Lecture 19.

Old problems with $\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$

Let R be the field of real numbers.

Let $N=\{1,2,\dots\}$. A subset $X\subseteq N$ is <u>cofinite</u> if $|N\setminus X|<\infty$. There exists an ultrafilter $\mathcal{F}$ in N, such that $\mathcal{F}$ contains all cofinite subsets of N.

The field $R^N/\mathcal{F}$ contains an element $\varepsilon>0$ such that $\varepsilon<\frac{1}{n}$ for all $n\in N$. Indeed, let

$\varepsilon=(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots)/\mathcal{F}$.

<u>Convergence with respect to a filter.</u>

Let a_n, $n\geq 1$, be a sequence of numbers. Let $\mathcal{F}$ be a filter in N.

<u>Def.</u> We say that $a=$ limit of a_n, $n\geq 1$, with respect to $\mathcal{F}$ if $\forall\varepsilon>0$ $\exists$ a large subset $X\in\mathcal{F}$ such that

$|a - a_n| < \varepsilon$ for all $n \in X$.

Example. The subsets $N_n = \{n, n+1, n+2, \ldots\}$, $n \geq 1$, form a filter in N. The usual limit is the limit with respect to this filter.

It is easy to see that a sequence can not have more than one limit. Indeed, let a, b be limits of a sequence $a_n, n \geq 1$, $a \neq b$. There exist large subsets $X_1, X_2 \in \mathcal{F}$:

$$|a - a_n| < \frac{1}{2}|a-b| \text{ for all } n \in X_1 ;$$

$$|b - a_n| < \frac{1}{2}|a-b| \text{ for all } n \in X_2 .$$

Then $X_1 \cap X_2 = \emptyset$, a contradiction.

Lemma. Let $\mathcal{F}$ be an ultrafilter in the set X. Let X_1, X_2 be subsets in X and $X_1 \cup X_2 \in \mathcal{F}$. Then $X_1 \in \mathcal{F}$ or $X_2 \in \mathcal{F}$.

Proof. If $X_1 \notin \mathcal{F}$, $X_2 \notin \mathcal{F}$ then $X \setminus X_1 \in \mathcal{F}$, $X \setminus X_2 \in \mathcal{F}$.

We have $(X \setminus X_1) \cap (X \setminus X_2) = X \setminus (X_1 \cup X_2) \in \mathcal{F}$.
This contradicts the assumption that
$X_1 \cup X_2 \in \mathcal{F}$. $\downarrow$

<u>Theorem</u>. Let $\mathcal{F}$ be an ultrafilter in N. Then every bounded sequence has a limit.

<u>Proof</u>. By the assumption there exists a segment $S_1 = [-b, b]$ such that all a_n's lie in S_1. Consider the segments $[-b, 0)$ and $[0, b]$.
By the lemma one of the subsets $\{n \mid a_n \in [-b, 0)\}$, $\{n \mid a_n \in [0, b]\}$ is large.

Let $\{n \mid a_n \in [-b, 0)\}$ be large, $S_2 = [-b, 0)$.
Again divide S_2 into two segments $[-b, -b/2)$ and $[-b/2, 0)$. One of the segments contains a large number of elements of the sequence.
Denote this segment as S_3 and so on,

$$S_1 \supset S_2 \supset \dots$$

For each i $\{n \mid a_n \in S_i\} \in \mathcal{F}$.

The intersection $\bigcap_{i \geq 1} S_i = \{a\}$ consists of one point. Then

$$\lim_{\mathcal{F}} a_n = a. \qquad \text{I}$$

<u>Metric Ultraproducts.</u> Let (M_n, d_n), $n \in N$, be a sequence of metric spaces. We assume that all these metric spaces have finite diameters all all these diameters are uniformly bounded. In other words, $\exists$ $A > 0$:

$$\forall n \quad \forall x, y \in M_n \quad d_n(x,y) \leq A.$$

Let $\mathcal{F}$ be a nonprinciple ultrafilter in N. Consider the ultraproduct $\prod_{n \in N} M_n / \mathcal{F}$. Define

$$d\left((x_n)_{n \in N}/\mathcal{F}, (y_n)_{n \in N}/\mathcal{F}\right) = \lim_{\mathcal{F}} d_n(x_n, y_n)$$

The limit on the right hand side exists because the sequence is bounded.

