Lecture 12. **Golod-Shafarevich Construction.**

Let $V = V_0 + V_1 + \cdots$ be a graded vector space, all homogeneous components V_i are finite dimensional. In other words V is a countable direct sum of finite dimensional subspaces. The **Hilbert Series** of V is defined as

$$H_V(t) = \sum_{i=0}^{\infty} (\dim_F V_i)\, t^i,$$

a formal series in t.

An algebra A is **graded** if

$$A = A_0 \dot{+} A_1 \dot{+} A_2 \dot{+} \cdots,$$

$$A_i A_j \subseteq A_{i+j}.$$

We assume also that all homogeneous components A_i are finite dimensional. Then $H_A(t) = \sum_{i=0}^{\infty} (\dim_F A_i)\, t^i$.

Ex. Let $A = F[x_1, \dots, x_m]$ be the algebra of polynomials, $\deg(x_i) = 1$, so $A_k = \operatorname{span}(x_{i_1} \cdots x_{i_k})$, $\dim_F A_k = P_n(k)$, the partition function $= \#$ of ways to present k as a sum of m summands.

Ex. Let $A = F\langle x_1, \dots, x_m \rangle$ be the free associative algebra, $A_k = \operatorname{Span}(x_{i_1} \cdots x_{i_k})$, $\dim_F A_k = m^k$, $H_A(t) = 1 + \sum_{k=1}^{\infty} m^k t^k = \dfrac{1}{1-mt}$

Convention. Given two formal infinite series $\sum_i a_i t^i$ and $\sum_i b_i t^i$, we say that

$\sum a_i t^i \le \sum b_i t^i$ if $a_i \le b_i$ for every i.

Let $R = R_2 + R_3 + \cdots \subset F\langle x_1, \ldots, x_m \rangle$ be a graded subspace. Then the algebra $A = \langle x_1, \ldots, x_m \mid R = 0 \rangle$ is graded. The ideal $id_{F\langle X\rangle}(R)$ has zero intersection with the subspace $F\cdot 1 + \sum_{i=1}^{m} F x_i$ of $F\langle X\rangle$, so

$$A = F\cdot 1 + \sum_{i=1}^{m} F x_i + A_2 + A_3 + \cdots,$$

$$H_A(t) = 1 + mt + \sum_{i=1}^{\infty} (\dim_F A_i)\, t^i.$$

Denote $a_i = \dim_F A_i$, $a_0 = 1$, $a_1 = m$.

Denote $r_i = \dim_F R_i$, $i \ge 2$.

Theorem (E. Golod - I. Shafarevich, 1964).

$$H_A(t)\left(1 - mt + H_R(t)\right) \geq 1.$$

Proof. The constant term of $H_A(t)(1-mt+H_R(t))$ is 1. So, we need to show that every coefficient of $H_A(t)\left(1-mt+H_R(t)\right)$ is ≥ 0. Let $k \geq 1$. The k-th coefficient of this formal series is

$$a_k - m a_{k-1} + \sum_{i+j=k} a_i r_j$$

Consider the ideal

$$I = id_{F\langle X\rangle}(R) = I_2 + I_3 + \cdots,$$

$$a_k = m^k - \dim_F I_k, \quad k \geq 0.$$

In each homogeneous component

$F\langle X\rangle_k$ choose a subspace B_k that is a complement to I_k,

$$F\langle X\rangle_k = B_k \dotplus I_k, \dim_F B_k = a_k.$$

Let $X = Fx_1 + \cdots + Fx_m$.

The homogeneous component I_k is spanned by uR_iv, where u, v are words of total length $k-i$.

If $v \neq 1$ then $uR_iv \subseteq I_{k-1}X$. If $v = 1$ then $uR_iv = uR_i \subseteq F\langle X\rangle_{k-i} R_i$ $= (I_{k-i} + B_{k-i}) R_i$.

We notice that

$$I_{k-i} R_i \subseteq I_{k-i} X^i \subseteq I_{k-1} X.$$

Hence,

$$I_K \subseteq I_{K-1} X + \sum_{i=2}^{K} B_{K-i} R_i.$$

Compare the dimensions:

$$m^k - a_K \leq (m^{K-1} - a_{K-1}) m + \sum_{i=2}^{K} a_{K-i} r_i,$$

$$a_K - a_{K-1} \cdot m + \sum_{i+j=K} a_i r_j \geq 0.$$

This completes the proof of the Golod-Shafarevich inequality.

Corollary: Let t_0 be a number lying between 0 an 1. Suppose that $H_R(t)$ converges at t_0 and $1 - m t_0 + H_R(t_0) < 0$. Then the algebra A is infinite dimensional.

Proof. We will show that the Hilbert series does not converge at t_0. If A

were finite dimensional then $H_A(t)$ would be a polynomial, hence would converge everywhere.

If $H_A(t)$ converges at t_0 then the formal Golod-Shafarevich inequality implies a numerical inequality

$$H_A(t_0)(1-mt_0+H_R(t_0)) \geq 1.$$

But $H_A(t_0)>0$, $1-mt_0+H_R(t_0)<0$, a contradiction.

Theorem (E. Golod, 1964). There exists a finitely generated infinite dimensional nil algebra.

Proof. Let $m \geq 2$. Let F be a countable field. Then the free algebra $F\langle x_1, x_2\rangle$ is countable. Consider the ideal $F_0\langle X\rangle$ of all elements of $F\langle X\rangle$ with zero constant term,

$F_0\langle X\rangle = \{f_1, f_2, \dots\}$. Let $\frac{1}{m} < t_0 < 1$.

