Lecture 18.

Ultrafilters and Ultraproducts.

Let X be an infinite set. Let $S(X)$ be the set of all subsets of X.

A system of subsets $\mathcal{F} \subseteq S(X)$ is called a filter if

(1) $A \in \mathcal{F}$, $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{F}$,

(2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,

(3) $\emptyset \notin \mathcal{F}$.

Example. Let A_0 be a nonempty subset of X. Then $\mathcal{F} = \{A \subseteq X \mid A_0 \subseteq A\}$ is a filter. It is called a principle filter and it is not interesting.

Example. $\mathcal{F} = \{A \subseteq X \mid |X \setminus A| < \infty\}$ cofinite subsets.

It is a nonprinciple filter and it is very interesting.

<u>Example</u>. (generalyation of the previous example).

$\mathcal{F} = \{ A \subseteq X \mid \operatorname{card}(X \setminus A) < \operatorname{card} X \}$.

Let $\mathcal{F}_1, \mathcal{F}_2 \subseteq S(X)$ be filters. We say that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ as subsets of $S(X)$.

<u>Lemma</u>. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ be an ascending chain of filters in X. Then $\mathcal{F} = \bigcup_i \mathcal{F}_i$ is a filter.

<u>Proof</u>. Let $A \in \mathcal{F}$, $A \subseteq B \subseteq X$. Then $A \in \mathcal{F}_i$ for some i. Then $B \in \mathcal{F}_i \subseteq \mathcal{F}$.

Let $A, B \in \mathcal{F}$. Then $A, B \in \mathcal{F}_i$ for some i and therefore $A \cap B \in \mathcal{F}_i \subseteq \mathcal{F}$.

Finally, $\emptyset \notin \mathcal{F}$ because $\emptyset \notin \mathcal{F}_i$ for each i. $\downarrow$

<u>Corollary</u>. Zorn Lemma $\Rightarrow$ every filter in X

is embeddable in a maximal filter.

<u>Lemma</u>: Let S be a system of subsets in X. Then S is embeddable in a filter if and only if $\forall\, A_1, \dots, A_n \in S \quad A_1 \cap \dots \cap A_n \neq \emptyset$.

<u>Proof</u>. If $S \subseteq \mathcal{F}$, $\mathcal{F}$ is a filter, then $A_1 \cap \dots \cap A_n \in \mathcal{F}$, hence $A_1 \cap \dots \cap A_n \neq \emptyset$.

Suppose that the intersection of every finite subsystem of S is $\neq \emptyset$. Let

$$\mathcal{F} = \{B \subseteq X \mid \text{there exist } A_1, \dots, A_n \in S,\ n \geq 1,$$

$$\text{such that } A_1 \cap \dots \cap A_n \subseteq B\}.$$

This already implies that every set in $\mathcal{F}$ is nonempty, and that $B \in \mathcal{F}$, $B \subseteq C \subseteq X \Rightarrow C \in \mathcal{F}$.

Let $B, C \in \mathcal{F}$, $A_1 \cap \dots \cap A_n \subseteq B$; $A_1' \cap \dots \cap A_m' \subseteq C$, $A_i, A_j' \in S$. Then $A_1 \cap \dots \cap A_n \cap A_1' \cap \dots \cap A_m' \subseteq B \cap C$, which implies $B \cap C \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter.

$\mathcal{J}$.

<u>Lemma</u>. A filter $\mathcal{F}$ in X is maximal if and only if for an arbitrary subset $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

<u>Proof</u>. Let $\mathcal{F}$ be a filter in X. Let $A \subseteq X$. On of the following statements must be wrong:

- there exists $B \in \mathcal{F} : A \cap B = \emptyset$,
- there exists $B \in \mathcal{F} : (X \setminus A) \cap B = \emptyset$

Indeed, if the first statement is correct, so there exists $B \in \mathcal{F} : A \cap B = \emptyset$, then $B \subseteq X \setminus A$. In this case $X \setminus A \in \mathcal{F}$ and the 2d statement can not be correct.

Similarly, if there exists $B \in \mathcal{F}$ such that $(X \setminus A) \cap B = \emptyset$ then $B \subseteq A$ and therefore $A \in \mathcal{F}$.

Suppose that $\mathcal{F}$ is a maximal filter. Let $A \subseteq X$.
If the statement $\cdot$ is wrong then A has nonempty intersection with every subset from $\mathcal{F}$.
Then the system $S = \mathcal{F} \cup \{A\}$ satisfies the assumptions of the previous lemma. Hence S is embeddable in a filter $\mathcal{F}'$. By maximality of the filter $\mathcal{F}$ we have $\mathcal{F} = \mathcal{F}'$, hence $A \in \mathcal{F}$.
If the statement $\cdot\cdot$ we wrong we would conclude that $X \setminus A \in \mathcal{F}$.

