>. Consider the direct sum of algebras $A \oplus B$. Let $A \ni a \xrightarrow{\varphi} a+0 \in A \oplus B$, $B \ni b \xrightarrow{\psi} 0+b \in A \oplus B$ be natural embeddings. There exists a morphism $U \to A \oplus B$ such that the diagrams

$$\begin{tikzcd}
a \arrow[r, "u_1"] \arrow[dr, bend right, "\varphi"'] & u_1(a) \arrow[d] \\
& a \oplus 0
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "u_2"] \arrow[dr, bend right, "\psi"'] & u_2(b) \arrow[d] \\
& 0+b
\end{tikzcd}$$

are commutative. Since φ, ψ are embeddings it implies that u_1, u_2 are embeddings as well. This completes the proof of the lemma.

Identifying $A \ni a$ with $u_1(a)$ we assume that $A < U$ and $B < U$; A and B generate U.

The algebra U is called the free product of the algebras A, B and denoted

$$U = A * B$$

Let $\{1, a_i\}_{i \in I}$, $\{1, b_j\}_{j \in J}$ be bases of the subalgebras A, B in $A * B$.

<u>Lemma</u>. The set $1, c_1 \cdots c_n, n \geq 1$, where $c_k \in \{a_i, b_j\}_{i \in I, j \in J}$, and two consequtive elements c_k, c_{k+1} do not lie in the same subalgebra A or B, is a basis of $A * B$.

<u>Proof</u>. Let $X = \{x_i\}_{i \in I}$, $Y = \{y_j\}_{j \in J}$.

Let $a_{i_1} a_{i_2} = \alpha^{0}_{i_1 i_2} \cdot 1 + \sum_p \alpha^{p}_{i_1 i_2} a_p$,

$b_{j_1} \,\S\, b_{j_2} = \beta^{0}_{j_1 j_2} \cdot 1 + \sum_q \beta^{q}_{j_1 j_2} b_q$; $\alpha^{0}_{i_1 i_2}$,
$\alpha^{p}_{i_1 i_2}$, $\beta^{0}_{j_1 j_2}$, $\beta^{q}_{j_1 j_2} \in F$. Consider the sets of defining relations

$$R_1(X) = \{ x_{i_1} x_{i_2} - \alpha^{0}_{i_1 i_2} \cdot 1 - \sum_p \alpha^{p}_{i_1 i_2} x_p \mid i_1, i_2 \in I \},$$

$$R_2(Y) = \{ y_{j_1} y_{j_2} - \beta^{0}_{j_1 j_2} \cdot 1 - \sum_q \beta^{q}_{j_1 j_2} y_q \mid j_1, j_2 \in J \}.$$

Then $A = \langle X \mid R_1(X) = 0 \rangle$, $B = \langle Y \mid R_2(Y) = 0 \rangle$.

The set $R_1(X) \dot{\cup} R_2(Y)$ is closed with respect to compositions. A word in $X \dot{\cup} Y$ is irreducible if and only if it does not contain subwords $x_{i_1} x_{i_2}$, $y_{j_1} y_{j_2}$. Now it suffices to refer to Theorem I.3.1.

Free Products of families of algebras.

Let $A_i = F\langle X_i \mid R_i(X_i) = 0\rangle$, $i \in I$, be a family of F-algebras.

<u>Objects</u>: (an associative algebra C with 1, homomorphisms $\varphi_i : A_i \to C$, $i \in I$).

<u>Morphisms</u>: $(C, \varphi_i) \to (C', \varphi_i')$ is a homomorphism $\chi : C \to C'$ such that all diagrams

$$\begin{tikzcd} C \arrow[r, "\chi"] & C' \\ A_i \arrow[u, "\varphi_i"] \arrow[ur, "\varphi_i'"'] & \end{tikzcd}, \quad i \in I,$$

are commutative.

<u>Free product</u> $\underset{i \in I}{\ast} A_i$ = universal object $(U, u_i : A_i \to U, i \in I)$.

For an arbitrary object $(C, \varphi_i : A_i \to C)$ there exists a unique morphism $(U, u_i) \to (C, \varphi_i)$.

This universal object is

$$U = F\langle \dot{\bigcup_{i\in I}} X_i \mid R_i(X_i) = 0,\ i \in I \rangle,$$

$$u_K : F\langle X_K \mid R_K(X_K) = 0 \rangle \to F\langle \dot{\bigcup_{i\in I}} X_i \mid R_i(X_i) = 0,\ i\in I \rangle$$

are the natural embeddings.

If $B_K \ni 1$ is a basis of A_K then products $1, c_1 \cdots c_n$, where each $c_i \in \bigcup_K (B_K \setminus \{1\})$ and two consequtive elements c_i, c_{i+1} do not in the same $B_K \setminus \{1\}$.