Clearly, the triangle inequality holds:

$$d(x,z) \le d(x,y) + d(y,z).$$

Is d a metric?

Not necessarily. There may be elements $(x_n)_n/\mathcal{F} \neq (y_n)_n/\mathcal{F}$, but $\lim_{\mathcal{F}} d_n(x,y) = 0$

So, d is a premetric.

Embeddings into ultraproducts of finite fields.

In this section we will prove:

Theorem (Lefschetz Principle). A commutative domain is embeddable in an ultraproduct of finite fields.

Lemma (on ultraproduct of ultraproducts). Let $A \cong \prod_{x\in X} A_x/\mathcal{F}$, let $A_x \cong \prod_{y\in Y_x} A_{xy}/\mathcal{F}_x$.

Here Y_x is a set, $\mathcal{F}_x$ is an ultraproduct in Y_x.

Let $\tilde{X} = \dot{\bigcup}_{x \in X} Y_x$. Large sets: let Ω be a large subset of X, i.e. $\Omega \in \mathcal{F}$. For each $q \in \Omega$ let Ω_q be a large subset in Y_q i.e. $\Omega_q \in \mathcal{F}_q$. Consider the system S of nonempty subsets of $\tilde{X}$: $\dot{\bigcup}_{q \in \Omega} \Omega_q$. Any finite collection of subsets from S has a nonempty intersection. Hence S is embeddable in a filter and some ~~an~~ ultrafilter $\tilde{\mathcal{F}}$ in the set $\tilde{X}$.

Then $\prod_{x \in X} A_x / \mathcal{F} \hookrightarrow \prod_{x \in X, y \in Y_x} A_{xy} / \tilde{\mathcal{F}}$.

Consider $a = (a_x)_{x \in X} / \mathcal{F}$, $a_x = (a_{xy})_{y \in Y_x} / \mathcal{F}_x$.

Let $\varphi(a) = (a_{xy})_{x \in X, y \in Y_x} / \tilde{\mathcal{F}}$.

<u>Exercise</u>: φ is an embedding.

Various Form of Nullstellensatz.

(1) Let F be an algebraically closed field; let $f_1, \dots, f_m, g \in F[x_1, \dots, x_n]$ be polynomials.

An ~~m~~-tuple $(\alpha_1, \dots, \alpha_n) \in F^n$ is a root of $f_1, \dots, f_m$ if $f_1(\alpha_1, \dots, \alpha_n) = \dots = f_m(\alpha_1, \dots, \alpha_n) = 0$.

Suppose that for an arbitrary root $\alpha = (\alpha_1, \dots, \alpha_n)$ of $f_1, \dots, f_m$ we have $g(\alpha_1, \dots, \alpha_n) = 0$. So, every root of $f_1, \dots, f_m$ is also a root of g.

Then there exists $k \geq 1$ such that

$g^k \in id(f_1, \dots, f_m)$. In other words, there exist polynomials $h_1, \dots, h_m \in F[x_1, \dots, x_n]$ such that

$$g^k = \sum_{i=1}^{m} f_i h_i.$$

(2) If $A \ni 1$ is a finitely generated commutative domain then the intersection of all maximal ideals of A is (0).

(3) If a field is finitely generated as a ring then it is finite.

<u>Lemma.</u> A finitely generated commutative domain is embeddable in an ultraproduct of finite fields.

<u>Proof.</u> By (2) there exists a system of epimorphisms $\varphi_i : A \to F_i, i \in I$, F_i is a field, $\bigcap_i \operatorname{Ker} \varphi_i = (0)$.

