Lecture 17.

Division Algebras.

In this section we will consider important examples of rings with Ore condition and examples of division algebras.

Recall that a ring (algebra) is called <u>left Noetherian</u> if every ascending chain of left ideals stabilizes.

<u>Theorem</u> . Let R be a Noetherian domain (domain means that there are no zero divisors: $ab=0 \Leftrightarrow a=0$ or $b=0$). Then R satisfies the Ore Condition with respect to $S=R\setminus\{0\}$.

<u>Proof</u>. We need to show that for any nonzero elements $a,b\in R$ $Ra\cap Rb\neq(0)$.

Let $a, b \in R$ be nonzero elements. Consider left ideals $L_n = Rb + Rba + \cdots + Rba^n$, $L_0 \subseteq L_1 \subseteq \ldots$. there exists $n \geq 1$ such that $L_{n-1} = L_n$, so an element $b^2 a^n$ from L_n lies in L_{n-1},

$$b^2 a^n = a_0 b + a_1 ba + \cdots + a_{n-1} b a^{n-1};$$

$a_i \in R$. Let $k \geq 0$ be a minimal index such that $a_k \neq 0$. Then

$$a_k b a^k = -a_{k+1} b a^{k+1} \cdots - a_{n-1} b a^{n-1} + b^2 a^n.$$

Cancelling a^k on the right we get $a_k b \in Ra$, so $Rb \cap Ra \neq (0)$.

Filtrations.

Let A be an F-algebra. An ascending chain of subspaces

$$A_0 \subseteq A_1 \subseteq \cdots$$

is called a filtration if

(1) $A_i A_j \subseteq A_{i+j}$

(2) $\bigcup_{i \geq 0} A_i = A$.

Consider the direct sum of factor spaces

$$gr(A) = A_0 \oplus A_1/A_0 \oplus A_2/A_1 \oplus \cdots$$

Define the multiplication

$$(a_i + A_{i-1})(b_j + A_{j-1}) = a_i b_j + A_{i+j-1}$$

It is easy to see that the product does not depend on a choice of representatives in cosets and therefore is well defined.

Sometimes the algebra A is called a <u>deformation</u> of the algebra $gr(A)$.

<u>Lemma</u>. (1) If the algebra $gr(A)$ is a domain then the algebra A is a domain as well.

(2) If the algebra $gr(A)$ is left (right) Noetherian then the algebra A is ~~l~~left (right) Noetherian as well.

<u>Proof</u>. (1) Let a, b be nonzero elements from A. Let $a \in A_i \setminus A_{i-1}$, $b \in A_j \setminus A_{j-1}$.

{We assume $A_{-1}=(0)$. Since $gr(A)$ is a domain we have $ab\in A_{i+j}\setminus A_{i+j-1}$, hence $ab\neq 0$.

(2) Let L be a left ideal of the algebra A. Let $\overline{L}_k=\{a+A_{k-1}\in A_k/A_{k-1},\ a\in L\cap A_k\}$, $k\geq 0$. It is easy to see that

$$\overline{L}=\sum_{k=0}^{\infty}\overline{L}_k$$

is a left ideal of the algebra $gr(A)$.

If $L\subseteq L'$ are left ideals in A then $\overline{L}\subseteq\overline{L}'$. Let us show that $\overline{L}=\overline{L}'$ implies $L=L'$. Indeed, choose $a\in L'$, $a\in A_k\setminus A_{k-1}$. We will use induction on k to show that $a\in L$.

The element $a + A_{k-1}$ lies in $\bar{L}_k' = \bar{L}_k$. Hence there exists an element $b \in L$, $b \in A_k \setminus A_{k-1}$, such that $a - b \in A_{k-1}$. By the induction assumption on k the element $a - b$ lies in L, hence $a \in L$. This completes the proof of the lemma.

Universal Enveloping Algebras.

