Lecture 15.

Tensor product of algebras.

Let A be an algebra over a field F. Let $K \supset F$ be a bigger field. Then K may be viewed as an F-algebra.

The algebra

$$A_K = A \otimes_F K$$

is an algebra over the field K. It is called a scalar extension of A. If $\{e_i\}_i$ is a basis of the F-algebra A then it is also a basis of the K-algebra A_K.

Central Simple Algebras.

If A is an algebra over a field F then it is also an algebra over all subfields of F

and, may be, over some overfields. There is a way to determine a "natural" field over which A is an algebra.

For an element $a \in A$ let $R_a : A \to A$, $x \to xa$, be the operator of right multiplication by a. Let $L_a : A \to A$, $x \to ax$, be the operator of left multiplication. Let $M(A)$ be the subalgebra of the algebra $Lin_F(A)$ of all linear transformations of the F-vector space A generated by all right and left multiplications. Let

$$C = \{\varphi \in Lin_F(A) \mid \varphi R_a = R_a \varphi, \ \varphi L_a = L_a \varphi$$
for all elements $a \in A\}$

be the centralizer of the subalgebra $M(A)$ in $Lin(A)$

Lemma: (1) If $A^2 = A$ then C is a commutative algebra.

(2) If the algebra A is prime, that is, for any two nonzero ideals $I, J \triangleleft A$ the product IJ is different from (0), then C is a commutative domain.

(3) If the algebra A is simple, that is, A does not have any ideals except for (0) and A, then C is a field.

Proof. (1) Let $\varphi, \psi \in C$, which means that φ and ψ commute with all right and left multiplications.

For arbitrary elements $a, b \in A$ we have $ab = R_b a$, $\varphi(ab) = \varphi R_b a = R_b \varphi a = \varphi(a) b$.

On the other hand $ab = L_a b$, $\varphi(ab) =$
$= \varphi L_a b = L_a \varphi b = a\varphi(b)$. We showed that

$$\varphi(ab) = \varphi(a)b = a\varphi(b).$$

If $\varphi, \psi \in C$ then $\varphi\psi(ab) = \varphi(a\psi(b))$
$= \varphi(a)\psi(b)$ and $\psi\varphi(ab) = \psi(\varphi(a)b) = \varphi(a)\psi(b)$,
so

$$(\varphi\psi - \psi\varphi)A^2 = (0).$$

Since $A^2 = A$ it follows that $(\varphi\psi - \psi\varphi)(A) =$
(0), so $\varphi\psi = \psi\varphi$. This proves the claim (1).

For an arbitrary element $\varphi \in C$ the image $\varphi(A)$ is an ideal of A. Indeed,
$\varphi(a)b = \varphi(ab)$ and $b\varphi(a) = \varphi(ba)$.

Now suppose that the algebra A is prime;

Let $\varphi, \psi \in C$ and $\varphi\psi = 0$. For elements $a, b \in A$ we have

$$(\varphi\psi)(ab) = \varphi(a)\psi(b) = 0.$$

Hence $\varphi(A)\psi(A) = (0)$. Since the algebra A is prime and both $\varphi(A)$ and $\psi(A)$ are ideals we conclude that $\varphi(A) = (0)$ or $\psi(A)$, i.e. $\varphi = 0$ or $\psi = 0$.

We have shown above that for arbitrary linear transformations $\varphi, \psi \in C$

$$(\varphi\psi - \psi\varphi)(A^2) = (0).$$

Let $\mu = \varphi\psi - \psi\varphi$, $\mu \in C$. We have

$$\mu(A^2) = \mu(A)A = (0).$$

As above, since $\mu(A)$ is an ideal of A and the algebra A is prime, we conclude that $\mu = 0$.

This proves (2).

Now suppose that the algebra A is simple. Since a simple algebra is prime the part (2) implies that C is a commutative domain. We need to show that an arbitrary nonzero element $\varphi \in C$ is invertible.

~~[illegible]~~ Since $\varphi(A)$ is a nonzero ideal it follows that $\varphi(A) = A$, so the linear transformation φ is surjective.

The kernel $\operatorname{Ker}\varphi$ is an ideal of the algebra A as well. Indeed, if $\varphi(a) = 0$ then

$\varphi(ab) = \varphi(a)b = 0$ and $\varphi(ba) = b\varphi(a) = 0$.

