2.7 Show that any open subset of a metric space (X, d) can be written as the union of open balls. [Hint: if U is open, then for each $x \in U$ there exists $r(x) > 0$ such that $x \in B(x, r(x)) \subseteq U$.]

Given any open subset $\mathcal{U}$ of (X, d), then for $\forall x \in \mathcal{U}$, $\exists\, r_x > 0$ s.t. $B(x, r_x) \subset \mathcal{U}$. Then we have $\bigcup_{x\in\mathcal{U}} B(x, r_x) \subset \mathcal{U}$. On the other hand, for $\forall y \in \mathcal{U}$, $y \in B(y, r_y) \subset \bigcup_{x\in\mathcal{U}} B(x, r_x)$ thus $\mathcal{U} \subset \bigcup_{x\in\mathcal{U}} B(x, r_x)$. Therefore $\mathcal{U} = \bigcup_{x\in\mathcal{U}} B(x, r_x)$, which is the union of some open balls. □

2.9 Show that if (X, d_X) and (Y, d_Y) are separable, then $(X \times Y, \varrho_p)$ is separable, where ϱ_p is any one of the metrics from Exercise 2.2.

Suppose $A \subset X$, $B \subset Y$ where A, B are both countable and $\overline{A} = X$, $\overline{B} = Y$. Consider $A \times B$ in $(X \times Y, \rho_p)$, which is also countable.

Given $\forall (x, y) \in X \times Y$, we have $x \in X = \overline{A}$, $y \in Y = \overline{B}$. Then there exist $\{a_n\} \subset A$, $\{b_n\} \subset B$ s.t. $a_n \to x$, $b_n \to y$. That is, $d_X(a_n, x) \to 0$ $d_Y(b_n, x) \to 0$

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $d_X(a_n, x) < \varepsilon$ whenever $n > N_1$; $\exists N_2 \in \mathbb{N}$ s.t. $d_Y(b_n, x) < \varepsilon$ whenever $n > N_2$. Take $N = \max\{N_1, N_2\}$, then for $n > N$,

1° $1 \le p < \infty$

$$\rho_p((a_n, b_n), (x, y)) = (d_X(a_n, x)^p + d_Y(b_n, y)^p)^{\frac{1}{p}} < (\varepsilon^p + \varepsilon^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}\varepsilon$$

2° $p = \infty$

$$\rho_p((a_n, b_n), (x, y)) = \max\{d_X(a_n, x), d_Y(b_n, y)\} < \varepsilon$$

Both implies that $\rho_p((a_n, b_n), (x, y)) \to 0$, thus $(x, y) \in \overline{A \times B}$.
Since $(x, y) \in X \times Y$ is arbitrary, $\overline{A \times B} = X \times Y$
Hence, $(X \times Y, \rho_p)$ is separable. □

2.10 Suppose that $\{F_\alpha\}_{\alpha\in\mathbb{A}}$ are a family of closed subsets of a compact metric space (X, d) with the property that the intersection of any finite number of the sets has non-empty intersection. Show that $\bigcap_{\alpha\in\mathbb{A}} F_\alpha$ is non-empty.

If $\bigcap_{\alpha\in\mathcal{A}} F_\alpha = \phi$, then $X = \bigcup_{\alpha\in\mathcal{A}} F_\alpha^c$. Notice that $\{F_\alpha^c\}_{\alpha\in\mathcal{A}}$ is an open covering of X, since X is compact, there exists a finite subcovering, denoted by $\{F_{\alpha_i}^c\}_{i=1}^N$. That is, $X = \bigcup_{i=1}^N F_{\alpha_i}^c$. This implies that $\bigcap_{i=1}^N F_{\alpha_i} = \phi$, contradicting with the property that the intersection of finitely many sets of $\{F_\alpha\}_{\alpha\in\mathcal{A}}$ is nonempty. Hence, $\bigcap_{\alpha\in\mathcal{A}} F_\alpha$ is nonempty. □

2.11 Suppose that (F_j) is a decreasing sequence $[F_{j+1} \subseteq F_j]$ of non-empty closed subsets of a compact metric space (X, d). Use the result of the previous exercise to show that $\cap_{j=1}^\infty F_j \neq \varnothing$.

For any finitely many sets of (F_j), denoted by $F_{i_1}, \dots, F_{i_n}$, where $i_1 < i_2 < \dots < i_n$, since (F_j) is decreasing, $\bigcap_{j=1}^n F_{i_j} = F_{i_n} \neq \phi$, thus (F_j) has the property as in the previous exercise.
Hence, $\bigcap_{j=1}^\infty F_j \neq \phi$. □

2.12 Show that if S is a closed subset of $\mathbb{R}$, then $\sup(S) \in S$.

We first show $\sup S \in S$ if S is bounded. If S is bounded, then S is compact, consider $f: S \to \mathbb{R}$ by setting $f(s) = s$ $(\forall s \in S)$. It is obvious that f is continuous, thus f attains its supremum. That is, $\sup f(S) \in f(S)$. Since $f(S) = S$, $\sup S \in S$.

1° S is upper bounded.
Take $s_0 \in S$, and define $S_0 = \{s \in S : s \ge s_0\}$
Then $S_0 = S \setminus (-\infty, s_0)$ which is closed and bounded.
From 1° we know $\sup S_0 \in S_0$. Thus $\sup S = \sup S_0 \in S_0 \subset S$.

2° S is not upper bounded.
Then $\sup S = \infty \in S$.
Hence $\sup S \in S$ whenever S is closed. □

2.13 Show that if $f : (X, d_X) \to (Y, d_Y)$ is a continuous bijection and X is compact, then f^{-1} is also continuous (i.e. f is a homeomorphism).

For any closed subset F of X, since X is compact, F is also compact. Then $(f^{-1})^{-1}(F) = f(F)$ is also compact (since f is a bijection), then $f(F)$ is closed. Hence, f^{-1} is continuous. □

2.14 Any compact metric space (X, d) is separable. Prove the stronger result that in any compact metric space there exists a countable subset $(x_j)_{j=1}^\infty$ with the following property: for any $\varepsilon > 0$ there is an $M(\varepsilon)$ such that for every $x \in X$ we have

$$d(x_j, x) < \varepsilon \qquad \text{for some } 1 \le j \le M(\varepsilon).$$

For each $n \in \mathbb{N}$, the collection $\{B(x, \frac{1}{n})\}_{x\in X}$ is an open covering of X. Since X is compact, it has a finite subcovering $\{B(x_i^{(n)}, \frac{1}{n})\}_{i=1}^{N_n}$.

Set $S_0 = 0$ $S_n = \sum_{i=1}^n N_i$ $(n \ge 1)$. We choose a countable sequence as $x_j = x_{j-S_{n-1}}^{(n)}$ $(S_{n-1} < i \le S_n)$. Then for $\forall \varepsilon > 0$, choose $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, and set $M(\varepsilon) = S_n$. For $\forall x \in X$, since $\{B(x_i^{(n)}, \frac{1}{n})\}_{i=1}^{N_n}$ covers X, $\exists\, 1 \le i \le N_n$ s.t. $d(x, x_i^{(n)}) < \frac{1}{n}$. That is, $d(x, x_{i+S_{n-1}}) < \frac{1}{n} < \varepsilon$ and $1 \le i + S_{n-1} \le N_n + S_{n-1} = S_n = M(\varepsilon)$. □