Let L be a Lie algebra and let U be the universal enveloping algebra of L. Then $U = F \cdot 1 + L + L \cdot L + \cdots$

The subspaces

$$U_k = F \cdot 1 + L + L \cdot L + \cdots + \underbrace{L \cdots L}_{k}, \quad U_0 = F \cdot 1,$$

form a filtration of the algebra U.

Let $\{e_i\}_{i\in I}$ be a basis of the algebra L. Then the vector space U_n/U_{n-1} is spanned by elements $e_{i_1}\cdots e_{i_n}+U_{n-1}=(e_{i_1}+U_0)\cdots(e_{i_n}+U_0)$. Hence the algebra $gr(U)$ is generated by elements $\bar{e}_i=e_i+F\cdot 1\in U_1/U_0$. Let $[e_i,e_j]=\sum_k \gamma_{ij}^k e_k$, $\gamma_{ij}^k\in F$. Then

$$[\bar{e}_i,\bar{e}_j]=\sum_k \gamma_{ij}^k e_k\in U_1.$$

Hence generators $\bar{e}_i$ of the algebra $gr(U)$ commute, hence $gr(U)$ is a commutative algebra, $[U_p,U_q]\subseteq U_{p+q-1}$.

Let $<$ be an order on the set I

satisfying minimality condition. By the Poincare-Birkhoff-Witt theorem ordered products $\bar{e}_{i_1} \cdots \bar{e}_{i_k}$, $i_1 \leq \cdots \leq i_k$, form a basis of U_k / U_{k-1}. Hence $gr(U) \cong F[\bar{e}_i, i \in I]$, the polynomial algebra on generators $\bar{e}_i$, $i \in I$.

We showed that $U(L)$ is a deformation of a polynomial algebra (this is equivalent the the PBW-theorem).

Since the polynomial algebra is a domain Lemma (1) implies that the universal enveloping algebra $U(L)$ is a domain as well.

If $\dim_F L < \infty$ then by Hilbert's theorem the polynomial algebra $U_{gr}(L)$ is Noetherian. By Lemma (2) the universal enveloping algebra $U(L)$ is Noetherian.

By Theorem the universal enveloping algebra $U(L)$ has a ring of fractions, which is a division algebra.

For an infinite dimensional Lie algebra L the algebra $U(L)$ may not satisfy the Ore condition. Nevertheless P.M. Cohn proved that $U(L)$ is embeddable in a division algebra (which is not the division algebra of fractions of $U(L)$).

Now we will consider another important example.

Weyl Algebras.

Let $F[x_1, \dots, x_n]$ be a polynomial algebra

Consider the algebra

$$A_n = F\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$$

of course, here x_i means the operator R_{x_i} of multiplication by x_i. Then

$$A_n \subseteq \mathrm{Lin}_F F[x_1, \dots, x_n]$$

If $a \in A$ and $d : A \to A$ is a <u>derivation</u> then

$$d R_a(x) = d(xa) = d(x)a + x d(a) = R_a d(x) + R_{d(a)}.$$

Hence,

$$[d, R_a] = R_{d(a)}.$$

This implies that $[\partial/\partial x_i, x_j] = \delta_{ij}$, the Kronecker symbol. Also,

$$[x_i, x_j] = [\partial/\partial x_i, \partial/\partial x_j] = 0.$$

Ordered products

$$x_{i_1} \cdots x_{i_p} (\partial/\partial x_{j_1}) \cdots (\partial/\partial x_{j_q}),$$

$$i_1 \leq \ldots \leq i_p, \; j_1 \leq \cdots \leq j_q,$$

form a basis of the algebra A.

Now consider the algebra presented by generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations $R = \{[x_i, x_j]; [y_i, y_j], [y_i, x_j] - \delta_{ij}\}$.