Since $\operatorname{Ker}\varphi \neq A$ we conclude that $\operatorname{Ker}\varphi = (0)$, so the linear transformation φ is injective.

we proved that the linear transformation φ has an inverse φ^{-1}. It is easy to see that φ^{-1} commutes with all right and left multiplications of A, so $\varphi^{-1} \in C$. This completes the proof of the lemma.

Clearly, $$F \subseteq C.$$

~~For a simple algebra A its centroid is the~~ The algebra C is called the <u>centroid</u> of the algebra A. If A is a simple algebra then the field C is the natural (maximal) field to consider A over.

If the algebra A contains 1 then the centroid C can be identified with the center of A

$\{ z \in A \mid az = za$ for an arbitrary element $a \in A \}$.

Indeed, let $\varphi \in C$. Then $z = \varphi(1)$ lies in the center of A : $\varphi(1)a = \varphi(1 \cdot a) = \varphi(a) = a\varphi(1)$. This equality implies also that the linear transformation φ is equal to the left (= right) multoplication by z.

<u>Theorem</u>. Let A be a simple central algebra over a field F, let B be a simple F-algebra with 1. Then the tensor product $A \otimes_F B$ is a simple algebra. If $A \ni 1$ and the algebra B is central then the algebra $A \otimes_F B$ is simple central.

To prove Theorem we will use the classical <u>Density Theorem</u> of N. Jacobson.

Let R be a ring, let V be a left R-module the centralizer $C({}_R V)$ is of the module V consists of endomorphisms φ of the abelian group V that commute with all multiplications $a : V \to V$, $a : v \to av$.

<u>The Schur Lemma</u> (one of the variants).

If the module ${}_R V$ is irreducible then the centralizer $\Delta = C({}_R V)$ is a division ring.

<u>Division ring</u> means that every nonzero

element is invertible, but Δ may be not commutative.

Since Δ consists of linear transformations of the vector space V, the module V can be viewed as a module over Δ = vector space over Δ.

The whole theory of linear dependence can be carried over to vector spaces over division rings.

Jacobson's Density Theorem. If elements $v_1, \ldots, v_n \in V$ are linearly independent over Δ and $w_1, \ldots, w_n \in V$ are arbitrary elements, $n \geq 1$, then there exists an element $a \in R$ such that $a v_i = w_i$, $1 \leq i \leq n$.

Let A be a simple F-algebra. The algebra A can be viewed as an irreducible module over the multiplication algebra $M(A) = \langle L_a, R_a, a \in A \rangle \subseteq Lin_F(A)$. Indeed

an ideal of A = $M(A)$-submodule.

The centralizer of this irreducible module is the centroid C. By lemma (3) C is a field. By Density Theorem if elements $a_1, \ldots, a_n \in A$ are linearly independent over C and $b_1, \ldots, b_n \in A$ are arbitrary elements then there exists an operator $P \in M(A)$ such that $Pa_i = b_i$, $1 \le i \le n$.

Proof of Theorem . Let I be a nonzero

ideal of $A\otimes_F B$. Let $0\neq u = a_1\otimes b_1+\cdots+a_n\otimes b_n\in I$ we assume that the elements $a_1,\dots,a_n\in A$ are linearly independent over the field F which the centroid of A. (Assume also that $b_1,\dots,b_n$ are $\neq 0$.) Let $a\in A$ be an arbitrary element of A. There exists an operator $P=\sum_i L_{x_i} R_{y_i}\in M(A)$, $x_i, y_i\in A$; such that $Pa_1=a$, $Pa_2=\cdots = Pa_n=0$.

The algebra B contains an identity element 1. Consider the operator

$$\tilde{P}=\sum_i L_{x_i\otimes 1}\, R_{y_i\otimes 1}\in M(A\otimes_F B).$$

We have $\tilde{P}u = Pa_1\otimes b_1+\cdots+Pa_n\otimes b_n$ $= a\otimes b_1\in I$.

Hence $A\otimes b_1 \subseteq I$.