Choose $\varepsilon > 0$ such that $1 - m t_0 + \varepsilon < 0$.

Choose $k \geq 1$ big enough so that

$$t_0^k / (1 - t_0) < \varepsilon.$$

We will construct a sequence $n_1 < n_2 < \dots$ with the following properties

1) $n_1 = k$

2) for every $i \geq 1$ n_{i+1} is greater than the degrees of all homogeneous

components of $f_1^{n_1}, f_2^{n_2}, \dots, f_i^{n_i}$.

Let R be the span of all homogenous components of elements $f_i^{n_i}$, $i \geq 1$. Then

$$R = R_k + R_{k+1} + \cdots$$

is a graded space, $R_i \subseteq F\langle X\rangle_i$, and for each i $\dim_P R_i = 0$ or 1.

Indeed, the minimal degree of a homogenous component of $f_i^{n_i}$ is greater than degrees of all homogenous components of $f_1^{n_1}, \dots, f_{i-1}^{n_{i-1}}$.

Hence, $H_R(t_0) \leq t_0^k + t_0^{k+1} + \cdots = \dfrac{t_0^k}{1-t_0} < \varepsilon$

and therefore $1 - m t_0 + H_R(t_0) < 0$.

Hence the graded algebra

$$A = \langle x_1, \dots, x_m \mid R = 0 \rangle$$

is infinite dimensional. We remark that here by $\langle X \mid R = 0 \rangle$ we mean $F_0\langle X \rangle / id_{F\langle X \rangle}(R)$, not $F\langle X \rangle / id_{F\langle X \rangle}(R)$ as before. For an arbitrary element f_i from $F_0\langle X \rangle$ we have $f_i^{n_i} \in id_{F\langle X \rangle}(R)$. Hence the algebra A is nil. This completes the proof of the theorem.

Remark. This proof works for a countable field F. However the theorem is true for algebras over any field.

Theorem (E. Golod, 1964). There exists a finitely generated infinite torsion group.

Proof. Let A be an infinite dimensional nil algebra over a field F of characteristic $p>0$, the algebra A is generated by elements $a_1, \dots, a_m$.

Let $\hat{A} = A + F\cdot 1$ be the unital hull of the algebra A,

$$(a+\alpha\cdot 1)(b+\beta\cdot 1) = (ab+\alpha b+\beta a)+\alpha\beta\cdot 1$$

Let $G(\hat{A})$ be the group of all invertible elements of $\hat{A}$.

For an arbitrary element $a \in A$ the element $1+a$ is invertible in $\hat{A}$. Moreover, it has finite order. Indeed, there exists $n(a) \geq 1$: $a^{n(a)} = 0$. Choose a p-power p^k such that $p^k \geq n(a)$. Then $(1+a)^{p^k} = 1 + a^{p^k} = 1$. We used the fact that all binomial coefficients $\binom{p^k}{i}$, $1 \leq i \leq p^k - 1$, are divisible by p.

Consider the subgroup G of $G(\hat{A})$ generated by elements $g_1 = 1 + a_1, \ldots, g_m = 1 + a_m$. This group is finitely generated and torsion. Our aim now is to prove that it is infinite.

Suppose that the group G is finite,

$|G| = n$. Then $G = \{1, g_{i_1} \cdots g_{i_k}, 1 \le i_1, \dots, i_k \le m, k < n\}$.

Indeed, consider a product $g_{i_1} \cdots g_{i_n}$ of length n. At least two elements out of $n+1$ elements $1, g_{i_1}, g_{i_1} g_{i_2}, \dots, g_{i_1} \cdots g_{i_n}$ are equal. It implies that some subproduct $g_{i_2} g_{i_{2+1}} \cdots g_{i_{2+e}} = 1$. We can cancel it in $g_{i_1} \cdots g_{i_n}$. We showed that every product of length n is equal to a shorter product.

Now we can conclude that every product $a_{i_1} \cdots a_{i_n}$ of length n is equal to a linear combination of shorter products of elements $a_1, \dots, a_m$. Indeed, let

$$(1+a_{i_1})\cdots(1+a_{i_n}) = (1+a_{j_1})\cdots(1+a_{j_r}),$$

$$r<n.$$

Then $a_{i_1}\cdots a_{i_n} + \sum(\text{products of length} < n)$

$= \sum(\text{products of length} \le r)$

Hence the algebra A is spanned by products $a_{i_1}\cdots a_{i_k}$, $k<n$. Hence, $\dim_F A < \infty$, a contradiction.

We proved that the group G is infinite. This completes the proof of the theorem.

The group that we constructed is residually-p. Indeed, assume for simplicity that the field F is finite, $\mathrm{char}\, F = p > 0$.

The homomorphism

$$\hat{A} \to \hat{A} / A_n + A_{n+1} + \cdots$$

gives rise to the homomorphism of multiplicative groups

$$G(\hat{A}) \xrightarrow{\varphi_n} G(\hat{A} / A_n + A_{n+1} + \cdots)$$

The ring $\hat{A} / A_n + A_{n+1} + \cdots$ is finite, hence $G(\hat{A} / A_n + A_{n+1} + \cdots)$ is a finite group. The image $\varphi_n(G)$ of the group G is a finite p-group since every element has an order p^k, $k \geq 1$.

$\operatorname{Ker} \varphi_n \subseteq 1 + A_n + A_{n+1} + \cdots$, hence $\bigcap_{n \geq 1} \operatorname{Ker} \varphi_n = \{1\}$. We proved that the group G is residually-p.