Lecture 10

Wreath Products.

§ Semidirect Products. Let G_1, G_2 be groups, $\varphi : G_1 \to \operatorname{Aut} G_2$ is a homomorphism. For elements $a \in G_1$, $b \in G_2$ we denote $\varphi(a) : b \to b^a$. Then we define a group structure on the Cartesian product $G_1 \times G_2$:

$$(a_1 b_1)(a_2 b_2) = a_1 a_2 \, b_1^{a_2} b_2$$

If $\varphi : G_1 \to \mathrm{Id}_{G_2}$ then this is just a direct product.

Wreath Product. Let A, B be groups. Consider the set $\operatorname{Fun}(B, A)$ of all mappings from B to A. The set

$\mathrm{Fun}(B, A)$ is a group isomorphic to the Cartesian product $A^{|B|}$.

For a fixed element $b_0 \in B$ and a function $f : B \to A$ define

$$f^{b_0}(b) = f(b_0 b).$$

The mapping $f \to f^{b_0}$ is an automorphism of the group $\mathrm{Fun}(B, A)$ and the resulting mapping

$$B \to \mathrm{Aut}\, \mathrm{Fun}(B, A)$$

is a homomorphism.

~~Indeed~~, We need to check that

$$(f^{b_1})^{b_2} = f^{b_1 b_2}$$

Indeed,

$$(f^{b_1})^{b_2}(b) = f^{b_1}(b_2 b) = f(b_1 b_2 b) = f^{b_1 b_2}(b).$$

Consider the semidirect product

$$A \operatorname{wr} B = B \cdot \operatorname{Fun}(B, A),$$

$$(b_1 f_1)(b_2 f_2) = b_1 b_2 f_1^{b_2} f_2 .$$

It is called the wreath product of the groups A, B. Sometimes it is denoted as $A \wr B$.

Let $\operatorname{Fun}_0(B, A) = \{ f \in \operatorname{Fun}(B, A) \mid f(b) \neq 1_A$ only for finitely many elements $b \in B \}$.

Clearly,

$$A \, \overline{\operatorname{wr}} \, B = B \cdot \operatorname{Fun}_0(B, A)$$ is a subgroup of $A \operatorname{wr} B$. It is called the restricted wreath product of A, B.

For an element $a \in A$ define a mapping $\mathring{a} \in \mathrm{Fun}(B, A)$:

$$\mathring{a}(b) = \begin{cases} a & \text{for } b = 1_B \\ 1_A & \text{for } b \neq 1_B \end{cases}$$

The mapping $A \to \mathrm{Fun}(B, A)$, $a \to \mathring{a}$ is an embedding. The subgroup $\mathring{A}$ is called the 1st copy of A, the conjugate subgroup $\mathring{A}^{b} = b^{-1} A^{\circ} b$ is called the b-th copy of A.

If $b_1 \neq b_2$ then elements from $\mathring{A}^{b_1}$ and $\mathring{A}^{b_2}$ commute with each other: if $x \in \mathring{A}^{b_1}$, $y \in \mathring{A}^{b_2}$ then $xy = yx$.

$$\mathrm{Fun}_0(B, A) = \prod_{b \in B} \mathring{A}^{b}.$$

If the group B is generated by elements $b_1, \dots, b_m$ and the group A is generated by elements $a_1, \dots, a_n$ then $A \overline{wr} B$ is generated by $\mathring{a}_1, \dots, \mathring{a}_n, b_1, \dots, b_m$. Hence a restricted wreath product of finitely generated groups is finitely generated.

Example. The restricted wreath product of two infinite cyclic groups is not finitely presented.

Proof. The restricted wreath product $\langle a \rangle \overline{wr} \langle b \rangle$ is generated by elements $\mathring{a}, b$. In these generators the group is defined by relations

$$G = \langle x, y \mid [y^{-i} x y^{i}, y^{-j} x y^{j}] = 1;\ i,j \in \mathbb{Z} \rangle$$

Equivalently, $\langle x, y \mid [y^{-i} x y^{i}, x] = 1,\ i \in \mathbb{Z} \rangle$

If the group G is finitely presented then a finite subset of this set of defining relations defines the group G, so

$$G = \langle x, y \mid [y^{-i} x y^{i}, x] = 1,\ i = 1, 2, \ldots, n-1 \rangle$$

for some $n \geq 1$.

Exercise. If $G = \langle X \mid R = 1 \rangle$, $|X| < \infty$, is finitely presented then there exists a finite subset $R_1 \subseteq R$, such that $G = \langle X \mid R_1 = 1 \rangle$.

We need to construct an example of a group with two elements x, y such that

$[y^{-i} x y^{i}, x] = 1$ for $i = 1, 2, \ldots, n-1$ but

$[y^{-n} x y^{n}, x] \neq 1$.