By (3) all the fields F_i are finite.

For a nonzero element $a \in A$ let $J_a = \{ i \in I \mid \varphi_i(a) \neq 0 \}$. Then $J_a \cap J_b = J_{ab}$.

Hence the system of subsets $\{ J_a \mid 0 \neq a \in A \}$

lies in some ultrafilter $\mathcal{F}$ in I. The mapping $A \to \prod_{i \in I} F_i / \mathcal{F}$ is an embedding, $a \to (\varphi_i(a))_{i \in I} / \mathcal{F}$. $\perp$

<u>Theorem</u>. A commutative domain is embeddable in an ultraproduct of finite fields.

<u>Proof</u>. By Malcev's Theorem $A \hookrightarrow \prod_{i \in I} A_i$, A_i are finitely generated commutative domains. By the lemma above each A_i is embeddable in an ultraproduct of finite fields. Hence by the "lemma about ultraproduct of ultraproducts" the domain A is embeddable in an ultraproduct of finite fields. $\perp$

Ax-Groethendieck Theorem.

Theorem. Let F be an algebraically closed field. Then every injective polynomial mapping $P: F^n \to F^n$, $(\alpha_1, \dots, \alpha_n) \to (P_1(\alpha_1, \dots, \alpha_n), \dots, P_n(\alpha_1, \dots, \alpha_n))$; $P_1, \dots, P_n \in F[x_1, \dots, x_n]$, is surjective.

For finite fields F it is obvious: every injective map of a finite set into itself is surjective.

Injectivity means that $P(X) = P(Y) \Rightarrow X = Y$. Thus every root of the system $P_1(X) - P_1(Y) = 0, \dots, P_n(X) - P_n(Y) = 0$ is a root of $x_1 - y_1$.

By the Nullstellensatz there exists $k_1 \geq 1$

and polynomials $h_{1i}(x,y)$, $1 \le i \le n$, such that

$$(x_1 - y_1)^{k_1} = \sum_{i=1}^{n} (P_i(x) - P_i(y))\, h_{1i}(x,y).$$

Similarly for any t, $1 \le t \le n$, there exists $k_t \ge 1$ and polynomials $h_{t1}, \dots, h_{tn}$:

$$(x_t - y_t)^{k_t} = \sum_{i=1}^{n} (P_i(x) - P_i(y))\, h_{ti}(x,y).$$

Without loss of generality : $k_1 = \dots = k_n = k$.

<u>Finitary Form of Injectivity</u>: there exists $k \ge 1$ and polynomials $h_{ij}(x,y)$; $1 \le i,j \le n$, such that

$$(x_t - y_t)^{k} = \sum_{i=1}^{n} (P_i(x) - P_i(y))\, h_{ti}(x,y) \quad (*)$$

for any t, $1 \le t \le n$.

<u>Non-surjectivity</u> : $\exists \alpha = (\alpha_1, \dots, \alpha_n)$ not an image of P. In other words, the system $P_1(x) - \alpha_1 = 0, \dots, P_n(x) - \alpha_n = 0$ does not have roots. By the Nullstellensatz there exist polynomials $Q_1(x), \dots, Q_n(x)$, such that

$$\sum_{i=1}^{n} (P_i(x) - \alpha_i) Q_i(x) = 1 \qquad (**).$$

This is a <u>finitary form of Non-Surjectivity</u>.

Let us show that for any field F $(*)$ and $(**)$ are not compatible.

$F \hookrightarrow \prod_{i \in I} K_i / \mathcal{F}$, $|K_i| < \infty$, finite fields.

If $(*)$ and $(**)$ hold in $\prod_{i \in I} K_i / \mathcal{F}$ then

(*) and (**) hold on a "large" number of fields K_i. But for these fields K_i it means that a certain polynomial map is injective but not surjective, a contradiction.