The set $\{b\in B \mid A\otimes b \subseteq I\}$ is an ideal in the algebra B. Since the algebra B is simple it follows that for an arbitrary element $b\in B$ we have $A\otimes b\subseteq I$, hence

$A\otimes_F B = I$.

We proved that the algebra $A\otimes_F B$ is simple.

Now suppose that $A\ni 1$ and the algebra B is simple and central.

Since $A\otimes_F B\ni 1\otimes 1$ we identify the centroid of $A\otimes_F B$ with its center.

Let an element $u = a_1\otimes b_1 + \cdots + a_n\otimes b_n$

lie in the center, the elements $a_1, \dots, a_n$ are linearly independent. For an arbitrary element $b \in B$ we have

$[x, 1 \otimes b] = \sum_{i=1}^{n} a_i \otimes [b_i, b] = 0$. Therefore $[b_1, b] = \cdots = [b_n, b] = 0$. We proved that the elements $b_1, \dots, b_n$ lie in the center of B, hence in $F \cdot 1$.

This implies that the element u can be represented as a simple tensor $u = a \otimes 1$, $a \in A$. It is easy to see that the element a lies in the center of the algebra A, that is, in $F \cdot 1$. This completes the proof of the theorem.

Brauer group. Fix a field F. Consider the class C of all finite dimensional central division algebras over F. We will define a multiplication operation in the class C.

Given two finite dimensional central division F-algebras A, B, consider the algebra $A \otimes_F B$. The algebra $A \otimes_F B$ is simple central and finite dimensional. By the celebrated Wedderburn Theorem $A \otimes_F B$ is isomorphic to a matrix algebra over a division algebra D

$$A \otimes_F B \cong M_n(D).$$

The center of the algebra $M_n(D)$ consists of diagonal matrices

$$n\left\{\begin{pmatrix} z & & & 0 \\ & z & & \\ & & \ddots & \\ 0 & & & z \end{pmatrix}\right.,$$

where z lies in the center of D. Hence centrality of the algebra $M_n(D)$ is equivalent to centrality of the division algebra D.

Define

$$A \cdot B = D$$

From the properties of tensor product it immediately follows that this multiplication is associative and commutative

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C), \quad A \otimes B \cong B \otimes A$$

The identity of this commutative semigroup is the field F itself,

$$A \otimes_F F \cong A$$

For a division algebra A define a new multiplication on the F-vector space A,

$$a * b = ba.$$

We will again get a finite dimensional central division algebra that is denoted as A^{op}.

We have

$$L_a L_b = L_{ab}$$

Hence the mapping $A \to M(A)$, $a \to L_a$, is an embedding. The subalgebra

$L_A = \{L_a, a \in A\}$ of $M(A)$ is isomorphic to the algebra A. On the other hand

$$R_a R_b = R_{ba}.$$

Hence the subalgebra $R_A = \{R_a, a \in A\}$ of $M(A)$ is isomorphic to the algebra A^{op}.

An arbitrary left multiplication L_a, $a \in A$, commutes with an arbitrary right multiplication R_b, $b \in A$. Hence

$$M(A) = L_A R_A$$

This implies that the mapping

$$A \otimes_F A^{op} \cong L_A \otimes_F R_A \to M(A), \; L_a \otimes R_b \to L_a R_b$$

is a homomorphism. Since $A \otimes_F A^{op}$ is a simple algebra it follows that this homomorphism is an isomorphism

$$A \otimes_F A^{op} \cong M(A).$$

Let $\dim_F A = d$. Let $e_1, \ldots, e_d$ be a basis of the F-algebra A. We have already explained above that for arbitrary elements $a_1, \dots$ $\dots, a_d \in A$ there exists an operator $P \in M(A)$ such that $P(e_i) = a_i$, $1 \le i \le d$ (Density Theorem). This implies that $M(A)$ is the algebra of all F-linear transformations of A,

$$M(A) \cong M_d(F).$$

Hence, $A \otimes_F A^{op} \cong M_d(F)$, $A \cdot A^{op} = F$. We showed that the division algebra A^{op} is the inverse of the division algebra A.

Hence C is an abelian group. It is called the <u>Brauer group</u> of the field F and denoted $Br(F)$.
If the field F is algebraically closed then $Br(F) = \{1\}$. But already $Br(\mathbb{R})$ is very interesting.