Lecture 6:

Chapter II. Free Groups and Free Products.

Recall that a group G is generated by a subset $X \subseteq G$ if every element $g \in G$ can be written as $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$, where $\varepsilon_i = \pm 1$; $r \geq 1$; $x_{i_1}, \ldots, x_{i_r} \in X$.

Def. We say that G is a **free group on the set of free generators X** if

- G is generated by X,
- for every group G' and every mapping $\varphi : X \to G'$ there exists a unique homomorphism $\bar{\varphi} : G \to G'$ that extends φ.

It is easy to see that any two free groups G_1 and G_2 on the same set X of free generators are isomorphic. Indeed, the identical mapping $G_1 \supseteq X \xrightarrow{id} X \subseteq G_2$ extends to a homomorphism $G_1 \xrightarrow{\varphi} G_2$. Similarly the identical mapping $G_2 \supseteq X \xrightarrow{id} X \subseteq G_1$ extends to a homomorphism $G_2 \xrightarrow{\psi} G_1$. The composition $\varphi\psi$ is identical on X, hence $\varphi\psi = id_{G_2}$. Similarly $\psi\varphi = id_{G_1}$. Hence φ, ψ are isomorphisms.

Now consider words in $X \cup X^{-1}$. A word $w = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$, where $x_{i_1}, \ldots, x_{i_k} \in X$, $\varepsilon_i = \pm 1$, is said to be <u>reduced</u> if it does not contain subwords $x_i x_i^{-1}, x_i^{-1} x_i$, i.e. x_i and x_i^{-1} do not stand together.

If $G = \langle X \rangle$ then every element in G can be represented as a reduced product of elements from X.

<u>Lemma</u> . If G is generated by X and every element in G can be represented as a reduced product of elements from X <u>uniquely</u> then G is a free group on the set of <u>free generators</u> X.

<u>Proof</u>. Suppose that G' is a group and we have a mapping $\varphi : X \to G'$. ~~Then~~ An arbitrary element $g \in G$ has a unique reduced form $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$. We map $\overline{\varphi}(g) = \varphi(x_{i_1})^{\varepsilon_1} \cdots \varphi(x_{i_k})^{\varepsilon_k}$. It is easy to check that $\overline{\varphi}$ is a homomorphism. This completes the proof of the lemma.

Def. Given a ring A with 1 let $A^* = \{a \in A \mid \exists\, b \in A : ab = ba = 1\}$ be the group of all invertible elements from A. This group is called the **multiplicative group of A.**

Proposition. (1) For an arbitrary nonempty set X a free group on the set of free generators X exists;

(2) an arbitrary element g of this free group can be represented as a reduced word in elements from X uniquely. This reduced product is called the **normal form of the element g.**

Proof. Let $X=\{x_i\}_{i\in I}$ be a set. By the Axiom of choice there exists an order on the set I with the minimality condition.

Consider the semigroup with 1:

$$S=\langle x_i, y_i, i\in I \mid x_i y_i = 1, y_i x_i = 1, i\in I\rangle$$

The only compositions are

$$(x_i y_i, y_i x_i)_{x_i y_i x_i} = (x_i y_i - 1)x_i - x_i(y_i x_i - 1)$$

$$=0,$$

$$(y_i x_i, x_i y_i)_{y_i x_i y_i} = 0.$$

Hence normal forms of elements in S are words that do not contain subwords $y_i x_i$, $x_i y_i$.

Remark. We did not have to specify if $x_i < y_i$

or $y_i < x_i$ because it is irrelevant.

It is easy to see that $x_i^{-1} = y_i$ in the semigroup S, hence S is a group. An arbitrary element reduces to normal form = a reduced product uniquely. Hence S is a free group on the set of free generators X. We proved (1). Moreover, in the group S an arbitrary element can be represented as a reduced product of elements from X uniquely.

If G is another free group on the set of ~~G~~ free generators X then $X \xrightarrow{id} X$ extends to an isomorphism $G \to S$, which proves (2).

Now we will assume that $|X| < \infty$, though in most cases it is irrelevant. We just don't want to deal with infinite cardinals.

An abelian group $A \supseteq X$ is called <u>free abelian</u> if $A = \langle X \rangle$ and every map $X \to B$ to an abelian group B extends to a homomorphism $A \to B$.

The uniqueness of this object follows as before. For existence, suppose that $|X| = n$. Let $\mathbb{Z}$ be the infinite cyclic group. Then $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}$ is a free abelian group on X.

<u>Lemma</u>. The groups $A(n) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}$ and

$A(m) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{m}$ are isomorphic if and only if $m=n$.

Proof. Let $A(n)\cong A(m)$. Then the groups

$A(n)/2A(n) \cong \underbrace{(\mathbb{Z}/2\mathbb{Z})\oplus\cdots\oplus(\mathbb{Z}/2\mathbb{Z})}_{n}$ and

$A(m)/2A(m) \cong \underbrace{(\mathbb{Z}/2\mathbb{Z})\oplus\cdots\oplus(\mathbb{Z}/2\mathbb{Z})}_{m}$ are

isomorphic,

$|A(n)/2A(n)| = 2^n$, $|A(m)/2A(m)| = 2^m$,

which completes the proof.

If $|X| = n$ then we denote the free group on the set of free generators X as $F_2(n)$.

Let G be a group and let $a, b \in G$. We call $[a,b] = a^{-1}b^{-1}ab$ the <u>commutator</u> of the elements a, b. Define the <u>commutator subgroup</u> $[G, G]$ to be the subgroup of G generated by all commutators $[a,b]$; $a, b \in G$. Then $G/[G,G]$ is an abelian group.