Let S be a nonabelian group;

$S \ni \overset{u,v}{\cancel{a,b}}$; ~~$ab \neq ba$.~~ $uv \neq vu$.

Consider the wreath product $S \operatorname{wr} \langle b \rangle$.

Let $x \in Fun_0(\langle b \rangle, S)$,

$x(1_B) = \overset{u}{\cancel{a}}$, $x(b^n) = v$, $x(b^i) = 1_S$ for

all other $i \neq 0, n$.

Let $1 \leq i \leq n-1$. The mapping

$b^{-i} x b$ is equal to u at b^{-i}, to

v at b^{n-i}, to 1_S at all other powers

of b. Then

$$[b^{-i} x b^{i}, x], \quad 0 \le i \le n-1.$$

But $(b^{-n} x b^{n})(1_B) = x(b^n) = v$. Hence x and $b^{-n} x b^{n}$ do not commute. This completes the proof. ~~of the lemma.~~

Theorem (Kalouzhnine, Krasner). Let G be a group with a normal subgroup A. Then the group G is embeddable in the wreath product $A \, wr(G/A)$.

Proof. For an element $g \in G$ let $\bar{g} \in G/A$ be its coset. We will define a homomorphism $G \to A \, wr(G/A)$, $g \to \bar{g} f_g$, $f_g : G/A \to A$. It remains to define the function f_g.

Let $s : G/A \to G$ be ~~the~~ a function that elects representatives.

For an element $b \in G/A$ define

$f_g(b) = ((\bar{g}b)^s)^{-1} g b^s$.

We have to check that

$(\bar{g}_1 f_{g_1})(\bar{g}_2 f_{g_2}) = \bar{g}_1 \bar{g}_2 f_{g_1}^{\bar{g}_2} f_{g_2} = \overline{g_1 g_2} f_{g_1 g_2}$

Choose $b \in G/A$.

$f_{g_1}^{\bar{g}_2}(b) = f_{g_1}(\bar{g}_2 b) = (\bar{g}_1 \bar{g}_2 b)^{s^{-1}} g_1 (\bar{g}_2 b)^s$

$f_{g_2}(b) = ((\bar{g}_2 b)^s)^{-1} g_2 b^s$.

Hence

$$f_{g_1}^{\bar g_2}(b)\, f_{g_2}(b) = \left((\bar g_1 \bar g_2 b)^s\right)^{-1} g_1 \cancel{(\bar g_2 b)^s} \cancel{\left((\bar g_2 b)^s\right)^{-1}} g_2 b^s$$

$$= \left((\bar g_1 \bar g_2 b)^s\right)^{-1} g_1 g_2 b^s ;$$

$$f_{g_1 g_2}(b) = \left((\overline{g_1 g_2}\, b)^s\right)^{-1} g_1 g_2 b^s .$$

The equality is straightforward.

Let us check that the homomorphism

$$g \to \bar g f_g$$

is an embedding. If the image is 1 then $g \in A$ and $f_g(b) = 1_A$ for every b.

Since $g \in A$ it follows that

$(\bar{g}b)^s = b^s$, hence $(b^s)^{-1} g\, b^s = 1$, $g = 1$. This completes the proof of the theorem.

Burnside's Problems.

In 1902 W. Burnside formulated two problems that essentially influenced the study of infinite groups in the 20th Century.

Problem 1. (General Burnside Problem).

Let G be a finitely generated group such that every element $g \in G$ has a finite order ($g^{n(g)} = 1$). Does it imply that $|G| < \infty$?

Problem 2. (The Burnside Problem). Let G be a finitely generated group. Suppose that there exists $d \geq 1 : \forall g \in G \ g^d = 1$. Does it imply that $|G| < \infty$?

A minimal such d is called the exponent of G.

Finite generation is essential: $G = \bigoplus_{n \geq 2} \mathbb{Z}(n)$ is an infinite torsion group.

Exponent $d=2$: $\forall g \in G \ g^2 = 1$, so $g^{-1} = g$. $\forall a, b \in G \ (ab)^{-1} = b^{-1}a^{-1}$, $ab = ba$, the group G is abelian. If $G = \langle a_1, \dots, a_m \rangle$ then $G = \{1, a_{i_1} a_{i_2} \dots a_{i_k}, \ i_1 < i_2 < \dots < i_k\}$, $|G| \leq 2^m$.

Exponent $d=3$ <u>W. Burnside</u>

$d=4$ <u>Sanov</u>

$d=6$ <u>M. Hall</u>

$d=5$ <u>Open</u>

<u>P.S. Novikov - S.I. Adian</u>, 1968 : examples of infinite finitely generated groups of finite exponent.