We proved that if $\mathcal{F}$ is a maximal filter then for any subset $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Now suppose that $\mathcal{F}$ is a filter and the property above holds. We need to prove that the filter $\mathcal{F}$ is maximal. Let $\mathcal{F}'$ be another filter in X, $\mathcal{F} \subset \mathcal{F}'$, $\mathcal{F} \neq \mathcal{F}'$. Let $A \in \mathcal{F}' \setminus \mathcal{F}$.

Then by our assumption $X \setminus A \in \mathcal{F}$.* Since $\mathcal{F} \subseteq \mathcal{F}'$ we conclude that $X \setminus A \in \mathcal{F}'$, so both A and $X \setminus A$ live in $\mathcal{F}'$, a contradiction.

<u>Def.</u> A maximal filter in X is called an <u>ultrafilter</u>.

Every filter embedds in an ultrafilter.

Let A_i, $i \in X$, be a family of groups (rings, algebras, fields, ...) indexed by an infinite set X.

Let $\prod\limits_{i \in X} A_i$ be a Cartesian product.

Let $\mathcal{F}$ be a filter in the set X. Define a relation in $\prod\limits_{i \in X} A_i$: given elements $(a_i)_{i \in X}$, $(b_i)_{i \in X}$ from $\prod\limits_{i \in X} A_i$ we say that $(a_i)_{i \in X} \sim (b_i)_{i \in X}$ if the set $\{ i \in X \mid a_i = b_i \}$ lies in $\mathcal{F}$.

<u>Intuition</u>: sets from $\mathcal{F}$ are "large" subsets of X. An intersection of two large subsets is a large subset. Two elements from $\prod_{i \in X} A_i$ are related if they coincide on a large set of indices.

<u>Lemma</u>. $\sim$ is an equivalence and a congruence.

<u>Proof</u>. Let us prove ~~that~~ transitivity. If $(a_i)_{i \in X} \sim (b_i)_{i \in X}$ and $(b_i)_{i \in X} \sim (c_i)_{i \in X}$ then $(a_i)_{i \in X}$ and $(b_i)_{i \in X}$ coincide on a large subset X_1, $(b_i)_{i \in X}$ and $(c_i)_{i \in X}$ coincide on a large subset X_2. Then $(a_i)_{i \in X}$, $(c_i)_{i \in X}$ coincide on a large subset $X_1 \cap X_2$.

Now let us prove that $(a_i)_{i \in X} \sim (a_i')_{i \in X}$, $(b_i)_{i \in X} \sim (b_i')_{i \in X}$ implies $(a_i b_i)_{i \in X} \sim (a_i' b_i')_{i \in X}$.

Here juxtaposition stands for any operation in groups, rings, ... A_i. Let $a_i = a_i'$ for $i \in X_1$, $b_i = b_i'$ for $i \in X_2$, the subset X_1, X_2 are large. Then $a_i b_i = a_i' b_i'$ for $i \in X_1 \cap X_2$. $\perp$

<u>Def.</u> The group (ring, algebra, ...) $\prod_{i \in X} A_i / \sim$ is called the <u>filtered product</u> of groups (rings, ...) A_i, $i \in X$.

<u>Def.</u> If $\mathcal{F}$ is an ultrafilter then we talk about the <u>ultraproduct</u> and denote it as

$$\prod_{i \in X} A_i / \mathcal{F}.$$

Elements are denoted as $(a_i)_{i \in X} / \mathcal{F}$.

<u>Łoś Theorem.</u> Let $\mathcal{F}$ be an ultrafilter. Let $\Phi(x_1, x_2, \dots)$ be a formula. Then Φ holds on

$\prod_{i\in X} A_i/\mathfrak{F}$ if and only if for a large subset $X_1 \subset X$

Φ holds on all A_i, $i \in X_1$.

I won't define "formulas" and prove this theorem. In each particular case the theorem is obvious.

Examples of formulas.

$\forall x, y \quad xy = yx$

$\forall x \ \exists y \quad xy = yx = 1$

$\forall x \quad (x=0) \vee (\exists y : xy = yx = 1) \Leftrightarrow$ every nonzero element is invertible.

1 and 0 are special elements ("operations in 0 variables")

Theorem. Let A_i, $i \in X$, be groups (rings, fields, F-algebras, division rings). Then

$\prod_{i\in X} A_i/\mathcal{F}$ is a group (rings, fields, ...).

Proof. We will prove it for the case of fields.

The operations on $\prod_{i\in X} A_i/\mathcal{F}$ are $(a_i)_{i\in X}/\mathcal{F}$
$+ (b_i)_{i\in X}/\mathcal{F} = (a_i+b_i)_{i\in X}/\mathcal{F}$. Multiplication
is defined similarly

commutativity, associativity are inherited

from $\prod_{i\in X} A_i$.

Let $a=(a_i)_{i\in X}/\mathcal{F}$, $a\neq 0$. Then the subset

$X_1=\{i\in X \mid a_i\neq 0\}$ is large (otherwise $a=0$).