Clearly, we have a homomorphism

$$F\langle X \mid R=0\rangle \to A_n, \quad x_i \to x_i,\ y_j \to \partial/\partial x_j$$

The system of relations R is closed with respect to compositions. Hence ordered products (= irreducible words)

$$x_{i_1}\cdots x_{i_p}\, y_{j_1}\cdots y_{j_q}, \quad i_1\le\cdots\le i_p,\ j_1\le\cdots\le j_q,$$

form a basis of $F\langle X \mid R=0\rangle$. This implies that the homomorphism above is an isomorphism,

$$A_n \cong F\langle X \mid R=0\rangle.$$

We remark also that

$$A_n \cong \langle x_1, \partial/\partial x_1\rangle \otimes_F \cdots \otimes_F \langle x_n, \partial/\partial x_n\rangle$$

$$\cong A_1 \otimes_F A_1 \otimes \cdots \otimes_F A_1.$$

Consider the filtration

$$V_k = \mathrm{Span}\left(x_{i_1}\cdots x_{i_p}\, \partial/\partial x_{j_1}\cdots \partial/\partial x_{j_q},\ p+q\le k\right)$$

It is easy to see that

$$[V_k, V_\ell] \subseteq V_{k+\ell-1},$$

hence the algebra $gr(A_n)$ is commutative an, in fact (see the basis!), isomorphic to the polynomial algebra in $2n$ variables. This implies that

(1) A_n is a deformation of a polynomial algebra,

(2) A_n is a Noetherian domain.

The ring of fractions of A_n is another example of an interesting division algebra.

New finite dimensional division algebra.

Until the beginning of the 20th century the only example of a division algebra that is not a field was the Hamilton's 4-dimensional algebra of quaternions.

Let F be a field, let φ be an automorphism of the field F, $|\varphi| = n$.

Consider the algebra D of Laurent series

$\sum_{k \geq -d} t^k \alpha_k$, $d \geq 1$, $\alpha_k \in F$, with multiplication

$$(t^p \alpha)(t^q \beta) = t^{p+q}(t^{-q} \alpha t^q)\beta,$$

$$t^{-q} \alpha t^q = \varphi^q(\alpha).$$

Lemma: (1) D is a division algebra.

(2) The center of D is

$Z = \{ \sum t^{kn} \alpha_{nk} \mid \alpha_{nk} \in F^{\langle \varphi \rangle}$, the subfield of fixed elements of $\varphi \}$.

(3) $|D : Z| = n^2$.

Proof. Let $a = \sum_{k \geq -d}^{\infty} t^k \alpha_k \neq 0$, $d \in \mathbb{Z}$. We assume that $\alpha_{-d} \neq 0$.

Then $a = t^{-d} \alpha_{-d} (1 + \sum_{i \geq 1} t^i (\dots))$. The elements t^{-d}, α_{-d} are invertible. Moreover,

$$\Big(1 + \sum_{i \geq 1} t^i (\dots)\Big)^{-1} = 1 - \sum_{i \geq 1} t^i (\dots) + \Big(\sum_{i \geq 1} t^i (\dots)\Big)^2 -$$

$$(\quad)^3 + \dots$$

We proved that D is a division algebra.

Let $\beta \in F$. The element β commutes with $\sum_{k=-d}^{\infty} t^k \alpha_k$ if and only if β commutes with each $t^k \alpha_k$, that is,

$$\beta t^k \alpha_k = t^k \varphi^k(\beta) \alpha_k = t^k \alpha_k \beta, \quad \varphi^k(\beta) = \beta.$$

Hence $\varphi^k = Id$, k is divisible by n.

Hence a central element looks as

$$z = \sum_k t^{kn} \alpha_{nk}$$

Now, $tz = \sum_k t^{kn+1} \alpha_{nk}$, $zt = \sum_k t^{kn} \alpha_{nk} t = \sum t^{kn+1} \varphi(\alpha_{nk})$. Hence $\alpha_{nk} = \varphi(\alpha_{nk})$, all the coefficients α_{nk} lie in the subfield $F^{\langle \varphi \rangle}$ of fixed elements.

We have $|F : F^{\langle 4 \rangle}| = n$. Let $\gamma_1, \ldots, \gamma_n$ be a basis of F over $F^{\langle 4 \rangle}$. Then

$$t^i \gamma_j, \quad 0 \le i \le n-1, \; 1 \le j \le n,$$

is a basis of D over Z.