If $g \in [F_2(n), F_2(n)]$ and $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$ is the reduced form of the element g then each inverse x_i^{-1}, $1 \le i \le n$, occurs in $x_{i_1}^{\varepsilon_1} \cdots x_{i_2}^{\varepsilon_2}$ as many times as x_i. Hence for $1 \le i \ne j \le n$ the element $x_i^{-1} x_j$ does not lie in $[F_2(n), F_2(n)]$. Hence the restriction of the ~~ma~~ homomorphism

$$F_2(n) \to F_2(n) / [F_2(n), F_2(n)]$$

to X is injective. Hence we will assume that $X \subset F_2(n) / [F_2(n), F_2(n)]$.

<u>Lemma</u>. $F_2(n)/[F_2(n), F_2(n)]$ is the free abelian group on the set of free generators X.

<u>Proof</u>. Let A be an abelian group and let $\varphi : X \to A$ be a mapping. This mapping extends to a homomorphism $F_2(n) \xrightarrow{\overline{\varphi}} A$. Since the group A is abelian all commutators $[a,b]$; $a,b \in F_2(n)$, are mapped to 1. Hence $[F_2(n), F_2(n)] \leq \operatorname{Ker} \overline{\varphi}$, which allows us to define a homomorphism

$F_2(n)/[F_2(n), F_2(n)] \xrightarrow{\overline{\varphi}} A$. This homomorphism extends φ. This completes the proof of the lemma.

<u>Lemma</u> . $Fr(n) \cong Fr(m)$ if and only if

$m = n$.

<u>Proof</u>. If $Fr(n) \cong Fr(m)$ then

$$Fr(n)/[Fr(n), Fr(n)] \cong Fr(m)/[Fr(m), Fr(m)].$$

Now it suffices to refer to Lemma .

Let G be a group and H a subgroup of G. Then G is equal to the disjoint union of cosets of H,

$$G = H \dot{\cup} Hg_1 \dot{\cup} Hg_2 \dot{\cup} \cdots$$

This partition corresponds to an equivalence relation defined by

$a \sim b$ if $ab^{-1} \in H$. For each coset elect a representative from among its elements, so that for each $g \in G$, $\bar{g}$ denotes the representative for Hg. For simplicity we will choose the representative for H to be 1.

Note that $g \cdot \bar{g}^{-1} \in H$ for every $g \in G$.

Then
$$(ab)(\bar{a}b)^{-1} = abb^{-1}\bar{a}^{-1} = a\bar{a}^{-1} \in H.$$

This implies that $\bar{a}b$ and ab are equivalent and thus
$$\overline{\bar{a}b} = \overline{ab}$$

Fix a set X of generators for G.

Let S be the set of elected representatives of all cosets of H. Let

$$Y = \{ sx\, \overline{sx}^{-1} \mid s \in S, x \in X \}$$

Lemma: The subset $Y \subseteq H$ generates H, $H = \langle Y \rangle$.

Proof. First, we will show that for arbitrary elements $s \in S$, $x \in X$ the element $sx^{-1}\, \overline{sx^{-1}}^{-1}$ lies in Y. Let $s_1 = \overline{sx^{-1}} \in S$. We have $s_1 x\, \overline{s_1 x}^{-1} \in Y$. Furthermore,

$$\overline{s_1 x} = \overline{\overline{sx^{-1}}\, x} = \overline{sx^{-1}x} = \overline{s} = s.$$ Hence

$$\left(s_1 x\, \overline{s_1 x}^{-1}\right)^{-1} = \overline{s_1 x}\, x^{-1}\, \overline{sx^{-1}}^{-1} = s x^{-1}\, \overline{sx^{-1}}^{-1},$$

which proves the claim.

An arbitrary element $h \in H$ can be represented as a product

$$h = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n}, \quad \varepsilon_i = \pm 1.$$

We will prove $h \in \langle Y \rangle$ for $n = 4$. The general case is analogous.

We have

$$h = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3} x_{i_4}^{\varepsilon_4} = \underbrace{1\, x_{i_1}^{\varepsilon_1}\, \overline{1 \cdot x_{i_1}^{\varepsilon_1}}^{\,-1}}_{\in\, Y \cup Y^{-1}} \cdot$$

$$\underbrace{\overline{x_{i_1}^{\varepsilon_1}}\, x_{i_2}^{\varepsilon_2}\, \overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2}}^{\,-1}}_{\in\, Y \cup Y^{-1}} \cdot \underbrace{\overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2}}\, x_{i_3}^{\varepsilon_3}\, \overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3}}^{\,-1}}_{\in\, Y \cup Y^{-1}} \cdot$$

$$\underbrace{\overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3}}\, x_{i_4}^{\varepsilon_4}\, \overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3} x_{i_4}^{\varepsilon_4}}^{\,-1}}_{\in\, Y \cup Y^{-1}} \cdot \underbrace{\overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3} x_{i_4}^{\varepsilon_4}}}_{=\,1}$$

The final term is $= 1$ because the

representative for H was chosen to be 1.

Corollary. If G is a finitely generated group and the index of a subgroup H in G is finite, then H is finitely generated.

Proof. If the group G is generated by m elements $x_1, \dots, x_m$ and $|G : H| = n$, $|S| = n$, then $\#\{s x_i \overline{s x_i}^{-1} \mid s \in S, 1 \le i \le m\}$

$\le mn$.

~~Now consider the free group $Fr(m)$ instead of an arbitrary group G, H is a subgroup of $Fr(m)$.~~