The element $(a_i^{-1})_{i\in X_1}/\mathcal{F}$ is the inverse of
a. We defined this element only on X_1. On
the small subset $X\setminus X_1$ we can define this
element in any (!) way. For example, by zeros.

Applications of Ultraproducts.

Local Theorems (A.I. Malcev)

Theorem. Every system (group, ring etc.) is embeddable in an ultraproduct of its finitely generated subsystems.

Proof. We will prove the theorem for groups. For other systems it is proved similarly.

Let G be a group. Let $S_0(G)$ be the system of finite nonempty subsets of G. For every $X \in S_0(G)$ let

$$S_X = \{Y \in S_0(G) \mid X \subseteq Y\}.$$

For any subsets $X_1, \ldots, X_n \in S_0(G)$ we have

$$S_{X_1} \cap \cdots \cap S_{X_n} \supseteq S_{X_1 \cup \cdots \cup X_n}.$$

Hence $S_{X_1} \cap \cdots \cap S_{X_n} \neq \emptyset$.

By one of the lemmas above the system of subsets $\{ S_X, X \in S_0(G) \}$ is embeddable in a filter and, therefore, in an ultrafilter $\mathcal{F}$ in the set $S_0(G)$.

For $X \in S_0(G)$ let $\langle X \rangle$ be the subgroup of G generated by the set X. Clearly, $\langle X \rangle$ is finitely generated. Consider the ultraproduct

$$\prod_{X \in S_0(G)} \langle X \rangle / \mathcal{F}$$

and the embedding

$$G \ni a \to \left(a \in \langle X \rangle \right)_{X \in S_{\{a\}}} / \mathcal{F}$$

The right hand side is defined only on a large subset $S_{\{a\}}$ of $S_0(G)$, but it is OK. You can extend the definition to the small set $S_0(G) \setminus S_{\{a\}}$ in an arbitrary way.

Exercise: check that this is an embedding.

This completes the proof of the theorem.

Theorem. An ultraproduct $\prod_{i\in X} K_i/\mathcal{F}$ of fields is a field. Moreover, either there exists a prime number p such that $\{i\in X \mid \operatorname{char} K_i = p\}$ is large or $\operatorname{char} K = 0$. In particular, for any ultrafilter $\mathcal{F}$ in the set $\mathbb{P}$ of prime numbers the ultraproduct $\prod_{p\in\mathbb{P}} (\mathbb{Z}/p\mathbb{Z})/\mathcal{F}$ has zero characteristic.

Proof. Obvious.

A group G is called <u>n-linear</u> if there exists a field K such that $G \hookrightarrow GL(n,K)$. A group is called <u>linear</u> if it is n-linear for some $n \geq 1$.

<u>Theorem</u> <u>(Malcev)</u> (1) If every finitely generated subgroup of G is n-linear then the group G is n-linear.

(2) If R is a domain and every finitely generated subring of R is embeddable in a division ring then R is embeddable in a division ring.

<u>Proof</u>. (1) We know that $G \hookrightarrow \prod_{i \in X} G_i / \mathcal{F}$, where G_i's are finitely generated subgroups of G. There exist embeddings $G_i \hookrightarrow GL(n, K_i)$, K_i's are fields.

Then $G \hookrightarrow \prod_{i \in X} GL(n, K_i) / \mathcal{F} \cong GL(n, \prod_{i \in X} K_i / \mathcal{F})$.

The latter isomorphism is an exercise.

(2) Let R be a domain, $R \hookrightarrow \prod_{i \in X} R_i / \mathcal{F}$, where R_i's are finitely generated subrings subrings of R. There exist embeddings $R_i \hookrightarrow D_i$, D_i's are division rings.

Then $G \hookrightarrow \prod_{i \in X} D_i / \mathcal{F} = D$. The ring D is a division ring ↓

A system of axioms is said to be <u>consistent</u> if it has a model.

<u>Compactness Theorem (Malcev)</u>. Given a family of axioms, if every finite collection of axioms is consistent, then the whole system is consistent.

<u>Proof</u>. Let Ω be a system of axioms. As above, let $S_0(\Omega)$ be the system of all nonempty subsets of Ω. For $X \in S_0(\Omega)$ let $S_X = \{Y \in S_0(\Omega) \mid X \subseteq Y\}$. Then there exists an ultrafilter $\mathcal{F}$ in the set $S_0(\Omega)$, such that all S_X's lie in $\mathcal{F}$.

Let A_X be a model for the set of axioms X. The ultraproduct $A = \prod_{X \in S_0(\Omega)} A_X / \mathcal{F}$ is a model for Ω. $\perp$

~~Nonstandard Analysis.~~

~~Old problems with~~ $\cancel{\dfrac{f(x+\varepsilon) - f(x)}{\varepsilon}}$.